INSTRUCTOR'S SOLUTIONS MANUAL

to accompany

CALCULUS

ONE AND SEVERAL VARIABLES

EIGHTH EDITION

SATURNINO SALAS

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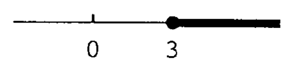
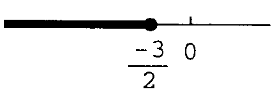
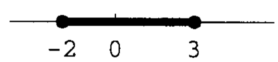
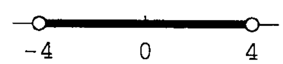
CHAPTER 1

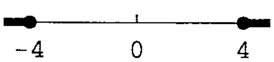
SECTION 1.2

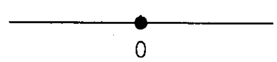
1. rational, complex
2. integer, rational, complex
3. rational, complex
4. irrational, complex
5. integer, rational, complex
6. irrational, complex
7. integer, rational, complex
8. rational, complex
9. integer, rational, complex
10. irrational, complex
11. $\frac{3}{4} = 0.75$
12. $0.33 < \frac{1}{3}$
13. $\sqrt{2} > 1.414$
14. $4 = \sqrt{16}$
15. $-\frac{2}{7} < -0.28517$
16. $\pi < \frac{22}{7}$
17. $|6| = 6$
18. $|-4| = 4$
19. $|3-7| = 4$
20. $|-5| - |-8| = -3$
21. $|-5| + |-8| = 13$
22. $|2-\pi| = \pi - 2$
23. $|5-\sqrt{5}| = 5-\sqrt{5}$
24. 

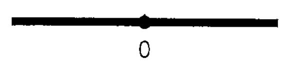
25. 

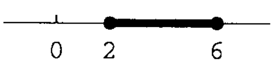
26. 

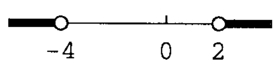
27. 

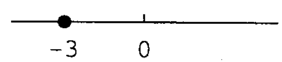
28. 

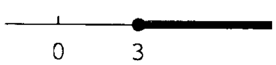
29. 

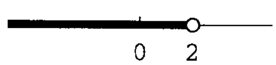
30. 

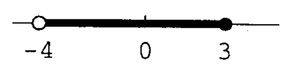
31. 

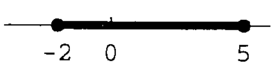
32. 

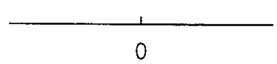
33. 

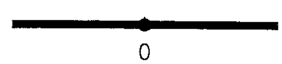
34. 

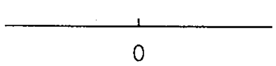
35. 

36. 

37. 

38. 

39. 

40. 

41. bounded, lower bound 0, upper bound 4
42. bounded above by 0
43. not bounded
44. bounded above by 4
45. not bounded
46. bounded; lower bound 0, upper bound 1
47. bounded above, upper bound $\sqrt{2}$
48. $9(x-\frac{2}{3})(x+\frac{2}{3})$
49. $x^2 - 10x + 25 = (x-5)^2$
50. $27(x-\frac{2}{3})(x^2+\frac{2}{3}x+\frac{4}{9})$
51. $8x^6 + 64 = 8(x^2+2)(x^4-2x^2+4)$
52. $4(x^2+\frac{1}{2})^2$
53. $4x^2 + 12x + 9 = (2x+3)^2$
54. $-3, 3$
55. $x^2 - x - 2 = (x-2)(x+1) = 0; \quad x = 2, -1$
56. $-\frac{1}{2}, 3$
57. $x^2 - 6x + 9 = (x-3)^2; \quad x = 3$

58. -4

59. $x^2 - 2x + 2 = 0$; no real zeros

60. no real zeros

61. $5! = 120$

62. $\frac{5!}{8!} = \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{1}{336}$

63. $\frac{8!}{3!5!} = \frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = 56$

64. $\frac{9!}{3!6!} = \frac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = 84$

65. $\frac{7!}{0!7!} = \frac{7!}{1\cdot 7!} = 1$

66. $\frac{p_1}{q_2} + \frac{p_2}{q_2} = \frac{p_1q_2 + p_2q_1}{q_1q_2}$, $p_1q_2 + p_2q_1$ and q_1q_2 are integers, and $q_1q_2 \neq 0$

67. Let r be a rational number and s an irrational number. Suppose $r+s$ is rational. Then $(r+s)-r = s$ is rational which contradicts the fact that s is irrational.

68. $\left(\frac{p_1}{q_1}\right)\left(\frac{p_2}{q_2}\right) = \frac{p_1p_2}{q_1q_2}$, p_1p_2 and q_1q_2 are integers, and $q_1q_2 \neq 0$

69. The product of a rational and an irrational number may either be rational or irrational;
$0\cdot\sqrt{2} = 0$ is rational, $1\cdot\sqrt{2} = \sqrt{2}$ is irrational.

70. $\sqrt{2} + 3\sqrt{2} = 4\sqrt{2}$ irrational; $\pi + (1-\pi) = 1$, rational.
$(\sqrt{2})(\sqrt{3}) = \sqrt{6}$ irrational; $(\sqrt{2})(3\sqrt{2} = 6$, rational.

71. Suppose that $\sqrt{2} = p/q$ where p and q are integers and $q \neq 0$. Assume that p and q have no common factors (other than ± 1). Then $p^2 = 2q^2$ and p^2 is even. This implies that $p = 2r$ is even. Thus $2q^2 = 4r^2$ which implies that q^2 is even, and hence q is even. It now follows that p and q are both even and contradicts the assumption that p and q have no common factors.

72. Assume $\sqrt{3} = \frac{p}{q}$, where p and q have no common factors. Then $3 = \frac{p^2}{q^2}$, so $p^2 = 3q^2$. Thus p^2 is divisible by 3, and therefore p is divisible by 3, say $p = 3a$. Then $9a^2 = 3q^2$, so $3a^2 = q^2$, where q must also be divisible by 3, contracting out assumption.

73. Let x be the length of a rectangle that has perimeter P. Then the width y of the rectangle is given by $y = \frac{1}{2}(P - x)$ and the area is

$$A = x\left(\frac{1}{2}P - x\right) = \left(\frac{P}{4}\right)^2 - \left(x - \frac{P}{4}\right)^2.$$

It now follows that the area is a maximum when $x = P/4$. Since $y = P/4$ when $x = P/4$, the rectangle of perimeter P having the largest area is a square.

74. Circle: perimeter $2\pi r = p \implies r = \frac{p}{2\pi} \implies \mathit{area} = \pi r^2 = \frac{p^2}{4\pi}$
square: perimeter $4x = p \implies x = \frac{p}{4} \implies$ area $= x^2 = \frac{p^2}{16} < \frac{p^2}{4\pi}$.
For an arbitrary rectangle, $p = 2(x+y)$, so $y = \frac{p}{2} - x$, and area $= xy = x(\frac{p}{2} - x)$. This is the equation of an inverted parabola with vertex (hence maximum value) at $x = \frac{p}{4}$. Thus $y = \frac{p}{4}$ and the rectangle is a square. The circle still has larger area.

PROJECT 1.2

1. $\dfrac{p}{q}$ terminates when q is of the form 2^m5^n for m, n nonnegative integers.

3. (a) $x = 13.201201\cdots$, $1000x = 13201.201201\cdots$. Therefore, $999x = 13188$ and $x = \dfrac{13188}{999}$.

(b) $2.777\cdots = \dfrac{25}{9}$.

(c) $x = 0.2323\cdots$, $100x = 23.2323\cdots$. Therefore, $99x = 23$ and $x = \dfrac{23}{99}$.

(d) $4.16\overline{3} = \dfrac{3477}{900}$

Let $x = 0.999\cdots$, Then $10x - x = 9.\overline{9} - 0.\overline{9}$, so $x = \dfrac{9}{9} = 1$

SECTION 1.3

1.
$$2 + 3x < 5$$
$$3x < 3$$
$$x < 1$$
Ans: $(-\infty, 1)$

2.
$$\tfrac{1}{2}(2x+3) < 6$$
$$2x + 3 < 12$$
$$2x < 9$$
Ans: $(-\infty, \frac{9}{2})$

3.
$$16x + 64 \le 16$$
$$16x \le -48$$
$$x \le -3$$
Ans: $(-\infty, -3]$

4.
$$3x + 5 > \tfrac{1}{4}(x-2)$$
$$12x + 20 > x - 2$$
$$11x > -22$$
$$x > -2$$
Ans: $(-2, \infty)$

5.
$$\tfrac{1}{2}(1+x) < \tfrac{1}{3}(1-x)$$
$$3(1+x) < 2(1-x)$$
$$3 + 3x < 2 - 2x$$
$$5x < -1$$
$$x < -\tfrac{1}{5}$$
Ans: $(-\infty, -\frac{1}{5})$

6.
$$3x - 2 \le 1 + 6x$$
$$-3x \le 3$$
$$x \ge -1$$
Ans: $[-1, \infty)$

7.
$$x^2 - 1 < 0$$
$$(x+1)(x-1) < 0$$
Ans: $(-1, 1)$

8.
$$x^2 + 9x + 20 < 0$$
$$(x+5)(x+4) < 0$$
Ans: $(-5, -4)$

9.
$$x(x-1)(x-2) > 0$$
Ans: $(0,1) \cup (2, \infty)$

10.
$$x(2x-1)(3x-5) \le 0$$
Ans: $(-\infty, 0] \cup [\frac{1}{2}, \frac{5}{3}]$

11.
$$x^3 - 2x^2 + x \ge 0$$
$$x(x-1)^2 \ge 0$$
Ans: $[0, \infty)$

12.
$$x^2 - 4x + 4 \le 0$$
$$(x-2)^2 \le 0$$
Ans: $\{2\}$

13.
$$\frac{1}{x} < x$$
$$x - \frac{1}{x} > 0$$
$$\frac{x^2-1}{x} > 0$$
$$x(x-1)(x+1) > 0 \qquad \text{(by 1.3.1)}$$
$$(x+1)\,x\,(x-1) > 0$$
Ans: $(-1, 0) \cup (1, \infty)$

14.
$$x + \frac{1}{x} \ge 1$$
$$x + \frac{1}{x} - 1 \ge 0$$
$$\frac{x^2 - x + 1}{x} \ge 0$$
$$x(x^2 - x + 1) > 0 \qquad \text{(by 1.3.1)}$$
$$x > 0$$
Ans: $(0, \infty)$

15.

$$\frac{x}{x-5} > \frac{1}{4}$$
$$\frac{x}{x-5} - \frac{1}{4} > 0$$
$$\frac{4x-(x-5)}{4(x-5)} > 0$$
$$\frac{3x+5}{4(x-5)} > 0$$
$$4(x-5)(3x+5) > 0 \quad \text{(by 1.3.1)}$$
$$(3x+5)(x-5) > 0$$
$$\text{Ans:} \quad \left(-\infty, -\frac{5}{3}\right) \cup (5, \infty)$$

16.

$$\frac{1}{3x-5} < 2$$
$$\frac{1}{3x-5} - 2 < 0$$
$$\frac{1-2(3x-5)}{3x-5} < 0$$
$$\frac{6-6x}{3x-5} < 0$$
$$6(1-x)(3x-5) < 0 \quad \text{(by 1.3.1)}$$
$$\text{Ans:} \quad (-\infty, 1) \cup (\frac{5}{3}, \infty)$$

17.

$$\frac{x^2-9}{x+1} > 0$$
$$(x+1)(x-3)(x+3) > 0 \quad \text{(by 1.4.1)}$$
$$(x+3)(x+1)(x-3) > 0$$
$$\textit{Ans}: (-3, -1) \cup (3, \infty)$$

18.

$$\frac{x^2}{x^2-4} < 0$$
$$\frac{x^2}{(x-2)(x+2)} < 0 \quad \text{(by 1.3.1)}$$
$$x^2(x-2)(x+2) < 0$$
$$\text{Ans:} \quad (-2, 0) \cup (0, 2)$$

19.

$$x^3(x-2)(x+3)^2 < 0$$
$$(x+3)^2 x(x-2) < 0$$
$$\text{Ans:} \quad (0, 2)$$

20.

$$x^3(x-3)(x+4)^2 > 0$$
$$\text{Ans:} \quad (3, \infty)$$

21.

$$x^2(x-2)(x+6) > 0$$
$$(x+6)x^2(x-2) > 0$$
$$\text{Ans:} \quad (-\infty, -6) \cup (2, \infty)$$

22.

$$7x(x-4)^2 < 0$$
$$\text{Ans:} \quad (-\infty, 0)$$

23.

$$\frac{1}{x-1} + \frac{4}{x-6} > 0$$
$$\frac{x-6+4(x-1)}{(x-1)(x-6)} > 0$$
$$\frac{5x-10}{(x-1)(x-6)} > 0$$
$$5(x-2)(x-1)(x-6) > 0$$
$$(x-1)(x-2)(x-6) > 0$$
$$\text{Ans:} (1, 2) \cup (6, \infty)$$

24.

$$\frac{3}{x-2} - \frac{5}{x-6} < 0$$
$$\frac{3(x-6)-5(x-2)}{(x-2)(x-6)} < 0$$
$$\frac{-2x-8}{(x-2)(x-6)} < 0$$
$$-2(x+4)(x-2)(x-6) < 0$$
$$\text{Ans:} \quad (-4, 2) \cup (6, \infty)$$

25.

$$\frac{2x-6}{x^2-6x+5}<0$$
$$2(x-3)(x-1)(x-5)<0$$
$$(x-1)(x-3)(x-5)<0$$
Ans: $(-\infty,1)\cup(3,5)$

26.

$$\frac{2x+8}{x^2+8x+7}>0$$
$$2(x+4)(x+7)(x+1)>0$$
Ans: $(-7,-4)\cup(-1,\infty)$

27. $(-2,2)$

28. $(-\infty,-1]\cup[1,\infty)$

29. $(-\infty,-3)\cup(3,\infty)$

30. $(0,2)$

31. $(\frac{3}{2},\frac{5}{2})$

32. $(-\frac{3}{2},\frac{5}{2})$

33. $(-1,0)\cup(0,1)$

34. $(-\frac{1}{2},0)\cup[0,\frac{1}{2})$

35. $(\frac{3}{2},2)\cup(2,\frac{5}{2})$

36. $(-\frac{3}{2},\frac{1}{2})\cup(\frac{1}{2},\frac{5}{2})$

37. $(-5,3)\cup(3,11))$

38. $(\frac{2}{3},\frac{8}{3}))$

39. $(-\frac{5}{8},-\frac{3}{8})$

40. $(\frac{1}{2},\frac{7}{10})$

41. $(-\infty,-4)\cup(-1,\infty)$

42. $(-\infty,-2)\cup(\frac{4}{3},\infty)$

43. $(-\infty,-\frac{8}{5})\cup(2,\infty)$

44. $|x-0|<2$ or $|x|<2$

45. $|x-0|<3$ or $|x|<3$

46. $|x-2|<2$

47. $|x-2|<5$

48. $|x-(-2)|<2$ or $|x+2|<2$

49. $|x-(-2)|<5$ or $|x+2|<5$

50. $|x-2|<1 \implies |2x-4|=2|x-2|<2,$ so $|2x-4|<A$ true for $A\geq 2$.

51. $|x-2|<A \implies 2|x-2|=|2x-4|<2A \implies |2x-4|<3$
provided that $0<A\leq\frac{3}{2}$

52. $|x+1|<A \implies |3x+3|=3|x+1|<3A \implies |3x+3|<4$
provided that $0<A\leq\frac{4}{3}$

53. $|x+1|<2 \implies 3|x+1|=|3x+3|<6 \implies |3x+3|<A$
provided that $A\geq 6$

54. $\dfrac{1}{x}<\dfrac{1}{\sqrt{x}}<1<\sqrt{x}<x.$

55. $x<\sqrt{x}<1<\dfrac{1}{\sqrt{x}}<\dfrac{1}{x}$

56. $\sqrt{\dfrac{x}{x+1}}<\sqrt{\dfrac{x+1}{x+2}}.$

57. If a and b have the same sign, then $ab>0$. Suppose that $a<b$. Then $a-b<0$ and

$$\frac{1}{b}-\frac{1}{a}=\frac{a-b}{ab}<0.$$

Thus, $(1/b)<(1/a)$.

58. $a^2 \le b^2 \implies b^2 - a^2 = (b+a)(b-a) \ge 0 \implies b - a \ge 0 \implies a \le b.$

59. With $a \ge 0$ and $b \ge 0$

$$b \ge a \implies b - a = (\sqrt{b} + \sqrt{a})(\sqrt{b} - \sqrt{a}) \ge 0 \implies \sqrt{b} - \sqrt{a} \ge 0 \implies \sqrt{b} \ge \sqrt{a}.$$

60. $|a - b| = |a + (-b)| \le |a| + |-b| = |a| + |b|.$

61. By the hint

$$\big|\,|a| - |b|\,\big|^2 = (|a| - |b|)^2 = |a|^2 - 2|a|\,|b| + |b|^2 = a^2 - 2|ab| + b^2$$

$$\le a^2 - 2ab + b^2 = (a-b)^2.$$

$$(ab \le |ab|)$$

Taking the square root of the extremes, we have

$$\big|\,|a| - |b|\,\big| \le \sqrt{(a-b)^2} = |a - b|.$$

62. If $a \ge 0$ and $b \ge 0:\ |a + b| = a + b = |a| + |b|.$

If $a < 0$ and $b < 0:\ |a + b| = -(a + b) = -a - b = |a| + |b|.$

If $a \ge 0$ and $b < 0$: If $a \ge |b|$ then $|a + b| = a - |b| < a + |b| = |a| + |b|.$

If $a < |b|$ then $|a + b| = |b| - a < |b| + a = |a| + |b|.$

Similarly, $a < 0, b \ge 0 \implies |a + b| < |a| + |b|.$

Thus equality holds iff a and b are of the same sign.

63. With $0 \le a \le b$

$$a\,(1 + b) = a + ab \le b + ab = b\,(1 + a).$$

Division by $(1 + a)(1 + b)$ gives

$$\frac{a}{1+a} \le \frac{b}{1+b}.$$

64. $\dfrac{a}{1+a} \le \dfrac{b+c}{1+b+c} = \dfrac{b}{1+b+c} + \dfrac{c}{1+b+c} \le \dfrac{b}{1+b} + \dfrac{c}{1+c}.$

by exercise 63

65. Suppose that $a < b$. Then

$$a = \frac{a+a}{2} \le \frac{a+b}{2} \le \frac{b+b}{2} = b.$$

$\dfrac{a+b}{2}$ is the midpoint of the line segment $\overline{ab}$.

66. First inequality: $a = (\sqrt{a})^2 \leq \sqrt{a}\sqrt{b} = \sqrt{ab}.$

Last inequality: $\frac{1}{2}(a+b) \leq \frac{1}{2}(b+b) = b.$

Middle inequality:

$$\begin{aligned} 0 &\leq (a-b)^2 = a^2 - 2ab + b^2 = (a+b)^2 - 4ab \\ 4ab &\leq (a+b)^2 \\ 2\sqrt{ab} &\leq (a+b) \quad \text{(by Exercise 59)} \\ \sqrt{ab} &\leq \frac{1}{2}(a+b) \end{aligned}$$

SECTION 1.4

1. $d(P_0, P_1) = \sqrt{(6-5)^2 + (-3-0)^2} = \sqrt{1+9} = \sqrt{10}$

2. $d(P_0, P_1) = \sqrt{(5-2)^2 + (5-2)^2} = 3\sqrt{2}$

3. $d(P_0, P_1) = \sqrt{[5-(-3)]^2 + (-2-2)^2} = \sqrt{64+16} = 4\sqrt{5}$

4. $d(P_0, P_1) = \sqrt{(-5)^2 + (12)^2} = 13$

5. $d(P_0, P_1) = \sqrt{(2-2)^2 + (-2-4)^2} = \sqrt{0+36} = 6$

6. $d(P_0, P_1) = \sqrt{(-4-2)^2 + (7-7)^2} = 6$

7. $\left(\frac{2+6}{2}, \frac{4+8}{2}\right) = (4, 6)$

8. $\left(\frac{3-1}{2}, \frac{-1+5}{2}\right) = (1, 2)$

9. $\left(\frac{2+7}{2}, \frac{-3-3}{2}\right) = (\frac{9}{2}, -3)$

10. $\left(\frac{5+5}{2}, \frac{-1+6}{2}\right) = (5, \frac{5}{2})$

11. $\left(\frac{\sqrt{3}+0}{2}, \frac{0+\sqrt{3}}{2}\right) = \frac{1}{2}(\sqrt{3}, \sqrt{3})$

12. $m = \left(\frac{a+3}{2}, \frac{3+a}{2}\right)$

13. $m = \frac{5-1}{(-2)-4} = \frac{4}{-6} = -\frac{2}{3}$

14. $m = \frac{-3-(-7)}{4-(-2)} = \frac{4}{6} = \frac{2}{3}$

15. $m = \frac{b-a}{a-b} = -1$

16. $m = \frac{-1-(-1)}{4-(-3)} = \frac{0}{7} = 0$

17. $m = \frac{0-y_0}{x_0-0} = -\frac{y_0}{x_0}$

18. $m = \frac{0-y_0}{0-x_0} = \frac{-y_0}{-x_0} = \frac{y_0}{x_0}$

19. Equation is in the form $y = mx + b$. Slope is 2; y-intercept is -4.

20. Rewrite as $5x = 6$, or $x = \frac{6}{5}$, This is a vertical line with slope undefined, no y-intercept.

21. Write equation as $y = \frac{1}{3}x + 2$. Slope is $\frac{1}{3}$; y-intercept is 2.

22. Write equation as $y = \frac{1}{2}x - \frac{4}{3}$. Slope is $\frac{1}{2}$, y-intercept is $-\frac{3}{4}$.

23. Write equation as $y = \frac{7}{3}x + \frac{4}{3}$. Slope is $\frac{7}{3}$; y-intercept is $\frac{4}{3}$.

24. Write equation as $y = \frac{3}{4}$; This is a horizontal line. Slop is 0, y-intercept is $\frac{3}{4}$.

25. $y = 5x + 2$ **26.** $y = 5x - 2$ **27.** $y = -5x + 2$ **28.** $y = -5x - 2$

29. $y = 3$ **30.** $y = -3$ **31.** $x = -3$ **32.** $x = 3$

33. Every line parallel to the x-axis has an equation of the form $y = a$ constant. In this case $y = 7$.

34. Every line parallel to the y-axis has an equation of the form $x = a$ constant. In this case $x = 2$.

35. The line $3y - 2x + 6 = 0$ has slope $\frac{2}{3}$. Every line parallel to it has that same slope. The line through $P(2, 7)$ with slope $\frac{2}{3}$ has equation $y - 7 = \frac{2}{3}(x - 2)$, which reduces to $3y - 2x - 17 = 0$.

36. The line $y - 2x + 5 = 0$ has slope 2. Every line perpendicular to it has the slope $-\frac{1}{2}$. The line through $P(2, 7)$ with slope $-\frac{1}{2}$ has equation $y - 7 = -\frac{1}{2}(x - 2)$, which reduces to $2y + x - 16 = 0$.

37. The line $3y - 2x + 6 = 0$ has slope $\frac{2}{3}$. Every line perpendicular to it has slope $-\frac{3}{2}$.
The line through $P(2, 7)$ with slope $-\frac{3}{2}$ has equation $y - 7 = -\frac{3}{2}(x - 2)$, which reduces to $2y + 3x - 20 = 0$.

38. The line $y - 2x + 5 = 0$ has slope 2. Every line parallel to it has slope 2.

The line through $P(2, 7)$ with slope 2 has equation $y - 7 = 2(x - 2)$, which reduces to

$y - 2x - 3 = 0$.

39. $\left(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right), \left(-\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}\right)$ [Substitute $y = x$ into $x^2 + y^2 = 1$.]

40. $\left(\dfrac{2}{\sqrt{1+m^2}}, \dfrac{2m}{\sqrt{1+m^2}}\right), \left(\dfrac{-2}{\sqrt{1+m^2}}, \dfrac{-2m}{\sqrt{1+m^2}}\right)$ [Substitute $y = mx$ into $x^2 + y^2 = 4$.]

41. $(3, 4)$ [Write $4x + 3y = 24$ as $y = \frac{4}{3}(6 - x)$ and substitute into $x^2 + y^2 = 25$.]

42. $(0, b), \left(\dfrac{-2mb}{1+m^2}, \dfrac{b(1-m^2)}{1+m^2}\right)$ [Substitute $y = mx + b$ into $x^2 + y^2 = b^2$.]

43. $(1, 1)$ **44.** $(-1, 2)$ **45.** $\left(-\frac{2}{23}, \frac{38}{23}\right)$ **46.** $\left(-\frac{17}{13}, -\frac{2}{13}\right)$

47. We select the side joining $A(1, -2)$ and $B(-1, 3)$ as the base of the triangle.

length of side AB : $\sqrt{29}$

equation of line through A and B : $5x + 2y - 1 = 0$

length of altitude from vertex $C\,(2, 4)$ to side AB : $\dfrac{|\,5(2) + 2(4) - 1|}{\sqrt{29}} = \dfrac{17}{\sqrt{29}}$

area of triangle: $\frac{1}{2}\left(\sqrt{29}\right)\left(\frac{17}{\sqrt{29}}\right) = \frac{17}{2}$

48. Let $A = (-1,1), B = (3,\sqrt{2}), C = (\sqrt{2},-1)$. Take $\overline{AB}$ as the base

Then $|AB| = \sqrt{(3+1)^2 + (\sqrt{2}-1)^2} = \sqrt{16 + (\sqrt{2}-1)^2}$.

The line through A and B is $y - 1 = \frac{\sqrt{2}-1}{3+1}(x+1)$, or $(1-\sqrt{2})x + 4y - 3 - \sqrt{2} = 0$.

The height of the triangle is the distance from C to that line,

$$\frac{|(1-\sqrt{2})\sqrt{2} - 4 - 3 - \sqrt{2}|}{\sqrt{(1-\sqrt{2})^2 + 4^2}} = \frac{9}{\sqrt{(1-\sqrt{2})^2 + 16}}.$$

So area$=\frac{1}{2}bh = \frac{1}{2}\sqrt{16 + (\sqrt{2}-1)^2}\frac{9}{\sqrt{16+(\sqrt{2}-1)^2}} = \frac{9}{2}$.

49. $(y+1)^2 = x + 1$; parabola, vertex at $(-1,1)$

50. $x^2 - 4x + 4 + y^2 + 6x + 9 = 3 + y + 9 \implies$ Circle of center at$(2,-3)$, radius 4

51. $2(x-2)^2 + 3(y+1)^2 = 6$, or $\frac{(x-2)^2}{3} + \frac{(y+1)^2}{2} = 1$; ellipse, center at $(2,-1)$

52. $x^2 + y^2 - 2x + 4y = -5 \implies (x-1)^2 + (y+2)^2 = 0$

Circle of radius 0 $\implies$ just one point $(1,-2)$.

53. $(y-2)^2 - 4(x-1)^2 = 4$, or $\frac{(y-2)^2}{4} - \frac{(x-1)^2}{1} = 1$; hyperbola, center at $(1,2)$

54. $x^2 + 4x = 4x + 8 \implies (x+2)^2 = 4(y+3)$, Parabola, vertex $(-2,-3)$.

55. $4(x-3)^2 - (y+2)^2 = 16$, or $\frac{(x-3)^2}{4} - \frac{y+2)^2}{16} = 1$; hyperbola, center at $(3,-2)$

56. $4x^2 - 8x + 4 + y^2 + 4y + 4 = 4 \implies 4(x-1)^2 + (y+2)^2 = 4$. ellipse, center at $(1,-2)$

57. Substitute $y = m(x-5) + 12$ into $x^2 + y^2 = 169$ and you get a quadratic in x that involves m. That quadratic has a unique solution iff $m = -\frac{5}{12}$. (A quadratic $ax^2 + bx + c = 0$ has a unique solution iff $b^2 - 4ac = 0$. This is clear from the general quadratic formula.)

58. $(x-1)^2 + (y+3)^2 = 25$, so center at $(1,-3)$. The radius through $(4,1)$ is the line with slope $\frac{1+3}{4-1} = -\frac{4}{3}$, so the tangent to the circle is $(y-1) = \frac{3}{4}(x-4)$, or $4y - 3x + 8 = 0$.

59. The slope of the line through the center of the circle and the point P is $\frac{3-(-1)}{-1-1} = -2$, so the slope of the tangent line is $\frac{1}{2}$. The equation for the tangent line to the circle at P is $(y+1) = \frac{1}{2}(x-1)$, or $x - 2y - 3 = 0$.

60. Slope of line segment is $\dfrac{-4-3}{3+1} = -\dfrac{7}{4}$. Midpoint $= \left(\dfrac{3-1}{2}, \dfrac{-4+3}{2}\right) = (1, -\frac{1}{2})$.

Equation of perpendicular bisector: $y + \dfrac{1}{2} = \dfrac{4}{7}(x-1)$.

61. midpoint of line segment $\overline{PQ}$: $\left(\dfrac{5}{2}\ \dfrac{5}{2}\right)$

slope of line segment $\overline{PQ}$: $\dfrac{13}{3}$

equation of the perpendicular bisector: $y - \dfrac{5}{2} = -\left(\dfrac{3}{13}\right)\left(x - \dfrac{5}{2}\right)$ or $3x + 13y - 40 = 0$

62. length of sides: $\overline{P_0P_1} : \sqrt{(-4+4)^2 + (-1-3)^2} = 4$, $\overline{P_0P_2} : \sqrt{(2+4)^2 + (1-3)^2} = \sqrt{40}$

$\overline{P_1P_2} : \sqrt{(2+4)^2 + (1+1)^2} = \sqrt{40}$. isosceles.

Slope of sides: $\overline{P_0P_1} : \dfrac{-1-3}{-4+4}$ vertical side. $\overline{P_0P_2} : \dfrac{1-3}{2+4} = -\dfrac{1}{3}$

$\overline{P_1P_2} : \dfrac{1+1}{2+4} = \dfrac{1}{3}$. Not right triangle.

63. $d(P_0, P_1) = \sqrt{(-2-1)^2 + (5-3)^2} = \sqrt{13}$, $d(P_0, P_2) = \sqrt{[-2-(-1)]^2 + (5-0)^2} = \sqrt{26}$,

$d(P_1, P_2) = \sqrt{[1-(-1)]^2 + (3-0)^2} = \sqrt{13}$.

Since $d(P_0, P_1) = d(P_1, P_2)$, the triangle is isosceles.

Since $[d(P_0, P_1)]^2 + [d(P_1, P_2)]^2 = [d(P_0, P_2)]^2$, the triangle is a right triangle.

64. length of sides: $\overline{P_0P_1} : \sqrt{(0+2)^2 + (7+1)^2} = \sqrt{68}$, $\overline{P_0P_2} : \sqrt{(3+2)^2 + (2+1)^2} = \sqrt{34}$

$\overline{P_1P_2} : \sqrt{(3-0)^2 + (2-7)^2} = \sqrt{34}$. Isosceles.

slope of sides: $\overline{P_0P_1} : \dfrac{7+1}{0+2} = 4$, $\overline{P_0P_2} : \dfrac{2+1}{3+2} = \dfrac{3}{5}$, $\overline{P_1P_2} : \dfrac{2-7}{3-0} = -\dfrac{5}{3}$. right triangle.

65. $d(P_0, P_1) = \sqrt{(3-1)^2 + (4-1)^2} = \sqrt{13}$, $d(P_0, P_2) = \sqrt{[3-(-2)]^2 + (4-3)^2} = \sqrt{26}$,

$d(P_1, P_2) = \sqrt{[1-(-2)]^2 + (1-3)^2} = \sqrt{13}$.

Since $d(P_0, P_1) = d(P_1, P_2)$, the triangle is isosceles.

Since $[d(P_0, P_1)]^2 + [d(P_1, P_2)]^2 = [d(P_0, P_2)]^2$, the triangle is a right triangle.

66. Length of side: $\sqrt{(4-0)^2 + (3-0)^2} = 5$, so we need a point (x, y) that is a distance 5 from both $(0,0)$ and $(4,3)$. Thus $x^2 + y^2 = 25$ and $(x-4)^2 + (y-3)^2 = 25$. From this we get $36(25 - x^2) = 25^2 - 400x + 64x^2$. Solving gives two possibilities for the third vertex:

$$\left(2 + \frac{1}{2}\sqrt{27},\ \frac{9 - 4\sqrt{27}}{6}\right), \left(2 - \frac{1}{2}\sqrt{27},\ \frac{9 + 4\sqrt{27}}{6}\right).$$

67. The coordinates of M are $\left(\dfrac{a}{2}, \dfrac{b}{2}\right)$; and

$d(M, (0, b)) = d(M, (0, a)) = d(M, (0, 0)) = \frac{1}{2}\sqrt{a^2 + b^2}$.

68. Let $A = (-1,-2), B = (2,1), C = (4,-3)$.

Midpoint $\overline{AB} = (\frac{1}{2}, -\frac{1}{2})$; distance to $C = \sqrt{(4-\frac{1}{2})^2 + (-3-\frac{1}{2})^2} = \frac{7}{2}\sqrt{2}$.

Midpoint $\overline{AC} = (\frac{2}{3}, -\frac{5}{2})$; distance to $B = \sqrt{(2-\frac{3}{2})^2 + (1+\frac{5}{2})^2} = \frac{5}{2}\sqrt{2}$.

Midpoint $\overline{BC} = (3,-1)$; distance to $A = \sqrt{(3+1)^2 + (-2+1)^2} = \sqrt{17}$.

69. Denote the points $(0,1)$, $(3,4)$ and $(-1,6)$ by A, B and C, respectively. The midpoints of the line segments $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ are $P\,(2,2)$, $Q\,(0,3)$ and $R\,(1,5)$, respectively.

An equation for the line through A and R is: $x = 1$.

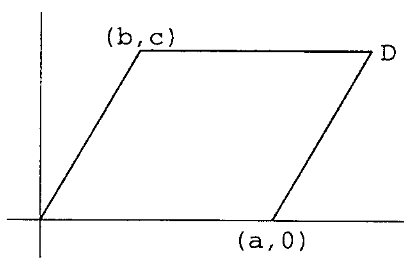
An equation for the line through B and Q is: $y = \frac{1}{3}x + 3$.

An equation for the line through C and P is: $y - 2 = -\frac{4}{3}(x-2)$.

These three lines intersect at the point $(1, \frac{10}{3})$.

70. The three midpoints are $\left(\frac{c}{2}, 0\right), \left(\frac{a+c}{2}, \frac{b}{2}\right)$, and $\left(\frac{a}{2}, \frac{b}{2}\right)$. Thus have equations $y = \frac{2b}{2a-c}\left(x - \frac{c}{2}\right)$, $y = \frac{b}{a+c}x$, and $y = \frac{b}{a-2c}(x-c)$. These three lines intersect at $\left(\frac{a+c}{3}, \frac{b}{3}\right)$.

71. Let $A\,(0,0)$ and $B\,(a,0)$, $a > 0$, be adjacent vertices of a parallelogram. If $C\,(b,c)$ is the vertex opposite B, then the vertex D opposite A has coordinates $(a+b, c)$ [see the figure].



The line through A and D has equation: $y = \frac{c}{a+b}x$.

The line through B and C has equation: $y = -\frac{c}{a-b}(x-a)$.

These lines intersect at the point $\left(\frac{a+b}{2}, \frac{c}{2}\right)$ which is the midpoint of each of the line segments $\overline{AD}$ and $\overline{BC}$.

72. The midpoints $M_1 = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$, $M_2 = \left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$, $M_3 = \left(\frac{x_3+x_4}{2}, \frac{y_3+y_4}{2}\right)$, $M_4 = \left(\frac{x_4+x_1}{2}, \frac{y_4+y_1}{2}\right)$. Slope of $\overline{M_1M_2} = \frac{y_3-y_1}{x_3-x_1}$; Slope of $\overline{M_3M_4} = \frac{y_1-y_3}{x_1-x_3} =$ slope of $\overline{M_1M_2}$.

Similarly, slope of $\overline{M_2M_3} =$ slope of $\overline{M_4M_1}$, so the quadrilateral is a parallelogram.

73. Since the relation between F and C is linear, $F = mC + b$ for some constants m and C. Setting $C = 0$ and $F = 32$ gives $b = 32$. Thus $F = mC + 32$. Now, letting $C = 100$ and $F = 212$ gives $m = (212-32)/100 = 9/5$. Therefore

$$F = \frac{9}{5}C + 32$$

The Fahrenheit and Centigrade temperatures are equal when

$$C = F = \frac{9}{5}C + 32$$

which implies $C = F = -40°$.

74. $K - 373 = \dfrac{373 - 273}{212 - 32}(F - 212) \implies K = \dfrac{5}{9}F + \dfrac{2297}{9}$

$K - 373 = \dfrac{373 - 273}{100 - 0}(C - 100) \implies K = C + 273,$ linear

SECTION 1.5

1. (a) $f(0) = 2(0)^2 - 3(0) + 2 = 2$ (b) $f(1) = 2(1)^2 - 3(1) + 2 = 1$

(c) $f(-2) = 2(-2)^2 - 3(-2) + 2 = 16$ (d) $f(\frac{3}{2}) = 2(3/2)^2 - 3(3/2) + 2 = 2$

2. (a) $-\dfrac{1}{4}$ (b) $\dfrac{1}{5}$ (c) $-\dfrac{5}{8}$ (d) $\dfrac{8}{25}$

3. (a) $f(0) = \sqrt{0^2 + 2 \cdot 0} = 0$ (b) $f(1) = \sqrt{1^2 + 2 \cdot 1} = \sqrt{3}$

(c) $f(-2) = \sqrt{(-2)^2 + 2(-2)} = 0$ (d) $f(\frac{3}{2}) = \sqrt{(3/2)^2 + 2(3/2)} = \frac{1}{2}\sqrt{21}$

4. (a) 3 (b) -1 (c) 11 (d) -3

5. (a) $f(0) = \dfrac{2 \cdot 0}{|0 + 2| + 0^2} = 0$ (b) $f(1) = \dfrac{2 \cdot 1}{|1 + 2| + 1^2} = \dfrac{1}{2}$

(c) $f(-2) = \dfrac{2 \cdot (-2)}{|-2 + 2| + (-2)^2} = -1$ (d) $f(\frac{3}{2}) = \dfrac{2 \cdot (3/2)}{|(3/2) + 2| + (3/2)^2} = \dfrac{12}{23}$

6. (a) 0 (b) $\dfrac{3}{4}$ (c) 0 (d) $\dfrac{21}{25}$

7. (a) $f(-x) = (-x)^2 - 2(-x) = x^2 + 2x$ (b) $f(1/x) = (1/x)^2 - 2(1/x) = \dfrac{1 - 2x}{x^2}$

(c) $f(a + b) = (a + b)^2 - 2(a + b) = a^2 + 2ab + b^2 - 2a - 2b$

8. (a) $f(-x) = -\dfrac{x}{x^2 + 1}$ (b) $f(\dfrac{1}{x}) = \dfrac{x}{x^2 + 1}$ (c) $f(a + b) = \dfrac{a + b}{(a + b)^2 + 1}$

9. (a) $f(-x) = \sqrt{1 + (-x)^2} = \sqrt{1 + x^2}$ (b) $f(1/x) = \sqrt{1 + (1/x)^2} = |x|/\sqrt{1 + x^2}$

(c) $f(a + b) = \sqrt{1 + (a + b)^2} = \sqrt{a^2 + 2ab + b^2 + 1}$

10. (a) $f(-x) = -\dfrac{x}{|x^2 + 1|}$ (b) $f(\dfrac{1}{x}) = \dfrac{1}{x|\frac{1}{x^2} - 1|}$ (c) $f(a + b) = \dfrac{a + b}{|(a + b)^2 - 1|}$

11. (a) $f(a + h) = 2(a + h)^2 - 3(a + h) = 2a^2 + 4ah + 2h^2 - 3a - 3h$

(b) $\dfrac{f(a + h) - f(a)}{h} = \dfrac{[2(a + h)^2 - 3(a + h)] - [2a^2 - 3a]}{h} = \dfrac{4ah + 2h^2 - 3h}{h} = 4a + 2h - 3$

12. (a) $f(a+h) = \dfrac{1}{a+h-2}$

(b) $\dfrac{f(a+h)-f(a)}{h} = \dfrac{\dfrac{1}{a+h-2} - \dfrac{1}{a-2}}{h} = \dfrac{-h}{h(a+h-2)(a-2)} = \dfrac{-1}{(a+h-2)(a-2)}$

13. $x = 1, 3$

14. $x = 0$

15. $x = -2$

16. $x = 5 \pm 2\sqrt{7}$

17. $x = -3, 3$

18. all $x > 0$

19. dom $(f) = (-\infty, \infty)$; range $(f) = [0, \infty)$

20. dom $(g) = (-\infty, \infty)$; range $(g) = [-1, \infty)$

21. dom $(f) = (-\infty, \infty)$; range $(f) = (-\infty, \infty)$

22. dom $(g) = [0, \infty)$; range $(g) = [5, \infty)$

23. dom $(f) = (-\infty, 0) \cup (0, \infty)$; range $(f) = (0, \infty)$

24. dom $(g) = (-\infty, 0) \cup (0, \infty)$; range $(g) = (-\infty, 0) \cup (0, \infty)$

25. dom $(f) = (-\infty, 1]$; range $(f) = [0, \infty)$

26. dom $(g) = [3, \infty)$; range $(g) = [0, \infty)$

27. dom $(f) = (-\infty, 7]$; range $(f) = [-1, \infty)$

28. dom $(g) = [1, \infty)$; range $(g) = [-1, \infty)$

29. dom $(f) = (-\infty, 2)$; range $(f) = (0, \infty)$

30. dom $(g) = (-2, 2)$; range $(g) = [\frac{1}{2}, \infty)$

31. horizontal line one unit above the x-axis.

32. horizontal line one unit below the x-axis.

33. line through the origin with slope 2.

34. line through $(0, 1)$ with slope 2.

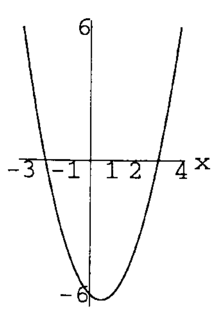
35. line through $(0, 2)$ with slope $\frac{1}{2}$.

36. line through $(0, -3)$ with slope $-\frac{1}{2}$.

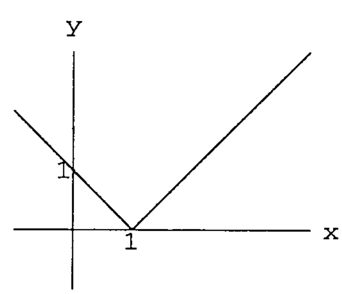
37. upper semicircle of radius 2 centered at the origin.

38. upper semicircle of radius 3 centered at the origin.

39. dom $(f) = (-\infty, \infty)$

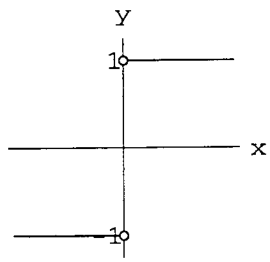


40. dom $(f) = (-\infty, \infty)$



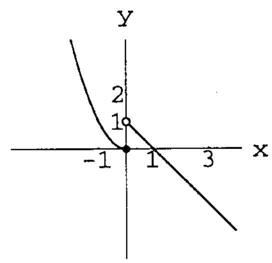
41. dom $(f) = (-\infty, 0) \cup (0, \infty)$;

range $(f) = \{-1, 1\}$.



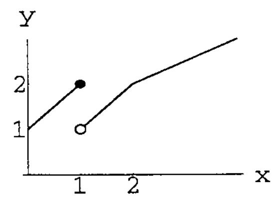
42. dom $(f) = (-\infty, \infty)$;

range $f = (-\infty, \infty)$.



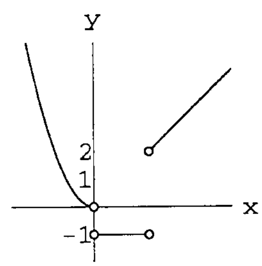
43. dom $(f) = [0, \infty)$;

range $(f) = [1, \infty)$.



44. dom $(f) = (-\infty, 0) \cup (0, 2) \cup (2\,\infty)$.

range $(f) = \{-1\} \cup (0, \infty)$.



45. The curve is the graph of a function: domain $[-2, 2]$, range $[-2, 2]$.

46. Not a function.

47. The curve is not the graph of a function; it fails the *vertical line test*.

48. Function; domain: $(-\infty, \infty)$, range: $(-1, 1)$

49. odd: $f(-x) = (-x)^3 = -x^3 = -f(x)$

50. even.

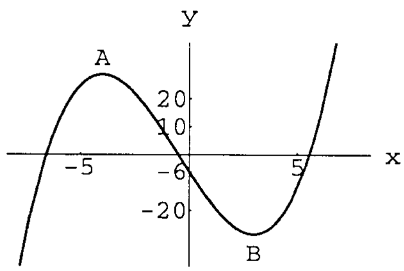
51. neither even nor odd: $g(-x) = -x(-x-1) = x(x+1)$; $g(-x) \neq g(x)$ and $g(-x) \neq -g(x)$

52. odd.

53. even.

54. odd.

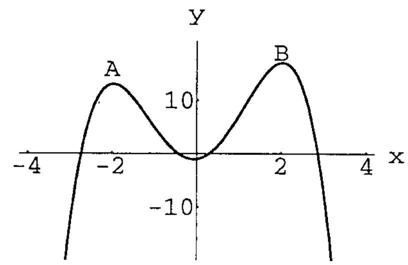
55. (a)



(b) $-6.566, -0.493, 5.559$

(c) $A\,(-4, 28.667)$, $\quad B\,(3, -28.500)$

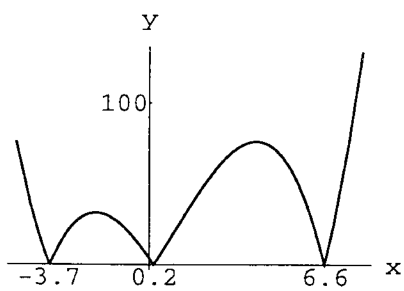
56. (a)



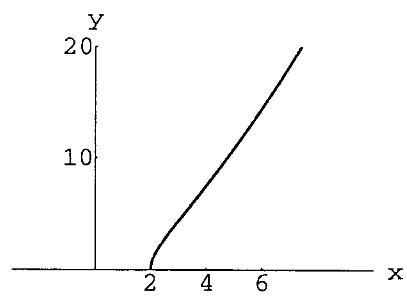
(b) $-2.739, -0.427, 0.298, 2.868$

(c) $A\,(-1.968, 13.016)$, $\quad B\,(2.032, 17.015)$

57. $-5 \leq x \leq 8, \quad 0 \leq y \leq 100$



58. $-10 \leq x \leq 10, -10 \leq y \leq 32$



59. $A = \dfrac{C^2}{4\pi}$, where C is the circumference; dom $(A) = [0, \infty)$

60. $A = 4\pi r^2 \implies r = \sqrt{\dfrac{A}{4\pi}} \implies V = \dfrac{4}{3}\pi r^3 = \dfrac{4}{3}\pi \left(\dfrac{A}{4\pi}\right)^{3/2} = \dfrac{A^{3/2}}{3\sqrt{4\pi}}$.

61. $V = s^{3/2}$, where s is the area of a face; dom $V = [0, \infty)$

62. $A = 6x^2 \implies V = x^3 = \left(\dfrac{a}{6}\right)^{3/2}$.

63. $S = 3d^2$, where d is the diagonal of a face; dom $(S) = [0, \infty)$

64. $d = \sqrt{3}x \implies V = x^3 = \left(\dfrac{d}{\sqrt{3}}\right)^3 = \dfrac{d^3\sqrt{3}}{9}$.

65. $A = \dfrac{\sqrt{3}}{4}\,x^2$, where x is the length of a side; dom $(A) = [0, \infty)$

66. $h = \sqrt{c^2 - x^2}$ so $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi x^2 \sqrt{c^2 - x^2}$.

67. Let y be the length of the rectangle. Then

$$x + 2y + \frac{\pi x}{2} = 15 \quad \text{and} \quad y = \frac{15}{2} - \frac{2+\pi}{4}x, \qquad 0 < x < \frac{30}{2+\pi}$$

The area $A = xy + \frac{1}{2}\pi(x/2)^2 = \left(\frac{15}{2} - \frac{2+\pi}{4}x\right)x + \frac{1}{8}\pi x^2 = \frac{15}{2}x - \frac{x^2}{2}\frac{\pi}{8}x^2 \quad 0 < x < \frac{30}{2+\pi}$.

68. $3x + 2y = 15 \implies y = \frac{1}{2}(15 - 3x)$. $\quad A = xy + \frac{1}{2}x\left(\frac{\sqrt{3}}{2}x\right) = \frac{1}{2}x(15-3x) + \frac{\sqrt{3}}{4}x^2$.

69. Let y be the length of the beam. Then $y = \sqrt{d^2 - x^2}$, $\; 0 < x < d$.

The cross-sectional area $A = x\sqrt{d^2 - x^2}$.

70. $2(2\pi r + h) = 30 \implies r = \frac{1}{2\pi}(15 - h)$. $\; V = \pi r^2 h = \frac{h}{4\pi}(15-h)^2$.

71. The coordinates x and y are related by the equation $\; y = -\frac{b}{a}(x - a), \; 0 \le x \le a$.

The area A of the rectangle is given by $\; A = xy = x\left[-\frac{b}{a}(x-a)\right] = bx - \frac{b}{a}x^2, \; 0 \le x \le a$.

72. Let $x = a$ be intercept. Then line is $y = \frac{5}{2-a}(x-a)$, with y-intercept $\frac{5a}{a-2}$,

so area is $\; A = \frac{1}{2}xy = \frac{1}{2}a\frac{5a}{a-2}$, $\;$ or in terms of x, $\; A = \frac{5x^2}{2(x-2)}$.

73. Let P be the perimeter of the square. Then the edge length of the square is $P/4$ and the area of the square is $A_s = (P/4)^2 = P^2/16$. Now, the circumference of the circle is $28 - P$ which implies that the radius is $\frac{1}{2\pi}(28 - \pi)$. Thus, the area of the circle is $A_c = \pi\left[\frac{1}{2\pi}(28-P)\right]^2 = \frac{1}{4\pi}(28-P)^2$. and the total area is $A_s + A_c = \frac{P^2}{16} + \frac{1}{4\pi}(28-P)^2, \;\; 0 \le P \le 28$.

74. By similar triangles, $\frac{r}{h} = \frac{10}{20}$, so $r = \frac{1}{2}h$. $\quad$ Then $\; V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$.

75. Set length plus girth equal to 108. $\;$ Then $\; l = 108 - 2\pi r$, and $\; V = (108 - 2\pi r)\pi r^2$.

SECTION 1.6

1. polynomial, degree 0

2. polynomial, degree 1

3. rational function

4. polynomial, degree 2

5. neither

6. polynomial, degree 4

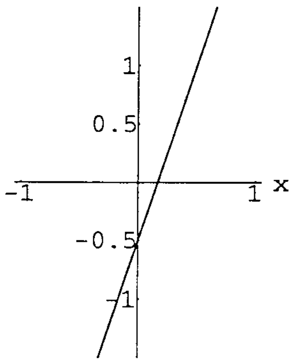
7. neither

8. rational function.

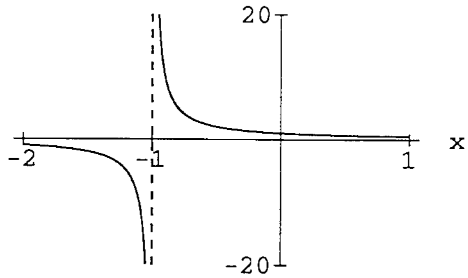
9. neither

10. $h(x) = \frac{1}{x^3 + 8}$, rational function

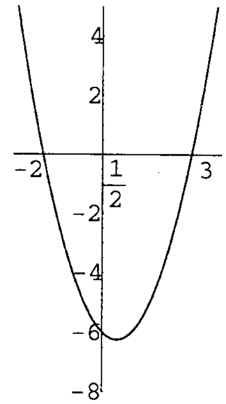
11. dom $(f) = (-\infty, \infty)$



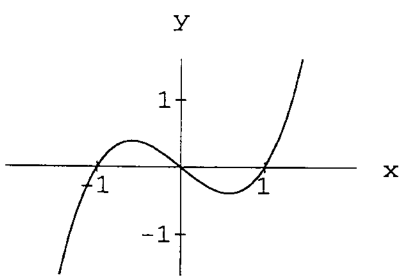
12. dom $(f) = (-\infty, -1) \cup (-1, \infty)$



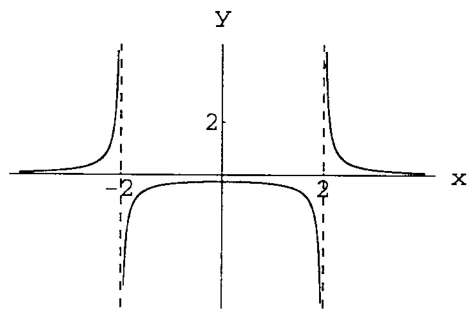
13. dom $(f) = (-\infty, \infty)$



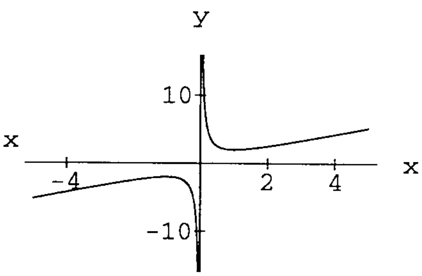
14. dom $(f) = (-\infty, \infty)$



15. dom $(f) = \{x : x \neq \pm 2\}$



16. dom $(g) = (-\infty, 0) \cup (0, \infty)$



17. $225\left(\frac{\pi}{180}\right) = \frac{5\pi}{4}$

18. $-210° = -\frac{7\pi}{6}$ rads

19. $(-300)\left(\frac{\pi}{180}\right) = -\frac{5\pi}{3}$

20. $450° = \frac{5\pi}{2}$ rads

21. $15\left(\frac{\pi}{180}\right) = \frac{\pi}{12}$

22. $3° = \frac{\pi}{60}$ rads

23. $\left(-\frac{3\pi}{2}\right)\left(\frac{180}{\pi}\right) = -270°$

24. 215°

25. $\left(\frac{5\pi}{3}\right)\left(\frac{180}{\pi}\right) = 300°$

26. $-330°$

27. $2\left(\frac{180}{\pi}\right) \cong 114.59°$

28. $-\frac{\sqrt{3}}{\pi}180°$

29. $\sin x = \frac{1}{2}$; $x = \pi/6, \quad 5\pi/6$

30. $2\pi/3, \quad 4\pi/3$

31. $\tan(x/2) = 1$; $x = \pi/2$

32. $\pi/2, \quad 3\pi/2$

33. $\cos x = \sqrt{2}/2$; $x = \pi/4, \quad 7\pi/4$

34. $4\pi/3, \quad 5\pi/6 \quad 5\pi/3 \quad 11\pi/6$

35. $\cos 2x = 0$; $x = \pi/4, \quad 3\pi/4, \quad 5\pi/4, \quad 7\pi/4$

36. $\frac{\pi}{3}, \quad \frac{4\pi}{3}$

37. $\sin 51° \cong 0.7772$

38. $\cos 17° \cong 0.9563$

39. $\sin(2.352) \cong 0.7101$

40. $\cos(-13.461°) \cong 0.6258$

41. $\tan 72.4° \cong 3.1524$

42. $\cot(-13.5°) \cong -4.1653$

43. $\tan(11.249) \cong -3.8611$

44. $\cot(7.311) \cong 0.6035$

45. $\sec(4.360) \cong -2.8974$

46. $\csc(-9.725) \cong 3.3814$

47. $\sin x = 0.5231; \quad x = 0.5505, \quad \pi - 0.5505$

48. $x = 2.5398$

49. $\tan x = 6.7192; \quad x = 1.4231, \quad \pi + 1.4231$

50. $x = 2.9678$

51. $\sec x = -4.4073; \quad x = 1.7997, \quad \pi + 1.7997$

52. $x = 0.0976$

53. $\text{dom}\,(f) = (-\infty, \infty)$; $\text{range}\,(f) = [0, 1]$

54. $\text{dom}\,(g) = (-\infty, \infty)$; $\text{range}\,(g) = \{1\}$

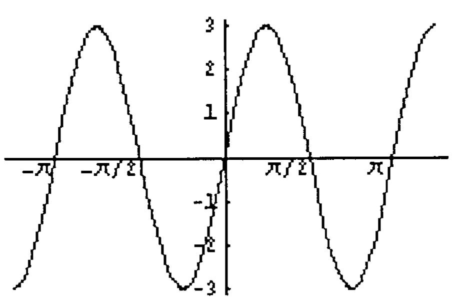
55. $\text{dom}\,(f) = (-\infty, \infty)$; $\text{range}\,(f) = [-2, 2]$

56. $\text{dom}\,(F) = (-\infty, \infty)$; $\text{range}\,(F) = [0, 2]$

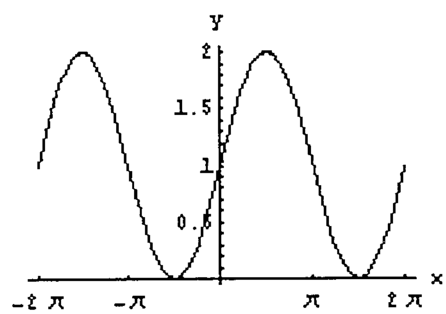
57. $\text{dom}(f) = \left(k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2}\right), \quad k = 0, \ \pm 1, \ \pm 2, \cdots$; $\text{range}\,(f) = [1, \infty)$

58. $\text{dom}\,(h) = (-\infty, \infty)$; $\text{range}\,(h) = [0, 1]$

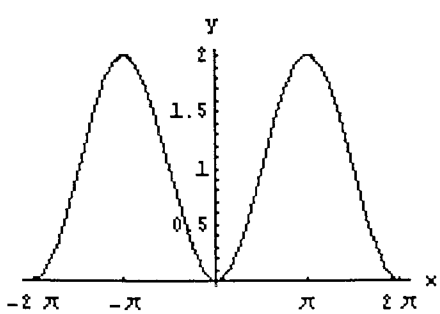
59.



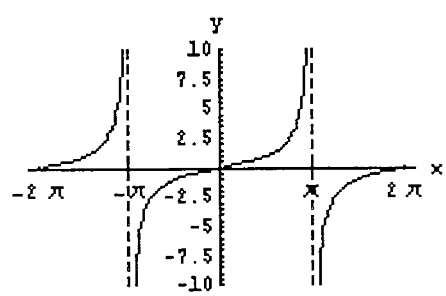
60.



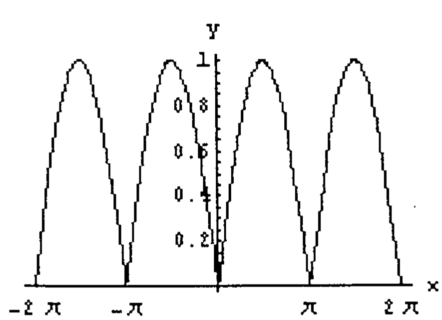
61.



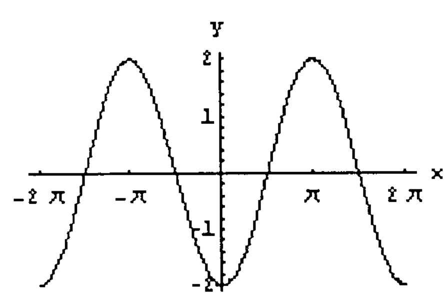
62.



63.



64.



65. odd **66.** odd **67.** even

68. even **69.** odd **70.** even

71. Assume that $\theta_2 > \theta_1$. Let $m_1 = \tan\theta_1$, $m_2 = \tan\theta_2$.. The angle α between l_1 and l_2 is the smaller of $\theta_2 - \theta_1$ and $180° - [\theta_2 - \theta_1]$. In the first case

$$\tan\alpha = \tan[\theta_2 - \theta_1] = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2\,\tan\theta_1} = \frac{m_2 - m_1}{1 + m_2 m_1} > 0$$

In the second case, $\tan\alpha = \tan[180° - (\theta_2 - \theta_1)] = -\tan(\theta_2 - \theta_1) = -\dfrac{m_2 - m_1}{1 + m_2 m_1} > 0$

Thus $\tan\alpha = \left|\dfrac{m_2 - m_1}{1 + m_2 m_1}\right|$

72. $(1,1)$; $\alpha \cong 39°$ $[m_1 = 4 = \tan\theta_1,\ \theta_1 \cong 76°;\ \ m_2 = \frac{3}{4} = \tan\theta_2,\ \theta_2 \cong 37°]$

73. $\left(\frac{23}{37}, \frac{116}{37}\right)$; $\alpha \cong 73°$ $[m_1 = -3 = \tan\theta_1,\ \theta_1 \cong 108°;\ \ m_2 = \frac{7}{10} = \tan\theta_2,\ \theta_2 \cong 35°]$

74. $\left(-\frac{2}{23}, \frac{38}{23}\right)$; $\alpha \cong 17°$ $[m_1 = 4 = \tan\theta_1,\ \theta_1 \cong 76°;\ \ m_2 = 19 = \tan\theta_2,\ \theta_2 \cong 87°]$

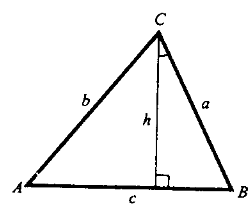
75. $\left(-\frac{17}{13}, -\frac{2}{13}\right)$; $\alpha \cong 82°$ $[m_1 = \frac{5}{6} = \tan\theta_1,\ \theta_1 \cong 40°;\ \ m_2 = -\frac{8}{5} = \tan\theta_2,\ \theta_2 \cong 122°]$

76. $\sin\theta = \dfrac{\sin\theta}{1}$, $\cos\theta = \dfrac{\cos\theta}{1}$, by similar triangles. All others follows.

77. $h = b\sin A = a\sin B$ (see figure)

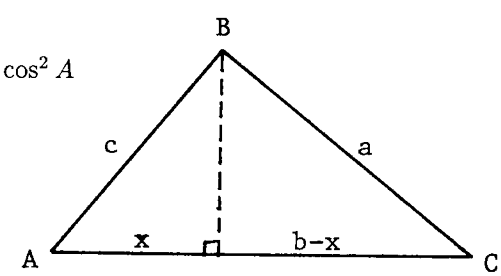
so $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$

Similarly, $\dfrac{\sin A}{a} = \dfrac{\sin C}{c}$



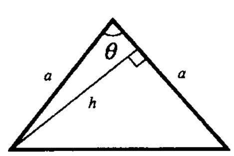
78. From the figure, $h = c\sin A, x = c\cos A$, so

$$\begin{aligned} a^2 &= h^2 + (b - x)^2 \\ &= c^2\sin^2 A + b^2 - 2bc\cos A + c^2\cos^2 A \\ &= b^2 + c^2 - 2bc\cos A \end{aligned}$$

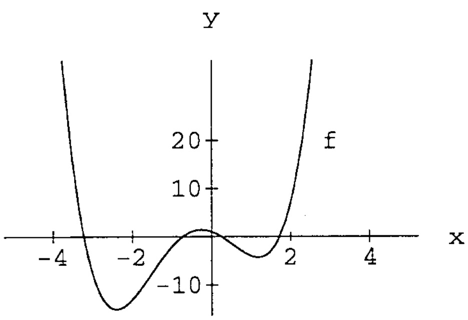


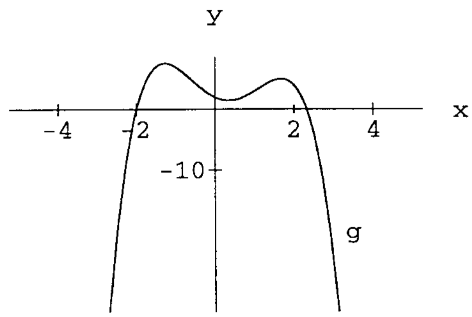
79. $A = \frac{1}{2}\,ah = \frac{1}{2}\,a^2\sin\theta$

(see figure)

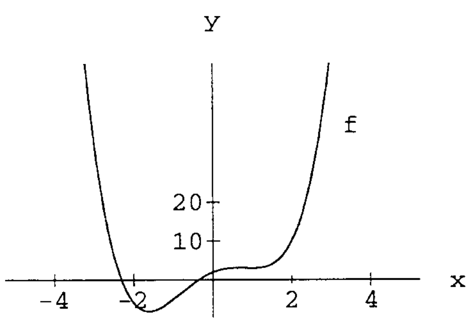


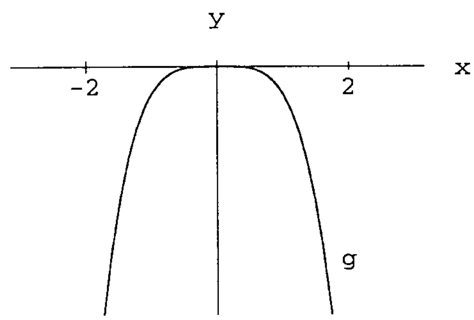
80. (a)


y
20
10
-10
f
x
-4
-2
2
4

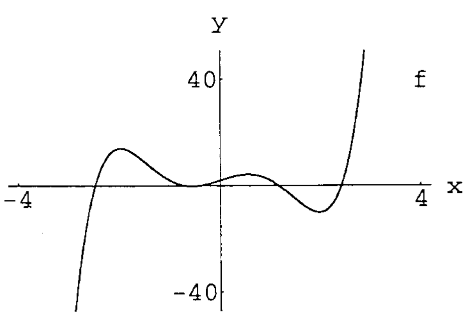

y
x
-4
-2
2
4
-10
g

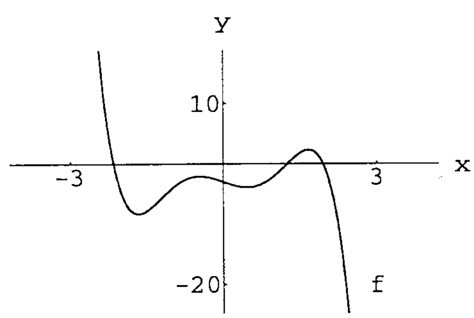
(c)


y
f
20
10
x
-4
-2
2
4

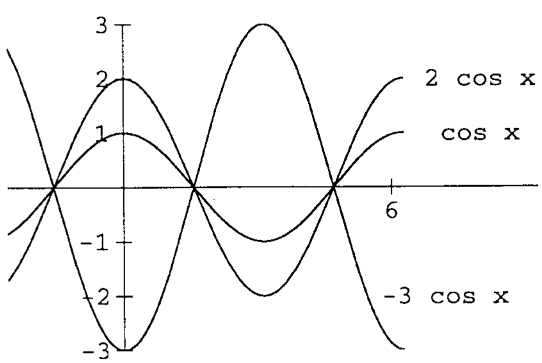

y
x
-2
2
g

81. (a)

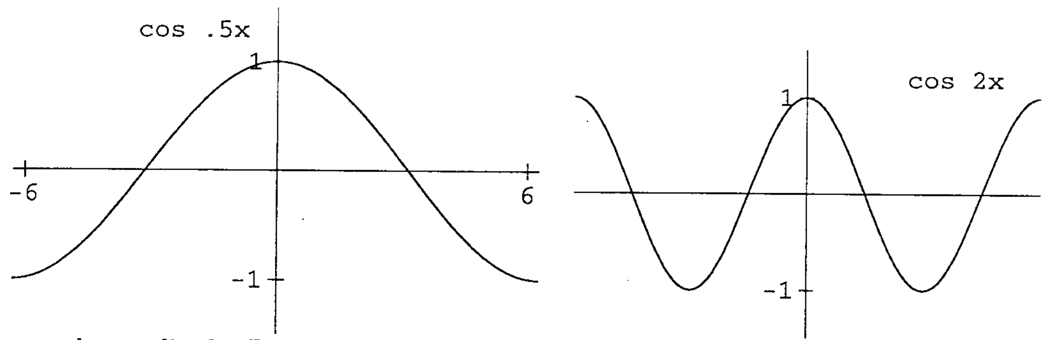

y
40
f
-4
4
x
-40


y
10
-3
3
x
-20
f

82. (a)

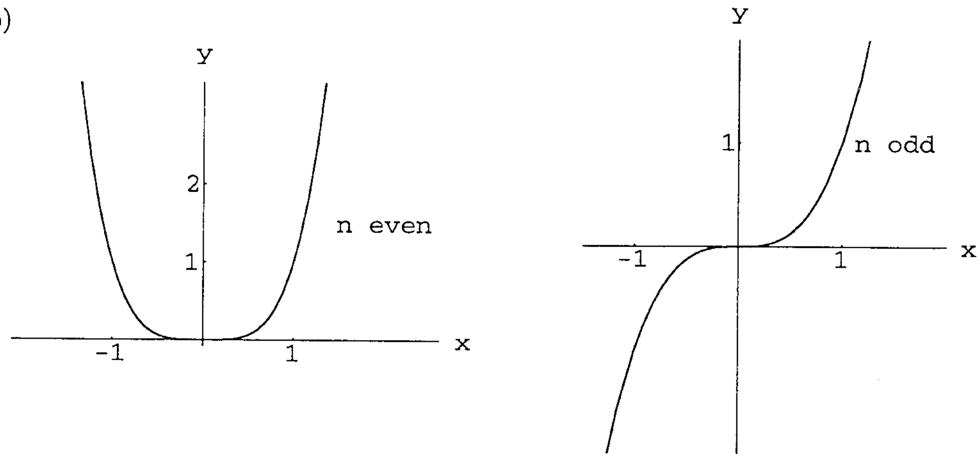

3
2
1
-1
-2
-3
6
2 cos x
cos x
-3 cos x

(b)



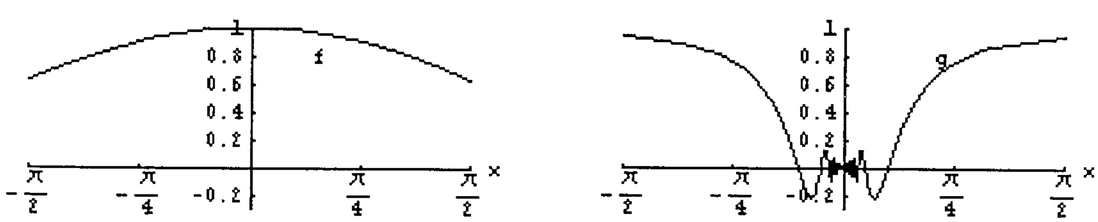
(c) A changes the amplitude; B stretches or compresses horizontally

83. (b)



(c) $f_k(x) \geq f_{k+1}(x)$ on $[0,1]$; $\quad f_{k+1}(x) \geq f_k(x)$ on $[1,\infty)$

84. (a)



(b) $f(x) \to 1$ as $x \to 0$. As $x \to 0$, $g(x)$ oscillates faster and faster, but with decreasing amplitudes; $g(x) \to 0$ as $x \to 0$.

SECTION 1.7

1. $(f+g)(2) = f(2) + g(2) = 3 + \frac{9}{2} = \frac{15}{2}$

2. $(f-g)(-1) = 6$

3. $(f \cdot g)(-2) = f(-2)g(-2) = 15 \cdot \frac{7}{2} = \frac{105}{2}$

4. $\dfrac{f}{g}(1) = 0$

5. $(2f - 3g)(\frac{1}{2}) = 2f(\frac{1}{2}) - 3g(\frac{1}{2}) = 2 \cdot 0 - 3 \cdot \frac{9}{4} = -\frac{27}{4}$

6. $\left(\dfrac{f+g}{f}\right)(-1) = 1$

7. $(f \circ g)(1) = f[g(1)] = f(2) = 3$

8. $(g \circ f)(1) = g(0)$, undefined.

9. $(f+g)(x) = f(x) + g(x) = x - 1; \quad \text{dom}\,(f+g) = (-\infty, \infty)$

$(f-g)(x) = f(x) - g(x) = 3x - 5; \quad \text{dom}\,(f-g) = (-\infty, \infty)$

$(f \cdot g)(x) = f(x)g(x) = -2x^2 + 7x - 6; \quad \text{dom}\,(f \cdot g) = (-\infty, \infty)$

$(f/g)(x) = \dfrac{2x-3}{2-x}; \quad \text{dom}\,(f/g) = \{x : \ x \neq 2\}$

10. $(f+g)(x) = f(x) + g(x) = x^2 + x - 1 + \dfrac{1}{x}; \quad \text{dom}\,(f+g) = (-\infty, 0) \cup (0, \infty)$

$(f-g)(x) = f(x) - g(x) = x^2 - x - 1 - \dfrac{1}{x}; \quad \text{dom}\,(f-g) = (-\infty, 0) \cup (0, \infty)$

$(f \cdot g)(x) = f(x)g(x) = \dfrac{x^4 - 1}{x}; \quad \text{dom}\,(f \cdot g) = (-\infty, 0) \cup (0, \infty)$

$(f/g)(x) = \dfrac{x^3 - x}{x^2 + 1}; \quad \text{dom}\,(f/g) = (-\infty, 0) \cup (0, \infty) \quad [g(0) \text{ is undefined.}]$

11. $(f+g)(x) = x + \sqrt{x-1} - \sqrt{x+1}; \quad \text{dom}\,(f+g) = [1, \infty)$

$(f-g)(x) = \sqrt{x-1} + \sqrt{x+1} - x; \quad \text{dom}\,(f-g) = [1, \infty)$

$(f \cdot g)(x) = \sqrt{x-1}\,\left(x - \sqrt{x+1}\right) = x\sqrt{x-1} - \sqrt{x^2-1}; \quad \text{dom}\,(f \cdot g) = [1, \infty)$

$(f/g)(x) = \dfrac{\sqrt{x-1}}{x - \sqrt{x+1}}; \quad \text{dom}\,(f/g) = \{x : x \geq 1 \ \text{ and } \ x \neq \tfrac{1}{2}(1+\sqrt{5})\}$

12. $(f+g)(x) = \sin^2 x + \cos 2x; \quad \text{dom}\,(f+g) = (-\infty, \infty)$

$(f-g)(x) = \sin^2 x - \cos 2x;; \quad \text{dom}\,(f-g) = (-\infty, \infty)$

$(f \cdot g)(x) = \sin^2 x \cos 2x; \quad \text{dom}\,(f \cdot g) = (-\infty, \infty)$

$(f/g)(x) = \dfrac{\sin^2 x}{\cos 2x}; \quad \text{dom}\,(f/g) = \{x : x \neq \dfrac{2n+1}{4}\pi, \ \ n = 0, \pm 1, \pm 2, \cdots\}$

13. (a) $(6f + 3g)(x) = 6(x + 1/\sqrt{x}) + 3(\sqrt{x} - 2/\sqrt{x}) = 6x + 3\sqrt{x}; \quad x > 0$

(b) $(f-g)(x) = x + 1/\sqrt{x} - (\sqrt{x} - 2/\sqrt{x}) = x + 3/\sqrt{x} - \sqrt{x}; \quad x > 0$

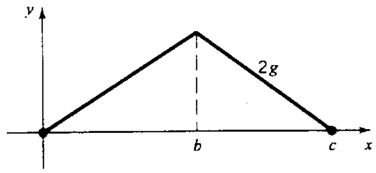
(c) $(f/g)(x) = \dfrac{x\sqrt{x} + 1}{x - 2}; \quad x > 0, \ x \neq 2$

14.

