M.C. Escher Kaleidocycles

kalos [beautiful] + *eîdos* [form] + *kŷklos* [ring]

Doris Schattschneider
Wallace Walker

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Sources of Illustrations:

The collection of the Haags Gemeentemuseum: Figures 16, 18, 19, 24, 30a, 33, 36, 38, 41, 49, 52.

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The National Gallery of Art, Rosenwald Collection: Figure 23.

The Collection of Michael S. Sachs, Inc., P.O. Box 2837, Westport, Connecticut 06880: Figures 20, 32, 44a, 46, 50, 54.

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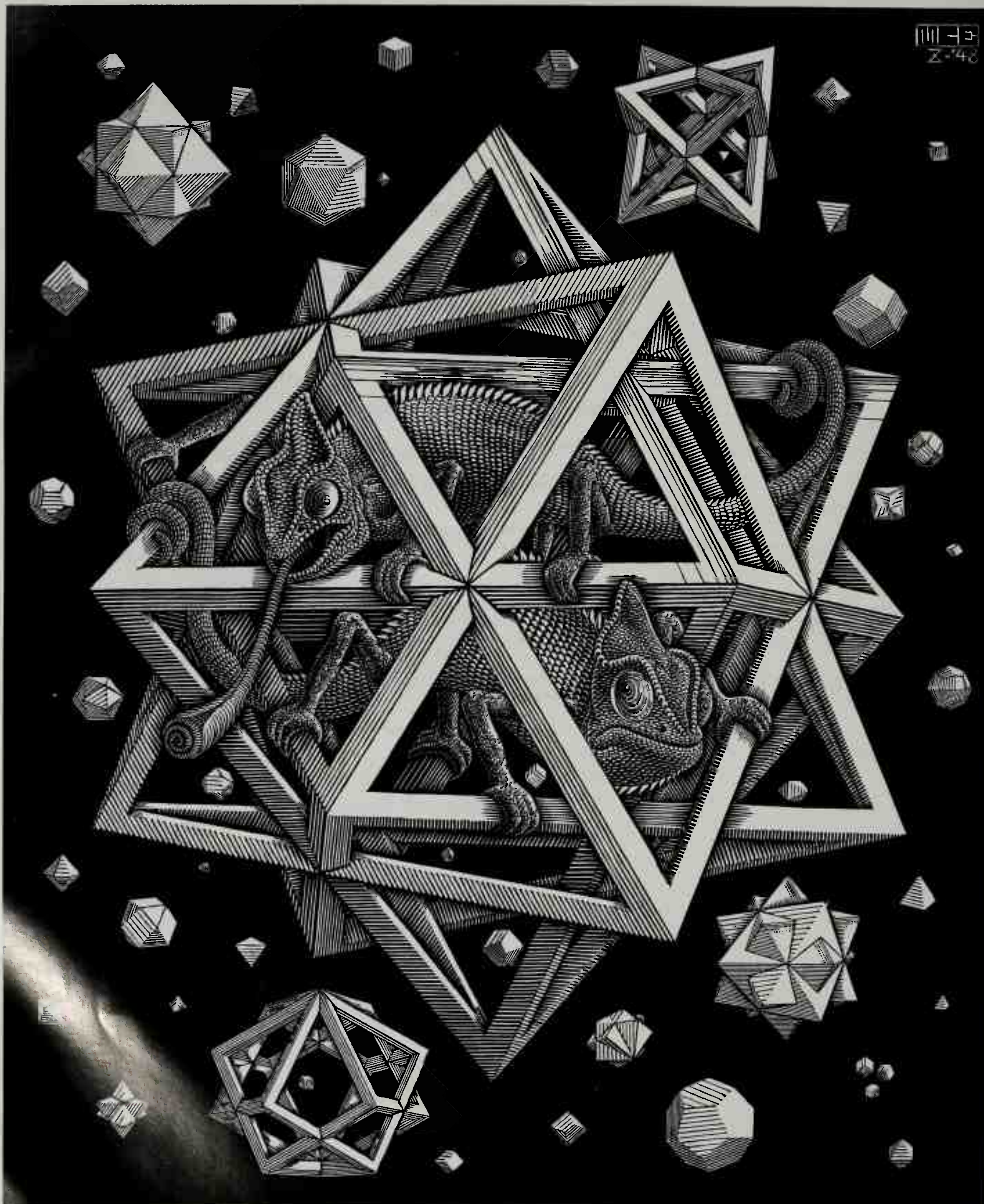
Note on Captions: M.C. Escher did not assign titles to his colored drawings and watercolors of periodic designs, but dated them and numbered them consecutively. In a few cases (*Three Elements, Shells and Starfish,* and *Heaven and Hell*) names came to be associated with the drawings. In the text, we identify each periodic design with Escher's number and date (month and year) and give a descriptive name.



1 *Stars,* wood engraving, 1948.

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3 *Reptiles,* lithograph, 1943.

In Three Dimensions: Extensions of M.C. Escher's Art

Doris Schattschneider

Everyone loves surprises. There are two types of surprises—the one is a happy accident or coincidence; the other is meticulously planned, perhaps cunningly disguised to appear natural, and brings double pleasure. It is often hard to say who has greater delight—the person who is surprised or the one who devised the magic.

The Dutch artist M.C. Escher (1898-1972) was an ingenious planner of many surprises of the second kind. His graphic art fairly bursts with cunningly planned visual surprises. At first glance, much of his work appears natural, yet, at second glance, the seemingly plausible is seen to be impossible and the viewer is drawn to look again and again as he discovers with delight the hidden surprises the work contains.

How did Escher do it? He was a genius of imagination, a skilled graphic craftsman, but the key to many of his surprising effects is mathematics. Not the mathematics of numbers and equations that most of us envision, but geometry in all aspects, both classical and modern. Escher could imagine the fantastic effects he wished to express graphically, but a necessary tool to capture these effects was mathematics. For this reason, he read technical works and corresponded with mathematicians and crystallographers—all the while disclaiming his ability to understand mathematics and yet visually expressing his understanding of the vital principles he needed.

The kaleidoscopically designed geometric forms in this collection are a continuation and extension of Escher's own work. Covered with adaptations of Escher's designs, they embody many of the themes dominant in his prints and are related to his own explorations of three-dimensional expression. In light of his work, it is not surprising that these creations required the collaboration of a mathematician and a graphic designer.

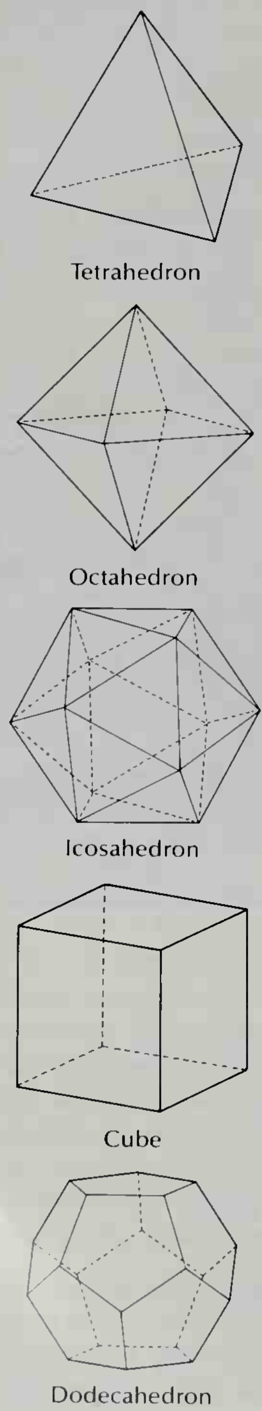
Your involvement is required also! A casual glance cannot reveal the surprises to be discovered in Escher's prints. So, too, the secrets to be discovered in our models are only revealed by your creating the forms, examining them, and yes, playing with them! Each geometric model begins as a flat design; you bring it to "life", providing the transition from its two-dimensional to its three-dimensional state. Once brought to life, the models provide many surprises for both hand and eye—the two-dimensional pattern gives few clues as to what you will see and feel when it takes shape in three dimensions.

Before assembling the models, you are invited to explore their various aspects: form, design, color and the relationship to Escher's own work. The journey will touch on many of the paths explored by Escher, as well as some newly discovered.

Geometric Solids

The *stars* which hang in Escher's black outer space (Figure 1) are geometric forms with the symmetry of faceted jewels. Surrounded by geometric models in his studio, Escher readily admitted his awe of these forms and included them in many of his prints.

Admired and exalted from earliest times, the Platonic solids (Figure 2) are the most perfectly symmetric of all convex polyhedra. How perfect? Each form is faceted with copies of a single regular polygon—all sides and all angles of all faces are equal. In addition, every corner (vertex) of the solid is the same—the same number of faces meet there, and the angle at which adjoining faces are inclined to each other is always the same. Mathematicians call these *regular* polyhedra. In each of them, every possible requirement that aspects of the form be alike (congruent) is fulfilled. These requirements are so severe that only five forms can meet them all.

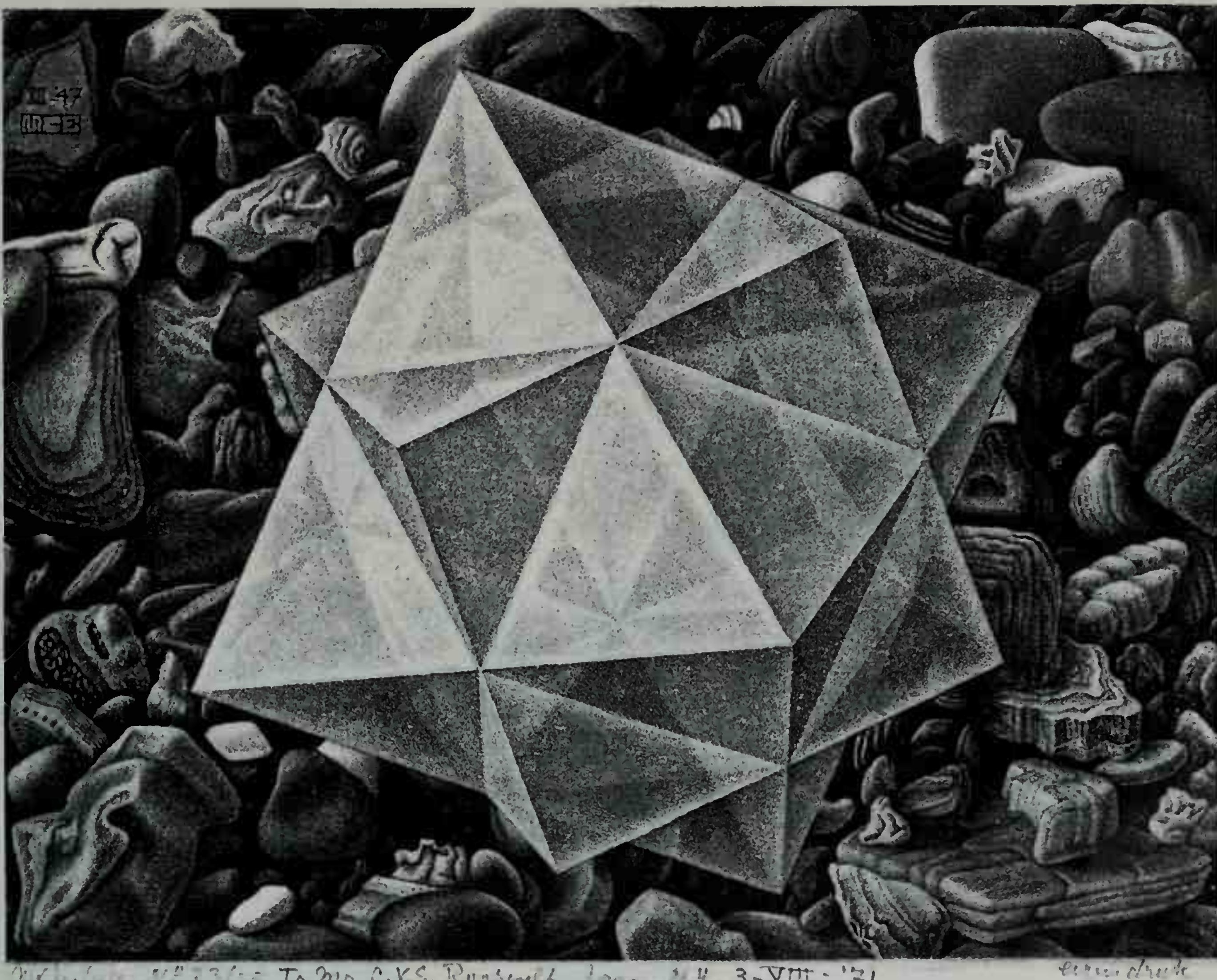


2 The five Platonic So[illegible]rahedron: four faces. Octahedron: eigh[illegible]cosahedron: twenty faces. Cube: six [illegible]odecahedron: twelve faces.

Their names come from the original Greek and tell us how many faces they possess. Three are faceted with equilateral triangles: *tetrahedron, octahedron, icosahedron.* The *cube*, faceted with squares at right angles to each other, is the most familiar form of all. Its name comes from the Greek word for "dice"—*cubos.* The *dodecahedron* is faceted with pentagons; it is perhaps the most admired since it is the least obvious of the solids to imagine. Escher's little dragon, at the pinnacle of his fanciful journey, gives a snort of power atop a dodecahedron in the print *Reptiles* (Figure 3).

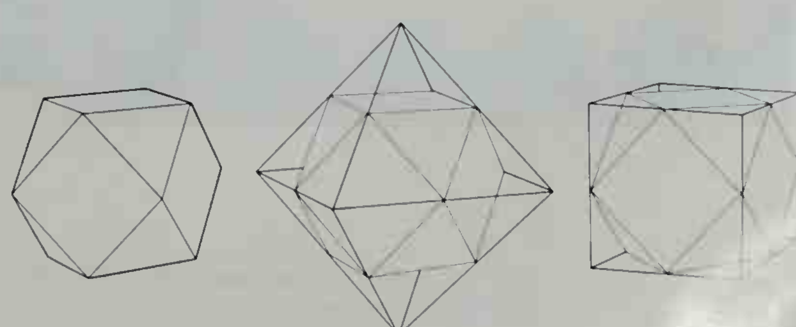
If just one of the requirements for a solid to be regular is removed, a large collection of other highly symmetric forms can be discovered. The Archimedean (or semi-regular) polyhedra allow two or more types of regular polygons as faces but fulfill all other requirements of regular polyhedra. Thus all sides of all faces are equal; every corner of the solid is the same (surrounded in the same manner by the faces which meet there). There are thirteen semi-regular solids; we have included just one in our collection.

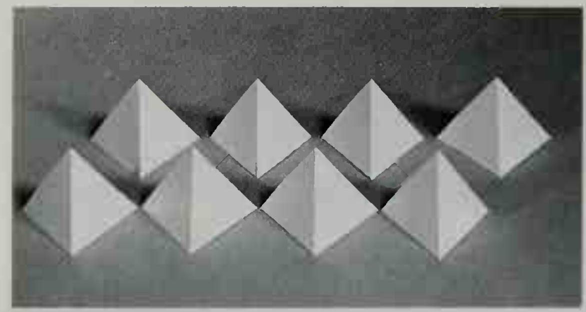
The cuboctahedron (cube + octahedron) has as its faces six squares and eight equilateral triangles (Figure 4). Each of its twelve vertices (corners) is surrounded by two triangles and two squares arranged so that the two squares separate the two triangles. The name of this solid suggests its derivation—it can arise either by slicing off corners of a cube or by slicing off corners of an octahedron. It can also arise as the solid which is the intersection of an octahedron with a cube as shown in Escher's print *Crystal* (Figure 5).



5 *Crystal,* mezzotint, 1947.

4 The Cuboctahedron can be obtained from a Cube or an Octahedron by slicing off their corners (always cutting through midpoints of the edges which meet at a corner). If all protruding corners are sliced off Escher's *Crystal* (Figure 5), which is an interlocked Cube and Octahedron, the Cuboctahedron is the resulting solid.





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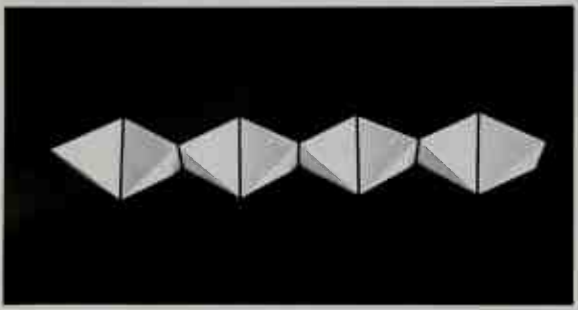


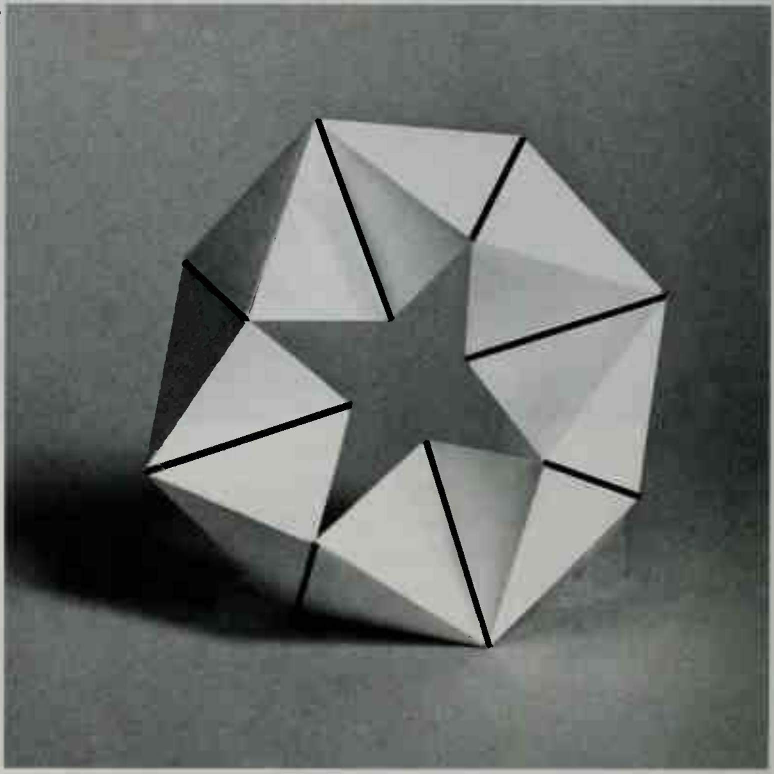
6 Linking together several Tetrahedra along their edges produces a ring of Tetrahedra. The Kaleidocycles can arise in this way.

Kaleidocycles

A Kaleidocycle is a three-dimensional ring made from a chain of tetrahedra. Begin with several tetrahedra, all exactly alike (Figure 6a). Hinge together two of these along an edge of each tetrahedron to begin a chain of tetrahedra, each one linked to an adjoining one along an edge (Figure 6b). When the chain of tetrahedra is long enough, the ends can be brought together to form a closed circle (Figure 6c). The hinges of the chain allow the ring to be turned through its center in a continous motion.

Contrary to the impression given by most textbooks, the discovery of new forms and new ideas is rarely the product of a predictable evolution. An unexpected discovery is made, and much later, the new knowledge is placed in its "natural" position with respect to the larger body of knowledge of which it becomes a part. So, too, with the discovery of Kaleidocycles. Although they naturally evolve in the manner described above, it was an extremely different analysis that led to their discovery.

6c

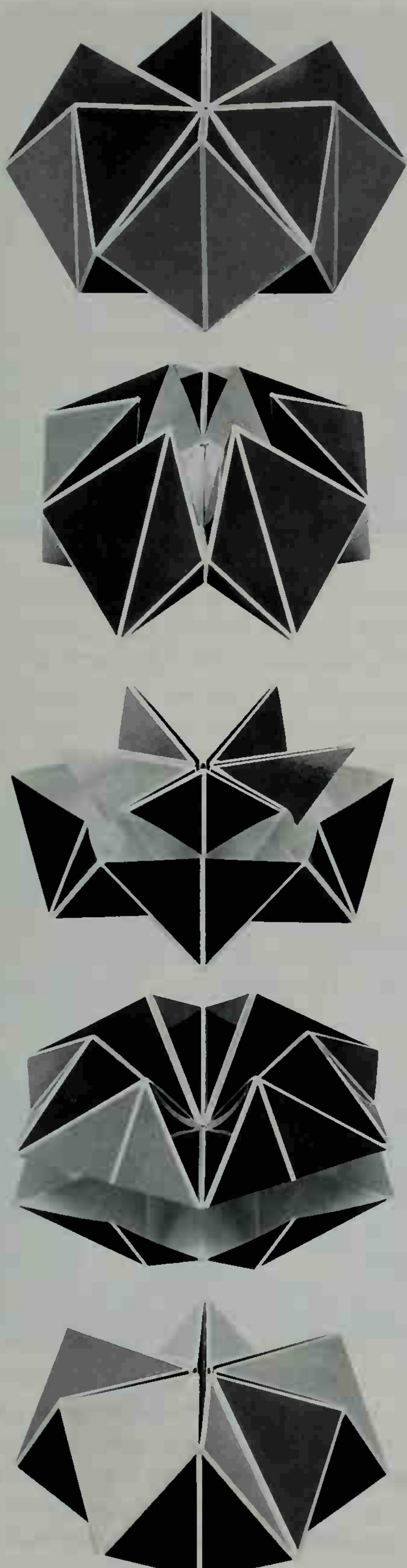


First came the form IsoAxis® (U.S. Patent no. 3302321), invented by Wallace Walker, a graphic designer. Walker created IsoAxis® as a solution to a structural paper design project in 1958 while a student at Cranbrook Academy of Art in Michigan. In two dimensions, IsoAxis® consists of a grid of sixty connected isosceles right triangles (Figure 7). In this two-dimensional state, the pattern gives no hint of its surprising three-dimensional form. When folded on the grid lines and assembled into a three-dimensional ring, IsoAxis® assumes a bold symmetrical shape. Finally, astonishingly, the form can be rotated through its center, at each turn transforming its appearance (Figure 8). After five rotations, it resumes its first shape and the cycle of transformation can begin again.