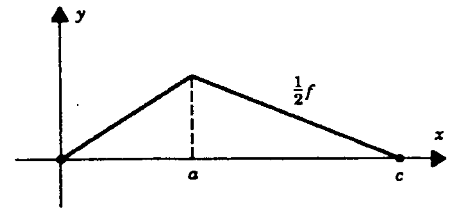
$$(f+g)(x) = \begin{cases} 1 - x, & x \leq 1 \\ 2x - 1, & 1 < x < 2 \\ 2x - 2, & x \geq 2 \end{cases} \qquad (f-g)(x) = \begin{cases} 1 - x, & x \leq 1 \\ 2x - 1, & 1 < x < 2 \\ 2x, & x \geq 2 \end{cases}$$

$$(f \cdot g)(x) = \begin{cases} 0, & x < 2 \\ 1 - 2x, & x \geq 2 \end{cases}$$

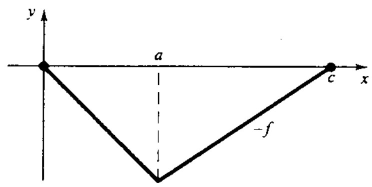
15.



16.

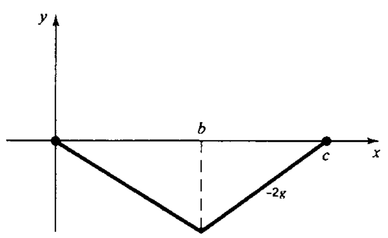


17.

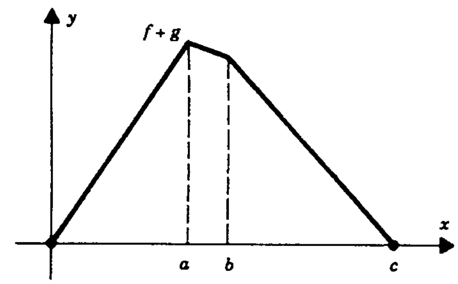


18. $y = 0$ i.e., the x-axis

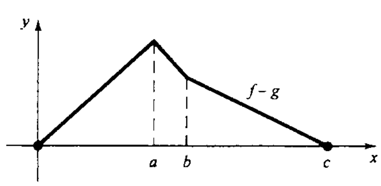
19.



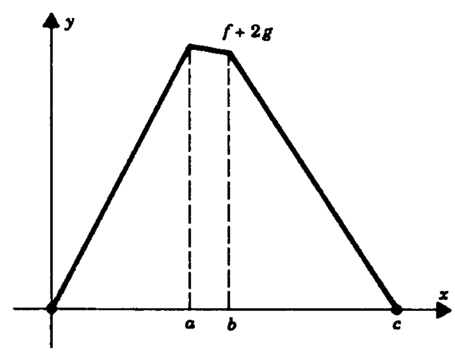
20.



21.



22.



23. $(f \circ g)(x) = 2x^2 + 5$; $\text{dom}\,(f \circ g) = (-\infty, \infty)$

24. $(f \circ g)(x) = (2x + 5)^2$; $\text{dom}\,(f \circ g) = (-\infty, \infty)$

25. $(f \circ g)(x) = \sqrt{x^2 + 5}$; $\text{dom}\,(f \circ g) = (-\infty, \infty)$

26. $(f \circ g)(x) = x + \sqrt{x}$; $\text{dom}\,(f \circ g) = [0, \infty)$

27. $(f \circ g)(x) = \dfrac{x}{x - 2}$; $\text{dom}\,(f \circ g) = \{x : x \neq 0, 2\}$

28. $(f \circ g)(x) = \dfrac{1}{x^2 - 1}$; $\text{dom}\,(f \circ g) = \{x \neq \pm 1\}$

29. $(f \circ g)(x) = \sqrt{1 - \cos^2 2x} = |\sin 2x|$; $\text{dom}\,(f \circ g) = (-\infty, \infty)$

30. $(f \circ g)(x) = \sqrt{1 - 2\cos x}$; $\text{dom}\,(f \circ g) = [-\pi/3, 5\pi/3]$

31. $(f \circ g \circ h)(x) = 4\,[\,g(h(x))] = 4\,[\,h(x) - 1\,] = 4(x^2 - 1)$; $\quad \text{dom}\,(f \circ g \circ h) = (-\infty, \infty)$

32. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x^2)) = f(4x^2) = 4x^2 - 1$

33. $(f \circ g \circ h)(x) = \dfrac{1}{g(h(x))} = \dfrac{1}{1/[2h(x) + 1]} = 2h(x) + 1 = 2x^2 + 1$; $\quad \text{dom}\, f \circ g \circ h) = (-\infty, \infty)$

34. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x^2)) = f\left(\dfrac{1}{2x^2 + 1}\right) = \dfrac{1/(2x^2 + 1) + 1}{1/(2x^2 + 1)} = 1 + (2x^2 + 1) = 2x^2 + 2$

35. Take $\; f(x) = \dfrac{1}{x}\;$ since $\;\dfrac{1 + x^4}{1 + x^2} = F(x) = f(g(x)) = f\left(\dfrac{1 + x^2}{1 + x^4}\right)$.

36. Take $f(x) = ax + b$ since $f(g(x)) = f(x^2) = ax^2 + b = F(x)$

37. Take $\; f(x) = 2\sin x\;$ since $\;2\sin 3x = F(x) = f(g(x)) = f(3x)$.

38. Take $f(x) = \sqrt{a^2 - x}$, since $f(g(x)) = f(-x^2) = \sqrt{a^2 - (-x^2)} = \sqrt{a^2 + x^2} = F(x)$.

39. Take $\; g(x) = \left(1 - \dfrac{1}{x^4}\right)^{2/3}\;$ since $\;\left(1 - \dfrac{1}{x^4}\right)^2 = F(x) = f(g(x)) = [\,g(x)]^3$.

40. Take $g(x) = a^2x^2 (x \neq 0)$, since $a^2x^2 + \dfrac{1}{a^2x^2} = F(x) = f(g(x)) = g(x) + \dfrac{1}{g(x)}$.

41. Take $\; g(x) = 2x^3 - 1$ (or $-(2x^3 - 1)$) $\;$ since $\;(2x^3 - 1)^2 + 1 = F(x) = f(g(x)) = [\,g(x)]^2 + 1$.

42. Take $g(x) = \dfrac{1}{x}\;$ since $\sin\dfrac{1}{x} = F(x) = f(g(x)) = \sin(g(x))$.

43. $(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x^2} = |x|$;

$(g \circ f)(x) = g(f(x)) = [f(x)]^2 = [\sqrt{x}\,]^2 = x, \quad x \geq 0$

44. $(f \circ g)(x) = f(g(x)) = 3g(x) + 1 = 3x^2 + 1, \quad (g \circ f)(x) = g(f(x)) = (f(x))^2 = (3x + 1)^2$

45. $(f \circ g)(x) = f(g(x)) = 1 - \sin^2 x = \cos x$; $\qquad (g \circ f)(x) = g(f(x)) = \sin f(x) = \sin(1 - x^2)$

46. $(f \circ g)(x) = f(g(x)) = 2g(x) = 2\dfrac{1}{2} = 1, \quad (g \circ f)(x) = g(f(x)) = \dfrac{1}{2}$

47. $(f \circ g)(x) = f(g(x)) = (x - 1) + 1 = x$; $\quad (g \circ f)(x) = g(f(x)) = \sqrt[3]{(x^3 + 1) - 1} = x$

48. even: $(fg)(-x) = f(-x)g(-x) = (-f(x))(-g(x)) = f(x)g(x) = (fg)(x)$.

49. fg is even since $(fg)(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x)$.

50. odd: $(fg)(-x) = f(-x)g(-x) = (-f(x))(g(x)) = -f(x)g(x) = -(fg)(x)$.

51. (a) If f is even, then $\quad f(x) = \begin{cases} -x, & -1 \leq x < 0 \\ 1, & x < -1. \end{cases}$

(b) If f is odd, then

$$f(x) = \begin{cases} x, & -1 \le x < 0 \\ -1, & x < -1. \end{cases}$$

52. (a) $f(x) = x^2 + x$, (b) $f(x) = -x^2 - x$

53. $g(-x) = f(-x) + f[-(-x)] = f(-x) + f(x) = g(x)$

54. $h(-x) = f(-x) - f[-(-x)] = f(-x) - f(x) = -[f(x) - f(-x)] = -h(x)$

55. $f(x) = \frac{1}{2}\underbrace{[f(x) + f(-x)]}_{\text{even}} + \frac{1}{2}\underbrace{[f(x) - f(-x)]}_{\text{odd}}$

56.

	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6
f_2	f_2	f_1	f_4	f_3	f_6	f_5
f_3	f_3	f_5	f_1	f_6	f_2	f_4
f_4	f_4	f_6	f_2	f_5	f_1	f_3
f_5	f_5	f_3	f_6	f_1	f_4	f_2
f_6	f_6	f_4	f_5	f_2	f_3	f_1

57. (a) For fixed b, varying a varies the x-coordinate of the vertex of the parabola.

(b) For fixed a, varying b varies the y-coordinate of the parabola

58. $a = \frac{1}{4}$, $b = -\frac{49}{16}$

59. (a) For $a > 0$, the graph of $f(x - a)$ is the graph of f shifted horizontally a units to the right; for $a < 0$, the graph of $f(x - a)$ is the graph of f shifted horizontally $|a|$ units to the left.

(b) For $b > 1$, the graph of $f(bx)$ is the graph of f compressed horizontally; for $0 < b < 1$, the graph of $f(bx)$ is the graph of f stretched horizontally; for $-1 < b < 0$, the graph of $f(bx)$ is the graph of f stretched horizontally and reflected in the y-axis; for $b < -1$, the graph of $f(bx)$ is the graph of f compressed horizontally and reflected in the y-axis.

(c) The graph of $f(x) + c$ is the graph of f shifted c units up if $c > 0$ and shifted $|c|$ units down if $c < 0$.

60. $f(b[x - a]) + c = b^3(x - a)^3 - 3b(x - a) + 1 + c$. By stretching and shifting, get x-intercepts at $-2, 1/2, 3$ with $a = 1/2, b = 2\sqrt{3}/5, c = -1$.

61. (a) For $A > 0$, the graph of Af is the graph of f scaled vertically by the factor A; for $A < 0$, the graph of Af is the graph of f scaled vertically by the factor $|A|$ and then reflected in the x-axis.

(b) See Exercise 59(b).

62. (a) The graph of $f(x-c)$ is the graph of $f(x)$ shifted c to the right (if $c<0$, this means the graph is shifted $|c|$ to the left.

(b) A changes the amplitude, B changes the period, C changes the phase.

PROJECT 1.7

1. (a) $(f\circ g)(x)=3[\frac{1}{3}(x+5)]-5=x$ and $(g\circ f)(x)=\frac{1}{3}[(3x-5)+5]=x$

(b) $(\sqrt[3]{x})^3=x=\sqrt[3]{x^3}$

(c) $(f\circ g)(x)=\dfrac{1}{\dfrac{1-x}{x}+1}=\dfrac{x}{(1-x)+x}=x$ and $(g\circ f)(x)=\dfrac{1-\dfrac{1}{x+1}}{\dfrac{1}{x+1}}=(x+1)-1=x$

2. (a) $f(x)=4x-7$

$$f[g(x)]=4[g(x)]-7=x \quad\Longrightarrow\quad g(x)=\frac{1}{4}x+\frac{7}{4}.$$

(b) $f(x)=1+3x^2$

$$f[g(x)]=1+3[g(x)]^3=x \quad\Longrightarrow\quad g(x)=(x-1)^{\frac{1}{3}}$$

(c) $f(x)=\dfrac{x+2}{x+1}$

$$f[g(x)]=\frac{g(x)+2}{g(x)+1}=x \quad\Longrightarrow\quad xg(x)+x=g(x)+2 \quad\Longrightarrow\quad g(x)=\frac{2-x}{x-1}.$$

If g is the inverse of f, then the graph of g is the reflection of the graph of f in the line $y=x$.

SECTION 1.8

1. Let S be the set of integers for which the statement is true. Since $2(1)\le 2^1$, S contains 1. Assume now that $k\in S$. This tells us that $2k\le 2^k$, and thus

$$2(k+1)=2k+2\le 2^k+2\le 2^k+2^k=2(2^k)=2^{k+1}.$$
$$(k\ge 1)$$

This places $k+1$ in S.

We have shown that

$$1\in S \quad\text{and that}\quad k\in S \quad\text{implies}\quad k+1\in S.$$

It follows that S contains all the positive integers.

2. Use $\quad 1+2(n+1)=1+2n+2\le 3^n+2<3^n+3^n=2\cdot 3^n<3^{n+1}.$

3. Let S be the set of integers for which the statement is true. Since $(1)(2) = 2$ is divisible by $2, 1 \in S$.

Assume now that $k \in S$. This tells us that $k(k+1)$ is divisible by 2 and therefore

$$(k+1)(k+2) = k(k+1) + 2(k+1)$$

is also divisible by 2. This places $k+1 \in S$.

We have shown that

$1 \in S$ and that $k \in S$ implies $k+1 \in S$.

It follows that S contains all the positive integers.

4. Use $\quad 1 + 3 + 5 + \cdots + (2(n+1) - 1) = n^2 + 2n + 1 = (n+1)^2$

5. Use

$$\begin{aligned} 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 &= \tfrac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\ &= \tfrac{1}{6}(k+1)[k(2k+1) + 6(k+1)] \\ &= \tfrac{1}{6}(k+1)(2k^2 + 7k + 6) \\ &= \tfrac{1}{6}(k+1)(k+2)(2k+3) \\ &= \tfrac{1}{6}(k+1)[(k+1)+1][2(k+1)+1]. \end{aligned}$$

6. Use

$$\begin{aligned} 1^3 + 2^3 + \cdots + n^3 + (n+1)^3 &= (1 + 2 + \cdots + n)^2 + (n+1)^3 \\ &= \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3 \quad \text{(by example 1)} \\ &= \frac{n^4 + 6n^3 + 13n^2 + 12n + 4}{4} \\ &= \left[\frac{(n+1)(n+2)}{2}\right]^2 \\ &= [1 + 2 + \cdots + n + (n+1)]^2 \end{aligned}$$

7. By Exercise 6 and Example 1

$$1^3 + 2^3 + \cdots + (n-1)^3 = [\tfrac{1}{2}(n-1)n]^2 = \tfrac{1}{4}(n-1)^2 n^2 < \tfrac{1}{4}n^4$$

and

$$1^3 + 2^3 + \cdots + n^3 = [\tfrac{1}{2}n(n+1)]^2 = \tfrac{1}{4}n^2(n+1)^2 > \tfrac{1}{4}n^4.$$

8. By Exercise 5,

$$1^2 + 2^2 + \cdots + (n-1)^2 = \tfrac{1}{6}(n-1)n(2n-1) < \tfrac{1}{3}n^3$$

$$1^2 + 2^2 + \cdots + n^2 = \tfrac{1}{6}n(n+1)(2n+1) > \tfrac{1}{3}n^3$$

9. Use

$$\begin{aligned} &\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} \\ &\qquad > \sqrt{n} + \frac{1}{\sqrt{n+1} + \sqrt{n}}\left(\frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1} - \sqrt{n}}\right) = \sqrt{n+1}. \end{aligned}$$

10. Use $$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(n+1)(n+2)}=\frac{n}{n+1}+\frac{1}{(n+1)(n+2)}=\frac{n(n+2)+1}{(n+1)(n+2)}=\frac{n+1}{n+2}$$

11. Let S be the set of integers for which the statement is true. Since

$$3^{2(1)+1}+2^{1+2}=27+8=35$$

is divisible by 7, we see that $1\in S$.

Assume now that $k\in S$. This tells us that

$$3^{2k+1}+2^{k+2} \text{ is divisible by 7.}$$

It follows that

$$\begin{aligned}3^{2(k+1)+1}+2^{(k+1)+2}&=3^2\cdot 3^{2k+1}+2\cdot 2^{k+2}\\&=9\cdot 3^{2k+1}+2\cdot 2^{k+2}\\&=7\cdot 3^{2k+1}+2(3^{2k+1}+2^{k+2})\end{aligned}$$

is also divisible by 7. This places $k+1\in S$.

We have shown that

$$1\in S \qquad \text{and that} \qquad k\in S \quad \text{implies} \quad k+1\in S.$$

It follows that S contains all the positive integers.

12. $n\geq 1:$ True for $n=1$. For the induction step, use

$$9^{n+1}-8(n+1)-1=9\cdot 9^n-8n-9-64n+64n=9(9^n-8n-1)+64n$$

13. For all positive integers $n\geq 2$,

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right)=\frac{1}{n}.$$

To see this, let S be the set of integers n for which the formula holds. Since $1-\frac{1}{2}=\frac{1}{2}$, $2\in S$. Suppose now that $k\in S$. This tells us that

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k}\right)=\frac{1}{k}$$

and therefore that

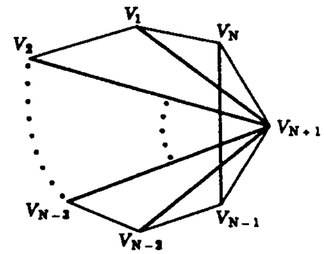
$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k}\right)\left(1-\frac{1}{k+1}\right)=\frac{1}{k}\left(1-\frac{1}{k+1}\right)=\frac{1}{k}\left(\frac{k}{k+1}\right)=\frac{1}{k+1}.$$

This places $k+1\in S$ and verifies the formula for $n\geq 2$.

14. The product is $\dfrac{n+1}{2n}$; use $$\frac{n+1}{2n}\left(1-\frac{1}{(n+1)^2}\right)=\frac{n+1}{2n}\left(\frac{n^2+2n}{(n+1)^2}\right)=\frac{n+2}{2(n+1)}$$

15. From the figure, observe that adding a vertex V_{N+1} to an N-sided polygon increases the number of diagonals by $(N-2)+1 = N-1$. Then use the identity

$$\tfrac{1}{2}N(N-3) + (N-1) = \tfrac{1}{2}(N+1)(N+1-3).$$



16. From the figure for Exercise 15, observe that adding a vertex (V_{N+1}) to an N-sided polygon increases the angle sum by 180°.

17. To go from k to $k+1$, take $A = \{a_1, \cdots, a_{k+1}\}$ and $B = \{a_1, \cdots, a_k\}$. Assume that B has 2^k subsets: $B_1, B_2, \cdots B_{2^k}$. The subsets of A are then $B_1, B_2, \cdots, B_{2^k}$ together with

$$B_1 \cup \{a_{k+1}\},\ B_2 \cup \{a_{k+1}\}, \cdots, B_{2^k} \cup \{a_{k+1}\}.$$

This gives $2(2^k) = 2^{k+1}$ subsets for A.

CHAPTER 2

SECTION 2.1

1.	(a) 2	(b) -1	(c) does not exist	(d) -3
2.	(a) -4	(b) -4	(c) -4	(d) 2
3.	(a) does not exist	(b) -3	(c) does not exist	(d) -3
4.	(a) 1	(b) does not exist	(c) does not exist	(d) 1
5.	(a) does not exist	(b) does not exist	(c) does not exist	(d) 1
6.	(a) 1	(b) -2	(c) does not exist	(d) -2
7.	(a) 2	(b) 2	(c) 2	(d) -1
8.	(a) 2	(b) 2	(c) 2	(d) 2
9.	(a) 0	(b) 0	(c) 0	(d) 0
10.	(a) does not exist	(b) does not exist	(c) does not exist	(d) 1

11. $c = 0, 6$	**12.** $c = 3$	**13.** -1	**14.** -3
15. 12	**16.** 5	**17.** 1	**18.** does not exist
19. $\frac{3}{2}$	**20.** 4	**21.** does not exist	**22.** does not exist

23. $\lim_{x\to 3} \dfrac{2x-6}{x-3} = \lim_{x\to 3} 2 = 2$

24. $\lim_{x\to 3} \dfrac{x^2-6x+9}{x-3} = \lim_{x\to 3} \dfrac{(x-3)^2}{x-3} = \lim_{x\to 3}(x-3) = 0$

25. $\lim_{x\to 3} \dfrac{x-3}{x^2-6x+9} = \lim_{x\to 3} \dfrac{x-3}{(x-3)^2} = \lim_{x\to 3} \dfrac{1}{x-3}$; does not exist

26. $\lim_{x\to 2} \dfrac{x^2-3x+2}{x-2} = \lim_{x\to 2} \dfrac{(x-1)(x-2)}{x-2} = \lim_{x\to 2}(x-1) = 1$

27. $\lim_{x\to 2} \dfrac{x-2}{x^2-3x+2} = \lim_{x\to 2} \dfrac{x-2}{(x-1)(x-2)} = \lim_{x\to 2} \dfrac{1}{x-1} = 1$

28. does not exist

29. does not exist

30. $\lim_{x\to 1} \left(x + \dfrac{1}{x}\right) = 2$

31. $\lim_{x\to 0} \dfrac{2x-5x^2}{x} = \lim_{x\to 0}(2-5x) = 2$

32. $\lim_{x\to 3} \dfrac{x-3}{6-2x} = \lim_{x\to 3} \dfrac{x-3}{2(3-x)} = \lim_{x\to 3}\left(-\dfrac{1}{2}\right) = -\dfrac{1}{2}$

33. $\lim_{x\to 1} \dfrac{x^2-1}{x-1} = \lim_{x\to 1} \dfrac{(x-1)(x+1)}{x-1} = \lim_{x\to 1}(x+1) = 2$

34. $\lim_{x\to1}\dfrac{x^3-1}{x^2-1}=\lim_{x\to1}(x-1)(x^2+x+1)(x-1)(x+1)=\lim_{x\to1}(x^2+x+1)=3$

35. 0 **36.** does not exist **37.** 1

38. 3 **39.** 16 **40.** 0

41. does not exist **42.** 2 **43.** does not exist

44. 2 **45.** 4 **46.** 0

47.

$$\lim_{x\to1}\frac{\sqrt{x^2+1}-\sqrt{2}}{x-1}=\lim_{x\to1}\frac{(\sqrt{x^2+1}-\sqrt{2})(\sqrt{x^2+1}+\sqrt{2})}{(x-1)(\sqrt{x^2+1}+\sqrt{2})}$$

$$=\lim_{x\to1}\frac{x^2-1}{(x-1)(\sqrt{x^2+1}+\sqrt{2})}=\lim_{x\to1}\frac{x+1}{\sqrt{x^2+1}+\sqrt{2}}=\frac{2}{2\sqrt{2}}=\frac{1}{\sqrt{2}}$$

48.

$$\lim_{x\to5}\frac{\sqrt{x^2+5}-\sqrt{30}}{x-5}=\lim_{x\to5}\frac{(\sqrt{x^2+5}-\sqrt{30})(\sqrt{x^2+5}+\sqrt{30})}{(x-5)(\sqrt{x^2+5}+\sqrt{30})}$$

$$=\lim_{x\to5}\frac{x^2-5}{(x-5)(\sqrt{x^2+5}+\sqrt{30})}=\lim_{x\to5}\frac{x+5}{\sqrt{x^2+5}+\sqrt{30}}=\frac{10}{2\sqrt{30}}=\frac{5}{\sqrt{30}}$$

49. $f(x)=x^2,\quad c=2,\quad f(2)=4$

$$\frac{f(2+h)-f(2)}{h}=\frac{(2+h)^2)-4}{h}=\frac{4+4h+h^2-4}{h}=4+h$$

$$\lim_{h\to0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to0}(4+h)=4$$

tangent line: $y-4=4(x-2)$ or $y=4x-4$

50. $f(x)=x^3+1,\quad c=1,\quad f(1)=2$

$$\frac{f(1+h)-f(1)}{h}=\frac{[(1+h)^3+1]-2}{h}=\frac{2+3h+3h^2+h^3-2}{h}=3+3h+h^2$$

$$\lim_{h\to0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to0}(3+3h+h^2)=3$$

tangent line: $y-2=3(x-1)$ or $y=3x-1$

51. $f(x)=1-2x+x^2,\quad c=-1,\quad f(-1)=4$

$$\frac{f(-1+h)-f(-1)}{h}=\frac{1-2(-1+h)+(-1+h)^2-4}{h}=\frac{4-4h+h^2-4}{h}=-4+h$$

$$\lim_{h\to0}\frac{f(-1+h)-f(-1)}{h}=\lim_{h\to0}(-4+h)=-4$$

tangent line: $y-4=-4(x+1)$ or $y=-4x$

52. $f(x)=1/x,\quad c=2,\quad f(1)=1/2$

$$\frac{f(2+h)-f(2)}{h}=\frac{\dfrac{1}{2+h}-\dfrac{1}{2}}{h}=\frac{\dfrac{-h}{2(2+h)}}{h}=\frac{-1}{2(2+h)}$$

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{-1}{2(2+h)}=-\frac{1}{4}$$

tangent line: $y-\frac{1}{2}=-\frac{1}{4}(x-2)$ or $y=-\frac{1}{4}x+1$

53. $f(x)=\sqrt{x},\quad c=1\quad f(1)=1$

$$\frac{f(1+h)-f(1)}{h}=\frac{\sqrt{1+h}-1}{h}=\frac{\sqrt{1+h}-1}{h}\cdot\frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}=\frac{h}{h(1+h)+1}=\frac{1}{\sqrt{1+h}+1}$$

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{1}{\sqrt{1+h}+1}=\frac{1}{2}$$

tangent line: $y-1=\frac{1}{2}(x-1)$ or $y=\frac{1}{2}x+\frac{1}{2}$

54.

$$\frac{f(0+h)-f(0)}{h}=\frac{|h|-0}{h}=\begin{cases}1 & \text{if } h>0\\ -1 & \text{if } h<0\end{cases}$$

Now, $\displaystyle\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=1$ and $\displaystyle\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=-1.$

Thus, $\displaystyle\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$ does not exist.

55. $f(x)=\sqrt[3]{x},\quad c=0,\quad f(0)=0$

$$\frac{f(0+h)-f(0)}{h}=\frac{\sqrt[3]{h}}{h}=\frac{1}{h^{2/3}}$$

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{1}{h^{2/3}}\quad\text{does not exist}$$

56. (a) For every nonzero integer n, $f(1/n\pi)=\sin\left(\dfrac{1}{1/n\pi}\right)=\sin(n\pi)=0.$

(b) For every integer n, $f\left(\dfrac{1}{2n\pi+(\pi/2)}\right)=\sin[2n\pi+(\pi/2)]=\sin(\pi/2)=1.$

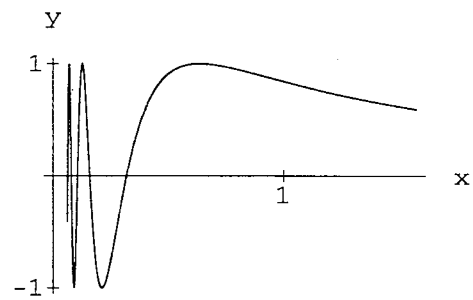
(c) For every integer n, $f\left(\dfrac{1}{2n\pi+(3\pi/2)}\right)=\sin[2n\pi+(3\pi/2)]=\sin(3\pi/2)=-1.$

57. (a)

$f(1/\pi) = f(2/\pi) = f(3/\pi) = f(4/\pi) = 0$

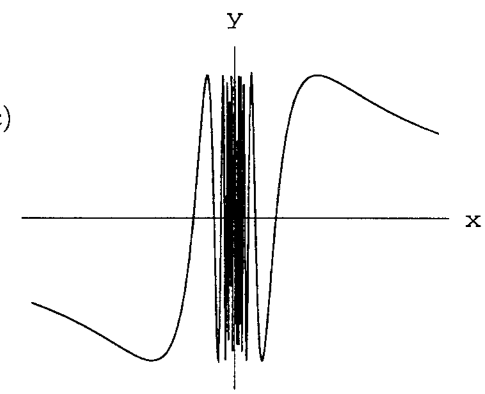
$f\left(\frac{1}{\pi/2}\right) = f\left(\frac{1}{5\pi/2}\right) = f\left(\frac{1}{9\pi/2}\right) = 1$

$f\left(\frac{1}{3\pi/2}\right) = f\left(\frac{1}{7\pi/2}\right) = f\left(\frac{1}{11\pi/2}\right) = -1$



(b) neither limit exists

(c)



58. 0 **59.** 2 **60.** 0.167

61. $\frac{3}{2}$ **62.** 0.693 **63.** 2.7182817

64.

x	3	3.1	3.14	3.141	3.1415	3.14159	3.141592
3^x	27	30.135327	31.489136	31.523749	31.541070	31.544189	31.544258

$3^\pi \cong 31.5443$

SECTION 2.2

1. $\frac{1}{2}$

2. $\lim_{x\to 0} \frac{x^2(1+x)}{2x} = \lim_{x\to 0} \frac{1}{2}x(1+x) = 0$

3. $\lim_{x\to 0} \frac{x(1+x)}{2x^2} = \lim_{x\to 0} \frac{1+x}{2x}$; does not exist

4. $\frac{4}{3}$

5. $\lim_{x\to 1} \frac{x^4-1}{x-1} = \lim_{x\to 1} (x^3+x^2+x+1) = 4$

6. does not exist

7. does not exist

8. $\lim\limits_{x\to 1}\dfrac{x^2-1}{x^2-2x+1}=\lim\limits_{x\to 1}\dfrac{(x-1)(x+1)}{(x-1)^2}=\lim\limits_{x\to 1}\dfrac{x+1}{x-1}$; does not exist

9. -1

10. does not exist

11. does not exist

12. -1

13. 0

14. 0

15. $\lim\limits_{x\to 2^+} f(x)=\lim\limits_{x\to 2^+}(x^2-x)=2$

16. $\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-} 1=1$

17. 1

18. 9

19. 1

20. 10

21. δ_1 and δ_2

22. ϵ_2 and ϵ_3

23. $\frac{1}{2}\epsilon$

24. $\delta=\frac{1}{5}\,\epsilon=0.1$

25. 2ϵ

26. $\delta=5\,\epsilon=0.5$

27. Since

$$|(2x-5)-3|=|2x-8|=2|x-4|,$$

we can take $\delta=\frac{1}{2}\epsilon$:

$$\text{if}\quad 0<|x-4|<\tfrac{1}{2}\epsilon\quad\text{then,}\quad |(2x-5)-3|=2|x-4|<\epsilon.$$

28. Since

$$|(3x-1)-5|=|3x-6|=3|x-2|,$$

we can take $\delta=\frac{1}{3}\epsilon$:

$$\text{if}\quad 0<|x-2|<\tfrac{1}{3}\epsilon\quad\text{then,}\quad |(3x-1)-5|=3|x-2|<\epsilon.$$

29. Since

$$|(6x-7)-11|=|6x-18|=6|x-3|,$$

we can take $\delta=\frac{1}{6}\epsilon$:

$$\text{if}\quad 0<|x-3|<\tfrac{1}{6}\epsilon\quad\text{then}\quad |(6x-7)-11|=6|x-3|<\epsilon.$$

30. Since

$$|(2-5x)-2|=|-5x|=5|x|,$$

we can take $\delta=\frac{1}{5}\epsilon$:

$$\text{if}\quad 0<|x|<\tfrac{1}{5}\epsilon\quad\text{then,}\quad |(2-5x)-2|=5|x|<\epsilon.$$

31. Since

$$\big||1-3x|-5\big|=\big||3x-1|-5\big|\le|3x-6|=3|x-2|,$$

we can take $\delta = \frac{1}{3}\epsilon$:

$$\text{if} \quad 0 < |x-2| < \tfrac{1}{3}\epsilon \quad \text{then} \quad \big||1-3x| - 5\big| \le 3|x-2| < \epsilon.$$

32. Take $\delta = \epsilon$: if $0 < |x-2| < \epsilon$ then $||x-2| - 0| = |x-2| < \epsilon$

33. Statements (b), (e), (g), and (i) are necessarily true.

34. Suppose $A \neq B$ and take $\epsilon = \frac{1}{2}|A-B|$. Then $|A-B| < \epsilon = \frac{1}{2}|A-B|$ which is impossible.

35. (i) $\displaystyle\lim_{x\to 3} \frac{1}{x-1} = \frac{1}{2}$ (ii) $\displaystyle\lim_{h\to 0} \frac{1}{(3+h)-1} = \frac{1}{2}$

(iii) $\displaystyle\lim_{x\to 3} \left(\frac{1}{x-1} - \frac{1}{2}\right) = 0$ (iv) $\displaystyle\lim_{x\to 3} \left|\frac{1}{x-1} - \frac{1}{2}\right| = 0$

36. (i) $\displaystyle\lim_{x\to 1} \frac{x}{x^2+2} = \frac{1}{3}$ (ii) $\displaystyle\lim_{h\to 0} \frac{1+h}{(1+h)^2+2} = \frac{1}{3}$

(iii) $\displaystyle\lim_{x\to 1} \left(\frac{x}{x^2+2} - \frac{1}{3}\right) = 0$ (iv) $\displaystyle\lim_{x\to 1} \left|\frac{x}{x^2+2} - \frac{1}{3}\right| = 0$

37. By (2.2.5) parts (i) and (iv) with $L = 0$

38. (a) Suppose $\displaystyle\lim_{x\to c} f(x) = L$, and let $\epsilon > 0$. Then there exists $\delta > 0$ such that

$$\text{if} \quad 0 < |x-c| < \delta, \quad \text{then} \quad |f(x) - L| < \epsilon.$$

But then we also have

$$\text{if} \quad 0 < |x-c| < \delta, \quad \text{then} \quad ||f(x)| - |L|| \le |f(x) - L| < \epsilon,$$

and therefore, $\displaystyle\lim_{x\to c} |f(x)| = |L|$

(b) (i) Set $f(x) = -2$ for all x, and let $c = 0$. Then

$$\lim_{x\to 0} |f(x)| = 2 = |2| \quad \text{but} \quad \lim_{x\to 0} f(x) = -2 \neq 2.$$

(ii) Set

$$f(x) = \begin{cases} -1 & \text{if} x < 0 \\ 1 & \text{if} x > 0, \end{cases} \qquad c = 0. \text{ Then}$$

$$\lim_{x\to 0} |f(x)| = 1 \quad \text{but} \quad \lim_{x\to 0} f(x) \quad \text{does not exist.}$$

39. Let $\epsilon > 0$. If

$$\lim_{x\to c} f(x) = L,$$

then there must exist $\delta > 0$ such that

$(*)$ $$\text{if} \quad 0 < |x-c| < \delta \quad \text{then} \quad |f(x) - L| < \epsilon.$$

Suppose now that

$$0 < |h| < \delta.$$

Then

$$0 < |(c+h) - c| < \delta$$

and thus by (*)

$$|f(c+h) - L| < \epsilon.$$

This proves that

$$\text{if} \quad \lim_{x\to c} f(x) = L \quad \text{then} \quad \lim_{h\to 0} f(c+h) = L.$$

If, on the other hand,

$$\lim_{h\to 0} f(c+h) = L,$$

then there must exist $\delta > 0$ such that

(**) $\qquad$ if $\; 0 < |h| < \delta \;$ then $\; |f(c+h) - L| < \epsilon.$

Suppose now that

$$0 < |x - c| < \delta.$$

Then by (**)

$$|f(c + (x-c)) - L| < \epsilon.$$

More simply stated,

$$|f(x) - L| < \epsilon.$$

This proves that

$$\text{if} \quad \lim_{h\to 0} f(c+h) = L \quad \text{then} \quad \lim_{x\to c} f(x) = L.$$

40. See comment after (2.2.8).

41. (a) Set $\delta = \epsilon\sqrt{c}$. By the hint,

$$\text{if} \quad 0 < |x-c| < \epsilon\sqrt{c} \quad \text{then} \quad |\sqrt{x} - \sqrt{c}| < \frac{1}{\sqrt{c}}|x-c| < \epsilon.$$

(b) Set $\delta = \epsilon^2$. If $\; 0 < x < \epsilon^2$, $\;$ then $\; |\sqrt{x} - 0| = \sqrt{x} < \epsilon.$

42. Take $\delta =$ minimum of 1 and $\epsilon/5$. If $0 < |x-2| < \delta$, then $|x-2| < \epsilon/5$, $|x+2| < 5$

and Therefore $\; |x^2 - 4| = |x-2|\,|x+2| < (\epsilon/5)(5) = \epsilon.$

$$|x^3 - 1| = |x^2 + x + 1|\,|x-1| < 7|x-1| < 7(\epsilon/7) = \epsilon.$$

43. Take $\delta =$ minimum of 1 and $\epsilon/7$. If $0 < |x-1| < \delta$, then $0 < x < 2$

and $\; |x-1| < \epsilon/7$. Therefore

$$|x^3 - 1| = |x^2 + x + 1|\,|x-1| < 7|x-1| < 7(\epsilon/7) = \epsilon.$$

44. Take $\delta = 2\epsilon$. If $\; 0 < |x-3| < \delta$, $\;$ then

$$\left|\sqrt{x+1} - 2\right| = \frac{|x-3|}{\sqrt{x+1}+2} < \frac{1}{2}|x-3| < \epsilon$$

45. Set $\delta = \epsilon^2$. If $3 - \epsilon^2 < x < 3$, then $-\epsilon^2 < x - 3$, $\; 0 < 3 - x < \epsilon^2$

and therefore $\; |\sqrt{3-x} - 0| < \epsilon.$

46. Take $\delta = \epsilon$. Suppose $\; 0 < |x - 0| < \delta$, $\;$ that is, suppose $\; 0 < |x| < \delta.$ $\;$ Then for x rational $|g(x)-0| = |x| < \delta = \epsilon$ and for x irrational $|g(x)-0| = 0 < \epsilon$

Thus, if $0 < |x-0| < \delta$, then $|g(x)-0| < \epsilon$, that is $\lim_{x\to 0} g(x) = 0$.

47. Suppose, on the contrary, that $\lim_{x\to c} f(x) = L$ for some particular c. Taking $\epsilon = \frac{1}{2}$, there must exist $\delta > 0$ such that

$$\text{if} \quad 0 < |x-c| < \delta, \quad \text{then} \quad |f(x)-L| < \tfrac{1}{2}.$$

Let x_1 be a rational number satisfying $0 < |x_1 - c| < \delta$ and x_2 an irrational number satisfying $0 < |x_2 - c| < \delta$. (That such numbers exist follows from the fact that every interval contains both rational and irrational numbers.) Now $f(x_1) = L$ and $f(x_2) = 0$. Thus we must have both

$$|1-L| < \tfrac{1}{2} \quad \text{and} \quad |0-L| < \tfrac{1}{2}.$$

From the first inequality we conclude that $L > \frac{1}{2}$. From the second, we conclude that $L < \frac{1}{2}$. Clearly no such number L exists.

48. We begin by assuming that $\lim_{x\to c^-} f(x) = L$ and showing that

$$\lim_{h\to 0} f(c-|h|) = L.$$

Let $\epsilon > 0$. Since $\lim_{x\to c^-} f(x) = L$, there exists $\delta > 0$ such that

$$(*) \qquad \text{if} \quad c-\delta < x < c \quad \text{then} \quad |f(x)-L| < \epsilon.$$

Suppose now that $0 < |h| < \delta$. Then $c - \delta < c - |h| < c$ and, by $(*)$,

$$|f(c-|h|)-L| < \epsilon.$$

Thus $\lim_{h\to 0} f(c-|h|) = L$.

Conversely we now assume that $\lim_{h\to 0} f(c-|h|) = L$. Then for $\epsilon > 0$ there exists $\delta > 0$ such that

$$(**) \qquad \text{if} \quad 0 < |h| < \delta \quad \text{then} \quad |f(c-|h|)-L| < \epsilon.$$

Suppose now that $c - \delta < x < c$. Then $0 < c - x < \delta$ so that, by $(**)$,

$$|f(c-(c-x))-L| = |f(x)-L| < \epsilon.$$

Thus $\lim_{x\to c^-} f(x) = L$.

49. We begin by assuming that $\lim_{x\to c^+} f(x) = L$ and showing that

$$\lim_{h\to 0} f(c+|h|) = L.$$

Let $\epsilon > 0$. Since $\lim_{x\to c^+} f(x) = L$, there exists $\delta > 0$ such that

$$(*) \qquad \text{if} \quad c < x < c+\delta \quad \text{then} \quad |f(x)-L| < \epsilon.$$

Suppose now that $0 < |h| < \delta$. Then $c < c + |h| < c + \delta$ and, by $(*)$,

$$|f(c+|h|)-L| < \epsilon.$$

Thus $\lim_{h\to 0} f(c+|h|) = L$.

Conversely we now assume that $\lim_{h\to 0} f(c+|h|) = L$. Then for $\epsilon > 0$ there exists $\delta > 0$ such that

$$(**) \qquad \text{if} \quad 0 < |h| < \delta \quad \text{then} \quad |f(c + |h|) - L| < \epsilon.$$

Suppose now that $c < x < c + \delta$. Then $0 < x - c < \delta$ so that, by (**),

$$|f(c + (x - c)) - L| = |f(x) - L| < \epsilon.$$

Thus $\lim_{x \to c^+} f(x) = L$.

50. Suppose that $\lim_{x \to c} f(x) = L$ and let $\epsilon > 0$. Then there exists $\delta > 0$ such that

$$\text{if} \quad 0 < |x - c| < \delta, \qquad \text{then} \qquad |f(x) - L| < \epsilon.$$

In particular, if $c - \delta < x < c$, then $0 < |x - c| < \delta$ and

$$|f(x) - L| < \epsilon \qquad \text{which implies} \qquad \lim_{x \to c^-} f(x) = L$$

A similar argument shows that $\lim_{x \to c^+} f(x) = L$.

Now suppose that $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$. Let $\epsilon > 0$. There exists $\delta_1 > 0$ such that if $c - \delta_1 < x < c$ then $|f(x) - L| < \epsilon$, and there exists $\delta_2 > 0$ such that if $c < x < c + \delta_2$, then $|f(x) - L| < \epsilon$. Choose $\delta = \min\{\delta_1, \delta_2\}$. Then,

$$0 < |x - c| < \delta \implies c - \delta_1 < x < c + \delta_2 \implies |f(x) - L| < \epsilon \implies \lim_{x \to c} f(x) = L$$

51. (a) Let $\epsilon = L$. Since $\lim_{x \to c} f(x) = L$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$ then

$$L - f(x) \le |L - f(x)| = |f(x) - L| < L$$

Therefore, $f(x) > L - L = 0$ for all $x \in (c - \delta, c + \delta)$; take $\gamma = \delta$.

(b) Let $\epsilon = -L$ and repeat the argument in part (a).

52. Counterexample: Set

$$f(x) = \begin{cases} x & \text{if } x \neq -1 \\ 1 & \text{if } x = -1. \end{cases}$$

Then $f(-1) = 1 > 0$ and $\lim_{x \to -1} f(x) = \lim_{x \to -1} x = -1$. By Exercise 51 (b), $f(x) < 0$ for all $x \neq -1$ in an interval of the form $(-1 - \gamma, -1 + \gamma)$, $\gamma > 0$.

53. (a) Let $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, and let $\epsilon > 0$. There exist positive numbers δ_1 and δ_2 such that

$$|f(x) - L| < \epsilon/2 \qquad \text{if} \qquad 0 < |x - c| < \delta_1$$

and

$$|g(x) - M| < \epsilon/2 \qquad \text{if} \qquad 0 < |x - c| < \delta_2$$

Let $\delta = \min(\delta_1, \delta_2)$. Then

$$M - L = M - g(x) + g(x) - f(x) + f(x) - L \ge [M - g(x)] + [f(x) - L]$$

$$\geq -\epsilon/2 - \epsilon/2 = -\epsilon$$

for all x such that $0 < |x - c| < \delta$. Since ϵ is arbitrary, it follows that $M \geq L$.

(b) No. For example, if $f(x) = x^2$ and $g(x) = |x|$ on $(-1, 1)$, then $f(x) < g(x)$ on $(-1, 1)$ except at $x = 0$, but $\lim_{x\to 0} x^2 = \lim_{x\to 0} |x| = 0$.

54. Let $\epsilon = 1$. Since $\lim_{x\to c} f(x) = L$, there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon = 1$. Thus,

$$-1 < f(x) - L < 1 \quad \text{or} \quad L - 1 < f(x) < L + 1$$

Take $B = \max\{|L - 1|, |L + 1|\}$. Then $|f(x)| < B$ for all x such that $0 < |x - c| < \delta$.

55. $f(x) = 2x^2 - 3x, \quad a = 2, \quad f(2) = 2$

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{2x^2 - 3x - 2}{x - 2}$$

$$= \lim_{x\to 2} \frac{(x-2)(2x+1)}{x - 2} = \lim_{x\to 2} (2x + 1) = 5$$

tangent line: $y - 2 = 5(x - 2)$ or $y = 5x - 8$.

56. $f(x) = x^3 + 1, \quad a = -1, \quad f(-1) = 0$

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to -1} \frac{x^3 + 1}{x + 1}$$

$$= \lim_{x\to -1} \frac{(x+1)(x^2 - x + 1)}{x + 1} = \lim_{x\to -1} (x^2 - x + 1) = 3$$

tangent line: $y - 0 = 3(x + 1)$ or $y = 3x + 3$.

57. $f(x) = \sqrt{x}, \quad a = 4, \quad f(4) = 2$

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x\to 4} \frac{\sqrt{x} - 2}{x - 2} \cdot \frac{\sqrt{x} + 2}{\sqrt{x} + 2}$$

$$= \lim_{x\to 4} \frac{x - 4}{(x - 4)(\sqrt{x} + 2)} = \lim_{x\to 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{4}$$

tangent line: $y - 2 = \frac{1}{4}(x - 4)$ or $y = \frac{1}{4}x + 1$.

58. $f(x) = 1/(x + 1), \quad a = 1, \quad f(1) = 1/2$

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to 1} \frac{\dfrac{1}{x+1} - \dfrac{1}{2}}{x - 1}$$

$$= \lim_{x\to 1} \frac{dfrac{-}(x-1)2(x+1)}{x - 1} = \lim_{x\to 1} frac{-}12(x+1) = -\frac{1}{4}$$

tangent line: $y - \frac{1}{2} = -\frac{1}{4}(x - 1)$ or $y = -\frac{1}{4}x + \frac{3}{4}$.

59. $$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{[5(x+h) + 2] - (5x + 2)}{h} = \lim_{h\to 0} \frac{5h}{h} = 5$$

60. $$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\left[2(x+h)^2-3(x+h)\right]-(2x^2-3x)}{h}$$

$$=\lim_{h\to 0}\frac{2x^2+4xh+2h^2-3x-3h-2x^2+3x}{h}$$

$$=\lim_{h\to 0}\frac{(4x-3)h+2h^2}{h}$$

$$=\lim_{h\to 0}(4x-3)+2h=4x-3$$

61. $$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\left[4(x+h)+5(x+h)^2\right]-(4x+5x^2)}{h}$$

$$=\lim_{h\to 0}\frac{4x+4h+5x^2+10xh+5h^2-4x-5x^2}{h}$$

$$=\lim_{h\to 0}\frac{h(4+10x)+5h^2}{h}=\lim_{h\to 0}(4+10x+5h)=4+10x$$

62. $$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\left[(x+h)^3+2(x+h)+4\right]-(x^3+2x+4)}{h}$$

$$=\lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3+2x+2h+4-x^3-2x-4}{h}$$

$$=\lim_{h\to 0}\frac{(3x^2+2)h+3xh^2+h^3}{h}$$

$$=\lim_{h\to 0}(3x^2+2)+3xh+h^2=3x^2+2$$

63. $$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\dfrac{1}{x+h+1}-\dfrac{1}{x+1}}{h}=\lim_{h\to 0}-\frac{h}{h(x+h+1)(x+1)}$$

$$=-\lim_{h\to 0}\frac{1}{(x+h+1)(x+1)}=-\frac{1}{(x+1)^2}$$

64. $$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{x+h+2}-\sqrt{x+2}}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{x+h+2}-\sqrt{x+2}}{h}\cdot\frac{\sqrt{x+h+2}+\sqrt{x+2}}{\sqrt{x+h+2}+\sqrt{x+2}}$$

$$=\lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h+2}+\sqrt{x+2}\right)}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{x+h+2}+\sqrt{x+2}}=\frac{1}{2\sqrt{x+2}}$$

65. $\lim_{x\to\frac{1}{3}} \frac{\cos\pi x - \cos(\pi/3)}{x-\frac{1}{3}} \cong -2.72070$

66. $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} \cong 0.5$

67. $\lim_{x\to 0} \frac{3^x - 3^0}{x} \cong 1.09861$

68. $\lim_{x\to 1} \frac{2^x - 2}{x-1} \cong 0.1.386$

PROJECT 2.2

1. For $\epsilon = 0.25$, take $\delta = 0.1$. For $\epsilon = 0.10$, take $\delta = 0.04$.

2. For $\epsilon = 0.5$, take $\delta = 0.75$. For $\epsilon = 0.1$, take $\delta = 0.19$.

3. For $\epsilon = 0.25$, take $\delta = 0.23$. For $\epsilon = 0.1$, take $\delta = 0.14$.

4. For $\epsilon = 0.5$, take $\delta = 0.25$. For $\epsilon = 0.1$, take $\delta = 0.05$.

SECTION 2.3

1. (a) 3 (b) 4 (c) -2 (d) 0 (e) does not exist (f) $\frac{1}{3}$

2. (a) 5 (b) -8 (c) does not exist (d) 0 (e) 4/5 (f) 9

3. $\lim_{x\to 4}\left(\frac{1}{x}-\frac{1}{4}\right)\left(\frac{1}{x-4}\right) = \lim_{x\to 4}\left(\frac{4-x}{4x}\right)\left(\frac{1}{x-4}\right) = \lim_{x\to 4}\frac{-1}{4x} = -\frac{1}{16}$; Theorem 2.3.2 does not apply since $\lim_{x\to 4}\frac{1}{x-4}$ does not exist.

4. Theorem 2.3.10 does not apply since $\lim_{x\to 3}(x^2+x-12) = 0$.

$$\lim_{x\to 3}\frac{x^2+x-12}{x-3} = \lim_{x\to 3}\frac{(x+4)(x-3)}{x-3} = \lim_{x\to 3}(x+4) = 7$$

5. 3 **6.** 49 **7.** -3

8. 9 **9.** 5 **10.** 2

11. does not exist **12.** $-\frac{23}{20}$ **13.** -1

14. 0 **15.** does not exist

16. $\lim_{h\to 0} h\left(1-\frac{1}{h}\right) = \lim_{h\to 0}(h-1) = -1$

17. $\lim_{h\to 0} h\left(1+\frac{1}{h}\right) = \lim_{h\to 0}(h+1) = 1$

18. $\lim_{x\to 2}\frac{x-2}{x^2-4} = \lim_{x\to 2}\frac{1}{x+2} = \frac{1}{4}$

19. $\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{x+2}{1} = 4$

20. $\displaystyle \lim_{x\to -2} \frac{(x^2-x-6)^2}{x+2} = \lim_{x\to -2} \frac{(x-3)^2(x+2)^2}{x+2} = \lim_{x\to -2} (x+3)^2(x+2) = 0$

21. $\displaystyle \lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x\to 4} \frac{x-4}{(x-4)(\sqrt{x}+2)} = \frac{1}{4}$

22. $\displaystyle \lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} = \lim_{x\to 1} \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{\sqrt{x}-1} = \lim_{x\to 1} (\sqrt{x}+1) = 2$

23. $\displaystyle \lim_{x\to 1} \frac{x^2-x-6}{(x+2)^2} = \lim_{x\to 1} \frac{(x+2)(x-3)}{(x+2)^2} = \lim_{x\to 1} \frac{x-3}{x+2} = -\frac{2}{3}$

24. $\displaystyle \lim_{x\to -2} \frac{(x^2-x-6)}{(x+2)^2} = \lim_{x\to -2} \frac{(x-3)(x+2)}{(x+2)^2} = \lim_{x\to -2} \frac{x+3}{x+2};$ does not exist

25. $\displaystyle \lim_{h\to 0} \frac{1-1/h^2}{1-1/h} = \lim_{h\to 0} \frac{h^2-1}{h^2-h} = \lim_{h\to 0} \frac{(h+1)(h-1)}{h(h-1)} = \lim_{h\to 0} \frac{h+1}{h};$ does not exist

26. $\displaystyle \lim_{h\to 0} \frac{1-1/h^2}{1+1/h^2} = \lim_{h\to 0} \frac{h^2-1}{h^2+1} = -1$

27. $\displaystyle \lim_{h\to 0} \frac{1-1/h}{1+1/h} = \lim_{h\to 0} \frac{h-1}{h+1} = -1$

28. $\displaystyle \lim_{h\to 0} \frac{1+1/h}{1+1/h^2} = \lim_{h\to 0} \frac{h^2+h}{h^2+1} = 0$

29. $\displaystyle \lim_{t\to -1} \frac{t^2+6t+5}{t^2+3t+2} = \lim_{t\to -1} \frac{(t+1)(t+5)}{(t+1)(t+2)} = \lim_{t\to -1} \frac{t+5}{t+2} = 4$

30. $\displaystyle \lim_{x\to 2^+} \frac{\sqrt{x^2-4}}{x-2} = \lim_{x\to 2^+} \frac{\sqrt{x+2}\,\sqrt{x-2}}{x-2} = \lim_{x\to 2^+} \frac{\sqrt{x+2}}{\sqrt{x-2}};$ does not exist

31. $\displaystyle \lim_{t\to 0} \frac{t+a/t}{t+b/t} = \lim_{t\to 0} \frac{t^2+a}{t^2+b} = \frac{a}{b}$

32. $\displaystyle \lim_{x\to 1} \frac{x^2-1}{x^3-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)(x^2+x+1)} = \lim_{x\to 1} \frac{x+1}{x^2+x+1} = \frac{2}{3}$

33. $\displaystyle \lim_{x\to 1} \frac{x^5-1}{x^4-1} = \lim_{x\to 1} \frac{(x-1)(x^4+x^3+x^2+x+1)}{(x-1)(x^3+x^2+x+1)} = \lim_{x\to 1} \frac{x^4+x^3+x^2+x+1}{x^3+x^2+x+1} = \frac{5}{4}$

34. $\displaystyle \lim_{h\to 0} h^2\left(1+\frac{1}{h}\right) = \lim_{h\to 0} (h^2+h) = 0$

35. $\displaystyle \lim_{h\to 0} h\left(1+\frac{1}{h^2}\right) = \lim_{h\to 0} \frac{h^2+1}{h};$ does not exist

36. $\displaystyle \lim_{x\to -4} \left(\frac{3x}{x+4}+\frac{8}{x+4}\right) = \lim_{x\to -4} \frac{3x+8}{x+4};$ does not exist

37. $\displaystyle \lim_{x\to -4} \left(\frac{2x}{x+4}+\frac{8}{x+4}\right) = \lim_{x\to -4} \frac{2x+8}{x+4} = \lim_{x\to -4} 2 = 2$

38. $\lim_{x\to -4}\left(\frac{2x}{x+4}-\frac{8}{x+4}\right)=\lim_{x\to -4}\frac{2x-8}{x+4}$; does not exist

39. (a) $\lim_{x\to 4}\left(\frac{1}{x}-\frac{1}{4}\right)=\lim_{x\to 4}\frac{4-x}{4x}=0$

(b) $\lim_{x\to 4}\left[\left(\frac{1}{x}-\frac{1}{4}\right)\left(\frac{1}{x-4}\right)\right]=\lim_{x\to 4}\left[\left(\frac{4-x}{4x}\right)\left(\frac{1}{x-4}\right)\right]=\lim_{x\to 4}\left(-\frac{1}{4x}\right)=-\frac{1}{16}$

(c) $\lim_{x\to 4}\left[\left(\frac{1}{x}-\frac{1}{4}\right)(x-2)\right]=\lim_{x\to 4}\frac{(4-x)(x-2)}{4x}=0$

(d) $\lim_{x\to 4}\left[\left(\frac{1}{x}-\frac{1}{4}\right)\left(\frac{1}{x-4}\right)^2\right]=\lim_{x\to 4}\frac{4-x}{4x(x-4)^2}=\lim_{x\to 4}\frac{1}{4x(4-x)}$; does not exist

40. (a) $\lim_{x\to 3}\frac{x^2+x+12}{x-3}$; does not exist

(b) $\lim_{x\to 3}\frac{x^2+x-12}{x-3}=\lim_{x\to 3}\frac{(x+4)(x-3)}{x-3}=\lim_{x\to 3}(x+4)=7$

(c) $\lim_{x\to 3}\frac{(x^2+x-12)^2}{x-3}=\lim_{x\to 3}\frac{(x+4)^2(x-3)^2}{x-3}=\lim_{x\to 3}(x+4)^2(x-3)=0$

(d) $\lim_{x\to 3}\frac{x^2+x-12}{(x-3)^2}=\lim_{x\to 3}\frac{(x+4)(x-3)}{(x-3)^2}=\lim_{x\to 3}\frac{x+4}{x-3}$; does not exist

41. (a) $\lim_{x\to 4}\frac{f(x)-f(4)}{x-4}=\lim_{x\to 4}\frac{(x^2-4x)-(0)}{x-4}=\lim_{x\to 4}x=4$

(b) $\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1}\frac{x^2-4x+3}{x-1}=\lim_{x\to 1}\frac{(x-1)(x-3)}{x-1}=\lim_{x\to 1}(x-3)=-2$

(c) $\lim_{x\to 3}\frac{f(x)-f(1)}{x-3}=\lim_{x\to 3}\frac{x^2-4x+3}{x-3}=\lim_{x\to 3}\frac{(x-1)(x-3)}{x-3}=\lim_{x\to 3}(x-1)=2$

(d) $\lim_{x\to 3}\frac{f(x)-f(2)}{x-3}=\lim_{x\to 3}\frac{x^2-4x+4}{x-3}$; does not exist

42. (a) $\lim_{x\to 3}\frac{f(x)-f(3)}{x-3}=\lim_{x\to 3}\frac{x^3-27}{x-3}=\lim_{x\to 3}(x^2+3x+9)=27$

(b) $\lim_{x\to 3}\frac{f(x)-f(2)}{x-3}=\lim_{x\to 3}\frac{x^3-8}{x-3}=\lim_{x\to 3}\frac{(x-2)(x^2+2x+4)}{x-3}$; does not exist

(c) $\lim_{x\to 3}\frac{f(x)-f(3)}{x-2}=\lim_{x\to 3}\frac{x^3-27}{x-2}=0$

(d) $\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1}\frac{x^3-1}{x-1}=\lim_{x\to 1}(x^2+x+1)=3$

43. $f(x)=1/x,\quad g(x)=-1/x$ with $c=0$

44. Set, for instance,

$$f(x) = \begin{cases} 0, & x < c \\ 1, & x > c, \end{cases} \qquad g(x) = \begin{cases} 1, & x < c \\ 0, & x > c \end{cases}$$

45. True. Let $\lim_{x\to c}[f(x)+g(x)] = L$. If $\lim_{x\to c} g(x) = M$ exists, then $\lim_{x\to c} f(x) = \lim_{x\to c}[f(x)+g(x)-g(x)] = L - M$ also exists. This contradicts the fact that $\lim_{x\to c} f(x)$ does not exist.

46. False, because $\lim_{x\to c} g(x) = \lim_{x\to c}[f(x)+g(x)-f(x)] = \lim_{x\to c}[f(x)+g(x)] - \lim_{x\to c} f(x)$. exists.

47. True. If $\lim_{x\to c} \sqrt{f(x)} = L$ exists, then $\lim_{x\to c} \sqrt{f(x)}\,\sqrt{f(x)} = L^2$ also exists.

48. False. Set $f(x) = -1$ for all x, $c = 0$.

49. False; for example set $f(x) = x$ and $c = 0$

50. False; for example, neither limit need exist: set $f(x) = \dfrac{1}{(x-3)^2}$ and $g(x) = \dfrac{2}{(x-3)^2}$.

51. False; for example, set $f(x) = 1 - x^2$, $g(x) = 1 + x^2$, and $c = 0$.

52. (a) If $f(x) \geq g(x)$ then $|f(x) - g(x)| = f(x) - g(x)$ and

$$\begin{aligned}\tfrac{1}{2}\,\{[f(x)+g(x)] + |f(x)-g(x)|\} &= \tfrac{1}{2}\,\{f(x)+g(x)+f(x)-g(x)\} \\ &= \tfrac{1}{2}\cdot 2f(x) = f(x) = \max\{f(x),\, g(x)\}.\end{aligned}$$

If $f(x) \leq g(x)$ then $|f(x) - g(x)| = -[f(x) - g(x)] = g(x) - f(x)$ and

$$\begin{aligned}\tfrac{1}{2}\,\{[f(x)+g(x)] + |f(x)-g(x)|\} &= \tfrac{1}{2}\,\{f(x)+g(x)+g(x)-f(x)\} \\ &= \tfrac{1}{2}\cdot 2g(x) = g(x) = \max\{f(x),\, g(x)\}.\end{aligned}$$

(b) $\min\{f(x),\, g(x)\} = \frac{1}{2}\,\{[f(x)+g(x)] - |f(x)-g(x)|\}$

53. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = L$, then

$$\begin{aligned}\lim_{x\to c} h(x) &= \lim_{x\to c} \tfrac{1}{2}\{[f(x)+g(x)] - |f(x)-g(x)|\} \\ &= \lim_{x\to c} \tfrac{1}{2}[f(x)+g(x)] - \lim_{x\to c} \tfrac{1}{f}(x) - g(x)| \\ &= \tfrac{1}{2}(L+L) - \tfrac{1}{2}(L-L) = L.\end{aligned}$$

A similar argument works for H.

54. (a) Let $\epsilon > 0$. Since $\lim_{x\to c} f(x) = L$, there exists $\delta_1 > 0$ such that

$$\text{if} \quad 0 < |x-c| < \delta_1 \quad \text{then} \qquad (*) \quad |f(x) - L| < \epsilon$$

Let $\delta_2 = \min\{|x_i - c| : 1 \le i \le n, x_i \ne c\}$. Then

$$(**) \quad f(x) = g(x) \qquad \text{if} \quad 0 < |x - c| < \delta_2$$

Now, choose $\delta = \min\{\delta_1, \delta_2\}$. Then, if $0 < |x - c| < \delta$

$|f(x) - L| < \epsilon$ by (*), and

$f(x) = g(x)$ by (**).

Therefore, if $0 < |x - c| < \delta$, then $|g(x) - L| < \epsilon \implies \lim_{x\to c} g(x) = L$.

(b) If $\lim_{x\to c} g(x)$ exists, then $\lim_{x\to c} f(x)$ must exist by part (a).

55. (a) Suppose on the contrary that $\lim_{x\to c} g(x)$ does exist. Let $L = \lim_{x\to c} g(x)$. Then

$$\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = 0 \cdot L = 0.$$

This contradicts the fact that $\lim_{x\to c} f(x)g(x) = 1$

(b) $\lim_{x\to c} g(x)$ exists since $\lim_{x\to c} g(x) = \lim_{x\to c} \dfrac{f(x)g(x)}{f(x)} = \dfrac{1}{L}$.

56. Suppose $\lim_{x\to c} f(x)$ does not exist. Let $g(x) = -f(x)$. Then $\lim_{x\to c} g(x)$ does not exist.

Now, $\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c}[f(x) - f(x)] = \lim_{x\to c} 0 = 0$ exists. This contradicts the hypothesis.

57. (a) $\displaystyle \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{x + h - x}{h} = 1$

(b) $$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h\to 0}(2x + h) = 2x$$

(c) $$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h\to 0}(3x^2 + 3xh + h^2) = 3x^2$$

(d) $$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h} = \lim_{h\to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h}$$

$$= \lim_{h\to 0}(4x^3 + 6x^2h + 4xh^2 + h^3) = 4x^3$$

(e) $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = nx^{n-1}$ for any positive integer n.

58. (a) $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$

(b) $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} = \lim_{h\to 0} \frac{-(2x+h)}{(x+h)^2x^2} = -\frac{2}{x^3}$

(c) $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{(x+h)^3} - \frac{1}{x^3}}{h} = \lim_{h\to 0} \frac{-(3x^2+3xh+h^2}{(x+h)^3x^3} = -\frac{3}{x^4}$

(d) $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{(x+h)^n} - \frac{1}{x^n}}{h} = -\frac{n}{x^{n+1}} = -nx^{-n-1}$

(e) From Exercise 57 (e) and part (d) of this Exercise, we have

$$\lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = nx^{n-1} \quad \text{for any nonzero integer } n.$$

If $n = 0$, $f(x) = x^0 = 1 \ (x \neq 0)$ and

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = 0 = 0\,x^{0-1} \ (x \neq 0)$$

Thus, the formula also holds for $n = 0$.

SECTION 2.4

1. (a) f is discontinuous at $x = -3,\ 0,\ 2,\ 6$

(b) at -3, neither; f is continuous from the right at 0; at 2 and 6, neither

2. g is continuous on $(-4,-1)$, $(-1,3]$, $(3,5)$, $(5,8]$

3. continuous

4. continuous

5. continuous

6. continuous

7. continuous

8. jump discontinuity

9. removable discontinuity

10. removable discontinuity

11. jump discontinuity

12. removable discontinuity

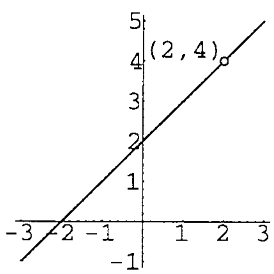
13. continuous

14. indefinite discontinuity

15. jump discontinuity

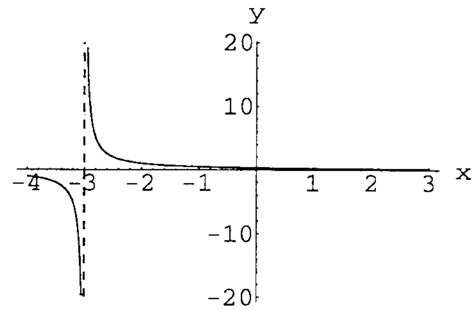
16. continuous

17.



removable discontinuity at 2

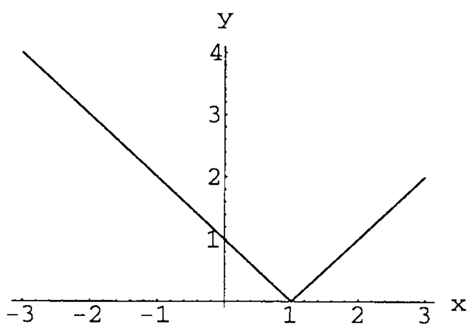
18.



removable discontinuity at 3

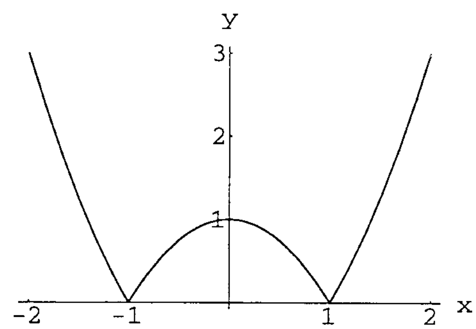
indefinite discontinuity at -3

19.



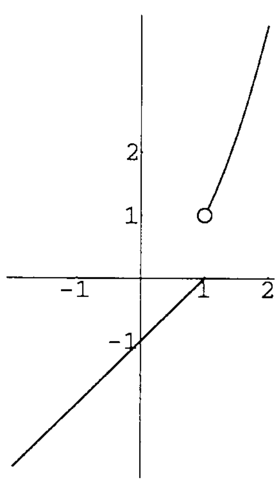
no discontinuities

20.



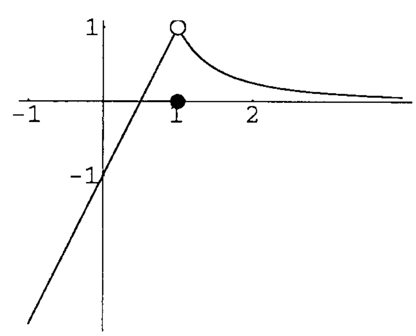
no discontinuities

21.



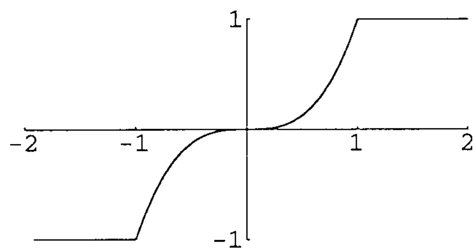
jump discontinuity at 1

22.



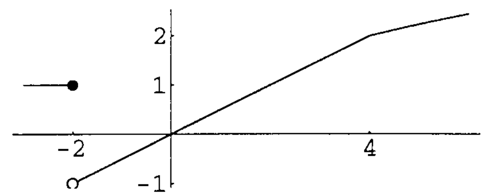
removable discontinuity at 1

23.



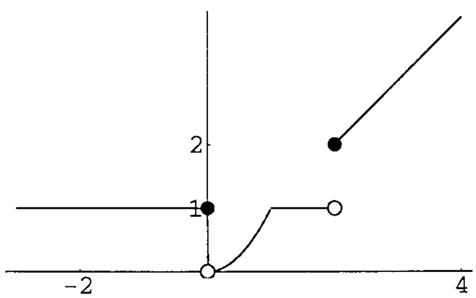
no discontinuities

24.



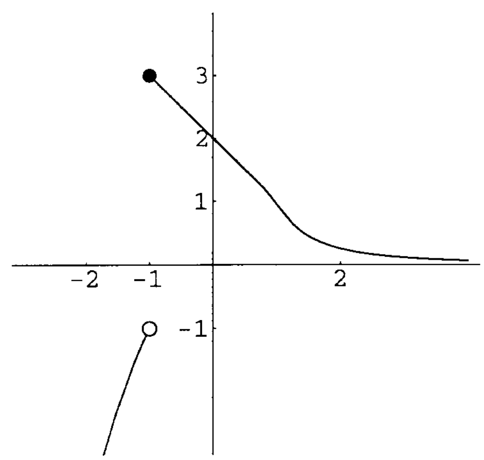
jump discontinuity at -2

25.



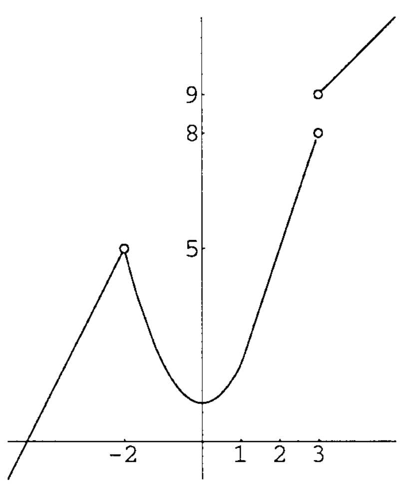
jump discontinuities at 0 and 2

26.



jump discontinuity at -1

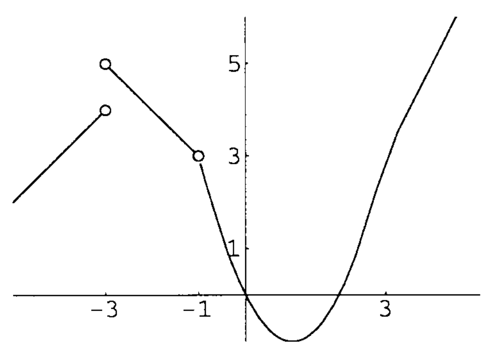
27.



removable discontinuity at -2;

jump discontinuity at 3

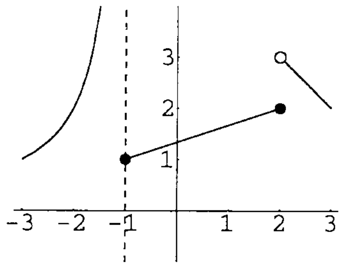
28.



jump discontinuity at -3

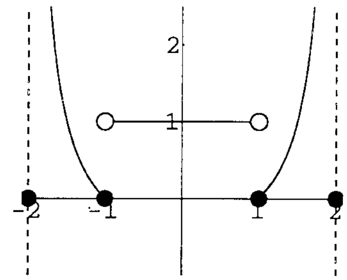
removable discontinuity at -1

29.



(One possibility)

30.



(One possibility)

31. $f(1) = 2$

32. impossible

33. impossible;

$\lim_{x\to 1^-} f(x) = -1; \quad \lim_{x\to 1^+} f(x) = 1$

34. f(1)=0

35. Since $\lim_{x\to 1^-} f(x) = 1$ and $\lim_{x\to 1^+} f(x) = A - 3 = f(1)$, take $A = 4$.

36. Since $\lim_{x\to 2^-} f(x) = 4A^2 = f(2)$ and $\lim_{x\to 2^+} f(x) = 2(1-A)$, we need $4A^2 = 2(1-A)$ or $2A^2 + A - 1 = 0$. This gives $A = \frac{1}{2},\ -1$.