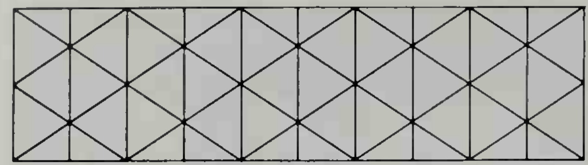
A mathematical mind is quick to ask: What relation does the two-dimensional grid have to the three-dimensional form it produces? What happens if the grid is altered? Investigation of these questions by the author, a mathematician, led to the discovery of an infinite family of three-dimensional forms. The Kaleidocycles in our collection are members of this family.

The grid of lines which forms the flat pattern of IsoAxis® is like an adjustable wooden gate or a hatrack which can be stretched or collapsed. Stretching or collapsing the IsoAxis® grid produces new flat patterns which can be folded up in the same manner as IsoAxis® to form a ring faceted with triangles. Surprisingly, these new forms also rotate through the center of the ring. *Kaleidocycles* seems an appropriate name for these highly symmetric forms which turn cartwheels in an endless cycle (Greek: *kalós* [beautiful] + *eîdos* [form] + *kŷklos* [ring]).

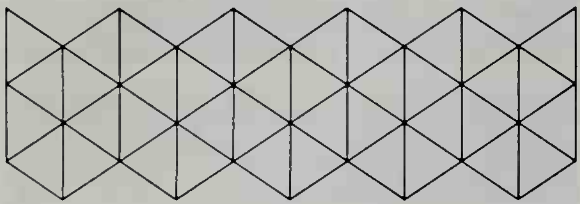
7 IsoAxis begins as a flat grid of isosceles triangles.



8 IsoAxis folds into the solid shown and then transforms as its facets are pushed through its center.

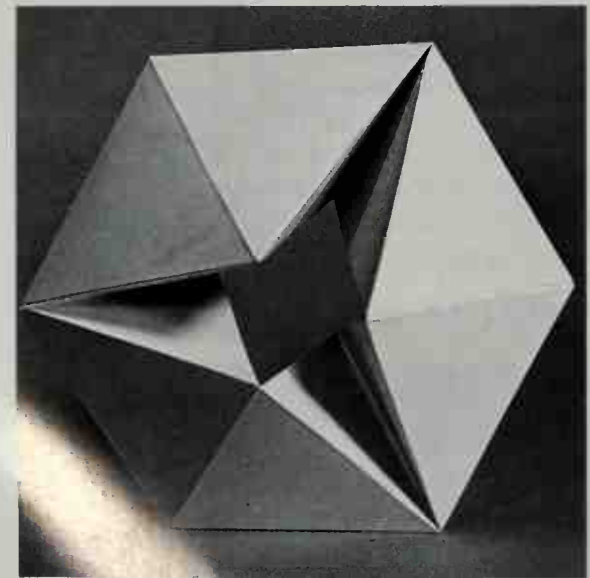


9 A grid produced by stretching the IsoAxis pattern.



10 This grid has all triangles congruent; when folded and assembled, it produces the same ring of Tetrahedra as the grid shown in Figure 9.

11a A Hexagonal Kaleidocycle.



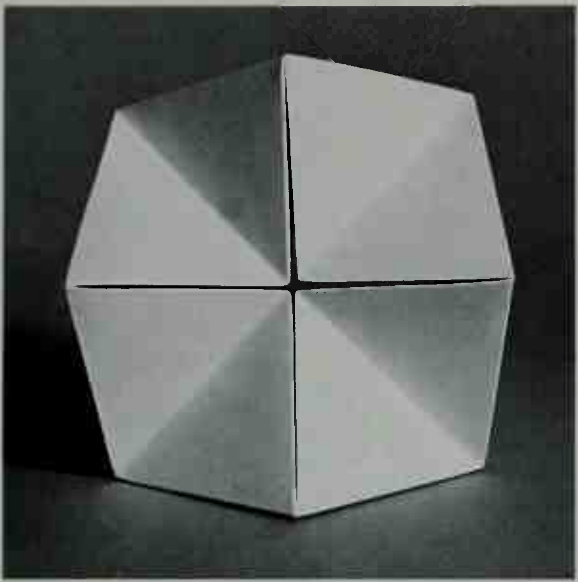
The Kaleidocycles created when the IsoAxis® grid is collapsed look like many-faceted folded-paper Danish lampshades. They have great flexibility, and seem to bloom like flowers when rotated.

When the IsoAxis® grid is stretched (Figure 9), all of the angles in the triangles formed are less than right angles. When this pattern is folded into three-dimensional form, it forms a ring of linked tetrahedra! By moving the small triangles at the top of the grid to join those at the bottom, yet another new grid (Figure 10) is formed which produces the same three-dimensional form. Each vertical "strip" of four congruent triangles folds into a single tetrahedron, and the vertical lines on the flat pattern become the hinges of the linked chain when it is folded. By adjusting the amount of stretch of the grid, or adding more triangles, an infinite variety of rings of linked tetrahedra can be created.

More questions were asked: How few tetrahedra can form a closed ring? How small can the hole in the center of the ring be made? Experimentation, guided by facts of plane and solid Euclidean geometry, answered these questions. In order to form a closed ring, at least six tetrahedra are required. The hole in the center can be made (theoretically, at least) as small as a point and still the tetrahedra will tumble through it. Finally, rules were worked out for the construction of triangles which produce a ring having a point as center hole.

Now many beautiful Kaleidocycles could be constructed. The first two in the family of forms having a point as a center hole have familiar outlines when viewed from above. The Kaleidocycle whose ring is formed by six tetrahedra has the outline of a regular hexagon (Figure 11a), white the ring of eight tetrahedra has the outline of a square (Figure 11b). As the number of tetrahedra in a Kaleidocycle is increased, these outlines become like many-petaled flowers or stars (Figure 12).

All the Kaleidocycles discovered thus far were beautifully symmetric. As they turned, the triangular faces met, matched, and then parted. Another question could not be ignored: Could other grids of triangles produce a different kind of Kaleidocycle? Again, experimentation and use of geometry produced an answer. A slanted grid of triangles would produce a twisted ring of tetrahedra (Figure 13). Each twisted Kaleidocycle has a jagged, uneven appearance and its tetrahedra seem to tumble through the center hole one at a time in sequence.



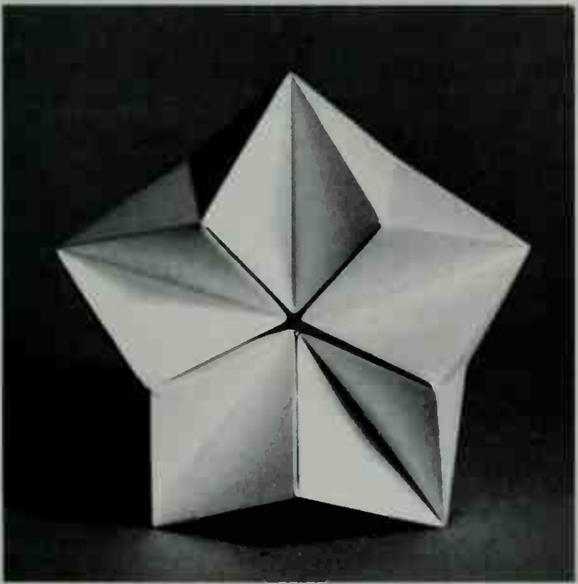
11b A Square Kaleidocycle.

The many-faceted Kaleidocycles seem to invite surface design. Many pleasing geometric effects can be produced by coloring the facets or adding lines to accentuate the kaleidoscopic nature of their movement.

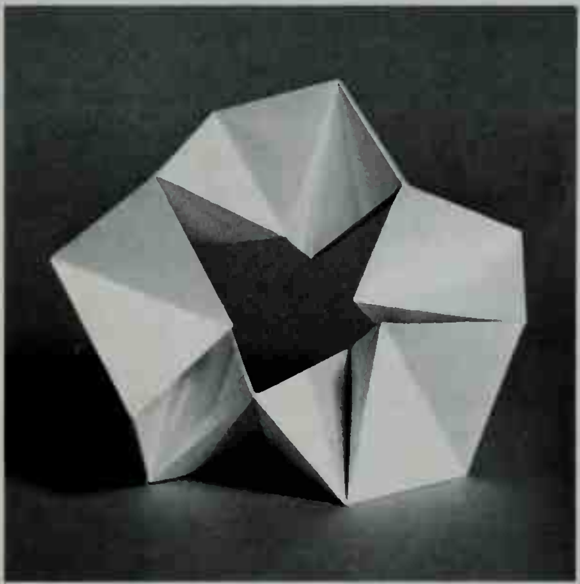
While investigating these forms, the author was also studying and teaching the mathematical art of repeating patterns. The designs of M.C. Escher were a major source of illustrations in this study. Fragments of these patterns appear in many of his prints. Observing his prints, it was clear that the Kaleidocycles captured two of Escher's dominant themes—a closed cycle and endless movement. *Reptiles* (Figure 3) and *Encounter* (Figure 14) are just two of many prints by Escher that contain part of an interlocking, repeating pattern and suggest an endless cycle of motion.

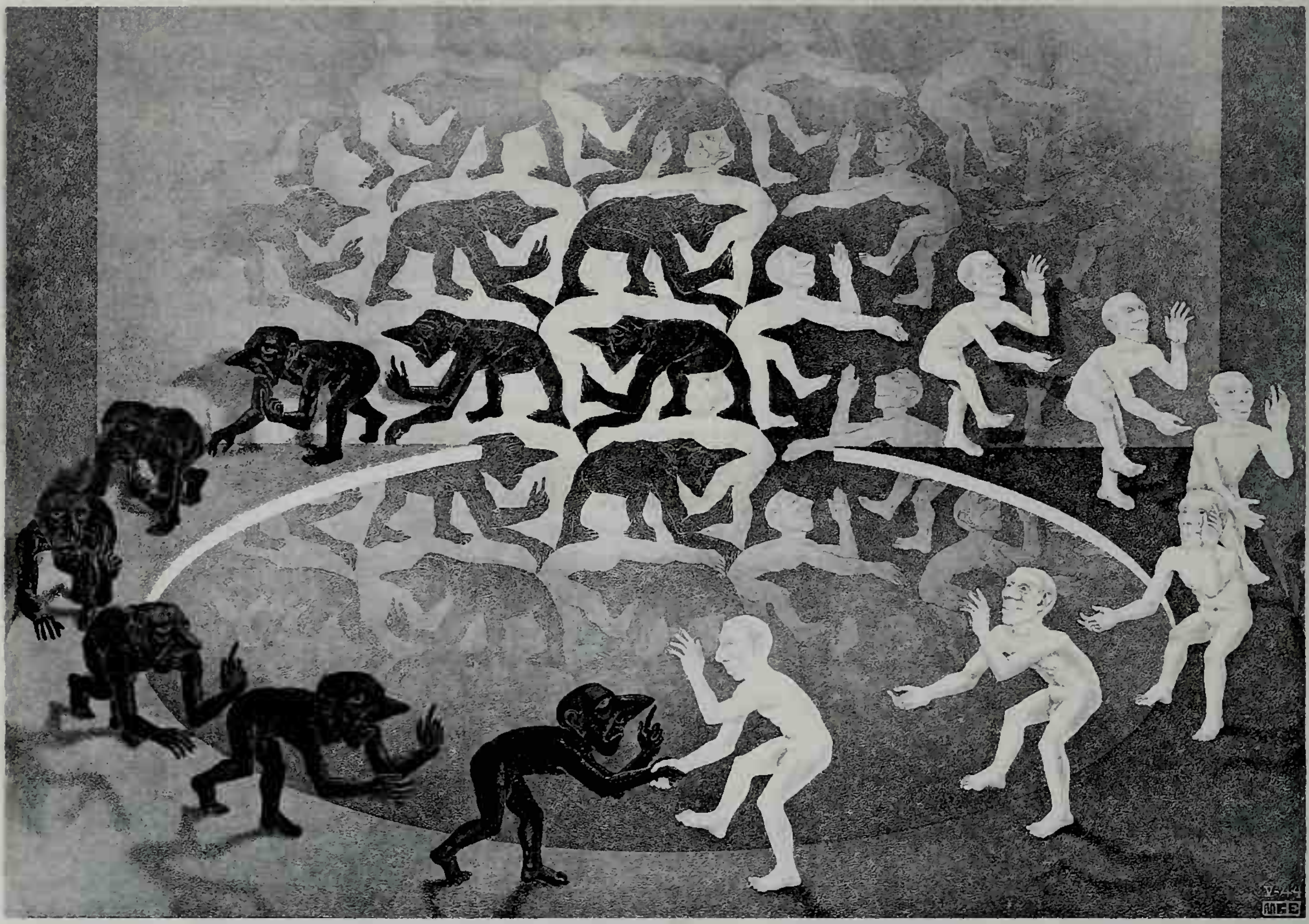
An idea for surface design on the Kaleidocycles came naturally—could they be covered continuously with Escher's designs? If so, they would simultaneously bring to life in three dimensions many aspects of Escher's work. Ultimately, the answer was yes, but in order to understand how it was accomplished, we must first discuss some aspects of repeating patterns.

12 A starlike Kaleidocycle of ten Tetrahedra.



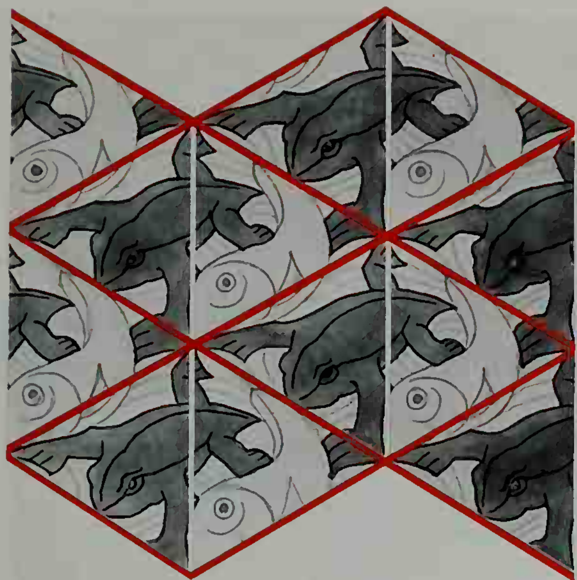
13 A Twisted Kaleidocycle.





14 *Encounter,* lithograph, 1944.

Repeating Designs



15 Periodic design 50, VII 1942. Fish and frog study for *Verbum* (Figure 47). Choose a point in a periodic pattern (the point chosen here is where the legs of three frogs meet), and find all repetitions of that point. This array of points is the *lattice* of the pattern. Joining rows of points forms a parallelogram grid (red lines); these parallelogram tiles are all alike and reproduce the original design. Bisecting the parallelograms (white lines) produces a triangular grid. Each parallelogram contains exactly one frog and one fish (even though the lines break up motifs.)

Everyone knows what it means to tile a floor: many tiles, all alike, or perhaps of a few different shapes, are fitted together like a jigsaw puzzle to cover the floor without gaps or overlaps. Although the tiles can be placed in a somewhat random manner, usually they form a pattern which repeats itself at regular intervals. Such a pattern is called a repeating or periodic tiling of the plane (plane tessellation). These tilings were a constant preoccupation of M.C. Escher—even he called his fascination "a hopeless mania."*

The formation of such tilings is a child's exercise if the tiles are restricted to the three shapes of equilateral triangle, square, and hexagon. Escher's self-imposed restriction on tilings was that the tiles be recognizable, animate forms (allowing, of course, fanciful imaginary creatures). He records his early struggles in producing such tilings, having no other guidance than his own intellect. Later, he became aware that mathematicians and crystallographers had abstractly analyzed all such patterns and formulated rules which every periodic pattern had to obey. This knowledge was the necessary key to free Escher from frustrating experimentation and allow the full force of his creative talent to bring forth creatures that he could be sure would interlock in a prescribed way. He filled notebooks with over one hundred and fifty color sketches of fanciful repeating patterns. In *Reptiles* (Figure 3) it is from the page of one of these notebooks that an Escher-created creature decides to escape, only to reenter its jigsaw world.

A repeating pattern by its very definition can be superimposed on itself by sliding it a certain distance in a prescribed direction. This motion is called by mathematicians a *translation* of points in the plane. If we pick a particular point in a pattern, then find every repetition of this same point by translations, a lattice, or an array of points, is formed. This lattice will have the same formation no matter what point was first chosen. Lines can be drawn through the parallel rows of lattice points to form a grid of parallelograms; a third set of parallel lines can bisect these to yield a grid of triangles (Figure 15). Every repeating pattern, no matter how strangely shaped its tiles, has an underlying grid of lines—and both the parallelogram grid and the triangular grid form tilings themselves, but of simple tiles, all alike. Although the lattice of points associated with each periodic pattern is unique, these points can be connected in many different ways, resulting in many different parallelogram and triangular grids. Mathematicians use these underlying geometric grids to analyze repeating patterns, choosing particular standard grids for this analysis.

*See Escher's preface to Caroline MacGillavry's book.

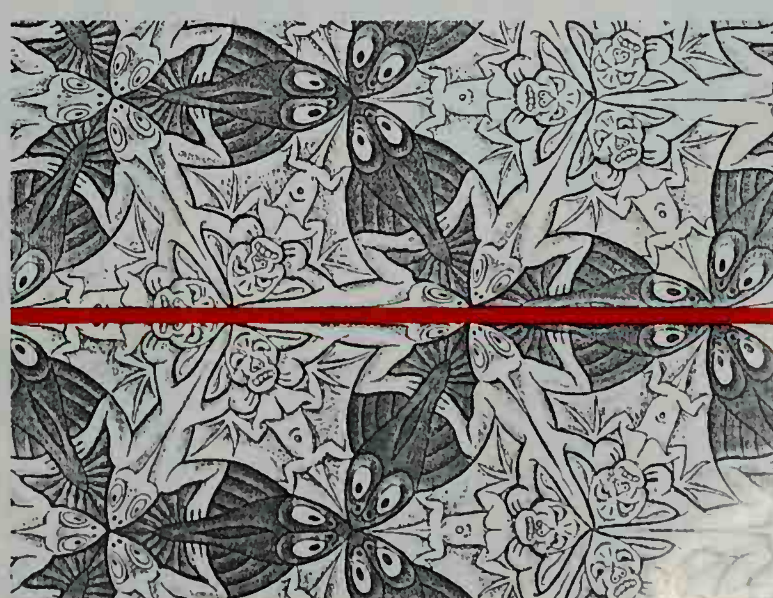
16 Periodic design 35, VII 1941. Lizards. Two kinds of squares mark centers of four-fold rotation of this pattern. A rotation of the pattern 90° about any of these centers will superimpose the outlines of the lizards. However, such a rotation about a center marked by a solid red square will superimpose lizards of opposite color, while a rotation about a square within a square will superimpose lizards of the same color. Red circles mark centers of two-fold rotation; a half-turn (180°) about these points superimposes lizards of opposite colors.

17 Wood engraving based on periodic design 99, VIII 1954. Flying fish. This design has centers of six-fold rotation ✸ , centers of three-fold rotation ▲ , and centers of two-fold rotation (find them). Rotating the pattern 60° about the centers of six-fold rotation superimposes the outlines of the fish, but interchanges the black and white fish. Rotating the pattern 120° about the centers of three-fold rotation superimposes fish of the same color.

A periodic pattern may be superimposed on itself by motions other than translations; such motions are called the *symmetries* of the pattern. Amazingly, there are only three other distinct motions which can be symmetries of a pattern. There may be a point in the pattern which can act like the center of a pinwheel. The pattern can be turned about this fixed point, and after less than a full turn, the pattern will be superimposed on itself. This motion is called a *rotation* (Figures 16 and 17). If a line can be drawn in the pattern so that the pattern on one side of the line is the mirror image of the pattern on the other side of the line, then the pattern can be superimposed on itself by flipping it over so that the line stays fixed (Figure 18). This motion is called a *reflection,* and the mirror line is called an axis of reflection. Finally, it may be necessary to combine the motions of translation and reflection to superimpose a pattern on itself—first sliding along a line, then flipping the pattern over this line (Figure 19). This motion is called a *glide-reflection.* The study of these motions is called *transformation geometry,* and its laws govern all repeating patterns.

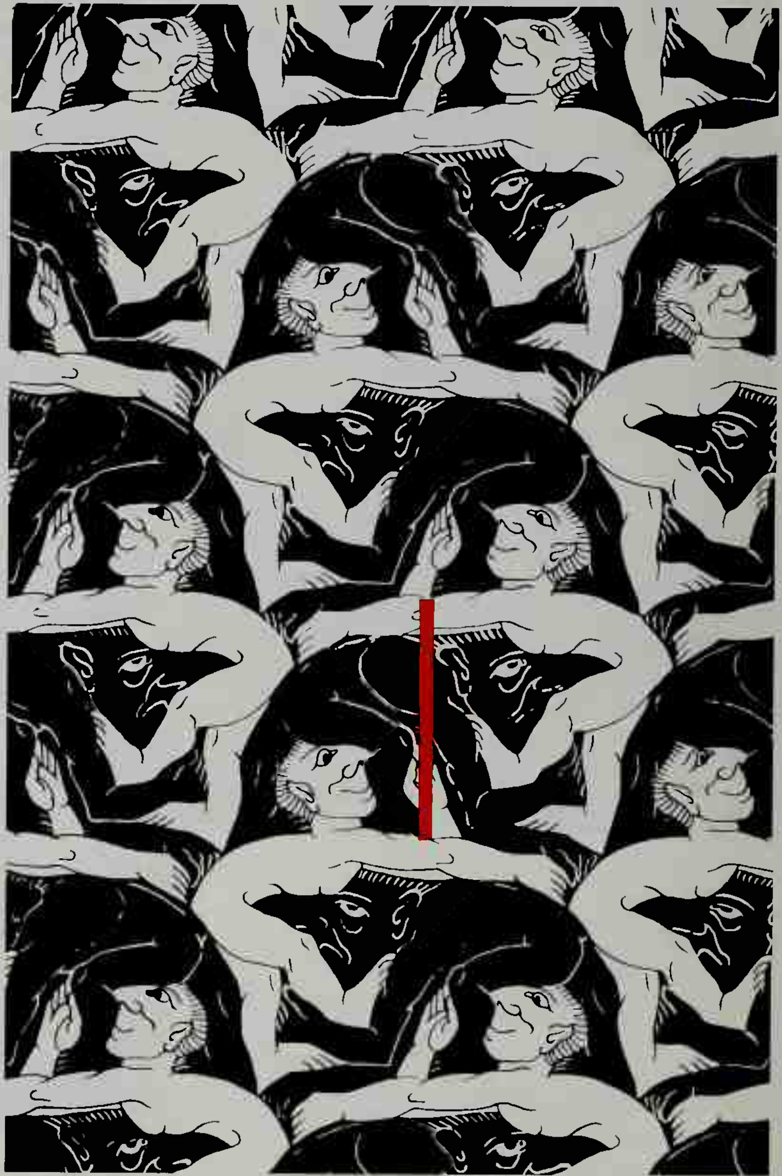
Escher colored his jigsaw designs so that adjacent creatures have different colors. Using this device, individual creatures (even when all have the same outline) are clearly delineated. In patterns where all creatures have the same shape, some symmetries may superimpose the outlines of creatures but interchange their colors, while other symmetries will superimpose creatures of the same color (Figures 16 and 17). Escher was a pioneer in the investigation of colored repeating designs, a field of inquiry which today is called *color symmetry.*

18 Periodic design 85, IV 1952. *Three Elements.* . An axis of reflection (red line) bisects this design into two parts which are mirror images of each other. The design contains a whole grid of such reflection axes which split each motif.



The actual process of creating a motif which will interlock with replicas of itself to fill the plane is, in Escher's words, "a complicated business." Mathematically, there are only seventeen distinct types of patterns (that is, patterns which have different symmetries), but artistically, there are an infinite number of possibilities. In creating a tile that will work, the artist must obey the constraints on its outline dictated by the possible symmetries of the design. Far more difficult is the creation of an outline which simultaneously defines the two shapes it will border in the design.

With this brief glimpse of the geometry which underlies repeating designs, we can now explain how the models in our collection were covered with Escher's designs.



19 Periodic design 63, II 1944. Study for *Encounter.* Glide this pattern over itself along the red "track" shown; flip the pattern over after traveling the length of the track. This glide reflection will take the optimists facing left, glide them upward, and then superimpose them on the next row of optimists facing right.

Surface Design of Solids

Escher himself experimented with covering the surfaces of three-dimensional objects with his repeating designs. In the essay "Approaches to Infinity,"* he notes that the flat designs represent the possibility of infinite repetition but only a fragment of this infinity can be captured on a sheet of paper. On the surface of a three-dimensional object, infinite repetition of design can be realized with only a finite number of figures – the pattern on a solid has neither beginning nor end. In his experimentation he covered a few cardboard models with his designs (Figure 20), but only one decorated solid was ever produced in finished form: a tin box icosahedron with an enamel design of shells and starfish which was commissioned by a Dutch firm to commemorate an anniversary (Figure 21).

Escher also created spheres whose carved surfaces are filled with a single repeated motif (Figure 22). It is likely that in covering these spheres with adaptations of his flat designs he first envisioned the pattern wrapped around a suitable solid such as a cube or octahedron, then projected the designs outward to the surface of a sphere surrounding the geometric solid.

*This is reproduced in *The World of M.C. Escher.*

20 This cardboard model of a Rhombic Dodecahedron (twelve diamond-shaped faces) was covered by Escher with a version of his design shown in Figure 18.

The cube and the three Platonic solids with triangular facets have flat patterns cut out from a grid of squares or equilateral triangles. Finding suitable designs to cover these solids is easy, for these grids are common for repeating designs. Covering these solids is not totally trivial, however, since when pieces of the repeating design are cut out to make the pattern for the solid, the designs which are brought together in folding up the pattern may no longer match.

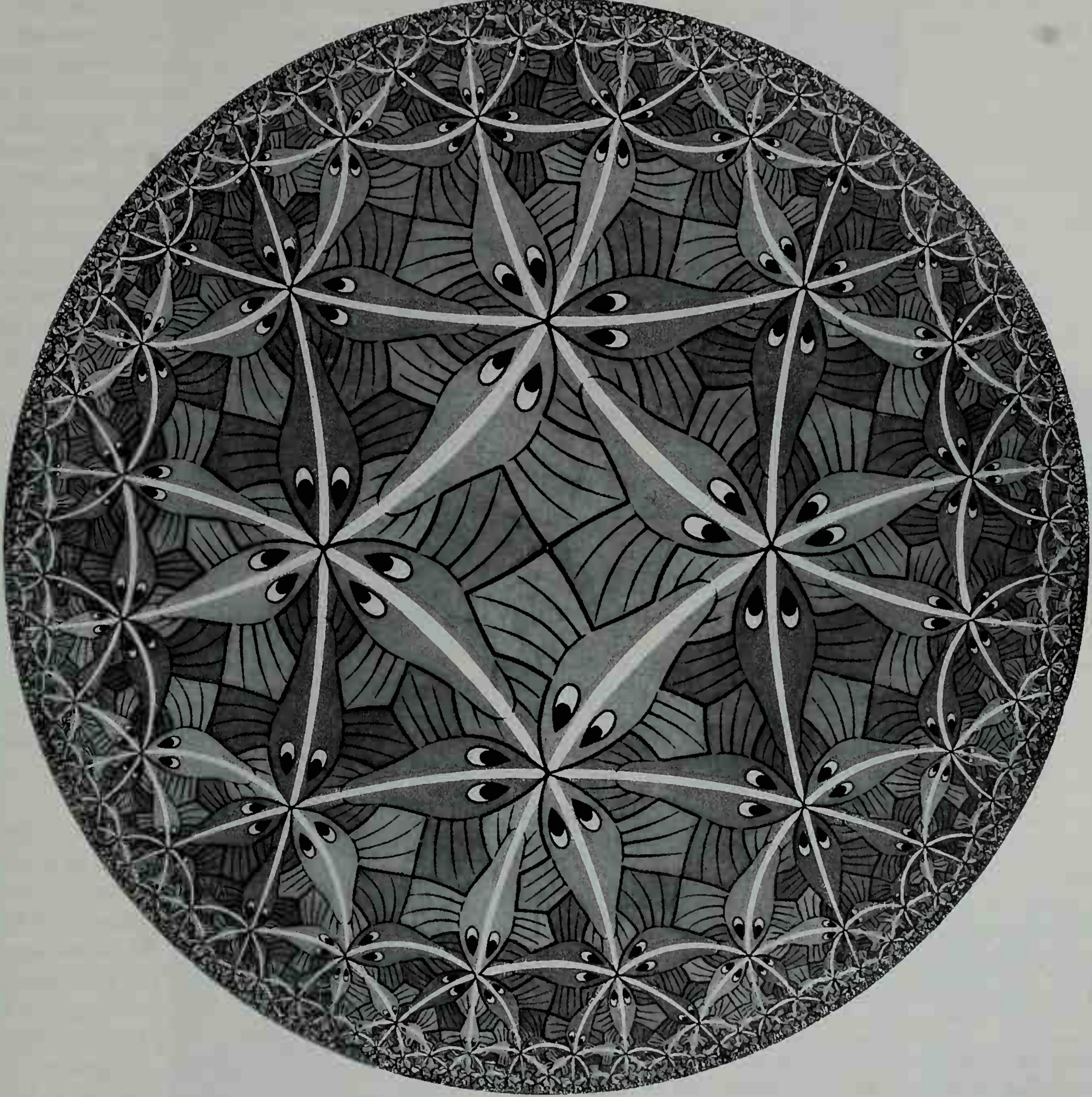
The cuboctahedron is also easily covered, provided designs filling both triangle and square grids can be matched. Escher had produced such designs as a study for his *Circle Limit III* (Figure 23). Although it was necessary to distort the triangular and square tiles to cover a representation of a special curved surface called a *hyperbolic plane,*† Escher's print is one in which matching squares and triangles alternate.

The surface design of the dodecahedron presented the greatest challenge of all the solids. Since regular pentagons are not the geometric grid of any repeating design, something more complicated than just cutting out a design and wrapping it around this solid was required. Here, familiarity with a special tiling of the plane by nonregular pentagons was the key to success. Combining this knowledge with the techniques that Escher must have used to cover his spheres achieved a successful covering of the dodecahedron with an Escher design. (Details of the designs on all of the solids appear later in the *Notes on the Models.*)

†A hyperbolic plane is a surface having the property that given a line and a point not on the line, at least two lines can be drawn through the point which do not intersect the line.

21 Escher designed this enameled tin box Icosahedron in 1963.

22 An ivory replica of Escher's *Sphere with Fish,* carved in 1962 by Masatoshi, a Japanese netsuke carver.



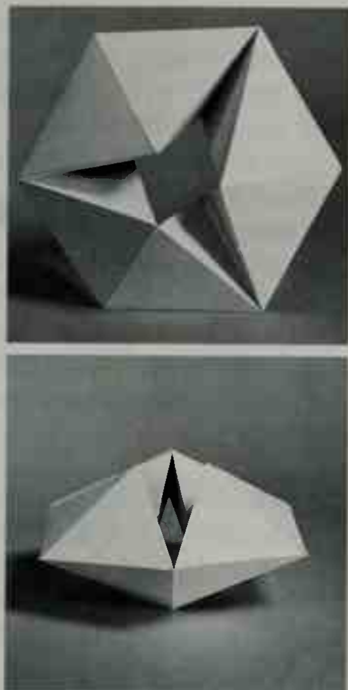
23 *Circle Limit III,* woodcut in four colors, 1959.

24 Repetition of periodic design 103, IV 1959. Fish. This rectangular portion of a periodic design of red, white and black fish has left and right edges matching, top and bottom edges matching. A cyclinder is formed when either pair of matching edges is brought together. If the design is creased along the grid lines of a Kaleidocycle, *both* pairs of matching edges can be brought together and the pattern is "wrapped around the Kaleidocycle.

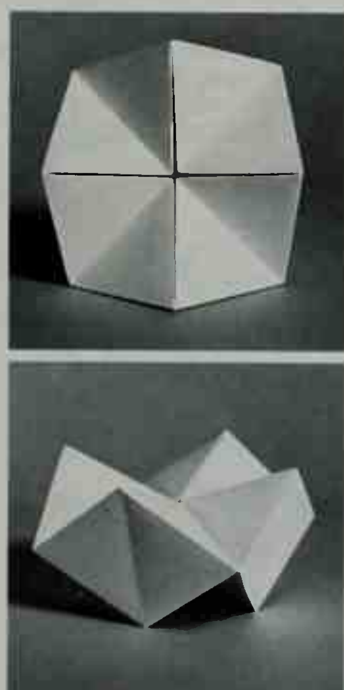
Surface Design of Kaleidocycles

In discussing the problem of representing true infinite repetition, Escher acknowledged that the drawings of repeating designs were inadequate for this representation, and he proposed a partial solution to the problem. If the flat design sketched on a rectangle of paper (Figure 24a) is lifted up and two opposite edges brought together, matching the design, then a cylinder is formed (Figure 24b). At least in the circular direction the design will have neither beginning nor end on this surface. But, of course, the cylinder has only finite height so the repetition abruptly stops at the top and the bottom of the cylinder. A form with true infinite repetition could be created by beginning with a flat rectangle, bringing together the top and bottom edges to form a cylinder, and then bringing together the ends of this cylinder to form a closed ring. An ordinary cylinder cannot have its ends brought together to form a ring without crushing the paper. The reason we can accomplish this feat in making a Kaleidocycle is because our paper has been scored and folded so that the first "cylinder" we form is creased into a chain of tetrahedra and it is easy to bring the ends of this chain together to form a closed ring without any crumpling of the paper (Figure 24c). Thus, the Kaleidocycle form has infinite repetition of its triangular facets in two distinct circular directions, like the surface of a doughnut. Continuously covering the surface of the Kaleidocycles with repeating designs accomplishes a solution to the problem proposed by Escher.

The actual covering of the Kaleidocycles with Escher's periodic designs was not as easily accomplished as the process outlined above might indicate. The hope that this could be realized was sparked by a simple observation that both the periodic designs and the patterns for the Kaleidocycles have a common geometric aspect. Each of the Kaleidocycles grows out of a flat grid of triangles—and such a grid is also associated with repeating patterns. Simply to superimpose a repeating design with a given grid onto the pattern of a Kaleidocycle formed from the same grid seemed to be an obvious thing to try, but, unfortunately, this was too naive to work. Viewed from above, a hexagonal Kaleidocycle appears to be six equal triangles forming a regular hexagon (Figure 25), and a square Kaleidocycle appears to be eight equal triangles forming a square (Figure 26). It *looks* as though a repeating design with a grid of equilateral triangles should cover the facets of the hexagonal Kaleidocycle: it *looks* as though a design with a grid of isosceles right triangles should cover the facets of the square Kaleidocycle. But our eyes are playing tricks, since in the top view, the triangular facets of these three-dimensional forms are



25 A top view and side view of a Hexagonal Kaleidocycle.



26 A top view and side view of a Square Kaleidocycle.

actually slanted. Side views of the models (Figures 25 and 26), or a closer inspection of their flat patterns, reveal that their triangular facets are not the same triangles which underlie the repeating designs.

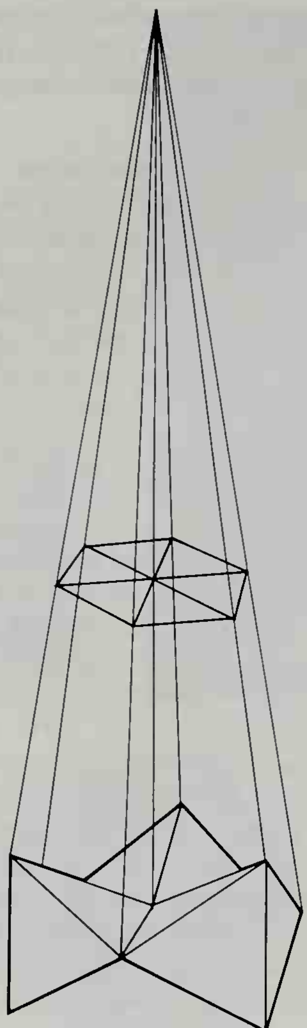
A camera projects the three-dimensional image it sees onto a flat plane. It is also possible to reverse this process and project a design in a flat plane onto a three-dimensional object. Why not *project* a design with equilateral triangles in its grid onto the hexagonal Kaleidocycle viewed from above? Similarly, why not project a pattern with right triangles onto the face of a square Kaleidocycle? In order to carry this out, another geometry was needed. *Projective geometry* tells us which properties of objects are not changed when the image of the object is projected onto another surface.

There are two essentially different ways to project an image. The first, called *central projection* (Figure 27a), is familiar to us in the way a slide projector works. A single-point source of light sends out a cone of light rays which stretch the image and carry it onto another surface. The other type of projection is technically harder to carry out, but mathematically just as natural as the first. In a *parallel projection* (Figure 27b), the image is projected to another surface by rays which are all parallel to each other, all moving in the same direction. Each type of projection preserves some properties and distorts some properties of the original image. The essential properties of the repeating designs which had to be preserved if the Kaleidocycles were to be covered continuously were preserved by a parallel projection (Figure 28) but not by a central projection. Thus, in theory at least, the problem of projecting the repeating patterns onto the surfaces of the Kaleidocycles was solved. (The actual process of producing a parallel projection can be done by a computer or via photography using a complicated system of lenses.)

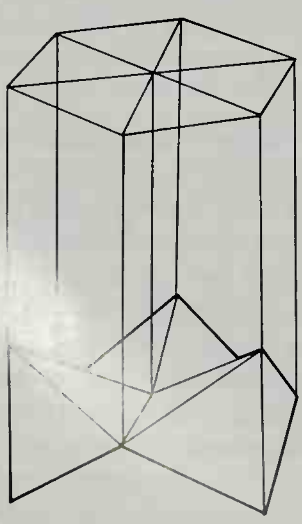
Now only one question remained: What patterns could be wrapped around the hexagonal or square Kaleidocycles? In turning these rings of tetrahedra, the triangular faces come together and then part. Many different edges match as the forms turn cartwheels through their endless cycle. Patterns suitable for covering the Kaleidocycles must have the proper symmetries so that the design will always match when edges come together. In addition, the design must fill the flat pattern of the Kaleidocycle so that top and bottom edges match and left and right edges match. Many patterns were found that fulfilled these requirements and so the theoretical possibility of continuously covered Kaleidocycles became a reality.

The actual process of producing the Kaleidocycles ready for you to assemble required a most exacting procedure. Although the sketches of repeating designs in Escher's notebooks appear to be quite precise, the hand-drawn repeating motifs contain tiny variations as they fill a page. When covering a Kaleidocycle, motifs from different parts of a pattern must match as the Kaleidocycle rotates, so extreme precision is necessary for the surface design. Each repeating design used on the solids and the Kaleidocycles was hand-drawn to these exacting requirements. Very literally, Wallace Walker and his assistants—Victoria Vebell, Robert McKee, and Robin McGrath—retraveled Escher's path in recreating the designs.

Once precise, the Kaleidocycle designs were stretched by a special photographic process to carry out the parallel projection described earlier. Finally, the designs were carefully hand-colored. In all cases, an attempt was made to match the colors of Escher's original sketches as closely as possible.

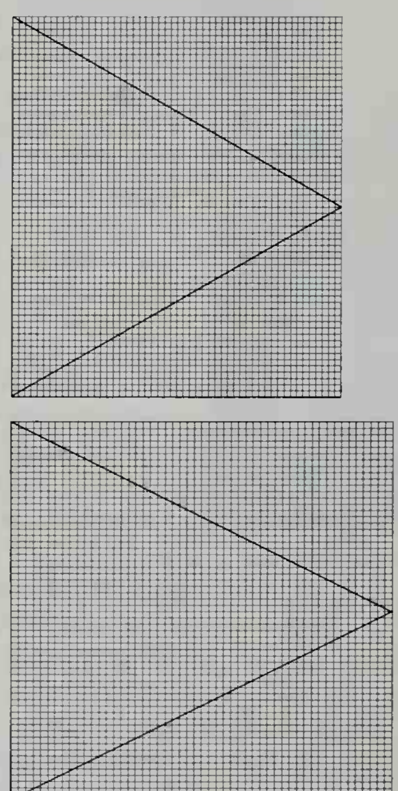


27a A Central Projection.