37. The function f is continuous at $x = 1$ iff

$$f(1) = \lim_{x\to 1^-} f(x) = A - B \quad \text{and} \quad \lim_{x\to 1^+} f(x) = 3$$

are equal; that is, $A - B = 3$. The function f is discontinuous at $x = 2$ iff

$$\lim_{x\to 2^-} f(x) = 6 \quad \text{and} \quad \lim_{x\to 2^+} f(x) = f(2) = 4B - A$$

are unequal; that is, iff $4B - A \neq 6$. More simply we have $A - B = 3$ with $B \neq 3$:

$$A - B = 3,\ 4B - A \neq 6 \implies A - B = 3,\ 3B - 3 \neq 6 \implies A - B = 3,\ B \neq 3.$$

38. Discontinuous at $x = 1$: $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x) \implies A - B \neq 3.$

Continuous at $x = 2$: $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) \implies 4B - A = 6.$

Now,

$$4B - A = 6, \text{ and } A - B \neq 3 \implies 4B - A = 6 \text{ and } B \neq 3.$$

39. $f(5) = \frac{1}{6}$

40. $f(5) = 0$

41. $f(5) = \frac{1}{3}$

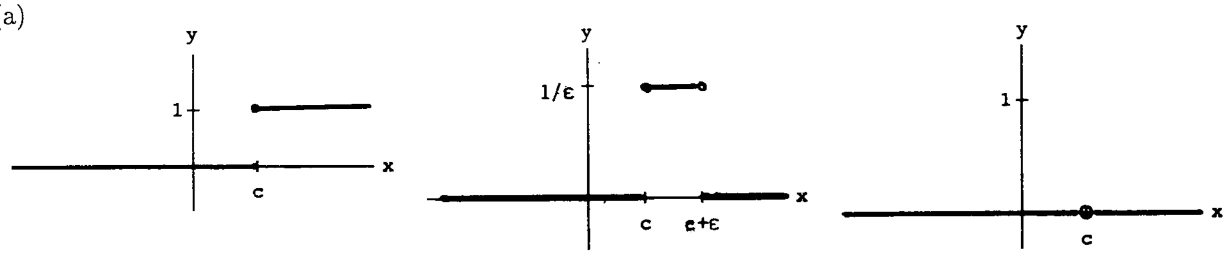
42. $f(5) = \frac{1}{4}$

43. nowhere; see Figure 2.1.8

44. continuous only at $x = 0$.

45. $x = 0, \quad x = 2,$ and all non-integral values of x

46. (a)



δ_c is "infinite" at c

(b) H_c is continuous on $(-\infty, c) \cup [c, \infty)$.

$P_{\epsilon,c}$ is continuous on $(-\infty, c) \cup [c, c+\epsilon) \cup [c+\epsilon, \infty)$.

δ_c is continuous on $(-\infty, c) \cup (c, \infty)$.

(c) $\lim_{x \to c^-} H_c(x) = 0, \quad \lim_{x \to c^+} H_c(x) = 1, \quad \lim_{x \to c} H_c(x)$ does not exist.

47. Refer to (2.2.5). Use the equivalence of (i) and (ii) setting $L = f(c)$.

48. (a) Let $\epsilon = f(c) > 0$. By the continuity of f at c, there exists $\delta > 0$ such that

$$|f(x) - f(c)| < f(c) \quad \text{for all } x \in (c - \delta, c + \delta).$$

This implies that $f(x) > 0$ for all $x \in (c - \delta, c + \delta)$.

(b) Apply the result in part (a) to $h(x) = -f(x)$.

(c) Apply the result in part (a) to $h(x) = g(x) - f(x)$.

49. Suppose that g does not have a non-removable discontinuity at c. Then either g is continuous at c or it has a removable discontinuity at c. In either case, $\lim g(x)$ as $x \to c$ exists. Since $g(x) = f(x)$ except at a finite set of points $x_1, x_2, \ldots, x_n$, $\lim f(x)$ exists as $x \to c$ by Exercise 54, Section 2.3.

50. (a) Choose any point c and let $\epsilon > 0$. Since f is continuous at c, there exists $\delta > 0$ such that

$$\text{if } |x - c| < \delta \quad \text{then} \quad |f(x) - f(c)| < \epsilon.$$

Now, since

$$\big|\,|f(x)| - |f(c)|\,\big| \le |f(x) - f(c)|,$$

it follows that

$$\big|\,|f(x)| - |f(c)|\,\big| < \epsilon \quad \text{if} \quad |x - c| < \delta.$$

Thus, $|f|$ is continuous at c.

(b)

$$\text{Let} \quad f(x) = \begin{cases} 1, & x \neq 1 \\ -1, & x = 1. \end{cases} \qquad \text{Then } f \text{ is not continuous at } x = 1.$$

However, $|f(x)| = 1$ for all x is continuous everywhere.

(c)

Let $f(x) = \begin{cases} 1, & x \text{ rational} \\ -1, & x \text{ irratiional.} \end{cases}$ Then f is nowhere continuous.

However, $|f(x)| = 1$ for all x is continuous everywhere.

51. By implication, f is defined on $(c-p, c+p)$. The given inequality implies that $B \geq 0$. If $B = 0$, then $f \equiv f(c)$ is a constant function and hence is continuous. Now assume that $B > 0$. Let $\epsilon > 0$ and let $\delta = \min\{\epsilon/B,\, p\}$. If $|x - c| < \delta$ then $x \in (c-p,\, c+p)$ and

$$|f(x) - f(c)| \leq B|x - c| < B \cdot \delta \leq B \cdot \frac{\epsilon}{B} = \epsilon$$

Thus, f is continuous at c.

52. Choose any point $c \in (a, b)$, and let $\epsilon > 0$. Now choosing $\delta = \epsilon$, we have

$$\text{if } |x - c| < \delta,\ x \in (a, b), \quad \text{then} \quad |f(x) - f(c)| \leq |x - c| < \delta = \epsilon.$$

Thus, f is continuous at c, and it follows that f is continuous on $(a, b$.

53. $\displaystyle\lim_{h\to 0} [f(c+h) - f(c)] = \lim_{h\to 0} \left[\frac{f(c+h) - f(c)}{h} \cdot h\right] = \lim_{h\to 0} \left[\frac{f(c+h) - f(c)}{h}\right] \cdot \lim_{h\to 0} h = L \cdot 0 = 0.$
Therefore f is continuous at c by Exercise 47.

54. Let $f_e(x) = \frac{1}{2}[f(x) + f(-x)]$ and $f_o(x) = \frac{1}{2}[f(x) - f(-x)]$. Then f_e is an even function, f_o is an odd function, each function is continuous on $(-\infty, \infty)$, and $f = f_e + f_o$.

SECTION 2.5

1. $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{x} = \lim_{x\to 0} 3\left(\frac{\sin 3x}{3x}\right) = 3(1) = 3$

2. $\displaystyle\lim_{x\to 0} \frac{2x}{\sin x} = 2 \lim_{x\to 0} \frac{\sin x}{x} = 2(1) = 2$

3. $\displaystyle\lim_{x\to 0} \frac{3x}{\sin 5x} = \lim_{x\to 0} \frac{3}{5}\left(\frac{5x}{\sin 5x}\right) = \frac{3}{5}(1) = \frac{3}{5}$

4. $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{2x} = \frac{3}{2} \lim_{x\to 0} \frac{\sin 3x}{3x} = \frac{3}{2}$

5. $\displaystyle\lim_{x\to 0} \frac{\sin 4x}{\sin 2x} = \lim_{x\to 0} \frac{4x}{2x} \cdot \frac{\sin 4x}{4x} \cdot \frac{2x}{\sin 2x} = 2(1)(1) = 2$

6. $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{5x} = \frac{3}{5} \lim_{x\to 0} \frac{\sin 3x}{3x} = \frac{3}{5}$

7. $\displaystyle\lim_{x\to 0} \frac{\sin x^2}{x} = \lim_{x\to 0} x\left(\frac{\sin x^2}{x^2}\right) = \lim_{x\to 0} x \cdot \lim_{x\to 0} \frac{\sin x^2}{x^2} = 0(1) = 0$

8. $\displaystyle\lim_{x\to 0} \frac{\sin x^2}{x^2} = 1$

9. $\displaystyle\lim_{x\to 0} \frac{\sin x}{x^2} = \lim_{x\to 0} \frac{(\sin x)/x}{x}$; does not exist

10. $\displaystyle\lim_{x\to 0} \frac{\sin^2 x^2}{x^2} = \lim_{x\to 0} (\sin x^2)\left(\frac{\sin x^2}{x^2}\right) = 0(1) = 0$

11. $\displaystyle\lim_{x\to 0} \frac{\sin^2 3x}{5x^2} = \lim_{x\to 0} \frac{9}{5}\left(\frac{\sin 3x}{3x}\right)^2 = \frac{9}{5}(1) = \frac{9}{5}$

12. $\lim_{x\to0}\frac{\tan^2 3x}{4x^2}=\lim_{x\to0}\frac{9}{4}\cdot\frac{1}{\cos^2 3x}\cdot\frac{\sin^2 3x}{(3x)^2}=\frac{9}{4}(1)(1)=\frac{9}{4}$

13. $\lim_{x\to0}\frac{2x}{\tan 3x}=\lim_{x\to0}\frac{2x\cos 3x}{\sin 3x}=\lim_{x\to0}\frac{2}{3}\left(\frac{3x}{\sin 3x}\right)\cos 3x=\frac{2}{3}(1)(1)=\frac{2}{3}$

14. $\lim_{x\to0}\frac{4x}{\cot 3x}=\lim_{x\to0}\frac{4x\sin 3x}{\cos 3x}=0$

15. $\lim_{x\to0}x\csc x=\lim_{x\to0}\frac{x}{\sin x}=1$

16. $\lim_{x\to0}\frac{\cos x-1}{2x}=-\frac{1}{2}\lim_{x\to0}\frac{1-\cos x}{x}=0$

17. $\lim_{x\to0}\frac{x^2}{1-\cos 2x}=\lim_{x\to0}\frac{x^2}{1-\cos 2x}\cdot\left(\frac{1+\cos 2x}{1+\cos 2x}\right)=\lim_{x\to0}\frac{x^2(1+\cos 2x)}{\sin^2 2x}$

$$=\lim_{x\to0}\frac{1}{4}\left(\frac{2x}{\sin 2x}\right)^2(1+\cos 2x)=\frac{1}{4}(1)(2)=\frac{1}{2}$$

18. $\lim_{x\to0}\frac{x^2-2x}{\sin 3x}=\lim_{x\to0}\left(\frac{x-2}{3}\right)\left(\frac{3x}{\sin 3x}\right)=\left(-\frac{2}{3}\right)(1)=-\frac{2}{3}$

19. $\lim_{x\to0}\frac{1-\sec^2 2x}{x^2}=\lim_{x\to0}\frac{-\tan^2 2x}{x^2}=\lim_{x\to0}\frac{-\sin^2 2x}{x^2\cos^2 2x}=\lim_{x\to0}\left[-4\left(\frac{\sin 2x}{2x}\right)^2\frac{1}{\cos^2 2x}\right]=-4$

20. $\lim_{x\to0}\frac{1}{2x\csc x}=\frac{1}{2}\lim_{x\to0}\frac{\sin x}{x}=\frac{1}{2}$

21. $\lim_{x\to0}\frac{2x^2+x}{\sin x}=\lim_{x\to0}(2x+1)\frac{x}{\sin x}=1$

22. $\lim_{x\to0}\frac{1-\cos 4x}{9x^2}=\lim_{x\to0}\left(\frac{1-\cos 4x}{9x^2}\right)\left(\frac{1+\cos 4x}{1+\cos 4x}\right)=\lim_{x\to0}\frac{1-\cos^2 4x}{9x^2(1+\cos 4x)}$

$$=\lim_{x\to0}\frac{\sin^2 4x}{9x^2(1+\cos 4x)}=\frac{16}{9}\lim_{x\to0}\frac{\sin^2 4x}{(4x)^2}\cdot\frac{1}{1+\cos 4x}=\frac{16}{9}\cdot1\cdot\frac{1}{2}=\frac{8}{9}$$

23. $\lim_{x\to0}\frac{\tan 3x}{2x^2+5x}=\lim_{x\to0}\frac{1}{x(2x+5)}\frac{\sin 3x}{\cos 3x}=\lim_{x\to0}\frac{3}{2x+5}\left(\frac{\sin 3x}{3x}\right)\frac{1}{\cos 3x}=\frac{3}{5}(1)(1)=\frac{3}{5}$

24. $\lim_{x\to0}x^2(1+\cot^2 3x)=\lim_{x\to0}\left(x^2+\frac{x^2\cos^2 3x}{\sin^2 3x}\right)=\lim_{x\to0}\left(x^2+\frac{(3x)^2}{\sin^2 3x}\cdot\frac{\cos^2 3x}{9}\right)=0+(1)\frac{1}{9}=\frac{1}{9}$

25. $\lim_{x\to0}\frac{\sec x-1}{x\sec x}=\lim_{x\to0}\frac{\frac{1}{\cos x}-1}{x\left(\frac{1}{\cos x}\right)}=\lim_{x\to0}\frac{1-\cos x}{x}=0$

26. $\lim_{x\to\pi/4}\frac{1-\cos x}{x}=\frac{1-\cos(\pi/4)}{\pi/4}=\frac{4-2\sqrt{2}}{\pi}$

27. $\frac{2\sqrt{2}}{\pi}$

28. $\lim_{x\to0}\frac{\sin^2 x}{x(1-\cos x)}=\lim_{x\to0}\frac{x}{1-\cos x}\cdot\frac{\sin^2 x}{x^2}$; does not exist

29. $\lim_{x\to\pi/2}\frac{\cos x}{x-\pi/2}=\lim_{h\to0}\frac{\cos\,(h+\pi/2)}{h}=\lim_{h\to0}\frac{-\sin h}{h}=-1$

$h=x-\pi/2$ $\qquad\cos\,(h+\pi/2)=\cos h\cos\pi/2-\sin h\sin\pi/2$

30. $\lim_{x\to\pi}\dfrac{\sin x}{x-\pi}=\lim_{x\to\pi}\dfrac{-\sin(x-\pi)}{x-\pi}=-1$

31. $\lim_{x\to\pi/4}\dfrac{\sin\,(x+\pi/4)-1}{x-\pi/4}=\lim_{h\to 0}\dfrac{\sin\,(h+\pi/2)-1}{h}=\lim_{h\to 0}\dfrac{\cos h-1}{h}=0$

$$h=x-\pi/4$$

32. $\lim_{x\to\pi/6}\dfrac{\sin[x+(\pi/3)]-1}{x-(\pi/6)}=\lim_{x\to\pi/6}\dfrac{\sin[x-(\pi/6)+(\pi/2)]-1}{x-(\pi/6)}=\lim_{x\to\pi/6}\dfrac{\cos[x-(\pi/6)]-1}{x-(\pi/6)}=0$

33. Equivalently we will show that $\lim_{h\to 0}\cos\,(c+h)=\cos c$. The identity

$$\cos\,(c+h)=\cos c\,\cos h-\sin c\,\sin h$$

gives

$$\lim_{h\to 0}\cos\,(c+h)=\cos c\left(\lim_{h\to 0}\cos h\right)-\sin c\left(\lim_{h\to 0}\sin h\right)=(\cos c)(1)-(\sin c)(0)=\cos c.$$

34. Let $\epsilon>0$. There exists $\delta_1>0$ such that

$$\text{if}\quad 0<|x|<\delta_1,\quad\text{then}\quad |f(x)-L|<\epsilon.$$

Take $\delta=\delta_1/|a|$:

$$\text{if}\quad 0<|x|<\delta,\quad\text{then}\quad 0<|ax|=|a|\,|x|<\delta_1\quad\text{and}\quad |f(ax)-L|<\epsilon.$$

35. $f(x)=\sin x;\quad a=\pi/4$

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\sin\left(\frac{\pi}{4}+h\right)-\sin\left(\frac{\pi}{4}\right)}{h}=\lim_{h\to 0}\frac{\sin\,(\pi/4)\cos h+\cos\,(\pi/4)\sin h-\sin\,(\pi/4)}{h}$$

$$=\lim_{h\to 0}\frac{-\sin\,(\pi/4)\ (1-\cos h)+\cos\,(\pi/4)\ \sin h}{h}$$

$$=-\sin\,(\pi/4)\ \lim_{h\to 0}\frac{1-\cos h}{h}+\cos\,(\pi/4)\ \lim_{h\to 0}\frac{\sin h}{h}=\cos\,(\pi/4)=\frac{\sqrt{2}}{2}$$

tangent line: $\quad y-\dfrac{\sqrt{2}}{2}=\dfrac{\sqrt{2}}{2}\left(x-\dfrac{\pi}{4}\right)$

36. $f(x)=\cos x;\quad a=\pi/3$

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\cos\left(\frac{\pi}{3}+h\right)-\cos\left(\frac{\pi}{3}\right)}{h}=\lim_{h\to 0}\frac{\cos\,(\pi/3)\cos h-\sin\,(\pi/3)\sin h-\cos\,(\pi/3)}{h}$$

$$=\lim_{h\to 0}\frac{-\cos\,(\pi/3)\ (1-\cos h)-sin\,(\pi/3)\ \sin h}{h}$$

$$=-\cos\,(\pi/3)\ \lim_{h\to 0}\frac{1-\cos h}{h}-\sin\,(\pi/3)\ \lim_{h\to 0}\frac{\sin h}{h}=-\ \sin\,(\pi/3)=-\ \frac{\sqrt{3}}{2}$$

tangent line: $\quad y-\dfrac{1}{2}=-\ \dfrac{\sqrt{3}}{2}\left(x-\dfrac{\pi}{3}\right)$

37. $f(x) = \cos 2x; \quad a = \pi/6$

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\cos 2\left(\frac{\pi}{6}+h\right) - \cos\left(2\frac{\pi}{6}\right)}{h} = \lim_{h\to 0} \frac{\cos(2h+\pi/3) - \cos(\pi/3)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(\pi/3)\cos 2h - \sin(\pi/3)\sin 2h - \cos(\pi/3)}{h}$$

$$= -\cos(\pi/3)\lim_{h\to 0} 2\,\frac{1-\cos 2h}{h} - \sin(\pi/3)\lim_{h\to 0} 2\,\frac{\sin 2h}{h}$$

$$= -\cos(\pi/3)\cdot 2\cdot 0 - \sin(\pi/3)\cdot 2\cdot 1 = -\sqrt{3}$$

tangent line: $y - \dfrac{1}{2} = -\sqrt{3}\left(x - \dfrac{\pi}{6}\right)$

38. $f(x) = \sin 3x; \quad a = \pi/2$

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\sin 3\left(\frac{\pi}{2}+h\right) - \sin\left(3\frac{\pi}{2}\right)}{h} = \lim_{h\to 0} \frac{\sin(3h+3\pi/2) - \sin(3\pi/2)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(3pi/2)\cos 3h + \cos(3\pi/2)\cos 3h - \sin(3\pi/2)}{h}$$

$$= \lim_{h\to 0} \frac{1-\cos 3h}{h} = 3\lim_{h\to 0} \frac{1-\cos 3h}{3h} = 3\cdot 0 = 0$$

tangent line: $y-(-1) = 0\,[x-(3\pi/2)]$ or $y = -1$

39. For $x \neq 0$, $|x\,\sin(1/x)| = |x|\,|\sin(1/x)| \leq |x|$. Thus,

$$-|x| \leq |x\,\sin(1/x)| \leq |x|$$

Since $\lim_{x\to 0}(-|x|) = \lim_{x\to 0}|x| = 0$, the result follows by the pinching theorem.

40. Since $0 \leq \cos^2[1/(x-\pi)] \leq 1$ for all $x \neq \pi$, we have $|(x-\pi)\,\cos^2[1/(x-\pi)]| \leq |x-\pi|$. Thus,

$$-|x-\pi| \leq |(x-\pi)\,\cos[1/(x-\pi)]| \leq |x-\pi|$$

Since $\lim_{x\to\pi}(-|x-\pi|) = \lim_{x\to\pi}|x-\pi| = 0$, the result follows by the pinching theorem.

41. For x close to 1(radian), $0 < \sin x \leq 1$. Thus,

$$0 < |x-1|\sin x \leq |x-1|$$

and the result follows by the pinching theorem.

42. Clearly, $|f(x)| \leq 1$ for all x. Therefore,

$$|x\,f(x)| \leq |x| \quad \text{which implies} \quad -|x| \leq xf(x) \leq |x| \quad \text{for all } x.$$

The result follows by the pinching theorem.

43. Suppose that there is a number B such that $|f(x)| \leq B$ for all $x \neq 0$. Then $|x\,f(x)| \leq B\,|x|$ and

$$-B\,|x| \leq x\,f(x) \leq B\,|x|$$

The result follows by the pinching theorem.

44. We have

$$f(x) = x \cdot \frac{f(x)}{x} \text{ and } \left|\frac{f(x)}{x}\right| \le B,\ x \ne 0.$$

Therefore, the result follows from Exercise 43.

45. Suppose that there is a number B such that $\left|\dfrac{f(x)-L}{x-c}\right| \le B$ for $x \ne c$. Then

$$0 \le |f(x) - L| = \left|(x-c)\,\frac{f(x)-L}{x-c}\right| \le B|x-c|$$

By the pinching theorem, $\lim_{x\to c} |f(x) - L| = 0$ which implies $\lim_{x\to c} f(x) = L$.

46. Let $\epsilon > 0$. Let $\delta > 0$ be such that for $|\delta - c| < p$,

$$\text{if } |x-c| < \delta \text{ then } |f(x)| < \frac{\epsilon}{B}.$$

In other words, $B|f(x)| < \epsilon$.

But $|g(x)| \le B$, so $|f(x)g(x)| \le B|f(x)|$. Thus $|f(x)g(x)| < \epsilon$.

SECTION 2.6

1. Let $f(x) = 2x^3 - 4x^2 + 5x - 4$. Then f is continuous on $[1,2]$ and $f(1) = -1 < 0$. $f(2) = 6 > 0$.

Thus by the intermediate-value theorem, there is a c in $[1,2]$ such that $f(c) = 0$.

2. Let $f(x) = x^4 - x - 1$. Then f is continuous on $[-1,1]$ and $f(-1) = 1 > 0$, $f(1) = -1 < 0$.

Thus by the intermediate-value theorem, there is a c in $[-1,1]$ such that $f(c) = 0$.

3. Let $f(x) = \sin x + 2\cos x - x^2$ Then f is continuous on $[0, \frac{\pi}{2}]$ and $f(0) = 2 > 0$, $f(\frac{\pi}{2}) = 1 - \frac{\pi^2}{4} < 0$.

Thus by the intermediate-value theorem, there is a c in $[0, \frac{\pi}{2}]$ such that $f(c) = 0$.

4. Let $f(x) = 2\tan x - x$. Then f is continuous on $[0, \frac{\pi}{4}]$ and $f(0) = 0 < 1$, $f(\frac{\pi}{4}) = 2 - \frac{\pi}{4} > 1$.

Thus by the intermediate-value theorem, there is a c in $[0, \frac{\pi}{4}]$ such that $f(c) = 1$.

5. Let $f(x) = x^2 - 2 + \frac{1}{2x}$. Then f is continuous on $[\frac{1}{4}, 1]$ and $f(\frac{1}{4}) = \frac{1}{16} > 0$, $f(1) = -\frac{1}{2} < 0$.

Thus by the intermediate-value theorem, there is a c in $[\frac{1}{4}, 1]$ such that $f(c) = 0$.

6. Let $f(x) = x^{\frac{5}{3}} + x^{\frac{1}{3}}$. Then f is continuous on $[-1,1]$ and $f(-1) = -2 < 1$, $f(1) = 2 > 1$.

Thus by the intermediate-value theorem, there is a c in $[-1,1]$ such that $f(c) = 1$.

7. Let $f(x) = x^3 - \sqrt{x+2}$. Then f is continuous on $[1,2]$ and $f(1) = 1 - \sqrt{3} < 0$, $f(2) = 6 > 0$.

Thus by the intermediate-value theorem, there is a c in $[1,2]$ such that $f(c) = 0$.

i.e. $c^3 = \sqrt{c+2}$.

8. Let $f(x) = \sqrt{x^2 - 3x} - 2$. Then f is continuous on $[3,5]$ and $f(3) = -2 < 0$, $f(5) = \sqrt{10} + 2 > 0$.

Thus by the intermediate-value theorem, there is a c in $[3,5]$ such that $f(c) = 0$.

9. Let $R(x) = (x-2)^2(10-2x)$. Then $R(x) = 0$ has solutions at $x = 2$ and $x = 5$.

Thus the intervals of interest are $(-\infty, 2), (2,5)$ and $(5, \infty)$.

By inspection, $R(x) > 0$ on $(-\infty, 2) \cup (2,5)$.

10. Let $R(x) = x(2x-1)(3x-5)$. Then $R(x) = 0$ has solutions at $x = 0$, $x = \frac{1}{2}$ and $x = \frac{5}{3}$.

Thus the intervals of interest are $(-\infty, 0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{5}{3})$ and $(\frac{5}{3}, \infty)$.

By inspection $R(x) \geq 0$ on $[0, \frac{1}{2}] \cup [\frac{5}{3}, \infty)$.

11. Let $R(x) = x^3 - 2x^2 + x$. Then $R(x) = 0$ has solutions at $x = 0$ and $x = 1$.

Thus the intervals of interest are $(-\infty, 0], [0,1]$ and $[1, \infty)$.

By inspection $R(x) \leq 0$ on $(-\infty, 0] \cup 1$.

12. Let $R(x) = \dfrac{2x-6}{x^2-6x+5}$. Then $R(x) = 0$ has a solution at $x = 3$ and is undefined at $x = 1$ and $x = 5$.

Thus the intervals of interest are $(-\infty, 1), (1,3), (3,5)$ and $(5, \infty)$.

By inspection $R(x) < 0$ on $(-\infty, 1) \cup (3,5)$.

13. Let $R(x) = \dfrac{1}{x-1} + \dfrac{4}{x-6}$. Then $R(x) = 0$ has a solutions at $x = 2$ and is undefined at $x = 1$ and $x = 6$.

Thus the intervals of interest are $(-\infty, 1), (1,2), (2,6)$ and $(6, \infty)$.

By inspection $R(x) > 0$ on $(1,2) \cup (6, \infty)$.

14. Let $R(x) = \dfrac{x^2-4x}{(x+2)^2}$. Then $R(x) = 0$ has solutions at $x = 0$ and $x = 4$. and is undefined at $x = -2$.

Thus the intervals of interest are $(-\infty, -2), (-2, 0), (0,4)$ and $(4, \infty)$.

By inspection $R(x) < 0$ on $(0,4)$.

15. $f(x)$ is continuous on $[0,1]$. $f(0) = 0 < 1$ and $f(1) = 4 > 1$.

Thus by the intermediate value theorem there is a c in $[0,1]$ such that $f(c) = 1$.

16. $f(x)$ is continuous on $[2,3]$. $f(2) = \frac{1}{2} > 0$ and $f(3) = -\frac{1}{2} < 0$,

Thus by the intermediate value theorem there is a c in $[2,3]$ (hence in $(1,4)$)such that $f(c) = 0$.

17. Let $f(x) = x^3 - 4x + 2$. Then $f(x)$ is continuous on $[-3, 3]$.

Checking the integer values on this interval,

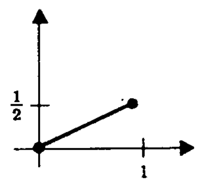
$$f(-3) = -13 < 0, \quad f(-2) = 2 > 0, \quad f(0) = 2 > 0, \quad f(1) = -1 < 0, \text{ and } f(2) = 2 > 0.$$

Thus by the intermediate value theorem there are roots in $(-3, -2), (0, 1)$ and $(1, 2)$.

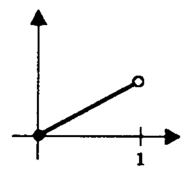
18. Let $f(x) = x^2$. Then $f(x)$ is continuous on $[1, 2]$, $f(1) = 1 < 2$ and $f(2) = 4 > 2$.

Thus by the intermediate value theorem there is a c in $[1, 2]$ such that $f(c) = 2$.

19.



20.



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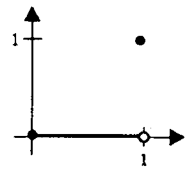
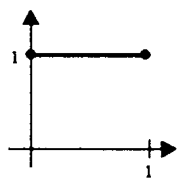
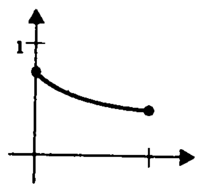


22. Impossible

23.



24.



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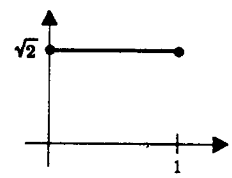

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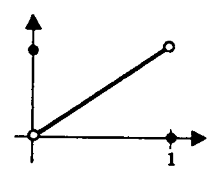
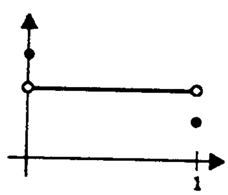


27. Impossible

28.



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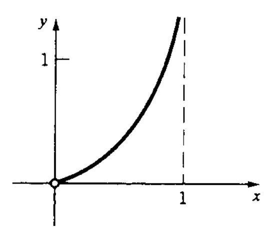


30. Impossible

31. Set $g(x) = x - f(x)$. Since g is continuous on $[0, 1]$ and $g(0) \le 0 \le g(1)$, there exists c in $[0, 1]$ such that $g(c) = c - f(c) = 0$.

32. (a) Let S be the set of positive integers for which the statement is true. Then $1 \in S$ by hypothesis. Now assume that $k \in S$. Then $a^k < b^k$ and

$$a^{k+1} = (a)a^k < (a)b^k < (b)b^k = b^{k+1}.$$

Therefore, $k + 1 \in S$ and S is the set of positive integers.

(b) Clearly 0 is the unique nth root of 0. Choose any positive number x and let $f(t) = t^n - x$. Since $f(0) = -x < 0$ and $f(t) \to \infty$ as $t \to \infty$, there exists a number $c > 0$ such that $f(c) = 0$. The number c is an nth root of x. The uniqueness follows from part (a).

33. Since f is bounded on $(-p, p)$, it follows from Exercise 43, Section 2.5, that $\lim_{x\to 0} xf(x) = 0$. Thus,

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} xf(x) = 0 = g(0)$$

which implies that g is continuous at 0.

34. Let $h(x) = f(x) - g(x)$. Then, h is continuous on $[a, b]$, and $h(a) = f(a) - g(a) < 0$, $h(b) = f(b) - g(b) > 0$. By the intermediate value theorem, there exists a number $c \in (a, b)$ such that $h(c) = 0$. Thus, $f(c) = g(c)$.

35. The cubic polynomial $P(x) = x^3 + ax^2 + bx + c$ is continuous on $(-\infty, \infty)$.. Writing P as

$$P(x) = x^3\left(1 + \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3}\right) \qquad x \neq 0$$

it follows that $P(x) < 0$ for large negative values of x and $P(x) > 0$ for large positive values of x. Thus there exists a negative number N such that $P(x) < 0$ for $x < N$, and a positive number M such that $P(x) > 0$ for $x > M$. By the intermediate-value theorem, P has a zero in $[N, M]$.

36. Think of the equator as being a circle and choose a reference point P and a positive direction. For example, choose P to be $0°$ longitude and let "eastward" be the positive direction. Using radian measure, let x, $0 \leq x \leq 2\pi$ denote the coordinate of a point x radians from P. Then, x and $x + \pi$ are diametrically opposite points on the equator. Let $T(x)$ be the temperature at the point x, and let $f(x) = T(x) - T(x + \pi)$. If $f(0) = 0$, then the temperatures at the points 0 and π are equal. If $f(x) \neq 0$, then $f(0) = T(0) - T(\pi)$ and $f(\pi) = T(\pi) - T(2\pi) = t(\pi) - T(0)$ have opposite sign. Thus, there exists a point $c \in (0, \pi)$ at which $f(c) = 0$, and $T(c) = T(c + \pi)$.

37. Let $A(r)$ denote the area of a circle with radius r, $r \in [0, 10]$. Then $A(r) = \pi r^2$ is continuous on $[0, 10]$, and $A(0) = 0$ and $A(10) = 100\pi \cong 314$. Since $0 < 250 < 314$ it follows from the intermediate value theorem that there exists a number $c \in (0, 10)$ such that $A(c) = 250$.

38. Let x and y be the dimensions of a rectangle in $\mathcal{R}$. Then, $2x + 2y = P$ and $y = \dfrac{P}{2} - x$. The area function

$$A(x) = xy = x\left(\frac{P}{2} - x\right) = \frac{P}{2}x - x^2, \quad x \in [0, P/2]$$

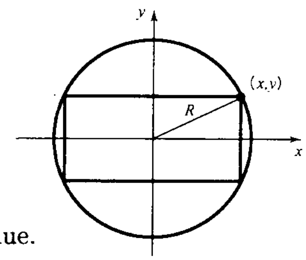
is continuous. Therefore, A has a maximum value on $[0, P/2]$. Since

$$A(x) = \frac{P}{2}x - x^2 = \frac{P^2}{16} - \left(x - \frac{P}{4}\right)^2$$

it is clear that the rectangle with maximum area has dimensions $x = y = \dfrac{P}{4}$.

39. Inscribe a rectangle in a circle of radius R and introduce a coordinate system as shown in the figure. Then the area of the rectangle is given by

$$A(x) = 4x\sqrt{R^2 - x^2}, \quad x \in [0, R].$$



Since A is continuous on $[0, R]$, A has a maximum value.

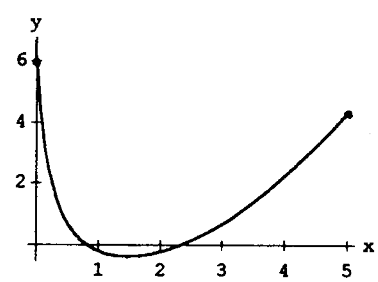
40. $f(0) = -4$, $f(1) = 2$. Thus, f has a zero in $(0,1)$ at $r = 0.771$.

41. $f(-3) = -9$, $f(-2) = 5$; $f(0) = 3$, $f(1) = -1$; $f(1) = -1$, $f(2) = 1$ Thus, f has a zero in $(-3, -2,)$ in $(0,1)$ and in $(1,2)$.
$r_1 = -2.4909$, $r_2 = 0.6566$, and $r_3 = 1.8343$

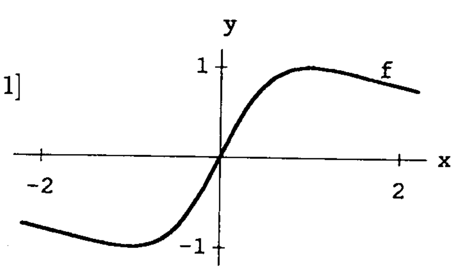
42. $f(-2) = -25$, $f(-1) = 1$; $f(0) = 1$, $f(1) = -1$; $f(1) = -1$, $f(2) = 27$ Thus, f has a zero in $(-2, -1,)$ in $(0,1)$ and in $(1,2)$.
$r_1 = -1.3888$, $r_2 = 0.3345$, and $r_3 = 1.2146$

43. $f(-2) = -5.6814$, $f(-1) = 1.1829$; $f(0) = 0.5$, $f(1) = -0.1829$; $f(1) = -0.1829$, $f(2) = 6.681$
Thus, f has a zero in $(-2, -1)$, in $(0,1)$ and in $(1,2)$.
$r_1 = -1.3482$, $r_2 = 0.2620$, and $r_3 = 1.0816$

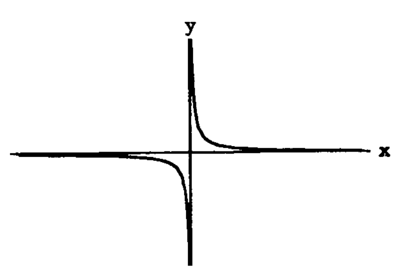
44. f is bounded.
$\max(f) = 6 \quad [f(0) = 6]$
$\min(f) \cong -0.376 \quad [f(1.46) \cong -0.376]$



45. f is bounded.
$\max(f) = 1 \quad [f(1) = 1]$
$\min(f) = -1 \quad [f(-1) = -1]$



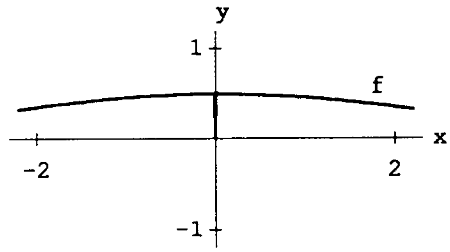
46. f is unbounded



47. f is bounded.

max $(f) = 0.5$

min $(f) \cong 0.3540$



PROJECT 2.6

1. $\dfrac{2-1}{2^n} < 0.01 \implies 2^n > 100 \implies n > 6.$

Thus, minimum number of iterations required is $n = 7$.

$$\sqrt{2} \simeq \frac{181}{128} \simeq 1.4140625.$$

$\dfrac{2-1}{2^n} < 0.0001 \implies 2^n > 10{,}000 \implies n > 13.$

Thus, minimum number of iterations required is $n = 14$.

2. $f(x) = x^3 + x - 9$; $f(1) = -7$ and $f(2) = 1$. Therefore, $f(c) = 0$ for some $c \in (1, 2)$.

A minimum of 7 iterations are required; $c \simeq \dfrac{245}{128} \simeq 1.9140625$.

Accurate to 7 decimal places, the root is $c \simeq 1.9201751$.

3. For $f(x) = x^2 - 2$, the first three iterations are:

$$c_1 = \frac{4}{3} \simeq 1.333...$$

$$c_2 = \frac{7}{5} = 1.4$$

$$c_3 = \frac{24}{17} \simeq 1.41176$$

For $f(x) = x^3 + x - 9$, the first three iterations are:

$$c_1 = \frac{15}{8} = 1.875$$

$$c_2 \simeq 1.918471$$

$$c_3 \simeq 1.920112$$

These approximations appear to converge more rapidly than the approximations obtained by the bisection method.

CHAPTER 3

SECTION 3.1

1. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{4-4}{h} = \lim_{h\to 0} 0 = 0$

2. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0$

3. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[2-3(x+h)]-[2-3x]}{h} = \lim_{h\to 0} \frac{-3h}{h} = \lim_{h\to 0} -3 = -3$

4. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[4(x+h)-1]-[4x+1]}{h} = \lim_{h\to 0} \frac{4h}{h} = \lim_{h\to 0} 4 = 4$

5. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[5(x+h)-(x+h)^2]-(5x-x^2)}{h}$

$$= \lim_{h\to 0} \frac{5h-2xh-h^2}{h} = \lim_{h\to 0}(5-2x-h) = 5-2x$$

6. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[2(x+h)^3+1]-[2x^3+1]}{h}$

$$= \lim_{h\to 0} \frac{2(x^3+3x^2h+3xh^2+h^3)-2x^3}{h} = \lim_{h\to 0} \frac{6x^2h+6xh^2+2h^4}{h}$$

$$= \lim_{h\to 0}(6x^2+6xh+2h^2) = 6x^2$$

7. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^4-x^4}{h}$

$$= \lim_{h\to 0} \frac{(x^4+4x^3h+6x^2h^2+4xh^3+h^4)-x^4}{h}$$

$$= \lim_{h\to 0}(4x^3+6x^2h+4xh^2+h^3) = 4x^3$$

8. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h+3}-\frac{1}{x+3}}{h}$

$$\lim_{h\to 0} \frac{(x+3)-(x+h+3)}{h(x+h+3)(x+3)} = \lim_{h\to 0} \frac{-h}{x(x+h+3)(x+3)}$$

$$\lim_{h\to 0} \frac{-1}{(x+h+3)(x+3)} = \frac{-1}{(x+3)^2}$$

9. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h-1}-\sqrt{x-1}}{h}$

$$= \lim_{h\to 0} \frac{(x+h-1)-(x-1)}{h(\sqrt{x+h-1}+\sqrt{x-1})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h-1}+\sqrt{x-1}} = \frac{1}{2\sqrt{x-1}}$$

10. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{[(x+h)^3 - 4(x+h)] - [x^3 - 4x]}{h}$

$= \lim_{h\to 0} \dfrac{3x^2h + 3xh^2 + h^3 - 4h}{h} = \lim_{h\to 0}(3x^2 + 3xh + h^2 - 4) = 3x^2 - 4$

11. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h}$

$= \lim_{h\to 0} \dfrac{x^2 - (x^2 + 2hx + h^2)}{hx^2(x+h)^2} = \lim_{h\to 0} \dfrac{-2x-h}{x^2(x+h)^2} = -\dfrac{2}{x^3}$

12. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\dfrac{1}{\sqrt{x+h}} - \dfrac{1}{\sqrt{x}}}{h}$

$\lim_{h\to 0} \dfrac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} = \lim_{h\to 0} \dfrac{\left(\sqrt{x} - \sqrt{x+h}\right)\left(\sqrt{x} + \sqrt{x+h}\right)}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x} + \sqrt{x+h}\right)} = \lim_{h\to 0} \dfrac{x - (x+h)}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x} + \sqrt{x+h}\right)}$

$\lim_{h\to 0} \dfrac{-1}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x} + \sqrt{x+h}\right)} = \dfrac{-1}{2x\sqrt{x}}$

13. $f'(2) = \lim_{h\to 0} \dfrac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \dfrac{(3h-1)^2 - 1}{h}$

$= \lim_{h\to 0} \dfrac{9h^2 - 6h}{h} = \lim_{h\to 0}(9h - 6) = -6$

14. $f'(2) = \lim_{h\to 0} \dfrac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \dfrac{[7(2+h) - (2+h)^2] - [7(2) - (2)^2]}{h}$

$= \lim_{h\to 0} \dfrac{3h - h^2}{h} = \lim_{h\to 0}(3 - h) = 3$

15. $f'(2) = \lim_{h\to 0} \dfrac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \dfrac{\dfrac{9}{6+h} - \dfrac{3}{2}}{h}$

$= \lim_{h\to 0} \dfrac{18 - 3(6+h)}{2h(6+h)} = \lim_{h\to 0} \dfrac{-3}{2(6+h)} = -\dfrac{1}{4}$

16. $f'(2) = \lim_{h\to 0} \dfrac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \dfrac{[5 - (2+h)^4] - 5 - 2^4]}{h}$

$= \lim_{h\to 0} \dfrac{-((4)2^3h + (6)2^2h^2 + (4)2h^3 + h^4}{h} = \lim_{h\to 0}(-32 - 24h - 8h^2) = -32$

17. $f'(2) = \lim_{h\to 0} \dfrac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \dfrac{(2 + h + \sqrt{4+2h}) - 4}{h}$

$= \lim_{h\to 0} \left(1 + \dfrac{\sqrt{4+2h} - 2}{h}\right) = \lim_{h\to 0} 1 + \dfrac{(4+2h) - 4}{h(\sqrt{4+2h} + 2)}$

$= \lim_{h\to 0} 1 + \dfrac{2}{\sqrt{4+2h} + 2} = \dfrac{3}{2}$

18. $$f'(2)=\lim_{h\to0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to0}\frac{\sqrt{6-(2+h)}-\sqrt{6-2}}{h}$$
$$=\lim_{h\to0}\frac{\sqrt{4-h}-2}{h}=\lim_{h\to0}\frac{\left(\sqrt{4-h}-2\right)\left(\sqrt{4-h}+2\right)}{h\left(\sqrt{4-h}+2\right)}$$
$$=\lim_{h\to0}\frac{4-h-4}{h\left(\sqrt{4-h}-2\right)}=\lim_{h\to0}\frac{-1}{\sqrt{4-h}+2}=-\frac{1}{4}$$

19. Slope of tangent at $(2,4)$ is $f'(2)=4$. Tangent $y-4=4(x-2)$; normal $y-4=-\frac{1}{4}(x-2)$.

20. Slope of tangent at $(4,2)$ is $f'(4)=\dfrac{1}{2\sqrt{4}}=\dfrac{1}{4}$. Tangent $y-2=\dfrac{1}{4},(x-4)$; normal $y-2=-4(x-4)$.

21. Slope of tangent at $(4,4)$ is $f'(4)=-3$. Tangent $y-4=-3(x-4)$; normal $y-4=\frac{1}{3}(x-4)$.

22. Slope of tangent at $(2,-3)$ is $f'(2)=-3(2)^2=-12$. Tangent $y+3=-12(x-2)$; normal $y+3=\frac{1}{12}(x-2)$.

23. Slope of tangent at $(-2,\dfrac{1}{4})$ is $\dfrac{1}{4}$. Tangent $y-\dfrac{1}{4}=\dfrac{1}{4}(x+2)$; normal $y-\dfrac{1}{4}=-4(x+2)$.

24. Slope of tangent at $(-1,\dfrac{1}{2})$ is $f'(-1)=-1/(-1+3)^2=-\dfrac{1}{4}$. Tangent $y-\dfrac{1}{2}=-\dfrac{1}{4}(x+1)$; normal $y-\dfrac{1}{2}=4(x+1)$.

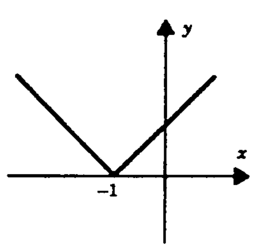
25. (a) f is not continuous at $c=-1$ and $c=1$; f has a removable discontinuity at $c=-1$ and a jump discontinuity at $c=1$.

(b) f is continuous but not differentiable at $c=0$ and $c=3$.

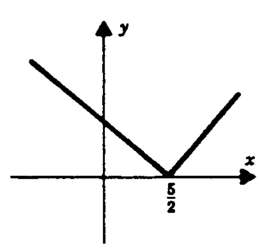
26. (a) f is not continuous at $c=2$; f has a jump discontinuity at 2

(b) f is continuous but not differentiable at $c=-2$ and $c=3$.

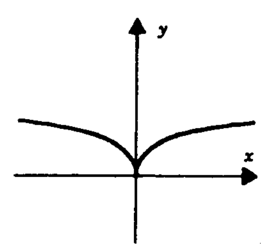
27. at $x=-1$



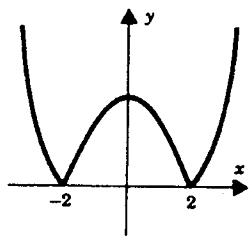
28. at $x=\frac{5}{2}$



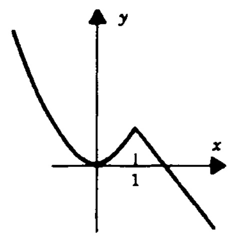
29. at $x=0$



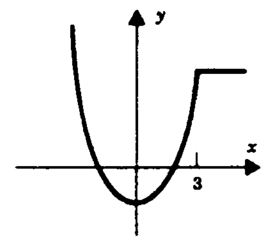
30. at $x = -2, 2$



31. at $x = 1$



32. at $x = 3$



33. $f'(1) = 4$

$$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{4(1+h)-4}{h} = 4$$

$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{2(1+h)^2+2-4}{h} = 4$$

34. $f'(1) = 6$

$$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{3(1+h)^2-3}{h} = 6$$

$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{[2(1+h)^3+1]-3}{h} = 6$$

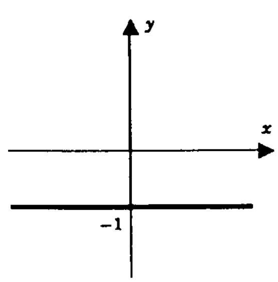
35. $f'(-1)$ does not exist

$$\lim_{h\to 0^-} \frac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0^-} \frac{h-0}{h} = 1$$

$$\lim_{h\to 0^+} \frac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0^+} \frac{h^2-0}{h} = 0$$

36. $f'(3)$ does not exist; f is not continuous at 3.

37.



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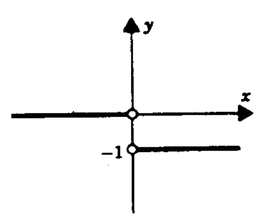

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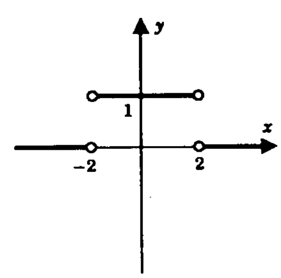

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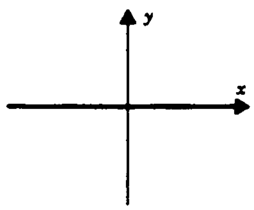

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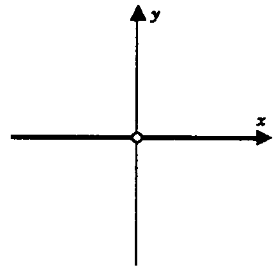

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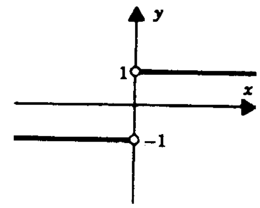


43. $f(x) = x^2; \quad c = 1$

44. $f(x) = -x^3; \quad c = -2$

45. $f(x) = \sqrt{x}; \quad c = 4$

46. $f(x) = x^{1/3}; \quad c = 8$

47. $f(x) = \cos x; \quad c = \pi$

48. $f(x) = \sin x; \quad c = \pi/6$

49. Since $f(1) = 1$ and $\lim\limits_{x\to 1^+} f(x) = 2$, f is not continuous at 1. Therefore, by (3.1.4), f is not differentiable at 1.

50. Continuity at $x = 1$: $\lim\limits_{x\to 1^-} f(x) = 1 = \lim\limits_{x\to 1^+} f(x) = A + B$. Thus $A + B = 1$.

Differentiability at $x = 1$:

$$\lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^-} \frac{(1+h)^3 - 1}{h} = 3 = \lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} = A$$

Therefore, $A = 3, \implies B = -2$.

51. (a) $f'(x) = \left\{\begin{array}{l} 2(x+1),\ x < 0 \\ 2(x-1),\ x > 0 \end{array}\right]$

(b) $$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{(h+1)^2 - 1}{h} = \lim_{h\to 0^-} (h+2) = 2,$$

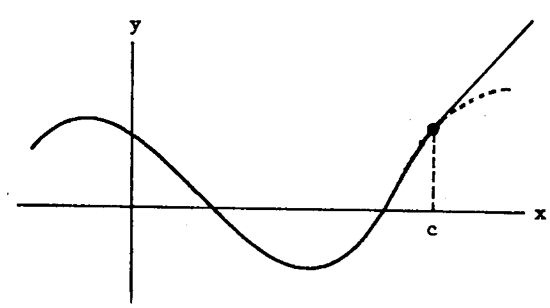
$$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{(h-1)^2 - 1}{h} = \lim_{h\to 0^+} (h-2) = -2.$$

52. (a) $$\lim_{h\to 0^-} \frac{g(c+h) - g(c)}{h} = \lim_{h\to 0^-} \frac{f(c+h) - f(c)}{h} = f'(c).$$

$$\lim_{h\to 0^+} \frac{g(c+h) - g(c)}{h} = \lim_{h\to 0^+} \frac{[f'(c)(c+h-c) + f(c)] - f(c)}{h} = f'(c).$$

Therefore, g is differentiable at c and $g'(c) = f'(c)$.

(b)



53. $f(x) = c$, c any constant

54. $f(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$

55. $f(x) = |x+1|$; or $f(x) = \begin{cases} 0, & x \neq -1 \\ 1, & x = -1 \end{cases}$

56. $f(x) = |x^2 - 1|$

57. $f(x) = 2x + 5$

58. $f(x) = -|x|$

59. (a) $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = f(2) = 2$ Thus, f is continuous at $x = 2$.

(b) $f'_-(2) = \lim_{h\to 0^-} \dfrac{f(2+h) - f(2)}{h} = \lim_{h\to 0^-} \dfrac{(2+h)^2 - (2+h) - 2}{h} = 3$

$f'_+(2) = \lim_{h\to 0^+} \dfrac{f(2+h) - f(2)}{h} = \lim_{h\to 0^+} \dfrac{2(2+h) - 2 - 2}{h} = 2$

(c) No, since $f'_-(2) \neq f'_+(2)$.

60. (a)
$$f'(x) = \lim_{h\to 0} \frac{(x+h)\sqrt{x+h} - x\sqrt{x}}{h}$$
$$= \lim_{h\to 0} \frac{x\left(\sqrt{x+h} - \sqrt{x}\right) + h\sqrt{x+h}}{h}$$
$$= \lim_{h\to 0} \frac{xh}{h\left(\sqrt{x+h} + \sqrt{x}\right)} + \lim_{h\to 0} \sqrt{x+h}$$
$$= \tfrac{1}{2}\sqrt{x} + \sqrt{x} = \tfrac{3}{2}\sqrt{x}$$

(b) $f'_+(0) = \lim_{h\to 0^+} \dfrac{(0+h)\sqrt{0+h} - 0}{h} = \lim_{h\to 0^+} \dfrac{h\sqrt{h}}{h} = 0.$

61. (a) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{\sqrt{1-(x+h)} - \sqrt{1-x}}{h}$

$= \lim_{h\to 0} \dfrac{-h}{h\left(\sqrt{1-(x+h)} + \sqrt{1-x}\right)} = \dfrac{-1}{2\sqrt{1-x}}$

(b) $f'_+(0) = \lim_{h\to 0^+} \dfrac{\sqrt{1-h} - 1}{h} = -\dfrac{1}{2}$

(c) $f'_-(1) = \lim_{h\to 0^-} \dfrac{\sqrt{1-(1+h)}}{h} = \lim_{h\to 0^-} \dfrac{\sqrt{-h}}{h} = \lim_{h\to 0^-} \dfrac{-1}{\sqrt{-h}}$ does not exist.

62. (a) $f'_-(0) = \lim_{h\to 0^-} \dfrac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \dfrac{(1-h^2) - 1}{h} = 0$

(b) $f'_+(0) = \lim_{h\to 0^-} \dfrac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \dfrac{h^2 - 1}{h}$ does not exist

(c) f is not differentiable at 0; f is not continuous at 0.

63. Suppose $f'_-(c) = \lim_{h\to 0^-} \dfrac{f(c+h) - f(c)}{h} = L = \lim_{h\to 0^+} \dfrac{f(c+h) - f(c)}{h} = f'_+(c).$

Then $\lim_{h\to 0}\dfrac{f(c+h)-f(c)}{h}=L$ exists and f is differentiable at c.

64. (a) No. Since f is not continuous at $x\neq 0$, f cannot be differentiable at $x\neq 0$.

(b) $$\lim_{h\to 0}\frac{f(h)-0}{h}=\begin{cases}\lim_{h\to 0}\dfrac{h}{h}=1, & h \text{ rational}\\ \lim_{h\to 0}\dfrac{0}{h}=0, & h \text{ irrational}\end{cases}$$

Therefore, $\lim_{h\to 0}\dfrac{f(0+h)-f(0)}{h}$ does not exist.

(c) $$\lim_{h\to 0}\frac{g(h)-0}{h}=\begin{cases}\lim_{h\to 0}\dfrac{h^2}{h}=0, & h \text{ rational}\\ \lim_{h\to 0}\dfrac{0}{h}=0, & h \text{ irrational}\end{cases}$$

Therefore, g is differentiable at 0 and $g'(0)=0$.

65. (a) Since $|\sin(1/x)|\leq 1$ it follows that

$$-x\leq f(x)\leq x \quad\text{and}\quad -x^2\leq g(x)\leq x^2$$

Thus $\lim_{x\to 0}f(x)=f(0)=0$ and $\lim_{x\to 0}g(x)=g(0)=0$, which implies that f and g are continuous at 0.

(b) $\lim_{h\to 0}\dfrac{h\sin(1/h)-0}{h}=\lim_{h\to 0}\sin(1/h)$ does not exist.

(c) $\lim_{h\to 0}\dfrac{h^2\sin(1/h)-0}{h}=\lim_{h\to 0}h\sin(1/h)=0.$ Thus g is differentiable at 0 and $g'(0)=0$.

66. $$f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{[(2+h)^3+1]-(2^3+1)}{h}=\lim_{h\to 0}\frac{12h+6h^2+h^3}{h}=12$$

$$f'(2)=\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2}\frac{(x^3+1)-9}{x-2}=\lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{x-2}=\lim_{x\to 2}(x^2+2x+4)=12$$

67. $$f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{[(1+h)^2-3(1+h)]-(-2)]}{h}=\lim_{h\to 0}\frac{-h+h^2}{h}=-1$$

$$f'(1)=\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1}\frac{(x^2-3x)-(-2)}{x-1}=\lim_{x\to 1}\frac{(x-2)(x-1)}{x-1}=\lim_{x\to 1}(x-2)=-1$$

68. $$f'(3)=\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}=\lim_{h\to 0}\frac{\sqrt{1+(3+h)}-2}{h}=\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}=\lim_{h\to 0}\frac{h}{h\sqrt{4+h}+2}=\frac{1}{4}$$

$$f'(3)=\lim_{x\to 3}\frac{f(x)-f(3)}{x-3}=\lim_{x\to 1}\frac{\sqrt{1+x}-2}{x-3}=\lim_{x\to 3}\frac{x-3}{(x-3)(\sqrt{1+x}+2}=\lim_{x\to 3}\frac{1}{\sqrt{1+x}+2}=\frac{1}{4}$$

69. $f'(-1) = \lim_{h\to 0} \dfrac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0} \dfrac{(-1+h)^{1/3} + 1}{h}$

$$= \lim_{h\to 0} \frac{(-1+h)^{1/3}+1}{h} \cdot \frac{(-1+h)^{2/3} - (-1+h)^{1/3} + 1}{(-1+h)^{2/3} - (-1+h)^{1/3} + 1}$$

$$= \lim_{h\to 0} \frac{h}{h\,(-1+h)^{2/3} - (-1+h)^{1/3} + 1} = \frac{1}{3}$$

$$f'(-1) = \lim_{x\to -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x\to -1} \frac{x^{1/3}+1}{x+1} = \lim_{x\to -1} \frac{x^{1/3}+1}{x+1} \cdot \frac{x^{2/3} - x^{1/3} + 1}{x^{2/3} - x^{1/3} + 1}$$

$$= \lim_{x\to -1} \frac{x+1}{(x+1)(x^{2/3} - x^{1/3} + 1)} = \frac{1}{3}$$

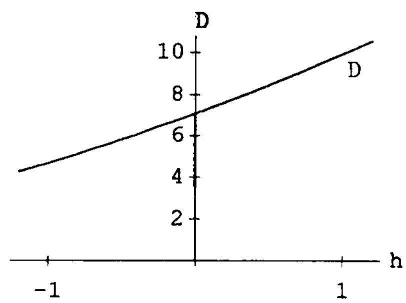
70. $f'(0) = \lim_{h\to 0} \dfrac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \dfrac{\dfrac{1}{h+2} - \dfrac{1}{2}}{h} = \lim_{h\to 0} \dfrac{-h}{2h(h+2)} = \lim_{h\to 0} \dfrac{-1}{2(h+2)} = -\dfrac{1}{4}$

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{\dfrac{1}{x+2} - \dfrac{1}{2}}{x} = \lim_{x\to 0} \frac{-x}{2x(x+2} = \lim_{x\to 0} \frac{-1}{2(x+2)} = -\frac{1}{4}$$

71. (a) $D = \dfrac{(2+h)^{5/2} - 2^{5/2}}{h}$ $\quad -1 \le h \le 1$

(b) $f'(2) \cong 7.071$

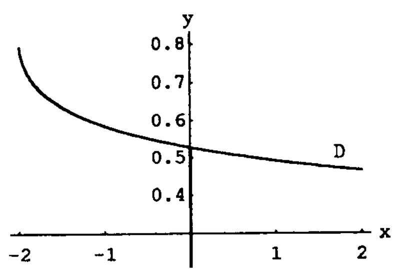
(c) $D(0.001) \cong 7.074, \quad D(-0.001) \cong 7.068$



72. (a) $D = \dfrac{(2+h)^{2/3} - 2^{2/3}}{h}$ $\quad -1 \le h \le 1$

(b) $f'(2) \cong 0.529$

(c) $D(0.001) \cong 0.52909, \quad D(-0.001) \cong 0.52918$

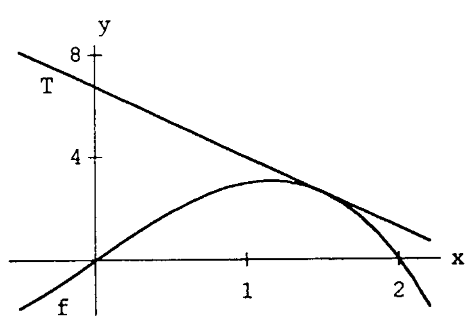


73. (a) Let $f(x) = 4x - x^3$.

Then $f'(x) = 4 - 3x^2$; $\quad f'(3/2) = \frac{11}{4}$

$T(x) = -\frac{11}{4}\left(x - \frac{3}{2}\right) + \frac{21}{8}$

(b)



(c) $(1.453, 1.547)$

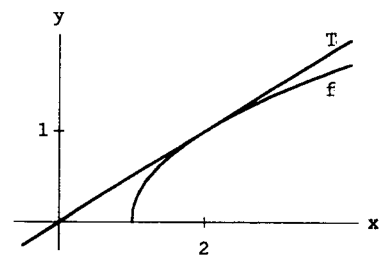
74. (a) Let $f(x) = \sqrt{x-1}$.

Then $f'(x) = \dfrac{1}{2sqrtx - 1}$; $f'(2) = \frac{1}{2}$

$T(x) = \frac{1}{2}(x-2) + 1$

(c) $(1.875, 2.125)$

(b)



SECTION 3.2

1. $F'(x) = -1$

2. $F'(x) = 2$

3. $F'(x) = 55x^4 - 18x^2$

4. $F'(x) = \dfrac{-6}{x^3}$

5. $F'(x) = 2ax + b$

6. $F'(x) = x^3 - x^2 + x - 1$

7. $F'(x) = 2x^{-3}$

8. $F'(x) = \dfrac{x^3(2x) - (x^2+2)3x^2}{x^6} = -\dfrac{x^2+6}{x^4}$

9. $G'(x) = (x^2-1)(1) + (x-3)(2x) = 3x^2 - 6x - 1$

10. $F'(x) = 1 + \dfrac{1}{x^2}$

11. $G'(x) = \dfrac{(1-x)(3x^2) - x^3(-1)}{(1-x)^2} = \dfrac{3x^2 - 2x^3}{(1-x)^2}$

12. $F'(x) = \dfrac{(cx-d)a - (ax-b)c}{(cx-d)^2} = \dfrac{bc - ad}{(cx-d)^2}$

13. $G'(x) = \dfrac{(2x+3)(2x) - (x^2-1)(2)}{(2x+3)^2} = \dfrac{2(x^2+3x+1)}{(2x+3)^2}$

14. $G'(x) = \dfrac{(x+1)(28x^3) - (7x^4+11)(1)}{(x+1)^2} = \dfrac{21x^4 + 28x^3 - 11}{(x+1)^2}$

15. $G'(x) = (x-1)(1) + (x-2)(1) = 2x - 3$

16. $G'(x) = \dfrac{(x+2)(4x) - (2x^2+1)(1)}{(x+2)^2} = \dfrac{2x^2 + 8x - 1}{(x+2)^2}$

17. $G'(x) = \dfrac{(x-2)(1/x^2) - (6-1/x)(1)}{(x-2)^2} = \dfrac{-2(3x^2 - x + 1)}{x^2(x-2)^2}$

18. $G'(x) = \dfrac{x^2(4x^3) - (1+x^4)2x}{x^4} = \dfrac{2(x^4-1)}{x^3}$

19. $G'(x) = (9x^8 - 8x^9)\left(1 - \dfrac{1}{x^2}\right) + \left(x + \dfrac{1}{x}\right)(72x^7 - 72x^8) = -80x^9 + 81x^8 - 64x^7 + 63x^6$

20. $G'(x) = \left(\dfrac{-1}{x^2}\right)\left(1 + \dfrac{1}{x^2}\right) + \left(1 + \dfrac{1}{x}\right)\left(\dfrac{-2}{x^3}\right) = -\dfrac{1}{x^2} - \dfrac{2}{x^3} - \dfrac{3}{x^4}$

21. $f'(x) = -x(x-2)^{-2}$; $f'(0) = -\frac{1}{4}$, $f'(1) = -1$

22. $f'(x) = 3x^3 + 2x$; $f'(0) = 0$, $f'(1) = 5$

23. $f'(x) = \dfrac{(1+x^2)(-2x) - (1-x^2)(2x)}{(1+x^2)^2} = \dfrac{-4x}{(1+x^2)^2}; \quad f'(0) = 0, \quad f'(1) = -1$

24. $f'(x) = \dfrac{(x+1)^2(4x+1) - (2x^2+x+1)2(x+1)}{(x+1)^4} = \dfrac{(x+1)(4x+1) - 2(2x^2+x+1)}{(x+1)^3};$

$f'(0) = -1, \quad f'(1) = \frac{1}{4}$

25. $f'(x) = \dfrac{(cx+d)a - (ax+b)c}{(cx+d)^2} = \dfrac{ad-bc}{(cx+d)^2}, \quad f'(0) = \dfrac{ad-bc}{d^2}, \quad f'(1) = \dfrac{ad-bc}{(c+d)^2}$

26. $f'(x) = \dfrac{(cx^2+bx+a)(2ax+b) - (ax^2+bx+c)(2cx+b)}{(cx^2+bx+a)^2}; \quad f'(0) = \dfrac{b(a-c)}{a^2}, \quad f'(1) = \dfrac{2(a-c)}{a+b+c}$

27. $f'(x) = xh'(x) + h(x); \quad f'(0) = 0h'(0) + h(0) = 0(2) + 3 = 3$

28. $f'(x) = 6xh(x) + 3x^2h'(x) - 5; \quad f'(0) = -5$

29. $f'(x) = h'(x) + \dfrac{h'(x)}{[h(x)]^2}, \quad f'(0) = h'(0) + \dfrac{h'(0)}{[h(0)]^2} = 2 + \dfrac{2}{3^2} = \dfrac{20}{9}$

30. $f'(x) = h'(x) + \dfrac{h(x) - xh'(x)}{h^2(x)}; \quad f'(0) = h'(0) + \dfrac{h(0)}{h^2(0)} = 2 + \dfrac{1}{3} = \dfrac{7}{3}$

31. $f'(x) = \dfrac{(x+2)(1) - x(1)}{(x+2)^2} = \dfrac{2}{(x+2)^2},$

slope of tangent at $(-4, 2) : f'(-4) = \frac{1}{2},$

equation for tangent: $y - 2 = \frac{1}{2}(x+4)$

32. $f'(x) = (x^3 - 2x + 1)(4) + (4x - 5)(3x^2 - 2) = 16x^3 - 15x^2 - 16x + 14$

slope of tangent at $(2, 15) : f'(2) = 50,$

equation for tangent: $y - 15 = 50(x - 2)$

33. $f'(x) = (x^2 - 3)(5 - 3x^2) + (5x - x^3)(2x);$

slope of tangent at $(1, -8) : f'(1) = (-2)(2) + (4)(2) = 4,$

equation for tangent: $y + 8 = 4(x - 1)$

34. $f'(x) = 2x + \dfrac{10}{x^2};$

slope of tangent at $(-2, 9) : f'(-2) = -\frac{3}{2},$

equation for tangent: $y - 9 = -\frac{3}{2}(x + 2)$

35. $f'(x) = (x - 2)(2x - 1) + (x^2 - x - 11)(1) = 3(x - 3)(x + 1);$

$f'(x) = 0$ at $x = -1, 3;$ $(-1, 27), (3, -5)$

36. $f'(x) = 2x + \dfrac{16}{x^2} = \dfrac{2(x^3+8)}{x^2}; \quad f'(x) = 0$ at $x = -2;$ $(-2, 12)$

37. $f'(x) = \dfrac{(x^2+1)(5) - 5x(2x)}{(x^2+1)^2} = \dfrac{5(1-x^2)}{(x^2+1)^2}, \quad f'(x) = 0$ at $x = \pm 1;$ $(-1, -5/2), (1, 5/2)$

38. $f'(x) = (x+2)(2x-2) + (x^2 - 2x - 8)(1) = 3x^2 - 12 = 3(x^2 - 4);$

$f'(x) = 0$ at $x = \pm 2$; $(-2, 0)$, $(2, -32)$

39. $f'(x) = 1 - 8/x^3$, $f'(x) = 0$ at $x = 2$; $(2, 3)$

40. $f'(x) = \dfrac{(x^2+4)(2x-2) - (x^2 - 2x + 4)(2x)}{(x^2+4)^2} = \dfrac{2(x^2-4)}{(x^2+4)^2};$

$f'(x) = 0$ at $x = \pm 2$; $\left(-2, \frac{3}{2}\right)$, $\left(2, \frac{1}{2}\right)$

41. slope of line 4; slope of tangent $f'(x) = -2x$; $-2x = 4$ at $x = -2$; $(-2, -10)$

42. slope of line 3/5; slope of tangent $f'(x) = 3x^2 - 3$;

perpendicular when $3x^2 - 3 = -\frac{5}{3}$; $x = \pm\frac{2}{3}$; $\left(-\frac{2}{3}, \frac{46}{27}\right)$, $\left(\frac{2}{3}, -\frac{46}{27}\right)$

43. slope of line $-1/5$; slope of tangent $3x^2 - 2x$;

perpendicular when $3x^2 - 2x = 5$; $x = -1, 5/3$; $(-1, -2)$, $\left(\frac{5}{3}, \frac{50}{27}\right)$

44. slope of line 3/2; slope of tangent $f'(x) = 4 - 2x$; $4 - 2x = \frac{3}{2}$ at $x = \frac{5}{4}$; $\left(\frac{5}{4}, \frac{55}{16}\right)$

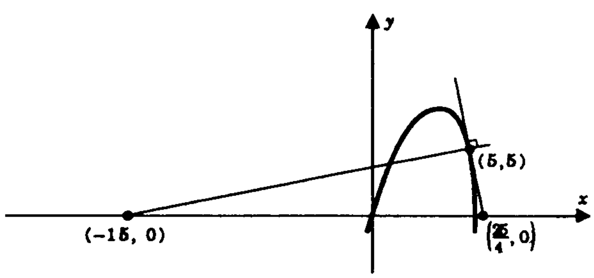
45. $f(x) = x^3 + x^2 + x + C$

46. $f(x) = x^4 - x^2 + 4x + C$

47. $f(x) = \dfrac{2x^3}{3} - \dfrac{3x^2}{2} + \dfrac{1}{x} + C$

48. $f(x) = \dfrac{x^5}{5} + \dfrac{x^4}{2} + \sqrt{x} + C$

49.



slope of tangent at $(5, 5)$ is $f'(5) = -4$

tangent $y - 5 = -4(x - 5)$ intersects

x-axis at $\left(\frac{25}{4}, 0\right)$

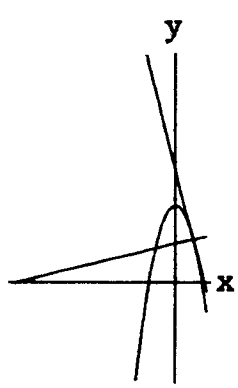
normal $y - 5 = \frac{1}{4}(x - 5)$ intersects

x-axis at $(-15, 0)$

area of triangle is

$\frac{1}{2}(5)(15 + \frac{25}{4}) = \frac{425}{8}$

50.



slope of tangent at $(2, 5)$ is $f'(2) = -4$

tangent $y - 5 = -4(x - 2)$ intersects

x-axis at $\left(\frac{13}{4}, 0\right)$

normal $y - 5 = \frac{1}{4}(x - 2)$ intersects

x-axis at $(-18, 0)$

area of triangle is

$\frac{1}{2}(5)(18 + \frac{13}{4}) = \frac{425}{8}$

51. If the point $(1, 3)$ lies on the graph, we have $f(1) = 3$ and thus

$$(*) \qquad A + B + C = 3.$$

If the line $4x + y = 8$ (slope -4) is tangent to the graph at $(2, 0)$, then

$f(2) = 0$ and $f'(2) = -4$. Thus,

$$(**) \qquad 4A + 2B + C = 0 \quad \text{and} \quad 4A + B = -4.$$

Solving the equations in $(*)$ and $(**)$, we find that $A = -1, \quad B = 0, \quad C = 4.$

52. First, $f(1) = 0$ and $f'(1) = 3 \implies A + B + C + D = 0$ and $3A + 2B + C = 3$

Next, $f(2) = 9$ and $f'(2) = 18 \implies 8A + 4B + 2C + D = 9$ and $12A + 4B + C = 18$

Solving these equations gives $A = 3,\ B = -6,\ C = 6,\ D = -3.$

53. Let $f(x) = ax^2 + bx + c$. Then $f'(x) = 2ax + b$ and $f'(x) = 0$ at $x = -b/2a$.

54. The derivative of p is the quadratic $p'(x) = 3ax^2 + 2bx + c$. Its discriminant is

$$D = (2b)^2 - 4(3a)(c) = 4b^2 - 12ac$$

(a) p has two horizontal tangents iff p' has two real roots iff $D > 0$.

(b) p has exactly one horizontal tangent iff p has only one real root iff $D = 0$.

(c) p has no horizontal tangent iff p has no real roots iff $D < 0$.

55. Let $f(x) = x^3 - x$. The secant line through $(-1, f(-1)) = (-1, 0)$ and $(2, f(2)) = (2, 6)$ has slope $m = \dfrac{6-0}{2-(-1)} = 2$. Now, $f'(x) = 3x^2 - 1$ and $3c^2 - 1 = 2$ implies $c = -1,\ 1$.

56. $m_{sec} = \dfrac{f(3) - f(1)}{3-1} = \dfrac{\frac{3}{4} - \frac{1}{2}}{2} = \frac{1}{8}$

$f'(x) = \dfrac{(x+1)(1) - x(1)}{(x+1)^2} = \dfrac{1}{(x+1)^2}; \quad f'(c) = \frac{1}{8} \implies c = -1 \pm 2\sqrt{2}$

57. Let $f(x) = 1/x,\ x > 0$. Then $f'(x) = -1/x^2$. An equation for the tangent line to the graph of f at the point $(a, f(a)),\ a > 0$, is $y = (-1/a^2)x + 2/a$. The y-intercept is $2/a$ and the x-intercept is $2a$. The area of the triangle formed by this line and the coordinate axes is: $A = \frac{1}{2}(2/a)(2a) = 2$ square units.

58. Let (x, y) be the point on the graph that the tangent line passes through. $f'(x) = 3x^2$, so $x^3 - 8 = 3x^2(x-2)$. Thus $x = 2$ or $x = -1$. The lines are $y - 8 = 12(x-2)$ and $y + 1 = 3(x+1)$.

59. Let (x, y) be the point on the graph that the tangent line passes through. $f'(x) = 3x^2 - 1$, so $x^3 - x - 2 = (3x^2 - 1)(x+2)$. Thus $x = 0$ or $x = -3$. The lines are $y = -x$ and $y + 24 = 26(x+3)$.

60. (a) $f(c) = c^3; \quad f'(x) = 3x^2$ and $f'(c) = 3c^2$. Tangent line: $y - c^3 = 3c^2(x - c)$ or $y = 3c^3x - 2c^3$.

(b) We solve the equation $3c^2x - 2c^3 = x^3$:

$$x^3 - 3c^2x + 2c^3 = 0 \implies (x-c)(x^2 + cx - 2c^2) = 0 \implies (x-c)^2(x+2c) = 0$$

Thus, the tangent line at $x = c,\ c \neq 0$ intersects the graph at $x = -2c$.

61. Since f and $f + g$ are differentiable, $g = (f + g) - f$ is differentiable. The functions $f(x) = |x|$ and $g(x) = -|x|$ are not differentiable at $x = 0$ yet their sum $f(x) + g(x) \equiv 0$ is differentiable for all x.

62. No. If f and fg are differentiable, then $g = \dfrac{fg}{f}$ will be differentiable where $f(x) \neq 0$.

63. Since

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)},$$

it follows from the product and reciprocal rules that

$$\left(\frac{f}{g}\right)'(x) = \left(f \cdot \frac{1}{g}\right)'(x) = f(x)\left(-\frac{g'(x)}{[g(x)]^2}\right) + f'(x) \cdot \frac{1}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$$

64.

$$\begin{aligned}(fgh)'(x) &= [(fg)(x) \cdot h(x)]' = (fg)(x)h'(x) + h(x)[(fg)(x)]' \\ &= f(x)g(x)h'(x) + h(x)[f(x)g'(x) + g(x)f'(x)] \\ &= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)\end{aligned}$$

65. $F'(x) = 2x\left(1 + \dfrac{1}{x}\right)(2x^3 - x + 1) + (x^2 + 1)\left(\dfrac{-1}{x^2}\right)(2x^3 - x + 1) + (x^2 + 1)\left(1 + \dfrac{1}{x}\right)(6x^2 - 1)$

66. $G'(x) = \dfrac{1}{2\sqrt{x}}\left(\dfrac{1}{1 + 2x}\right)(x^2 + x + 1) + \sqrt{x}\left(\dfrac{-2}{(1 + 2x)^2}\right)(x^2 + x + 1) + \sqrt{x}\left(\dfrac{1}{1 + 2x}\right)(2x + 1)$

67. $g(x) = [f(x)]^2 = f(x) \cdot f(x)$

$g'(x) = f(x)f'(x) + f(x)f'(x) = 2f(x)f'(x)$

68. Let $g(x) = [f(x)]^n$, where n is a positive integer. Let S be the set of positive integers for which $g'(x) = n[f(x)]^{n-1}f'(x)$. Then, $1 \in S$. Assume that the positive integer $k \in S$ and let $g(x) = [f(x)]^{k+1}$. Then

$g(x) = f(x)[f(x)]^k$ and $g'(x) = f(x)(k)[f(x)]^{k-1}f'(x) + [f(x)]^k f'(x) = (k + 1)[f(x)]^k f'(x)$

Thus, $k + 1 \in S$, and the result holds for all positive integers n.

To show that the result holds for all negative integers, write $g(x) = [f(x)]^n$, (n negative) as

$$g(x) = \frac{1}{[f(x)]^{-n}}, \quad \text{where } -n \text{ is a positive integer}$$

and use the quotient rule together with the result above.

69. $g'(x) = 3(x^3 - 2x^2 + x + 2)^2(3x^2 - 4x + 1)$

70. $g'(x) = 10\left[\dfrac{x^2}{1 + 2x}\right]^9 \dfrac{(1 + 2x)2x - x^2(2)}{(1 + 2x)^2} = 20\,\dfrac{x^{19}(1 + x)}{(1 + 2x)^{11}}$

71. (a) $f'_+(-1) = \displaystyle\lim_{h \to 0^+} \frac{f(-1 + h) - f(-1)}{h} = \lim_{h \to 0^+} \frac{(-1 + h)^2 - 4(-1 + h) + 2 - 7}{h}$

$$= \lim_{h \to 0^+} \frac{-6h + h^2}{h} = -6$$

$$f'_-(3) = \lim_{h\to 0^-} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0^-} \frac{(3+h)^2-4(3+h)+2+1}{h}$$
$$= \lim_{h\to 0^-} \frac{2h+h^2}{h} = 2$$

(b) If $g(x) = x^2 - 4x + 2$ then $g'(x) = 2x - 4$, and $g'(-1) = -6$, $g'(3) = 2$.

72. $f'_+(a) = \lim\limits_{x\to a^+} \dfrac{f(x)-f(a)}{x-a} = \lim\limits_{x\to a^+} \dfrac{g(x)-g(a)}{x-a} = g'(a)$

$f'_-(b) = \lim\limits_{x\to b^-} \dfrac{f(x)-f(b)}{x-b} = \lim\limits_{x\to b^-} \dfrac{g(x)-g(b)}{x-b} = g'(b)$

73. We want f to be continuous at $x = 2$. That is, we want

$$\lim_{x\to 2^-} f(x) = f(2) = \lim_{x\to 2^+} f(x).$$

This gives

(1) $$8A + 2B + 2 = 4B - A.$$

We also want

$$\lim_{x\to 2^-} f'(x) = \lim_{x\to 2^+} f'(x).$$

This gives

(2) $$12A + B = 4B.$$

Equations (1) and (2) together imply that $A = -2$ and $B = -8$.

74. First, we need, $\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^+} f(x)$, $\implies$ $A + B = -B - A + 4$ or $A + B = 2$.

Next, we need, $\lim\limits_{x\to -1^-} f'(x) = \lim\limits_{x\to -1^+} f'(x)$ $\implies$ $-2A = 5B + A$ or $3A + 5B = 0$.

Solving these equations gives $A = 5$, $B = -3$.

75. (a) $\dfrac{\sin(0+0.001)-\sin 0}{0.001} \cong 0.99999$ $\quad \dfrac{\sin(0-0.001)-\sin 0}{-0.001} \cong 0.99999$

$\dfrac{\sin[(\pi/6)+0.001]-\sin(\pi/6)}{0.001} \cong 0.86578$ $\quad \dfrac{\sin[(\pi/6)-0.001]-\sin(\pi/6)}{-0.001} \cong 0.86628$

$\dfrac{\sin[(\pi/4)+0.001]-\sin(\pi/4)}{0.001} \cong 0.70675$ $\quad \dfrac{\sin[(\pi/4)-0.001]-\sin(\pi/4)}{-0.001} \cong 0.70746$

$\dfrac{\sin[(\pi/3)+0.001]-\sin(\pi/3)}{0.001} \cong 0.49957$ $\quad \dfrac{\sin[(\pi/3)-0.001]-\sin(\pi/3)}{-0.001} \cong 0.50043$

$\dfrac{\sin[(\pi/2)+0.001]-\sin(\pi/2)}{0.001} \cong -0.0005$ $\quad \dfrac{\sin[(\pi/2)-0.001]-\sin(\pi/2)}{-0.001} \cong 0.0005$

(b) $\cos 0 = 1$, $\cos(\pi/6) \cong 0.866025$, $\cos(\pi/4) \cong 0.707107$, $\cos(\pi/3) = 0.5$, $\cos(\pi/2) = 0$

(c) If $f(x) = \sin x$ then $f'(x) = \cos x$.

76. (c) If $f(x) = \cos x$ then $f'(x) = -\sin x$.

77. (a) $\dfrac{2^{0+0.001}-2^0}{0.001} \cong 0.69339$ $\quad\dfrac{2^{0-0.001}-2^0}{-0.001} \cong 0.69291$

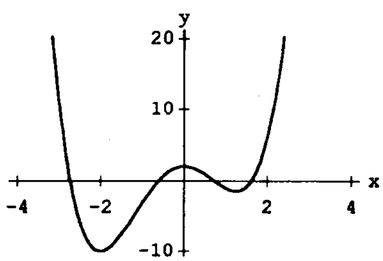
$\dfrac{2^{1+0.001}-2^1}{0.001} \cong 1.38678$ $\quad\dfrac{2^{1-0.001}-2^1}{-0.001} \cong 1.38581$

$\dfrac{2^{2+0.001}-2^2}{0.001} \cong 2.77355$ $\quad\dfrac{2^{2-0.001}-2^2}{-0.001} \cong 2.77163$

$\dfrac{2^{3+0.001}-2^3}{0.001} \cong 5.54710$ $\quad\dfrac{2^{3-0.001}-2^0}{-0.001} \cong 5.54326$

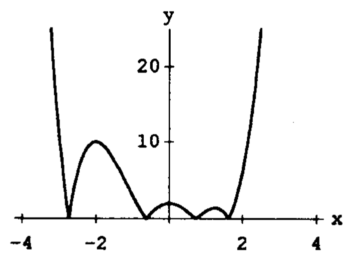
(b) $\dfrac{f'(x)}{f(x)} \cong 0.693$ (c) If $f(x)=2^x$ then $f'(x)=2^x K$, where $K \cong 0.693$.

78. (a)



$x=-2,\ 0,\ \frac{5}{4}$

(b)



$x_1=-2.731,\quad x_2=-0.619,$

$x_3=0.730,\quad x_4=1.619$

SECTION 3.3

1. $\dfrac{dy}{dx}=12x^3-2x$

2. $\dfrac{dy}{dx}=2x-8x^{-5}$

3. $\dfrac{dy}{dx}=1+\dfrac{1}{x^2}$

4. $\dfrac{dy}{dx}=\dfrac{(1-x)2-2x(-1)}{(1-x)^2}=\dfrac{2}{(1-x)^2}$

5. $\dfrac{dy}{dx}=\dfrac{(1+x^2)(1)-x(2x)}{(1+x^2)^2}=\dfrac{1-x^2}{(1+x^2)^2}$

6. $y=x^3-x^2-2x;\ \dfrac{dy}{dx}=3x^2-2x-2$

7. $\dfrac{dy}{dx}=\dfrac{(1-x)2x-x^2(-1)}{(1-x)^2}=\dfrac{x(2-x)}{(1-x)^2}$

8. $y=\dfrac{2x-x^2}{3+3x};\ \dfrac{dy}{dx}=\dfrac{(3+3x)(2-2x)-(2x-x^2)(3)}{(3+3x)^2}=\dfrac{2-2x-x^2}{3(1+x)^2}$

9. $\dfrac{dy}{dx}=\dfrac{(x^3-1)3x^2-(x^3+1)3x^2}{(x^3-1)^2}=\dfrac{-6x^2}{(x^3-1)^2}$

10. $\dfrac{dy}{dx}=\dfrac{(1+x)2x-x^2(1)}{(1+x)^2}=\dfrac{x^2+2x}{(1+x)^2}$

11. $\dfrac{d}{dx}(2x-5)=2$

12. $\dfrac{d}{dx}(5x+2)=5$

13. $\dfrac{d}{dx}[(3x^2-x^{-1})(2x+5)]=(3x^2-x^{-1})2+(2x+5)(6x+x^{-2})=18x^2+30x+5x^{-2}$

14. $\dfrac{d}{dx}\left[(2x^2+3x^{-1})(2x-3x^{-2})\right]=(2x^2+3x^{-1})(2+6x^{-3})+(2x-3x^{-2})(4x-3x^{-2})=12x^2+27x^{-4}$

15. $\dfrac{d}{dt}\left(\dfrac{t^4}{2t^3-1}\right)=\dfrac{(2t^3-1)4t^3-t^4(6t^2)}{(2t^3-1)^2}=\dfrac{2t^3(t^3-2)}{(2t^3-1)^2}$

16. $\dfrac{d}{dt}\left(\dfrac{2t^3+1}{t^4}\right)=\dfrac{d}{dt}\left(\dfrac{2}{t}+\dfrac{1}{t^4}\right)=-\dfrac{2}{t^2}-\dfrac{4}{t^5}=-\dfrac{2(t^3+2)}{t^5}$

17. $\dfrac{d}{du}\left(\dfrac{2u}{1-2u}\right)=\dfrac{(1-2u)2-2u(-2)}{(1-2u)^2}=\dfrac{2}{(1-2u)^2}$

18. $\dfrac{d}{du}\left(\dfrac{u^2}{u^3+1}\right)=\dfrac{(u^3+1)(2u)-u^2(3u^2)}{(u^3+1)^2}=\dfrac{u(2-u^3)}{(u^3+1)^2}$

19. $\dfrac{d}{du}\left(\dfrac{u}{u-1}-\dfrac{u}{u+1}\right)=\dfrac{(u-1)(1)-u}{(u-1)^2}-\dfrac{(u+1)(1)-u}{(u+1)^2}$

$$=-\frac{1}{(u-1)^2}-\frac{1}{(u+1)^2}=-\frac{2(1+u^2)}{(u^2-1)^2}$$

20. $\dfrac{d}{du}\left[u^2(1-u^2)(1-u^3)\right]=\dfrac{d}{du}[u^2-u^4-u^5+u^7]=2u-4u^3-5u^4+7u^6$

21. $\dfrac{d}{dx}\left(\dfrac{x^3+x^2+x+1}{x^3-x^2+x-1}\right)=\dfrac{(x^3-x^2+x-1)(3x^2+2x+1)-(x^3+x^2+x+1)(3x^2-2x+1)}{(x^3-x^2+x-1)^2}$

$$=\frac{-2(x^4+2x^2+1)}{(x^2+1)^2(x-1)^2}=\frac{-2}{(x-1)^2}$$

22. $\dfrac{d}{dx}\left(\dfrac{x^3+x^2+x-1}{x^3-x^2+x+1}\right)=\dfrac{(x^3-x^2+x+1)(3x^2+2x+1)-(x^3+x^2+x-1)(3x^2-2x+1)}{(x^3-x^2+x+1)^2}$

$$=\frac{-2x^4+8x^2+2)}{(x^3-x^2+x+1)^2}$$

23. $\dfrac{dy}{dx}=(x+1)\dfrac{d}{dx}[(x+2)(x+3)]+(x+2)(x+3)\dfrac{d}{dx}(x+1)$

$=(x+1)(2x+5)+(x+2)(x+3)$

At $x=2$, $\dfrac{dy}{dx}=(3)(9)+(4)(5)=47.$

24. $\dfrac{dy}{dx}=(x+1)(x^2+2)(3x^2)+(x+1)(x^3+3)(2x)+(x^2+2)(x^3+3)(1)$

At $x = 2$, $\dfrac{dy}{dx} = 3(6)(12) + 3(11)(4) + (6)(11) = 414$.

25. $$\frac{dy}{dx} = \frac{(x+2)\dfrac{d}{dx}\left[(x-1)(x-2)\right] - (x-1)(x-2)(1)}{(x+2)^2}$$

$$= \frac{(x+2)(2x-3) - (x-1)(x-2)}{(x+2)^2}$$

At $x = 2$, $\dfrac{dy}{dx} = \dfrac{4(1) - 1(0)}{16} = \dfrac{1}{4}$.

26. $y = \dfrac{x^4 - x^2 - 2}{x^2 + 2}$; $\dfrac{dy}{dx} = \dfrac{(x^2+2)(4x^3 - 2x) - (x^4 - x^2 - 2)(2x)}{(x^2+2)^2}$

At $x = 2$, $\dfrac{dy}{dx} = \dfrac{6(28) - (10)4}{36} = \dfrac{32}{9}$

27. $f'(x) = 21x^2 - 30x^4,\ f''(x) = 42x - 120x^3$

28. $f'(x) = 10x^4 - 24x^3 + 2,\ f''(x) = 40x^3 - 72x^2$

29. $f'(x) = 1 + 3x^{-2},\ f''(x) = -6x^{-3}$

30. $f'(x) = 2x + 2x^{-3},\ f''(x) = 2 - 6x^{-4}$

31. $f(x) = 2x^2 - 2x^{-2} - 3,\quad f'(x) = 4x + 4x^{-3},\quad f''(x) = 4 - 12x^{-4}$

32. $f(x) = 4x - 9x^{-1},\quad f'(x) = 4 + 9x^{-2},\quad f''(x) = -18x^{-3}$

33. $\dfrac{dy}{dx} = x^2 + x + 1$, $\dfrac{d^2y}{dx^2} = 2x + 1$, $\dfrac{d^3y}{dx^3} = 2$

34. $\dfrac{dy}{dx} = 2 + 10x$, $\dfrac{d^2y}{dx^2} = 10$, $\dfrac{d^3y}{dx^3} = 0$

35. $\dfrac{dy}{dx} = 8x - 20$, $\dfrac{d^2y}{dx^2} = 8$, $\dfrac{d^3y}{dx^3} = 0$

36. $\dfrac{dy}{dx} = \frac{1}{2}x^2 - \frac{1}{2}x + 1$, $\dfrac{d^2y}{dx^2} = x - \frac{1}{2}$, $\dfrac{d^3y}{dx^3} = 1$

37. $\dfrac{dy}{dx} = 3x^2 + 3x^{-4}$, $\dfrac{d^2y}{dx^2} = 6x - 12x^{-5}$, $\dfrac{d^3y}{dx^3} = 6 + 60x^{-6}$

38. $\dfrac{dy}{dx} = 3x^2 - 2x^{-2}$, $\dfrac{d^2y}{dx^2} = 6x + 4x^{-3}$, $\dfrac{d^3y}{dx^3} = 6 - 12x^{-4}$

39. $$\frac{d}{dx}\left[x\frac{d}{dx}(x - x^2)\right] = \frac{d}{dx}\left[x(1-2x)\right] = \frac{d}{dx}\left[x - 2x^2\right] = 1 - 4x$$

40. $$\frac{d^2}{dx^2}\left[(x^2 - 3x)\frac{d}{dx}(x + x^{-1})\right] = \frac{d^2}{dx^2}\left[(x^2 - 3x)(1 - x^{-2})\right]$$

$$= \frac{d^2}{dx^2}\left[x^2 - 3x - 1 + 3x^{-1}\right]$$

$$= \frac{d}{dx}(2x - 3 - 3x^{-2}) = 2 + 6x^{-3}$$

41. $\frac{d^4}{dx^4}[3x - x^4] = \frac{d^3}{dx^3}[3 - 4x^3] = \frac{d^2}{dx^2}[-12x^2] = \frac{d}{dx}[-24x] = -24$

42.
$$\frac{d^5}{dx^5}[ax^4 + bx^3 + cx^2 + dx + e] = \frac{d^4}{dx^4}[4ax^3 + 3bx^2 + 2cx + d]$$
$$= \frac{d^3}{dx^3}[12ax^2 + 6bx + 2c]$$
$$= \frac{d^2}{dx^2}[24ax + 6b] = \frac{d}{dx}[24a] = 0$$

43. $\frac{d^2}{dx^2}\left[(1+2x)\frac{d^2}{dx^2}(5 - x^3)\right] = \frac{d^2}{dx^2}[(1+2x)(-6x)] = \frac{d^2}{dx^2}\left[-6x - 12x^2\right] = -24$

44.
$$\frac{d^3}{dx^3}\left[\frac{1}{x}\frac{d^2}{dx^2}[x^4 - 5x^2]\right] = \frac{d^3}{dx^3}\left[\frac{1}{x}(12x^2 - 10)\right]$$
$$= \frac{d^3}{dx^3}[12x - 10x^{-1}] = \frac{d^2}{dx^2}[12 + 10x^{-2}]$$
$$= \frac{d}{dx}[-20x^{-}3] = 60x^{-4}$$

45. $y = x^4 - \frac{x^3}{3} + 2x^3 + C$

46. $y = \frac{x^2}{2} + \frac{1}{x^2} + 3x + C$

47. $y = x^5 - \frac{1}{x^4} + C$

48. $y = \frac{2x^6}{3} + \frac{5}{3x^3} - 2x + C$

49. Let $p(x) = ax^2 + bx + c$. Then $p'(x) = 2ax + b$ and $p''(x) = 2a$. Now

$$p''(1) = 2a = 4 \Longrightarrow a = 2; \quad p'(1) = 2(2)(1) + b = -2 \Longrightarrow b = -6;$$
$$p(1) = 2(1)^2 - 6(1) + c = 3 \Longrightarrow c = 7$$

Thus $p(x) = 2x^2 - 6x + 7$.

50.
$p(x) = ax^3 + bx^2 + cx + d$ $\qquad$ $p'''(-1) = 6 \Longrightarrow a = 1$

$p'(x) = 3ax^2 + 2bx + c$ $\qquad$ $p''(-1) = -2 \Longrightarrow b = 2$

$p''(x) = 6ax + 2b$ $\qquad$ $p'(-1) = 3 \Longrightarrow c = 4$

$p'''(x) = 6a$ $\qquad$ $p(-1) = 0 \Longrightarrow d = 3$

Therefore, $p(x) = x^3 + 2x^2 + 4x + 3$.

51. (a) If $k = n$, $\quad f^{(n)}(x) = n!$ $\qquad$ (b) If $k > n$, $\quad f^n(x) = 0$.

(c) If $k < n$, $\quad f^{(n)}(x) = n(n-1)(n-2)\cdots(n-k+1)x^{n-k}$.

52. (a) $\dfrac{d^n}{dx^n} = n!\,a_n$ (b) $\dfrac{d^k}{dx^k} = 0$ if $k > n$.

53. Let $f(x) = \begin{cases} x^2 & x \geq 0 \\ 0 & x \leq 0 \end{cases}$

(a) $f'_+(0) = \lim\limits_{h\to 0^+} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h\to 0^+} \dfrac{h^2 - 0}{h} = 0$ and

$f'_-(0) = \lim\limits_{h\to 0^-} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h\to 0^-} \dfrac{0}{h} = 0$

Therefore, f is differentiable at 0 and $f'(0) = 0$.

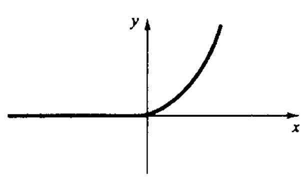
(b) $f'(x) = \begin{cases} 2x & x \geq 0 \\ 0 & x \leq 0 \end{cases}$

(c) $f''_+(0) = \lim\limits_{h\to 0^+} \dfrac{f'(0+h) - f'(0)}{h} = \lim\limits_{h\to 0^+} \dfrac{2h - 0}{h} = 2$ and

$f''_-(0) = \lim\limits_{h\to 0^-} \dfrac{f'(0+h) - f'(0)}{h} = \lim\limits_{h\to 0^-} \dfrac{0}{h} = 0$

Since $f''_+(0) \neq f''_-(0)$, $f''(0)$ does not exist.

(d)



54. Let $g(x) = \begin{cases} x^3 & x \geq 0 \\ 0 & x < 0 \end{cases}$

(a) $g'_+(0) = \lim\limits_{h\to 0^+} \dfrac{g(0+h) - g(0)}{h} = \lim\limits_{h\to 0^+} \dfrac{h^3 - 0}{h} = 0$ and

$g'_-(0) = \lim\limits_{h\to 0^-} \dfrac{g(0+h) - g(0)}{h} = \lim\limits_{h\to 0^-} \dfrac{0}{h} = 0$

Therefore, g is differentiable at 0 and $g'(0) = 0$.

$g'(x) = \begin{cases} 3x^2 & x \geq 0 \\ 0 & x \leq 0 \end{cases}$

$g''_+(0) = \lim\limits_{h\to 0^+} \dfrac{g'(0+h) - g,(0)}{h} = \lim\limits_{h\to 0^+} \dfrac{3h^2 - 0}{h} = 0$ and

$$g''_{-}(0) = \lim_{h\to 0^-} \frac{g'(0+h) - g'(0)}{h} = \lim_{h\to 0^-} \frac{0}{h} = 0$$

Therefore, g' is differentiable at 0 and $g''(0) = 0$.

(b) $g'(x) = \begin{cases} 3x^2 & x \geq 0 \\ 0 & x \leq 0 \end{cases}$

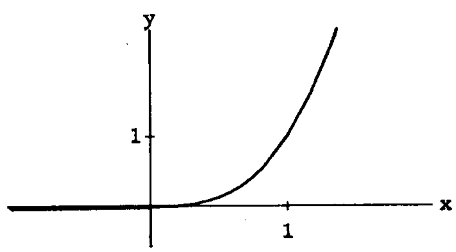
$g''(x) = \begin{cases} 6x & x \geq 0 \\ 0 & x \leq 0 \end{cases}$

(c) $g'''_{+}(0) = \lim_{h\to 0^+} \frac{g''(0+h) - g''(0)}{h} = \lim_{h\to 0^+} \frac{6h-0}{h} = 6$ and

$g'''_{-}(0) = \lim_{h\to 0^-} \frac{g''(0+h) - g''(0)}{h} = \lim_{h\to 0^-} \frac{0}{h} = 0$

Since $g'''_{+}(0) \neq g'''_{-}(0)$, $g'''(0)$ does not exist.

(d)



55. It suffices to give a single counterexample. For instance, if $f(x) = g(x) = x$, then $(fg)(x) = x^2$ so that $(fg)''(x) = 2$ but $f(x)g''(x) + f''(x)g(x) = x\cdot 0 + 0\cdot x = 0$.

56.
$$\begin{aligned}\frac{d}{dx}[f(x)g'(x) - f'(x)g(x)] &= [f(x)g''(x) + f'(x)g'(x)] - [f'(x)g'(x) + f''(x)g(x)] \\ &= f(x)g''(x) - f''(x)g(x)\end{aligned}$$

57. $f''(x) = 6x$; (a) $x = 0$ (b) $x > 0$ (c) $x < 0$

58. $f''(x) = 12x^2$; (a) $x = 0$ (b) all $x \neq 0$ (c) none

59. $f''(x) = 12x^2 + 12x - 24$; (a) $x = -2, 1$ (b) $x < -2$, $x > 1$ (c) $-2 < x < 1$

60. $f''(x) = 12x^2 + 18x - 12$; (a) $x = -2, \frac{1}{2}$ (b) $x < -2$, $x > \frac{1}{2}$ (c) $-2 < x < \frac{1}{2}$

61. The result is true for $n = 1$:

$$\frac{d^1y}{dx^1} = \frac{dy}{dx} = -x^{-2} = (-1)^1 1!\, x^{-1-1}.$$

If the result is true for $n = k$:

$$\frac{d^k y}{dx^k} = (-1)^k k!\, x^{-k-1}$$

then the result is true for $n = k + 1$:

$$\frac{d^{k+1} y}{dx^{k+1}} = \frac{d}{dx}\left[\frac{d^k y}{dx^k}\right] = \frac{d}{dx}\left[(-1)^k k!\, x^{-(k+1)}\right] = (-1)^{(k+1)}(k+1)!\, x^{-(k+1)-1}.$$

62. $y' = -2x^{-3}, \quad y'' = 6x^{-4}, \quad y''' = -24x^{-5}; \quad y^{(n)} = (-1)^n (n+1)! x^{-(n+2)}$

Let S be the set of positive integers for which the result holds. Then $1 \in S$. Assume that the positive integer $k \in S$. Now,

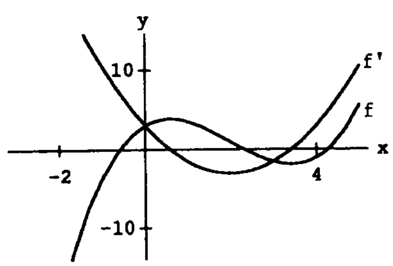
$$y^{k+1} = \frac{d}{dx} y^{(k)} = \frac{d}{dx}\left[(-1)^k (k+1)! x^{-(k+2)}\right]$$
$$= -(-1)^k (k+2)(k+1)! x^{-(k+2)-1} = (-1)^{k+1}(k+2)! x^{-(k+3)}$$

Thus, $k + 1 \in S$, and S is the set of positive integers.

63. $\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$

64. (a) $f(x) = \frac{1}{2}x^3 - 3x^2 + 3x + 3; \quad f'(x) = \frac{3}{2}x^2 - 6x + 3$

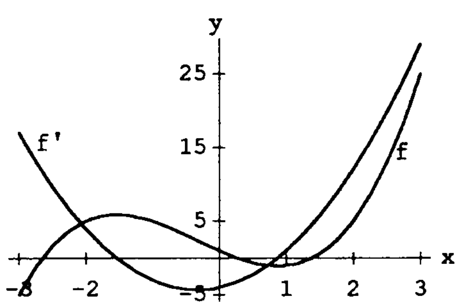
(b)



(c) $x_1 \cong 0.586, \quad x_2 \cong 3.414$

65. (a) $f(x) = x^3 + x^2 - 4x + 1; \quad f'(x) = 3x^2 + 2x - 4.$

(b)

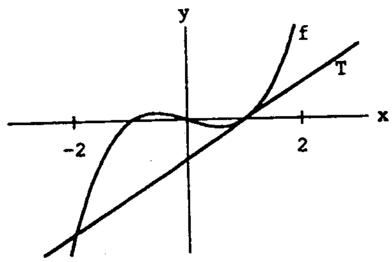


(c) The graph is "falling" when $f'(x) < 0$;

The graph is "rising" when $f'(x) > 0$.

66. $f'(x) = 3x^2 - 1,\ f'(1) = 2;$ tangent: $y = 2(x - 1).$

(b)



(c) $x = -2$

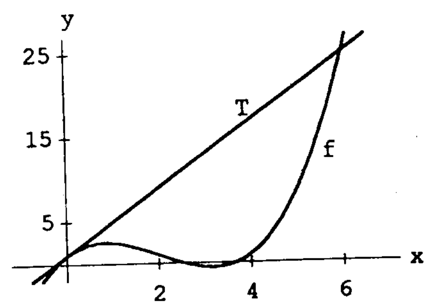
67. (a) Let $f(x) = \frac{1}{2}x^3 - 3x^2 + 4x + 1.$

Then $f'(x) = \frac{3}{2}x^2 - 6x + 4$ and

$f'(0) = 4.$

Tangent line at $x = 0:$ $y = 4x + 1$

(b)



(c) Solving $\frac{1}{2}x^3 - 3x^2 + 4x + 1 = 4x + 1$ for x gives $x = 6;$ the graph and the tangent line intersect at $(6, 25).$

PROJECT 3.3

1. $g(x) = f^4(x) = f(x)f^3(x)$

$g'(x) = f(x)[f^3(x)]' + f^3(x)f'(x) = f(x)3f^2(x)f'(x) + f^3(x)f'(x) = 4f^3(x)f'(x).$

2. Let S be the set of positive integers for which

$$[f^n(x)]' = nf^{n-1}(x)f'(x)$$

We know that $1, 2, 3, 4 \in S.$ Assume that the positive integer $k \in S;$

that is, assume$[f^k(x)]' = kf^{k-1}(x)f'(x).$ Let $g(x) = f^{k+1}(x) = f(x)f^k(x).$ Then

$$\begin{aligned} g'(x) &= f(x)[f^k(x)]' + f^k(x)f'(x) \\ &= f(x)kf^{k-1}(x)f'(x) + f^k(x)f'(x) \\ &= (k+1)f^k(x)f'(x). \end{aligned}$$

and so $k + 1 \in S.$ Thus, we conclude by mathematical induction

that the result holds for all positive integers $n.$

3. We know the result holds for all positive integers. If $k = 0$, then $g(x) = [f(x)] = 1$ (provided $f(x) \neq 0$) and $g'(x) = 0 = 0[f(x)]^{-1}.$ If k is a negative integer, then

$$g(x) = \frac{1}{f^n(x)}, \quad (f(x) \neq 0)$$

where $n = -k$ is a positive integer. Thus

$$g'(x) = \frac{-nf^{n-1}(x)f'(x)}{f^{2n}(x)} \text{ (reciprocal rule) } = \frac{-nf'(x)}{f^{n+1}} = -nf^{-n-1}(x)f'(x) = kf^{k-1}(x)f'(x).$$

Thus the result holds for all integers n.

4. The result holds for $n = 1, 2, 3$. Assume that it holds for the integer n and consider:

$$(f \cdot g)^{(n+1)}(x) = [(f \cdot g)^n(x)]'$$

$$= [f^{(n)}(x)g(x) + nf^{(n-1)}(x)g'(x) + \cdots + \tbinom{n}{k}f^{(n-k)}(x)g^{(k)}(x) + \cdots + f(x)g^{(n)}(x)]'.$$

The result follows after showing that

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1} \quad \text{for } k = 0, 1, ..., n$$

SECTION 3.4

1. $A = \pi r^2$, $\frac{dA}{dr} = 2\pi r$. When $r = 2$, $\frac{dA}{dr} = 4\pi$. 2. $V = s^3$, $\frac{dV}{ds} = 3s^2$. When $s = 4$, $\frac{dV}{ds} = 48$.

3. $A = \frac{1}{2}z^2$, $\frac{dA}{dz} = z$. When $z = 4$, $\frac{dA}{dz} = 4$. 4. $\frac{dy}{dx} = -x^{-2}$. When $x = -1$, $\frac{dy}{dx} = -1$.

5. $y = \frac{1}{x(1+x)}$, $\frac{dy}{dx} = \frac{-(2x+1)}{x^2(1+x)^2}$. At $x = 2$, $\frac{dy}{dx} = -\frac{5}{36}$.

6. $\frac{dy}{dx} = 3x^2 - 24x + 45 = 3(x-3)(x-5)$; $\frac{dy}{dx} = 0$ at $x = 3, 5$.

7. $V = \frac{4}{3}\pi r^3$, $\frac{dV}{dr} = 4\pi r^2 =$ the surface area of the ball.

8. $S = 4\pi r^2$; $\frac{dS}{dr} = 8\pi r$ and $\frac{dS}{dr} = 8\pi r_0$ at $r = r_0$. $\frac{dS}{dr} = 1 \implies r_0 = \frac{1}{8\pi}$.

9. $y = 2x^2 + x - 1$, $\frac{dy}{dx} = 4x + 1$; $\frac{dy}{dx} = 4$ at $x = \frac{3}{4}$. Therefore $x_0 = \frac{3}{4}$.

10. (a) $A = \pi\left(\frac{d}{2}\right)^2 = \frac{\pi}{4}d^2$; $A' = \frac{\pi}{2}d$ (b) $A = \pi\left(\frac{C}{2\pi}\right)^2 = \frac{C^2}{4\pi}$; $\frac{dA}{dC} = \frac{C}{2\pi}$

11. (a) $w = s\sqrt{2}$, $V = s^3 = \left(\frac{w}{\sqrt{2}}\right)^3 = \frac{\sqrt{2}}{4}w^3$, $\frac{dV}{dw} = \frac{3\sqrt{2}}{4}w^2$.

(b) $z^2 = s^2 + w^2 = 3s^2$, $z = s\sqrt{3}$. $V = s^3 = \left(\frac{z}{\sqrt{3}}\right)^3 = \frac{\sqrt{3}}{9}z^3$, $\frac{dV}{dz} = \frac{\sqrt{3}}{3}z^2$.

12. $A = bh = (\text{constant}) \implies h = \frac{A}{b}$; $\frac{dh}{db} = -\frac{A}{b^2} = -\frac{bh}{b^2} = -\frac{h}{b}$

13. (a) $\frac{dA}{d\theta} = \frac{1}{2}r^2$ (b) $\frac{dA}{dr} = r\theta$

(c) $\theta = \dfrac{2A}{r^2}$ so $\dfrac{d\theta}{dr} = \dfrac{-4A}{r^3} = \dfrac{-4}{r^3}\left(\dfrac{1}{2}r^2\theta\right) = \dfrac{-2\theta}{r}$

14. (a) $\dfrac{dA}{dh} = 2\pi r$ (b) $\dfrac{dA}{dr} = 2\pi(2r+h)$

(c) $h = \dfrac{A}{2\pi r} - r;\quad \dfrac{dh}{dr} = -\dfrac{A}{2\pi r^2} - 1 = \dfrac{-2\pi r(r+h) - 2\pi r^2}{2\pi r^2} = -\dfrac{2r+h}{r}$

15. $y = ax^2 + bx + c,$ $z = bx^2 + ax + c.$

$\dfrac{dy}{dx} = 2ax + b,$ $\dfrac{dz}{dx} = 2bx + a.$

$\dfrac{dy}{dx} = \dfrac{dz}{dx}$ iff $2ax + b = 2bx + a$. With $a \neq b$, this occurs only at $x = \dfrac{1}{2}$.

16. rate of change $= f(1)g(1)h'(1) + f(1)g'(1)h(1) + f'(1)g(1)h(1) = 0 + 0 + (1)(2)(-2) = -4$

17. $x(5) = -6;\ v(t) = 3 - 2t$ so $v(5) = -7$ and speed $= 7;\ a(t) = -2$ so $a(5) = -2.$

18. $x(3) = -12;\ v(t) = 5 - 3t^2$ so $v(3) = -22$ and speed $= 22;\ a(t) = -6t$ so $a(3) = -18.$

19. $x(1) = 6;\ v(t) = -18/(t+2)^2$ so $v(1) = -2$ and speed $= 2,$

$a(t) = 36/(t+2)^3$ so $a(1) = 4/3.$

20. $x(3) = 1;\ v(t) = -6/(t+3)^2$ so $v(3) = 1/6 =$ speed; $a(t) = -12/(t+3)^3$ so $a(3) = -1/18.$

21. $x(1) = 0,\ v(t) = 4t^3 + 18t^2 + 6t - 10$ so $v(1) = 18$ and speed $= 18,$

$a(t) = 12t^2 + 36t + 6$ so $a(1) = 54.$

22. $x(2) = -20;\ v(t) = 4t^3 - 18t^2$ so $v(2) = -4$ and speed $= 4,$

$a(t) = 12t^2 - 18$ so $a(2) = 30.$

23. $v(t) = 3t^2 - 6t + 3 = 3(t-1)^2 \geq 0;$ the object never changes direction.

24. $v(t) = 1 - \dfrac{3}{(t+1)^2};$ the object changes direction (from left to right) at $t = -1 + \sqrt{3}$.

25. $v(t) = 1 - \dfrac{5}{(t+2)^2};$ the object changes direction (from left to right) at $t = -2 + \sqrt{5}$.

26. $v(t) = 4t^3 - 12t^2 + 8t;$ the object changes direction at $t = 0$ (left to right), $t = 1$ (right to left), $t = 2$ (left to right).

27. A **28.** C **29.** A **30.** C **31.** A and B

32. B **33.** A **34.** A **35.** A and C **36.** B

37. The object is moving right when $v(t) > 0$. Here,

$v(t) = 4t^3 - 36t^2 + 56t = 4t(t-2)(t-7);\quad v(t) > 0$ when $0 < t < 2$ and $7 < t$.

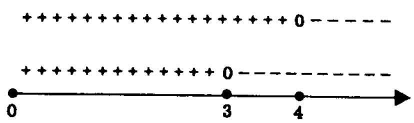
38. The object is moving left when $v(t) < 0$. Here,

$$v(t) = 3t^2 - 24t + 21 = 3(t-7)(t-1); \quad v(t) < 0 \text{ when } 1 < t < 7.$$

39. The object is speeding up when $v(t)$ and $a(t)$ have the same sign.

$v(t) = 5t^3(4-t)$ sign of $v(t)$:

$a(t) = 20t^2(3-t)$ sign of $a(t)$:

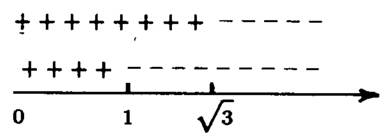


Thus, $0 < t < 3$ and $4 < t$.

40. The object is slowing down when $v(t)$ and $a(t)$ have opposite sign.

$v(t) = 12t - 4t^3 = 4t(3-t^2)$ sign of $v(t)$:

$a(t) = 12(1-t)$ sign of $a(t)$:

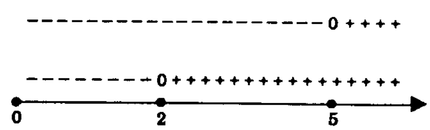


Thus, $1 < t < \sqrt{3}$.

41. The object is moving left and slowing down when $v(t) < 0$ and $a(t) > 0$.

$v(t) = 3(t-5)(t+1)$ sign of $v(t)$:

$a(t) = 6(t-2)$ sign of $a(t)$:

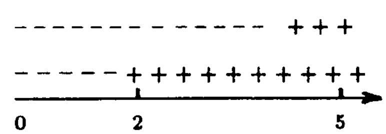


Thus, $2 < t < 5$.

42. The object is moving right and slowing down when $v(t) > 0$ and $a(t) < 0$.

$v(t) = 3(t-5)(t+1)$ sign of $v(t)$:

$a(t) = 6(t-2)$ sign of $a(t)$:

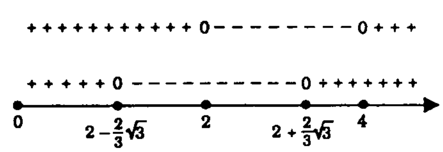


This never happens.

43. The object is moving right and speeding up when $v(t) > 0$ and $a(t) > 0$.

$v(t) = 4t(t-2)(t-4)$ sign of $v(t)$:

$a(t) = 4(3t^2 - 12t + 8)$ sign of $a(t)$:

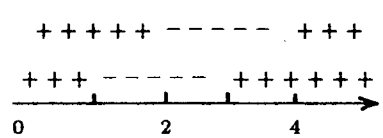


Thus, $0 < t < 2 - \frac{2}{3}\sqrt{3}$ and $4 < t$.

44. The object is moving left and speeding up when $v(t) < 0$ and $a(t) < 0$.

$v(t) = 4t(t-2)(t-4)$ sign of $v(t)$:

$a(t) = 4(3t^2 - 12t + 8)$ sign of $a(t)$:



Thus, $2 < t < 2 + \frac{2}{3}\sqrt{3}$.

45. Since $v_0 = 0$ the equation of motion is

$$y(t) = -16t^2 + y_0.$$

We want to find y_0 so that $y(6) = 0$. From

$$0 = -16(6)^2 + y_0$$

we get $y_0 = 576$ feet.

46. The equation of motion is: $y(t) = -4.9t^2 + y_0$. Therefore, the velocity is given by $v(t) = -9.8t$.

Since the object hits the ground at 98 m/sec., we have $-9.8t = -98$, and $t = 10$.

Therefore, $y(10) = 0 = -4.9(10)^2 + y_0$ and $y_0 = 490$ meters.

47. The object's height and velocity at time t are given by

$$y(t) = -\frac{1}{2}gt^2 + v_0 t \qquad \text{and} \qquad v(t) = -gt + v_0$$

Since the object's velocity at its maximum height is 0, it takes v_0/g seconds to reach

maximum height, and

$$y(v_0/g) = -\tfrac{1}{2}g(v_0/g)^2 + v_0(v_0/g) = v_0^2/2g \quad \text{or} \quad v_0^2/19.6 \ \text{(meters)}$$

48. Since $y_0 = 0$, we have $y(t) = -16t^2 + v_0 t = t(-16t + v_0)$. Now,

$$y(8) = 0 \quad \Longrightarrow \quad v_0 = (16)8 = 128 \quad \Longrightarrow \quad \text{the initial velocity was 128 ft/sec.}$$

49. At time t, the object's height is $y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0$, and its velocity is $v(t) = -gt + v_0$. Suppose that $y(t_1) = y(t_2)$, $t_1 \neq t_2$. Then

$$-\tfrac{1}{2}gt_1^2 + v_0 t_1 + y_0 = -\tfrac{1}{2}gt_2^2 + v_0 t_2 + y_0$$

$$\tfrac{1}{2}g(t_2^2 - t_1^2) = v_0(t_2 - t_1)$$

$$gt_2 + gt_1) = 2v_0$$

From this equation, we get $-(-gt_1 + v_0) = -gt_2 + v_0$ and so $|v(t_1)| = |v(t_2)|$.

50. Since $y_0 = 0$, we have $y(t) = -4.9t^2 + v_0 t = t(v_0 - 4.9t)$ The object hits the ground at $t = v_0/4.9$ sec., that is, the object is in the air for $v_0/4.9$ sec. At its maximum height, the velocity of the object is 0. Since $v(t) = -9.8t + v_0$, we have $-9.8t + v_0 = 0$ and $t = v_0/9.8 = \frac{1}{2}(v_0/4.9)$. The result follows from this.

51. In the equation

$$y(t) = -16t^2 + v_0 t + y_0$$

we take $v_0 = -80$ and $y_0 = 224$. The ball first strikes the ground when

$$-16t^2 - 80t + 224 = 0;$$

that is, at $t = 2$. Since

$$v(t) = y'(t) = -32t - 80,$$

we have $v(2) = -144$ so that the speed of the ball the first time it strikes the ground is 144 ft/sec. Thus, the speed of the ball the third time it strikes the ground is $\frac{1}{4}\left[\frac{1}{4}(144)\right] = 9$ ft/sec.

52. Since $y_0 = 0$, we have $y(t) = -16t^2 + v_0 t$.

$$y(2) = 64 \quad \Longrightarrow \quad -16(2)^2 + 2v_0 = 64 \quad \Longrightarrow \quad v_0 = 64 \quad \text{and} \quad y(t) = -16t^2 + 64t$$

Now, at the maximum height, $v(t) = -32t + 64 = 0 \quad \Longrightarrow \quad t = 2$. We already know the height at $t = 2$, namely 64 ft.

53. The equation is $y(t) = -16t^2 + 32t$. (Here $y_0 = 0$ and $v_0 = 32$.)

(a) We solve $y(t) = 0$ to find that the stone strikes the ground at $t = 2$ seconds.

(b) The stone attains its maximum height when $v(t) = 0$. Solving

$v(t) = -32t + 32 = 0$, we get $t = 1$ and, thus, the maximum height is $y(1) = 16$ feet.

(c) We want to choose v_0 in

$$y(t) = -16t^2 + v_0 t$$

so that $y(t_0) = 36$ when $v(t_0) = 0$ for some time t_0.

From $v(t) = -32t + v_0 = 0$ we get $t_0 = v_0/32$ so that

$$-16\left(\frac{v_0}{32}\right)^2 + v_0\left(\frac{v_0}{32}\right) = 36, \quad \text{or} \quad \frac{{v_0}^2}{64} = 36.$$

Thus, $v_0 = 48$ ft/sec.

54. (a) Measuring height from the water surface, we have $y(t) = -16t^2 + y_0$, since $v_0(0) = 0$.

If the stone hits the water 3 seconds later, then $y(3) = -16(3)^2 + y_0 = 0$. so $y_0 = 144$.

(b) It takes $y_0/1080$ seconds for the sound of the splash to reach the man so the stone hits the at time $t = 3 - y_0/1080$. Thus,

$$y(t) = -16\left(3 - \frac{y_0}{1080}\right)^2 + y_0 = 0 \quad \Longrightarrow \quad y_0 \cong 132.47 \text{ ft.}$$

55. For all three parts of the problem the basic equation is

$$y(t) = -16t^2 + v_0 t + y_0$$

with

$$(*) \qquad y(t_0) = 100 \quad \text{and} \quad y(t_0 + 2) = 16$$

for some time $t_0 > 0$.

We are asked to find y_0 for a given value of v_0.

From (*) we get

$$\begin{aligned} 16 - 100 &= y(t_0 + 2) - y(t_0) \\ &= [-16(t_0+2)^2 + v_0(t_0+2) + y_0] - [-16{t_0}^2 + v_0 t_0 + y_0] \\ &= -64t_0 - 64 + 2v_0 \end{aligned}$$

so that

$$t_0 = \tfrac{1}{32}(v_0 + 10).$$

Substituting this result in the basic equation and noting that $y(t_0) = 100$, we have

$$-16\left(\frac{v_0+10}{32}\right)^2 + v_0\left(\frac{v_0+10}{32}\right) + y_0 = 100$$

and therefore

(**) $$y_0 = 100 - \frac{{v_0}^2}{64} + \frac{25}{16}.$$

We use (**) to find the answer to each part of the problem.

(a) $v_0 = 0$ so $y_0 = \frac{1625}{16}$ ft (b) $v_0 = -5$ so $y_0 = \frac{6475}{64}$ ft (c) $v_0 = 10$ so $y_0 = 100$ ft

56. Let $v_0 > 0$ be the initial velocity. The equation of motion prior to the impact is: $y(t) = -16t^2 - v_0 t + 4$. The ball hits the ground at time $t = \dfrac{\sqrt{v_0^2+256} - v_0}{32}$ with velocity $v = \sqrt{v_0^2 + 256}$. The equation of motion following the impact is: $y(t) = -16t^2 + \dfrac{\sqrt{v_0^2+256}}{2}t$. It reaches its maximum height at time $T = \dfrac{\sqrt{v_0^2+256}}{64}$. Now, $y(T) = 4 \implies v_0 = 16\sqrt{3}$.

57. Let $y/_0 > 0$ be the initial height. The equation of motion becomes:

$0 = -16(8)^2 + 5(8) + y_0,$ so $y_0 = 984$ ft.

58. Using $0 = -16t^2 - 5t + 984,$ yields $t = \dfrac{123}{16}$ or about 7.7 sec.

59. $C(x) = 200 + 0.02x + 0.0001x^2,$ $C'(x) = 0.02 + 0.002x$

Marginal cost at $x = 100$ units: $C'(100) = 0.04$

Actual cost of 101st unit: $C(101) - C(100) = 0.0401$

60. $C(x) = 1000 + 2x + 0.02x^2 + 0.0001x^3,$ $C'(x) = 2 + 0.04x + 0.0003x^2$

Marginal cost at $x = 100$ units: $C'(100) = 9$

Actual cost of 101st unit: $C(101) - C(100) = 9.05$

61. $C(x) = 200 + 0.01x + \dfrac{100}{x},$ $C'(x) = 0.01 - \dfrac{100}{x^2}$

Marginal cost at $x = 100$ units: $C'(100) = 0$

Actual cost of producing the 101st unit: $C(101) - C(100) = 0$

62. $C(x) = 2000 + 2\sqrt{x}, \quad C'(x) = \dfrac{1}{\sqrt{x}}$

Marginal cost at $x = 100$ units: $C'(100) = 0.1$

Actual cost of producing the 101st unit: $C(101) - C(100) = 0.0998$

63. $C(x) = 1000 + 25x - \dfrac{x^2}{10}, \qquad C'(x) = 25 - \dfrac{x}{5}$

Marginal cost of producing 10 motors: $C'(10) = 23$

Actual cost of producing the 10th motor: $C(11) - C(10) = 22.90$

64. (a) $R'(x) = 24 + 10x - x^2$, and $R'(x) > 0$ on $(0, 12)$

(b) $R''(x) = 10 - 2x$, so $R'(x)$ reaches a maximum when $10 - 2x = 0$, or $x = 5$ units.

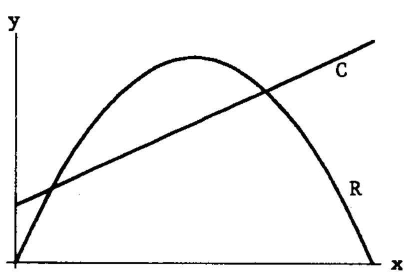
65. (a) Profit function: $P(x) = R(x) - C(x) = 20x - \dfrac{x^2}{50} - (4x + 1400) = 16x - \dfrac{x^2}{50} - 1400.$

Break-even points: $16x - \dfrac{x^2}{50} - 1400 = 0$ so $x^2 - 800x + 70,000 = 0$

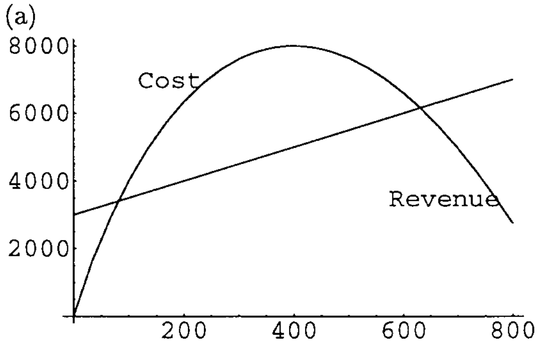
Thus $x = 100$, or $x = 700$ units.

(b) $P'(x) = 16 - \dfrac{x}{25}; \quad P'(x) = 0 \implies x = 400$ units.

(c)

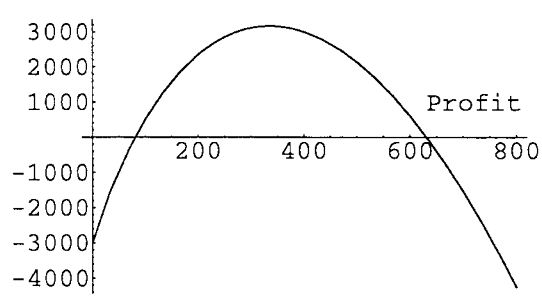


66. (a)



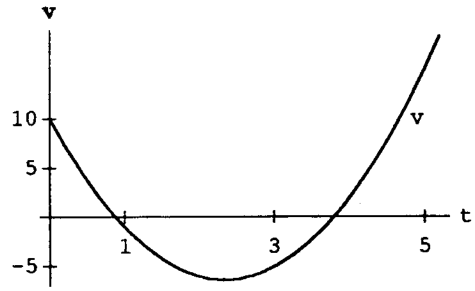
Break-even points at $x = 81.11$ and $x = 631.19$

(b)



Maximum profit at $x = 336.11$

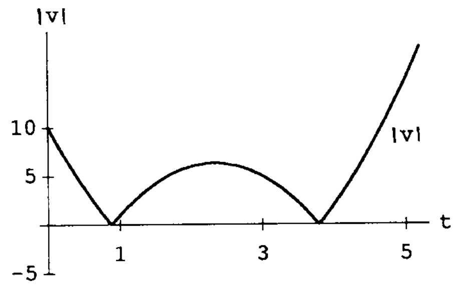
67. (a) $v(t) = 3t^2 - 14t + 10, \ 0 \leq t \leq 5$



(b) The object is moving to the right when $0 < t < 0.88$ and when $3.79 < t < 5$.

The object is moving to the left when $0.88 < t < 3.79$

(c)



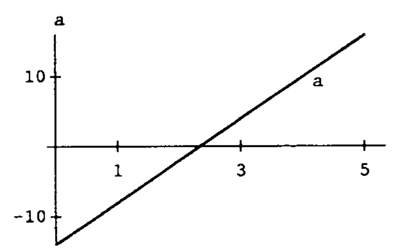
The object stops at times $t \cong 0.88$ and $t \cong 3.79$.

The maximum speed is $v \cong 6.33$ at $t \cong 2.33$.

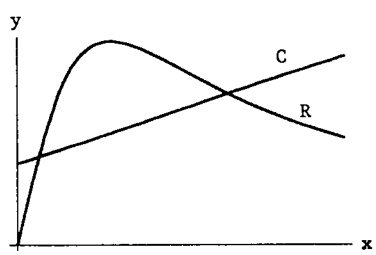
(d) $a(t) = 6t - 14$

The object is speeding up when $v(t)$ and $a(t)$ have the same sign: $0.88 < t < 2.33$ and $3.79 < t < 5$.

The object is slowing down when $v(t)$ and $a(t)$ have opposite sign: $0 < t < 0.88$ and $2.33 < t < 3.79$.

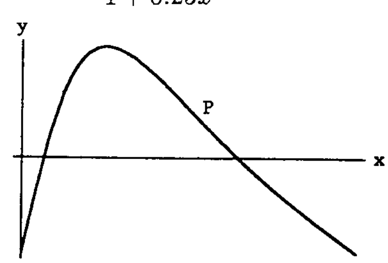


68. (a)



$x_1 \cong 0.46, \quad x_2 \cong 4.53$

(b) $P(x) = \dfrac{10x}{1 + 0.25x^2} - 4 - 0.75x$



175 units

SECTION 3.5

1. $f(x) = x^4 + 2x^2 + 1, \quad f'(x) = 4x^3 + 4x = 4x(x^2 + 1)$

$f(x) = (x^2 + 1)^2, \quad f'(x) = 2(x^2 + 1)(2x) = 4x(x^2 + 1)$

2. $f(x) = x^6 - 2x^3 + 1, \quad f'(x) = 6x^5 - 6x^2 = 6x^2(x^3 - 1)$

$f(x) = (x^3 - 1)^2, \quad f'(x) = 2(x^3 - 1)(3x^2) = 6x^2(x^3 - 1)$

3. $f(x) = 8x^3 + 12x^2 + 6x + 1, \quad f'(x) = 24x^2 + 24x + 6 = 6(2x + 1)^2$

$f(x) = (2x + 1)^3, \quad f'(x) = 3(2x + 1)^2(2) = 6(2x + 1)^2$

4. $f(x) = x^6 + 3x^4 + 3x^2 + 1, \quad f'(x) = 6x^5 + 12x^3 + 6x = 6x(x^2 + 1)^2$

$f(x) = (x^2 + 1)^3, \quad f'(x) = 3(x^2 + 1)^2(2x) = 6x(x^2 + 1)^2$

5. $f(x) = x^2 + 2 + x^{-2}, \quad f'(x) = 2x - 2x^{-3} = 2x(1 - x^{-4})$

$f(x) = (x + x^{-1})^2, \quad f'(x) = 2(x + x^{-1})(1 - x^{-2}) = 2x(1 + x^{-2})(1 - x^{-2}) = 2x(1 - x^{-4})$

6. $f(x) = 9x^4 - 12x^3 + 4x^2, \quad f'(x) = 36x^3 - 36x^2 + 8x = 4x(3x - 2)(3x - 1)$

$f(x) = (3x^2 - 2x)^2, \quad f'(x) = 2(3x^2 - 2x)(6x - 2) = 4x(3x - 2)(3x - 1)$

7. $f'(x) = -1(1 - 2x)^{-2} \frac{d}{dx}(1 - 2x) = 2(1 - 2x)^{-2}$

8. $f'(x) = 5(1 + 2x)^4 \frac{d}{dx}(1 + 2x) = 10(1 + 2x)^4$

9. $f'(x) = 20(x^5 - x^{10})^{19} \frac{d}{dx}(x^5 - x^{10}) = 20(x^5 - x^{10})^{19}(5x^4 - 10x^9)$

10. $f'(x) = 3(x^2 + x^{-2})^2 \frac{d}{dx}(x^2 + x^{-2}) = 6(x^2 + x^{-2})(x - x^{-3})$

11. $f'(x) = 4\left(x - \frac{1}{x}\right)^3 \frac{d}{dx}\left(x - \frac{1}{x}\right) = 4\left(x - \frac{1}{x}\right)^3\left(1 + \frac{1}{x^2}\right)$

12. $f'(x) = 3(x + x^{-1})^2 \frac{d}{dx}(x + x^{-1}) = 3(x + x^{-1})(1 - x^{-2})$

13. $f'(x) = 4(x - x^3 - x^5)^3 \frac{d}{dx}(x - x^3 - x^5) = 4(x - x^3 - x^5)^3(1 - 3x^2 - 5x^4)$

14. $f(t) = (1 + t)^{-4}; \quad f'(t) = -4(1 + t)^{-5} \frac{d}{dt}(1 + t) = -4(1 + t)^{-5}$

15. $f'(t) = 100(t^2 - 1)^{99} \frac{d}{dt}(t^2 - 1) = 200t(t^2 - 1)^{99}$

16. $f'(t) = 3(t - t^2)^2 \frac{d}{dt}(t - t^2) = 3(t - t^2)^2(1 - 2t)$

17. $f'(t) = 4(t^{-1} + t^{-2})^3 \frac{d}{dt}(t^{-1} + t^{-2}) = 4(t^{-1} + t^{-2})^3(-t^{-2} - 2t^{-3})$

18. $f'(x) = 3\left(\frac{4x + 3}{5x - 2}\right)^2 \frac{d}{dx}\left(\frac{4x + 3}{5x - 2}\right) = 3\left(\frac{4x + 3}{5x - 2}\right)^2\left[\frac{(5x - 2)4 - (4x + 3)5}{(5x - 2)^2}\right] = -\frac{69(4x + 3)^2}{(5x - 2)^4}$

19. $f'(x) = 4\left(\dfrac{3x}{x^2+1}\right)^3 \dfrac{d}{dx}\left(\dfrac{3x}{x^2+1}\right) = 4\left(\dfrac{3x}{x^2+1}\right)^3\left[\dfrac{(x^2+1)3-3x(2x)}{(x^2+1)^2}\right] = \dfrac{324x^3(1-x^2)}{(x^2+1)^5}$

20. $f'(x) = 3\left[(2x+1)^2+(x+1)^2\right]^2 \dfrac{d}{dx}\left[(2x+1)^2+(x+1)^2\right]$

$= 3\left[(2x+1)^2+(x+1)^2\right]^2\left[2(2x+1)(2)+2(x+1)(1)\right]$

$= 6\left[(2x+1)^2+(x+1)^2\right]^2(5x+3)$

21. $f'(x) = -\left(\dfrac{x^3}{3}+\dfrac{x^2}{2}+\dfrac{x}{1}\right)^{-2}\dfrac{d}{dx}\left(\dfrac{x^3}{3}+\dfrac{x^2}{2}+\dfrac{x}{1}\right) = -\left(\dfrac{x^3}{3}+\dfrac{x^2}{2}+x\right)^{-2}(x^2+x+1)$

22. $f'(x) = 2[(6x+x^5)^{-1}+x]\dfrac{d}{dx}[(6x+x^5)^{-1}+x] = 2[(6x+x^5)^{-1}+x][1-(6x+x^5)^{-2}(6+5x^4)]$

23. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \dfrac{-2u}{(1+u^2)^2}\cdot(2)$

At $x = 0$, we have $u = 1$ and thus $\dfrac{dy}{dx} = \dfrac{-4}{4} = -1.$

24. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (1-u^{-2})\cdot 4(3x+1)^3(3)$

At $x = 0$, we have $u = 1$ and thus $\dfrac{dy}{dx} = 0.$

25. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \dfrac{(1-4u)2-2u(-4)}{(1-4u)^2}\cdot 4(5x^2+1)^3(10x) = \dfrac{2}{(1-4u)^2}\cdot 40x(5x^2+1)^3$

At $x = 0$, we have $u = 1$ and thus $\dfrac{dy}{dx} = \dfrac{2}{9}(0) = 0.$

26. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (3u^2-1)\cdot\left(\dfrac{-1}{(1+x)^2}\right)$

At $x = 0$, we have $u = 1$ and thus $\dfrac{dy}{dx} = 2(-2) = -4.$

27. $\dfrac{dy}{dt} = \dfrac{dy}{du}\dfrac{du}{dx}\dfrac{dx}{dt} = \dfrac{(1+u^2)(-7)-(1-7u)(2u)}{(1+u^2)^2}(2x)(2)$

$= \dfrac{7u^2-2u-7}{(1+u^2)^2}(4x) = \dfrac{4x(7x^4+12x-2)}{(x^4+2x^2+2)^2} = \dfrac{4(2t-5)[7(2t-5)^4+12(2t-5)^2-2]}{[(2t-5)^4+2(2t-5)^2+2]^2}$

28. $\dfrac{dy}{dt} = \dfrac{dy}{du}\dfrac{du}{dx}\dfrac{dx}{dt} = 2u\left(\dfrac{(1+x^2)(-7)-(1-7x)(2x)}{(1+x^2)^2}\right)(5)$

$= 10u\cdot\dfrac{7x^2-2x-7}{(1+x^2)^2} = \dfrac{10[1-7(5t+2)][7(5t+2)^2-2(5t+2)-7]}{[1+(5t+2)^2]^2}$

29. $\dfrac{dy}{dx} = \dfrac{dy}{ds}\dfrac{ds}{dt}\dfrac{dt}{dx} = 2(s+3)\cdot\dfrac{1}{2\sqrt{t-3}}\cdot(2x)$

At $x=2$, we have $t=4$ so that $s=1$ and thus $\frac{dy}{dx}=2(4)\frac{1}{2\cdot 1}(4)=16$.

30. $\frac{dy}{dx}=\frac{dy}{ds}\frac{ds}{dt}\frac{dt}{dx}=\frac{2}{(1-s)^2}\left(1+\frac{1}{t^2}\right)\frac{1}{2\sqrt{x}}$

At $x=2$, we have $t=\sqrt{2}$ and $s=\sqrt{2}/2$. Thus $\frac{dy}{dx}=\frac{2}{(1-\sqrt{2}/2)^2\left(1+\frac{1}{2}\right)}\frac{1}{2\sqrt{2}}=6+\frac{9}{2}\sqrt{2}$.

31. $(f\circ g)'(0)=f'(g(0))g'(0)=f'(2)g'(0)=(1)(1)=1$

32. $(f\circ g)'(1)=f'(g(1))g'(1)=f'(1)g'(1)=(1)(0)=0$

33. $(f\circ g)'(2)=f'(g(2))g'(2)=f'(2)g'(2)=(1)(1)=1$

34. $(g\circ f)'(0)=g'(f(0))f'(0)=g'(1)f'(0)=(0)(2)=0$

35. $(g\circ f)'(1)=g'(f(1))f'(1)=g'(0)f'(1)=(1)(1)=1$

36. $(g\circ f)'(2)=g'(f(2))f'(2)=g'(1)f'(2)=(0)(1)=0$

37. $(f\circ h)'(0)=f'(h(0))h'(0)=f'(1)h'(0)=(1)(2)=2$

38. $(f\circ h\circ g)'(1)=f'(h(g(1)))\,h'(g(1))g'(1)=f'(2)h'(1)g'(1)=(1)(1)(0)=0$

39. $(g\circ f\circ h)'(2)=g'(f(h(2)))\,f'(h(2))h'(2)=g'(1)f'(0)h'(2)=(0)(2)(2)=0$

40. $(g\circ h\circ f)'(0)=g'(h(f(0)))\,h'(f(0))f'(0)=g'(2)h'(1)f'(0)=(1)(1)(2)=2$

41. $f'(x)=4(x^3+x)^3(3x^2+1)$

$f''(x)=3(4)(x^3+x)^2(3x^2+1)^2+4(x^3+x)^3(6x)=12(x^3+x)^2[(3x^2+1)^2+2x(x^3+x)]$

42. $f'(x)=10(x^2-5x+2)^9(2x-5)$

$$f''(x)=9(10)(x^2-5x+2)^8(2x-5)^2+10(x^2-5x+2)^9(2)$$
$$=(10)(x^2-5x+2)^8\left[9(2x-5)^2+2(x^2-5x+2)\right]$$

43. $f'(x)=3\left(\frac{x}{1-x}\right)^2\cdot\frac{1}{(1-x)^2}=\frac{3x^2}{(1-x)^4}$

$$f''(x)=\frac{6x(1-x)^4-3x^2(4)(1-x)^3(-1)}{(1-x)^8}=\frac{6x(1+x)}{(1-x)^5}$$

44. $f'(x)=\frac{1}{2\sqrt{x^2+1}}(2x)=\frac{x}{\sqrt{x^2+1}}$

$$f''(x)=\frac{\sqrt{x^2+1}(1)-x\frac{x}{\sqrt{x^2+1}}}{\left(\sqrt{x^2+1}\right)^2}=\frac{1}{(x^2+1)^{3/2}}$$

45. $2xf'(x^2+1)$

46. $f'\left(\frac{x-1}{x+1}\right)\frac{d}{dx}\left(\frac{x-1}{x+1}\right)=\frac{2}{(x+1)^2}f'\left(\frac{x-1}{x+1}\right)$

47. $2f(x)f'(x)$

48. $\frac{[f(x)+1]f'(x)-[f(x)-1]f'(x)}{[f(x)+1]^2}=\frac{2f'(x)}{[f(x)+1]^2}$

49. $f'(x)=-4x(1+x^2)^{-3}$; (a) $x=0$ (b) $x<0$ (c) $x>0$

50. $f'(x)=2(1-x^2)(-2x)=-4x(1-x^2)$; (a) $x=-1,0,1$ (b) $-1<x<0,\ x>1$

(c) $x<-1,\ 0<x<1$

51. $f'(x)=\frac{1-x^2}{(1+x^2)^2}$; (a) $x=\pm1$ (b) $-1<x<1$ (c) $x<-1,\quad x>1$

52. $f'(x)=(1-x^2)^3+x(3)(1-x^2)^2(-2x)=(1-x^2)^2(1-7x^2)$;

(a) $x=\pm1,\quad x=\pm\frac{1}{7}\sqrt{7}$ (b) $-\frac{1}{7}\sqrt{7}<x<\frac{1}{7}\sqrt{7}$

(c) $x<-1,\quad -1<x<-\frac{1}{7}\sqrt{7},\quad \frac{1}{7}\sqrt{7}<x<1,\quad x>1$

53. $v(t)=5(t+1)(t-9)^2(t-3)$; the object changes direction (from left to right) at $t=3$.

54. $v(t)=(t-8)^3+t(3)(t-8)^2=(t-8)^2(4t-8)$; the object changes direction (from left to right) at $t=2$.

55. $v(t)=12t^3(t^2-12)^3(t^2-4)$; the object changes direction (from right to left) at $t=2$ and (from left to right) at $t=2\sqrt{3}$.

56. $v(t)=3(t^2-8t+15)^2(2t-8)$; the object changes direction (from left to right) at $t=4$.

57. $\frac{n!}{(1-x)^{n+1}}$

58. $\frac{(-1)^{n+1}n!}{(1+x)^{n+1}}$

59. $n!\,b^n$

60. $\frac{(-1)^n n!\,ab^n}{(bx+c)^{n+1}}$

61. $y=(x^2+1)^3+C$

62. $y=\frac{(x^2-1)^2}{2}+C$

63. $y=(x^3-2)^2+C$

64. $y=\frac{(x^3+2)^3}{3}+C$

65. $L'(x)=\frac{1}{x^2+1}\cdot 2x=\frac{2x}{x^2+1}$

66. $H'(x)=2f(x)f'(x)-2g(x)g'(x)=2f(x)g(x)-2g(x)f(x)=0$

67. $T'(x)=2f(x)\cdot f'(x)+2g(x)\cdot g'(x)=2f(x)\cdot g(x)-2g(x)\cdot f(x)=0$

68. (a) Suppose f is even: $[f(x)]'=[f(-x)]'=f'(-x)(-1)=-f'(-x)$; thus $f'(-x)=-f'(x)$.

(b) Suppose f is odd: $[f(x)]' = -[f(-x)]' = -f'(-x)(-1) = f'(-x)$; thus $f'(-x) = f'(x)$.

69. Suppose $p(x) = (x-a)^2 q(x)$, where $q(a) \neq 0$. Then

$$p'(x) = 2(x-a)q(x) + (x-a)^2 q'(x) \quad \text{and} \quad p''(x) = 2q(x) + 4(x-a)q'(x) + (x-a)^2 q''(x),$$

and it follows that $p(a) = p'(a) = 0$, and $p''(a) \neq 0$.

Now suppose that $p(a) = p'(a) = 0$ and $p''(a) \neq 0$.

$$p(a) = 0 \quad \Longrightarrow \quad p(x) = (x-a)g(x) \quad \text{for some polynomial } g.$$

Then $p'(x) = g(x) + (x-a)g'(x)$ and

$$p'(a) = 0 \quad \Longrightarrow \quad g(a) = 0 \ \text{ and so} g(x) = (x-a)q(x) \text{for some polynomial} q.$$

Therefore, $p(x) = (x-a)^2 q(x)$. Finally, $p''(a) \neq 0$ implies $q(a) \neq 0$.

70. Suppose $p(x) = (x-a)^3 q(x)$, where $q(a) \neq 0$. Then

$$p'(x) = 3(x-a)^2 q(x) + (x-a)^3 q'(x)$$

$$p''(x) = 6(x-a)q(x) + 6(x-a)^2 q'(x) + (x-a)^3 q''(x)$$

$$p'''(x) = 6q(x) + 18(x-a)q'(x) + 9(x-a)^2 q''(x) + (x-a)^3 q'''(x)$$

and it follows that $p(a) = p'(a) = p''(a) = 0$, $p'''(a) \neq 0$.

Now suppose that $p(a) = p'(a) = p''(a) = 0$ and $p'''(a) \neq 0$.

$$p(a) = 0 \quad \Longrightarrow \quad p(x) = (x-a)g(x) \quad \text{for some polynomial } g.$$

Then $p'(x) = g(x) + (x-a)g'(x)$ and

$$p'(a) = 0 \quad \Longrightarrow \quad g(a) = 0 \ \text{ and so } \ g(x) = (x-a)h(x) \ \text{ for some polynomial } h.$$

Therefore, $p(x) = (x-a)^2 h(x)$. Now $p''(x) = 2h(x) + 4(x-a)h'(x) + (x-a)^2 h''(x)$ and

$$p''(a) = 0 \quad \Longrightarrow \quad h(a) = 0 \ \text{ and so } \ h(x) = (x-a)q(x) \ \text{ for some polynomial } q.$$

Therefore, $p(x) = (x-a)^3 q(x)$. Finally, $p'''(a) \neq 0$ implies $q(a) \neq 0$.

71. Let p be a polynomial function of degree n. The number a is a root of p of multiplicity k, $(k < n)$ if and only if $p(a) = p'(a) = \cdots = p^{(k-1)}(a) = 0$ and $p^{(k)}(a) \neq 0$.

72. Let $y = g(x)$ and suppose that $f(y) = f[g(x)] = x$. Differentiating this equation, we have

$$\frac{d}{dx}(f(y) = f'(y)g'(x) = 1 \quad \Longrightarrow \quad f'(y) = \frac{1}{g'(x)} \text{ provided } g'(x) \neq 0$$

73. $\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt} = (3x^2 - 3)(4t - 1)$

At $t = 2$, $x(2) = 8$ and $\dfrac{dy}{dt} = [3(8)^2 - 3][4(2) - 1] = 1323.$

74. $A = \dfrac{\sqrt{3}}{4}x^2$, where $x = \dfrac{2\sqrt{3}}{3}h$. Now

$$\frac{dA}{dh} = \frac{dA}{dx}\frac{dx}{dh} = \frac{\sqrt{3}}{2}x \cdot \frac{2\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}h; \quad \frac{dA}{dh} = 4 \text{ when } h = 2\sqrt{3}$$

75. $V = \frac{4}{3}\pi r^3$ and $\dfrac{dr}{dt} = 2$ cm/sec. By the chain rule, $\dfrac{dV}{dt} = \dfrac{dV}{dr}\dfrac{dr}{dt} = 4\pi r^2 \dfrac{dr}{dt} = 8\pi r^2.$

At the instant the radius is 10 centimeters, the volume is increasing at the rate

$\dfrac{dV}{dt} = 8\pi(10)^2 = 800\pi \ \text{cm}^3/\text{sec}.$

76. $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$, and $\dfrac{dV}{dt} = 200.$

$$\begin{aligned}\frac{dS}{dt} &= \frac{dS}{dr} \cdot \frac{dr}{dV} \cdot \frac{dV}{dt} \\ &= 8\pi r \cdot \frac{1}{4\pi r^2} \cdot 200 \\ &= \frac{400}{r} \\ &= 80 \text{ when } r = 5\end{aligned}$$

The surface area is increasing 80 cm^2/sec. at the instant the radius is 5 centimeters.