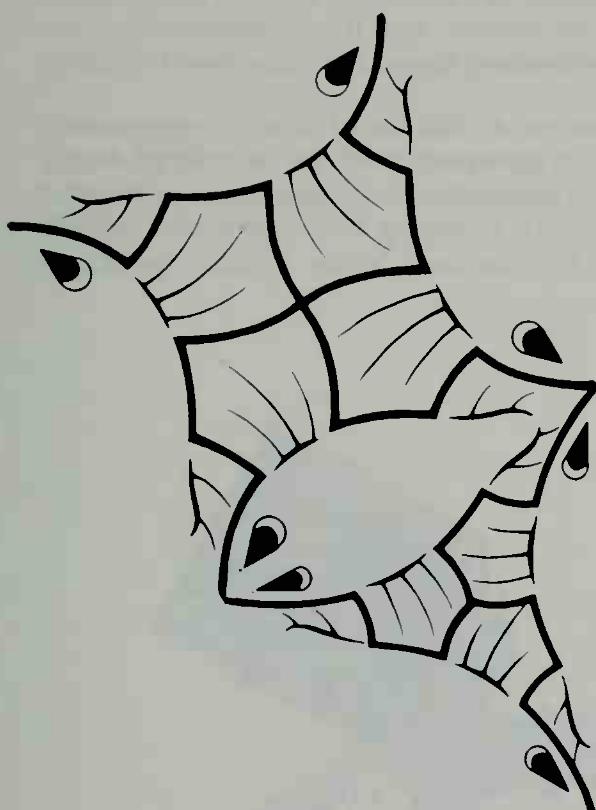
27b A Parallel Projection.

28 An equilateral triangle drawn on ordinary squared graph paper (top). The same triangle, parallel-projected by a computer onto the face triangle of a Hexagonal Kaleidocycle (bottom). Note that the distance between parallel lines in the graph grid remains unchanged in the vertical direction and is stretched uniformly in the horizontal direction.



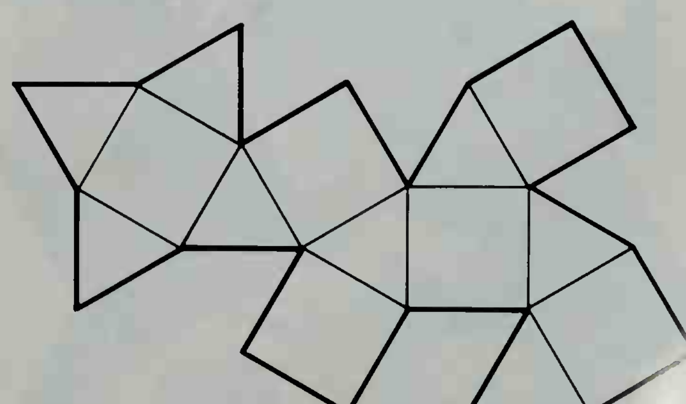
Coloring the Designs

Escher had a rigid criterion for coloring all of his repeating patterns. Any two adjacent motifs must have different colors. Only by color contrast can a single motif be distinguished in a design filled with replicas of this motif.



Mapmakers are usually required to color each country with a single color and use enough different colors so that countries which border each other have different colors. Strange as it may seem, the problems that arise in meeting these coloring requirements are in the domain of mathematics. Given a design (a map of countries, a geometric design, a tiling), mathematicians ask questions like these: How few colors can I use to meet the coloring requirements of the map? How many different ways can I color it? Can it be colored so that certain color combinations must occur? These questions are often surprisingly hard to answer if the design is complicated or the coloring must meet stringent conditions. Combinatorics, graph theory, and topology are all branches of modern mathematics in which these questions are considered.

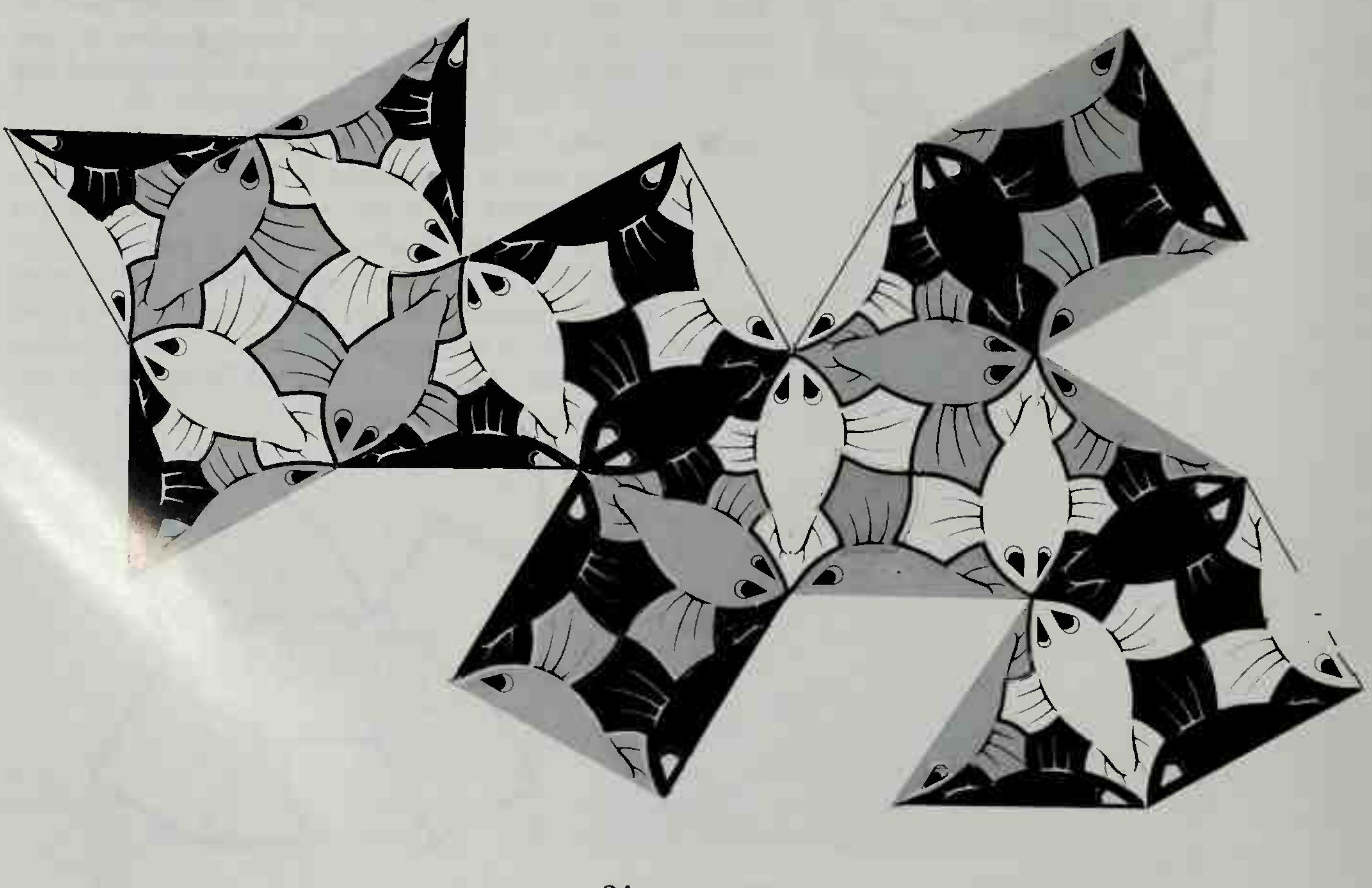
The problem of determining the smallest number of colors that suffice to map-color *any* design drawn on a plane or surface of a sphere was unsolved for over one hundred years, although many skilled mathematicians attempted to answer it. Many believed that four colors was the answer since no one was able to produce a map that required five colors. Only in 1976 was the answer proved to be correct. (Mathematicans K. Appel and W. Haken of the University of Illinois used tens of thousands of computer operations to support their proof.) Although four colors are sufficient to map-color any plane design, a repeating design is most pleasingly colored if the coloring emphasizes the symmetries of the design. Escher experimented with and categorized many of these special types of coloring long before the subject became a field of investigation for crystallographers and mathematicians.



In adapting Escher's designs to the surface of the geometric models, the criterion that the design must be map-colored was maintained. When the flat patterns of the solids are cut out from the geometric grids underlying Escher's periodic designs, naturally some portions of the design are cut out. When a pattern is folded up to form a solid, different portions of the periodic design are brought together, and in some cases, this causes adjacent motifs on the solid to have the same color. In order to have the design on these solids map-colored, some adjustments in Escher's coloring had to be made. Sometimes a design which required only three colors in the plane demanded a fourth color when applied to the surface of a geometric solid. In three cases—the cube, the icosahedron, and the cuboctahedron—the map-coloring requirement forced a rearrangement of colors.

On all of the models, an additional coloring requirement was also met: Each patterned solid can be colored *evenly*, that is to say, each different color is used the same number of times. Finding such a coloring for the cube covered with twelve identical fish has been left as a puzzle for you to solve.

29 The fish design on the Cuboctahedron can be map-colored evenly using a minimum of three colors. In the coloring here, each color (white, gray and black) is used on exactly eight of the twenty-four fish which cover the Cuboctahedron. Other interesting even map-colorings of this design are possible using four colors; we chose one of these for our model.



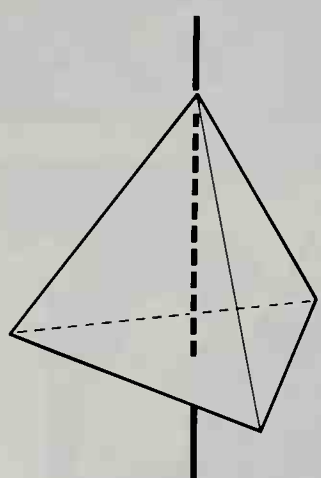
30 Periodic design 56, XI 1942. Reptiles.

Geometric Solids

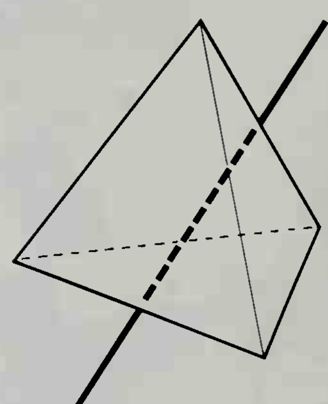
Tetrahedron

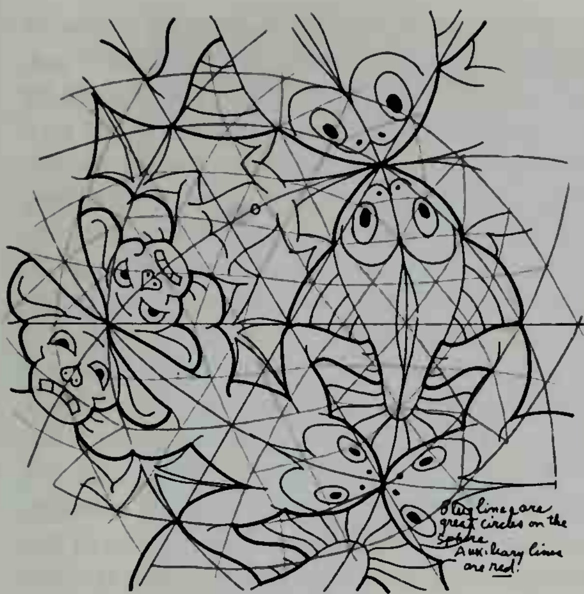
This design of reptiles was a natural choice to cover the tetrahedron. Joining the points of six-fold rotation forms a natural grid of equilateral triangles (one of which is shown in Figure 30), and the tetrahedron pattern consisting of four of these triangles easily wraps the design around the solid with both outlines and colors of the creatures matching. On the surface of the solid the centers of rotation of the flat design become centers of rotation of the solid. For example, if an axis is inserted through the tetrahedron so that it pierces the center of one triangular face and the vertex opposite that face, then the tetrahedron can be rotated 120° about that axis and be superimposed on itself (Figure 30a). This rotation of the tetrahedron about such an axis is called a three-fold rotational symmetry of the tetrahedron. You will see three reptiles turning around each of the points where a three-fold rotational axis can pierce the tetrahedron. The midpoint of each edge of the tetrahedron is a center of two-fold rotation, both for the periodic design and for the tetrahedron. A two-fold rotation axis for the tetrahedron will pierce through the midpoints of two nonadjacent edges (Figure 30b).

30a Axis of three-fold rotational symmetry.



30b Axis of two-fold rotational symmetry.





31 Escher's drawing with instructions and a plastic sketch ball for a carving of an ivory sphere with his *Three Elements* design. Clearly visible is the spherical triangle which contains half of each of the three motifs. The equilateral triangle is the projection of one face of our Octahedron onto the surface of a sphere.

Octahedron

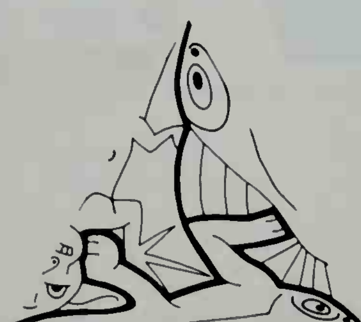
The repeating design depicting the "Three Elements" of earth, water, and sky (Figure 18) was used by Escher as he experimented with surface design on three-dimensional forms. Each motif in the pattern approximately fills a diamond shape (rhombus) of two joined equilateral triangles. By altering the dimensions of these diamonds, they can become the faces of a rhombic dodecahedron. In this way, Escher wrapped a distortion of the flat periodic design around the twelve-sided solid (Figure 20). He also covered a prism with this design.

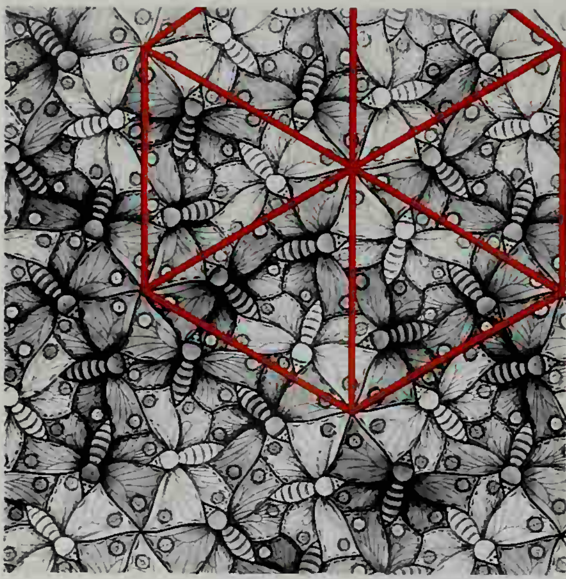
In 1963, at the suggestion of C.V.S. Roosevelt, an avid collector of his work, Escher provided a detailed drawing and plastic ball covered with the design to guide the Japanese craftsman Masatoshi as he carved the design onto the surface of an ivory sphere (Figure 31).

The most obvious adaptation of this design to the surface of a geometric solid is to cover an octahedron. No distortion of the flat pattern is required. Each equilateral triangle containing the interlocked halves of the three motifs is a face of the octahedron (Figure 31). Twelve motifs in all, four of each kind, cover the octahedron, and the four replicas of each motif form a square path traveling around adjoining edges of the octahedron.

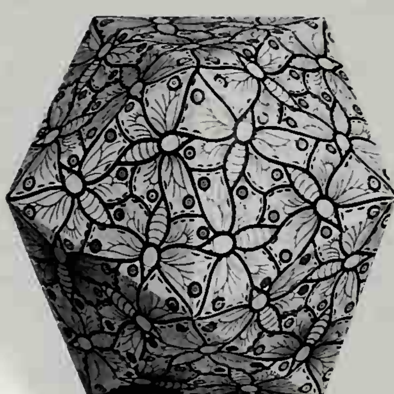
If the octahedron is surrounded by a sphere and this design is centrally projected onto the surface of the sphere, Escher's decorated ball is the result. Under this projection the square path followed by replicas of a single motif becomes a great circle on the sphere.

Escher was very specific in his instructions as to the coloring of the carved ivory sphere and included samples of the three colors he wished to be used. The coloring of our octahedron with the *Three Elements* design is Escher's choice for the carved sphere. Note that this coloring differs from the coloring of the original periodic design; we have used the latter in our covering of a hexagonal Kaleidocycle with the same design.





32 Periodic design 70, III 1948. Butterflies. Pink and green butterflies alternate around the six-fold center of rotation ✳. These two colors are sufficient to map-color this portion of this design in the plane. However, one of the six butterflies must be cut out in order to adapt this design to the surface of an Icosahedron (the center of rotation becomes a vertex of the Icosahedron). Thus at this vertex, two butterflies of the same color come together. In order to have adjacent butterflies on the Icosahedron always have different colors, four colors are necessary to color the design on the solid.

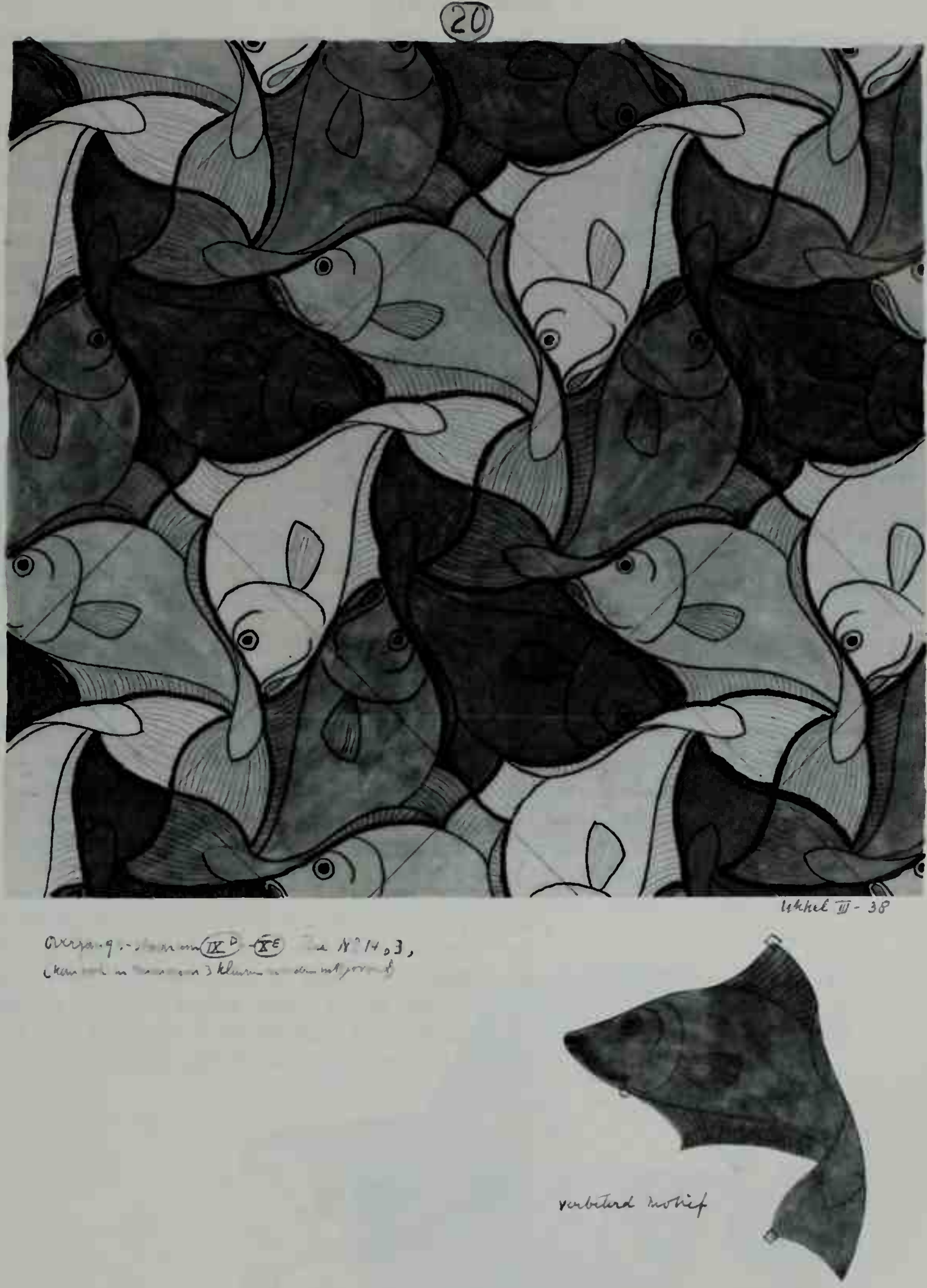


Icosahedron

Escher's periodic design of butterflies (Figure 32) is one of his most intricate and carefully colored. Any of Escher's patterns which contain centers of six-fold rotation could be adapted to cover the icosahedron, but the beautiful butterfly design proved to be a special challenge. The flat design requires three colors for recognition of adjacent butterflies, and Escher provided a coloring in which just two of the three colors alternate about each six-fold center of rotation (where six wing tips meet). In addition, Escher emphasized the missing third color from the butterflies about a given center of six-fold rotation by using this third color for the wing markings (the small circles) on the butterflies whirling about that point. These rotation centers become the vertices of the icosahedron when the flat pattern of the icosahedron is cut out from the grid of equilateral triangles which underlies the butterfly design. Thus, just five of the six butterflies whirl around each vertex of the solid and the coloring must be altered so that adjacent butterflies have different colors.