77. $KE = \frac{1}{2}mv^2; \quad \dfrac{d(KE)}{dt} = \dfrac{d(KE)}{dv} \cdot \dfrac{dv}{dt} = mv\dfrac{dv}{dt}.$

78. (a) $\dfrac{dF}{dt} = \dfrac{dF}{dr} \cdot \dfrac{dr}{dt} = -\dfrac{2k}{r^3} \cdot (49 - 9.8t) - \dfrac{2k}{(49t - 4.9t^2)^3}(49 - 9.8t), \ 0 \leq t \leq 10$

(b) $\dfrac{dF}{dt}(3) = -\dfrac{2k}{(102.9)^3}(19.6); \quad \dfrac{dF}{dt}(7) = \dfrac{2k}{(102.9)^3}(19.6).$

SECTION 3.6

1. $\dfrac{dy}{dx} = -3\sin x - 4\sec x \tan x$

2. $\dfrac{dy}{dx} = 2x\sec x + x^2 \sec x \tan x$

3. $\dfrac{dy}{dx} = 3x^2 \csc x - x^3 \csc x \cot x$

4. $\dfrac{dy}{dx} = 2\sin x \cos x$

5. $\dfrac{dy}{dt} = -2\cos t \sin t$

6. $\dfrac{dy}{dt} = 6t \tan t + 3t^2 \sec^2 t$

7. $\dfrac{dy}{du} = 4\sin^3\sqrt{u}\,\dfrac{d}{du}(\sin\sqrt{u}) = 4\sin^3\sqrt{u}\,\cos\sqrt{u}\,\dfrac{d}{du}(\sqrt{u}) = 2u^{-1/2}\sin^3\sqrt{u}\,\cos\sqrt{u}$

8. $\dfrac{dy}{du} = \csc u^2 - 2u^2 \csc u^2 \cot u^2$

9. $\dfrac{dy}{dx} = \sec^2 x^2 \dfrac{d}{dx}(x^2) = 2x\sec^2 x^2$

10. $\frac{dy}{dx} = -\frac{1}{2\sqrt{x}}\sin\sqrt{x}$

11. $\frac{dy}{dx} = 4[x + \cot \pi x]^3[1 - \pi \csc^2 \pi x]$

12. $\frac{dy}{dx} = 3(x^2 - \sec 2x)^2(2x - 2\sec 2x\tan 2x) = 6(x^2 - \sec 2x)^2(x - \sec 2x\tan 2x)$

13. $\frac{dy}{dx} = \cos x,\ \frac{d^2y}{dx^2} = -\sin x$

14. $\frac{dy}{dx} = -\sin x,\ \frac{d^2y}{dx^2} = -\cos x$

15. $\frac{dy}{dx} = \frac{(1+\sin x)(-\sin x) - \cos x(\cos x)}{(1+\sin x)^2} = \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2} = -(1+\sin x)^{-1}$

$\frac{d^2y}{dx^2} = (1+\sin x)^{-2}\frac{d}{dx}(1+\sin x) = \cos x\,(1+\sin x)^{-2}$

16. $\frac{dy}{dx} = 3\tan^2(2\pi x)\sec^2(2\pi x)(2\pi) = 6\pi\tan^2(2\pi x)\sec^2(2\pi x)$

$\frac{d^2y}{dx^2} = 6\pi(2)\tan(2\pi x)\sec^2(2\pi x)\sec^2(2\pi x)(2\pi) + 6\pi\tan^2(2\pi x)(2)\sec(2\pi x)[\sec(2\pi x)\tan(2\pi x)](2\pi)$

$= 24\pi^2\tan(2\pi x)\sec^2(2\pi x)[\sec^2(2\pi x) + \tan^2(2\pi x)]$

17. $\frac{dy}{du} = 3\cos^2 2u\,\frac{d}{du}(\cos 2u) = -6\cos^2 2u\sin 2u$

$\frac{d^2y}{du^2} = -6[\cos^2 2u\,\frac{d}{du}(\sin 2u) + \sin 2u\,\frac{d}{du}(\cos^2 2u)]$

$= -6[2\cos^3 2u + \sin 2u\,(-4\cos 2u\sin 2u)] = 12\cos 2u\,[2\sin^2 2u - \cos^2 2u]$

18. $\frac{dy}{dt} = 5\sin^4(3t)\cos(3t)(3) = 15\sin^4(3t)\cos(3t)$

$\frac{d^2y}{dt^2} = 15(4)\sin^3(3t)\,3\cos^2(3t) + 15\sin^4(3t)[-3\sin(3t)] = 45\sin^3(3t)[4\cos^2(3t) - sin^2(3t)]$

19. $\frac{dy}{dt} = 2\sec^2 2t,\quad \frac{d^2y}{dt^2} = 4\sec 2t\,\frac{d}{dt}(\sec 2t) = 8\sec^2 2t\tan 2t$

20. $\frac{dy}{du} = -4\csc^2 4u;\quad \frac{d^2y}{du^2} = -4(2)\csc(4u)[-\csc(4u)\cot(4u)(4)] = 32\csc^2(4u)\cot(4u)$

21. $\frac{dy}{dx} = x^2(3\cos 3x) + 2x\sin 3x$

$\frac{d^2y}{dx^2} = [x^2(-9\sin 3x) + 2x(3\cos 3x)] + [2x(3\cos 3x) + 2(\sin 3x)]$

$= (2 - 9x^2)\sin 3x + 12x\cos 3x$

22. $\frac{dy}{dx} = \frac{(1-\cos x)\cos x - \sin x\,(-[-\sin x])}{(1-\cos x)^2} = \frac{\cos x - 1}{(1-\cos x)^2} = \frac{-1}{1-\cos x}$

$$\frac{d^2y}{dx^2} = \sin x(1-\cos x)^{-2}$$

23. $y = \sin^2 x + \cos^2 x = 1$ so $\dfrac{dy}{dx} = \dfrac{d^2y}{dx^2} = 0$

24. $y = \sec^2 x - \tan^2 x = 1$ so $\dfrac{dy}{dx} = \dfrac{d^2y}{dx^2} = 0$

25. $\dfrac{d^4}{dx^4}(\sin x) = \dfrac{d^3}{dx^3}(\cos x) = \dfrac{d^2}{dx^2}(-\sin x) = \dfrac{d}{dx}(-\cos x) = \sin x$

26. $\dfrac{d^4}{dx^4}(\cos x) = \dfrac{d^3}{dx^3}(-\sin x) = \dfrac{d^2}{dx^2}(-\cos x) = \dfrac{d}{dx}(\sin x) = \cos x$

27. $$\begin{aligned}\frac{d}{dt}\left[t^2\frac{d^2}{dt^2}(t\cos 3t)\right] &= \frac{d}{dt}\left[t^2\frac{d}{dt}(\cos 3t - 3t\sin 3t)\right]\\ &= \frac{d}{dt}[t^2(-3\sin 3t - 3\sin 3t - 9t\cos 3t)]\\ &= \frac{d}{dt}[-6t^2\sin 3t - 9t^3\cos 3t]\\ &= (-18t^2\cos 3t - 12t\sin 3t) + (27t^3\sin 3t - 27t^2\cos 3t)\\ &= (27t^3 - 12t)\sin 3t - 45t^2\cos 3t\end{aligned}$$

28. $$\begin{aligned}\frac{d}{dt}\left[t\frac{d}{dt}(cost^2)\right] &= \frac{d}{dt}\left[-t\,\sin t^2(2t)\right] = \frac{d}{dt}\left[-2t^2\,\sin t^2\right]\\ &= -4t\,\sin t^2 - 2t^2\cos t^2(2t) = -4t(\sin t^2 + t^2\cos t^2)\end{aligned}$$

29. $\dfrac{d}{dx}[f(\sin 3x)] = f'(\sin 3x)\dfrac{d}{dx}(\sin 3x) = 3\cos 3x f'(\sin 3x)$

30. $\dfrac{d}{dx}(\sin[f(3x)] = \cos[f(3x)]f'(3x)(3) = 3f'(3x)\cos[f(3x)]$

31. $\dfrac{dy}{dx} = \cos x$; slope of tangent at $(0,0)$ is 1; tangent: $y = x$.

32. $\dfrac{dy}{dx} = \sec^2 x$; slope of tangent at $(\pi/6, \sqrt{3}/3)$ is $sec^2(\pi/6) = 4/3$;

tangent: $y - \frac{1}{3}\sqrt{3} = \frac{4}{3}\left(x - \frac{1}{6}\pi\right)$

33. $\dfrac{dy}{dx} = -\csc^2 x$; slope of tangent at $(\frac{\pi}{6}, \sqrt{3})$ is -4, an equation for tangent: $y - \sqrt{3} = -4(x - \frac{\pi}{6})$.

34. $\dfrac{dy}{dx} = -\sin x$; slope of tangent at $(0,1)$ is 0; tangent: $y = 1$.

35. $\dfrac{dy}{dx} = \sec x\tan x$, slope of tangent at $(\frac{\pi}{4}, \sqrt{2})$ is $\sqrt{2}$, an equation for tangent is $y - \sqrt{2} = \sqrt{2}(x - \frac{\pi}{4})$.

36. $\frac{dy}{dx} = -\csc x \cot x$, slope of tangent at $(\pi/3, 2\sqrt{3}/3)$ is $-2/3$;

tangent: $y - \frac{2}{3}\sqrt{3} = -\frac{2}{3}\left(x - \frac{1}{3}\pi\right)$.

37. $\frac{dy}{dx} = -\sin x;\quad x = \pi$

38. $\frac{dy}{dx} = \cos x;\quad x = \frac{1}{2}\pi,\ x = \frac{3}{2}\pi$

39. $\frac{dy}{dx} = \cos x - \sqrt{3}\sin x;\quad \frac{dy}{dx} = 0$ gives $\tan x = \frac{1}{\sqrt{3}};\quad x = \frac{\pi}{6},\ \frac{7\pi}{6}$

40. $\frac{dy}{dx} = -\sin x - \sqrt{3}\cos x;\quad \frac{dy}{dx} = 0$ gives $\tan x = -\sqrt{3};\quad x = \frac{2\pi}{3},\ \frac{5\pi}{3}$

41. $\frac{dy}{dx} = 2\sin x\cos x = \sin 2x;\quad x = \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}$

42. $\frac{dy}{dx} = -2\sin x\cos x = -\sin 2x;\quad x = \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}$

43. $\frac{dy}{dx} = \sec^2 x - 2;\quad \frac{dy}{dx} = 0$ gives $\sec x = \pm\sqrt{2};\quad x = \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{7\pi}{4}$

44. $\frac{dy}{dx} = -3\csc^2 x + 4;\quad \frac{dy}{dx} = 0$ gives $\csc x = \pm\frac{2}{\sqrt{3}};\quad x = \frac{\pi}{3},\ \frac{2\pi}{3},\ \frac{4\pi}{3},\ \frac{5\pi}{3}$

45. $\frac{dy}{dx} = 2\sec x\tan x + \sec^2 x$; since $\sec x$ is never zero, $\frac{dy}{dx} = 0$ gives

$2\tan x + \sec x = 0$ so that $\sin x = -1/2;\quad x = \frac{7\pi}{6},\ \frac{11\pi}{6}$

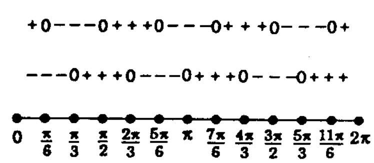
46. $\frac{dy}{dx} = -\csc^2 x + 2\csc x\cot x$; since $\csc x$ is never zero, $\frac{dy}{dx} = 0$ gives

$2\cot x - \csc x = 0$ so that $\cos x = 1/2;\quad x = \frac{\pi}{3},\ \frac{5\pi}{3}$

47. We want $v(t) > 0$ and $a(t) > 0$.

$v(t) = 3\cos 3t$ sign of $v(t)$:

$a(t) = -9\sin 3t$ sign of $a(t)$:

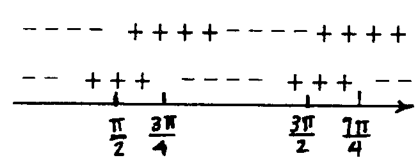


Thus, $\pi < t < \frac{2\pi}{3},\quad \frac{7\pi}{6} < t < \frac{4\pi}{3},\quad \frac{11\pi}{6} < t < 2\pi$.

48. We want $v(t) > 0$ and $a(t) > 0$.

$v(t) = -2\sin 2t$ sign of $v(t)$:

$a(t) = -4\cos 2t$ sign of $a(t)$:

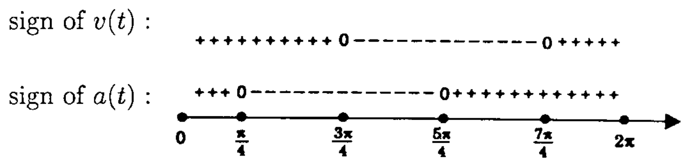


Thus, $\frac{\pi}{2} < t < \frac{3\pi}{4},\quad \frac{3\pi}{2} < t < \frac{7\pi}{4}$.

49. We want $v(t) > 0$ and $a(t) > 0$.

$v(t) = \cos t + \sin t$

$a(t) = -\sin t + \cos t$

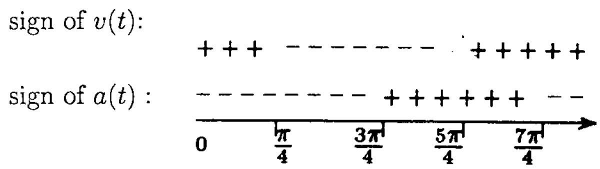


Thus, $0 < t < \dfrac{\pi}{4}$ and $\dfrac{7\pi}{4} < t < 2\pi$.

50. We want $v(t) > 0$ and $a(t) > 0$.

$v(t) = \cos t - \sin t$

$a(t) = -\sin t - \cos t$

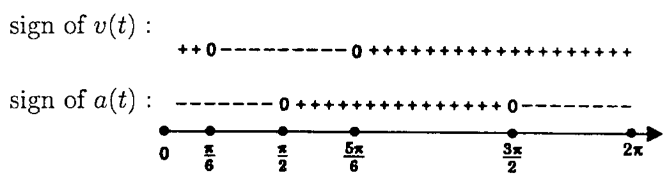


Thus, $\dfrac{5\pi}{4} < t < \dfrac{7\pi}{4}$.

51. We want $v(t) > 0$ and $a(t) > 0$.

$v(t) = 1 - 2\sin t$

$a(t) = -2\cos t$

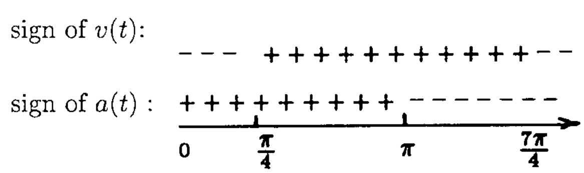


Thus, $\dfrac{5\pi}{6} < t < \dfrac{3\pi}{2}$.

52. We want $v(t) > 0$ and $a(t) > 0$.

$v(t) = 1 - \sqrt{2}\,\cos t$

$a(t) = \sqrt{2}\,\sin t$



Thus, $\dfrac{\pi}{4} < t < \pi$.

53. (a) $\dfrac{dy}{dt} = \dfrac{dy}{du}\dfrac{du}{dx}\dfrac{dx}{dt} = (2u)(\sec x \tan x)\pi = 2\pi \sec^2 \pi t \tan \pi t$

(b) $y = \sec^2 \pi t - 1, \quad \dfrac{dy}{dt} = 2\sec \pi t\,(\sec \pi t \tan \pi t)\pi = 2\pi \sec^2 \pi t \tan \pi t$

54. (a) $\dfrac{dy}{dt} = \dfrac{dy}{du}\dfrac{du}{dx}\dfrac{dx}{dt} = 3\left[\dfrac{1}{2}(1+u)\right]^2\left(\dfrac{1}{2}\right)(-\sin x)(2) = 3\left[\dfrac{1}{2}(1+cos2t)\right]^2(-\sin 2t)$

$=3(\cos^4 t)(-2\sin t\,\cos t) = -6\cos^5 t\,\sin t$

(b) $y = \left[\frac{1}{2}(1+\cos 2t)\right]^3 = \cos^6 t;\quad \frac{dy}{dt} = 6\cos^5 t(-\sin t) = -6\cos^5 t\sin t$

55. (a) $\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dx}\frac{dx}{dt} = 4\left[\frac{1}{2}(1-u)\right]^3\left(-\frac{1}{2}\right)(-\sin x)(2) = 4\left[\frac{1}{2}(1-\cos 2t)\right]^3 \sin 2t$

$= 4\sin^6 t\,(2\sin t\cos t) = 8\sin^7 t\cos t$

(b) $y = \left[\frac{1}{2}(1-\cos 2t)\right]^4 = \sin^8 t,\quad \frac{dy}{dt} = 8\sin^7 t\cos t$

56. (a) $\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dx}\frac{dx}{dt} = (-2u)(-\csc x\ \cot x)(3) = (-2\ \csc(3t)(-\ \csc(3t)\ \cot(3t)3$

$= 6\ \csc^2(3t)\ \cot(3t)$

(b) $y = 1 - csc^2(3t);\quad \frac{dy}{dt} = -2\ \csc(3t)[-\csc(3t)\ \cot(3t)](3) = 6\csc^2(3t)\ \cot(3t)$

57. $\frac{d^n}{dx^n}(\cos x) = \begin{cases} (-1)^{(n+1)/2}\sin x, & n \text{ odd} \\ (-1)^{n/2}\cos x, & n \text{ even} \end{cases}$

58. (a) $\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{-\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x$

(b) $\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{-1}{\cos^2 x}(-\sin x) = \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} = \sec x\ \tan x$

(c) $\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{-1}{\sin^2 x}(\cos x) = \frac{-1}{\sin x}\cdot\frac{\cos x}{\sin x} = -\csc x\ \cot x$

59. $\frac{d}{dx}(\cos x) = \frac{d}{dx}(\sin(\frac{\pi}{2}x)) = -\cos(\frac{\pi}{2}x) = -\sin x.$

60. Differentiating both sides, $2\cos 2x = 2(cos^2 x - sin^2 x)$. Thus $\cos 2x = cos^2 x - sin^2 x$.

61. $f'(0) = \lim_{h\to 0}\frac{\sin(0+h)-\sin 0}{h} = \lim_{h\to 0}\frac{\sin h}{h} = \lim_{x\to 0}\frac{\sin x}{x}$

62. $f'(0) = \lim_{h\to 0}\frac{\cos(0+h)-\cos 0}{h} = \lim_{h\to 0}\frac{\cos h - 1}{h} = \lim_{x\to 0}\frac{\cos x - 1}{x}$

63. $f(x) = 2\sin x + 3\cos x + C$

64. $f(x) = \tan x + \cot x + C$

65. $f(x) = \sin 2x + \sec x + C$

66. $f(x) = \frac{-\cos 3x}{3} - \frac{\csc 2x}{2} + C$

67. $f(x) = \sin(x^2) + \cos 2x + C$

68. $f(x) = \frac{\tan(x^3)}{3} + \sec 2x + C$

69. (a) $f'(x) = \sin(1/x) + x\ \cos(1/x)(-1/x^2)$

$= \sin(1/x) - (1/x)\cos(1/x)$

$$g'(x) = 2x\ \sin(1/x) + x^2 \cos(1/x)(-1/x^2)$$
$$= 2x\ \sin(1/x) - \cos(1/x)$$

(b) $\lim\limits_{x\to 0} g'(x) = \lim\limits_{x\to 0} [2x\ \sin(1/x) - \cos(1/x)] = -\lim\limits_{x\to 0} \cos(1/x)$ does not exist

70. (a) f must be continuous at 0:

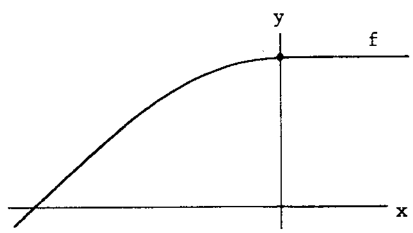
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \cos x = 1, \quad \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (ax+b) = b; \text{ thus } b = 1$$

Differentiable at 0 :

$$\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{\cos h - 1}{h} = 0,$$
$$\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{ah+1-1}{h} = a$$

Therefore, f is differentiable at 0 if $a = 0$ and $b = 1$.

(b)



71. (a) Continuity:

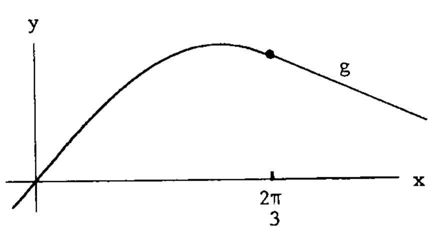
$$\lim_{x\to(2\pi/3^-} \sin x = \frac{\sqrt{3}}{2}, \quad \lim_{x\to(2\pi/3^-} (ax+b) = \frac{2\pi a}{3} + b; \quad \text{thus} \quad \frac{2\pi a}{3} + b = \frac{\sqrt{3}}{2}$$

Differentiability:

$$\lim_{x\to(2\pi/3^+} \cos x = -\frac{1}{2}, \quad \lim_{x\to(2\pi/3^+} (a) = a; \quad \text{thus} \quad a = -\frac{1}{2}$$

Therefore, f is differentiable at $2\pi/3$ if $a = -\frac{1}{2}$ and $b = \frac{1}{2}\sqrt{3} + \frac{1}{3}\pi$

(b)



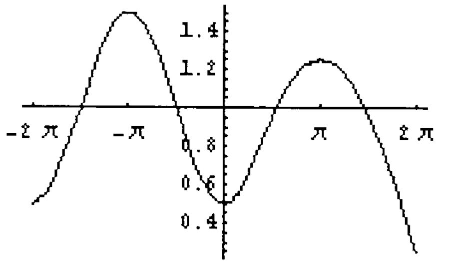
72. (a) Setting right and left-hand derivatives equal to each other at $x = \dfrac{\pi}{3}$ yields

$-a \sin \dfrac{\pi}{3} = \dfrac{1}{2} \cos \dfrac{\pi}{6}$, so $a = -\dfrac{1}{2}$.

Setting right and left-hand values of f equal to each other at $\dfrac{\pi}{3}$ yields

$1 - \dfrac{1}{2} \cos \dfrac{\pi}{3} = b + \sin \dfrac{\pi}{3}$, so $b = \dfrac{1}{4}$.

(b)



73. Let $y(t) = A\,\sin\omega t + B\,\cos\omega t$. Then

$$y'(t) = \omega\,A\,\cos\omega t - \omega\,B\,\sin\omega t \quad \text{and} \quad y''(t) = -\omega^2 A\,\sin\omega t - \omega^2 B\,\cos\omega t$$

Thus,

$$\frac{d^2y}{dt^2} + \omega^2 y = 0.$$

74. (a) $\theta = a\,\cos(\omega t + \phi_0)$; $\theta' = -\,a\omega\,\sin(\omega t + \phi_0)$; $\theta'' = -\,a\omega^2\cos(\omega t + \phi_0)$

Thus, θ satisfies the equation.

(b)

$$\begin{aligned} \theta &= a\,\cos(\omega t + \phi_0) \\ &= a\,\cos(\omega t)\cos\phi_0 - a\,\sin(\omega t)\sin\phi_0 \\ &= A\,\sin(\omega t) + B\,\cos(\omega t) \text{ where } A = -a\,\sin\phi_0,\ \ B = a\,\cos\phi_0 \end{aligned}$$

75. $A = \frac{1}{2}c^2\sin x$; $\quad \dfrac{dA}{dx} = \frac{1}{2}c^2\cos x$

76. $c = \sqrt{a^2 + b^2 - 2ab\,\cos x}$; $\quad \dfrac{dc}{dx} = \dfrac{1}{2\sqrt{a^2 + b^2 - 2ab\,\cos x}}(2ab\,\sin x) = \dfrac{ab\,\sin x}{\sqrt{a^2 + b^2 - 2ab\,\cos x}}$

77. (a)

θ	5	1	0.1	0.01	0.001
$\dfrac{\sin\theta}{\theta}$	0.01743	0.01745	0.01745	0.01745	0.01745

(b) $\dfrac{\pi}{180} \cong 0.01745$

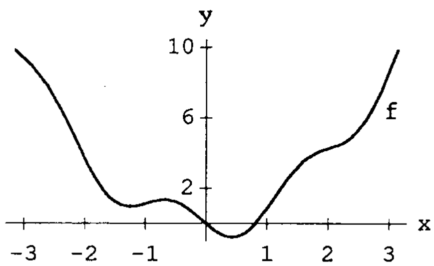
78. Let $D(h) = \dfrac{f(0+h) - f(0)}{h} = \dfrac{\cos^2 h - 1}{h}$. Then

$D(0.1) \cong -0.0005$ $\qquad D(0.01) \cong 0$ $\qquad D(0.001) \cong 0$

$D(-0.1) \cong 0.0005$ $\qquad D(-0.01) \cong 0$ $\qquad D(0.001) \cong 0$

By the chain rule, $f'(x) = -\,2x\,\sin x^2$, and $f'(0) = 0$.

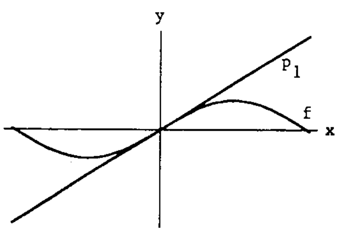
79. (a)



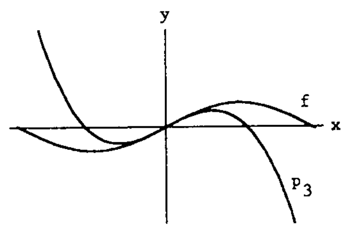
(b) $f(x) = 0$ at $x = 0$ and $x \cong 0.81$

(c) $f'(x) = 0$ at $x \cong -1.25$, $x \cong -0.68$, and $x \cong 0.43$

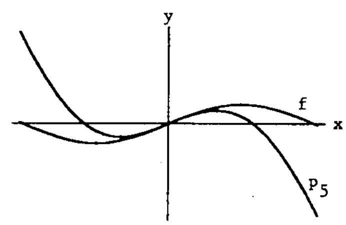
80. (a)



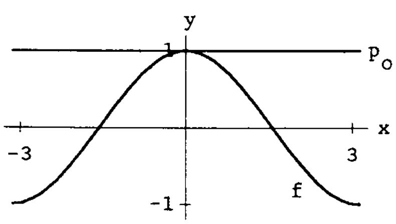
(b)



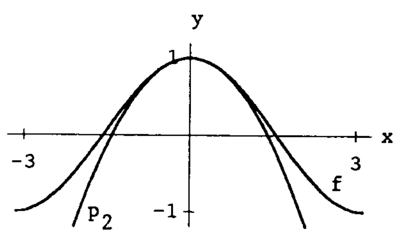
(c)



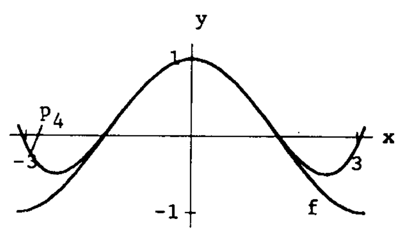
81. (a)



(b)



(c)



SECTION 3.7

1.

$$x^2 + y^2 = 4$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

2.

$$x^3 + y^3 - 3xy = 0$$

$$3x^2 + 3y^2\frac{dy}{dx} - 3\left(y + x\frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

3.

$$4x^2 + 9y^2 = 36$$

$$8x + 18y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-4x}{9y}$$

4.

$$\sqrt{x} + \sqrt{y} = 4$$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

5.

$$x^4 + 4x^3y + y^4 = 1$$

$$4x^3 + 12x^2y + 4x^3\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x^3 + 3x^2y}{x^3 + y^3}$$

6.

$$x^2 - x^2y + xy^2 + y^2 = 1$$

$$2x - 2xy - x^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2xy - 2x - y^2}{2xy + 2y - x^2}$$

7.

$$(x - y)^2 - y = 0$$

$$2(x - y)\left(1 - \frac{dy}{dx}\right) - \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2(x - y)}{2(x - y) + 1}$$

8.

$$(y + 3x)^2 - 4x = 01$$

$$2(y + 3x)\left(\frac{dy}{dx} + 3\right) - 4 = 0$$

$$\frac{dy}{dx} = -3 + \frac{2}{y + 3x}$$

9.

$$\sin(x + y) = xy$$

$$\cos(x + y)\left(1 + \frac{dy}{dx}\right) = x\frac{dy}{dx} + y$$

$$\frac{dy}{dx} = \frac{y - \cos(x + y)}{\cos(x + y) - x}$$

10. $\tan xy = xy$; $\sec^2(xy)\left(y + x\dfrac{dy}{dx}\right) = y + x\dfrac{dy}{dx}$; $\dfrac{dy}{dx} = -\dfrac{y}{x}$

11.

$$y^2 + 2xy = 16$$

$$2y\frac{dy}{dx} + 2x\frac{dy}{dx} + 2y = 0$$

$$(x + y)\frac{dy}{dx} + y = 0.$$

Differentiating a second time, we have

$$(x + y)\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(2 + \frac{dy}{dx}\right) = 0.$$

Substituting $\dfrac{dy}{dx} = \dfrac{-y}{x+y}$, we have

$$(x+y)\frac{d^2y}{dx^2} - \frac{y}{(x+y)}\left(\frac{2x+y}{x+y}\right) = 0, \quad \frac{d^2y}{dx^2} = \frac{2xy+y^2}{(x+y)^3} = \frac{16}{(x+y)^3}.$$

12.

$$x^2 - 2xy + 4y^2 = 3$$

$$2x - 2y - 2x\frac{dy}{dx} + 8y\frac{dy}{dx} = 0$$

$$x - y + (4y - x)\frac{dy}{dx} = 0.$$

Differentiating a second time, we have

$$1 - \frac{dy}{dx} + \left(4\frac{dy}{dx} - 1\right)\frac{dy}{dx} + (4y - x)\frac{d^2y}{dx^2} = 0.$$

Substituting $\dfrac{dy}{dx} = \dfrac{y-x}{4y-x}$, we have

$$\frac{d^2y}{dx^2} = -\frac{3(x^2 - 2xy + 4y^2)}{(4y-x)^3} = \frac{-9}{(4y-x)^3}.$$

13.

$$y^2 + xy - x^2 = 9$$

$$2y\frac{dy}{dx} + x\frac{dy}{dx} + y - 2x = 0.$$

Differentiating a second time, we have

$$\left[2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2}\right] + \left[x\frac{d^2y}{dx^2} + \frac{dy}{dx}\right] + \frac{dy}{dx} - 2 = 0$$

$$(2y + x)\frac{d^2y}{dx^2} + 2\left[\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} - 1\right] = 0.$$

Substituting $\dfrac{dy}{dx} = \dfrac{2x-y}{2y+x}$, we have

$$(2y+x)\frac{d^2y}{dx^2} + 2\left[\frac{(2x-y)^2 + (2x-y)(2y+x) - (2y+x)^2}{(2y+x)^2}\right] = 0$$

$$\frac{d^2y}{dx^2} = \frac{10(y^2 + xy - x^2)}{(2y+x)^3} = \frac{90}{(2y+x)^3}.$$

14.

$$x^2 - 3xy = 18$$

$$2x - 3y - 3x\frac{dy}{dx} = 0.$$

Differentiating a second time, we have

$$2 - 3\frac{dy}{dx} - 3\frac{dy}{dx} - 3\frac{d^2y}{dx^2} = 0$$

Substituting $\dfrac{dy}{dx} = \dfrac{2x-3y}{3x}$, we have

$$\frac{d^2y}{dx^2} = -\frac{6(x-3y)}{9x^2} = -\frac{6(x^2-3xy)}{9x^3} = -\frac{6(18)}{9x^3} = -\frac{12}{x^3}$$

15.

$$4\tan y = x^3$$

$$4\sec^2 y\,\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3}{4}x^2\cos^2 y$$

$$\frac{d^2y}{dx^2} = \frac{3}{2}x\cos^2 y + \frac{3}{4}x^2\left(2\cos y(-\sin y)\frac{dy}{dx}\right)$$

$$= \frac{3}{2}x\cos^2 y - \frac{9}{8}x^4\sin y\cos^3 y$$

16.

$$\sin^2 x + \cos^2 y = 1$$

$$2\sin x\cos x - 2\cos y\sin y\frac{dy}{dx} = 0$$

$$\sin 2x - \sin 2y\,\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{\sin 2x}{\sin 2y}$$

$$\frac{d^2y}{dx^2} = \frac{\sin 2y(\cos 2x)2 - \sin 2x(\cos 2y)2\dfrac{dy}{dx}}{\sin^2 2y}$$

Substituting $\dfrac{dy}{dx} = \dfrac{\sin 2x}{\sin 2y}$ and using the double angle formulas, we find that

$$\frac{d^2y}{dx^2} = \frac{4\left[\cos^2 x\cos^2 y(\sin^2 y - \sin^2 x) - \sin^2 x\sin^2 y(\cos^2 y - \cos^2 x)\right]}{\sin^2 2y} = 0$$

since $\sin^2 x = \sin^2 y$ and $\cos^2 x = \cos^2 y$ from the original equation.

17. $x^2 - 4y^2 = 9, \qquad 2x - 8y\dfrac{dy}{dx} = 0.$

At $(5,2)$, we get $\dfrac{dy}{dx} = \dfrac{5}{8}$. Then,

$$2 - 8\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = 0.$$

At $(5,2)$ we get

$$2 - 8\left[2\frac{d^2y}{dx^2} + \frac{25}{64}\right] = 0 \quad \text{so that} \quad \frac{d^2y}{dx^2} = -\frac{9}{128}.$$

18. $x^2 + 4xy + y^3 + 5 = 0, \qquad 2x + 4y + 4x\dfrac{dy}{dx} + 3y^2\dfrac{dy}{dx} = 0.$

At (2,-1), we get $4 - 4 + 8\dfrac{dy}{dx} + 3\dfrac{dy}{dx} = 0$ so $\dfrac{dy}{dx} = 0.$ Differentiating again,

$$2+4\frac{dy}{dx}+4\frac{dy}{dx}+4x\frac{d^2y}{dx^2}+6y\left(\frac{dy}{dx}\right)^2+3y^2\frac{d^2y}{dx^2}=0$$

At (2,-1) we get $2+11\frac{d^2y}{dx^2}=0$ so $\frac{d^2y}{dx^2}=-\frac{2}{11}$

19. $\cos(x+2y)=0 \qquad -\sin(x+2y)\left(1+2\frac{dy}{dx}\right)=0.$

At $(\pi/6,\ \pi/6)$, we get $\frac{dy}{dx}=-1/2$. Then,

$$-\cos(x+2y)\left(1+2\frac{dy}{dx}\right)^2-\sin(x+2y)\left(2\frac{d^2y}{dx^2}\right)=0.$$

At $(\pi/6,\ \pi/6)$, we get

$$-\cos\frac{\pi}{2}(0)^2-\sin\frac{\pi}{2}\left(2\frac{d^2y}{dx^2}\right)=0 \quad \text{so that} \quad \frac{d^2y}{dx^2}=0.$$

20. $x=\sin^2 y \qquad 1=2\sin y\cos y\frac{dy}{dx}=\sin 2y\frac{dy}{dx}.$

At $(1/2,\pi/4)$, we get $\frac{dy}{dx}=1$. Differentiating again,

$$0=2\cos 2y\left(\frac{dy}{dx}\right)^2+\sin 2y\frac{d^2y}{dx^2}.$$

At $(1/2,\pi/4)$, we get $\frac{d^2y}{dx^2}=0$.

21.

$$2x+3y=5$$

$$2+3\frac{dy}{dx}=0$$

slope of tangent at $(-2,3)$: $-2/3$

tangent: $y-3=-\frac{2}{3}(x+2)$

normal: $y-3=\frac{3}{2}(x+2)$

22.

$$9x^2+4y^2=72$$

$$18x+8y\frac{dy}{dx}=0$$

slope of tangent at $(2,3)$: $-\frac{3}{2}$

tangent: $y-3=-\frac{3}{2}(x-2)$

normal: $y-3=\frac{2}{3}(x-2)$

23.

$$x^2+xy+2y^2=28$$

$$2x+x\frac{dy}{dx}+y+4y\frac{dy}{dx}=0$$

slope of tangent at $(-2,-3)$: $-1/2$

tangent: $y+3=-\frac{1}{2}(x+2)$

normal: $y+3=2(x+2)$

24.

$$x^3-axy+3ay^2=3a^3$$

$$3x^2-ay-ax\frac{dy}{dx}+6ay\frac{dy}{dx}=0$$

slope of tangent at (a,a): $-\frac{2}{5}$

tangent: $y-a=-\frac{2}{5}(x-a)$

normal: $y-a=\frac{5}{2}(x-a)$

25.

$$x=\cos y$$

$$1=-\sin y\frac{dy}{dx}$$

26.

$$\tan xy=x$$

$$\sec^2(xy)\left(y+x\frac{dy}{dx}\right)=1$$

slope of tangent at $\left(\frac{1}{2}, \frac{\pi}{3}\right)$: $\frac{-2}{\sqrt{3}}$

tangent: $y - \frac{\pi}{3} = -\frac{2}{\sqrt{3}}\left(x - \frac{1}{2}\right)$

normal: $y - \frac{\pi}{3} = \frac{\sqrt{3}}{2}\left(x - \frac{1}{2}\right)$

slope of tangent at $(1, \pi/4)$: $\frac{2-\pi}{4}$

tangent: $y - \frac{\pi}{4} = \frac{2-\pi}{4}(x-1)$

normal: $y - \frac{\pi}{4} = -\frac{4}{2-\pi}(x-1)$

27. $\frac{dy}{dx} = \frac{1}{2}(x^3+1)^{-1/2}\frac{d}{dx}(x^3+1) = \frac{3}{2}x^2(x^3+1)^{-1/2}$

28. $\frac{dy}{dx} = \frac{1}{3}(x+1)^{-2/3}$

29. $\frac{dy}{dx} = x\left(\frac{1}{2}(x^2+1)^{-1/2}(2x)\right) + (x^2+1)^{1/2} = (1+2x^2)(x^2+1)^{-1/2}$

30. $\frac{dy}{dx} = 2x(x^2+1)^{1/2} + x^2\left(\frac{1}{2}\right)(x^2+1)^{-1/2}(2x) = \frac{3x^3+2x}{\sqrt{x^2+1}}$

31. $\frac{dy}{dx} = \frac{1}{4}(2x^2+1)^{-3/4}\frac{d}{dx}(2x^2+1) = x(2x^2+1)^{-3/4}$

32. $\frac{dy}{dx} = \frac{1}{3}(x+1)^{-2/3}(x+2)^{2/3} + (x+1)^{1/3}\left(\frac{2}{3}\right)(x+2)^{-1/3} = \frac{3x+4}{3(x+1)^{2/3}(x+2)^{1/3}}$

33. $\frac{dy}{dx} = \sqrt{2-x^2}\left[\frac{-x}{\sqrt{3-x^2}}\right] + \sqrt{3-x^2}\left[\frac{-x}{\sqrt{2-x^2}}\right] = \frac{x(2x^2-5)}{\sqrt{2-x^2}\,\sqrt{3-x^2}}$

34. $\frac{dy}{dx} = \frac{3}{2}(x^4-x+1)^{1/2}(4x^3-1) = \frac{3}{2}\sqrt{x^4-x+1}\,(4x^3-1)$

35. $\frac{d}{dx}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) = \frac{d}{dx}(x^{1/2}+x^{-1/2}) = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} = \frac{1}{2}x^{-3/2}(x-1)$

36. $\frac{d}{dx}\left(\sqrt{\frac{3x+1}{2x+5}}\right) = \frac{1}{2}\left(\frac{3x+1}{2x+5}\right)^{-1/2}\left(\frac{(2x+5)3-(3x+1)2}{(2x+5)^2}\right) = \frac{13}{2(2x+5)^2}\sqrt{\frac{2x+5}{3x+1}}$

37. $\frac{d}{dx}\left(\frac{x}{\sqrt{x^2+1}}\right) = \frac{d}{dx}\left(x(x^2+1)^{-1/2}\right)$

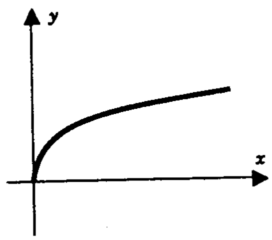
$$= x\left(-\frac{1}{2}(x^2+1)^{-3/2}(2x)\right) + (x^2+1)^{-1/2} = (x^2+1)^{-3/2}$$

38. $\frac{d}{dx}\left(\frac{\sqrt{x^2+1}}{x}\right) = \frac{x\left(\frac{1}{2}\right)(x^2+1)^{-1/2}(2x) - (x^2+1)^{1/2}}{x^2} = \frac{\frac{x}{\sqrt{x^2+1}} - \sqrt{x^2+1}}{x^2} = \frac{-1}{x^2\sqrt{x^2+1}}$

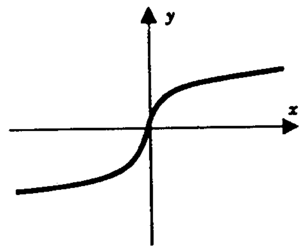
39. $\frac{d}{dx}(x^{1/3}+x^{-1/3}) = \frac{1}{3}x^{-2/3} - \frac{1}{3}x^{-4/3} = \frac{1}{3}x^{-4/3}(x^{2/3}-1)$

40. $\frac{d}{dx}\left(\sqrt{\frac{ax+b}{cx+d}}\right) = \frac{1}{2}\left(\frac{ax+b}{cx+d}\right)^{-1/2}\left(\frac{(cx+d)a-(ax+b)c}{(cx+d)^2}\right) = \frac{ad-bc}{2(cx+d)^2}\sqrt{\frac{cx+d}{ax+b}}$

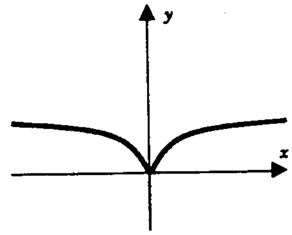
41. (a)



(b)



(c)



42. $y = (a^2 + x^2)^{1/2}; \quad \dfrac{dy}{dx} = \dfrac{1}{2}(a^2 + x^2)^{-1/2}(2x) = x(a^2 + x^2)^{-1/2};$

$$\frac{d^2y}{dx^2} = (a^2 + x^2)^{-1/2} - \frac{1}{2}x(a^2 + x^2)^{-3/2}(2x) = \frac{a^2}{(a^2 + x^2)^{3/2}}$$

43. $y = (a + bx)^{1/3}; \quad \dfrac{dy}{dx} = \dfrac{b}{3}(a + bx)^{-2/3}; \quad \dfrac{d^2y}{dx^2} = \dfrac{-2b^2}{9}(a + bx)^{-5/3}$

44. $y = x(a^2 - x^2)^{1/2}; \quad \dfrac{dy}{dx} = (a^2 - x^2)^{1/2} + \frac{1}{2}x(a^2 - x^2)^{-1/2}(-2x) = (a^2 - x^2)^{1/2} - x^2a^2 - x^2)^{-1/2}$

$$\frac{d^2y}{dx^2} = \tfrac{1}{2}(a^2 - x^2)^{-1/2}(-2x) - 2x(a^2 - x^2)^{-1/2} - x^2\left(-\tfrac{1}{2}\right)(a^2 - x^2)^{-3/2}(-2x) = \frac{x(2x^2 - 3a^2)}{(a^2 - x^2)^{3/2}}$$

45. $y = \sqrt{x}\,\tan\sqrt{x}; \quad \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}\tan\sqrt{x} + \sqrt{x}\,\sec^2\sqrt{x}\left(\dfrac{1}{2\sqrt{x}}\right) = \dfrac{1}{2\sqrt{x}}\tan\sqrt{x} + \dfrac{1}{2}\sec^2\sqrt{x}$

$$\frac{d^2y}{dx^2} = \frac{2\sqrt{x}\,\sec^2\sqrt{x}\,(1/2\sqrt{x}) - \tan\sqrt{x}(1/\sqrt{x})}{4x} + \sec\sqrt{x}\,\sec\sqrt{x}\,\tan\sqrt{x}(1/2\sqrt{x})$$

$$= \frac{\sqrt{x}\,\sec^2\sqrt{x} - \tan\sqrt{x} + 2x\,\sec^2\sqrt{x}\tan\sqrt{x}}{4x\sqrt{x}}$$

46. $y = \sqrt{x}\,\sin\sqrt{x}; \quad \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}\sin\sqrt{x} + \sqrt{x}\,\cos\sqrt{x}\left(\dfrac{1}{2\sqrt{x}}\right) = \dfrac{1}{2\sqrt{x}}\sin\sqrt{x} + \dfrac{1}{2}\cos\sqrt{x}$

$$\frac{d^2y}{dx^2} = \frac{2\sqrt{x}\cos\sqrt{x}(1/2\sqrt{x}) - \sin\sqrt{x}(1/\sqrt{x})}{4x} - \frac{1}{2}\sin\sqrt{x}(1/2\sqrt{x})$$

$$= \frac{\sqrt{x}\cos\sqrt{x} - \sin\sqrt{x}}{4x^{3/2}} - \frac{\sin\sqrt{x}}{4\sqrt{x}}$$

47. Differentiation of $x^2 + y^2 = r^2$ gives $2x + 2y\dfrac{dy}{dx} = 0$ so that the slope of the normal line is

$$\frac{-1}{dy/dx} = \frac{y}{x} \quad (x \neq 0).$$

Let (x_0, y_0) be a point on the circle. Clearly, if $x_0 = 0$, the normal line, $x = 0$, passes through the origin. If $x_0 \neq 0$, the normal line is

$$y - y_0 = \frac{y_0}{x_0}(x - x_0), \quad \text{which simplifies to} \quad y = \frac{y_0}{x_0}x,$$

a line through the origin.

48. $y^2 = x;\quad 2y\dfrac{dy}{dx} = 1;\quad \dfrac{dy}{dx} = \dfrac{1}{2y}$

When $x = a,\quad y = \pm\sqrt{a}$.

Slope of tangent at $(a, \sqrt{a})$: $\dfrac{1}{2\sqrt{a}}$

Equation of tangent line: $y - \sqrt{a} = \dfrac{1}{2\sqrt{a}}(x - a)$

x-intercept: $-\sqrt{a} = \dfrac{1}{2\sqrt{a}}(x - a) \implies x = -a$

Slope of tangent at $(a, -\sqrt{a})$: $-\dfrac{1}{2\sqrt{a}}$

Equation of tangent line: $y + \sqrt{a} = -\dfrac{1}{2\sqrt{a}}(x - a)$

x-intercept: $\sqrt{a} = -\dfrac{1}{2\sqrt{a}}(x - a) \implies x = -a$

49. For the parabola $y^2 = 2px + p^2$, we have $2y\dfrac{dy}{dx} = 2p$ and the slope of a tangent is given by $m_1 = p/y$.

For the parabola $y^2 = p^2 - 2px$, we obtain $m_2 = -p/y$ as the slope of a tangent. The parabolas intersect at the points $(0, \pm p)$. At each of these points $m_1 m_2 = -1$; the parabolas intersect at right angles.

50. For $y = 2x$, the slope is $m_1 = 2$. For $x^2 - xy + 2y^2 = 28$, we have

$$2x - y - x\frac{dy}{dx} + 4y\frac{dy}{dx} = 0 \quad \text{so} \quad \frac{dy}{dx} = m_2 = \frac{y - 2x}{4y - x}$$

At a point of intersection of the line and the curve, we have $m_2 = 0$ since $y = 2x$. Thus

$$\tan\alpha = |-m_1| = 2 \implies \alpha \cong 1.107(\text{radians}) \cong 63.4^\circ$$

51. For $y = x^2$ we have $m_1 = \dfrac{dy}{dx} = 2x$; for $x = y^3$ we have $3y^2\dfrac{dy}{dx} = 1$ or $m_2 = \dfrac{dy}{dx} = 1/3y^2$.

At $(1, 1)$, $m_1 = 2$, $m_2 = 1/3$ and

$$\tan\alpha = \left|\frac{m_1 - m_2}{1 - m_1 m_2}\right| = \left|\frac{2 - (1/3)}{1 + 2(1/3)}\right| = 1 \quad \Rightarrow \alpha = \frac{\pi}{4}$$

At $(0, 0)$, $m_1 = 0$ and m_2 is undefined. Thus $\alpha = \pi/2$.

52. $(x - 1)^2 + y^2 = 10;\quad 2(x - 1) + 2y\dfrac{dy}{dx} = 0,\quad m_1 = \dfrac{dy}{dx} = -\dfrac{x - 1}{y}$

$x^2 + (y - 2)^2 = 5;\quad 2x + 2(y - 2)\dfrac{dy}{dx} = 0,\quad m_2 = \dfrac{dy}{dx} = -\dfrac{x}{y - 2}$

The circles intersect at the points $(-2, 1)$ and $(2, 3)$.

At $(-2, 1)$: $m_1 = 3$, $m_2 = -2$ and $\tan\alpha = \left|\dfrac{3 + 2}{1 + (3)(-2)}\right| = 1$. Thus $\alpha = \dfrac{\pi}{4}$.

At $(2, 3)$: $m_1 = -1/3$, $m_2 = -2$ and $\tan\alpha = \left|\dfrac{(-1/3) + 2}{1 + (-1/3)(-2)}\right| = 1$. Thus $\alpha = \dfrac{\pi}{4}$.

53. The hyperbola and the ellipse intersect at the points $(\pm 3, \pm 2)$. For the hyperbola, $\dfrac{dy}{dx} = \dfrac{x}{y}$ and for the ellipse $\dfrac{dy}{dx} = -\dfrac{4x}{9y}$. The product of these slopes is $-\dfrac{4x^2}{9y^2}$. This product is -1 at each of the points of intersection. Therefore the hyperbola and ellipse are orthogonal.

54. The curves intersect at the points $(\pm 1, 1)$. For the ellipse, $\dfrac{dy}{dx} = -\dfrac{3x}{2y}$ and for $y^3 = x^2$ we have $\dfrac{dy}{dx} = -\dfrac{2x}{3y^2}$. The product of these slopes is $-\dfrac{6x^2}{6y^3} = -\dfrac{x^2}{y^3}$. This product is -1 at each of the points of intersection. Therefore the curves are orthogonal.

55. For the circles, $\dfrac{dy}{dx} = -\dfrac{x}{y}$, $y \neq 0$, and for the straight lines, $\dfrac{dy}{dx} = m = \dfrac{y}{x}$, $x \neq 0$. Since the product of the slopes is -1, it follows that the two families are orthogonal trajectories.

56. For the parabolas, $m_1 = \dfrac{dy}{dx} = \dfrac{1}{2ay}$, $y \neq 0$, and for the ellipses, $m_2 = \dfrac{dy}{dx} = -\dfrac{2x}{y}$, $y \neq 0$. Let (x_0, y_0) be a point of intersection of a parabola and an ellipse. Then

$$m_1 \cdot m_2 = \frac{1}{2ay_0} \cdot \left(-\frac{2x_0}{y_0}\right) = -\frac{x_0}{ay_0^2} = -1 \text{ since } x_0 = ay_0^2.$$

57. The line $x + 2y + 3 = 0$ has slope $m = -1/2$. Thus, a line perpendicular to this line will have slope 2. A tangent line to the ellipse $4x^2 + y^2 = 72$ has slope $m = \dfrac{dy}{dx} = -\dfrac{4x}{y}$. Setting $-\dfrac{4x}{y} = 2$ gives $y = -2x$. Substituting into the equation for the ellipse, we have

$$4x^2 + 4x^2 = 72 \quad \Rightarrow \quad 8x^2 = 72 \quad \Rightarrow \quad x = \pm 3$$

It now follows that $y = \mp 6$ and the equations of the tangents are:

at $(3, -6)$: $\quad y + 6 = 2(x - 3)$ or $y = 2x - 12$;

at $(-3, 6)$: $\quad y - 6 = 2(x + 3)$ or $y = 2x + 12$.

58. The line $2x + 5y - 4 = 0$ has slope $m = -2/5$. A tangent line to the hyperbola $4x^2 - y^2 = 36$ has slope $\dfrac{dy}{dx} = \dfrac{4x}{y}$. Therefore a normal line to the hyperbola will have slope $m = -\dfrac{y}{4x}$. Setting $-\dfrac{y}{4x} = -\dfrac{2}{5}$ gives $y = 8x/5$. Substituting this into the equation for the hyperbola, we have

$$4x^2 - \frac{64}{25}x^2 = 36 \quad \Longrightarrow \quad x = \pm 5$$

It now follows that $y = 8$ when $x = 5$ and $y = -8$ when $x = -5$. The equations of the normals are:

at $(5, 8)$: $\quad y - 8 = -\frac{2}{5}(x - 5)$ or $y = -\frac{2}{5}x + 10$;

at $(-5, -8)$: $\quad y + 8 = -\frac{2}{5}(x + 5)$ or $y = -\frac{2}{5}x - 10$.

59. Differentiate the equation $(x^2 + y^2)^2 = x^2 - y^2$ implicitly with respect to x:

$$2(x^2 + y^2)\left(2x + 2y\,\frac{dy}{dx}\right) = 2x - 2y\,\frac{dy}{dx}$$

Now set $dy/dx = 0$. This gives

$$2x(x^2+y^2) = x$$
$$x^2+y^2 = \frac{1}{2} \quad (x \neq 0)$$

Substituting this result into the original equation, we get

$$x^2 - y^2 = \frac{1}{4}$$

Now

$$\begin{array}{l} x^2+y^2 = 1/2 \\ x^2-y^2 = 1/4 \end{array} \Rightarrow x = \pm\frac{\sqrt{6}}{4}, \quad y = \pm\frac{\sqrt{2}}{4}$$

Thus, the points on the curve at which the tangent line is horizontal are:

$$(\sqrt{6}/4, \sqrt{2}/4), \quad (\sqrt{6}/4, -\sqrt{2}/4), \quad (-\sqrt{6}/4, \sqrt{2}/4), \quad (-\sqrt{6}/4, -\sqrt{2}/4).$$

60. (a) $x^{2/3} + y^{2/3} = a^{2/3}$; $\quad \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$, and $\dfrac{dy}{dx} = -\left(\dfrac{y}{x}\right)^{1/3}$

Thus, the slope at (x_1, y_1), $x_1 \neq 0$ is: $m = -\left(\dfrac{y_1}{x_1}\right)^{1/3}$

(b) $m = 0: \quad -\left(\dfrac{y_1}{x_1}\right)^{1/3} = 0 \quad \Longrightarrow \quad y_1 = 0$ and $x_1 = \pm a$; $\quad (a, 0),\ (-a, 0)$

$m = 1: \quad -\left(\dfrac{y_1}{x_1}\right)^{1/3} = 1 \quad \Longrightarrow \quad y_1 = -x_1$ and $x_1 = \pm\frac{1}{4}a\sqrt{2}$;

$$\left(\frac{1}{4}a\sqrt{2}, -\frac{1}{4}a\sqrt{2}\right), \ \left(-\frac{1}{4}a\sqrt{2}, \frac{1}{4}a\sqrt{2}\right)$$

$m = -1: \quad -\left(\dfrac{y_1}{x_1}\right)^{1/3} = -1 \quad \Longrightarrow \quad y_1 = x_1$ and $x_1 = \pm\frac{1}{4}a\sqrt{2}$;

$$\left(\frac{1}{4}a\sqrt{2}, \frac{1}{4}a\sqrt{2}\right), \ \left(-\frac{1}{4}a\sqrt{2}, -\frac{1}{4}a\sqrt{2}\right)$$

61. Differentiate the equation $x^{1/2} + y^{1/2} = c^{1/2}$ implicitly with respect to x:

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\frac{dy}{dx} = 0 \quad \text{which implies} \quad \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/2}$$

An equation for the tangent line to the graph at the point (x_0, y_0) is

$$y - y_0 = -\left(\frac{y_0}{x_0}\right)^{1/2}(x - x_0)$$

The x- and y-intercepts of this line are

$$a = (x_0y_0)^{1/2} + x_0 \quad \text{and} \quad b = (x_0y_0)^{1/2} + y_0 \quad \text{respectively.}$$

Now

$$a + b = 2(x_0y_0)^{1/2} + x_0 + y_0 = \left(x_0^{1/2} + y_0^{1/2}\right)^2 = c.$$

62. The circle has equation $x^2 + (y-a)^2 = 1$ and $2x + 2(y-a)\dfrac{dy}{dx} = 0$. Thus $\dfrac{dy}{dx} = -\dfrac{x}{y-a}$.

A tangent line to the parabola has slope $\dfrac{dy}{dx} = 4x$. Now

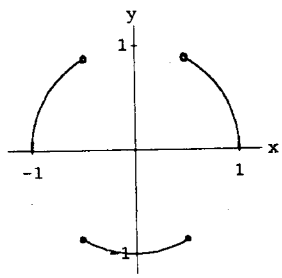
$$4x = -\frac{x}{y-a} \quad \Longrightarrow \quad x(4y-4a+1)=0 \quad \Longrightarrow \quad 4y-4a+1=0 \text{ since } x \neq 0$$

It follows that

$$y = a - \frac{1}{4} \quad \Longrightarrow \quad x = \pm\frac{\sqrt{15}}{4} \quad \Longrightarrow \quad y = \frac{15}{8}$$

Points of intersection: $\left(\pm\frac{\sqrt{15}}{4}, \frac{15}{8}\right)$

63. (a)



(b) $2x + 2y\frac{dy}{dx} = 0$ and $\frac{dy}{dx} = -\frac{x}{y}$

At $(-\sqrt{3}/2, 1/2)$, $\frac{dy}{dx} = \sqrt{3}$.

At $(\sqrt{3}/2, 1/2)$, $\frac{dy}{dx} = -\sqrt{3}$.

At $(0,-1)$, $\frac{dy}{dx} = 0$.

(c) $y = -\sqrt{1-x^2}$ for $-\frac{1}{2} \leq x \leq \frac{1}{2}$

$$y'_+(-1/2) = \lim_{h\to 0^+} \frac{y(-\frac{1}{2}+h) - y(-\frac{1}{2})}{h} = \lim_{h\to 0^+} \frac{-\sqrt{1-(-\frac{1}{2}+h)^2} + \frac{\sqrt{3}}{2}}{h}$$

$$= \lim_{h\to 0^+} \frac{-\sqrt{3+4h-4h^2}+\sqrt{3}}{2h} = -\frac{1}{\sqrt{3}}$$

$$y'_-(1/2) = \lim_{h\to 0^-} \frac{y(\frac{1}{2}+h) - y(\frac{1}{2})}{h} = \lim_{h\to 0^-} \frac{-\sqrt{1-(\frac{1}{2}+h)^2} - \frac{\sqrt{3}}{2}}{h}$$

$$= \lim_{h\to 0^-} \frac{-\sqrt{3-4h-4h^2}+\sqrt{3}}{2h} = \frac{1}{\sqrt{3}}$$

64. By numerical work, $L \cong 98.6$. Note that the given limit is $f'(1)$ where $f(x) = x^{98.6}$.

$$f'(x) = (98.6)x^{97.6}, \quad f'(1) = 98.6$$

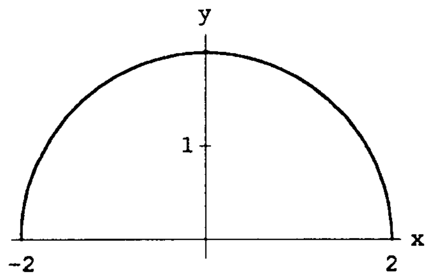
65. By numerical work, $f'(16) \cong 0.375$; from (3.7.1)

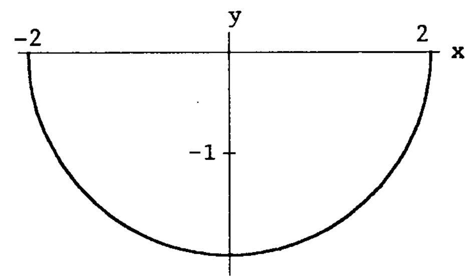
$$f'(x) = \frac{3}{4}x^{-1/4}, \quad \text{and} \quad f'(16) = \tfrac{3}{8} = 0.375.$$

66. By numerical work $L \cong 0.125$;

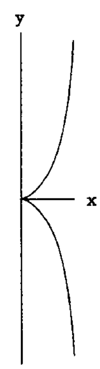
$$\lim_{x\to 32} \frac{x^{2/5}-4}{x^{4/5}-16} = \lim_{x\to 32} \frac{1}{x^{2/5}+4} = \frac{1}{(32)^{2/5}+4}\frac{1}{8} = 0.125$$

67. $x = t, \quad y = \sqrt{4 - t^2}$ $\qquad$ $x = t, \quad y = -\sqrt{4 - t^2}$

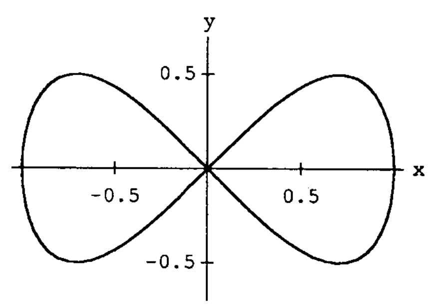




68. $(2 - x)y^2 = x^3$



69. (a) The graph of $x^4 = x^2 - y^2$ is:



(b) Differentiate the equation $x^4 = x^2 - y^2$ implicitly with respect to x :

$$4x^3 = 2x - 2y\,\frac{dy}{dx}$$

Now set $dy/dx = 0$. This gives $4x^3 = 2x$ which implies $x = \pm\frac{\sqrt{2}}{2}$.

SECTION 3.8

1. $x + 2y = 2, \quad \dfrac{dx}{dt} + 2\dfrac{dy}{dt} = 0$

(a) If $\dfrac{dx}{dt} = 4$, then $\dfrac{dy}{dt} = -2$ units/sec. $\qquad$ (b) If $\dfrac{dy}{dt} = -2$, then $\dfrac{dx}{dt} = 4$ units/sec.

2. $x^2 + y^2 = 25, \quad 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$ and $\dfrac{dx}{dt} = -\dfrac{y}{x}\dfrac{dy}{dt}$.

At the point $(3, 4)$, $\dfrac{dy}{dt} = -2$. Therefore, $\dfrac{dx}{dt} = \dfrac{8}{3}$; the x-coordinate is increasing at the rate of 8/3 units per second.

3. $y^2 = 4(x+2), \quad 2y\dfrac{dy}{dt} = 4\dfrac{dx}{dt} \quad$ and $\quad \dfrac{dx}{dt} = \frac{1}{2}y\dfrac{dy}{dt}$

At the point $(7,6)$, $\quad \dfrac{dy}{dt} = 3$. Therefore $\quad \dfrac{dx}{dt} = \frac{1}{2}\cdot 6\cdot 3 = 9$ units/sec.

4. We are given $\dfrac{dx}{dt} = 2$. Also $4y = (x+2)^2$ so $4\dfrac{dy}{dt} = 2(x+2)\dfrac{dx}{dt}$ or $\dfrac{dy}{dt} = \frac{1}{2}(x+2)\dfrac{dx}{dt}$.

At $x = 2$, $\dfrac{dy}{dt} = 4$. The distance from a point on the parabola to the point $(-2,0)$ is given by

$S = \sqrt{(x+2)^2 + y^2} = \sqrt{4y + y^2}$ since $(x+2)^2 = 4y$. Now

$$\frac{dS}{dt} = \frac{1}{2}(4y+y^2)^{-1/2}(4+4y)\frac{dy}{dt} = \frac{2+y}{\sqrt{4y+y^2}}\frac{dy}{dt}.$$

Therefore, at the point $(2,4)$, $\quad \dfrac{dS}{dt} = \dfrac{6}{\sqrt{32}}4 = 3\sqrt{2}.$

5. Let $s = \sqrt{x^2+y^2}$ denote the distance to the origin at time t. Since $x = 4\cos t$ and $y = 2\sin t$, we have

$$s(t) = \sqrt{16\cos^2 t + 4\sin^2 t} = \sqrt{12\cos^2 t + 4}$$

$$\frac{ds}{dt} = \frac{1}{2}(12\cos^2 t + 4)^{-1/2}(-24\cos t\sin t)$$

$$= \frac{-12\cos t\sin t}{\sqrt{12\cos^2 t + 4}}$$

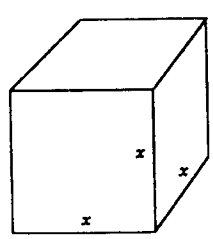
At $t = \pi/4$, $\quad \dfrac{ds}{dt} = \dfrac{-12\cos(\pi/4)\sin(\pi/4)}{\sqrt{12\cos^2(\pi/4) + 4}} = -\frac{3}{5}\sqrt{10}.$

6. $y = x\sqrt{x} = x^{3/2}; \quad \dfrac{dy}{dt} = \frac{3}{2}x^{1/2}\dfrac{dx}{dt}.$

Now $\dfrac{dx}{dt} = \dfrac{dy}{dt} = z \neq 0, \quad \Longrightarrow \quad \frac{3}{2}x^{1/2} = 1 \quad \Longrightarrow \quad x = \frac{4}{9}$ and $y = \frac{8}{27}$.

Both coordinates are changing at the same rate at the point $(4/9, 8/27)$.

7.



Find $\dfrac{dx}{dt}$ and $\dfrac{dS}{dt}$ when $\quad V = 27\text{m}^3$

given that $\quad \dfrac{dV}{dt} = -2\text{m}^3/\text{min}.$

$(*) \quad V = x^3, \quad S = 6x^2$

Differentiation of equations $(*)$ gives

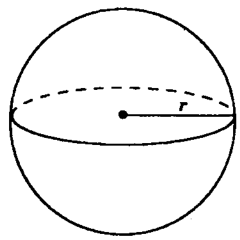
$$\frac{dV}{dt} = 3x^2\frac{dx}{dt} \quad \text{and} \quad \frac{dS}{dt} = 12x\frac{dx}{dt}.$$

When $V = 27$, $x = 3$. Substituting $x = 3$ and $dV/dt = -2$, we get

$$-2 = 27\frac{dx}{dt} \quad \text{so that} \quad \frac{dx}{dt} = -2/27 \quad \text{and} \quad \frac{dS}{dt} = 12(3)\left(\frac{-2}{27}\right) = -8/3.$$

The rate of change of an edge is $-2/27$ m/min; the rate of change of the surface area is $-8/3$ m^2/min.

8.



Find $\dfrac{dr}{dt}$ and $\dfrac{dS}{dt}$ when $r = 10$ ft given that $\dfrac{dV}{dt} = 8$ ft^3/min.

(∗) $V = \frac{4}{3}\pi r^3, \quad S = 4\pi r^2$

Differentiation of equations (∗) with respect to t gives

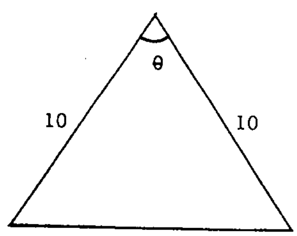
$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} \quad \text{and} \quad \frac{dS}{dt} = 8\pi r\frac{dr}{dt}.$$

Substituting $r = 10$ and $dV/dt = 8$, we get

$$8 = 4\pi(10)^2\frac{dr}{dt} \quad \text{so that} \quad \frac{dr}{dt} = \frac{1}{50\pi} \quad \text{and} \quad \frac{dS}{dt} = 8\pi(10)\frac{1}{50\pi} = \frac{8}{5}.$$

The radius is increasing $\dfrac{1}{50\pi}$ ft/min; the surface area is increasing $\dfrac{8}{5}$ ft^2/min.

9.



(a) $A = \frac{1}{2}\cdot 10\cdot 10\cdot \sin\theta = 50\sin\theta$ (see the figure)

(b) $\dfrac{d\theta}{dt} = 10° = \frac{10}{360}(2\pi) = \dfrac{\pi}{18}$ radians

$\dfrac{dA}{dt} = 50\cos\theta\,\dfrac{d\theta}{dt}$

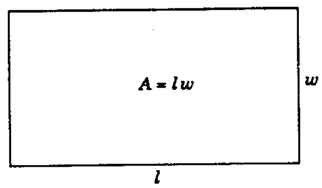
At the instant $\theta = 60° = \pi/3$ radians, $\dfrac{dA}{dt} = 50\cos(\pi/3)\,\dfrac{\pi}{18} \cong 4.36$ cm^2/min

(c) $\dfrac{dA}{d\theta} = 50\cos\theta = 0 \quad \Rightarrow \theta = \pi/2;$ the triangle has maximum area when $\theta = \pi/2$.

10. The area of an equilateral triangle of side x is given by

$$A = \frac{1}{2}x\left(\frac{x\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}x^2. \quad \text{Thus} \quad \frac{dA}{dt} = \frac{\sqrt{3}}{2}x\frac{dx}{dt}.$$

When $x = \alpha$, $\dfrac{dx}{dt} = k$, and $\dfrac{dA}{dt} = \frac{\sqrt{3}}{2}\alpha k$cm^2/min.

11. 

We will find the values of l for which $\dfrac{dA}{dt} < 0$ given that $\dfrac{dl}{dt} = 1$ cm/sec and $P = 2(l + w) = 24$.

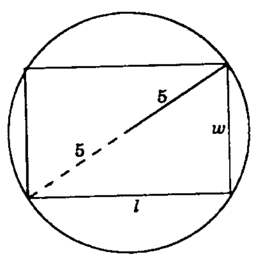
We combine $A = lw$ and $l + w = 12$ to write $A = 12l - l^2$. Differentiating with respect to t, we have

$$\frac{dA}{dt} = 12\frac{dl}{dt} - 2l\frac{dl}{dt} = 2(6 - l)\frac{dl}{dt}.$$

Since $\dfrac{dl}{dt} = 1$, $\dfrac{dA}{dt} < 0$ for $l > 6$. The area of the rectangle starts to decrease when the length is 6 cm.

12. We have $2x + 2y = 24$, or $x + y = 12$. Thus, $A = xy = x(12 - x) = 12x - x^2$.

When $A = 32$, $x = 4$ or $x = 8$, and it follows that $\dfrac{dA}{dt} = (12 - 2x)\dfrac{dx}{dt} = \pm 4\dfrac{dx}{dt}$.

13. 

Find $\dfrac{dA}{dt}$ when $l = 6$ in. given that $\dfrac{dl}{dt} = -2$ in./sec.

By the Pythagorean theorem

$$l^2 + w^2 = 100.$$

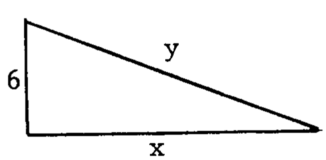
Also, $A = lw$. Thus, $A = l\sqrt{100 - l^2}$. Differentiation with respect to t gives

$$\frac{dA}{dt} = l\left(\frac{-l}{\sqrt{100 - l^2}}\right)\frac{dl}{dt} + \sqrt{100 - l^2}\,\frac{dl}{dt}.$$

Substituting $l = 6$ and $dl/dt = -2$, we get

$$\frac{dA}{dt} = 6\left(\frac{-6}{8}\right)(-2) + (8)(-2) = -7.$$

The area is decreasing at the rate of 7 in.^{2}/sec.

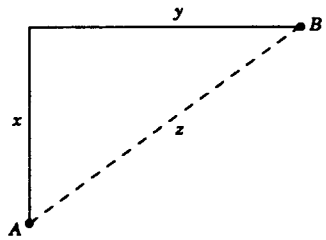
14. 

$x^2 + 6^2 = y^2$; $2x\dfrac{dx}{dt} = 2y\dfrac{dy}{dt}$

If $y = 30$ ft and $\dfrac{dx}{dt} = 8$ ft/min, then

$$\frac{dy}{dt} = \frac{x}{y}\frac{dy}{dt} = \frac{\sqrt{(30)^2 - 36}}{30}\,8 = \tfrac{16}{5}\sqrt{6}\,\text{ft/min}$$

15.



Compare $\frac{dy}{dt}$ to $\frac{dx}{dt} = -13$ mph given that $z = 16$ and $\frac{dz}{dt} = -17$ when $x = y$.

By the Pythagorean theorem $x^2 + y^2 = z^2$. Thus,

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}.$$

Since $x = y$ when $z = 16$, we have $x = y = 8\sqrt{2}$ and

$$2(8\sqrt{2})(-13) + 2(8\sqrt{2})\frac{dy}{dt} = 2(16)(-17).$$

Solving for dy/dt, we get

$$-13\sqrt{2} + \sqrt{2}\frac{dy}{dt} = -34 \quad \text{or} \quad \frac{dy}{dt} = \frac{1}{\sqrt{2}}(13\sqrt{2} - 34) \cong -11.$$

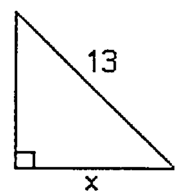
Thus, boat A wins the race.

16. $V = \frac{4}{3}\pi r^3$, so $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 4\pi r^2(-\frac{1}{5})$. Thus at $r = 12$ we have $\frac{dV}{dt} = -\frac{576}{5}\pi$ cm^3/min.

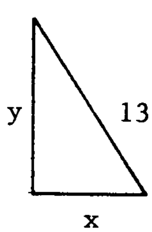
17. We want to find $\frac{dA}{dt}$ when $\frac{dx}{dt} = 2$ and $x = 12$.

$A = \frac{1}{2}x\sqrt{169 - x^2}$, so $\frac{dA}{dt} = \left[\frac{1}{2}\sqrt{169 - x^2} - \frac{x^2}{2\sqrt{169 - x^2}}\right]\frac{dx}{dt}$

When $\frac{dx}{dt} = 2$ and $x = 12$, $\frac{dA}{dt} = -\frac{119}{5} ft^2$/sec.



18.



$x^2 + y^2 = (13)^2$; $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ and $\frac{dy}{dt} = -\frac{x}{2y}$ since $\frac{dx}{dt} = 0.5$. When $x = 5$, $y = 12$ and $\frac{dy}{dt} = -\frac{5}{24}$; the top of the ladder is dropping 5/24 ft/sec.

19. We want to find dV/dt when $V = 1000$ ft^3 and $P = 5$ lb/in.^{2} given that $dP/dt = -0.05$ lb/in.^{2}/hr. Differentiating $PV = C$ with respect to t, we get

$$P\frac{dV}{dt} + V\frac{dP}{dt} = 0 \quad \text{so that} \quad 5\frac{dV}{dt} + 1000(-0.05) = 0. \quad \text{Thus, } \frac{dV}{dt} = 10.$$

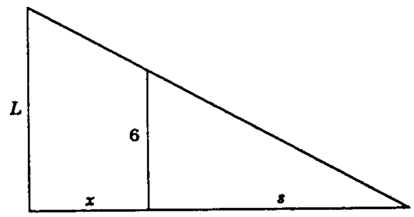
The volume increases at the rate of 10 ft^3/hr.

20. $PV^{1.4} = C$; $V^{1.4}\frac{dP}{dt} + (1.4)PV^{0.4}\frac{dV}{dt} = 0$ and $\frac{dP}{dt} = -\frac{1.4P}{V}\frac{dV}{dt}$.

With $V = 10,\ P = 50$ and $\dfrac{dV}{dt} = -1$, we have $\dfrac{dP}{dt} = -\dfrac{(1.4)50}{10}(-1) = 7$

The pressure is increasing 7lb/in^2/sec.

21.



Find $\dfrac{ds}{dt}$ when $x = 3$ ft (and $s = 4$ ft)

given that $\dfrac{dx}{dt} = 400$ ft/min.

By similar triangles

$$\frac{L}{x+s} = \frac{6}{s}.$$

Substitution of $x = 3$ and $s = 4$ gives us $\dfrac{L}{7} = \dfrac{6}{4}$ so that the lamp post is

$L = 10.5$ ft tall. Rewriting

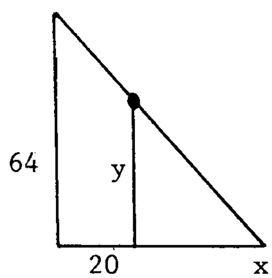
$$\frac{10.5}{x+s} = \frac{6}{s} \quad \text{as} \quad s = \frac{4}{3}x$$

and differentiating with respect to t, we find that

$$\frac{ds}{dt} = \frac{4}{3}\frac{dx}{dt} = \frac{1600}{3}.$$

The shadow lengthens at the rate of 1600/3 ft/min.

22.



$y(t) = -16t^2;\quad \dfrac{dy}{dt} = -32t$

By similar triangles, $\dfrac{y}{x} = \dfrac{64}{20+x}$.

Thus, $y = \dfrac{64x}{20+x}$.

Now, $\dfrac{dy}{dt} = \dfrac{(20+x)(64) - 64x}{(20+x)^2}\dfrac{dx}{dt} = \dfrac{1280}{(20+x)^2} \quad \Longrightarrow \quad \dfrac{dx}{dt} = \dfrac{(20+x)^2}{1280}\dfrac{dy}{dt}$.

At $t = 1$, $y = 48$, $x = 60$, and $\dfrac{dy}{dt} = -32$. Therefore, $\dfrac{dx}{dt} = \dfrac{(80)^2}{1280}(-32) = -160$ ft/sec.

23. Let $W(t) = 150\left(1 + \frac{1}{4000}r\right)^{-2}$. We want to find dW/dt when $r = 400$ given that

$dr/dt = 10$ mi/sec. Differentiating with respect to t, we get

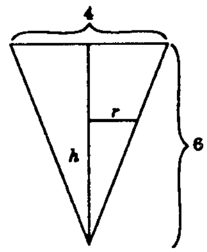
$$\frac{dW}{dt} = -300\left(1 + \frac{1}{4000}r\right)^{-3}\left(\frac{1}{4000}\right)\frac{dr}{dt}$$

Now set $r = 400$ and $dr/dt = 10$. Then

$$\frac{dW}{dt} = -300\left(1 + \frac{400}{4000}\right)^{-3}\frac{10}{4000} \cong 0.5634\text{lbs/sec}$$

24. $M = \dfrac{m}{\sqrt{1 - v^2/c^2}};\quad \dfrac{dM}{dt} = m\left(-\frac{1}{2}\right)\left(1 - v^2/c^2\right)^{-3/2}\left(-2v/c^2\right)\dfrac{dv}{dt} = \dfrac{mv}{c^2\left(1 - v^2/c^2\right)^{3/2}}\dfrac{dv}{dt}.$

If $v=\dfrac{c}{2}$ and $\dfrac{dv}{dt}=\dfrac{c}{100}$, then $\dfrac{dM}{dt}=\dfrac{m(c/2)}{c^2\left(1-c^2/4c^2\right)}\dfrac{c}{100}=\dfrac{\sqrt{3}}{225}m.$

25. 

Find $\dfrac{dh}{dt}$ when $h=3$ in.

given that $\dfrac{dV}{dt}=-\dfrac{1}{2}$ cu in./min.

By similar triangles

$$r=\tfrac{1}{3}h.$$

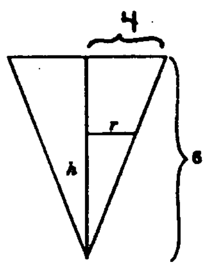
Thus $V=\frac{1}{3}\pi r^2h=\frac{1}{27}\pi h^3$. Differentiating with respect to t , we get

$$\frac{dV}{dt}=\frac{1}{9}\pi h^2\frac{dh}{dt}.$$

When $h=3$,

$$-\frac{1}{2}=\frac{1}{9}\pi(9)\frac{dh}{dt}\quad\text{and}\quad\frac{dh}{dt}=-\frac{1}{2\pi}.$$

The water level is dropping at the rate of $1/2\pi$ inches per minute.

26. 

$V=\dfrac{1}{3}\pi r^2h$ and $\dfrac{r}{h}=\dfrac{4}{6}$ (similar triangles)

so $V=\dfrac{4}{27}\pi h^3$. Thus $\dfrac{dV}{dt}=\dfrac{4}{9}\pi h^2\dfrac{dh}{dt}$,

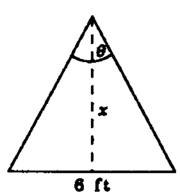
so at $\dfrac{dh}{dt}=0.5$ and $h=2$,

$\dfrac{dV}{dt}=\dfrac{8\pi}{9}$ cubic ft per sec.

27. $\dfrac{dV}{dt}=4\pi r^2\dfrac{dr}{dt}$ *and* $\dfrac{dSA}{dt}=8\pi r\dfrac{dr}{dt}$. Thus when $\dfrac{dSA}{dt}=4$ and $\dfrac{dr}{dt}=0.1$ we get $r=5\pi$ and $\dfrac{dV}{dt}=10\pi^3$ cubic cm/min.

28. $V=\pi rh^2-\frac{1}{3}\pi h^3$; $\dfrac{dV}{dt}=2\pi rh\dfrac{dh}{dt}-\pi h^2\dfrac{dh}{dt}$, and $\dfrac{dh}{dt}=\dfrac{2}{\pi(14h-h^2)}$ since $r=7$and $\dfrac{dV}{dt}=2$.

(a) When $h=7/2$, $\dfrac{dh}{dt}=\dfrac{8}{147\pi}$ in./sec. (b) When $h=7$, $\dfrac{dh}{dt}=\dfrac{2}{49\pi}$ in./sec.

29. 

Find $\dfrac{d\theta}{dt}$ when $x=4$ ft

given that $\dfrac{dx}{dt}=2$in./min.

$(*)\qquad \tan\dfrac{\theta}{2}=\dfrac{3}{x}$

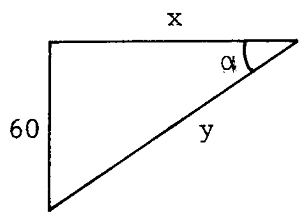
Differentiation of $(*)$ with respect to t gives

$$\frac{1}{2}\sec^2\frac{\theta}{2}\frac{d\theta}{dt}=-\frac{3}{x^2}\frac{dx}{dt}\quad\text{or}\quad\frac{d\theta}{dt}=-\frac{6}{x^2}\cos^2\frac{\theta}{2}\frac{dx}{dt}.$$

Note that $dx/dt = 2$ in./min$=1/6$ ft/min. When $x = 4$, we have $\cos\theta/2 = 4/5$ and thus

$$\frac{d\theta}{dt} = -\frac{6}{16}\left(\frac{4}{5}\right)^2\left(\frac{1}{6}\right) = -\frac{1}{25}.$$

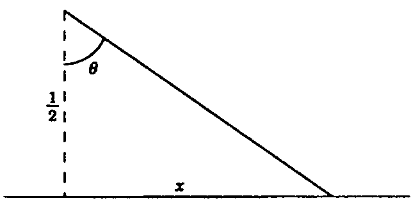
The vertex angle decreases at the rate of 0.04 rad/min.

30. 

$\tan\alpha = \dfrac{60}{x};\quad \sec^2\alpha\dfrac{d\alpha}{dt} = -\dfrac{60}{x^2}\dfrac{dx}{dt}$

Now, $\sec^2\alpha = \left(\dfrac{y}{x}\right)^2$ so $\dfrac{d\alpha}{dt} = -\dfrac{60}{y^2}\dfrac{dx}{dt}$.

With $y = 100$, and $\dfrac{dx}{dt} = -10$, we have $\dfrac{d\alpha}{dt} = -\dfrac{60}{(100)^2}(-10) = 0.06$ rad/min.

31. 

Find $\dfrac{dx}{dt}$ when $x = 1$ mi

given that $\dfrac{d\theta}{dt} = 2\pi$ rad/min.

$(*)\qquad \tan\theta = \dfrac{x}{1/2} = 2x$

Differentiation of $(*)$ with respect to t gives

$$\sec^2\theta\,\frac{d\theta}{dt} = 2\frac{dx}{dt}.$$

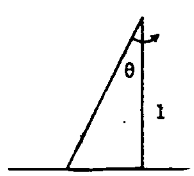
When $x = 1$, we get $\sec\theta = \sqrt{5}$ and thus $\dfrac{dx}{dt} = 5\pi$.

The light is traveling at 5π mi/min.

32. (a) We have $\tan\theta = x$, so $\sec^2\theta\dfrac{d\theta}{dt} = \dfrac{dx}{dt}$. Switching to radians, $\dfrac{d\theta}{dt} = 4\pi$.

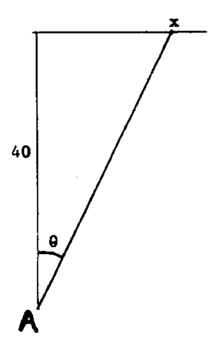
Thus at $\theta = \dfrac{\pi}{4}$, $\dfrac{dx}{dt} = 8\pi$ mi/min.

(b) At $\theta = 0$, $\dfrac{dx}{dt} = 4\pi$ mi/min.



33. We have $\tan\theta = \dfrac{x}{40}$, so $\sec^2\theta\dfrac{d\theta}{dt} = \dfrac{1}{40}\dfrac{dx}{dt}$, and $\dfrac{dx}{dt} = 4$.

At $t = 15, x = 60$ and $\sec\theta = \dfrac{\sqrt{5200}}{40}$, so $\dfrac{d\theta}{dt} = \dfrac{2}{65}$ rad/sec.

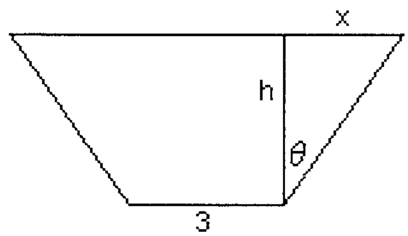


34. We have $V = h\pi r^2$, so $V = 500\pi$ cm^3. Thus $h = \frac{500}{r^2}$, and $\frac{dh}{dt} = \frac{-1000}{r^3}\frac{dr}{dt}$.

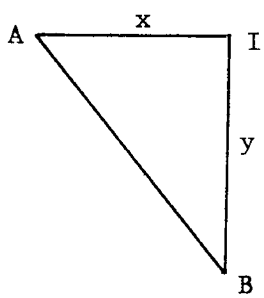
At $r = 5$ and $\frac{dr}{dt} = \frac{1}{2}$, we have $\frac{dh}{dt} = -4$ cm/sec.

35. We have $\sin\theta = \frac{4}{5}$ so $\tan\theta = \frac{4}{3} = \frac{x}{h}$. Thus $x = \frac{4}{3}h$. $V = 12\left(\frac{3+2x+3}{2}h\right) = 36h + 16h^2$.

Thus $\frac{dv}{dt} = (36+32h)\frac{dh}{dt}$, so at $\frac{dV}{dt} = 10$ and $h = 2$, $\frac{dh}{dt} = \frac{1}{10}$ ft/min.



36.



$\tan\alpha = \frac{y}{x}$; $\sec^2\alpha\frac{d\alpha}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}$

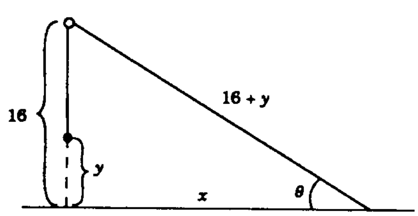
$$\frac{d\alpha}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}\cdot\frac{x^2}{x^2+y^2}$$
$$= \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2+y^2}$$

Now $\frac{dx}{dt} = -30$ mph $= -44$ ft/sec and $\frac{dy}{dt} = -22.5$ mph $= -33$ ft/sec.

At $x = 300$, $y = 400$, we have

$$\frac{d\alpha}{dt} = \frac{300(-33) - 400(-44)}{(300)^2 + (400)^2} = 0.0308 \text{ the angle is increasing } 0.0308 \text{ rad/sec.}$$

37.



Find $\frac{d\theta}{dt}$ when $y = 4$ ft

given that $\frac{dx}{dt} = 3$ ft/sec.

$\tan\theta = \frac{16}{x}$, $x^2 + (16)^2 = (16+y)^2$

Differentiating $\tan\theta = 16/x$ with respect to t, we obtain

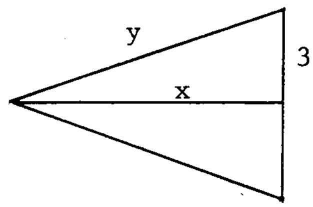
$$\sec^2\theta\frac{d\theta}{dt} = \frac{-16}{x^2}\frac{dx}{dt} \quad \text{and thus} \quad \frac{d\theta}{dt} = \frac{-16}{x^2}\cos^2\theta\frac{dx}{dt}.$$

From $x^2 + (16)^2 = (16 + y)^2$ we conclude that $x = 12$, when $y = 4$. Thus

$$\cos\theta = \frac{x}{16+y} = \frac{12}{20} = \frac{3}{5} \quad \text{and} \quad \frac{d\theta}{dt} = \frac{-16}{(12)^2}\left(\frac{3}{5}\right)^2(3) = \frac{-3}{25}.$$

The angle decreases at the rate of 0.12 rad/sec.

38.



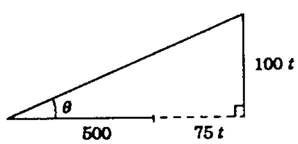
$$\tan(\alpha/2) = \frac{3}{x}; \quad \sec^2(\alpha/2)\tfrac{1}{2}\frac{d\alpha}{dt} = -\frac{3}{x^2}\frac{dx}{dt},$$

and $\dfrac{d\alpha}{dt} = -\dfrac{6}{y^2}\dfrac{dx}{dt}$

Initially, $y = 5$ so $x = 4$. 8 seconds later $x = 12$ so $y = \sqrt{(12)^2 + (3)^2} = \sqrt{153}$. Therefore,

$$\frac{d\alpha}{dt} = -\frac{6}{y^2}\frac{dx}{dt} = -\frac{6}{153}(1) = -\frac{6}{153}; \text{ the angle is decreasing } -6/153 \text{ rad/sec.}$$

39.



Find $\dfrac{d\theta}{dt}$ when $t = 6$ min.

$$\tan\theta = \frac{100t}{500+75t} = \frac{4t}{20+3t}$$

Differentiation with respect to t gives

$$\sec^2\theta\frac{d\theta}{dt} = \frac{(20+3t)4 - 4t(3)}{(20+3t)^2} = \frac{80}{(20+3t)^2}.$$

When $t = 6$

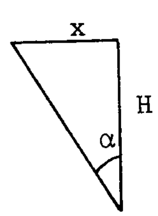
$$\tan\theta = \tfrac{24}{38} = \tfrac{12}{19} \quad \text{and} \quad \sec^2\theta = 1 + \left(\tfrac{12}{19}\right)^2 = \tfrac{505}{361}$$

so that

$$\frac{d\theta}{dt} = \frac{80}{(20+3t)^2}\cdot\frac{1}{\sec^2\theta} = \frac{80}{(38)^2}\cdot\frac{361}{505} = \frac{4}{101}.$$

The angle increases at the rate of 4/101 rad/min.

40.



$$\tan\alpha = \frac{x}{H}; \quad \sec^2\alpha\frac{d\alpha}{dt} = \frac{1}{H}\frac{dx}{dt}$$

$$\frac{d\alpha}{dt} = \frac{1}{H}\frac{dx}{dt}\cos^2\alpha = \frac{H}{H^2+x^2}\frac{dx}{dt}$$

We have $H = 2$ mi. and $\dfrac{dx}{dt} = 400$ mph. After 2 seconds, $x = 400\left(\dfrac{2}{3600}\right) = \dfrac{2}{9}$ miles.

$$\frac{d\alpha}{dt} = \frac{2}{2^2 + (2/9)^2}(400) = \frac{200(81)}{82} \text{ rad/hr } = \frac{9}{164} \text{ rad/sec.}$$

PROJECT 3.8

1. length of arc $= r\theta$, speed $= \dfrac{d}{dt}[r\theta] = r\dfrac{d\theta}{dt} = r\omega$

2. $v = r\omega$ so $KE = \frac{1}{2}mr^2\omega^2$.

3. We know that $d\theta/dt = \omega$ and, at time t, $\theta = \theta_0$. Therefore $\theta = \omega t + \theta_0$. It follows that

$$x(t) = r\cos(\omega t + \theta_0) \quad \text{and} \quad y(t) = r\sin(\omega t + \theta_0).$$

$x(t) = r\,\cos(\omega t + \theta_0), \quad y(t) = r\,\sin(\omega t + \theta_0)$

$v(t) = x'(t) = -r\omega\,\sin(\omega t + \theta_0) = -\omega\, y(t)$

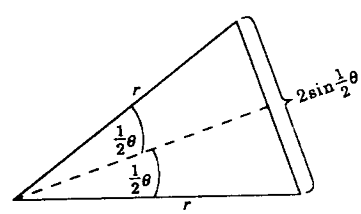
$a(t) = -r\omega^2\,\cos(\omega t + \theta_0) = -\omega^2\, x(t)$

$v(t) = y'(t) = r\omega\,\cos(\omega t + \theta_0) = \omega\, x(t)$

$a(t) = -r\omega^2\,\sin(\omega t + \theta_0) = -\omega^2\, y(t)$

4. For the sector $A = \dfrac{1}{2}r^2\theta, \quad \dfrac{dA}{dt} = \dfrac{1}{2}r^2\dfrac{d\theta}{dt} = \dfrac{1}{2}r^2\omega$ is constant.

For triangle T



$$A = \tfrac{1}{2}(2r\sin\tfrac{1}{2}\theta)(r\cos\tfrac{1}{2}\theta)$$

$$= \tfrac{1}{2}r^2(2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta) = \tfrac{1}{2}r^2\sin\theta,$$

$$\frac{dA}{dt} = \frac{1}{2}r^2\cos\theta\,\frac{d\theta}{dt} = \frac{1}{2}r^2\omega\cos\theta \quad \text{varies with } \theta.$$

For segment S

$$A = \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2\sin\theta = \tfrac{1}{2}r^2(\theta - \sin\theta),$$

$$\frac{dA}{dt} = \frac{1}{2}r^2\left(\frac{d\theta}{dt} - \cos\,\frac{d\theta}{dt}\right) = \frac{1}{2}r^2\omega(1 - \cos\theta) \quad \text{varies with } \theta.$$

5. From Exercise 4, $\dfrac{dA_T}{dt} = \frac{1}{2}r^2\omega\cos\theta$ and $\dfrac{dA_S}{dt} = \frac{1}{2}r^2\omega - \frac{1}{2}r^2\omega\cos\theta$

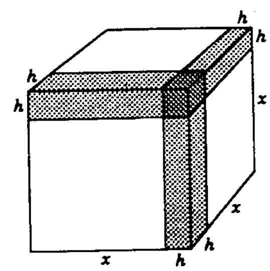
Now,

$$\frac{1}{2}r^2\omega\cos\theta = \frac{1}{2}r^2\omega - \frac{1}{2}r^2\omega\cos\theta \implies \cos\theta = \frac{1}{2} \implies \theta = \frac{\pi}{3}.$$

SECTION 3.9

1.

$$\begin{aligned} \Delta V &= (x+h)^3 - x^3 \\ &= (x^3 + 3x^2h + 3xh^2 + h^3) - x^3 \\ &= 3x^2h + 3xh^2 + h^3, \\ dV &= 3x^2h, \\ \Delta V - dV &= 3xh^2 + h^3 \qquad \text{(see figure)} \end{aligned}$$



2. The area of the ring can be thought of as the increase of the area of a disk as the radius increases from r to $r+h$.

$$A = \pi r^2; \quad \text{so} \quad dA = 2\pi r\,dr = 2\pi rh$$

The exact area is: $pi(r+h)^2 - \pi r^2 = \pi(r^2 + 2rh + h^2) - \pi r^2 = 2\pi rh + \pi h^2$.

3. $f(x) = x^{1/3}, \quad x = 1000, \quad h = 10, \quad f'(x) = \frac{1}{3}x^{-2/3}$

$\sqrt[3]{1010} = f(x+h) \cong f(x) + hf'(x) = \sqrt[3]{1000} + 10\left[\frac{1}{3}(1000\right]^{-2/3}) = 10\frac{1}{30}$

4. $f(x) = \sqrt{x}, \quad x = 121, \quad h = 4, \quad f'(x) = \dfrac{1}{2\sqrt{x}}$

$\sqrt{125} = f(x+h) \cong f(x) + hf'(x) = \sqrt{121} + 4\dfrac{1}{2\sqrt{121}} = 11 + \dfrac{2}{11} = \dfrac{123}{11} \cong 11.1888$

5. $f(x) = x^{1/4}, \quad x = 16, \quad h = -1, \quad f'(x) = \frac{1}{4}x^{-3/4}$

$(15)^{1/4} = f(x+h) \cong f(x) + hf'(x) = (16)^{1/4} + (-1)\left[\frac{1}{4}(16)^{-3/4}\right] = 1\frac{31}{32}$

6. $f(x) = \dfrac{1}{\sqrt{x}}, \quad x = 25, \quad h = -1, \quad f'(x) = \dfrac{-1}{2x^{3/2}}$

$\dfrac{1}{\sqrt{24}} = f(x+h) \cong f(x) + hf'(x) = \dfrac{1}{5} + (-1)\dfrac{-1}{2(5)^3} = \dfrac{51}{250} = 0.204$

7. $f(x) = x^{1/5}, \quad x = 32, \quad h = -2, \quad f'(x) = \frac{1}{5}x^{-4/5}$

$(30)^{1/5} = f(x+h) \cong f(x) + hf'(x) = (32)^{1/5} + (-2)\left[\frac{1}{5}(32)^{-4/5}\right] = 1.975$

8. $f(x) = x^{2/3}, \quad x = 27, \quad h = -1, \quad f'(x) = \frac{2}{3}x^{-1/3}$

$(26)^{2/3} = f(x+h) \cong f(x) + hf'(x) = (27)^{2/3} + (-1)\left[\frac{2}{3}(27)^{-1/3}\right] = \dfrac{79}{9} \cong 8.778$

9. $f(x) = x^{3/5}, \quad x = 32, \quad h = 1, \quad f'(x) = \frac{3}{5}x^{-2/5}$

$(33)^{3/5} = f(x+h) \cong f(x) + hf'(x) = (32)^{3/5} + (1)\left[\frac{3}{5}(32)^{-2/5}\right] = 8.15$

10. $f(x) = x^{-1/5}, \quad x = 32, \quad h = 1, \quad f'(x) = -\frac{1}{5}x^{-6/5}$

$(33)^{-1/5} = f(x+h) \cong f(x) + hf'(x) = (32)^{-1/5} - (1)\left[\frac{1}{5}(32)^{-6/5}\right] = \dfrac{159}{320} \cong 0.497$

11. $f(x) = \sin x, \quad x = \dfrac{\pi}{4}, \quad h = \dfrac{\pi}{180}, \quad f'(x) = \cos x$

$\sin 46° = f(x+h) \cong f(x) + hf'(x) = \sin\dfrac{\pi}{4} + \dfrac{\pi}{180}\cos\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}\left(1 + \dfrac{\pi}{180}\right) \cong 0.719$

12. $f(x) = \cos x, \quad x = \dfrac{\pi}{3}, \quad h = \dfrac{\pi}{90}, \quad f'(x) = -\sin x$

$\cos 62° = f(x+h) \cong f(x) + hf'(x) = \cos\dfrac{\pi}{3} + \dfrac{\pi}{90}\left(-\sin\dfrac{\pi}{3}\right) = \dfrac{1}{2} - \left(\dfrac{\pi}{90}\right)\left(\dfrac{\sqrt{3}}{2}\right) \cong 0.470$

13. $f(x) = \tan x, \quad x = \dfrac{\pi}{6}, \quad h = \dfrac{-\pi}{90}, \quad f'(x) = \sec^2 x$

$\tan 28° = f(x+h) \cong f(x) + hf'(x) = \tan\dfrac{\pi}{6} + \left(\dfrac{-\pi}{90}\right)\left(\dfrac{4}{3}\right) = \dfrac{\sqrt{3}}{3} - \dfrac{2\pi}{135} \cong 0.531$

14. $f(x) = \sin x, \quad x = \dfrac{\pi}{4}, \quad h = -\dfrac{\pi}{90}, \quad f'(x) = \cos x$

$\sin 43° = f(x+h) \cong f(x) + hf'(x) = \sin\dfrac{\pi}{4} + \left(-\dfrac{\pi}{180}\right)\cos\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2} - \left(\dfrac{\pi}{180}\right)\left(\dfrac{\sqrt{2}}{2}\right) \cong 0.682$

15. $f(2.8) \cong f(3) + (-0.2)f'(3) = 2 + (-0.2)(2) = 1.6$

16. $f(5.4) \cong f(5) + (0.4)f'(5) = 1 + (0.4)(3) = 2.2$

17. $V(x) = \pi x^2 h;$ volume $= V(r+t) - V(r) \cong tV'(r) = 2\pi rht$

18. Error in diameter $= 0.3 \implies$ error in radius $= 0.15$.

(a) $dS = 8\pi rh = 8\pi(8)(0.15) \cong 9.6\pi \text{ cm}^2$

(b) $dV = 4\pi r^2 h = 4\pi(8)^2(0.15) \cong 38.4 \text{ cm}^3$

19. $V(x) = x^3, \quad V'(x) = 3x^2, \quad \Delta V \cong dV = V'(10)h = 300h$

$|dV| \le 3 \implies |300h| \le 3 \implies |h| \le 0.01,$ error ≤ 0.01 feet

20. (a) Let $f(x) = \sqrt{x}$. Then $f'(x) = \dfrac{1}{2\sqrt{x}}$ and $\sqrt{x+1} - \sqrt{x} \cong (1)f'(x) = \dfrac{1}{2\sqrt{x}}$.

Now, $\dfrac{1}{2\sqrt{x}} < 0.01 = \dfrac{1}{100} \implies \sqrt{x} > 50 \implies x > 2500.$

(b) Let $f(x) = x^{1/4}$. Then $f'(x) = \frac{1}{4}x^{-3/4}$ and $\sqrt[4]{x+1} - \sqrt[4]{x} \cong (1)f'(x) = \frac{1}{4}x^{-3/4}$.

Now, $\frac{1}{4}x^{-3/4} < 0.002 = \dfrac{2}{1000} \implies x^{3/4} > 125 \implies x > 625.$

21. $V(r) = \frac{2}{3}\pi r^3$ and $dr = 0.01.$

$$V(r+0.01) - V(r) \cong V'(r)(0.01) = 2\pi r^2(0.01)$$

$$= 2\pi(600)^2(0.01) \quad (50 \text{ ft} = 600 \text{ in})$$

$$= 22619.5 \text{ in}^3 \quad \text{or} \quad 98 \text{ gallons (approx.)}$$

22. $V(r) = \frac{4}{3}\pi r^3;\quad \dfrac{dV}{dr} = 4\pi r^2.$

Now, $V(r+h) - V(r) = 8 \times (10)6 \cong \dfrac{dV}{dr}h = 4\pi r^2 h$. Therefore,

$$h = \frac{8 \times (10)^6}{4\pi(4000)^2} \cong 0.0398 \text{ (miles)} \cong 210 \text{ (feet)}$$

23. $P = 2\pi\sqrt{\dfrac{L}{g}}$ implies $P^2 = 4\pi^2\dfrac{L}{g}$

Differentiating with respect to t, we have

$$2P\frac{dP}{dt} = \frac{4\pi^2}{g}\cdot\frac{dL}{dt} = \frac{P^2}{L}\cdot\frac{dL}{dt} \quad \text{since} \quad \frac{P^2}{L} = \frac{4\pi^2}{g}.$$

Thus $\dfrac{dP}{P} = \dfrac{1}{2}\cdot\dfrac{dL}{L}$

24. $\dfrac{dP}{P} = -15 \text{ sec/hour} = -\dfrac{15}{3600}$. By Exercise 23,

$$\frac{1}{2}\frac{dL}{L} = -\frac{15}{3600} \quad \text{and} \quad dL = -\frac{30}{3600}L = -\frac{1}{120}L$$

With $L = 90$, we have $dL = -90/120 = -0.75$; the pendulum should be shortened 0.75 cm to 89.25 cm.

25. $L = 3.26$ ft, $P = 2$ sec, and $dL = 0.01$ ft

$$\frac{dP}{P} = \frac{1}{2}\cdot\frac{dL}{L}$$

$$dP = \frac{1}{2}\cdot\frac{dL}{L}\cdot P = \frac{1}{2}\cdot\frac{0.01}{3.26}\cdot 2 \quad dP \cong 0.00307 \text{ sec}$$

26. Each edge increases by 0.1%; take $h = 0.001x$.

$S = 6x^2$, $dS = 12xh$, and $\dfrac{dS}{S} = \dfrac{12x(0.001x)}{6x^2} = 0.002 = 0.2\%.$

$A = x^3$, $dA = 3x^2h$, and $\dfrac{dA}{A} = \dfrac{3x^2(0.001x)}{x^3} = 0.003 = 0.3\%.$

27. $A(x) = \frac{1}{4}\pi x^2, \quad dA = \frac{1}{2}\pi x h, \quad \frac{dA}{A} = 2\frac{h}{x}$

$\frac{dA}{A} \le 0.01 \iff 2\frac{h}{x} \le 0.01 \iff \frac{h}{x} \le 0.005$ within $\frac{1}{2}\%$

28. (a) Let $y = x^n$. Then $dy = nx^{n-1}h$.

To get $\frac{dy}{y} = \frac{nx^{n-1}h}{x^n} < 0.01$, we need $\frac{h}{x} < \frac{0.01}{n}$, that is, within $\frac{1}{n}\%$.

(b) Let $y = x^{1/n}$. Then $dy = \frac{1}{n}x^{(1-n)/n}h$.

To get $\frac{dy}{y} = \frac{\frac{1}{n}x^{(1-n)/n}h}{x^{1/n}} < 0.01$, we need $\frac{h}{x} < (0.01)n$, that is, within $n\%$.

29. (a) $x_{n+1} = \frac{1}{2}x_n + 12\left(\frac{1}{x_n}\right)$ (b) $x_4 \cong 4.89898$

30. (a) $x_{n+1} = \frac{1}{2}x_n + \frac{17}{2}\left(\frac{1}{x_n}\right)$ (b) $x_4 \cong 4.12311$

31. (a) $x_{n+1} = \frac{2}{3}x_n + \frac{25}{3}\left(\frac{1}{x_n}\right)^2$ (b) $x_4 \cong 2.92402$

32. (a) $x_{n+1} = \frac{4}{5}x_n + 6\left(\frac{1}{x_n}\right)^4$ (b) $x_4 \cong 1.97435$

33. (a) $x_{n+1} = \frac{x_n \sin x_n + \cos x_n}{\sin x_n + 1}$ (b) $x_4 \cong 0.73909$

34. (a) $x_{n+1} = \frac{x_n \cos x_n - \sin x_n - x_n^2}{\cos x_n - 2x_n}$ (b) $x_4 \cong 0.87673$

35. (a) $x_{n+1} = \frac{2x_n \cos x_n - 2\sin x_n}{2\cos x_n - 1}$ (b) $x_4 \cong 1.89549$

36. (a) $x_{n+1} = \frac{2x_n^3 - 1}{3x_n^2 - 4}$ (b) $x_4 \cong 1.86081$

37. Let $f(x) = x^{1/3}$. Then $f'(x) = \frac{1}{3}x^{-2/3}$. The Newton-Raphson method applied to this function gives:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}} = -2x_n.$$

Choose any $x_1 \neq 0$. Then $x_2 = -2x_1$, $x_3 = -2x_2 = 4x_1$, $\cdots$,

$x_n = -2x_{n-1} = (-1)^{n-1}2^n x_1$, $\cdots$.

38. (a) f is a continuous function, and $f(1) = -2 < 0$, $f(2) = 3 > 0$. Thus, f has a root in $(1, 2)$.

(b) $f'(x) = 6x^2 - 6x$ and $f'(1) = 0$. Therefore, $x_1 = 1$ will fail to generate values that will approach the root in $(1,2)$.

(c) $x_{n+1}=x_n-\dfrac{2x_n^3-3x_n^2-1}{6x_n^2-6x_n}$;

$x_1=2,\ x_2=1.75,\ x_3=1,68254,\ x_4=1.67768,\quad f(x_4)\cong 0.00020.$

39. (a) Let $f(x)=x^4-2x^2-\frac{17}{16}$. Then $f'(x)=4x^3-4x$. The Newton-Raphson method applied to this function gives:

$$x_{n+1}=x_n-\frac{x_n^4-2x_n^2-\frac{17}{16}}{4x_n^3-4x_n}$$

If $x_1=\frac{1}{2}$, then $x_2=-\frac{1}{2},\quad x_3=\frac{1}{2},\ \cdots x_n=(-1)^{n-1}\,\frac{1}{2},\ \cdots.$

(b) $x_1=2,\quad x_2=1.71094,\quad x_3=1.58569,\quad x_4=1.56165;\quad f(x_4)=0.00748$

40. (a) $f(x)=x^2-a;\quad f'(x)=2x$. Substituting into (3.9.3), we have

$$x_{n+1}=x_n-\frac{x_n^2-a}{2x_n}=\frac{x_n^2+a}{2x_n}=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right),\quad n\geq 1$$

(b) Let $a=5,\ \ x_1=2,$ and $x_{n+1}=\dfrac{1}{2}\left(x_n+\dfrac{a}{x_n}\right),\quad n\geq 1.$

Then $x_2=2.25,\ x_3=2.23611,\ x_4=2.23607,$ and $f(x_4)\cong 0.000009045.$

41. (a) Let $f(x)=x^k-a$. Then $f'(x)=kx^{k-1}$. The Newton-Raphson method applied to this function gives:

$$x_{n+1}=x_n-\frac{x_n^k-a}{kx_n^{k-1}}=x_n-\frac{1}{k}\,x_n+\frac{1}{k}\,\frac{a}{x_n^{k-1}}$$
$$=\frac{1}{k}\left[(k-1)x_n+\frac{a}{x_n^{k-1}}\right]$$

(b) Let $a=23,\ \ k=3$ and $x_1=3$. Then

$x_1=3,\quad x_2=2.85185,\quad x_3=2.84389,\quad x_4=2.84382;\quad f(x_4)=-0.00114$

42. (a) Let $f(x)=\frac{1}{x}-a$. Then $f'(x)=-\frac{1}{x^2}$. The Newton-Raphson method applied to this function gives:

$$x_{n+1}=x_n-\frac{\frac{1}{x_n}-a}{-\frac{1}{x_n^2}}=x_n+x_n-ax_n^2$$
$$=2x_n-ax_n^2$$

(b) Let $a = 2.7153$, and $x_1 = 0.3$. Then

$x_2 = 0.35562, \quad x_3 = 0.36785, \quad x_4 = x_5 = 0.36828,$

Thus $\frac{1}{2.7153} \simeq 0.36828$.

43. (a) and (b)

44. $\lim_{h\to 0} g(h) = \lim_{h\to 0} \frac{g(h)}{h} \cdot h = \left(\lim_{h\to 0} \frac{g(h)}{h}\right)\left(\lim_{h\to 0} h\right) = 0$

45. $\lim_{h\to 0} \frac{g_1(h) + g_2(h)}{h} = \lim_{h\to 0} \frac{g_1(h)}{h} + \lim_{h\to 0} \frac{g_2(h)}{h} = 0 + 0 = 0$

$$\lim_{h\to 0} \frac{g_1(h)g_2(h)}{h} = \lim_{h\to 0} h\,\frac{g_1(h)g_2(h)}{h^2} = \left(\lim_{h\to 0} h\right)\left(\lim_{h\to 0} \frac{g_1(h)}{h}\right)\left(\lim_{h\to 0} \frac{g_2(h)}{h}\right) = (0)(0)(0) = 0$$

46. (a) $g(h) = f(x+h) - f(x) - mh$.

(b) $\lim_{h\to 0} \frac{g(h)}{h} = \lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h} - m\right] = f'(x) - m = 0$ iff $m = f'(x)$.

PROJECT 3.9

1. Let $F(x) = f(x) - x$. Then F is continuous on $[a, b]$, and $F(b) = f(b) - b \leq 0$. If either $f(a) - a = 0$ or $f(b) - b = 0$, then a and/or b is a fixed point. If $f(a) - a \neq 0$ and $f(b) - b \neq 0$, Then $F(a) > 0$ and $F(b) < 0$, and, by the intermediate value theorem, there exists at least one $c \in (a, b)$ such that $F(c) = f(c) - c = 0$. The number c is a fixed point of f.

2. (a) Let $F(x) = f(x) - x = 2x^3 - 4x - 3 - x = 2x^3 - 5x - 3$. Then $F(1) = -6 < 0$ and $F(2) = 3 > 0$. Therefore, F has a root in $(1, 2)$ which implies that f has a fixed point in $(1, 2)$.

$F'(x) = 6x^2 - 5; \quad x_{n+1} = x_n - \dfrac{2x_n^3 - 5x_n - 3}{6x_n^2 - 5}$

$x_1 = 2,\ x_2 = 1.8421,\ x_3 = 1.8231,\ x_4 = 1.8229;\ \ f(x_4) \cong 1.8233$.

(b) Let $F(x) = \frac{1}{2}\cos x - x$. Then $F(0) = \frac{1}{2} > 0$ and $F(\pi/2) = -\pi/2 < 0$. Thus F has a root in $(0, \pi/2)$ which implies that f has a fixed point in $(0, \pi/2)$.

$F'(x) = -\frac{1}{2}\sin x - 1$ and

$$x_{n+1} = x_n - \frac{\frac{1}{2}\cos x_n - x_n}{-\frac{1}{2}\sin x_n - 1} = x_n + \frac{\cos x_n - 2x_n}{\sin x_n + 2}$$

$x_1 = 0, \quad x_2 = 0.5, \quad x_3 = 0.4506, \quad x_4 = 0.4502; \quad f(x_4) = 0.45018$

(c) Let $F(x) = \frac{2}{3}\sin x + 1 - x$. Then $F(0) = 1 > 0$ and $F(2) \cong -0.3938 < 0$. Thus F has a root in $(0, 2)$ which implies that f has a fixed point in $(0, 2)$.

$F'(x) = \frac{2}{3}\cos x - 1$ and

$$x_{n+1} = x_n - \frac{\frac{2}{3}\sin x_n + 1 - x_n}{-\frac{2}{3}\cos x_n - 1}$$

$x_1 = 2, \quad x_2 = 1.6917, \quad x_3 = 1.6640, \quad x_4 = 1.6638; \quad f(x_4) = 1.6638$

CHAPTER 4

SECTION 4.1

1. f is differentiable on $(0,1)$, continuous on $[0,1]$; and $f(0)=f(1)=0$.

$f'(c)=3c^2-1;\quad 3c^2-1=0 \Longrightarrow c=\dfrac{\sqrt{3}}{3}\quad \left(-\dfrac{\sqrt{3}}{3}\notin(0,1)\right)$

2. f is differentiable on $(-2,2)$, continuous on $[-2,2]$; and $f(-2)=f(2)=0$.

$f'(c)=4c^3-4c;\quad 4c(c^2-1)=0 \Longrightarrow c=0,\ \pm 1$

3. f is differentiable on $(0,2\pi)$, continuous on $[0,2\pi]$; and $f(0)=f(2\pi)=0$.

$f'(c)=2\cos 2c;\quad 2\cos 2c=0 \Longrightarrow 2c=\dfrac{\pi}{2}+n\pi,\ \text{ and } c=\dfrac{\pi}{4}+\dfrac{n\pi}{2},\quad n=0,\pm 1,\pm 2\ldots$

Thus, $c=\dfrac{\pi}{4},\ \dfrac{3\pi}{4},\ \dfrac{5\pi}{4},\ \dfrac{7\pi}{4}$

4. f is differentiable on $(0,8)$, continuous on $[0,8]$; and $f(0)=f(8)=0$.

$f'(c)=\frac{2}{3}c^{-1/3}-\frac{2}{3}c^{-2/3}=\dfrac{2}{3}\dfrac{1-c}{c^{2/3}}\quad f'(c)=0 \quad\Longrightarrow\quad c=1.$

5. $f'(c)=2c,\quad \dfrac{f(b)-f(a)}{b-a}=\dfrac{4-1}{2-1}=3;\qquad 2c=3\quad\Longrightarrow\quad c=3/2$

6. $f'(c)=\dfrac{3}{2\sqrt{c}}-4,\quad \dfrac{f(b)-f(a)}{b-a}=\dfrac{-10-(-1)}{4-1}=-3;\qquad \dfrac{3}{2\sqrt{c}}-4=-3\quad\Longrightarrow\quad c=9/4$

7. $f'(c)=3c^2,\quad \dfrac{f(b)-f(a)}{b-a}=\dfrac{27-1}{3-1}=13;\quad 3c^2=13\ \Longrightarrow\ c=\dfrac{1}{3}\sqrt{39}\quad \left(-\dfrac{1}{3}\sqrt{39}\text{ is not in }[a,b]\right)$

8. $f'(c)=\frac{2}{3}c^{-1/3},\quad \dfrac{f(b)-f(a)}{b-a}=\dfrac{4-1}{8-1}=\frac{3}{7};\quad \frac{2}{3}c^{-1/3}=\frac{3}{7}\quad\Longrightarrow\quad c=\dfrac{(14)^3}{9^3}$

9. $f'(c)=\dfrac{-c}{\sqrt{1-c^2}},\quad \dfrac{f(b)-f(a)}{b-a}=\dfrac{0-1}{1-0}=-1;\qquad \dfrac{-c}{\sqrt{1-c^2}}=-1\quad\Longrightarrow\quad c=\dfrac{1}{2}\sqrt{2}$

$(-\dfrac{1}{2}\sqrt{2}$ is not in $[a,b])$

10. $f'(c)=3c^2-3,\quad \dfrac{f(b)-f(a)}{b-a}=\dfrac{-2-2}{1-(-1)}=-2;\quad 3c^2-3=-2\quad\Longrightarrow\quad c=\pm\dfrac{\sqrt{3}}{3}$

11. f is continuous on $[-1,1]$, differentiable on $(-1,1)$ and $f(-1)=f(1)=0$.

$f'(x)=\dfrac{-x(5-x^2)}{(3+x^2)^2\sqrt{1-x^2}},\quad f'(c)=0$ for c in $(-1,1)$ implies $c=0$.

12. (a) $f'(x)=\frac{2}{3}x^{-1/3}=\dfrac{2}{3x^{1/3}}\neq 0$ for all $x\in(-1,1)$.

(b) $f'(0)$ does not exist. Therefore, f is not differentiable on $(-1,1)$.

13. No. By the mean-value theorem there exists at least one number $c \in (0,2)$ such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{3}{2} > 1.$$

14. No, by Rolle's theorem: $f(2) = f(3) = 1$ but there is no value $c \in (2,3)$ such that $f'(c) = 0$.

15. f is everywhere continuous and everywhere differentiable except possibly at $x = -1$.

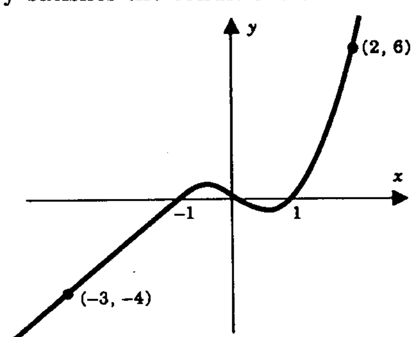
f is continuous at $x = -1$: as you can check,

$$\lim_{x \to -1^-} f(x) = 0, \quad \lim_{x \to -1^+} f(x) = 0, \quad \text{and} \quad f(-1) = 0.$$

f is differentiable at $x = -1$ and $f'(-1) = 2$: as you can check,

$$\lim_{h \to 0^-} \frac{f(-1+h) - f(-1)}{h} = 2 \quad \text{and} \quad \lim_{h \to 0^+} \frac{f(-1+h) - f(-1)}{h} = 2.$$

Thus f satisfies the conditions of the mean-value theorem on every closed interval $[a,b]$.



$$f'(x) = \begin{cases} 2, & x \le -1 \\ 3x^2 - 1, & x > -1 \end{cases}$$

$$\frac{f(2) - f(-3)}{2 - (-3)} = \frac{6 - (-4)}{2 - (-3)} = 2.$$

$$f'(c) = 2 \text{ with } c \in (-3, 2) \quad \text{iff} \quad c = 1 \text{ or } -3 < c \le -1.$$

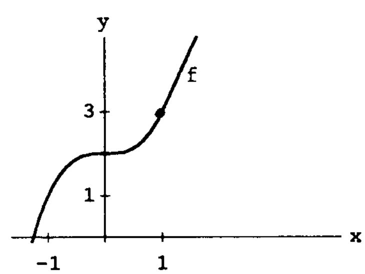
16. f is continuous and differentiable everywhere;

$$f'(x) = \begin{cases} 3x^2, & x \le 1 \\ 3, & x > 1 \end{cases}$$

$$\frac{f(2) - f(-1)}{2 - (-1)} = \frac{6 - 1}{3} = \frac{5}{3}$$

For $c \le 1$, $f'(c) = 3c^2 = \frac{5}{3} \implies c = \pm \frac{\sqrt{5}}{3}$

For $c > 1$, $f'(c) = 3 \ne \frac{5}{3}$



17. Let $f(x) = Ax^2 + Bx + C$. Then $f'(x) = 2Ax + B$. By the mean-value theorem

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{(Ab^2 + Bb + C) - (Aa^2 + Ba + C)}{b - a}$$

$$= \frac{A(b^2 - a^2) + B(b - a)}{b - a} = A(b + a) + B$$

Therefore, we have

$$2Ac + B = A(b + a) + B \implies c = \frac{a + b}{2}$$

18. $\dfrac{f(1)-f(-1)}{1-(-1)}=1$ and $f'(x)=-1/x^2<0$; f is not continuous at 0.

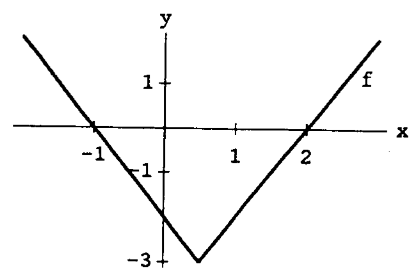
19. $\dfrac{f(1)-f(-1)}{1-(-1)}=0$ and $f'(x)$ is never zero. This result does not violate the mean-value theorem since f is not differentiable at 0; the theorem does not apply.

20. $f(x)=\begin{cases} 2x-4, & x\geq 1/2 \\ -2x-2, & x<1/2 \end{cases}$

$f'(x)=\begin{cases} 2, & x\geq 1/2 \\ -2, & x<1/2 \end{cases}$

$f'(x)\neq 0$ for all $x\neq \frac{1}{2}$.

Rolle's theorem is not violated because f is not differentiable at $x=\frac{1}{2}$.



21. Set $P(x)=6x^4-7x+1$. If there existed three numbers $a<b<c$ at which $P(x)=0$, then by Rolle's theorem $P'(x)$ would have to be zero for some x in $(a,\,b)$ and also for some x in $(b,\,c)$. This is not the case: $P'(x)=24x^3-7$ is zero only at $x=(7/24)^{1/3}$.

22. Set $P(x)=6x^5+13x+1$. Note that $P(-1)<0$ and $P(0)>0$. By the intermediate-value theorem, the equation $P(x)=0$ has at least one real root c. If this equation had another real root d, then by Rolle's theorem $P'(x)$ would have to be zero for some x between c and d. This is not the case: $P'(x)=30x^4+13$ is never zero.

23. Set $P(x)=x^3+9x^2+33x-8$. Note that $P(0)<0$ and $P(1)>0$. Thus, by the intermediate-value theorem, there exists some number c between 0 and 1 at which $P(x)=0$. If the equation $P(x)=0$ had an additional real root, then by Rolle's theorem there would have to be some real number at which $P'(x)=0$. This is not the case: $P'(x)=3x^2+18x+33$ is never zero since the discriminant $b^2-4ac=(18)^2-12(33)<0$.

24. (a) Suppose that f has two zeros, $x_1,\ x_2\ \in(a,b)$. Then, f is differentiable on (x_1,x_2) and continuous on $[x_1,x_2]$. By Rolle's theorem, f' has a zero in (x_1,x_2) which contradicts the hypothesis.

(b) If f had three zeros in (a,b), then, by Rolle's' theorem, f' would have at least two zeros in (a,b) and f'' would have at least one zero in (a,b) which contradicts the hypothesis.

25. Let c and d be two consecutive roots of the equation $P'(x)=0$. The equation $P(x)=0$ cannot have two or more roots between c and d for then, by Rolle's theorem, $P'(x)$ would have to be zero somewhere between these two roots and thus between c and d. In this case c and d would no longer be consecutive roots of $P'(x)=0$.

26. If $f(x)=0$ at $a_1,\ a_2,\ \ldots,a_n$ then by Rolle's theorem, $f'(x)$ is zero at some number $b_1\in(a_1,a_2)$, at some number $b_2\in(a_2,a_3)$, $\ldots$, at some number $b_{n-1}\in(a_{n-1},a_n)$; $f''(x)$, in turn, must be zero at some number $c_1\in(b_1,b_2)$, at some number $b_2\in(b_2,b_3)$, $\ldots$, at some number $c_{n-2}\in(b_{n-2},b_{n-1})$.

27. Suppose that f has two fixed points $a,\ b \in I$, with $a < b$. Let $g(x) = f(x) - x$. Then $g(a) = f(a) - a = 0$ and $g(b) = f(b) - b = 0$. Since f is differentiable on I, we can conclude that g is differentiable on (a, b) and continuous on $[a, b]$. By Rolle's theorem, there exists a number $c \in (a, b)$ such that $g'(c) = f'(c) - 1 = 0$ or $f'(c) = 1$. This contradicts the assumption that $f'(x) < 1$ on I.

28. Set $P(x) = x^3 + ax + b$. It is obvious that for x sufficiently large, $P(x) > 0$ and for x sufficiently large negative, $P(x) < 0$. Thus, by the intermediate-value theorem, the equation $P(x) = 0$ has at least one real root.

If $a \geq 0$, then $P'(x) = 3x^2 + a$ is positive, except possibly at 0, where it remains nonnegative. It follows that P is everywhere increasing and therefore it cannot take on the value 0 more than once.

Suppose now that $a < 0$. Then $-\frac{1}{3}\sqrt{3}\,|a|$ and $\frac{1}{3}\sqrt{3}\,|a|$ are consecutive roots of the equation $P'(x) = 0$ and thus, by Exercise 17, P cannot take on the value zero more than once between these two numbers.

29. (a) $f'(x) = 3x^2 - 3 > 0$ for all x in $(-1, 1)$. Also, f is differentiable on $(-1, 1)$ and continuous on $[-1, 1]$. Thus there cannot be a and b in $(-1, 1)$ such that $f(a) = f(b) = 0$, or they would contradict Rolle's theorem.

(b) When $f(x) = 0$, $b = 3x - x^3 = x(3 - x^2)$. When x is in $(-1, 1)$, then $|x(3 - x^2)| < 2$. Thus $|b| < 2$.

30. $f'(x) = 3x^2 - 3a^2 > 0$ for all x in$(-a, a)$. Also, f is differentiable on $(-a, a)$ and continuous on $[-a, a]$. Thus there cannot be b and c in $(-a, a)$ such that $f(b) = f(c) = 0$, or they would contradict Rolle's theorem.

31. For $p(x) = x^n + ax + b$, $p'(x) = nx^{n-1} + a$, which has at most one real zero for n even $\left(x = -\frac{a}{n}^{\frac{1}{n-1}}\right)$. If there were more than two distinct real roots of $p(x)$, then by Rolle's theorem there would be more than one zero of $p'(x)$. Thus there are at most two distinct real roots of $p(x)$.

32. For $p(x) = x^n + ax + b$, $p'(x) = nx^{n-1} + a$, which has at most two real zeros for n odd . If there were more than three distinct real roots of $p(x)$, then by Rolle's theorem there would be more than two zeros of $p'(x)$. Thus there are at most three distinct real roots of $p(x)$.

33. If $x_1 = x_2$, then $|f(x_1) - f(x_2)|$ and $|x_1 - x_2|$ are both 0 and the inequality holds. If $x_1 \neq x_2$, then by the mean-value theorem

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} = f'(c)$$

for some number c between x_1 and x_2. Since $|f'(c)| \leq 1$:

$$\left|\frac{f(x_1) - f(x_2)}{x_1 - x_2}\right| \leq 1 \quad \text{and thus} \quad |f(x_1) - f(x_2)| \leq |x_1 - x_2|.$$

34. See the proof of Theorem 4.2.2.

35. Set, for instance, $f(x) = \begin{cases} 1, & a < x < b \\ 0, & x = a,\ b \end{cases}$

36. (a) Let $f(x) = \cos x$. Choose any numbers x and y, (assume $x < y$). By the mean-value theorem,

there is a number c between x and y such that

$$\frac{f(y) - f(x)}{y - x} = f'(c) \quad \Rightarrow \quad \frac{|\cos y - \cos x|}{|y - x|} = |-\sin c| \leq 1 \quad \Rightarrow \quad |\cos x - \cos y| \leq |x - y|$$

(b) Same argument as part (a) with $f(x) = \sin x$.

37. (a) By the mean-value theorem, there exists a number $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$. If $f'(x) \leq M$ for all $x \in (a, b)$, then it follows that

$$f(b) \leq f(a) + M(b - a)$$

(b) If $f'(x) \geq m$ for all $x \in (a, b)$, then it follows that

$$f(b) \geq f(a) + m(b - a)$$

(c) If $|f'(x)| \leq L$ on (a, b), then $-L \leq f'(x) \leq L$ on (a, b) and the result follows from parts (a) and (b).

38. Suppose $g(x) \neq 0$ for all $x \in [a, b]$ and let $h(x) = \dfrac{f(x)}{g(x)}$. Then h is defined on $[a, b]$ and $h(a) = h(b) = 0$. Therefore, by Rolle's theorem, there exists a number $c \in (a, b)$ such that

$$h'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{g^2(c)} = 0$$

Thus $g(c)f'(c) - f(c)g'(c) = 0$ which contradicts the given condition $f(x)g'(x) - g(x)f'(x) \neq 0$ for all $x \in I$. Thus, g has at least one zero in (a, b).

By reversing the roles of f and g, the same argument can be used to show that g cannot have two (or more) zeros on (a, b).

39. Let $f(x) = \cos x$ and $g(x) = \sin x$ on $I = (-\infty, \infty)$. Then

$$f(x)g'(x) - g(x)f'(x) = \cos^2 x + \sin^2 x = 1 \text{ for all } x \in I$$

The result follows from Exercise 38.

40. We prove the result for $h > 0$. The proof for $h < 0$ is similar. If f is differentiable on $(x, x + h)$, it is continuous there and thus, by the hypothesis at x and $x + h$ continuous on $[x, x + h]$. By the mean-value theorem, there exists c in $x, x + h)$ for which

$$\frac{f(x + h) - f(x)}{x + h - x} = f'(c).$$

Multiplying through by $(x + h) - x = h$, we have

$$f(x + h) - f(x) = f'(c)h.$$

Since c is between x and $x + h$, c can be written

$$c = x + \theta h \quad \text{with} \quad 0 < \theta < 1.$$

41.

$$f'(x_0) = \lim_{y\to 0} \frac{f(x_0+y)-f(x_0)}{y} = \lim_{y\to 0} \frac{f'(x_0+\theta y)y}{y} = \lim_{y\to 0} f'(x_0+\theta y)$$

(by the hint)

$$= \lim_{x\to x_0} f'(x) = L$$

(by 2.2.5)

42. Suppose that $f(a) = f(b) = k$, and let $g(x) = f(x) - k$. Then g is differentiable on (a, b), continuous on $[a, b]$, and $g(a) = g(b) = 0$. Therefore, by Rolle's theorem, there exists at least one number $c \in (a, b)$ such that $g'(c) = 0$. Since $g'(x) = f'(x)$, it follows that $f'(c) = 0$.

43. Using the hint, F is continuous on $[a, b]$, differentiable on (a, b), and $F(a) = F(b)$. Thus by Exercise 42, there is a c in (a, b) such that $F'(c) = 0$.
Thus $[f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) = 0$ and $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)}$.

44. Let $f(t)$ be a function telling the car's velocity after t hours. Then $f(0) = 30$ and $f(\frac{1}{4}) = 60$. f is differentiable on $(1, \frac{1}{4})$ and continuous on $[1, \frac{1}{4}]$, so by the mean-value theorem there is a c in $(1, \frac{1}{4})$ such that $f'(c) = \dfrac{60-30}{0-t_1} = 120$. i.e. The acceleration at time c was 120 mph.

45. Let $f_1(t)$ and $f_2(t)$ be the positions of the cars at time t. Consider $f(t) = f_1(t) - f_2(t)$. Let T be the time the cars finish the race. Then $f(t)$ satisfies the hypothesis of Exercise 42, so there is a c in $(0, T)$ such that $f'(c) = 0$. Hence $f_1'(t) = f_2'(t)$, so the cars had the same velocity at time c.

46. The driver must have exceeded the speed limit at least once during the trip. Let $s(t)$ denote the driver's position at time t, with $s(0) = 0$ and $s(1.75) = 120$. Then, by the mean-value theorem, there exists at least one number $c \in (0, 1.75)$ such that

$$v(c) = s'(c) = \frac{s(1.75)-s(0)}{1.75-0} = \frac{120}{1.75} = 68.57$$

47. Let $s(t)$ denote the distance that the car has traveled in t seconds since applying the brakes, $0 \le t \le 6$. Then $s(0) = 0$ and $s(6) = 280$. Assume that s is differentiable on $(0, 6)$ and continuous on $[0, 6]$. Then, by the mean-value theorem, there exists a time $c \in (0, 6)$ such that

$$s'(c) = v(c) = \frac{s(6)-s(0)}{6-0} = \frac{280}{6} \cong 46.67 \text{ ft/sec}$$

Now $v(0) \ge v(c) = 46.7$ ft/sec. Thus, the driver must have been exceeding the speed limit (44 ft/sec) at the instant he applied his brakes.

48. By the mean-value theorem, there exists a number $c \in (x, x + \Delta x)$ such that

$$f'(c) = \frac{f(x+\Delta x)-f(x)}{x+\Delta x - x} \quad\Longrightarrow\quad \Delta f = f(x+\Delta x) - f(x) = f'(x)\Delta x$$

49. Let $f(x) = \sqrt{x}$. Then $f'(x) = \dfrac{1}{2\sqrt{x}}$. Using Exercise 48, we have

$$\sqrt{65} = \sqrt{64+1} = f(64+1) \cong f(64) + f'(64)(1) = \sqrt{64} + \frac{1}{2\sqrt{64}} = 8.0625$$

50. (a) If $f(x) = 0$ had two solutions, then $f'(x) = 0$ would have at least one solution. But

$$f'(x) = 3x^2 - 6x + 6 = 3(x-1)^2 + 3 > 0 \quad \text{for all } x.$$

(b) $f(2) = -4 < 0$ and $f(3) = 6 > 0$. Therefore, f has a zero in $(2,3)$.

(c) $x_{n+1} = x_n - \dfrac{x_n^3 - 3x_n^2 + 6x_n - 12}{3x_n^2 - 6x_n + 6}$; $x_1 = 2$, $x_2 = 2.6667$, $x_3 = 2.5229$

51. (a) Let $f(x) = 1 + 4x - 2\cos x$, $x \in I = (-\infty, \infty)$. If f had two (or more) zeros on I, then, by Rolle's theorem, f' would have to have a zero on I. But, $f'(x) = 4 + 2\sin x > 0$ on I. Thus f has at most one zero on I.

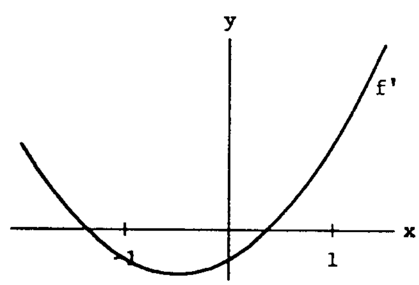
(b) $f(0) = -1$ and $f(1) \cong 3.92$. Thus f has a zero in $(0,1)$.

(b) $x_{n+1} = x_n - \dfrac{1 + 4x_n - 2\cos x_n}{4 + 2\sin x_n}$; $x_1 = 0$, $x_2 = 0.25$, $x_3 \cong 0.2361$

52. $f(x) = 2x^3 + 3x^2 - 3x - 2$ is differentiable on $(-2,1)$, continuous on $[-2,1]$, and $f(-2) = f(1) = 0$.

$f'(x) = 6x^2 + 6x - 3$

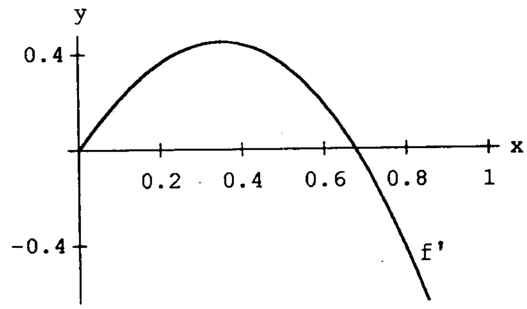
$f'(c) = 0$ at $c_1 \cong -1.366$, $c_2 \cong 0.366$



53. $f(x) = 1 - x^3 - \cos(\pi x/2)$ is differentiable on $(0,1)$, continuous on $[0,1]$, and $f(0) = f(1) = 0$.

$f'(x) = -3x^2 + \frac{\pi}{2}\sin(\pi x/2)$

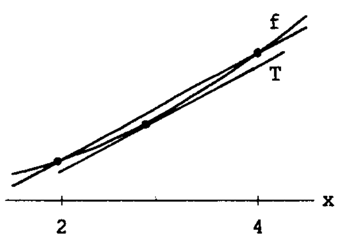
$f'(c) = 0$ at $c \cong 0.676$



54. $f(x) = x^2 + \frac{1}{x}$; $f'(x) = 2x - \frac{1}{x^2}$

$g(x) = 2x - \frac{1}{x^2} - \frac{f(4) - f(2)}{4-2} = 2x - \frac{1}{x^2} - \frac{47}{8}$

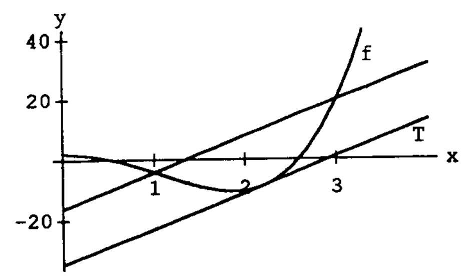
$g(c) = 0$ at $c \cong 2.993$



55. $f(x) = x^4 - 7x^2 + 2$; $f'(x) = 4x^3 - 14x$

$g(x) = 4x^3 - 14x - \frac{f(3) - f(1)}{3-1} = 4x^3 - 14x - 12$

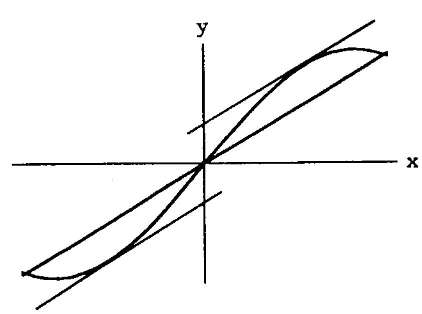
$g(c) = 0$ at $c \cong 2.205$



56. $f(x) = x\cos x + 4\sin x$; $f'(x) = \cos x - x\sin x + 4\cos x = 5\cos x - x\sin x$ and

$$g(x) = 5\cos x - x\sin x - \frac{f(\pi/2) - f(-\pi/2)}{\pi} = 5\cos x - x\sin x - \frac{8}{\pi}$$

$g(c) = 0$ at $c_1 \cong -0.872$, $c_2 \cong 0.872$



SECTION 4.2

1. $f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x+1)(x-1)$

f increases on $(-\infty, -1]$ and $[1, \infty)$, decreases on $[-1, 1]$

2. $f'(x) = 3x^2 - 6x = 3x(x-2)$

f increases on $(-\infty, 0]$ and $[2, \infty)$, decreases on $[0, 2]$

3. $f'(x) = 1 - \dfrac{1}{x^2} = \dfrac{x^2-1}{x^2} = \dfrac{(x+1)(x-1)}{x^2}$

f increases on $(-\infty, -1]$ and $[1, \infty)$, decreases on $[-1, 0)$ and $(0, 1]$ (f is not defined at 0)

4. $f'(x) = 3(x-3)^2$; f increases on $(-\infty, \infty)$.

5. $f'(x) = 3x^2 + 4x^3 = x^2(3+4x)$

f increases on $\left[-\frac{3}{4}, \infty\right)$, decreases on $\left(-\infty, -\frac{3}{4}\right]$

6. $f'(x) = 3x^2 + 6x + 2$

f increases on $\left(-\infty, -1-\frac{1}{3}\sqrt{3}\right]$ and $\left[-1-\frac{1}{3}\sqrt{3}, \infty\right)$, decreases on $\left[-1-\frac{1}{3}\sqrt{3}, -1+\frac{1}{3}\sqrt{3}\right]$

7. $f'(x) = 4(x+1)^3$

f increases on $[-1, \infty)$, decreases on $(-\infty, -1]$

8. $f'(x) = \dfrac{2(x^3+1)}{x^3}$

f increases on $[-\infty, -1]$ and $(0, \infty)$, decreases on $[-1, 0)$

9. $f(x) = \begin{cases} \dfrac{1}{2-x}, & x < 2 \\ \dfrac{1}{x-2}, & x > 2 \end{cases}$ $\qquad f'(x) = \begin{cases} \dfrac{1}{(2-x)^2}, & x < 2 \\ \dfrac{-1}{(x-2)^2}, & x > 2 \end{cases}$

f increases on $(-\infty, 2)$, decreases on $(2, \infty)$ (f is not defined at 2)

10. $f'(x) = \dfrac{(1+x^2) - x(2x)}{(1+x^2)^2}$; f increases on $[-1, 1]$, decreases on $(-\infty, -1]$ and $[1, \infty)$

11. $f'(x) = -\dfrac{4x}{(x^2-1)^2}$

f increases on $(-\infty, -1)$ and $(-1, 0]$, decreases on $[0, 1)$ and $(1, \infty)$ (f is not defined at ± 1)

12. $f'(x) = \dfrac{(x^2+1)(2x) - x^2(2x)}{(x^2+1)^2} = \dfrac{2x}{(x^2+1)^2}$

f increases on $[0, \infty)$, decreases on $(-\infty, 0]$

13. $f(x) = \begin{cases} x^2 - 5, & x < -\sqrt{5} \\ -\left(x^2 - 5\right), & -\sqrt{5} \le x \le \sqrt{5} \\ x^2 - 5, & \sqrt{5} < x \end{cases}$ $\qquad f'(x) = \begin{cases} 2x, & x < -\sqrt{5} \\ -2x, & -\sqrt{5} < x < \sqrt{5} \\ 2x, & \sqrt{5} < x \end{cases}$

f increases on $[-\sqrt{5}, 0]$ and $[\sqrt{5}, \infty)$, decreases on $(-\infty, -\sqrt{5}]$ and $[0, \sqrt{5}]$

14. $f'(x) = x^2(2)(1+x) + (x+1)^2(2x) = 2x(x+1)(2x+1)$

f increases on $[-1, -1/2]$ and $[2, \infty)$, decreases on $(-\infty, -1]$ and $[-1/2, 0]$

15. $f'(x) = \dfrac{2}{(x+1)^2}$; f increases on $(-\infty, -1)$ and $(-1, \infty)$ (f is not defined at -1)

16. $f'(x) = 2x - \dfrac{32}{x^3} = \dfrac{2(x^4-16)}{x^3} = \dfrac{2(x-2)(x+2)(x^2+4)}{x^3}$

f increases on $[-2, 0)$ and $[2, \infty)$, decreases on $(-\infty, -2]$ and $(0, 2]$

17. $f'(x) = -\dfrac{7\left(1-\sqrt{x}\right)^6}{\sqrt{x}\left(1+\sqrt{x}\right)^8}$; f decreases on $[0, \infty)$

18. $f'(x) = \dfrac{-1}{2(1+x)^2}\sqrt{\dfrac{1+x}{2+x}}$

f decreases on $(-\infty, -2]$ and $(-1, \infty)$; (f is not defined on $(-2, -1]$

19. $f'(x) = \dfrac{x}{(2+x^2)^2}\sqrt{\dfrac{2+x^2}{1+x^2}}$ f increases on $[0, \infty)$, decreases on $(-\infty, 0]$

20. $f(x) = \begin{cases} x^2 - x - 2, & x \le -1 \\ -x^2 + x + 2, & -1 < x < 2 \\ x^2 - x - 2, & x \ge 2 \end{cases}$ $f'(x) = \begin{cases} 2x - 1, & x \le -1 \\ -2x + 1, & -1 < x < 2 \\ 2x - 1, & x \ge 2 \end{cases}$

f increases on $\left[-1, \frac{1}{2}\right]$ and $[2, \infty)$, decreases on $(-\infty, -1]$ and $[1/2\,2]$

21. $f'(x) = \dfrac{-3}{2x^2}\sqrt{\dfrac{x}{3-x}}$; f decreases on $(0, 3]$

22. $f'(x) = 1 + \cos x \ge 0$; f increases on $[0, 2\pi]$

23. $f'(x) = 1 + \sin x \ge 0$; f increases on $[0, 2\pi]$

24. $f'(x) = -2\cos x \sin x = -2\sin 2x$; f increases on $[\pi/2, \pi]$, decreases on $[0, \pi/2]$

25. $f'(x) = -2\sin 2x - 2\sin x = -2\sin x\,(2\cos x + 1)$; f increases on $\left[\frac{2}{3}\pi, \pi\right]$, decreases on $\left[0, \frac{2}{3}\pi\right]$

26. $f'(x) = 2\sin x\cos x - \sqrt{3}\cos x = \cos x\left(2\sin x - \sqrt{3}\right)$

f increases on $\left[\frac{1}{3}\pi, \frac{1}{2}\pi\right]$ and $\left[\frac{2}{3}\pi, \pi\right]$, decreases on $\left[0, \frac{1}{3}\pi\right]$ and $\left[\frac{1}{2}\pi, \frac{2}{3}\pi\right]$

27. $f'(x) = \sqrt{3} + 2\sin 2x$; f increases on $\left[0, \frac{2}{3}\pi\right]$ and $\left[\frac{5}{6}\pi, \pi\right]$, decreases on $\left[\frac{2}{3}\pi, \frac{5}{6}\pi\right]$

28. $\dfrac{d}{dx}\left(\dfrac{x^3}{3} - x\right) = f'(x) \implies f(x) = \dfrac{x^3}{3} - x + C$

$f(0) = 1 \implies C = 1$. Thus, $f(x) = \frac{1}{3}x^3 - x + 1$.

29. $\frac{d}{dx}\left(\frac{x^3}{3}-x\right)=f'(x) \implies f(x)=\frac{x^3}{3}-x+C$

$f(1)=2 \implies 2=\frac{1}{3}-1+C$, so $C=\frac{8}{3}$. Thus, $f(x)=\frac{1}{3}x^3-x+\frac{8}{3}$.

30. $\frac{d}{dx}\left(x^2-5x\right)=f'(x) \implies f(x)=x^2-5x+C$

$f(2)=4 \implies 4=4-10+C$, so $C=10$. Thus, $f(x)=x^2-5x+10$.

31. $\frac{d}{dx}\left(x^5+x^4+x^3+x^2+x\right)=f'(x) \implies f(x)=x^5+x^4+x^3+x+C$

$f(0)=5 \implies 5=0+C$, so $C=5$. Thus, $f(x)=x^5+x^4+x^3+x^2+x+5$.

32. $\frac{d}{dx}2x^{-2}=f'(x) \implies f(x)=-2x^{-2}+C$

$f(1)=0 \implies 0=-2+C$, so $C=2$. Thus, $f(x)=-2x^{-2}+2,\ x>0$.

33. $\frac{d}{dx}\left(\frac{3}{4}x^{4/3}-\frac{2}{3}x^{3/2}\right)=f'(x) \implies f(x)=\frac{3}{4}x^{4/3}-\frac{2}{3}x^{3/2}+C$

$f(0)=1 \implies 1=0+C$, so $C=1$. Thus, $f(x)=\frac{3}{4}x^{4/3}-\frac{2}{3}x^{3/2}+1,\ x\geq 0$.

34. $\frac{d}{dx}\left(-\frac{1}{4}x^{-4}-\frac{25}{4}x^{4/5}\right)=f'(x) \implies f(x)=-\frac{1}{4}x^{-4}-\frac{25}{4}x^{4/5}+C$

$f(1)=0 \implies 0=-\frac{1}{4}-\frac{25}{2}+C$, so $C=\frac{13}{2}$. Thus, $f(x)=-\frac{1}{4}x^{-4}-\frac{25}{4}x^{4/5}+\frac{13}{2},\ x>0$.