It is quickly discovered that four colors are required on the solid in order to fulfill this condition. The challenge was to make the new coloring as balanced as possible. Although the coloring achieved may appear somewhat random, it possesses unusual balance. Around each vertex of the solid just three of the four colors are used. Following Escher, the wing markings of the butterflies whirling about a vertex contain the fourth color not used to color the butterflies about that vertex. Every possible combination of using two colors twice and a third color once to color the five butterflies around one vertex occurs. (For example, using the colors green, blue, and pink, there are exactly three different colorings of vertices of the solid: (1) green twice, blue twice, pink once; (2) green twice, pink twice, blue once; (3) blue twice, pink twice, green once.) Finally, it is an even coloring, that is to say, of the total of sixty butterflies that cover the solid, exactly fifteen of each color occur.

This is the only model in which a new color had to be introduced. Escher used green, blue, and pink to color his flat design of butterflies. Yellow was added to solve this three-dimensional coloring puzzle.



33 Periodic design 20, III 1938. Fish.

34 Escher's instructions for carving an ivory replica of his *Sphere with Fish*. His outline on the photograph of his carved beechwood sphere shows a projection of one face of our Cube. He notes "The eight points A and B (of which only four are visible) are angular points of a cube."

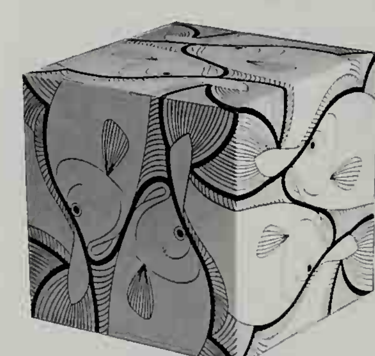
Cube

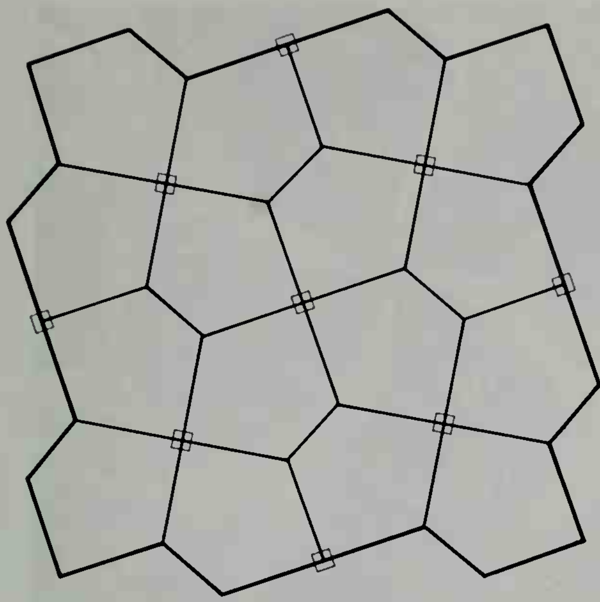
Directly related to Escher's own "Sphere with Fish," this cube can be viewed as an intermediate stage in transforming a two-dimensional tiling of the plane (Figure 33) to a tiling on the surface of a sphere. This single fish tile fills the plane in a pattern which has two different points of four-fold rotation: four fish whirl around the point where their tails meet and another four fish spin around the point where their back fins meet. Escher has symbolized with small squares these centers of rotation on his single fish tile. Joining these points of the pattern forms a square grid from which we cut out the pattern of a cube. Now fold up the cube—it is covered with twelve fish and at each corner of the cube three fish whirl. If we could inflate this cube into a sphere, it would become Escher's own "Sphere with Fish."

It is quite possible that Escher devised his own carved wooden sphere in this manner. In giving instructions (Figure 34) for a small replica of the sphere to be carved from ivory (Figure 22), Escher noted that the three-fold points of symmetry on the sphere should be where an inscribed cube would touch the surface of the sphere.

In his notebook, Escher remarks below his repeating design with fish that it is possible to use just three colors to map-color the design. However, he chose to use four colors since this coloring was more compatible with the symmetry of the design. The design wrapped around the cube demands four colors to be map-colored. Escher's carved sphere with fish is uncolored, and so we have left our cube uncolored. You are challenged to solve the following coloring puzzle:

Color the fish on the cube so that all of the following conditions are met: (1) Each fish is one color. (2) No two adjacent fish have the same color. (3) Exactly four colors are used. (4) Each color is used to color exactly three of the twelve fish. (Yes, it can be done!) When you have accomplished this, you have an example of an even coloring of a "map" on a sphere which requires four colors.





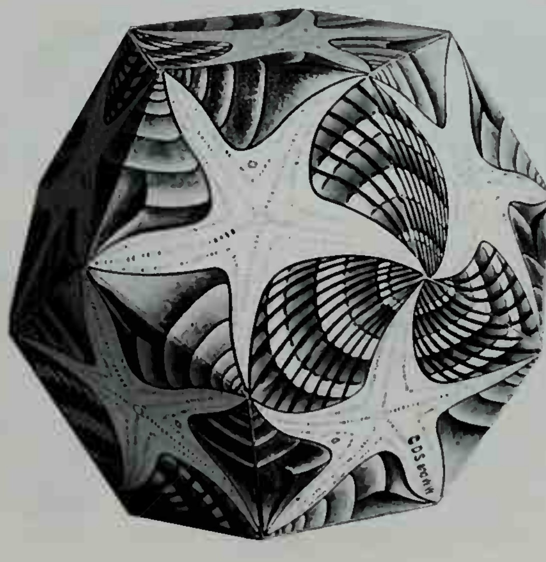
35 One of Escher's favorite geometric patterns was the tiling by congruent pentagons.

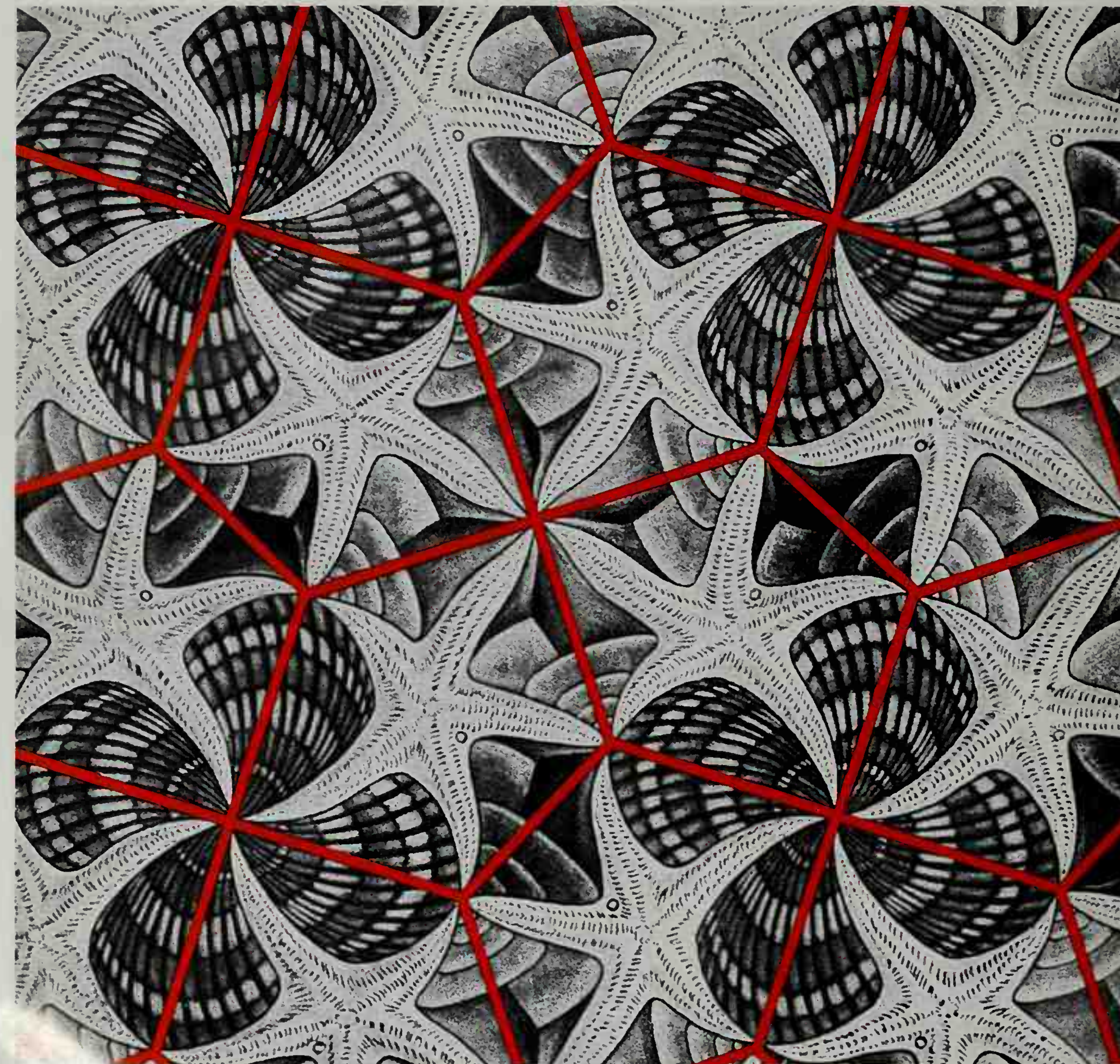
Dodecahedron

Regular pentagons cannot be used as tiles to fill the plane—there will always be gaps in such a tiling. How, then, could the dodecahedron be covered with a repeating design?

One of Escher's favorite geometric patterns was the tiling by congruent pentagons shown (Figure 35). These pentagons are not regular since their angles are not all equal. On the pattern we have marked the centers of four-fold rotation: these can be joined to form a square grid of lines. From this grid we can cut out the flat pattern of a cube and fold it up into a solid cube.

Escher's periodic design "Shells and Starfish" is based on this geometric pentagonal tiling, with each starfish occupying one pentagon (Figure 36a). To cover our dodecahedron with this design, we first wrapped it around the cube (Figure 36b). Examining the pentagon-covered cube, two exciting observations indicate how to solve the original problem. First exactly twelve pentagons are in the design covering the cube—and the dodecahedron has twelve pentagonal faces. Second, we know it is possible to inscribe a cube inside a dodecahedron so that each of its edges lies on a face of the dodecahedron, and each of its corners is at a vertex of the dodecahedron (Figure 36c). The lines of the pentagon pattern drawn on our cube are in the currect position so as to appear as though the edges of the dodecahedron have been projected onto the cube. The cube in Figure 36b was thus inscribed in a dodecahedron, and then the pattern was projected outward onto the surface of the dodecahedron. In this way the pattern was preserved (though distorted from its two-dimensional beginning), and continuously covered the most unusual Platonic solid.





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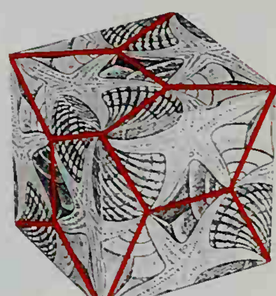

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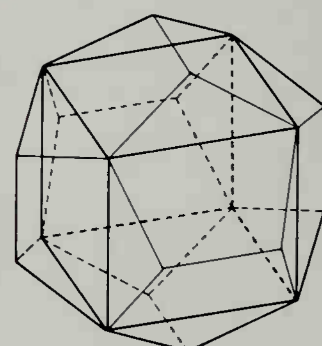
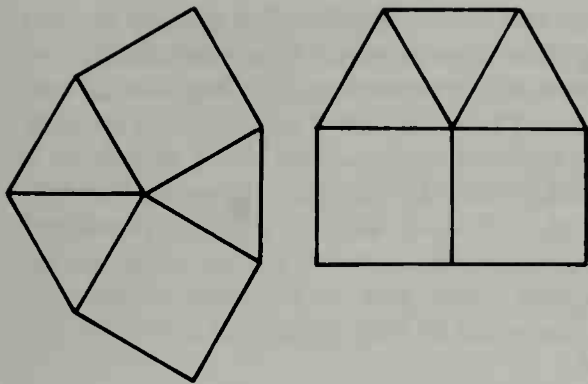


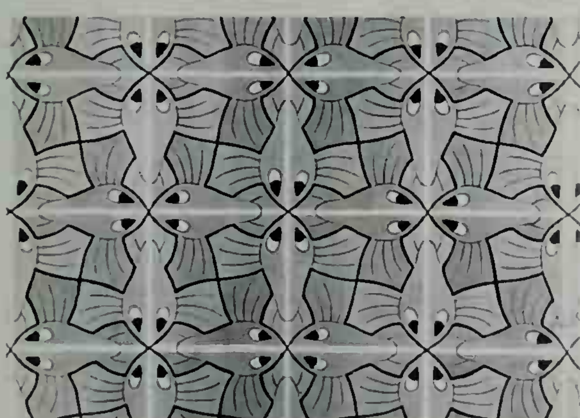
c

36 Periodic [illegible] VIII 1941. *Shells and Starfish,* with a network of pentagons outlined. The Cube is covered with the shells and starfish pattern, and the network of pentagons containing the design is projected outward from the surface of the Cube to the surface of a Dodecahedron containing the Cube.



37 Three triangles and two squares fit exactly around a point in two distinct ways.

38 Periodic designs 122 and 123, IV 1964. Study for *Circle Limit III*.



Cuboctahedron

A floor can be tiled in a variety of ways using equal-sided triangles and squares so that their edges are matched. Three triangles and two squares fit exactly around a point in two distinct ways (Figure 37). In any tiling using both types of tiles, one or both of these arrangements must occur.

A single page in Escher's notebook contains a triangular tiling and a square tiling with the same fish motif (Figures 38a and b). If we attempt to use both Escher's triangular and square fish tiles to fill a plane, we quickly note that in either arrangement of the tiles around a point it is impossible to have all of the fish "match" (Figure 39). (This is because an odd number of tiles surrounds a point.) However, a flat pattern of a cuboctahedron can be cut out from a plane-filling design of squares and triangles (Figure 40) in which the odd nonmatching tile is omitted at each vertex so the fish will match when the pattern is wrapped around the solid cuboctahedron.

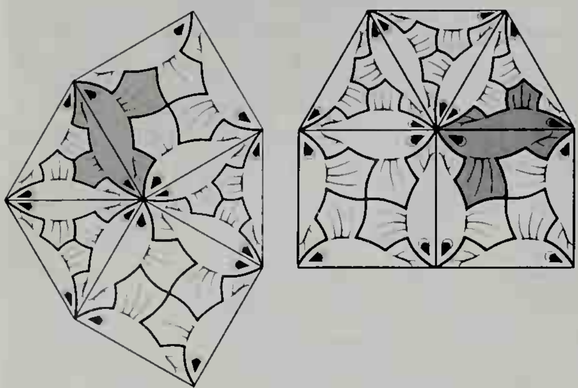
Circle Limit III (Figure 23) was Escher's solution to combining these tiles in a two-dimensional representation. He distorted them mathematically so they could fill a non-Euclidean plane with an even number of tiles surrounding each vertex, and thus achieved a matching of tiles.

Escher's periodic fish tiling using all square tiles requires only two colors; the one with all triangular tiles needs three. In combining the two types of tiles in *Circle Limit III,* Escher used four colors for the design. The map-coloring criterion did not force him to do this, but he wanted to emphasize the "traffic flow" of fish swimming along the curved paths of the design. In his coloring, all fish along one path are of the same color and are surrounded by fish of different colors. This scheme made four colors necessary.

There are many pleasing ways to color the fish design covering the cuboctahedron, all of which meet the map-coloring requirement and use the colors "evenly."

The minimum number of colors necessary to map-color the design on our cuboctahedron is three, and a mathematically balanced coloring using just three colors is shown in Figure 29. In this coloring each triangular tile contains all three colors and each square tile contains just two of the three colors. Each color is used on exactly eight of the twenty-four fish covering the solid.

If we follow Escher and choose to use four colors for the design, two particularly interesting colorings re[illegible] follow a chosen aspect of Escher's coloring of C[illegible]

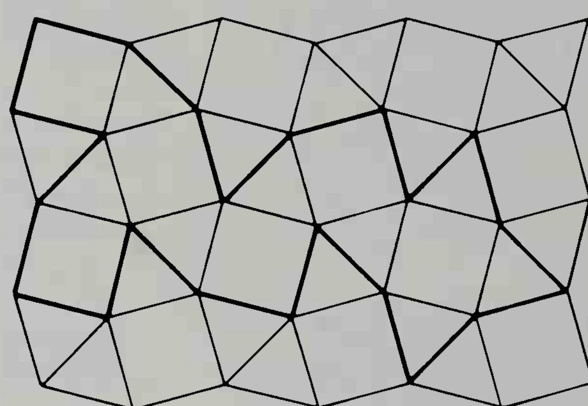


III. To emphasize the "traffic flow" of fish around natural paths encircling the cuboctahedron is one possibility. On the surface of the cuboctahedron there are four natural hexagonal paths formed by six adjacent edges which encircle the solid. Using Escher's idea, we can color all fish on a single path the same color. In doing so, all twenty-four fish are colored, six of each color, and the coloring produced has all possible circular arrangements of four colors occurring on the square faces and all possible circular arrangements of three out of four colors occurring on the triangular faces.

Another possibility is to create an even four-coloring of the design in which square faces contain just two colors and triangular faces, three colors, as in *Circle Limit III*. Such a coloring is the one we have chosen for our pattern. Visually, this coloring appears to be the more closely related to the two-dimensional coloring by Escher.

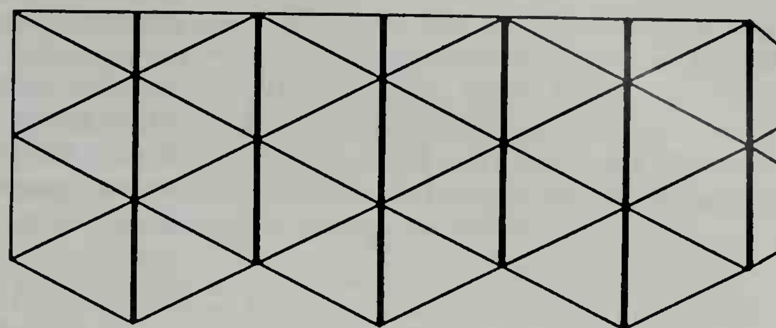
39 Square and triangular tiles can fill the plane, but Escher's fish tiles will not all match in such a planar tiling.

40 A Cuboctahedron pattern can be cut from the plane tiling, resulting in a matching fish design on the three-dimensional solid.



Hexagonal Kaleidocycles

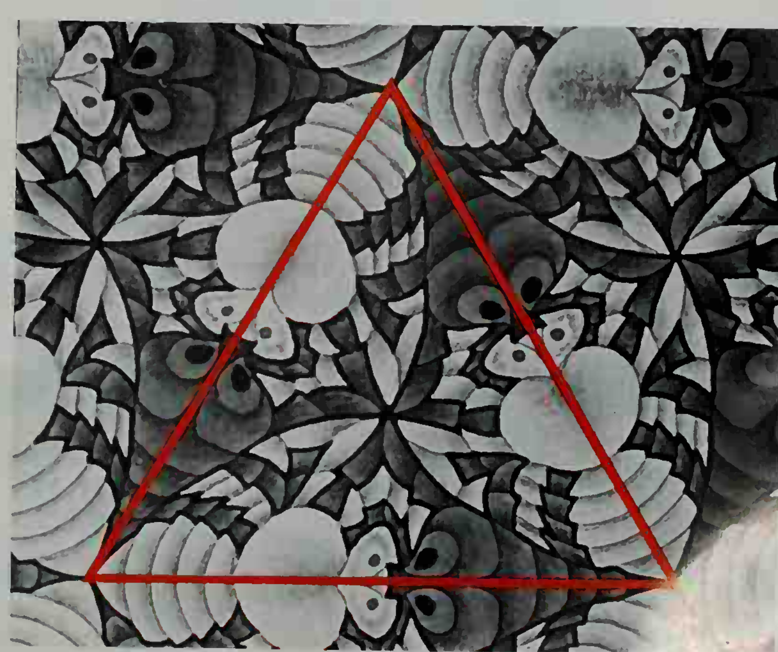
Every pattern which decorates these Kaleidocycles begins with an underlying grid of equilateral triangles. All the patterns have three-fold centers of rotation; some have two-fold and thus six-fold centers of rotation as well. Some have reflection symmetries. Look for these symmetries as you examine the assembled models.