35. $\frac{d}{dx}\left(2x-\cos x\right)=f'(x) \implies f(x)=2x-\cos x+C$

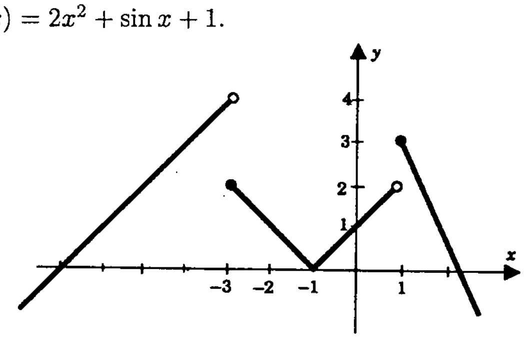
$f(0)=3 \implies 3=0-1+C$, so $C=4$. Thus, $f(x)=2x-\cos x+4$.

36. $\frac{d}{dx}\left(2x^2+\sin x\right)=f'(x) \implies f(x)=2x^2+\sin x+C$

$f(0)=1 \implies 1=0+C$, so $C=1$. Thus, $f(x)=2x^2+\sin x+1$.

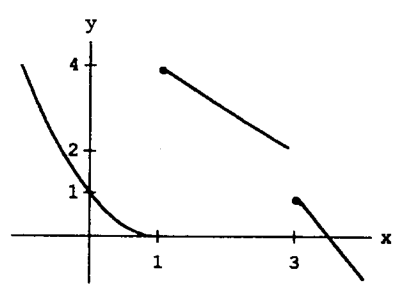
37. $f'(x)=\begin{cases} 1, & x<-3 \\ -1, & -3<x<-1 \\ 1, & -1<x<1 \\ -2, & 1<x \end{cases}$

f increases on $(-\infty,-3)$ and $[-1,1]$;
decreases on $[-3,-1]$ and $[1,\infty)$



38. $f'(x)=\begin{cases} 2(x-1), & x<1 \\ -1, & 1<x<3 \\ -2, & x>3 \end{cases}$

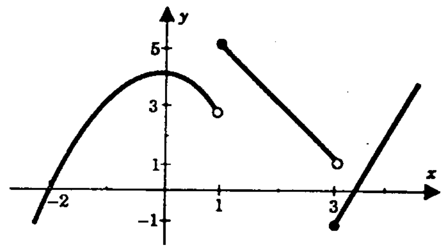
f decreases on $(-\infty,1)$ and $[1,\infty)$



39. $f'(x) = \begin{cases} -2x, & x < 1 \\ -2, & 1 < x < 3 \\ 3, & 3 < x \end{cases}$

f increases on $(-\infty, 0]$ and $[3, \infty)$;

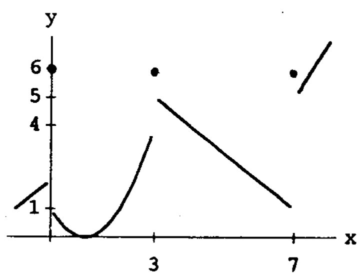
decreases on $[0, 1)$ and $[1, 3]$



40. $f'(x) = \begin{cases} 1, & x < 0 \\ 2(x-1), & 0 < x < 3 \\ -1, & 3 < x < 7 \\ 2, & x > 7 \end{cases}$

f increases on $(-\infty, 0]$, $[1,3]$, $(7, \infty)$;

decreases on $[0, 1]$ and $[3, 7)$



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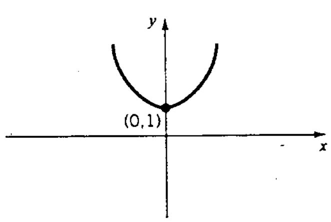

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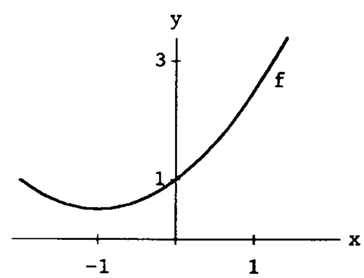

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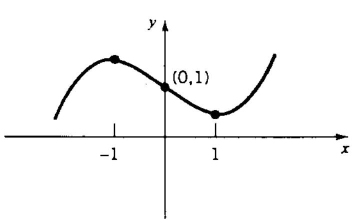

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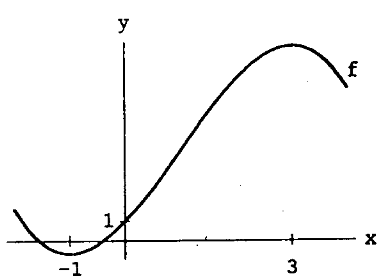

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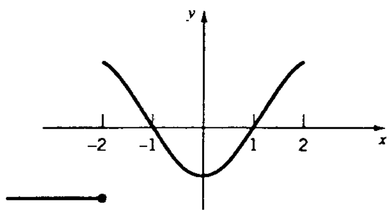

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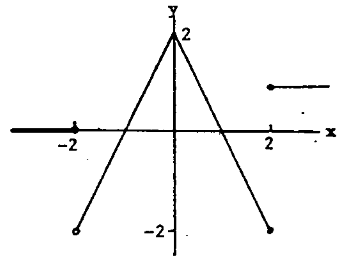

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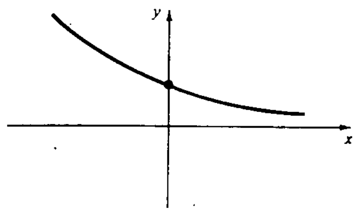
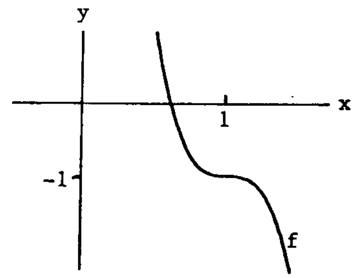


48.



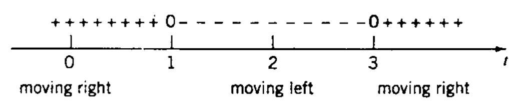
49. Not possible; f is increasing, so $f(2)$ must be greater than $f(-1)$.

50. Not possible; by the intermediate-value theorem, f must have a zero in $(3,5)$.

51. Let $x(t) = t^3 - 6t^2 + 9t + 2$. Then

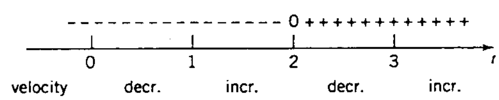
$v(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3)$

sign of v :



$a(t) = 6t - 12 = 6(t-2)$

sign of a :

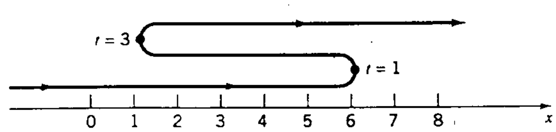


sign of v : + + + 0 − − − − − − 0 + + +

sign of a : − − − − − − − 0 + + + + + + +

1 2 3 x

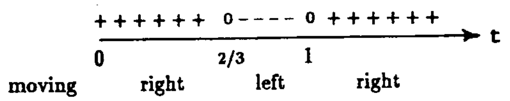
speed decr incr decr incr



52. Let $x(t) = (2t-1)(t-1)^2$. Then

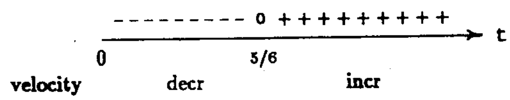
$v(t) = 2(t-1)(3t-2)$

sign of v :



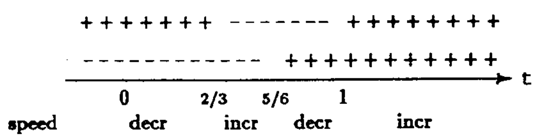
$a(t) = 2(6t-5)$

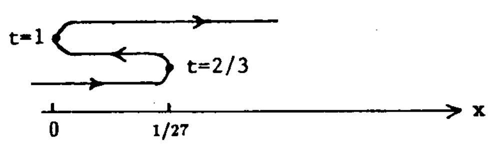
sign of a :



sign of v :

sign of a :

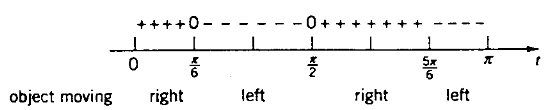




53. Let $x(t) = 2\sin 3t, \quad t \in [0, \pi]$. Then

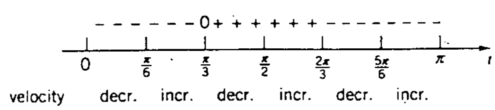
$v(t) = 6\cos 3t$

sign of v :



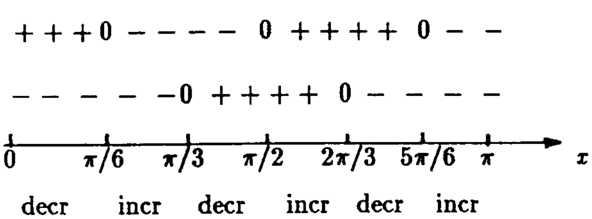
$a(t) = -18\sin 3t$

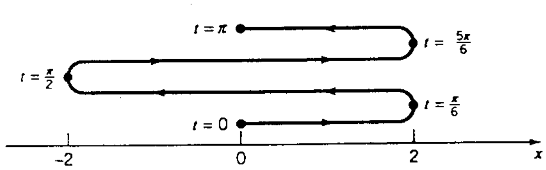
sign of a :



sign of v :

sign of a :

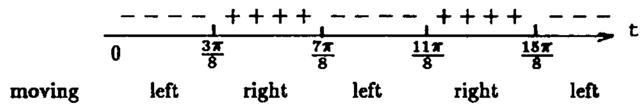




54. Let $x(t) = 3\cos\left(2t + \frac{\pi}{4}\right)$, $t \in [0, 2\pi]$. Then

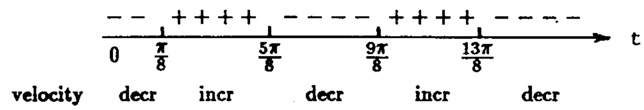
$v(t) = -6\sin\left(2t + \frac{\pi}{4}\right)$

sign of v :



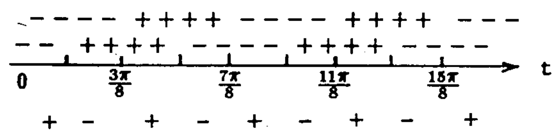
$a(t) = -12\cos\left(2t + \frac{\pi}{4}\right)$

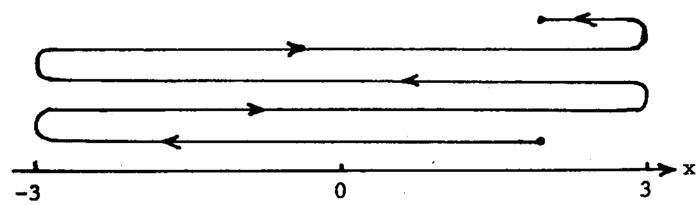
sign of a :



sign of v :

sign of a :





55. (a) $M \le L \le N$ (b) none (c) $M = L = N$

56. $\lim\limits_{x \to b^-} f(x) \le f(b)$ and $f(b) \ge \lim\limits_{x \to b^+} f(x)$

57. Suppose there is a c in (a, b) such that $f'(c) < 0$. Then by theorem 4.1.2, $f(c+h) < f(c)$ for h sufficiently small. This contradicts the fact that f is increasing on (a, b). Thus $f'(c) \ge 0$ for all c in (a, b).

58. A proof is outlined just below the statement of the theorem.

59. (a) $f'(x) = 2\sec x(\sec x \tan x) = 2\sec^2 x \tan x$ and $g'(x) = 2\tan x \sec^2 x$.

Therefore, $f'(x) = g'(x)$ for all $x \in I$.

(b) Evaluating $\sec^2 x - \tan^2 x = C$ at $x = 0$ gives $C = 1$.

60. $L'(x) = \dfrac{1}{x}$; $L'(ax) = \dfrac{1}{ax}\dfrac{d}{dx}(ax) = \dfrac{1}{ax}(a) = \dfrac{1}{x}$.

Therefore, $L(ax) - L(x) = C$ for some constant C. Now set $x = 1$. Then

$$L(a) - L(1) = C \quad \Longrightarrow \quad C = L(a) \text{ since } L(1) = 0$$

Thus, $L(ax) = L(a) + L(x)$ and the result follows by letting $x = b$.

61. Let f and g be functions such that $f'(x) = -g(x)$ and $g'(x) = f(x)$. Then:

(a) Differentiating $f^2(x) + g^2(x)$ with respect to x, we have

$$2f(x)f'(x) + 2g(x)g'(x) = -2f(x)g(x) + 2g(x)f(x) = 0.$$

Thus, $f^2(x) + g^2(x) = C$ (constant).

(b) $f(a) = 1$ and $g(a) = 0$ implies $C = 1$. The functions $f(x) = \cos(x - a)$, $g(x) = \sin(x - a)$ have these properties.

62. Suppose that f increases on (a, b) and is continuous on $[a, b]$. To show that f increases on $[a, b]$, we need only show that

$$f(a) < f(x) < f(b) \quad \text{for all} \quad x \in (a, b).$$

Suppose on the contrary that there exists $x_0 \in (a, b)$ for which $f(x_0) < f(a)$. Since f is continuous from the right at a and $f(a) - f(x_0) > 0$, there exists $\delta > 0$ such that

(*) $$|f(x) - f(a)| < f(a) - f(x_0) \quad \text{for all} \quad x \in (a, a + \delta).$$

Now choose $x_1 \in (a, a + \delta)$ with $x_1 < x_0$. On the one hand, by (*),

$$|f(x_1) - f(a)| < f(a) - f(x_0)$$

$$f(a) - f(x_1 < f(a) - f(x_0)$$

$$f(x_0) < f(x_1)$$

On the other hand, since f increases on (a, b),

$$f(x_1) < f(x_0)$$

The assumption that there exists $x_0 \in (a, b)$ for which $f(x_0) < f(a)$ has led to a contradiction. We can therefore conclude that

$$f(a) < f(x) \quad \text{for all} \quad x \in (a, b).$$

A similar argument shows that

$$f(x) < f(b) \quad \text{for all} \quad x \in (a, b).$$

63. Let $f(x) = x - \sin x$. Then $f'(x) = 1 - \cos x$.

(a) $f'(x) \geq 0$ for all $x \in (-\infty, \infty)$ and $f'(x) = 0$ only at $x = \dfrac{\pi}{2} + n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

It follows from Theorem 4.2.3 that f is increasing on $(-\infty, \infty)$.

(b) Since f is increasing on $(-\infty, \infty)$ and $f(0) = 0 - \sin 0 = 0$, we have:

$$f(x) > 0 \text{ for all } x > 0 \Rightarrow x > \sin x \text{ on } (0, \infty);$$

$$f(x) < 0 \text{ for all } x < 0 \Rightarrow x < \sin x \text{ on } (-\infty, 0).$$

64. (a) Let $h(x) = f(x) - g(x)$. Then $h'(x) = f'(x) - g'(x) > 0$ on $(0, c)$, and h is increasing on $(0, c)$. Since $h(0) = f(0) - g(0) = 0$, it follows that $h(x) > 0$ on $(0, c)$. Thus, $f(x) > g(x)$ on $(0, c)$.

(b) Again let $h(x) = f(x) - g(x)$. Then h is increasing on $(-c, 0)$ which implies that $h(x) < 0$ on this interval since $h(0) = 0$. Therefore, $f(x) < g(x)$ on $(-c, 0)$.

65. Let $f(x) = \tan x$ and $g(x) = x$ for $x \in [0, \pi/2)$. Then $f(0) = g(0) = 0$ and $f'(x) = \sec^2 x > g'(x) = 1$ for $x \in (0, \pi/2)$. Thus, $\tan x > x$ for $x \in (0, \pi/2)$ by Exercise 64(a).

66. Let $f(x) = \cos x - \left(1 - \frac{1}{2}x^2\right)$ for $x \in [0, \infty)$. Then $f(0) = 0$ and $f'(x) = \sin x + x = x - \sin x > 0$ for $x \in (0, \infty)$ by Exercise 63 (b). Thus, $f(x) > 0$ for $x \in (0, \infty)$ which implies $\cos x > 1 - \frac{1}{2}x^2$ on $(0, \infty)$.

67. Choose an integer $n > 1$. Let $f(x) = (1 + x)^n$ and $g(x) = 1 + nx$, $x > 0$. Then, $f(0) = g(0) = 1$ and $f'(x) = n(1 + x)^{n-1} > g'(x) = n$ since $(1 + x)^{n-1} > 1$ for $x > 0$. The result follows from Exercise 64(a).

68. Let $f(x) = \sin x - \left(x - \frac{1}{6}x^3\right)$. Then $f(0) = 0$ and $f'(x) = \cos x - \left(1 - \frac{1}{2}x^2\right) > 0$ by Exercise 66. Therefore, $f(x) > f(0) = 0$ for all $x \in (0, \infty)$ which implies $\sin x > x - \frac{1}{6}x^3$ on $(0, \infty)$.

69. $4° \cong 0.06981$ radians. By Exercises 63 and 68,

$$0.6981 - \frac{(0.6981)^3}{6} = 0.06975 < \sin 4° < 0.6981$$

70. (a) Let $f(x) = \cos x - (1 - \frac{1}{2}x^2 + \frac{1}{24}x^4)$. Then $f(0) = 0$ and $f'(x) = -\sin x + x - \frac{x^3}{6} < 0$ by Exercise 66. Therefore, $f(x) < f(0) = 0$ on all $x \in (0, \infty)$, which implies $\cos x < 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$ on $(0, \infty)$.

(b) $6° = \frac{\pi}{30}$. Using this for x in $1 - \frac{1}{2}x^2 < \cos x < 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$,

$\Longrightarrow$ $0.994517 < \cos 6° < 0.994522$.

71. Let $f(x) = 3x^4 - 10x^3 - 4x^2 + 10x + 9$, $x \in [-2, 5]$. Then $f'(x) = 12x^3 - 30x^2 - 8x + 10$.

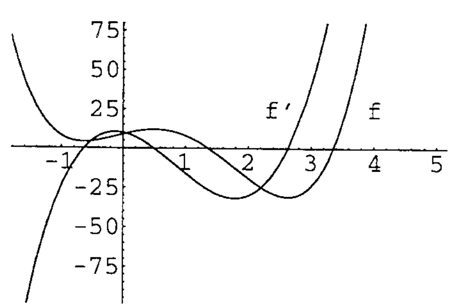
$f'(x) = 0$ at $x \cong -0.633,\ 0.5,\ 2.633$

f is decreasing on $[-2, -0.633]$

and $[0.5, 2.633]$

f is increasing on $[-0.633, 0.5]$

and $[2.633, 5]$

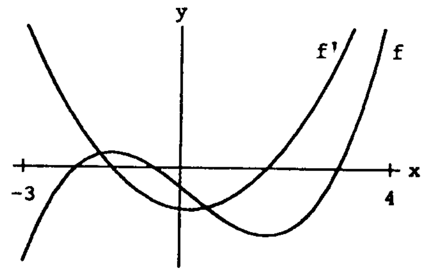


72. Let $f(x) = 2x^3 - x^2 - 13x - 6, \quad x \in [-3, 4]$. Then $f'(x) = 6x^2 - 2x - 13$.

$f'(x) = 0$ at $x \cong -1.315,\ 1.648$

f is decreasing on $[-1.315, 1.648]$

f is increasing on $[-3, -1.315]$ and $[1.648, 4]$



73. Let $f(x) = x\cos x - 3\sin 2x, \quad x \in [0, 6]$. Then $f'(x) = \cos x - x\sin x - 6\cos 2x$.

$f'(x) = 0$ at $x \cong 0.770,\ 2.155,\ 3.798,\ 5.812$

f is decreasing on $[0, 0.770],\ [2.155, 3.798]$

and $[5.812, 6]$

f is increasing on $[0.770, 2.155]$

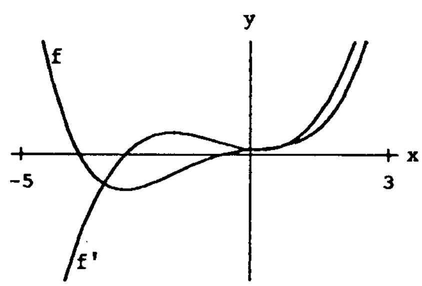
and $[3.798, 5.812]$

74. Let $f(x) = x^4 + 3x^3 - 2x^2 + 4x + 4, \quad x \in [-5, 3]$. Then $f'(x) = 4x^3 + 9x^2 - 4x + 4$.

$f'(x) = 0$ at $x \cong -2.747$

f is decreasing on $[-5, -2.747]$

f is increasing on $[-2.747, 3]$



PROJECT 4.2

1.

$$\begin{aligned}\frac{d}{dt}\left[mgy + \tfrac{1}{2}mv^2\right] &= mg\frac{dy}{dt} + \frac{1}{2}m\frac{d}{dt}(v^2)\\ &= mgv + \frac{1}{2}m\left[2v\frac{dv}{dt}\right]\\ &= mgv + mv\frac{dv}{dt}\\ &= mgv + mv(-g) \quad (\text{since} \quad dv/dt = a = -g)\\ &= mgv - mgv = 0\end{aligned}$$

2. By Problem 1, $mgy + \frac{1}{2}mv^2 = C$ (constant). Since $v = 0$ at height $y = y_0$, we have $C = mgy_0$. Thus,

$$mgy_0 = mgy + \tfrac{1}{2}mv^2 \quad \text{and} \quad |v| = \sqrt{2g(y_0 - y)}$$

3. $y(t) = \frac{1}{2}\,gt^2 + y_0 \quad \Longrightarrow \quad gt = \sqrt{2g(y_0 - y)}$

$v(t) = y'(t) = -gt$ Therefore, $|v(t)| = \sqrt{2g(y_0 - y)}$.

4. Set $y_0 = 150$ $y = 0$ and $g = 9.8$ in the equation $|v| = \sqrt{2g(y_0 - y)}$. Then

$$|v| = \sqrt{2(9.8)(150)} \cong 54.22 \text{ m/sec}$$

SECTION 4.3

1. $f'(x) = 3x^2 + 3 > 0$; no critical nos, no local extreme values

2. $f'(x) = 8x^3 - 8x = 8x(x^2 - 1)$; critical nos $-1, 0, 1$

$f''(x) = 24x^2 - 8$; $f''(-1) = f''(1) = 16 > 0$, $f''(0) = -8 < 0$;

$f(0) = 6$ local max, $f(-1) = 4$ local min, $f(1) = 4$ local min

3. $f'(x) = 1 - \dfrac{1}{x^2}$; critical nos $-1, 1$

$f''(x) = \dfrac{2}{x^3}$, $f''(-1) = -2$, $f''(1) = 2$ $f(-1) = -2$ local max, $f(1) = 2$ local min

4. $f'(x) = 2x + \dfrac{6}{x^3} = \dfrac{2x^4 + 6}{x^3}$; no critical nos (note: 0 is not in the domain of f), no local extreme values

5. $f'(x) = 2x - 3x^2 = x(2 - 3x)$; critical nos $0, \frac{2}{3}$

$f''(x) = 2 - 6x$; $f''(0) = 2$, $f''\left(\frac{2}{3}\right) = -2$

$f(0) = 0$ local min, $f\left(\frac{2}{3}\right) = \frac{4}{27}$ local max

6. $f'(x) = -2(1 - x)(1 + x) + (1 - x)^2 = (x - 1)(3x + 1)$; critical nos $-\frac{1}{3}, 1$

$f''(x) = (1 + 3x) + 3(x - 1) = 2(3x - 1)$; $f''\left(-\frac{1}{3}\right) = -4$, $f''(1) = 4$

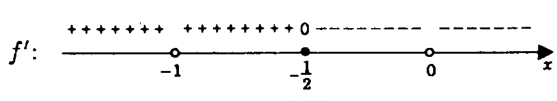
$f\left(-\frac{1}{3}\right) = \frac{32}{27}$ local max, $f(1) = 0$ local min

7. $f'(x) = \dfrac{2}{(1 - x)^2}$; no critical nos, no local extreme values

8. $f'(x) = \dfrac{(2 + x)(-3) - (2 - 3x)(1)}{(2 + x)^2} = -\dfrac{8}{(2 + x)^2}$; no critical nos (note: -2 is not in the domain of f), no local extreme values

9. $f'(x) = -\dfrac{2(2x + 1)}{x^2(x + 1)^2}$; critical no $-\dfrac{1}{2}$

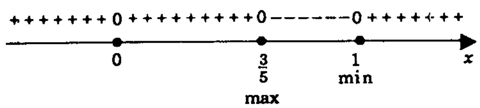
$f\left(-\frac{1}{2}\right) = -8$ local max



10. $f(x) = \begin{cases} x^2 - 16, & x < -4 \\ 16 - x^2, & -4 \le x < 4 \\ x^2 - 16, & x \ge 4 \end{cases} \qquad f'(x) = \begin{cases} 2x, & x < -4 \\ -2x, & -4 < x < 4 \\ 2x, & x > 4 \end{cases}$

critical nos $-4, 0, 4$; $f(-4) = f(4) = 0$ local minima, $f(0) = 16$ local max

11. $f'(x) = x^2(5x-3)(x-1)$; critical nos $0, \frac{3}{5}, 1$

f': 

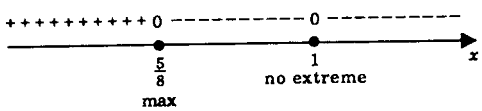
$f\left(\dfrac{3}{5}\right) = \dfrac{2^2 3^3}{5^5}$ local max

$f(1) = 0$ local min

no local extreme at 0

12. $f'(x) = 3\left(\dfrac{x-2}{x+2}\right)^2 \dfrac{4}{(x+2)^2} \geq 0$; critical no 2, no local extreme values

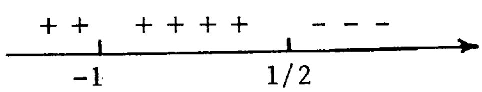
13. $f'(x) = (5-8x)(x-1)^2$; critical nos $\frac{5}{8}, 1$

f': 

$f\left(\frac{5}{8}\right) = \frac{27}{2048}$ local max

no local extreme at 1

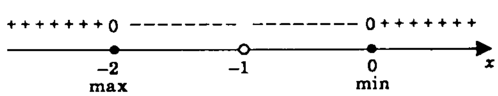
14. $f'(x) = -(1+x)^3 + (1-x)(3)(1+x)^2 = 2(1+x)^2(1-2x)$; critical nos $-1, \frac{1}{2}$

f' : 

$f\left(\frac{1}{2}\right) = \frac{27}{16}$ local max

no local extreme at -1

15. $f'(x) = \dfrac{x(2+x)}{(1+x)^2}$; critical nos $-2, 0$

f': 

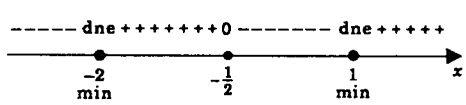
$f(-2) = -4$ local max

$f(0) = 0$ local min

16. $f(x) = \begin{cases} \dfrac{-x}{1-x}, & x < 0 \\ \dfrac{x}{1+x}, & x \geq 0 \end{cases}$ $\quad f'(x) = \begin{cases} \dfrac{-1}{(1-x)^2}, & x < 0 \\ \dfrac{1}{(1+x)^2}, & x > 0 \end{cases}$

critical no 0, $f(0) = 0$ local min

17. $f'(x) = \begin{cases} 2x+1, & x < -2, x > 1 \\ -(2x+1), & -2 < x < 1 \end{cases}$; critical nos $-2, -\frac{1}{2}, 1$

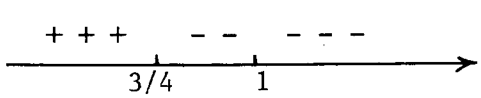
f': 

$f(-2) = 0$ local min

$f\left(-\frac{1}{2}\right) = \frac{9}{4}$ local max

$f(1) = 0$ local min

18. $f'(x) = (1-x)^{1/3} - \frac{1}{3}x(1-x)^{-2/3} = \dfrac{3-4x}{3(1-x)^{2/3}}$; critical nos $\frac{3}{4}, 1$

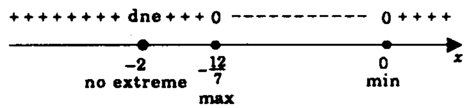
f' : 

$f(3/4) = \dfrac{3}{4^{4/3}}$ local max

no local extreme at 1

19. $f'(x) = \frac{1}{3}x(7x+12)(x+2)^{-2/3}$; critical nos $-2,\ -\frac{12}{7},\ 0$

f':

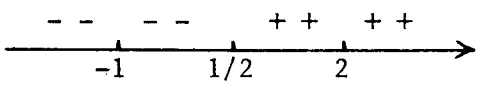


$f(-\frac{12}{7}) = \frac{144}{49}\left(\frac{2}{7}\right)^{1/3}$ local max

$f(0) = 0$ local min

20. $f'(x) = \dfrac{-1}{(x+1)^2} + \dfrac{1}{(x-2)^2} = \dfrac{3(2x-1)}{(x+1)^2(x-2)^2}$; critical no $\frac{1}{2}$

f' :

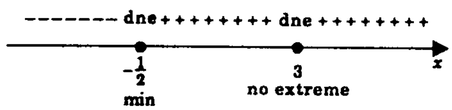


$f\left(\frac{1}{2}\right) = \frac{4}{3}$ local min

21. $f(x) = \begin{cases} 2-3x, & x \le -\frac{1}{2} \\ x+4, & -\frac{1}{2} < x < 3 \\ 3x-2, & 3 \le x \end{cases}$ $\qquad f'(x) = \begin{cases} -3, & x < -\frac{1}{2} \\ 1, & -\frac{1}{2} < x < 3 \\ 3, & 3 < x \end{cases}$

critical nos $-\frac{1}{2}$, 3

f':



$f\left(-\frac{1}{2}\right) = \frac{7}{2}$ local min

no local extreme at 3

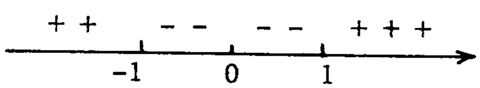
22. $f'(x) = \frac{7}{3}\,x^{4/3} - \frac{7}{3}\,x^{-2/3} = \dfrac{7}{3}\,\dfrac{x^2-1}{x^{2/3}}$; critical nos $-1, 0, 1$

$f(-1) = 6$ local max,

$f(1) = -6$ local min

no local extreme at 0

f' :

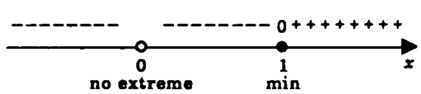


23. $f'(x) = \frac{2}{3}x^{-4/3}(x-1)$; critical nos 0, -1

$f(1) = 3$ local min

no local extreme at 0

f':

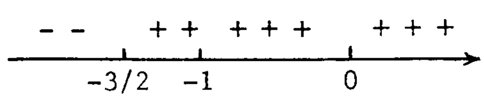


24. $f'(x) = \dfrac{(x+1)3x^2 - x^3}{(x+1)^2} = \dfrac{x^2(2x+3)}{(x+1)^2}$; critical nos $-\frac{3}{2}$, 0

$f\left(-\frac{3}{2}\right) = \frac{27}{4}$ local min

no local extreme at 0

f' :

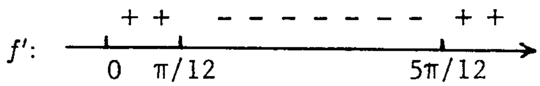


25. $f'(x) = \cos x - \sin x$; critical nos $\frac{1}{4}\pi,\ \frac{5}{4}\pi$

$f''(x) = -\sin x - \cos x, \quad f''\left(\frac{1}{4}\pi\right) = -\sqrt{2}, \quad f''\left(\frac{5}{4}\pi\right) = \sqrt{2}$

$f(\frac{1}{4}\pi) = \sqrt{2}$ local max, $\quad f\left(\frac{5}{4}\pi\right) = -\sqrt{2}$ local min

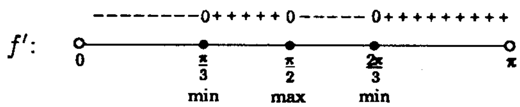
26. $f'(x) = 1 - 2\sin 2x$; critical nos $\frac{\pi}{12}, \frac{5\pi}{12}$



$f\left(\frac{1}{12}\pi\right) = \frac{\pi}{12} + \frac{\sqrt{3}}{2}$ local max

$f\left(\frac{5}{12}\pi\right) = \frac{5\pi}{12} - \frac{\sqrt{3}}{2}$ local min

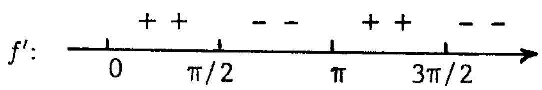
27. $f'(x) = \cos x\,(2\sin x - \sqrt{3})$; critical nos $\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{2}{3}\pi$



$f\left(\frac{1}{3}\pi\right) = f\left(\frac{2}{3}\pi\right) = -\frac{3}{4}$ local mins

$f\left(\frac{1}{2}\pi\right) = 1 - \sqrt{3}$ local max

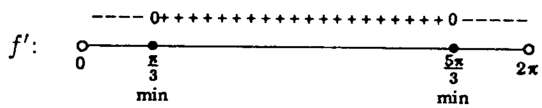
28. $f'(x) = 2\sin x\cos x$; critical nos $\frac{1}{2}\pi, \pi, \frac{3}{2}\pi$



$f\left(\frac{1}{2}\pi\right) = 1 = f\left(\frac{3}{2}\pi\right)$ local max

$f(0) = 0$ local min

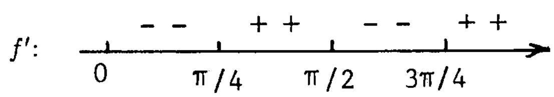
29. $f'(x) = \cos^2 x - \sin^2 x - 3\cos x + 2 = (2\cos x - 1)(\cos x - 1)$ critical pts $\frac{1}{3}\pi, \frac{5}{3}\pi$



$f\left(\frac{1}{3}\pi\right) = \frac{2}{3}\pi - \frac{5}{4}\sqrt{3}$ local min

$f\left(\frac{5}{3}\pi\right) = \frac{10}{3}\pi + \frac{5}{4}\sqrt{3}$ local max

30. $f'(x) = 6\sin^2 x\cos x - 3\cos x = 3\cos x(2\sin^2 x - 1)$; critical nos $\frac{1}{4}\pi, \frac{1}{2}\pi, \frac{3}{4}\pi$



$f\left(\frac{1}{4}\pi\right) = f\left(\frac{3}{4}\pi\right) = -\sqrt{2}$ local mins

$f\left(\frac{1}{2}\pi\right) = -1$ local max

31. (i) f increases on $(c-\delta, c]$ and decreases on $[c, c+\delta)$.

(ii) f decreases on $(c-\delta, c]$ and increases on $[c, c+\delta)$.

(iii) If $f'(x) > 0$ on $(c-\delta, c) \cup (c, c+\delta)$, then, since f is continuous at c, f increases on $(c-\delta, c]$ and also on $[c, c+\delta)$. Therefore, in this case, f increases on $(c-\delta, c+\delta)$. A similar argument shows that, if $f'(x) < 0$ on $(c-\delta, c) \cup (c, c+\delta)$, then f decreases on $(c-\delta, c+\delta)$.

32. Set $g(x) = -f(-x)$ and apply the proof of the second derivative test already given.

33. Solving $f'(x) = 2ax + b = 0$ gives a critical point at $x = -\frac{b}{2a}$. Since $f''(x) = 2a$, f has a local maximum at $-\frac{b}{2a}$ if $a < 0$ and a local minimum at $-\frac{b}{2a}$ if $a > 0$.

34. Setting $f'(x) = 3ax^2 + 2bx + c = 0$ and checking the discriminant, we get

(1) 2 local extrema if $b^2 > 3ac$

(2) 1 local extrema if $b^2 = 3ac$

(3) 0 local extrema if $b^2 < 3ac$

35.

$$P(x) = x^4 - 8x^3 + 22x^2 - 24x + 4$$
$$P'(x) = 4x^3 - 24x^2 + 44x - 24$$
$$P''(x) = 12x^2 - 48x + 44$$

Since $P'(1) = 0$, $P'(x)$ is divisible by $x - 1$. Division by $x - 1$ gives

$$P'(x) = (x-1)\left(4x^2 - 20x + 24\right) = 4(x-1)(x-2)(x-3).$$

The critical pts are 1, 2, 3. Since

$$P''(1) > 0, \quad P''(2) < 0, \quad P''(3) > 0,$$

$P(1) = -5$ is a local min, $P(2) = -4$ is a local max, and $P(3) = -5$ is a local min.

Since $P'(x) < 0$ for $x < 0$, P decreases on $(-\infty, 0]$. Since $P(0) > 0$, P does not take on the value 0 on $(-\infty, 0]$.

Since $P(0) > 0$ and $P(1) < 0$, P takes on the value 0 at least once on $(0, 1)$. Since $P'(x) < 0$ on $(0, 1)$, P decreases on $[0, 1]$. It follows that P takes on the value zero only once on $[0, 1]$.

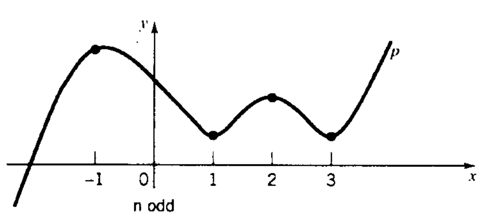
Since $P'(x) > 0$ on $(1, 2)$ and $P'(x) < 0$ on $(2, 3)$, P increases on $[1, 2]$ and decreases on $[2, 3]$. Since $P(1)$, $P(2)$, $P(3)$ are all negative, P cannot take on the value 0 between 1 and 3.

Since $P(3) < 0$ and $P(100) > 0$, P takes on the value 0 at least once on $(3, 100)$. Since $P'(x) > 0$ on $(3, 100)$, P increases on $[3, 100]$. It follows that P takes on the value zero only once on $[3, 100]$.

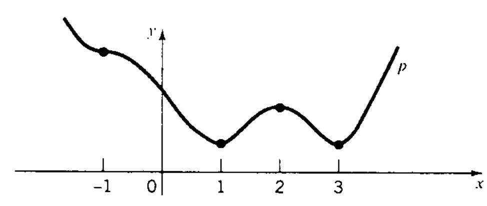
Since $P'(x) > 0$ on $(100, \infty)$, P increases on $[100, \infty)$. Since $P(100) > 0$, P does not take on the value 0 on $[100, \infty)$.

36. f has a local maximum at $x = 0$; f has a local minimum at $x = -1$ and $x = 2$.

37. (a)



(b)



38. Let $f(x) = Ax^2 + Bx + C$. Then $f'(x) = 2Ax + B$.

$f(-1) = 3 \implies A - B + C = 3$; $f(3) = -1 \implies 9A + 3B + C = -1$

Since f has a minimum at $x = 2$, $f'(2) = 4A + B = 0$

Solving for A, B, C, we get $A = \frac{1}{2}$, $B = -2$, $C = \frac{1}{2}$.

39. Let $f(x) = \dfrac{ax}{x^2 + b^2}$. Then $f'(x) = \dfrac{a\left(b^2 - x^2\right)}{\left(b^2 + x^2\right)^2}$. Now

$$f'(0) = \frac{a}{b^2} = 1 \Rightarrow a = b^2 \quad \text{and} \quad f'(x) = \frac{b^2\left(b^2 - x^2\right)}{\left(b^2 + x^2\right)^2}$$

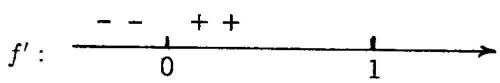
$$f'(-2) = \frac{b^2\left(b^2 - 4\right)}{\left(b^2 + 4\right)^2} = 0 \Rightarrow b = \pm 2$$

Thus, $a = 4$ and $b = \pm 2$.

40. (a) $f(x) = x^p(1-x)^q$, $p, q \geq 2$; $f'(x) = x^{p-1}(1-x)^{q-1}[p - (p+q)x]$

$$f'(x) = 0 \quad \Longrightarrow \quad x = 0,\ x = 1,\ x = \frac{p}{p+q}$$

(b) p even, $p - 1$ odd:



f has a local min at $x = 0$

(c) q even, $q - 1$ odd:

f has a local min at $x = 1$

(d) $f''\left(\dfrac{p}{p+q}\right) = -(p+q)\left(\dfrac{p}{p+q}\right)^{p-1}\left(\dfrac{q}{p+q}\right)^{q-1} < 0 \quad \Rightarrow \quad f$ has a local max at $x = \dfrac{p}{p+q}$.

41. Let δ be any positive number and consider f on the interval $(-\delta, \delta)$. Let n be a positive integer such that

$$0 < \frac{1}{\frac{\pi}{2} + 2n\pi} < \delta \quad \text{and} \quad 0 < \frac{1}{\frac{-\pi}{2} + 2n\pi} < \delta.$$

Then

$$f\left(\frac{1}{\frac{\pi}{2} + 2n\pi}\right) > 0 \quad \text{and} \quad f\left(\frac{1}{\frac{-\pi}{2} + 2n\pi}\right) < 0.$$

Thus f takes on both positive and negative values in every interval centered at 0 and it follows that f cannot have a local maximum or minimum at 0.

42. If P is a maximum at $x = x_0$, then $P'(x_0) = 0$. But, $P'(x_0) = R(x_0) - C(x_0)$. Therefore, $R'(x_0) = C'(x_0)$.

43. The function $D(x) = \sqrt{x^2 + [f(x)]^2}$ gives the distance from the origin to the point $(x, f(x))$ on the graph of f. Since the graph of f does not pass through the origin,

$$D'(x) = \frac{x + f(x)f'(x)}{\sqrt{x^2 + [f(x)]^2}}$$

is defined for all $x \in \text{dom}(f)$. Suppose that D has a local extreme value at c. Then

$$D'(c) = \frac{c + f(c)f'(c)}{\sqrt{c^2 + [f(c)]^2}} = 0 \Rightarrow c + f(c)f'(c) = 0 \text{ and } f'(c) = -\frac{c}{f(c)}$$

Suppose that $c \neq 0$. The slope of the line through $(0,0)$ and $(c, f(c))$ is given by $m_1 = \frac{f(c)}{c}$ and the slope of the tangent line to the graph of f at $x = c$ is given by $m_2 = f'(c) = -\frac{c}{f(c)}$. Since $m_1 m_2 = -1$, these two lines are perpendicular. If $c = 0$, then the tangent line to the graph of f is horizontal and the line through $(0,0)$ and $(0, f(0))$ is vertical.

44. If p is a polynomial of degree n, then p' has degree $n-1$. This implies that p' has at most $n-1$ zeros, and it follows that p has at most $n-1$ local extreme values.

45. (a) Let $f(x) = x^4 - 7x^2 - 8x - 3$. Then $f'(x) = 4x^3 - 14x - 8$ and $f''(x) = 12x^2 - 14$. Since $f'(2) = -4 < 0$ and $f'(3) = 58 > 0$, f' has at least one zero in $(2,3)$. Since $f''(x) > 0$ for $x \in (2,3)$, f' is increasing on this interval and so it has exactly one zero. Thus, f has exactly one critical number c in $(2,3)$.

(b) $c \cong 2.1091$; f has a local minimum at c.

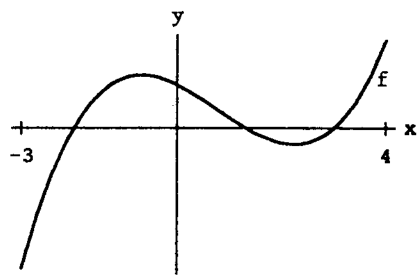
46. (a) Let $f(x) = x \cos x$. Then $f'(x) = \cos x - x \sin x$ and $f''(x) = -2 \sin x - x \cos x$. Since $f'(0) = 1 > 0$ and $f'(\pi/2) = -\pi/2 < 0$, f' has at least one zero in $(0, \pi/2)$. Since $f''(x) < 0$ for $x \in (0, \pi/2)$, f' is decreasing on this interval and so it has exactly one zero. Thus, f has exactly one critical number c in $(0, \pi/2)$.

(b) $c \cong 0.8603$; f has a local maximum at c.

47. (a) Let $f(x) = \sin x + \frac{x^2}{2} - 2x$. Then $f'(x) = \cos x + x - 2$ and $f''(x) = -\sin x + 1$. Since $f'(2) = -0.4161 < 0$ and $f'(3) = 0.01 > 0$, f' has at least one zero in $(2,3)$. Since $f''(x) > 0$ for $x \in (2,3)$, f' is increasing on this interval and so it has exactly one zero. Thus, f has exactly one critical number c in $(2,3)$.

(b) $x_{n+1} = x_n - \frac{\cos x_n + x_n - 2}{-\sin x_n + 1}$; $x_1 = 3$, $x_2 = 2.9883$, $x_3 = 2.9883$. Thus $c \cong 2.9883$; f has a local minimum at c.

48. (a)

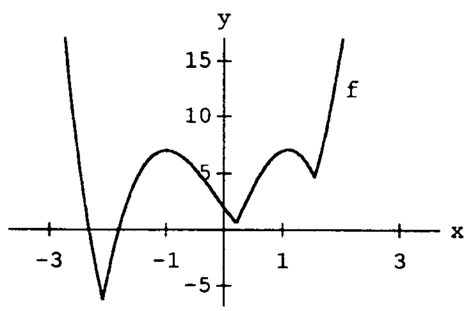


critical numbers: $x_1 \cong -0.962,\ x_2 \cong 2.248$

local extreme values: $f(-0.692) \cong 29.342,\ f(2.248) \cong -8.766$

(b) f is increasing on $[-3, -0.692]$, and $[2.248, 4]$; f is decreasing on $[-0.692, 2.248]$

49. (a)



critical numbers: $x_1 \cong -2.085,\ x_2 \cong -1,\ x_3 \cong 0.207,\ x_4 \cong 1.096,\ x_5 = 1.544$

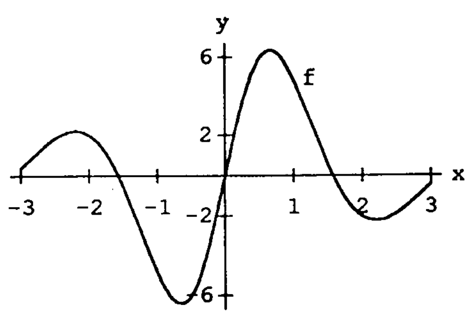
local extreme values: $f(-2.085) \cong -6.255,\ f(-1) = 7,\ f(0.207) \cong 0.621,\ f(1.096) \cong 7.097,$

$f(1.544) \cong 4.635$

(b) f is increasing on $[-2.085, -1]$, $[0.207, 1.096]$, and $[1.544, 4]$

f is decreasing on $[-4, -2.085]$, $[-1, 0.207]$, and $[1.096, 1.544]$

50. (a)



critical numbers: $x_1 \cong -2.204,\ x_2 \cong -0.654,\ x_3 \cong 0.654,\ x_4 \cong 2.204$

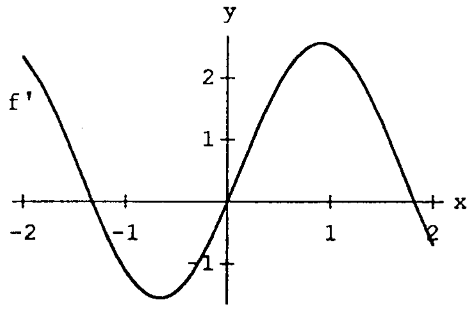
local extreme values: $f(-2.204) \cong 2.226,\ f(-0.654) \cong -6.634,\ f(0.654) \cong 6.634,$

$f(2.204) \cong -2.226$

(b) f is increasing on $[-3. - 2.204]$, $[-0.654, 0.654]$, and $[2.204, 3]$

f is decreasing on $[-2.204, -0.654]$, and $[0.654, 2.204]$

51.



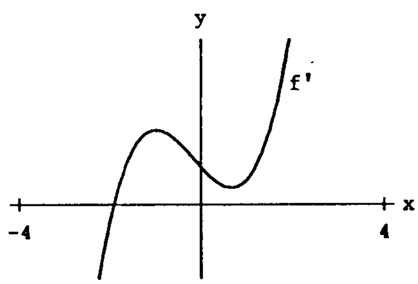
critical numbers of f : $x_1 \cong -1.326$, $x_2 = 0$, $x_3 \cong 1.816$

$f''(-1.326) \cong -4 < 0 \quad \Rightarrow \; f$ has a local maximum at $x = -1.326$

$f''(0) = 4 > 0 \quad \Rightarrow \; f$ has a local minimum at $x = 0$

$f''(1.816) \cong -4 \quad \Rightarrow \; f$ has a local maximum at $x = 1.816$

52.

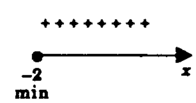


critical number of f : $x_1 \cong -1.935$

$f''(-1.935) \cong 14.60 > 0 \quad \Rightarrow \; f$ has a local minimum at $x = -1.935$

SECTION 4.4

1. $f'(x) = \frac{1}{2}(x+2)^{-1/2}, \quad x > -2;$ $\qquad f'$:

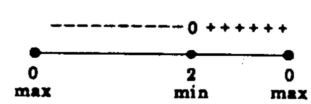


critical no -2;

$f(-2) = 0$ endpt and abs min; as $x \to \infty$, $f(x) \to \infty$; so no abs max

2. $f'(x) = 2x - 3$; critical no $\frac{3}{2}$; $f\left(\frac{3}{2}\right) = -\frac{1}{4}$ local and abs min

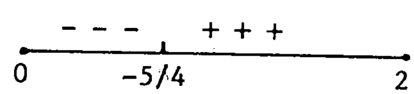
3. $f'(x) = 2x - 4, \quad x \in (0,3);$ $\qquad f'$:



critical nos 0, 2, 3;

$f(0) = 1$ endpt and abs max, $f(2) = -3$ local and abs min, $f(3) = -2$ endpt max

4. $f'(x) = 4x + 5, \quad x \in (-2,0);$ $\qquad f'$:



critical nos $-2, -\frac{5}{4}, 0$;

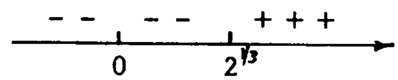
$f(-2) = -3$ endpt max, $f\left(-\frac{5}{4}\right) = -\frac{33}{8}$ local and abs min, $f(0) = -1$ endpt and abs max

5. $f'(x) = 2x - \dfrac{1}{x^2} = \dfrac{2x^3 - 1}{x^2},\ x \neq 0;\quad f'(x) = 0$ at $x = 2^{-1/3}$

critical no $2^{-1/3}$; $\quad f''(x) = 2 + \dfrac{2}{x^3},\quad f''(2^{-1/3}) = 6$

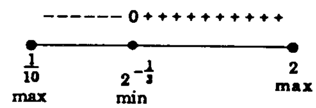
$f(2^{-1/3}) = 2^{-2/3} + 2^{1/3} = 2^{-2/3} + 2 \cdot 2^{-2/3} = 3 \cdot 2^{-2/3}$ local min

6. $f'(x) = 1 - \dfrac{2}{x^3}$

f' :


critical no $2^{1/3}$; $\quad f\left(2^{1/3}\right) = 3(2)^{-2/3}$ local min

7. $f'(x) = \dfrac{2x^3 - 1}{x^2},\quad x \in \left(\dfrac{1}{10}, 2\right);$

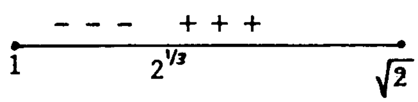
f' :


critical nos $\frac{1}{10}$, $2^{-1/3}$, 2;

$f\left(\frac{1}{10}\right) = 10\frac{1}{100}$ endpt and abs max, $\quad f\left(2^{-1/3}\right) = 3 \cdot 2^{-2/3}$ local and abs min,

$f(2) = 4\frac{1}{2}$ endpt max

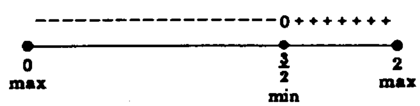
8. $f'(x) = 1 - \dfrac{2}{x^3},\quad x \in (1, \sqrt{2})$

f' :


critical nos $1, 2^{1/3}, \sqrt{2}$; $\quad f(1) = 2$ endpt and abs max

$f\left(2^{1/3}\right) = 3(2)^{-2/3}$ local and abs min, $f\left(\sqrt{2}\right) = \sqrt{2} + \frac{1}{2}$ endpt max

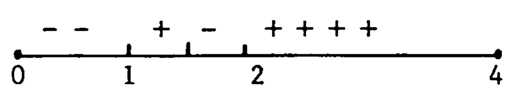
9. $f'(x) = 2x - 3,\quad x \in (0, 2);$

f' :


critical nos $0, \frac{3}{2}, 2$;

$f(0) = 2$ endpt and abs max, $\quad f\left(\frac{3}{2}\right) = -\frac{1}{4}$ local and abs min,

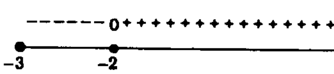
$f(2) = 0$ endpt max

10. $f'(x) = 2(x-1)(x-2)(2x-3);\quad x \in (0, 4)$ $\quad f'$:


critical nos $0, 1, \frac{3}{2}, 2, 4$; $\quad f(0) = 4$ endpt max, $\quad f(1) = 0$ local and abs min,

$f\left(\frac{3}{2}\right) = \frac{1}{16}$ local max, $f(2) = 0$ local and abs min, $f(4) = 36$ endpt and abs max

11. $f'(x) = \dfrac{(2-x)(2+x)}{(4+x^2)^2},\quad x \in (-3, 1);$

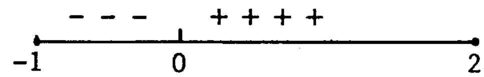
f' :


critical nos $-3, -2, 1$;

$f(-3) = -\frac{3}{13}$ endpt max, $\quad f(-2) = -\frac{1}{4}$ local and abs min,

$f(1) = \frac{1}{5}$ endpt and abs max

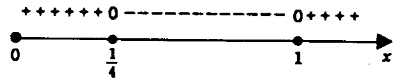
12. $f'(x) = \dfrac{2x}{(1+x^2)^2}, \quad x \in (-1, 2)$ f' :



critical nos $-1, 0, 2$; $f(-1) = -\frac{1}{2}$ endpt max, $f(0) = 0$ local and abs min,

$f(2) = \frac{4}{5}$ endpt and abs max

13. $f'(x) = 2\left(x - \sqrt{x}\right)\left(1 - \dfrac{1}{2\sqrt{x}}\right), \quad x > 0;$ f' :

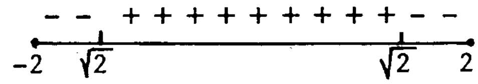


critical nos $0, \frac{1}{4}, 1$;

$f(0) = 0$ endpt and abs min, $f\left(\frac{1}{4}\right) = \frac{1}{16}$ local max, $f(1) = 0$ local and abs min;

as $x \to \infty$, $f(x) \to \infty$; so no abs max

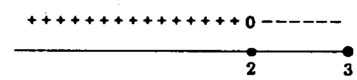
14. $f'(x) = \dfrac{2(2-x^2)}{(4-x^2)^{1/2}}, \quad x \in (-2, 2)$ f' :



critical nos $-2, -\sqrt{2}, \sqrt{2}, 2$; $f(-2) = 0$ endpt max, $f\left(-\sqrt{2}\right) = -2$ local and abs min,

$f\left(\sqrt{2}\right) = 2$ local and abs max, $f(2) = 0$ endpt min

15. $f'(x) = \dfrac{3(2-x)}{2\sqrt{3-x}}, \quad x < 3$ f' :

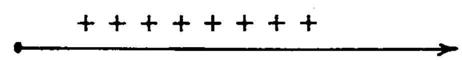


critical nos $2, 3$;

$f(2) = 2$ local and abs max, $f(3) = 0$ endpt min;

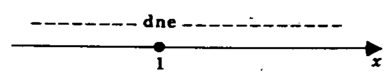
as $x \to -\infty$, $f(x) \to -\infty$; so no abs min

16. $f'(x) = \frac{1}{2}\left(x^{-1/2} + x^{-3/2}\right), \quad x > 0$ f' :



no critical nos; no extreme values.

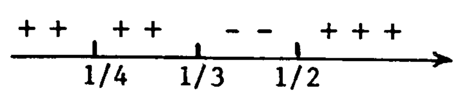
17. $f'(x) = -\frac{1}{3}(x-1)^{-2/3}, \quad x \neq 1;$ f' :



critical no 1;

no local extremes; $\left.\begin{array}{ll} \text{as} \quad x \to \infty, & f(x) \to -\infty \\ \text{as} \quad x \to -\infty, & f(x) \to \infty \end{array}\right\}$ no abs extremes

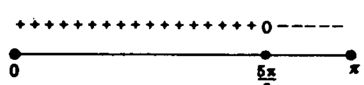
18. $f'(x) = \dfrac{8}{3}\,\dfrac{3x-1}{(4x-1)^{2/3}(2x-1)^{1/3}};$ f' :



critical nos $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}$; no extreme value at $\frac{1}{4}$

$f\left(\frac{1}{3}\right) = \frac{1}{3}$ local max, $f\left(\frac{1}{2}\right) = 0$ local min

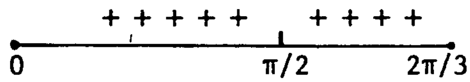
19. $f'(x) = \sin x\left(2\cos x + \sqrt{3}\right), x \in (0, \pi);$ f' :



critical nos $0, \frac{5}{6}\pi, \pi$;

$f(0) = -\sqrt{3}$ endpt and abs min, $f\left(\frac{5}{6}\pi\right) = \frac{7}{4}$ local and abs max, $f(\pi) = \sqrt{3}$ endpt min

20. $f'(x) = -\csc^2 x + 1, \quad x \in (0, 2\pi/3)$; f' :



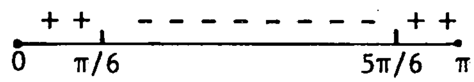
critical nos $\frac{1}{2}\pi, \frac{2}{3}\pi$ (note: 0 is not in the domain)

No extreme value at $\frac{1}{2}\pi$, $f\left(\frac{2}{3}\pi\right) = \frac{1}{3}\left(2\pi - \sqrt{3}\right)$ endpt and abs max

21. $f'(x) = -3\sin x\left(2\cos^2 x + 1\right) < 0, \quad x \in (0, \pi)$; critical nos $0, \pi$;

$f(0) = 5$ endpt and abs max, $f(\pi) = -5$ endpt and abs min

22. $f'(x) = 2\cos 2x - 1, \quad x \in (0, \pi)$: f' :



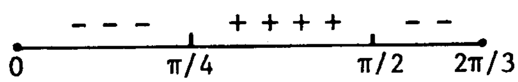
critical nos $0, \frac{1}{6}\pi, \frac{5}{6}\pi, \pi$; $f(0) = 0$ endpt min, $f\left(\frac{1}{6}\pi\right) = \frac{1}{2}\sqrt{3} - \frac{1}{6}\pi$ local and abs max,

$f\left(\frac{5}{6}\pi\right) = -\frac{1}{2}\sqrt{3} - \frac{5}{6}\pi$ local and abs min, $f(\pi) = -\pi$ endpt max

23. $f'(x) = \sec^2 x - 1 \geq 0, \quad x \in \left(-\frac{1}{3}\pi, \frac{1}{2}\pi\right)$; critical nos $-\frac{1}{3}\pi, 0$;

$f\left(-\frac{1}{3}\pi\right) = \frac{1}{3}\pi - \sqrt{3}$ endpt and abs min, no abs max

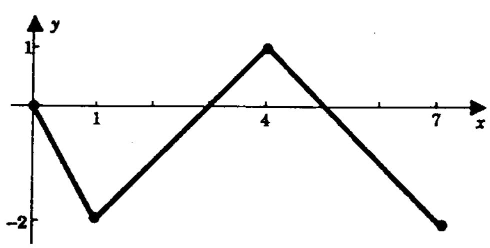
24. $f'(x) = 2\sin x\cos x(2\sin^2 x - 1), \quad x \in \left(0, \frac{2}{3}\pi\right)$; f' :



critical nos $0, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{2}{3}\pi$; $f(0) = 0$ endpt and abs max, $f\left(\frac{1}{4}\pi\right) = -\frac{1}{4}$ local and abs min,

$f\left(\frac{1}{2}\pi\right) = 0$ local and abs max, $f\left(\frac{2}{3}\pi\right) = -\frac{3}{16}$ endpt min

25.



$$f'(x) = \begin{cases} -2, & 0 < x < 1 \\ 1, & 1 < x < 4 \\ -1, & 4 < x < 7 \end{cases}$$

critical nos $0, 1, 4, 7$

$f(0) = 0$ endpt max, $f(1) = -2$ local and abs min,

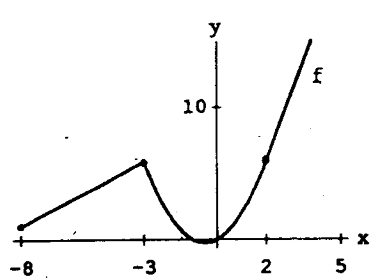
$f(4) = 1$ local and absolute max, $f(7) = -2$ endpt and abs min

26. $$f'(x) = \begin{cases} 1, & -8 < x < -3 \\ 2x + 1, & -3 < x \leq 2 \\ 5, & 2 < x < 5 \end{cases}$$

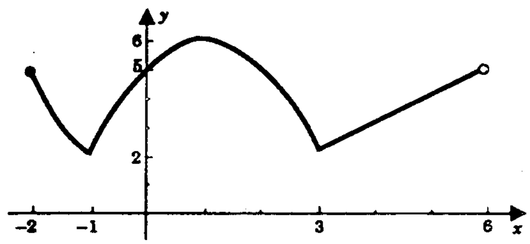
critical nos $-8, -3, -\frac{1}{2}$;

$f(-8) = 1$ endpt min, $f(-3) = 6$ local max

$f\left(-\frac{1}{2}\right) = -\frac{1}{4}$ local and abs min



27.



$$f'(x) = \begin{cases} 2x, & -2 < x < -1 \\ 2 - 2x, & -1 < x < 3 \\ 1, & 3 < x < 6 \end{cases}$$

critical nos $-2, -1, 1, 3$

$f(-2) = 5$ endpt max, $f(-1) = 2$ local and abs min,

$f(1) = 6$ local and abs max, $f(3) = 2$ local and abs min

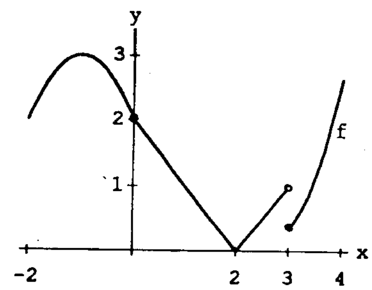
28. $$f'(x) = \begin{cases} -2x - 2, & -2 < x < 0 \\ -1, & 0 < x < 2 \\ 1, & 2 < x < 3 \\ (x-2)^2, & 3 < x < 4 \end{cases}$$

critical nos $-2, -1, 0, 2, 3, 4$;

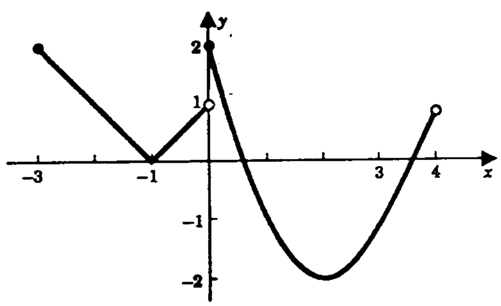
$f(-2) = 2$ endpt min, $f(-1) = 3$ local and abs max,

$f(0)$ not an extreme value, $f(2) = 0$ local and abs min,

$f(3) = \frac{1}{3}$ local min, $f(4) = \frac{8}{3}$ endpt max



29.



$$f'(x) = \begin{cases} -1, & -3 < x < -1 \\ 1, & -1 < x < 0 \\ 2x - 4, & 0 < x < 3 \\ 2, & 3 \leq x < 4 \end{cases}$$

critical nos $-3, -1, 0, 2$

$f(-3) = 2$ endpt and abs max, $f(-1) = 0$ local min,

$f(0) = 2$ local and abs max, $f(2) = -2$ local and abs min

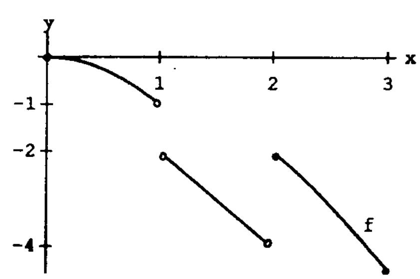
30. $$f'(x) = \begin{cases} -2x, & 0 < x < 1 \\ -2, & 1 < x < 2 \\ -x, & 2 < x < 3 \end{cases}$$

Note: 1 is not in the domain of f.

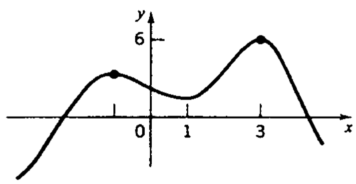
critical nos $0, 2, 3$;

$f(0) = 0$ endpt and abs max, $f(2) = -2$ local max,

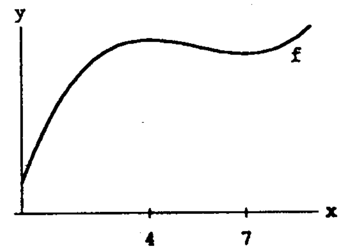
$f(3) = -\frac{9}{2}$ endpt and abs min



31.



32.



33. $f(-3) = 0$ and $f'(x) > 0$ on $(-3, -1)$

$\Rightarrow\; f(-1) > 0.$

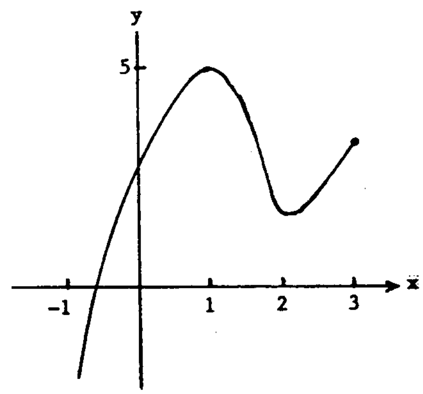
$f(3) = 0$ and $f'(x) > 0$ on $(1, 3)$ $\Rightarrow f(1) < 0.$

It now follows that f has a zero on $(-1, 1)$,

contradicting the fact that $f(x) \neq 0$ for

$x \in (-3, 3)$.

34.



35. Let $p(x) = x^3 + ax^2 + bx + c$. Then $p'(x) = 3x^2 + 2ax + b$ is a quadratic with discriminant $\Delta = 4a^2 - 12b = 4(a^2 - 3b)$. If $a^2 \leq 3b$, then $\Delta \leq 0$. This implies that $p'(x)$ does not change sign on $(-\infty, \infty)$ and hence p is either increasing on $(-\infty, \infty)$ (if $a \leq 0$) or decreasing ($a \geq 0$). In either case, p has no extreme values. On the other hand, if $a^2 - 3b > 0$, then $\Delta > 0$ and p' has two real zeros, c_1 and c_2, from which it follows that p has extreme values at c_1 and c_2. Thus, if p has no extreme values, then we must have $a^2 - 3b \leq 0$.

36. $f(x) = (1 + x)^r - (1 + rx), \quad x \geq -1.$

$f'(x) = r\left[(1 + x)^{r-1} - 1\right]; \quad f'(x) = 0 \quad \Longrightarrow \quad x = 0$

$f''(x) = r(r - 1)(1 + x)^{r-2}; \quad f''(0) = r(r - 1) > 0 \quad \Longrightarrow \quad f$ has a local minimum at $x = 0$

By Theorem 4.4.3, $f(0) = 0$ is the absolute minimum of f.

37. By contradiction. If f is continuous at c, then, by the first-derivative test (4.3.4), $f(c)$ is not a local maximum.

38. Since $f(c) \geq f(x)$ for all x in some open interval around c, and likewise $f(c) \leq f(x)$ for all x in some open interval around c, it follows that f must be constant on some open interval containing c.

39. If f is not differentiable on $(a,\, b)$, then f has a critical point at each point c in $(a,\, b)$ where $f'(c)$ does not exist. If f is differentiable on $(a,\, b)$, then by the mean-value theorem there exists c in $(a,\, b)$ where $f'(c) = [f(b) - f(a)]/(b - a) = 0$. This means c is a critical point of f.

40. We give a proof by contradiction. Suppose for no c in (c_1, c_2) is $f(c)$ a local minimum. By Theorem 2.6.2, f has a minimum on $[c_1, c_2]$ and, thus, this minimum must occur at c_1 or c_2. Suppose that $f(c_1)$ is an endpoint minimum. Then for some $\delta_1 > 0$,

$$(*) \qquad f(x) \geq f(c), \quad x \in [c_1, c_1 + \delta_1).$$

Since $f(c_1)$ is a local maximum, there exists $\delta_2 > 0$ such that

$$(**) \qquad f(x) \leq f(c_1), \quad x \in (c_1 - \delta_2, c_1 + \delta_2).$$

Set $\delta = \min[\delta_1, \delta_2]$. From (*) and (**), it follows that

$$f(x) = f(c_1), \quad x \in (c_1, c_1 + \delta).$$

This means that f has a local minimum on (c_1, c_2). The argument at c_2 is similar.

41. Let M be a positive number. Then

$$\begin{aligned} P(x) - M &\geq a_n x^n - \left(|a_{n-1}|x^{n-1} + \cdots + |a_1|x + |a_0| + M\right) \quad \text{for} \quad x > 0 \\ &\geq a_n x^n - \left(|a_{n-1}| + \cdots + |a_1| + |a_0| + M\right) \quad \text{for} \quad x > 1 \end{aligned}$$

It now follows that

$$P(x) - M \geq 0 \qquad \text{for} \quad x \geq K = \left(\frac{|a_{n-1}| + \cdots + |a_1| + |a_0| + M}{a_n}\right)^{1/n} + 1.$$

42. (a) $x(t) = A\sin(\omega t + \phi_0)$; $\quad x'(t) = \omega A\cos(\omega t + \phi_0)$; $\quad x''(t) = -\omega^2 A\sin(\omega t + \phi_0)$

Thus, $x''(t) + \omega^2 x(t) = 0$.

(b) Absolute max: $|A|$, absolute min: $-|A|$

43. Let R be a rectangle with its diagonals having length c, and let x be the length of one of its sides. Then the length of the other side is $y = \sqrt{c^2 - x^2}$ and the area of R is given by

$$A(x) = x\sqrt{c^2 - x^2}$$

Now

$$\begin{aligned} A'(x) &= \sqrt{c^2 - x^2} - \frac{x^2}{\sqrt{c^2 - x^2}} \\ &= \frac{c^2 - 2x^2}{\sqrt{c^2 - x^2}}, \end{aligned}$$

and

$$A'(x) = 0 \Longrightarrow x = \frac{\sqrt{2}}{2}c$$

It is easy to verify that A has a maximum at $x = \frac{\sqrt{2}}{2}c$. Since $y = \frac{\sqrt{2}}{2}c$ when $x = \frac{\sqrt{2}}{2}c$, it follows that the rectangle of maximum area is a square

44. Let $S = x^3 + y^3$ where $x + y + 16$. Then $S(x) = x^3 + (16 - x)^3$ and

$S'(x) = 3x^2 - 3(16 - x)^2 = 96(x - 8); \quad S'(x) = 0 \implies x = 8 \implies y = 8$

$S''(x) = 6x + 6(16 - x); \quad S''(8) = 96 \implies S$ has a local minimum at $x = 8$

It now follows that $S(8)$ is the absolute minimum of S.

45. Setting $R'(x) = \dfrac{v^2 \cos 2x}{16} = 0$, gives $x = \dfrac{\pi}{4}$. Since $R''(\frac{\pi}{4}) = -\dfrac{v^2}{8} < 0$, $x = \dfrac{\pi}{4}$ is a maximum.

46. Cut the wire into two pieces, one of length x and the other of length $L - x$. Suppose that the wire of length x is used to form the equilateral triangle, and the other piece is used to form the square. Then the area of the triangle is $\sqrt{3}\,x^2/36$, and the area of the square is $(L - x^2)/16$. Now, let

$$S(x) = \frac{\sqrt{3}}{36}\,x^2 + \frac{1}{16}\,(L - x)^2$$

Then

$$S'(x) = \frac{\sqrt{3}}{18}\,x - \frac{1}{8}\,(L - x) = \frac{4\sqrt{3} + 9}{72}\,x - \frac{1}{8}\,L$$

Setting $S'(x) = 0$ we find that

$$x = \frac{9}{4\sqrt{3} + 9}\,L. \cong 0.5650\,L$$

Now,

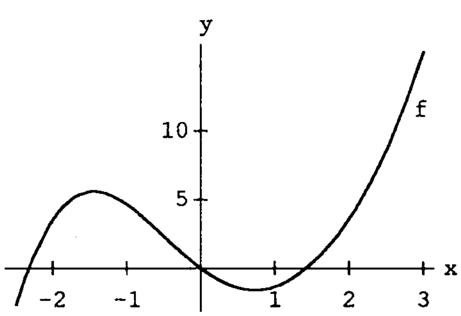
$$S(0) = \frac{1}{16}\,L^2 = 0.0625L^2 \quad \text{(absolute maximum)}$$

$$S\left(\frac{9}{4\sqrt{3} + 9}\,L\right) \cong 0.0390L^2 \quad \text{(absolute minimum)}$$

$$S(L) = \frac{\sqrt{3}}{36}\,L^2 = 0.0481L^2$$

To maximize the sum of the areas, use the wire to form a square; to minimize the sum, use $x \cong 0.5650\,L$ to form the triangle and the remainder to form the square.