Bugs

This pattern (Figure 41) appears almost abstract, like a rich brocade, when viewed from a distance—only close examination reveals the interlocked bugs. Each triangle, like the one outlined, covers a single face of the Kaleidocycle. At the center of this triangle is a three-fold center of rotation; the edges of the triangle are reflection axes for the pattern. Turning the Kaleidocycles matches mirror images of bugs. This completes the design, and reveals another three-fold center of rotation at the center hole of the closed ring.

41 Periodic design 54, X 1942. Bugs.



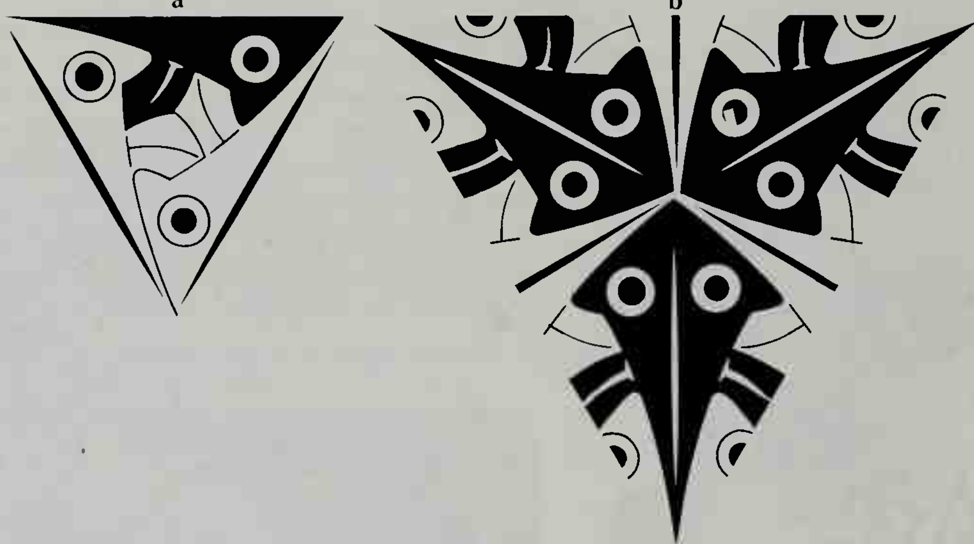
Fish

The symmetries of this repeating design (Figure 24) are exactly the same as those of "Bugs" (Figure 41), if we ignore coloring. Map-coloring each design reveals its difference: The bug design with two different creatures requires only two colors, while the design with just one motif requires three colors. Without color, the design formed by the fish has an underlying grid of triangular tiles like the small one shown in Figure 42a.

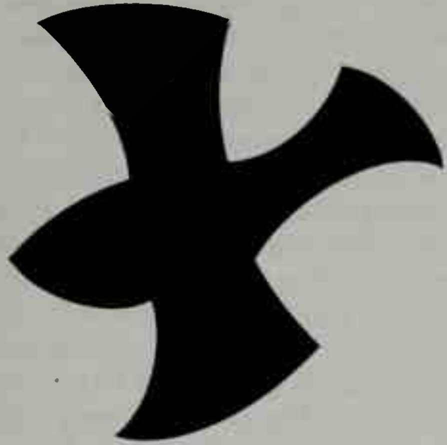
In order to have this 3-color fish pattern wrap around the Kaleidocycle, it is necessary to use a larger triangular tile (Figure 42b) for each face of the Kaleidocycle. At the center of each face is a three-fold center of rotation which is also the point of intersection of three reflection axes of the pattern.

42 Although a triangle of three half-fish generates the design in the plane, the larger triangle is needed as the face of a continously covered kaleidocycle.

a b



Bird / Fish

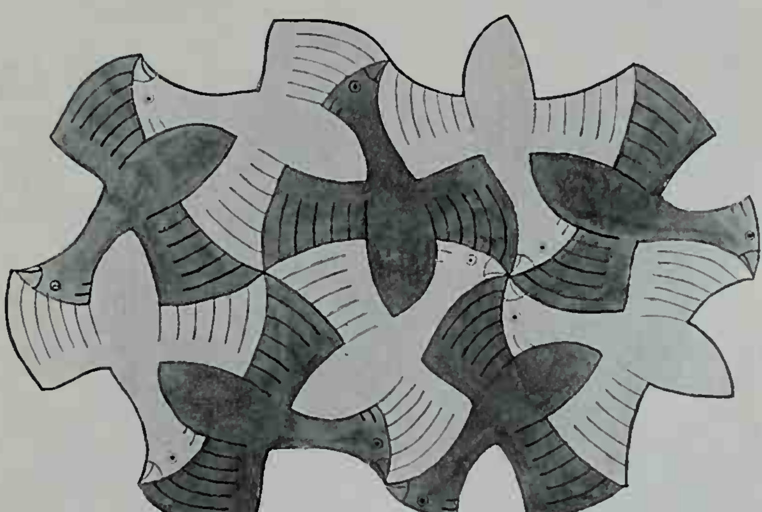


43 A silhouette of a bird in flight or a soaring fish?

Look at the shape in Figure 43. What do you see? A silhouetted bird in flight? Yes, and look again with the imagination of a child. It is a soaring fish. One shape, yet two creatures can occupy that outline. The shape is a tile which can fill the plane, and thus the pattern it creates can contain both creatures, one merging with the other (Figure 44).

How we perceive or interpret outlines—the fact that one silhouette can have many interpretations—provides part of the magical surprise of Escher's work. As you turn this Kaleidocycle, you will witness a seeming metamorphosis of birds into fish and back again in an endless cycle. Yet if the detail of the creatures were blotted out, an allover pattern of one tile would be all that remained.

44a Periodic design 44, XII 1941. Birds. Here the silhouette from Figure 43 is a bird.

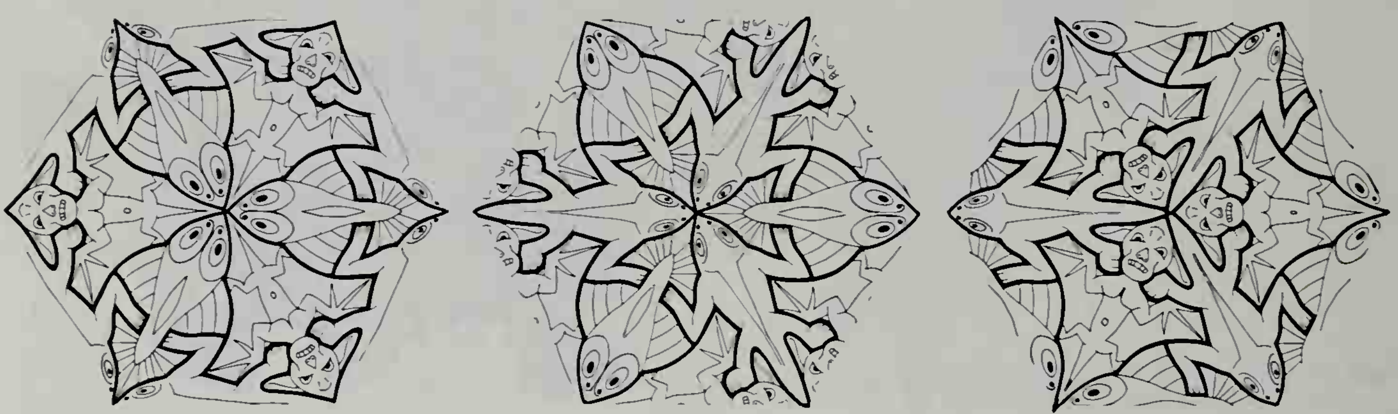


44b The silhouette becomes a flying fish in this woodcut from Escher's *Regelmatige vlakverdeling* (Regular Division of the Plane, 1958.) (See also Figure 17.) Escher's study for this woodcut contains the note "See No. 44".

Three Elements

Escher's "Three Elements" design (Figure 18) has three different centers of three-fold rotation. At one center, the heads of three fish meet; at another, the heads of three lizards meet; at the third, the heads of three bats meet (Figure 45). We can animate this design by choosing as faces for the Kaleidocycle the small triangles of the pattern which form the faces of our decorated octahedron (Figure 31). Imitating the action of a Kaleidoscope, each turn of this Kaleidocycle clicks a new image into view. The design which covers three faces of this model cannot be continued onto the fourth face because a fact from the geometry governing the periodic design comes into conflict with a fact governing the three-dimensional form: The periodic design repeats the triangles we have chosen as faces for our model in strips of six, but the tetrahedra which make up the model have as their flat pattern triangles in strips of four. Thus, we have designed the fourth face of our model to contain Escher's name.

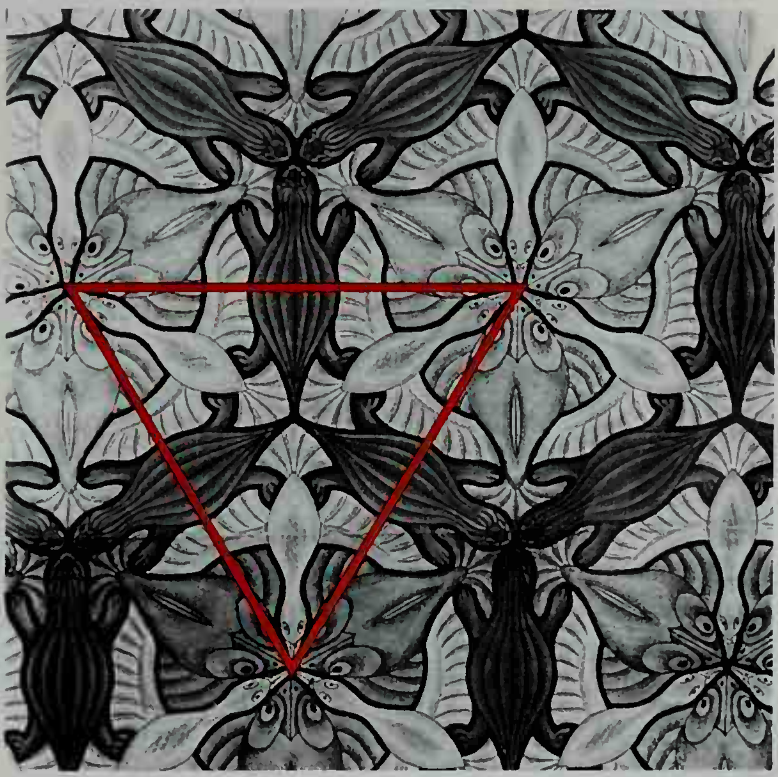
45 Water, land, and air—each of the three elements is the central focus in a hexagon.

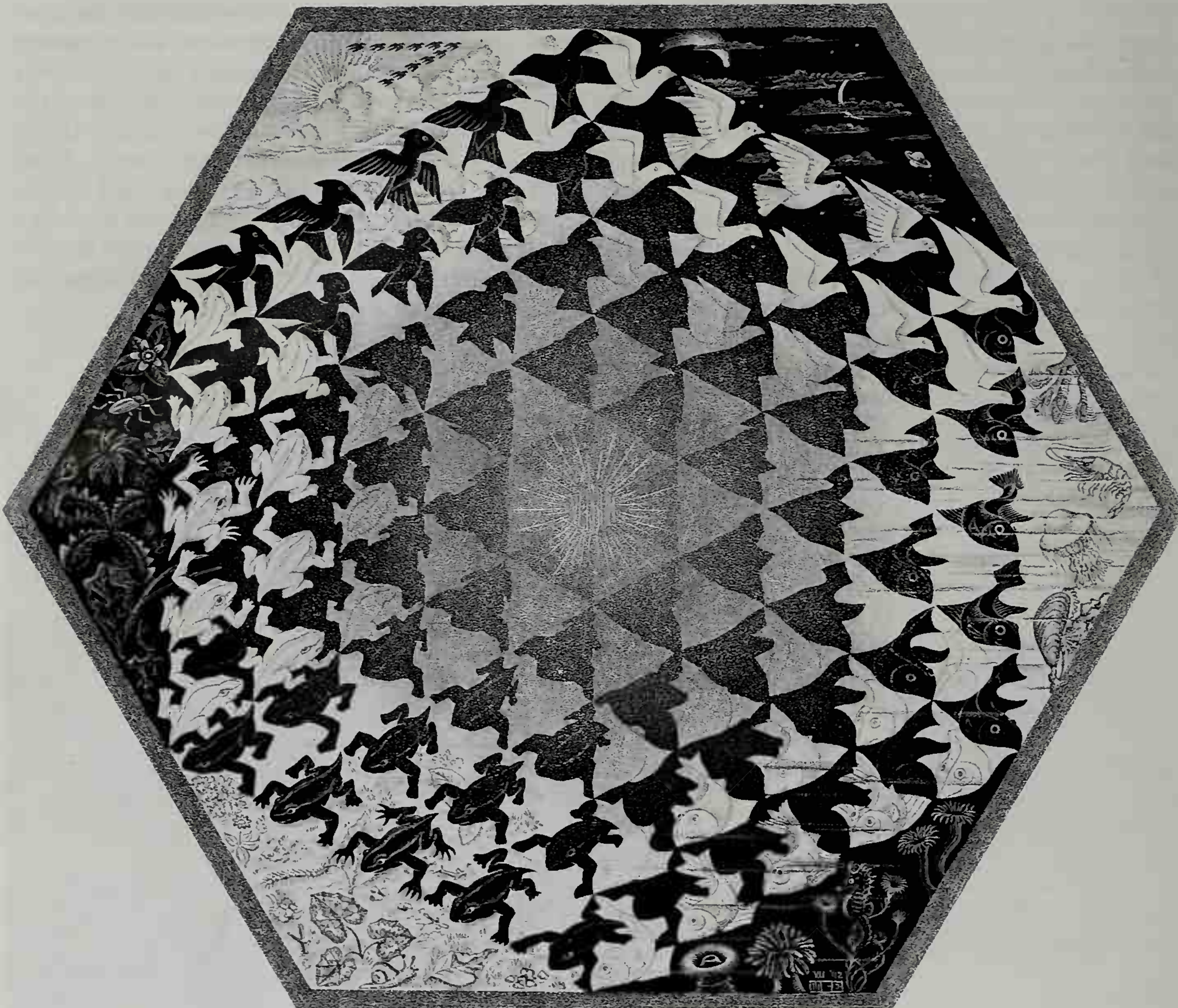


Fish / Duck / Lizard

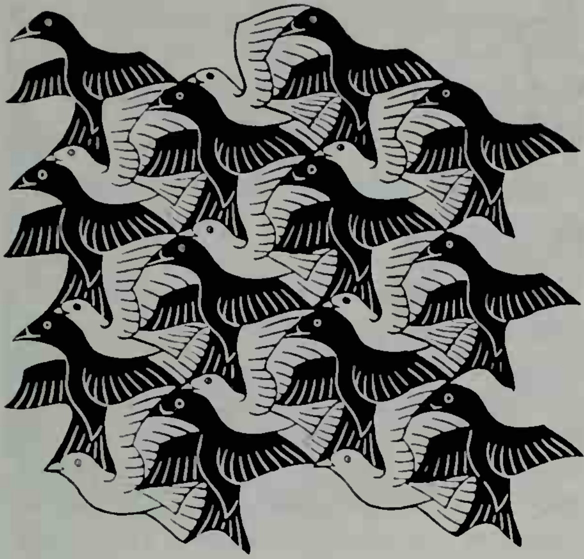
This repeating design (Figure 46) is mathematically the same as "Three Elements" (Figure 18). (Its creatures also represent air, land, and water.) Although we could have used this design to cover a Kaleidocycle in the same manner as the "Three Elements" model described above, we decided to wrap this design continuously around a Kaleidocycle. To accomplish this, a triangle larger than the one used for the "Three Elements" model (Figure 31) is necessary to design the faces of the Kaleidocycle. The triangles formed by joining the points of the design where the heads of three fish meet successfully cover the Kaleidocycle.

46 Periodic design 69, III 1948. Fish, Duck, Lizard. (First version of *Three Elements.*)





47 *Verbum,* lithograph, 1942.



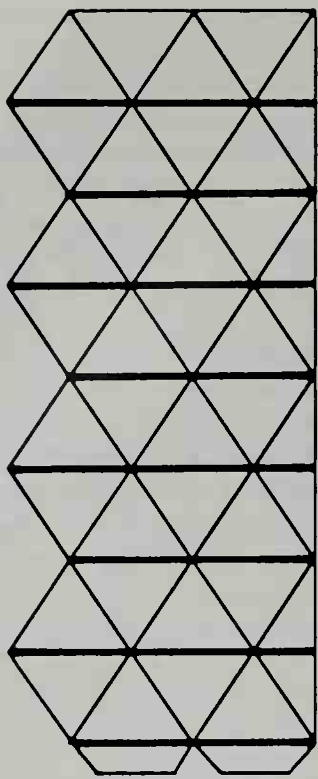
48 The periodic design on the cover of *Regelmatige vlakverdeling* appears in the interlocked bird portion of *Verbum* at the top of the print. See Figure 15 for the periodic design of the frog and fish portion at the bottom of the print.

Verbum

Fragments of Escher's repeating patterns most often occur in prints which depict metamorphosis. The lithograph *Verbum* (creation) is one of the most skillful of these (Figure 47). The interlocking pairs of creatures were first sketched as simple plane-filling designs (Figures 48, 15) in which parallel rows of the creatures form a grid of intersecting lines. However, in incorporating these designs into *Verbum,* Escher used a subtle mathematical device to capture the themes of explosion, evolution, interdependence and cycle.

Verbum contains two distinct kinds of development. From the center outward, vague amorphous shapes gradually evolve into recognizable creatures which escape into their natural element. In a circular sweep around the hexagonal ring, creatures metamorphose—bird into fish into frog into bird, tracing the ecological cycle of air, water, land. This double system of development, consisting of rays emanating from a single center and concentric circles about that center, is familiar to mathematicians in the polar coordinate system.

The design has been adapted to the Kaleidocycle so that beginning with the explosion at the center of the print, each inward turn of the Kaleidocycle produces a multiplication and evolution of the creatures, culminating in the total view of the print.



Square Kaleidocycles

Each of the patterns adapted to cover the surface of a square Kaleidocycle has as its underlying grid a mesh of squares like ordinary graph paper. All designs have centers of two-fold rotation; some also have reflection axes. At the center "hole" of each Kaleidocycle will occur a center of four-fold rotation.

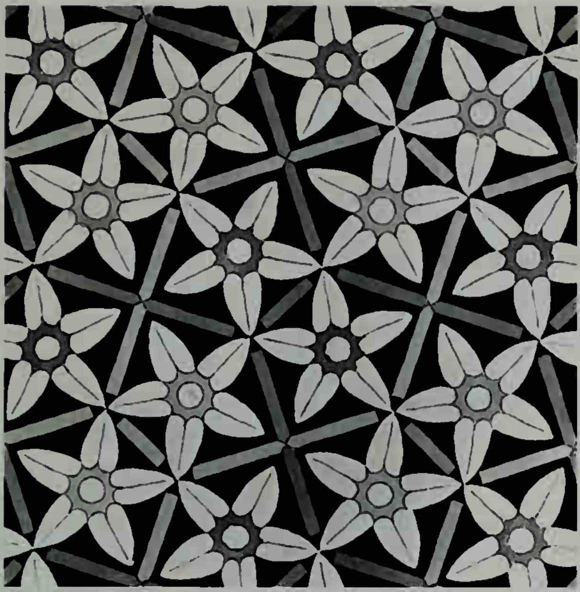
Watch closely as you turn these models—each turn will change the image you view.

Shells and Starfish

This pattern has been discussed in describing the decoration of the dodecahedron (Figure 36a). This time we join the points in the pattern where four shells of the same kind meet to form a square grid. Each square is bisected into right triangles. (One of these bisected squares outlined in Figure 49.) Each of these triangles becomes a face of the Kaleidocycle. As you turn the Kaleidocycle watch the change of shells occur at the center point.

49 Periodic design 42, VIII 1941. *Shells and Starfish.*

Flowers

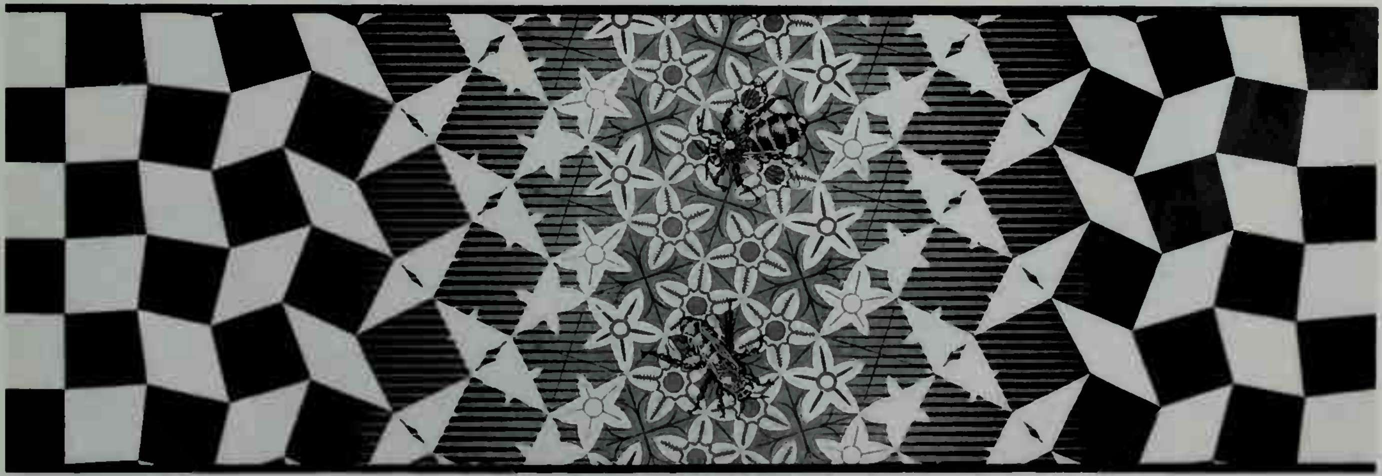


50 Periodic design 132, XII 1967. Flowers.

The geometric network of pentagons in this design (Figure 50) is boldly visible. It is the same grid which invisibly underlies "Shells and Starfish" (see Figure 36a). Escher used this design as a fragment in his extended *Metamorphosis* print (Figure 51); see *The Graphic Work of M.C. Escher* for a reproduction of the complete 700-centimeter-long print.

A startling change in design takes place as you transform this Kaleidocycle to its three-dimensional form. The design on the flat pattern is created by a red grid of hexagons intersecting at right angles a blue grid of hexagons; thus, each pentagon formed by these super-imposed grids has some red edges and some blue edges. When viewing the design on the three-dimensional Kaleidocycle, however, you will see that the pentagons now have all edges of the same color! Each turn of the Kaleidocycle changes the color of the pentagons and their direction of rotation about the center point.

51 Fragment of *Metamorphosis III*, woodcut, 1967-1968.

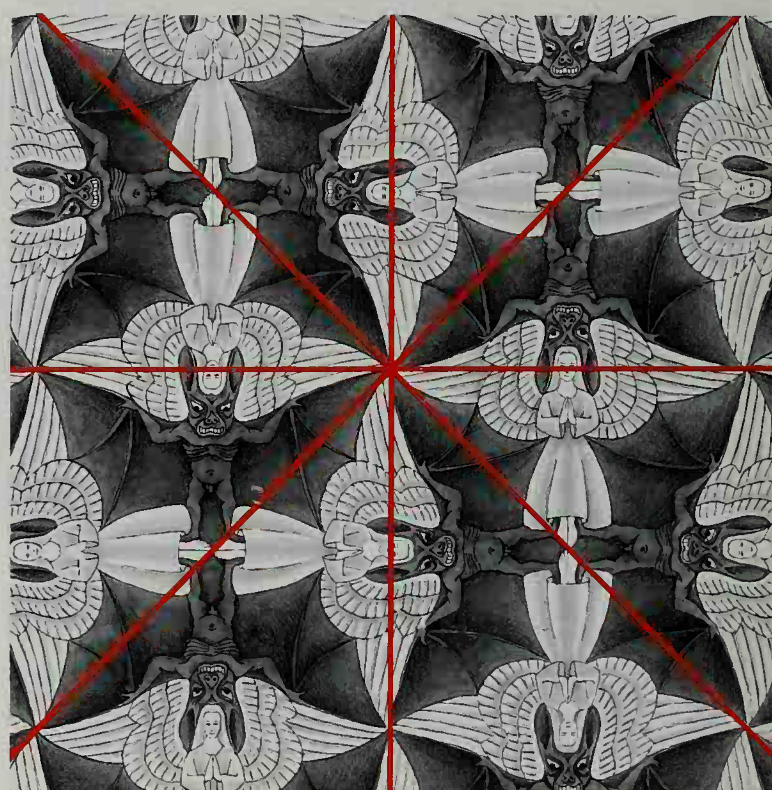


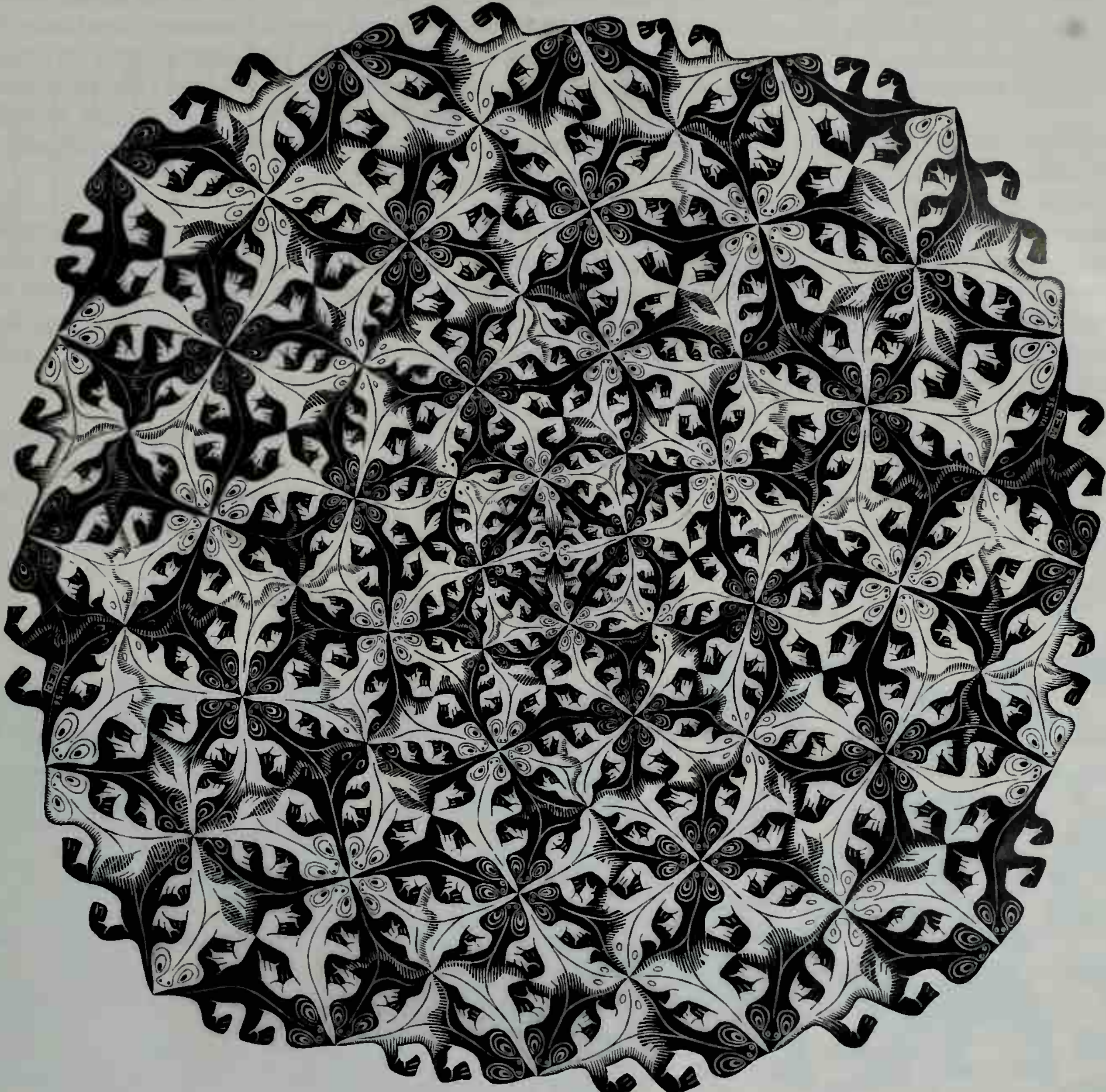
Heaven and Hell

Interlocking motifs depicting opposites is a device frequently used by Escher to portray the inability to recognize one without the other. "Heaven and Hell" is such a design (Figure 52). (The print *Encounter* [Figure 14] interlocks a pessimist and an optimist.) Although this periodic design of angels and devils was never incorporated in this form into a print, Escher produced a version of "Heaven and Hell" in *Circle Limit IV,* which is a hyperbolic tessellation. He also used the periodic design as the basis for covering a sphere with a carved version of interlocked angels and devils. (If the periodic design is wrapped around a cube, as we have done with the cube with fish, the projection of the cube onto a sphere surrounding it is exactly Escher's carved "Heaven and Hell" sphere.)

To cover the Kaleidocycle, wingtips of angels and devils are joined to form the outline of right triangles contained in a square; these triangles are the faces of the Kaleidocycle. With each turn of the Kaleidocycle the angels and devils change their direction of rotation about the center point of the model—rotating clockwise, then counterclockwise.

52 Periodic design 45, Christmas 1941. *Heaven and Hell.*





53 *Division*, woodcut 1956.

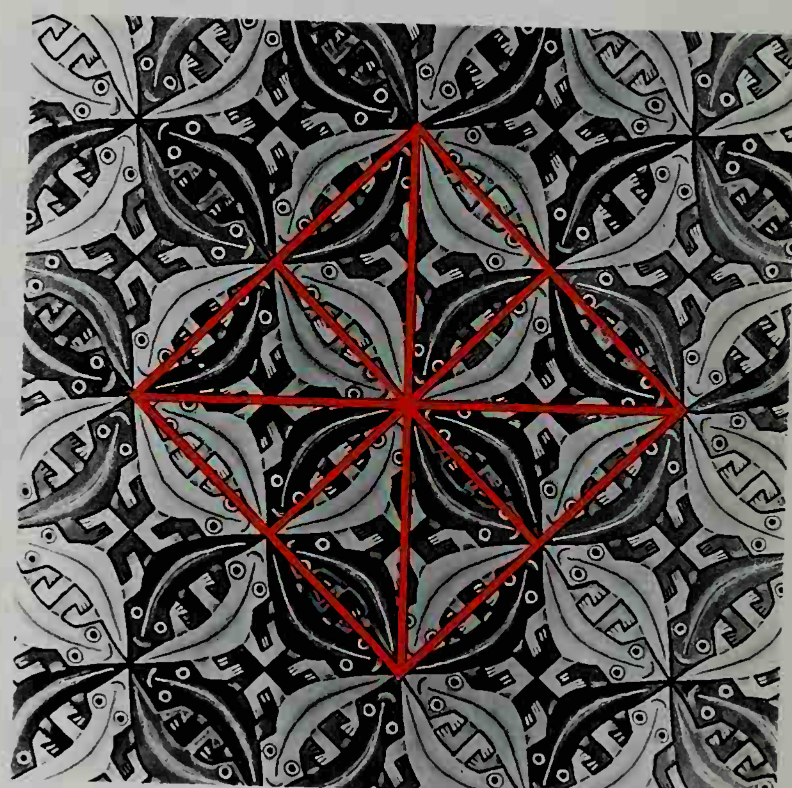
Lizards

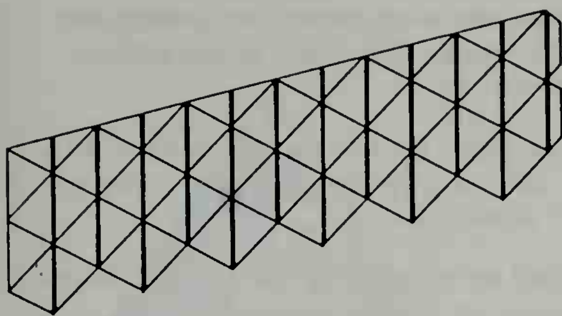
Escher's print *Division* (Figure 53) may at first glance appear to be a periodic design of lizards. However, closer examination reveals many ambiguities (indeed,"divisions" of lizards) and no true repetition of the motifs. His colored drawing of lizards from which the print is derived (Figure 16) is truly periodic, and he also produced a four color version of this same design (Figure 54).

To cover a Kaleidocycle with this design, we form a square grid by joining the centers of four-fold rotation (where heads of four lizards meet), and then bisect these squares into isosceles triangles. The outlined square containing eight triangles (Figure 54) is the view you will see looking at the completed model from above.

From a distance, Escher's four-color design appears to be interlocked circles of different colors. We have map-colored our decorated model with a rearrangement of Escher's four colors. Each turn of the Kaleidocycle brings a different coloration into view.

54 Periodic design 118, IV 1963. Lizards (four-color variation of design.) See Figure 16.





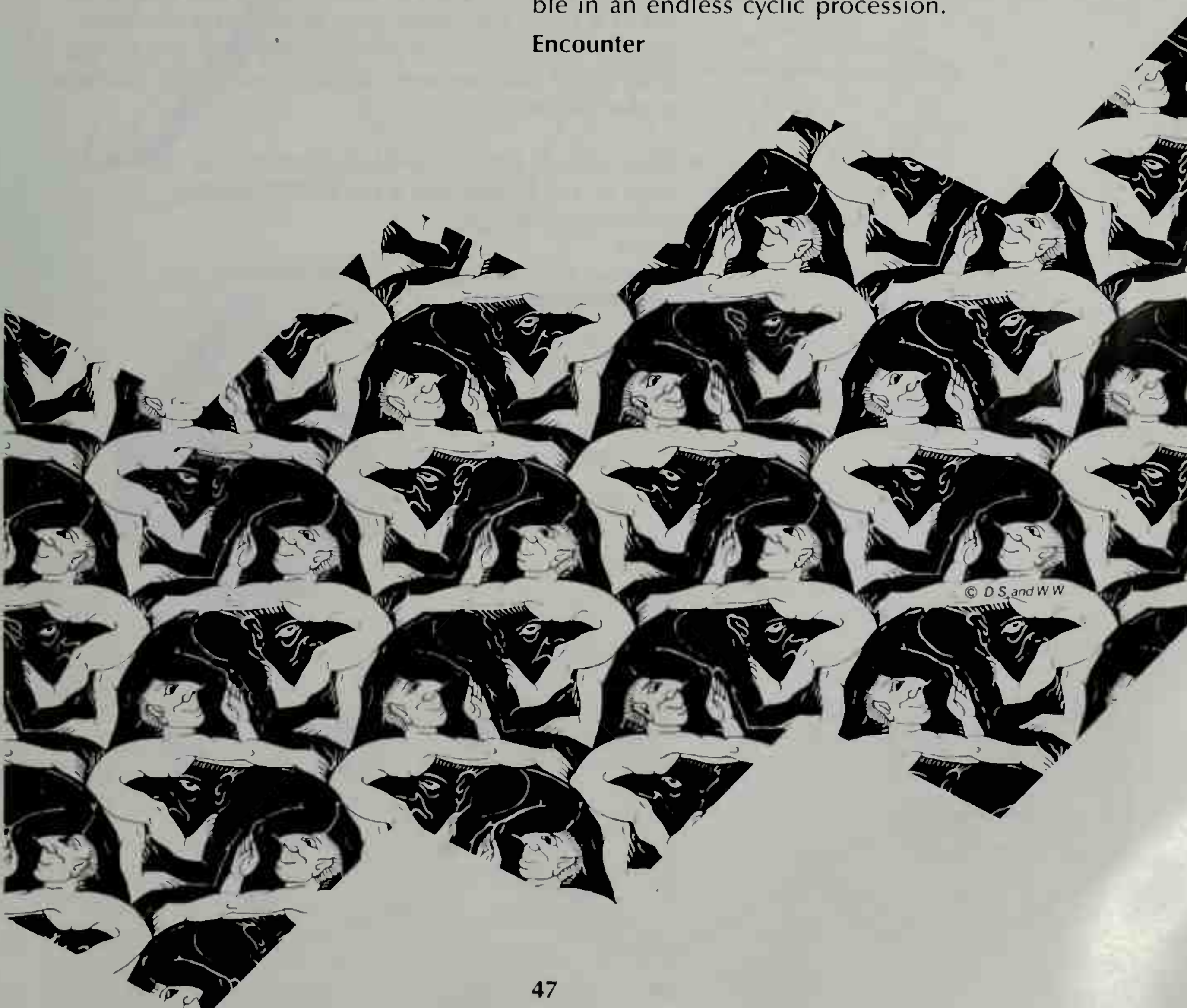
Twisted Kaleidocycle

The interlocking design (Figure 19) from which emerge the pessimist and optimist in the print *Encounter* (Figure 14) cannot be adapted to a symmetric Kaleidocycle; however, it easily wraps around the surface of a twisted Kaleidocycle.

The lattice of points which underlies this periodic design is one of rectangles (to see this, find all repetitions of a chosen point in the pattern – for instance, the tip of the nose of a pessimist facing right). To cover the twisted model, a slanted grid of triangles was super-imposed on the periodic design so that top and bottom edges would match and right and left edges would match. The dimensions of the rectangles in the lattice of the periodic design dictated the edge length of the triangles and the steepness of slant in the flat pattern for the model.

As you turn this jagged ring, the interlocked figures will tumble in an endless cyclic procession.

Encounter



Assembly Instructions for All Models

1 Carefully pull away the extra paper surrounding the flat pattern of the model. The edges of the pattern should be clean-cut.

2 Follow directions below for folding the patterns on the score lines. Take care not to crease the pattern except on these score lines.

3 Before gluing, fold up each model as directed, to test that you understand its proper assembly.

4 Use a quick-drying glue suitable for paper, such as a clear cement or white glue. Do *not* use an "instant bond" glue, since you will need to slide the glued edges to obtain a perfect match of the design. Paste or other thick glue is also unsuitable.

5 Apply a small amount of glue to tabs and glue one tab at a time. Rub finger over glued seam to obtain tight bond, without air pockets. Wipe off any excess glue which may squeeze out of a seam onto the face of the model.

6 Glue tabs to the *inside* of each model so that each edge of the model has a perfectly matched continuous design.

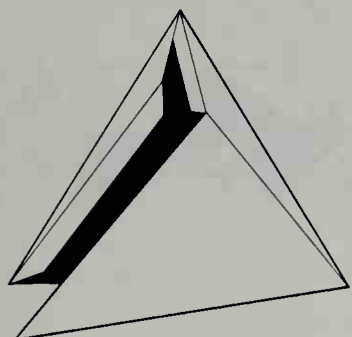
7 Let glue dry thoroughly before handling the completed models.

Assembly of Solid Models

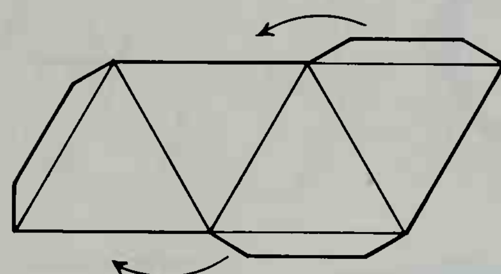
Fold the pattern *back to back* along *all* scored lines, including those which adjoin tabs. Fold up each model as indicated in the diagrams below and then glue each tab to the inside of the adjoining face. In every case carefully match the design at the joined edges. Rub the seams which have been glued in order to obtain a secure bond.

Tetrahedron

Fold up and glue tabs to inside of edges as shown in Figure 55. Glue remaining edge.

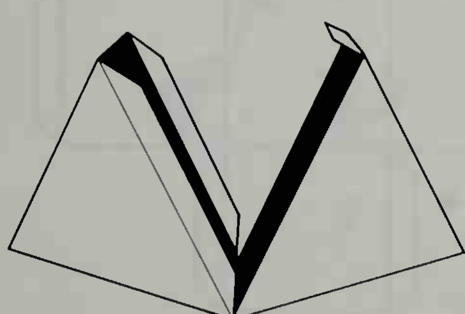


55 Tetrahedron Assembly.

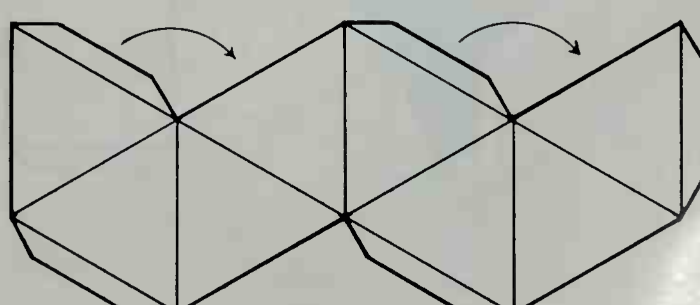


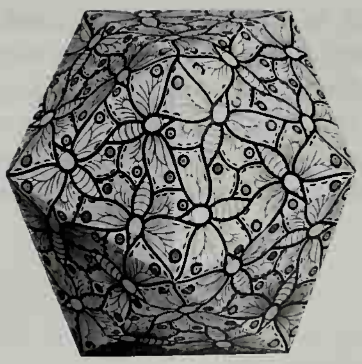
Octahedron

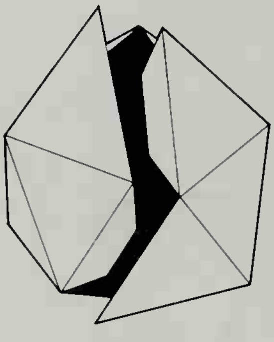
Join two halves of model, gluing tab A to matching edge to form flat pattern shown in Figure 56. Fold up and glue tabs to inside of edges as shown: this forms two hinged pyramids. Join the matching edges of the pyramids.



56 Octahedron Assembly.



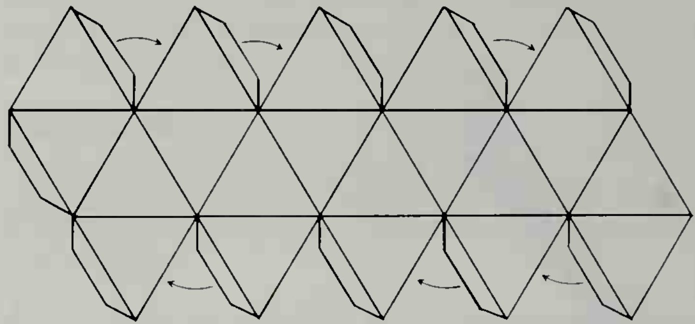


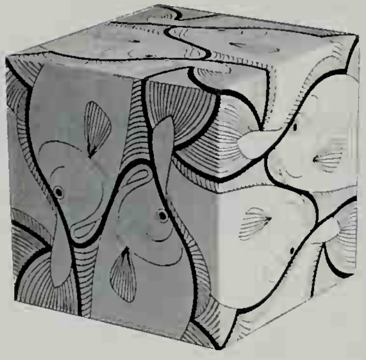


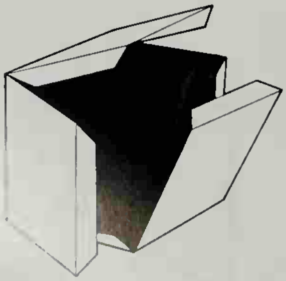
Icosahedron

Join two halves of model, gluing tab A to matching edge to form flat pattern shown in Figure 57. Fold up and glue tabs to inside of edges as shown: this forms two joined "caps." Join the matching edges of the caps.

57 Icosahedron Assembly.



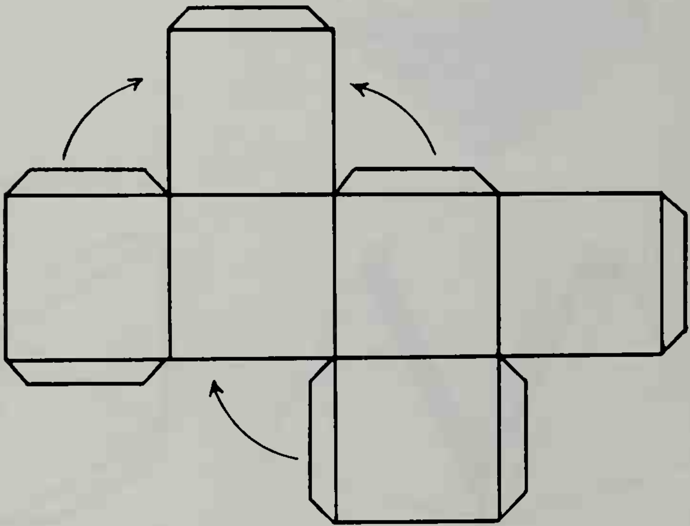




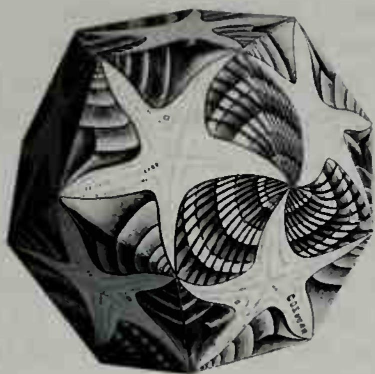
Cube

Join two halves of model, gluing tab A to matching edge to form flat pattern shown in Figure 58. Fold up and glue tabs to inside of edges as shown. Now complete by joining edges of the box.

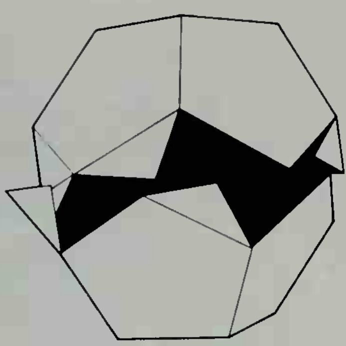
58 Cube Assembly.



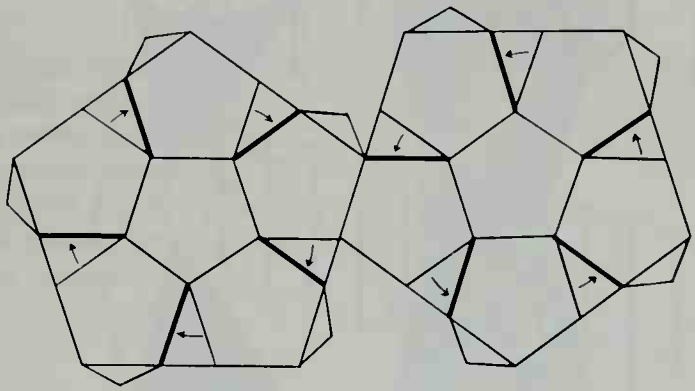
Dodecahedron



Join two halves of model, gluing tab A to matching edge to form flat pattern shown in Figure 59. Fold up and glue tabs to inside of edges of the "petals" as shown; this forms two hinged "caps." Join the matching edges of the caps.

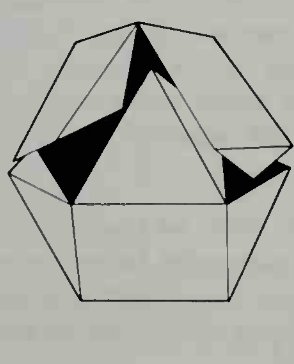


59 Dodecahedron Assembly.

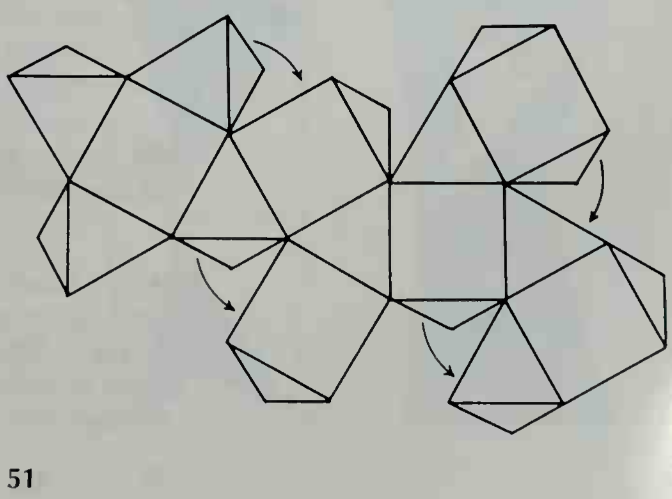


Cuboctahedron

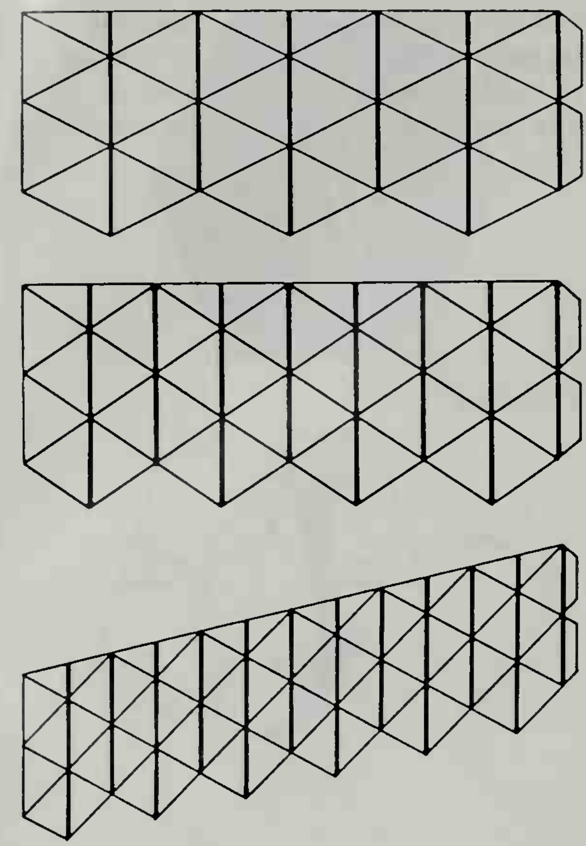
Fold up and glue tabs to inside of edges as shown in Figure 60; this forms two hinged "caps." Join matching edges of the caps, always joining a triangle to a square.



60 Cuboctahedron Assembly.



Assembly of Kaleidocycles



61 Hexagonal Kaleidocycle Grid. Square Kaleidocycle Grid. Twisted Kaleidocycle Grid.

Figure 61 shows grids of the three types of Kaleidocycles. The two halves of the twisted Kaleidocycle should be joined, gluing tab A to matching edge. Thick lines on these diagrams are *vertical:* other lines are *diagonal.*

All patterns are folded on the scored lines as follows: Fold *face to face* on all *vertical* scored lines, including those which adjoin tabs. Fold *back to back* on all *diagonal* scored lines. The folded pattern will begin to naturally curl into shape.

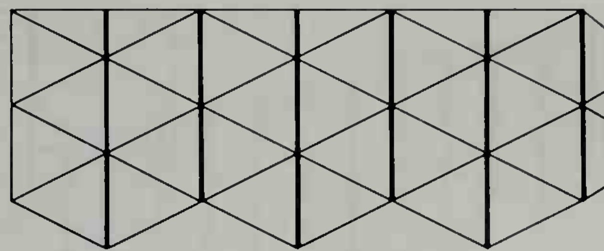
Gently cup the pattern in your hands so that the bottom triangles come around to meet the triangular blank white tabs at the top of the pattern. Put glue on these white tabs and glue the triangles to these tabs. Match designs exactly and make sure that all seams are completely sealed. You now have a chain of linked tetrahedra. Let glue dry before proceeding to the next step.

Hold the chain of tetrahedra in both hands and bring its ends together to form a ring. You may need to turn the ring to accomplish this. The double tab on one end will be fitted inside the slot at the other end of this ring (For the twisted model, the ring must be twisted to accomplish this.)

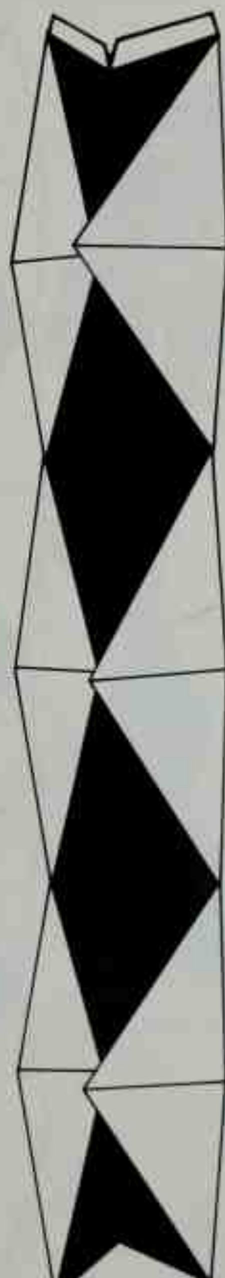
Put glue on each of the outer sides of the double tab and slide it into the open slot at the other end of the ring. Adjust its position for a perfect match of design. Turn the ring so that you can apply pressure with your finger to seal the seam. Wipe off excess glue and let the seam set for several seconds, holding the model in position so that the seam does not separate. Now gently turn the model over and rotate it slightly to bring into view the seam on the opposite side. Match the design and seal this seam, again holding the model so that the seam does not separate.

When these seams have "set," *let the model dry thoroughly* (preferably overnight) before attempting to rotate it. When dry, it can be rotated in a continuous cycle of motion by pushing the points of the tetrahedra through the center hole.

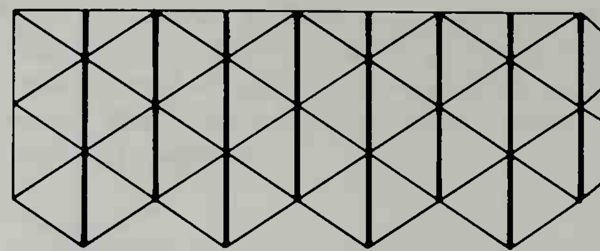
Hexagonal Kaleidocycles



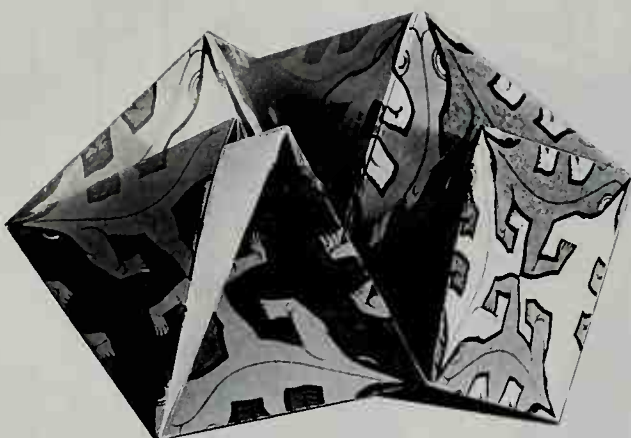
62 Hexagonal Kaleidocycle Assembly.

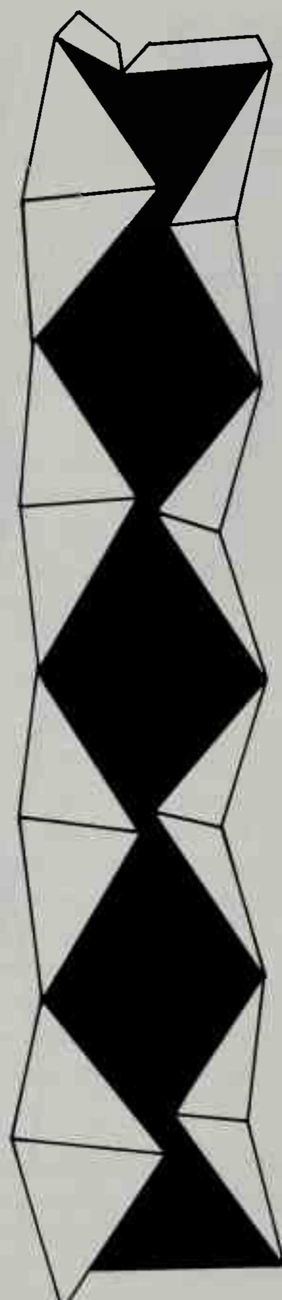


Square Kaleidocycles

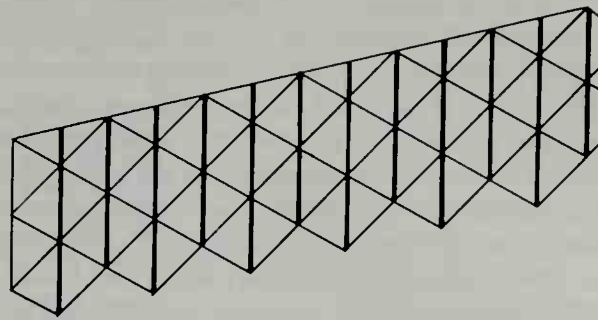


63 Square Kaleidocycle Assembly.

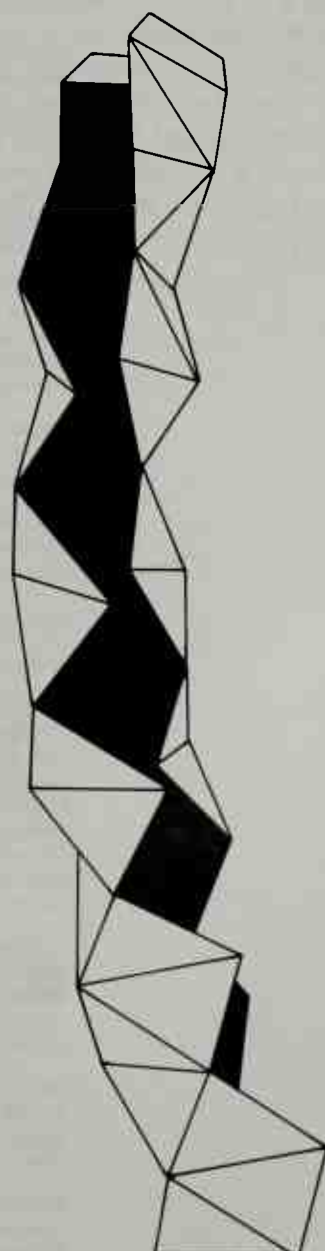




Twisted Kaleidocycle



64 Twisted Kaleidocycle Assembly.



Sources for More Information

Beard, Col. R.S., *Patterns in Space,* Creative Publications, Palo Alto, California, 1973. Many two- and three-dimensional mathematical designs and constructions. Two rotating rings of tetrahedra are described.

Bool, F.H., Kist, J.R., Locher, J.L. and Wierda, F., *M.C. Escher, His Life and Complete Graphic Work,* Harry N. Abrams, New York, 1982. The most complete presentation of Escher's work, including the full text and plates from Escher's book *Regelmatige vlakverdeling* (Regular Division of the Plane), 1958. Extensive quotes from Escher's diaries and letters personalize the excellent biography.

Coxeter, H.S.M., *Introduction to Geometry,* Second Edition, John Wiley & Sons, New York, 1969. Covers the wide range of geometries. College level. (Through correspondence with Coxeter, Escher learned the mathematical rules which govern hyperbolic tessellations; *Circle Limit III* is an example.)

Coxeter, H.S.M., Emmer, M., Penrose, R. and Teuber, M. (eds.), *M.C. Escher: Art and Science,* North Holland, Amsterdam, 1986. Essays by a variety of authors on many aspects of Escher's work.

Cundy, H.N. and Rollett, A.P., *Mathematical Models,* Second Edition, Oxford University Press, Oxford, 1961. Detailed information on the Platonic and Archimedean solids, and many other beautiful geometric models.

Ernst, Bruno, *The Magic Mirror of M.C. Escher,* Random House, New York, 1976; paperback edition, Ballantine Books, New York, 1977. Discusses all aspects of Escher's work, including Escher's use of mathematics to obtain many of his surprising effects.

Escher, M.C., *The Graphic Work of M.C. Escher,* Ballantine Books, New York, 1971. A collection of Escher's work with notes by the artist.

Grünbaum, B. and Shephard, G.C., *Tilings and Patterns,* W.H. Freeman & Co., New York, 1986. Detailed study of the mathematical aspects of tilings, including Escher's work.

Locher, J.L. (ed.), *The World of M.C. Escher,* Harry N. Abrams, Inc., New York, 1971. A collection of Escher's work with essays concerning various aspects of it.

MacGillavry, Caroline, *Fantasy and Symmetry, The Periodic Drawings of M.C. Escher,* Harry N. Abrams, Inc., New York, 1976. Reprint of 1965 edition, published for International Union of Crystallography by A. Oosthoek's Uitgeversmaatschappij, Utrecht, under title: *Symmetry Aspects of M.C. Escher's Periodic Drawings.* Contains 41 of Escher's periodic drawings. Originally written to interest beginning crystallography students in the laws which underlie repeating designs and their colorings.

O'Daffer, P.G. and Clemens, S.R., *Geometry: An Investigative Approach,* Addison-Wesley, Menlo Park, California, 1976. Includes wide range of elementary information on Platonic solids, transformation geometry, and repeating designs, with Escher's work used frequently for illustration.

Senechal, M. and Fleck, G. (eds.) *Shaping Space,* Birkhäuser, Boston, 1987. Essays, teaching ideas, problems—all related to the subject of polyhedra. The authors include mathematicians, artists, scientists and teachers.

Weninger, Magnus J., *Polyhedron Models,* Cambridge University Press, Cambridge, 1971. Construction of Platonic, Archimedean, and other symmetric geometric solids.

Authors

Doris Schattschneider is a professor of mathematics at Moravian College in Bethlehem, Pennsylvania. Her dual interest in geometry and art led naturally to the study of M.C. Escher's work. Active as a teacher, lecturer, editor and writer, she has authored several articles on tiling and on Escher's pioneering mathematical investigations. Two recent articles appear as chapters in the books *Symmetry: Unifying Human Understanding,* edited by István Hargittai, Pergamon, 1986, and *M.C. Escher: Art and Science,* North-Holland, 1986.

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