47.



critical nos: $x_1 = -1.452$, $x_2 = 0.760$

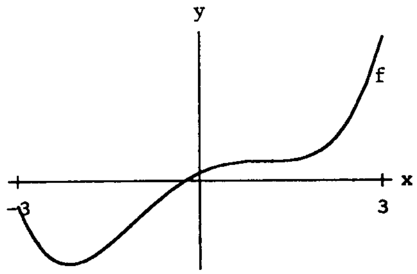
$f(-1.452)$ local maximum

$f(0.727)$ local minimum

$f(3)$ absolute maximum

$f(-2.5)$ absolute minimum

48.



critical nos: $x_1 = -2.179,\ x_2 = 1,\ x_3 = 1.158$

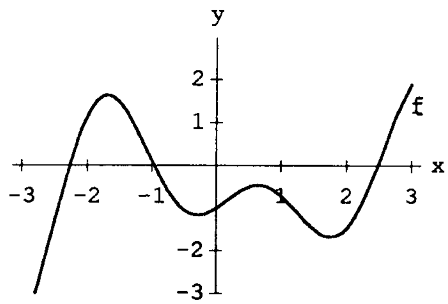
$f(1)$ local maximum

$f(-2.179),\ f(1.158)$ local minima

$f(3)$ absolute maximum

$f(-2.179)$ absolute minimum

49.



critical numbers: $x_1 = -1.683,\quad x_2 = -0.284,$

$x_3 = 0.645,\quad x_4 = 1.760$

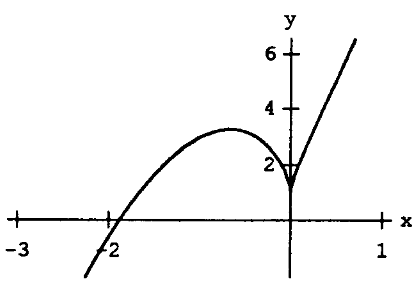
$f(-1.683), f(0.645)$ local maxima

$f(-0.284), f(1.760)$ local minima

$f(\pi)$ absolute maximum

$f(-\pi)$ absolute minimum

50.



critical nos: $x_1 = -0.684,\ x_2 = 0$

$f(-0.684)$ local maximum

$f(0)$ local minimum

$f(1)$ absolute maximum

$f(-3)$ absolute minimum

SECTION 4.5

1. Set $P = xy$ and $y = 40 - x$. We want to maximize

$$P(x) = x(40 - x), \quad 0 < x < 40.$$

$$P'(x) = 40 - 2x, \quad P'(x) = 0 \quad \Longrightarrow \quad x = 20.$$

Since P increases on (0, 20] and decreases on [20, 40), the abs max of P occurs when $x = 20$. Then, $y = 20$ and $xy = 400$.

The maximal value of xy is 400.

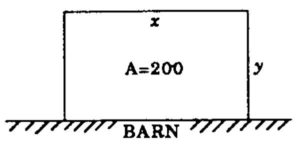
2. Set $A = xy$ and $2x + 2y = 24$ or $y = 12 - x$. We want to maximize

$$A(x) = x(12 - x), \quad 0 \le x \le 12.$$

$$A'(x) = 12 - 2x, \quad P'(x) = 0 \quad \Longrightarrow \quad x = 6.$$

Since A increases on $[0, 6]$ and decreases on $[6, 12]$, the abs max of A occurs when $x = 6$. Then, $y = 6$. The dimensions of the rectangle having perimeter 24 and maximum area are: 6×6.

3.



Minimize P

$P = x + 2y, \quad 200 = xy, \quad y = 200/x$

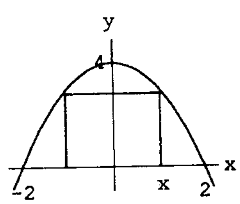
$$P(x) = x + \frac{400}{x}, \quad 0 < x.$$

$$P'(x) = 1 - \frac{400}{x^2}, \quad P'(x) = 0 \quad \Longrightarrow \quad x = 20.$$

Since P decreases on $(0, 20]$ and increases on $[20, \infty)$, the abs min of P occurs when $x = 20$.

To minimize the fencing, make the garden 20 ft (parallel to barn) by 10 ft.

4.



Maximize A

$A = 2xy, \quad y = 4 - x^2$

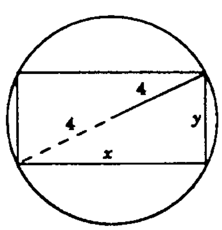
$$A(x) = 2x(4 - x^2) = 8x - 2x^3, \quad 0 \le x \le 2.$$

$$A'(x) = 8 - 6x^2, \quad P'(x) = 0 \quad \Longrightarrow \quad x = \frac{2}{\sqrt{3}}.$$

Since A increases on $[0, 2/\sqrt{3}]$ and decreases on $[2/\sqrt{3}, 2]$, the abs max of A occurs when $x = 2/\sqrt{3}$.

The maximal area is $\frac{32}{9}\sqrt{3}$.

5.



Maximize A

$A = xy, \quad x^2 + y^2 = 8^2, \quad y = \sqrt{64 - x^2}$

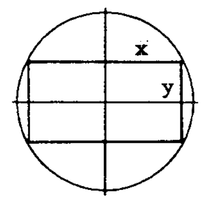
$$A(x) = x\sqrt{64 - x^2}, \quad 0 < x < 8.$$

$$A'(x) = \sqrt{64 - x^2} + x\left(\frac{-x}{\sqrt{64 - x^2}}\right) = \frac{64 - 2x^2}{\sqrt{64 - x^2}}, \qquad A'(x) = 0 \quad \Longrightarrow \quad x = 4\sqrt{2}.$$

Since A increases on $(0, 4\sqrt{2}]$ and decreases on $[4\sqrt{2}, 8)$, the abs max of A occurs when $x = 4\sqrt{2}$. Then, $y = 4\sqrt{2}$ and $xy = 32$.

The maximal area is 32.

6. Maximize S



$S = w^2 l, \quad w = 2y, \quad l = 2x \quad \text{where} \quad x^2 + y^2 = \frac{9}{4}$

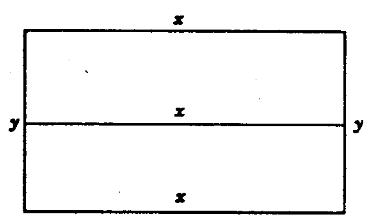
$S(x) = 8\left(\frac{9}{4} - x^2\right)x = 2x(9 - 4x^2), \quad 0 \le x \le \frac{3}{2}.$

$S'(x) = 18 - 24x^2, \quad S'(x) = 0 \quad \Longrightarrow \quad x = \frac{\sqrt{3}}{2}.$

Since S increases on $[0, \sqrt{3}/2]$ and decreases on $[\sqrt{3}/2, 3/2]$, the abs max of S occurs when $x = \sqrt{3}/2$.

The dimensions of the strongest beam are: $\sqrt{6} \times \sqrt{3}$.

7.



Maximize A

$A = xy, \quad 2y + 3x = 600, \quad y = \frac{600 - 3x}{2}$

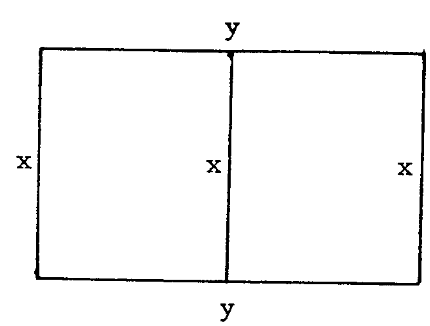
$A(x) = x\left(300 - \frac{3}{2}x\right), \quad 0 < x < 200.$

$A'(x) = 300 - 3x, \quad A'(x) = 0 \quad \Longrightarrow \quad x = 100.$

Since A increases on $(0, 100]$ and decreases on $[100, 200)$, the abs max of A occurs when $x = 100$. Then, $y = 150$.

The playground of greatest area measures 100 ft by 150 ft. (The fence divider is 100 ft long.)

8. Minimize $C = 300y + 400x$



$A = 5000 = xy \quad \Rightarrow \quad y = \frac{5000}{x}; p$

$C(x) = \frac{1,500,000}{x} + 400x, \quad x > 0;$

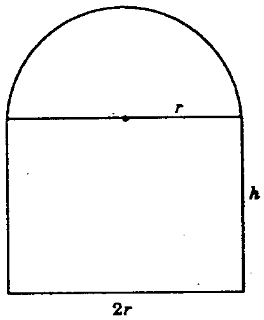
$C'(x) = -\frac{1,500,000}{x^2} + 400;$

$C'(x) = 0 \quad \Longrightarrow \quad x \cong 61.24.$

$C''(x) = \frac{3,000,000}{x^3}; \quad C''(61.24) > 0.$

C has an abs min at $x = 61.24$. The dimensions that will minimize the cost are: $x = 61.24$, $y = 81.65$.

9.



Maximize L

To account for the semi-circular portion admitting less light per square foot, we multiply its area by 1/3.

$$L = 2rh + \frac{1}{3}\left(\frac{\pi r^2}{2}\right),$$

$$2r + 2h + \pi r = 24, \quad h = \tfrac{1}{2}(24 - 2r - \pi r)$$

$$L = 2r\left(\frac{24 - 2r - \pi r}{2}\right) + \frac{1}{6}\pi r^2$$

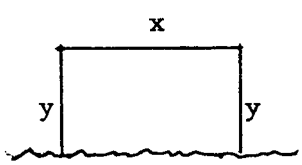
$$L(r) = 24r - \left(2 + \frac{5}{6}\pi\right)r^2, \quad 0 < r < \frac{30}{2+\pi}.$$

$$L'(r) = 24 - \left(4 + \frac{5}{3}\pi\right)r, \quad L'(r) = 0 \implies r = \frac{72}{12 + 5\pi}.$$

Since $L''(r) < 0$ for all r in the domain of L, the local max at $r = 72/(12 + 5\pi)$ is the abs max.

For the window that admits the most light, take the radius of the semicircle as $\dfrac{72}{12+5\pi} \cong 2.6$ ft and the height of the rectangular portion as $\dfrac{72 + 24\pi}{12 + 5\pi} \cong 5.32$ ft.

10.



Maximize A

$$A = xy, \quad x + 2y = 800$$

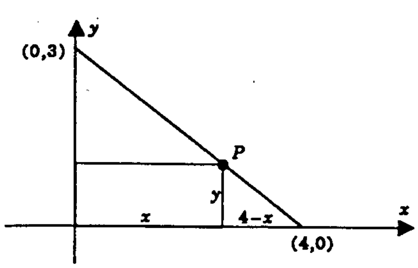
$$A(y) = (800 - 2y)y = 800y - 2y^2, \quad 0 \le y \le 400.$$

$$A'(y) = 800 - 4y, \quad A'(y) = 0 \implies y = 200.$$

Since A increases on $[0, 200]$ and decreases on $[200, 400]$, the abs max of A occurs when $y = 200$.

The dimensions of the field of maximum area are: 200×400.

11.



Maximize A

$$A = xy, \quad \frac{3}{4} = \frac{y}{4-x} \qquad \text{(similar triangles)}$$

$$y = \tfrac{3}{4}(4 - x)$$

$$A(x) = \frac{3x}{4}(4-x), \quad 0 < x < 4.$$

$$A'(x) = 3 - \frac{3x}{2}, \quad A'(x) = 0 \quad \Longrightarrow \quad x = 2.$$

Since A increases on $(0, 2]$ and decreases on $[2, 4)$, the abs max of A occurs when $x = 2$.

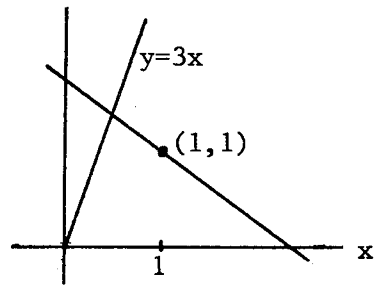
To maximize the area of the rectangle, take P as the point $\left(2, \frac{3}{2}\right)$.

12. The equation of the third side is: $y = mx + (1-m)$.

The base of the triangle is: $b = \dfrac{m-1}{m}$.

The two lines intersect when $3x = mx + (1-m)$;

$$\Longrightarrow \quad x = \frac{1-m}{3-m} \quad \Longrightarrow \quad h = \frac{3(1-m)}{3-m}.$$



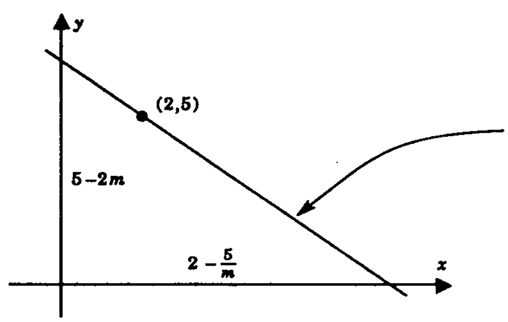
We want to minimize

$$A(m) = \frac{1}{2}\,\frac{m-1}{m}\,\frac{3(1-m)}{3-m} = -\frac{3}{2}\,\frac{(1-m)^2}{3m-m^2}, \quad m < 0.$$

$$A'(m) = -\frac{3}{2}\,\frac{(m+3)(m-1)}{(3m-m^2)^2}, \quad A'(m) = 0 \quad \Longrightarrow \quad m = -3.$$

The area of the triangle is a minimum when the slope of the line is -3.

13.



Minimize A

$A = \frac{1}{2}(x\text{-intercept})\ (y\text{-intercept})$

Equation of line: $\quad y - 5 = m(x-2)$

x-intercept: $\; 2 - \dfrac{5}{m}$

y-intercept: $\; 5 - 2m$

$$A = \frac{1}{2}\left(2 - \frac{5}{m}\right)(5-2m) = 10 - 2m - \frac{25}{2m}$$

$$A(m) = 10 - 2m - \frac{25}{2m}, \quad m < 0.$$

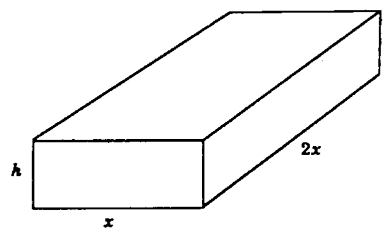
$$A'(m) = -2 + \frac{25}{2m^2}, \quad A'(m) = 0 \quad \Longrightarrow \quad m = -\frac{5}{2}.$$

Since $A''(m) = -25/m^3 > 0$ for $m < 0$, the local min at $m = -5/2$ is the abs min.

The triangle of minimal area is formed by the line of slope $-5/2$.

14. Since $\displaystyle\lim_{m\to 0^-} A(m) = +\infty$, no minimum exists.

15.



Maximize V

$V = 2x^2h, \quad 2\left(2x^2 + xh + 2xh\right) = 100, \quad h = \dfrac{50 - 2x^2}{3x}$

$V = 2x^2\left(\dfrac{50 - 2x^2}{3x}\right)$

$V(x) = \frac{100}{3}x - \frac{4}{3}x^3, \quad 0 < x < 5.$

$V'(x) = \frac{100}{3} - 4x^2, \quad V'(x) = 0 \quad \Longrightarrow \quad x = \frac{5}{3}\sqrt{3}.$

Since $V''(x) = -8x < 0$ on $(0, 5)$, the local max at $x = \frac{5}{3}\sqrt{3}$ is the abs max.

The base of the box of greatest volume measures $\frac{5}{3}\sqrt{3}$ in. by $\frac{10}{3}\sqrt{3}$ in.

16. With no top, we have $2x^2 + 2xh + 4xh = 100$, or $h = \dfrac{50 - x^2}{3x}$.

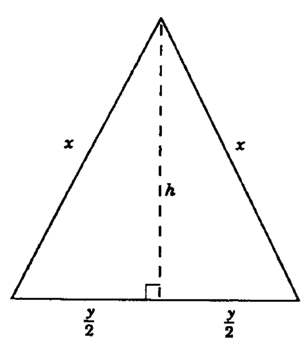
Maximize $V(x) = 2x^2\left(\dfrac{50 - x^2}{3x}\right) = \dfrac{2x}{3}\left(50 - x^2\right), \quad 0 < x < 5\sqrt{2}.$

$V'(x) = \dfrac{100}{3} - 2x^2, \quad V'(x) = 0 \quad \Longrightarrow \quad x = \dfrac{5}{3}\sqrt{6}.$

Since $V''(x) = -4x < 0$ on $(0, 5\sqrt{2})$, the local max at $x = \frac{5}{3}\sqrt{6}$ is the abs max.

The base of the box of greatest volume measures $\frac{5}{3}\sqrt{6}$ in. by $\frac{10}{3}\sqrt{6}$ in.

17.



Maximize A

$A = \frac{1}{2}hy$

$2x + y = 12 \quad \Longrightarrow \quad y = 12 - 2x$

Pythagorean Theorem:

$h^2 + \left(\dfrac{y}{2}\right)^2 = x^2 \quad \Longrightarrow \quad h = \sqrt{x^2 - \left(\dfrac{y}{2}\right)^2}$

Thus, $h = \sqrt{x^2 - (6 - x)^2} = \sqrt{12x - 36}.$

$A(x) = (6 - x)\sqrt{12x - 36}, \quad 3 < x < 6.$

$A'(x) = -\sqrt{12x - 36} + (6 - x)\left(\dfrac{6}{\sqrt{12x - 36}}\right) = \dfrac{72 - 18x}{\sqrt{12x - 36}},$

$A'(x) = 0 \quad \Longrightarrow \quad x = 4.$

Since A increases on $(3, 4]$ and decreases on $[4, 6)$, the abs max of A occurs at $x = 4$.

The triangle of maximal area is equilateral with side of length 4.

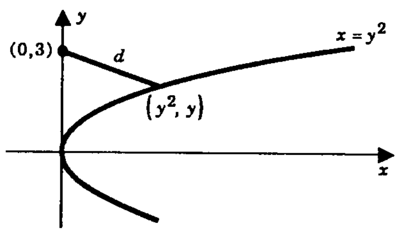
18. It is sufficient to minimize the square of the distance:

$$S = (x-0)^2 + (y-6)^2 = 8y + (y-6)^2 \quad \text{since} \quad x^2 = 8y \quad \text{where} \quad y \geq 0.$$

$$S'(y) = 2y - 4, \quad S'(y) = 0 \quad \Longrightarrow \quad y = 2.$$

The points on the parabola that are closest to $(0, 6)$ are: $(4, 2)$ and $(-4, 2)$.

19.



Minimize d

$d = \sqrt{(y^2-0)^2 + (y-3)^2}$

The square-root function is increasing;

d is minimal when $D = d^2$ is minimal.

$D(y) = y^4 + (y-3)^2, \quad y$ real.

$D'(y) = 4y^3 + 2(y-3) = (y-1)\left(4y^2+4y+6\right), \quad D'(y) = 0$ at $y = 1$.

Since $D''(y) = 12y^2 + 2 > 0$, the local min at $y = 1$ is the abs min.

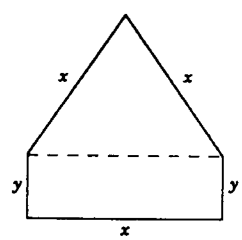
The point $(1, 1)$ is the point on the parabola closest to $(0, 3)$.

20. $f(x) = Ax^{-1/2} + Bx^{1/2}, \quad f(9) = 6 \quad \Longrightarrow \quad \frac{1}{3}A + 3B = 6.$

$f'(x) = \dfrac{-A}{2x^{3/2}} + \dfrac{B}{2x^{1/2}}; \quad f'(9) = 0 \quad \Longrightarrow \quad \dfrac{-A}{54} + \dfrac{B}{6} = 0.$

Solving the two equations gives: $A = 9, \ B = 1$.

21.



Maximize A

$$A = xy + \frac{\sqrt{3}}{4}x^2, \quad 30 = 3x + 2y, \quad y = \frac{30-3x}{2}$$

$A(x) = 15x - \dfrac{3}{2}x^2 + \dfrac{\sqrt{3}}{4}x^2, \quad 0 < x < 10.$

$A'(x) = 15 - 3x + \dfrac{\sqrt{3}}{2}x, \quad A'(x) = 0 \quad \Longrightarrow \quad x = \dfrac{30}{6-\sqrt{3}} = \dfrac{10}{11}\left(6+\sqrt{3}\right).$

Since $A''(x) = -3 + \dfrac{\sqrt{3}}{2} < 0$ on $(0, 10)$, the local max at $x = \dfrac{10}{11}\left(6+\sqrt{3}\right)$ is the abs max.

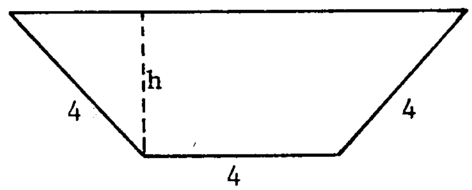
The pentagon of greatest area is composed of an equilateral triangle with side $\frac{10}{11}\left(6+\sqrt{3}\right) \cong 7.03$ in. and rectangle with height $\frac{15}{11}\left(5-\sqrt{3}\right) \cong 4.46$ in.

22. To maximize the area, use the cross-section that is wider at the top.

$A(h) = 4h + h\sqrt{16-h^2}, \quad 0 \le h \le 4;$

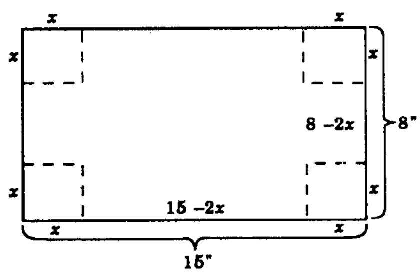
$A'(h) = 4 + \dfrac{16-2h^2}{\sqrt{16-h^2}};$

$A'(h) = 0 \quad \Longrightarrow \quad h = 2\sqrt{3}.$



The depth of the gutter that has maximum carrying capacity is: $2\sqrt{3}$ inches.

23.



Maximize V

$V = x(8-2x)(15-2x)$

$$\left.\begin{array}{r} x > 0 \\ 8-2x > 0 \\ 15-2x > 0 \end{array}\right\} \quad \Longrightarrow \quad 0 < x < 4$$

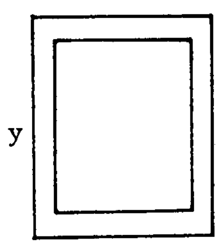
$V(x) = 120x - 46x^2 + 4x^3, \quad 0 < x < 4.$

$V'(x) = 120 - 92x + 12x^2 = 4(3x-5)(x-6), \quad V'(x) = 0$ at $x = \frac{5}{3}$.

Since V increases on $\left(0, \frac{5}{3}\right]$ and decreases on $\left[\frac{5}{3}, 4\right)$, the abs max of V occurs when $x = \frac{5}{3}$.

The box of maximal volume is made by cutting out squares 5/3 inches on a side.

24.



Minimize P

$P = 2x + 2y;$

$(x-4)(y-6) = 81; \quad y = \dfrac{81}{x-4} + 6.$

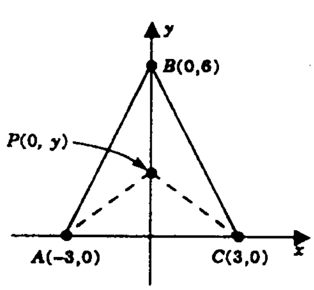
$P(x) = 2x + 2\left(\dfrac{81}{x-4} + 6\right), \quad x > 4.$

$P'(x) = 2 - \dfrac{162}{(x-4)^2}, \quad P'(x) = 0 \quad \Longrightarrow \quad x = 13.$

Since $P''(x) = \dfrac{324}{(x-4)^3} > 0$ when $x > 4$, $x = 13$ is the abs min.

The most economical page has dimensions: width 13 cm, length 15 cm.

25.



Minimize $\overline{AP} + \overline{BP} + \overline{CP} = S$

length $AP = \sqrt{9+y^2}$

length $BP = 6 - y$

length $CP = \sqrt{9+y^2}$

$S(y) = 6 - y + 2\sqrt{9+y^2}, \quad 0 \le y \le 6.$

$S'(y) = -1 + \dfrac{2y}{\sqrt{9+y^2}}, \quad S'(y) = 0 \implies y = \sqrt{3}.$

Since

$$S(0) = 12, \quad S\left(\sqrt{3}\right) = 6 + 3\sqrt{3} \cong 11.2, \quad \text{and} \quad S(6) = 6\sqrt{5} \cong 13.4,$$

the abs min of S occurs when $y = \sqrt{3}$.

To minimize the sum of the distances, take P as the point $\left(0, \sqrt{3}\right)$.

26. Refer to Exercise 25. Here we want to minimize

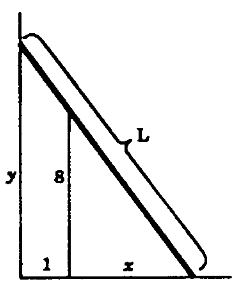
$S(y) = 3 - y + 2\sqrt{36+y^2}, \quad 0 \le y \le 3.$

$S'(y) = -1 + \dfrac{2y}{\sqrt{36+y^2}}, \quad S'(y) = 0 \implies y = \sqrt{12} > 3.$

Thus, the minimum must occur at one of the endpoints: $S(0) = 15, \quad S(3) = 2\sqrt{45} < S(0).$

To minimize the sum of the distances, take $P = (0, 3)$.

27.



Minimize L

$L^2 = y^2 + (x+1)^2.$

By similar triangles $\dfrac{y}{x+1} = \dfrac{8}{x}, \quad y = \dfrac{8}{x}(x+1).$

$$L^2 = \left[\left(\frac{8}{x}\right)(x+1)\right]^2 + (x+1)^2 = (x+1)^2\left(\frac{64}{x^2} + 1\right)$$

Since L is minimal when L^2 is minimal, we consider the function

$$f(x) = (x+1)^2\left(\frac{64}{x^2} + 1\right), \quad x > 0.$$

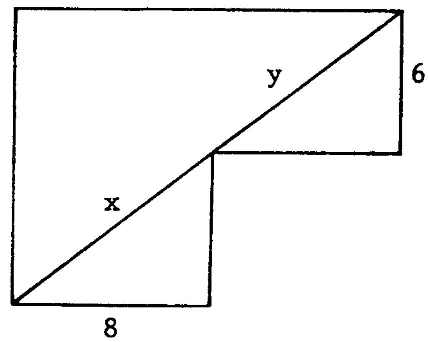
$$f'(x) = 2(x+1)\left(\frac{64}{x^2} + 1\right) + (x+1)^2\left(\frac{-128}{x^3}\right)$$

$$= \frac{2(x+1)}{x^3}\left[x^3 - 64\right], \quad f'(x) = 0 \implies x = 4.$$

Since f decreases on $(0, 4]$ and increases on $[4, \infty)$, the abs min of f occurs when $x = 4$.

The shortest ladder is $5\sqrt{5}$ ft long.

28. Maximize $L = x + y$.



By similar triangles, $\dfrac{y}{6} = \dfrac{x}{\sqrt{x^2 - 64}}$

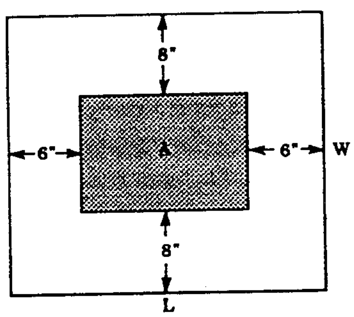
$L(x) = x + \dfrac{6x}{\sqrt{x^2 - 64}}, \quad x > 8$

$L'(x) = 1 - \dfrac{384}{(x^2 - 64)^{3/2}}$

$L'(x) = 0 \quad \Longrightarrow \quad x = \sqrt{64 + (384)^{2/3}} \cong 10.81$

$L(10.81) \cong 19.73$; the longest ladder is approximately 19.7 ft.

29.



Maximize A

(We use feet rather than inches to reduce arithmetic.)

$A = (L - 1)\left(W - \frac{4}{3}\right)$

$LW = 27 \quad \Longrightarrow \quad W = \dfrac{27}{L}$

$A = (L - 1)\left(\dfrac{27}{L} - \dfrac{4}{3}\right) = \dfrac{85}{3} - \dfrac{27}{L} - \dfrac{4}{3}L$

$A(L) = \dfrac{85}{3} - \dfrac{27}{L} - \dfrac{4}{3}L, \quad 1 < L < \dfrac{81}{4}.$

$A'(L) = \dfrac{27}{L^2} - \dfrac{4}{3}, \quad A'(L) = 0 \quad \Longrightarrow \quad L = \dfrac{9}{2}.$

Since $A''(L) = -54/L^3 < 0$ for $1 < L < \frac{81}{4}$, the max at $L = \frac{9}{2}$ is the abs max.

The banner has length $9/2$ ft $= 54$ in. and height 6 ft $= 72$ in.

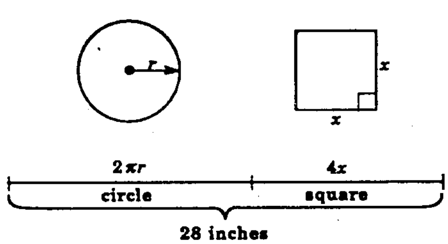
30. Assume $V = \dfrac{1}{3}\pi r^2 h = 1$, or $h = \dfrac{3}{\pi r^2}$.

Minimize surface area $S = \pi r\sqrt{r^2 + h^2} = \pi r\sqrt{r^2 + \dfrac{9}{\pi^2 r^4}} = \dfrac{\sqrt{\pi^2 r^6 + 9}}{r}$.

$$\frac{dS}{dr} = \frac{2\pi^2 r^6 - 9}{r^2\sqrt{\pi^2 r^6 + 9}} = 0 \quad \Longrightarrow \quad r = \left(\frac{9}{2\pi^2}\right)^{\frac{1}{6}} = \frac{3^{\frac{1}{3}}}{2^{\frac{1}{6}}\pi^{\frac{1}{6}}}$$

$$\Longrightarrow \quad h = \frac{3}{\pi\left(\dfrac{9}{2\pi^2}\right)^{\frac{1}{3}}} = \frac{3^{\frac{1}{3}}2^{\frac{1}{3}}}{\pi^{\frac{1}{3}}} = \sqrt{2}r.$$

31.



Find the extreme values of A

$A = \pi r^2 + x^2$

$2\pi r + 4x = 28 \quad \Longrightarrow \quad x = 7 - \frac{1}{2}\pi r.$

$$A(r) = \pi r^2 + \left(7 - \frac{1}{2}\pi r\right)^2, \quad 0 \le r \le \frac{14}{\pi}.$$

Note: the endpoints of the domain correspond to the instances when the string is not cut: $r = 0$ when no circle is formed, $r = 14/\pi$ when no square is formed.

$$A'(r) = 2\pi r - \pi\left(7 - \frac{1}{2}\pi r\right), \quad A'(r) = 0 \implies r = \frac{14}{4+\pi}.$$

Since $A''(r) = 2\pi + \pi^2/2 > 0$ on $(0, 14/\pi)$, the abs min of A occurs when $r = 14/(4+\pi)$ and the abs max of A occurs at one of the endpts: $A(0) = 49$, $A(14/\pi) = 196/\pi > 49$.

(a) To maximize the sum of the two areas, use all of the string to form the circle.

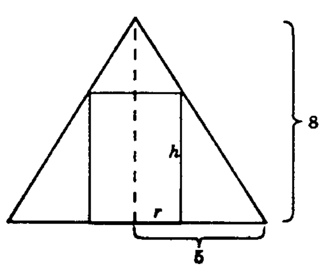
(b) To minimize the sum of the two areas, use $2\pi r = 28\pi/(4+\pi) \cong 12.32$ inches of string for the circle.

32. Maximize $V = x^2 h$ given that $x^2 + 4xh = 12 \implies h = \dfrac{12 - x^2}{4x}$.

$V(x) = x^2\left(\dfrac{12-x^2}{4x}\right) = 3x - \frac{1}{4}x^3, \quad 0 < x < \sqrt{12}.$

$V'(x) = 3 - \frac{3}{4}x^2, \quad V'(x) = 0 \implies x = 2.$ Since V increases on $(0, 2]$ and decreases on $[2, \sqrt{12})$, V has an abs max at $x = 2$; the maximum volume is $V(2) = 4$ cu ft.

33.



Maximize V

$V = \pi r^2 h$

By similar triangles

$$\frac{8}{5} = \frac{h}{5-r} \quad \text{or} \quad h = \frac{8}{5}(5-r).$$

$$V(r) = \frac{8\pi}{5} r^2(5-r), \quad 0 < r < 5.$$

$$V'(r) = \frac{8\pi}{5}\left(10r - 3r^2\right), \quad V'(r) = 0 \implies r = 10/3.$$

Since V increases on $(0,\ 10/3]$ and decreases on $[10/3,\ 5)$, the abs max of V occurs when $r = 10/3$.

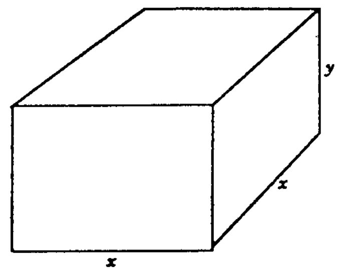
The cylinder with maximal volume has radius $10/3$ and height $8/3$.

34. Maximize $A = 2\pi r h = \dfrac{16\pi}{5} r(5-r), \quad 0 < r < 5, \quad \left(h = \frac{8}{5}(5-r) \text{ from Exercise 33}\right).$

$A'(r) = \dfrac{16\pi}{5}(5 - 2r), \quad A'(r) = 0 \implies r = \frac{5}{2}$

The curved surface is a maximum when $r = \frac{5}{2}$, $h = 4$.

35.



Minimize C

In dollars,

$$C = \text{cost base} + \text{cost top} + \text{cost sides}$$
$$= .35\,(x^2) + .15\,(x^2) + .20(4xy)$$
$$= \tfrac{1}{2}x^2 + \tfrac{4}{5}xy$$

Volume $= x^2y = 1250 \quad y = \dfrac{1250}{x^2}$

$$C(x) = \frac{1}{2}x^2 + \frac{1000}{x}, \quad x > 0.$$

$$C'(x) = x - \frac{1000}{x^2}, \quad C'(x) = 0 \quad \Longrightarrow \quad x = 10.$$

Since $C''(x) = 1 + 2000/x^3 > 0$ for $x > 0$, the local min of C at $x = 10$ is the abs min.

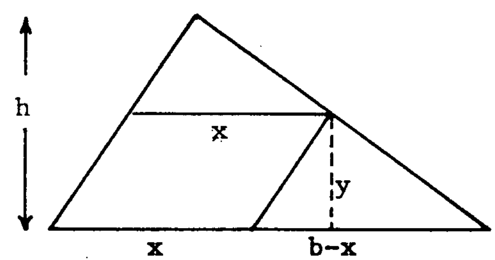
The least expensive box is 12.5 ft tall with a square base 10 ft on a side.

36. Maximize $A = xy$. By similar triangles

$$\frac{y}{b-x} = \frac{h}{b}, \text{ so}$$

$$A(x) = \frac{h}{b}\,x(b-x), \quad 0 \le x \le b.$$

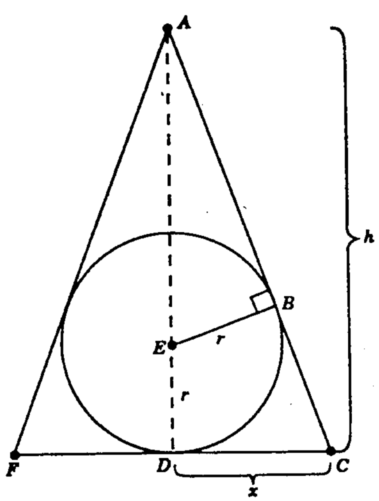
$$A'(x) = \frac{h}{b}\,(b - 2x), \quad A'(x) = 0 \quad \Rightarrow \quad x = \frac{b}{2}.$$



Since A is increasing on $[0, b/2]$ and decreasing on $[b/2, b]$, A has an abs max at $x = b/2$

$A(b/2) = \frac{1}{4}\,hb = \frac{1}{2}$ area of triangle ABC.

37.



Minimize A

$$A = \tfrac{1}{2}(h)(2x) = hx$$

Triangles ADC and ABE are similar:

$$\frac{AD}{DC} = \frac{AB}{BE} \quad \text{or} \quad \frac{h}{x} = \frac{AB}{r}.$$

Pythagorean Theorem:

$$r^2 + (AB)^2 = (h - r)^2.$$

Thus

$$r^2 + \left(\frac{hr}{x}\right)^2 = (h - r)^2.$$

Solving this equation for h we find that

$$h = \frac{2x^2 r}{x^2 - r^2}.$$

$$A(x) = \frac{2x^3 r}{x^2 - r^2}, \quad x > r.$$

$$A'(x) = \frac{(x^2 - r^2)(6x^2 r) - 2x^3 r(2x)}{(x^2 - r^2)^2} = \frac{2x^2 r(x^2 - 3r^2)}{(x^2 - r^2)^2},$$

$$A'(x) = 0 \quad \Longrightarrow \quad x = r\sqrt{3}.$$

Since A decreases on $(r, r\sqrt{3}]$ and increases on $[r\sqrt{3}, \infty)$, the local min at $x = r\sqrt{3}$ is the abs min of A. When $x = r\sqrt{3}$, we get $h = 3r$ so that $FC = 2r\sqrt{3}$ and $AF = FC = \sqrt{h^2 + x^2} = 2r\sqrt{3}$.

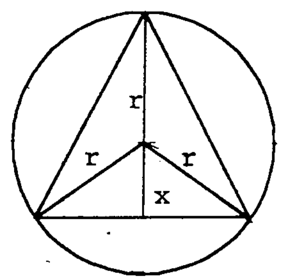
The triangle of least area is equilateral with side of length $2r\sqrt{3}$.

38. Maximize $A(x) = \frac{1}{2}(r + x)2\sqrt{r^2 - x^2}$

$$= (r + x)\sqrt{r^2 - x^2}, \quad 0 \le x \le r.$$

$$A'(x) = \frac{r^2 - rx - 2x^2}{\sqrt{r^2 - x^2}};$$

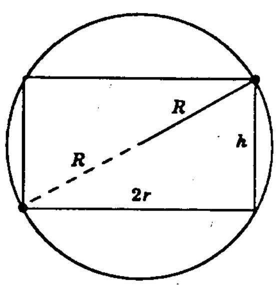
$$A'(x) = 0 \quad \Longrightarrow \quad x = \frac{r}{2}.$$



Since A increases on $[0, r/2]$ and decreases on $[r/2, r]$, A has an abs max at $x = r/2$;

$$A(r/2) = \frac{3\sqrt{3}}{4}\, r^2.$$

39.



Maximize V

$V = \pi r^2 h$

By the Pythagorean Theorem,

$(2r)^2 + h^2 = (2R)^2$

so

$h = 2\sqrt{R^2 - r^2}$.

$$V(r) = 2\pi r^2 \sqrt{R^2 - r^2}, \quad 0 < r < R.$$

$$V'(r) = 2\pi \left[2r\sqrt{R^2 - r^2} - \frac{r^3}{\sqrt{R^2 - r^2}}\right] = \frac{2\pi r(2R^2 - 3r^2)}{\sqrt{R^2 - r^2}}$$

$$V'(r) = 0 \quad \Longrightarrow \quad r = \tfrac{1}{3}R\sqrt{6}.$$

Since V increases on $(0, \frac{1}{3}R\sqrt{6}]$ and decreases on $[\frac{1}{3}R\sqrt{6}, R)$, the local max at $r = \frac{1}{3}R\sqrt{6}$ is the abs max.

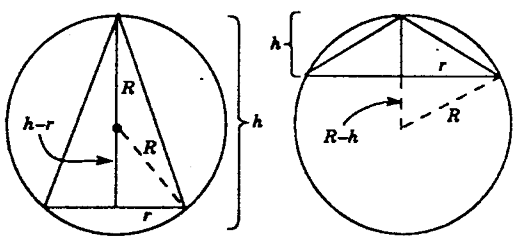
The cylinder of maximal volume has base radius $\frac{1}{3}R\sqrt{6}$ and height $\frac{2}{3}R\sqrt{3}$.

40. Maximize $A = 2\pi rh = 4\pi r\sqrt{R^2 - r^2}$, $0 \le r \le R$, $\left(h = 2\sqrt{R^2 - r^2} \text{ from Exercise 39}\right)$.

$$A'(r) = \frac{4\pi(R^2 - 2r^2)}{\sqrt{R^2 - r^2}}, \quad A'(r) = 0 \quad \Longrightarrow \quad r = \frac{R}{\sqrt{2}}.$$

The curved surface is a maximum when $r = \dfrac{R}{\sqrt{2}}$, $h = R\sqrt{2}$.

41.



Case 1 : $h \ge R$ Case 2 : $h \le R$

Maximize V

$V = \frac{1}{3}\pi r^2 h$

Pythagorean Theorem

Case 1 : $(h - R)^2 + r^2 = R^2$

Case 2 : $(R - h)^2 + r^2 = R^2$

In both cases

$r^2 = R^2 - (R - h)^2 = 2hR - h^2.$

$$V(h) = \tfrac{1}{3}\pi\left(2h^2R - h^3\right), \quad 0 < h < 2R.$$

$$V'(h) = \frac{1}{3}\pi\left(4hR - 3h^2\right), \quad V'(h) = 0 \quad \text{at} \quad h = \frac{4R}{3}.$$

Since V increases on $\left(0, \frac{4}{3}R\right]$ and decreases on $\left[\frac{4}{3}R, 2R\right)$, the local max at $h = \frac{4}{3}R$ is the abs max.

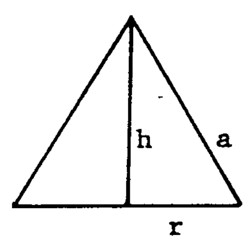
The cone of maximal volume has height $\frac{4}{3}R$ and radius $\frac{2}{3}R\sqrt{2}$.

42. Maximize $V = \frac{1}{3}\pi r^2 h$, where $r^2 + h^2 = a^2$.

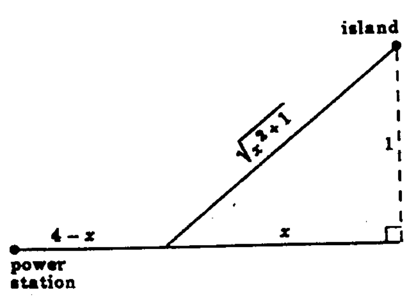
$V(h) = \frac{1}{3}\pi(a^2 - h^2)h, \quad 0 \le h \le a,$

$V'(h) = \frac{1}{3}\pi(a^2 - 3h^2), \quad V'(h) = 0 \quad \Rightarrow \quad h = \dfrac{a}{\sqrt{3}}.$

Maximum volume $V\left(a/\sqrt{3}\right) = \frac{2}{27}\pi a^3\sqrt{3}.$



43.



Minimize C

In units of \$10,000,

$$C = \begin{matrix}\text{cost of cable} \\ \text{underground}\end{matrix} + \begin{matrix}\text{cost of cable} \\ \text{under water}\end{matrix}$$

$$= 3(4 - x) \quad + \quad 5\sqrt{x^2 + 1}.$$

Clearly, the cost is unnecessarily high if

$$x > 4 \quad \text{or} \quad x < 0.$$

$$C(x) = 12 - 3x + 5\sqrt{x^2 + 1}, \quad 0 \le x \le 4.$$

$$C'(x) = -3 + \frac{5x}{\sqrt{x^2 + 1}}, \quad C'(x) = 0 \quad \Longrightarrow \quad x = 3/4.$$

Since the domain of C is closed, the abs min can be identified by evaluating C at each critical point:

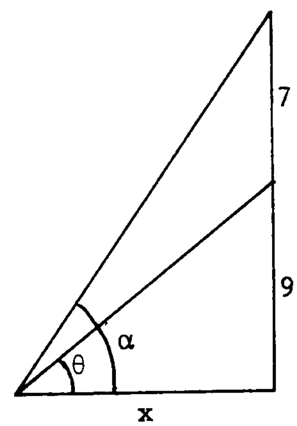
$$C(0) = 17, \quad C\left(\tfrac{3}{4}\right) = 16, \quad C(4) = 5\sqrt{17} \cong 20.6.$$

The minimum cost is \$160,000.

44. Maximize $\alpha - \theta$.

Since the tangent function is an increasing function on $[0, \pi/2)$, it suffices to maximize $\tan(\alpha - \theta)$.

$$\tan(\alpha - \theta) = \frac{\tan\alpha - \tan\theta}{1 + \tan\alpha\,\tan\theta}; \quad \tan\alpha = \frac{16}{x}, \quad \tan\theta = \frac{9}{x}.$$



Thus, we maximize $T(x) = \dfrac{\dfrac{16}{x} - \dfrac{9}{x}}{1 + \dfrac{16}{x}\cdot\dfrac{9}{x}} = \dfrac{7x}{x^2 + 144}, \quad x > 0.$

$$T'(x) = \frac{7(144 - x^2)}{(x^2 + 144)^2}, \quad T'(x) = 0 \implies x = 12.$$

Stand 12 ft from the wall for the most favorable view.

45. $P'(\theta) = \dfrac{-mW(m\cos\theta - \sin\theta)}{(m\sin\theta + \cos\theta)^2}$; P is minimized when $\tan\theta = m$.

46. $R(\theta) = \dfrac{2v^2}{g\cos^2\alpha}\cos\theta\,\sin(\theta - \alpha), \quad 0 < \theta < \tfrac{1}{2}\pi$

$$R'(\theta) = \frac{2v^2}{g\cos^2\alpha}\left[-\sin\theta\,\sin(\theta - \alpha) + \cos\theta\,\cos(\theta - \alpha)\right]$$

$$= \frac{2v^2}{g\cos^2\alpha}\cos(2\theta - \alpha)$$

$$R'(\theta) = 0 \implies 2\theta - \alpha = \tfrac{1}{2}\pi \implies \theta = \tfrac{1}{4}\pi + \tfrac{1}{2}\alpha.$$

47. Minimize $I = \dfrac{a}{x^2} + \dfrac{b}{(s - x)^2}$.

$$I'(x) = -\frac{2a}{x^3} + \frac{2b}{(s - x)^3}, \quad I'(x) = 0 \implies x = \frac{a^{\frac{1}{3}}s}{a^{\frac{1}{3}} + b^{\frac{1}{3}}}.$$

48. Minimize $D = (y - y_1)^2 + (x - x_1)^2$, where $y = -\dfrac{1}{b}(ax - c)$.

$$D' = 2\left(-\frac{1}{b}(ax - c) - y_1\right)\left(-\frac{a}{b}\right) + 2(x - x_1) = 0$$

$$\implies x = \frac{b^2x_1 - ac - aby_1}{a^2 + b^2} \text{ and thus } y = \frac{a^2y_1 - bc - abx_1}{a^2 + b^2}.$$

Thus $d = \sqrt{D} = \sqrt{\left(\dfrac{a^2y_1 - bc - abx_1}{a^2 + b^2} - y_1\right)^2 + \left(\dfrac{b^2x_1 - ac - aby_1}{a^2 + b^2} - x_1\right)^2}$

$$= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$

49. The slope of the line through (a, b) and $(x, f(x))$ is $\dfrac{f(x)-b}{x-a}$.

Let $D(x) = [x-a]^2 + [b-f(x)]^2$. Then $D'(x) = 0$

$$\implies \quad 2[x-a] + 2[b-f(x)]f"(x) = 0$$

$$\implies \quad f'(x) = -\frac{x-a}{b-f(x)}.$$

50. Let $P = (x_1, x_1^2)$ and $Q = (x_2, x_2^2)$. The slope of the line PQ is $\dfrac{x_1^2 - x_2^2}{x_1 - x_2} = x_1 + x_2$. This is perpendicular to the slope of the tangent to the parabola through point P, which is $2x_1$. Thus $x_1 + x_2 = -\dfrac{1}{2x_1} \implies x_2 = -x_1 - \dfrac{1}{2x_1}$. Now minimize:

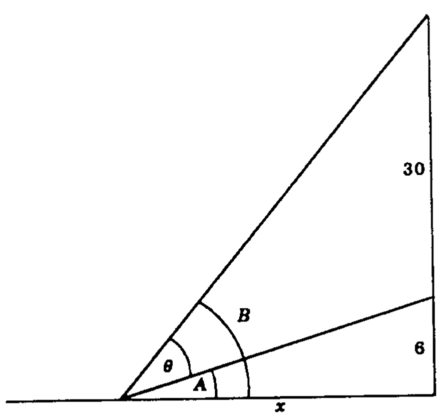
$$D = (x_1^2 - x_2^2)^2 + (x_1 - x_2)^2 = (x_1^2 - x_2^2)^2(1 + (x_1 + x_2)^2)$$
$$= \left(2x_1 + \frac{1}{2x_1}\right)^2 \left(1 + \frac{1}{4x_1^2}\right)$$

Thus $D' = \left(2x_1 + \dfrac{1}{2x_1}\right)^2 \left(-\dfrac{1}{2x_1^3}\right) + 2\left(2x_1 + \dfrac{1}{2x_1}\right)\left(2 - \dfrac{1}{2x_1^2}\right)\left(1 + \dfrac{1}{4x_1^2}\right) = 0$

$$\implies \quad \left(2x_1 + \frac{1}{2x_1}\right)\left(\frac{1}{2x_1^3}\right) = 2\left(2 - \frac{1}{2x_1^2}\right)\left(1 + \frac{1}{4x_1^2}\right)$$

$$\implies \quad x_1 = \pm\frac{\sqrt{2}}{2}, \text{ and } y_1 = \frac{1}{2}.$$

51.



Maximize θ

Since the tangent function increases on $[0, \pi/2)$, we can maximize θ by maximizing $\tan\theta$.

$$\tan\theta = \tan(B - A)$$
$$= \frac{\tan B - \tan A}{1 + \tan B \tan A}$$
$$= \frac{36/x - 6/x}{1 + (36/x)(6/x)} = \frac{30x}{x^2 + 216}.$$

Thus, we consider

$$f(x) = \frac{30x}{x^2 + 216}, \quad x \ge 0.$$

$$f'(x) = \frac{(x^2+216)\,30 - 30x(2x)}{(x^2+216)^2} = \frac{30\,(216 - x^2)}{(x^2+216)^2},$$

$$f'(x) = 0 \quad \implies \quad x = 6\sqrt{6}.$$

Since f increases on $[0, 6\sqrt{6}]$ and decreases on $[6\sqrt{6}, \infty)$, the local max at $x = 6\sqrt{6}$ is the abs max. The observer should sit $6\sqrt{6}$ ft from the screen.

52. Let x be the number of passengers and R the revenue in dollars.

$$R(x) = \begin{cases} 37x, & 16 \le x \le 35 \\ \left[37 - \frac{1}{2}(x - 35)\right] x, & 35 < x \le 48; \end{cases}$$

$$R'(x) = \begin{cases} 37, & 16 < x < 35 \\ \frac{109}{2} - x, & 35 < x < 48. \end{cases}$$

The critical points are $x = 16$, 35, and 48. From $R(16) = 592$, $R(35) = 1295$, and $R(48) = 1464$ we conclude that the revenue is maximized by taking a full load of 48 passengers.

53. Let x be the number of customers and P the net profit in dollars. Then $0 \le x \le 250$ and

$$P(x) = \begin{cases} 12x, & 0 \le x \le 50 \\ [12 - 0.06(x - 50)]x, & 50 < x \le 250; \end{cases}$$

$$P'(x) = \begin{cases} 12, & 0 \le x \le 50 \\ 62x - 0.06x^2, & 50 < x \le 250. \end{cases}$$

The critical numbers are $x = 0$, $x = 50$, $x = 125$, and $x = 250$. From $P(0) = 0$, $P(50) = 600$, $P(125) = 937.50$, and $P(250) = 0$, we conclude that the net profit is maximized by servicing 125 customers.

54. Maximize $P(x) = 2cy + cx$, where c is the price of low-grade steel and $y = \left(\dfrac{40 - 5x}{10 - x}\right)$.

$$P(x) = 2c\left(\frac{40 - 5x}{10 - x}\right) + cx$$

$$P'(x) = 2c\left[\frac{(10 - x)(-5) - (40 - 5x)(-1)}{(10 - x)^2}\right] + c,$$

$$P'(x) = 0 \quad \Longrightarrow \quad x = 10 - \sqrt{20} \text{ or about } 5\tfrac{1}{2} \text{ tons.}$$

55. $A'(x) = \dfrac{xC'(x) - C(x)}{x^2}, \quad A'(x) = 0 \quad \Longrightarrow \quad C'(x) = \dfrac{C(x)}{x}.$

56. Minimize $SA = 2\pi rh + 2\pi r^2$, where $2r \le h < 6$ and $\pi r^2 h = 16\pi$ (hence $h = \dfrac{16}{r^2}$).

Thus $SA = \dfrac{32\pi}{r} + 2\pi r^2$. Differentiating, $\quad SA' = -\dfrac{32\pi}{r^2} + 4\pi r = 0$

$\Longrightarrow \quad r^3 = 8$, so $r = 2$ feet and $h = 4$ feet. Thus no minimum exists.

57. $y = mx - \frac{1}{800}(m^2+1)x^2$. When $y = 0$, $x = \frac{800m}{m^2+1}$.

Differentiating x with respect to m, $x' = \frac{800 - 800m^2}{(m^2+1)^2} = 0$

$\implies m = 1.$

58. When $x = 300$, $y = 300m - \frac{225}{2}(m^2+1)$.

Differentiating y with respect to m, $y' = 300 - 225m = 0$

$\implies m = \frac{4}{3}.$

59. Driving at x mph, the trip takes $\frac{300}{x}$ hours and uses $\left(2 + \frac{1}{600}x^2\right)\frac{300}{x}$ gallons of fuel.

Thus the expences are $E = 1.35\left(2 + \frac{1}{600}x^2\right)\frac{300}{x} - 13\left(\frac{300}{x}\right) = 0.675x - \frac{3090}{x}.$

Differentiating, $E' = 0.675 + \frac{3090}{x^2}$, which never equals zero.

Thus the minimal expenses occur at the endpoints:

$x = 35$ or $x = 55$.

Evaluating E at these points shows that the mnimal expenses occur when the truck is driven at 55 mph.

60. We want to maximize the ratio $\frac{\text{income}}{\text{cost}} = \frac{200,000n}{1,000,000n + 100,000(1 + 2 + \cdots + n - 1) + 5,000,000}$

$= \frac{2n}{10n + \frac{1}{2}(n-1)n + 50} = \frac{4n}{n^2 + 19n + 100}.$

Let $f(x) = \frac{4x}{x^2 + 19x + 100}$, $x > 0$

Then $f'(x) = \frac{(x^2 + 19x + 100)4 - 4x(2x + 19)}{(x^2 + 19x + 100)^2} = \frac{4(100 - x^2)}{(x^2 + 19x + 100)^2} = 0$

$\implies x = 10.$

Since $f'(x) > 0$ for $x < 0$, $f'(x) < 0$ for $x > 10$, f has an absolute maximum at $x = 10$.

A ten story building provides the greatest return on investment.

PROJECT 4.5

1. Distance over water: $\sqrt{36 + x^2}$.

Distance over land: $12 - x$.

Total energy: $E(x) = W\sqrt{36 + x^2} + L(12 - x)$.

2. $W = 1.5L$, so $E(x) = 1.5L\sqrt{36 + x^2} + L(12 - x)$, for $0 \le x \le 12$.

$$E'(x) = \frac{1.5Lx}{\sqrt{36+x^2}} - L = 0 \quad\Longrightarrow\quad x = \frac{12}{\sqrt{5}} \simeq 5.36.$$

$E'(x) < 0$ on $(0, \frac{12}{\sqrt{5}})$ and $E'(x) > 0$ on $(\frac{12}{\sqrt{5}}, 12)$, so E has an absolute minimum at $\frac{12}{\sqrt{5}}$.

3. (a) $W = kL,\ k > 1,$ so $E(x) = kL\sqrt{36+x^2} + L(12-x),$ for $0 \le x \le 12$.

$$E'(x) = \frac{kLx}{\sqrt{36+x^2}} - L = 0 \quad\Longrightarrow\quad x = \frac{6}{\sqrt{k^2-1}}.$$

$E'(x) < 0$ on $(0, \frac{6}{\sqrt{k^2-1}})$ and $E'(x) > 0$ on $(\frac{6}{\sqrt{k^2-1}}, 12)$, so E has an absolute minimum at $\frac{6}{\sqrt{k^2-1}}$.

(b) As k increases, x decreases.

As $k \to 1^+$, x increases.

(c) $x = 12 \quad\Longrightarrow\quad k = \frac{\sqrt{5}}{2} \simeq 1.12.$

(d) No

SECTION 4.6

1. (a) f is increasing on $[a,b],\ [d,n];$ f is decreasing on $[b,d],\ [n,p]$.

(b) The graph of f is concave up on $(c,k),\ (l,m);$

The graph of f is concave down on $(a,c),\ (k,l),\ (m,p)$.

The x-coordinates of the points of inflection are: $x=c,\ x=k,\ x=l,\ x=m$.

2. (a) g is increasing on $[a,b],\ [c,e],\ [m,n];$ g is decreasing on $[b,c],\ [e,m]$.

(b) The graph of g is concave up on $(a,b),\ (b,d);$

The graph of g is concave down on $(d,m),\ (m,n)$.

The x-coordinate of the point of inflection is: $x = d$.

3. $f'(x) = -x^{-2}, \quad f''(x) = 2x^{-3};$

concave down on $(-\infty, 0)$, concave up on $(0, \infty)$; no pts of inflection

4. $f'(x) = 1 - x^{-2}, \quad f''(x) = 2x^{-3};$

concave down on $(-\infty, 0)$, concave up on $(0, \infty)$; no pts of inflection

5. $f'(x) = 3x^2 - 3, \quad f''(x) = 6x;$

concave down on $(-\infty, 0)$, concave up on $(0, \infty)$; pt of inflection $(0, 2)$

6. $f'(x) = 4x - 5, \quad f''(x) = 4;$ concave up on $(-\infty, \infty)$

7. $f'(x) = x^3 - x, \quad f''(x) = 3x^2 - 1;$

concave up on $\left(-\infty, -\frac{1}{3}\sqrt{3}\right)$ and $\left(\frac{1}{3}\sqrt{3}, \infty\right)$, concave down on $\left(-\frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}\right)$;

pts of inflection $\left(-\frac{1}{3}\sqrt{3}, -\frac{5}{36}\right)$ and $\left(\frac{1}{3}\sqrt{3}, -\frac{5}{36}\right)$

8. $f'(x) = 3x^2 - 4x^3, \quad f''(x) = 6x - 12x^2 = 6x(1-2x);$

concave down on $(-\infty, 0)$, and $\left(\frac{1}{2}, \infty\right)$, concave up on $\left(0, \frac{1}{2}\right)$;

pts of inflection $(0,0)$, $\left(\frac{1}{2}, \frac{1}{16}\right)$

9. $f'(x) = -\dfrac{x^2+1}{(x^2-1)^2}, \quad f''(x) = \dfrac{2x\,(x^2+3)}{(x^2-1)^3};$

concave down on $(-\infty, -1)$ and $(0, 1)$, concave up on $(-1, 0)$ and on $(1, \infty)$;

pt of inflection $(0, 0)$

10. $f'(x) = \dfrac{-4}{(x-2)^2}, \quad f''(x) = \dfrac{8}{(x-2)^3};$

concave down on $(-\infty, 2)$, concave up on $(2, \infty)$; no pts of inflection

11. $f'(x) = 4x^3 - 4x, \quad f''(x) = 12x^2 - 4;$

concave up on $\left(-\infty, -\frac{1}{3}\sqrt{3}\right)$ and $\left(\frac{1}{3}\sqrt{3}, \infty\right)$, concave down on $\left(-\frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}\right)$;

pts of inflection $\left(-\frac{1}{3}\sqrt{3}, \frac{4}{9}\right)$ and $\left(\frac{1}{3}\sqrt{3}, \frac{4}{9}\right)$

12. $f'(x) = \dfrac{6(1-x^2)}{(x^2+1)^2}, \quad f''(x) = \dfrac{12x(x^2-3)}{(x^2+1)^3};$

concave down on $\left(-\infty, -\sqrt{3}\right)$ and $\left(0, \sqrt{3}\right)$, concave up on $\left(-\sqrt{3}, 0\right)$ and $\left(\sqrt{3}, \infty\right)$;

pts of inflection $\left(-\sqrt{3}, -\frac{3}{2}\sqrt{3}\right)$, $(0,0)$, $\left(\sqrt{3}, \frac{3}{2}\sqrt{3}\right)$

13. $f'(x) = \dfrac{-1}{\sqrt{x}\,(1+\sqrt{x})^2}, \quad f''(x) = \dfrac{1+3\sqrt{x}}{2x\sqrt{x}\,(1+\sqrt{x})^3};$

concave up on $(0, \infty)$; no pts of inflection

14. $f'(x) = \frac{1}{5}(x-3)^{-4/5}, \quad f''(x) = -\frac{4}{25}(x-3)^{-9/5};$

concave up on $(-\infty, 3)$, concave down on $(3, \infty)$; pt of inflection $(3, 0)$

15. $f'(x) = \frac{5}{3}(x+2)^{2/3}, \quad f''(x) = \frac{10}{9}(x+2)^{-1/3};$

concave down on $(-\infty, -2)$, concave up on $(-2, \infty)$; pt of inflection $(-2, 0)$

16. $f'(x) = \dfrac{4-2x^2}{(4-x^2)^{1/2}}, \quad f''(x) = \dfrac{2x(x^2-6)}{(4-x^2)^{3/2}}$ Note: $\text{dom}(f) = [-2, 2]$

concave up on $(-2, 0)$, concave down on $(0, 2)$; pt of inflection $(0, 0)$

17. $f'(x) = 2\sin x \cos x = \sin 2x, \quad f''(x) = 2\cos 2x;$

concave up on $\left(0, \frac{1}{4}\pi\right)$ and $\left(\frac{3}{4}\pi, \pi\right)$, concave down on $\left(\frac{1}{4}\pi, \frac{3}{4}\pi\right)$;

pts of inflection $\left(\frac{1}{4}\pi,\ \frac{1}{2}\right)$ and $\left(\frac{3}{4}\pi,\ \frac{1}{2}\right)$

18. $f'(x) = -4\cos x\sin x - 2x, \quad f''(x) = -4(\cos^2 x - \sin^2 x) - 2 = -4\cos 2x - 2;$

concave down on $\left(0,\ \frac{1}{3}\pi\right)$ and $\left(\frac{2}{3}\pi,\ \pi\right)$, concave up on $\left(\frac{1}{3}\pi,\ \frac{2}{3}\pi\right)$;

pts of inflection $\left(\dfrac{1}{3}\pi,\ \dfrac{9-2\pi^2}{18}\right)$ and $\left(\dfrac{2}{3}\pi,\ \dfrac{9-8\pi^2}{18}\right)$

19. $f'(x) = 2x + 2\cos 2x, \quad f''(x) = 2 - 4\sin 2x;$

concave up on $\left(0,\ \frac{1}{12}\pi\right)$ and on $\left(\frac{5}{12}\pi,\ \pi\right)$, concave down on $\left(\frac{1}{12}\pi,\ \frac{5}{12}\pi\right)$;

pts of inflection $\left(\dfrac{1}{12}\pi,\ \dfrac{72+\pi^2}{144}\right)$ and $\left(\dfrac{5}{12}\pi,\ \dfrac{72+25\pi^2}{144}\right)$

20. $f'(x) = 4\sin^3 x\cos x, \quad f''(x) = 4\sin^2 x[3\cos^2 x - \sin^2 x];$

concave up on $\left(0,\ \frac{1}{3}\pi\right)$ and $\left(\frac{2}{3}\pi,\ \pi\right)$, concave down on $\left(\frac{1}{3}\pi,\ \frac{2}{3}\pi\right)$;

pts of inflection $\left(\frac{1}{3}\pi,\ \frac{9}{16}\right)$ and $\left(\frac{2}{3}\pi,\ \frac{9}{16}\right)$

21. $f(x) = x^3 - 9x$

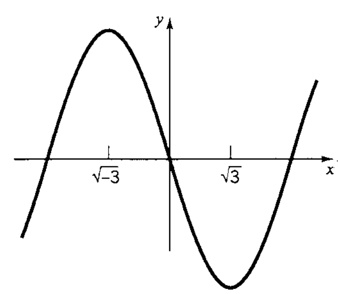
(a) $f'(x) = 3x^2 - 9 = 3(x^2 - 3)$

$f'(x) \geq 0 \Rightarrow\ x \leq -\sqrt{3}$ or $x \geq \sqrt{3}$;

$f'(x) \leq 0\ \Rightarrow\ -\sqrt{3} \leq x \leq \sqrt{3}$.

Thus, f is increasing on $(-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$

and decreasing on $[-\sqrt{3}, \sqrt{3}]$.



(b) $f(-\sqrt{3}) \cong 10.39$ is a local maximum;

$f(\sqrt{3}) \cong -10.39$ is a local minimum.

(c) $f''(x) = 6x$;

The graph of f is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$.

(d) point of inflection: $(0, 0)$

22. $f(x) = 3x^4 + 4x^3 + 1$

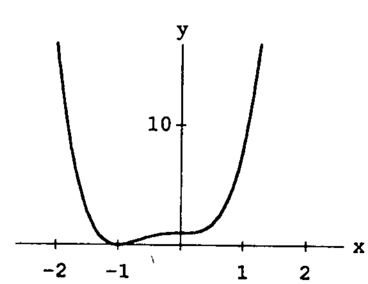
(a) $f'(x) = 12x^3 + 12x^2 = 12x^2(x+1)$

$f'(x) \geq 0 \Rightarrow\ x \geq -1$;

$f'(x) \leq 0\ \Rightarrow\ x \leq -1$.

Thus, f is increasing on $[-1, \infty)$

and decreasing on $(-\infty, -1]$.



(b) $f(-1) = 0$ is a local minimum;

no local maxima.

(c) $f''(x) = 36x^2 + 24x = 36x\left(x + \frac{2}{3}\right)$;

The graph of f is concave up on $\left(-\infty, -\frac{2}{3}\right)$ and $(0, \infty)$; and concave down on $\left(-\frac{2}{3}, 0\right)$.

(d) points of inflection: $\left(-\frac{2}{3}, \frac{11}{27}\right)$, $(0, 1)$

23. $f(x) = \dfrac{2x}{x^2+1}$

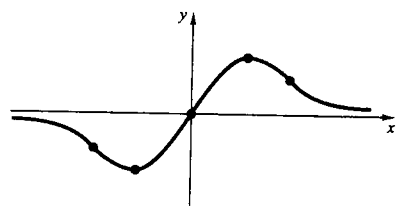
(a) $f'(x) = -\dfrac{2(x+1)(x-1)}{(x^2+1)^2}$

$f'(x) \geq 0 \Rightarrow -1 \leq x \leq 1$;

$f'(x) \leq 0 \Rightarrow x \leq -1$ or $x \geq 1$.

Thus, f is increasing on $[-1, 1]$;

and decreasing on $(-\infty, -1] \cup [1, \infty)$.



(b) $f(-1) = -1$ is a local minimum;

$f(1) = 1$ is a local maximum.

(c) $f''(x) = \dfrac{4x(x+\sqrt{3})(x-\sqrt{3})}{(x^2+1)^3}$

$f''(x) > 0 \Rightarrow x \leq -\sqrt{3}$ or $x \geq \sqrt{3}$;

$f''(x) < 0 \Rightarrow -\sqrt{3} < x < \sqrt{3}$.

The graph of f is concave up on $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$ and concave down on $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$.

(d) points of inflection: $(-\sqrt{3}, -\sqrt{3}/2)$, $(0, 0)$, $(\sqrt{3}, \sqrt{3}/2)$

24. $f(x) = x^{1/3}(x-6)^{2/3}$

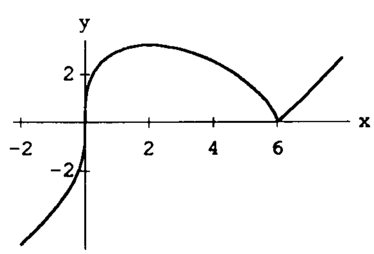
(a) $f'(x) = \dfrac{x-2}{x^{2/3}(x-6)^{1/3}}$

$f'(x) \geq 0 \Rightarrow x \leq 2,\ x \neq 0$, or $x > 6$;

$f'(x) \leq 0 \Rightarrow 2 \leq x < 6$.

Thus, f is increasing on $(-\infty, 2] \cup [6, \infty)$

and decreasing on $[2, 6]$.



(b) $f(2) = 2(4)^{1/3}$ is a local maximum;

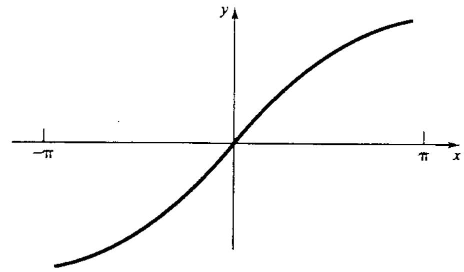
$f(6) = 0$ is a local minimum.

(c) $f''(x) = \dfrac{-8}{x^{5/3}(x-6)^{4/3}}$;

The graph of f is concave down on $(0, \infty)$ and concave down up $(-\infty, 0)$.

(d) point of inflection: $(0, 0)$

25. $f(x) = x + \sin x, \quad x \in [-\pi, \pi]$



(a) $f'(x) = 1 + \cos x$

$f'(x) > 0$ on $(-\pi, \pi)$

Thus, f is increasing on $[-\pi, \pi]$.

(b) No local extrema

(c) $f''(x) = -\sin x$

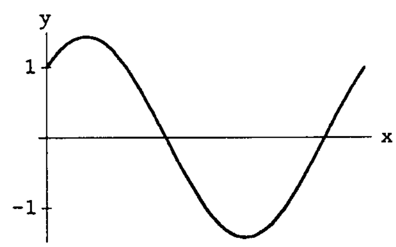
$f''(x) > 0$ for $x \in (-\pi, 0)$;

$f''(x) < 0$ for $x \in (0, \pi)$.

The graph of f is concave up on $(\pi, 0)$ and concave down on $(0, \pi)$.

(d) point of inflection: $(0, 0)$

26. $f(x) = \sin x + \cos x, \quad x \in [0, 2\pi]$



(a) $f'(x) = \cos x - \sin x$

$f'(x) \geq 0 \quad \Rightarrow \quad 0 \leq x \leq \frac{1}{4}\pi$ or $\frac{5}{4}\pi \leq x \leq 2\pi$

$f'(x) \leq 0 \quad \Rightarrow \quad \frac{1}{4}\pi \leq x \leq \frac{5}{4}\pi$

Thus, f is increasing on $\left[0, \frac{1}{4}\pi\right] \cup \left[\frac{5}{4}\pi, 2\pi\right]$;

f is decreasing on $\left[\frac{1}{4}\pi, \frac{5}{4}\pi\right]$.

(b) $f(\pi/4) = \sqrt{2}$ is a local maximum;

$f(5\pi/4) = -\sqrt{2}$ is a local minimum.

(c) $f''(x) = -\sin x - \cos x$

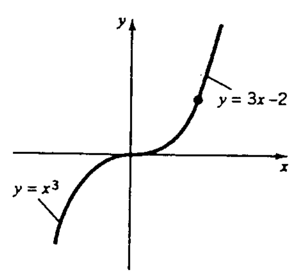
$f''(x) > 0 \quad \Rightarrow \quad \frac{3}{4}\pi < x < \frac{7}{4}\pi$;

$f''(x) < 0 \quad \Rightarrow \quad 0 < x < \frac{3}{4}\pi$ or $\frac{7}{4}\pi < x < 2\pi$.

The graph of f is concave up on $\left(\frac{3}{4}\pi, \frac{7}{4}\pi\right)$ and concave down on $\left(0, \frac{3}{4}\pi\right) \cup \left(\frac{7}{4}\pi, 2\pi\right)$.

(d) points of inflection: $\left(\frac{3}{4}\pi, 0\right)$, $\left(\frac{7}{4}\pi, 0\right)$

27. $f(x) = \begin{cases} x^3, & x < 1 \\ 3x - 2, & x \geq 1. \end{cases}$



(a) $f'(x) = \begin{cases} 3x^2, & x < 1 \\ 3, & x \geq 1; \end{cases}$

$f'(x) > 0$ on $(-\infty, 0) \cup (0, \infty)$

Thus, f is increasing on $(-\infty, \infty)$.

(b) No local extrema

(c) $f''(x) = \begin{cases} 6x, & x < 1 \\ 0, & x \geq 1; \end{cases}$

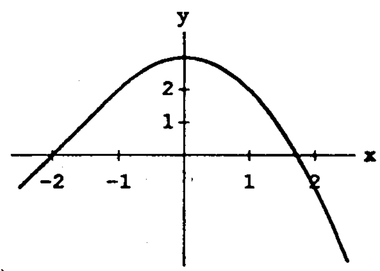
$f''(x) > 0$ for $x \in (0,1)$; $f''(x) < 0$ for $x \in (-\infty, 0)$.

Thus, the graph of f is concave up on $(0,1)$ and concave down on $(-\infty, 0)$.

The graph of f is a straight line for $x \geq 1$.

(d) point of inflection: $(0,0)$

28. $f(x) = \begin{cases} 2x+4, & x \leq -1 \\ 3-x^2, & x > -1. \end{cases}$



(a) $f'(x) = \begin{cases} 2, & x \leq -1 \\ -2x, & x > -1; \end{cases}$

$f'(x) \geq 0$ on $(-\infty, 0]$;

$f'(x) \leq 0$ on $[0, \infty)$.

Thus, f is increasing on $(-\infty, 0]$ and decreasing on $[0, \infty)$.

(b) $f(0) = 3$ is a local maximum.

(c) $f''(x) = \begin{cases} 0, & x < -1 \\ -2, & x > -1; \end{cases}$

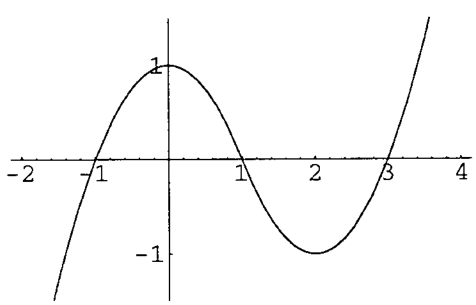
$f''(x) < 0$ for $x \in (-1, \infty)$.

Thus, the graph of f is concave down on $(-1, \infty)$.

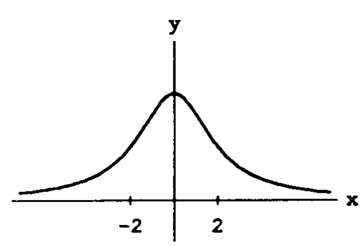
The graph of f is a straight line for $x \leq -1$.

(d) There are no points of inflection.

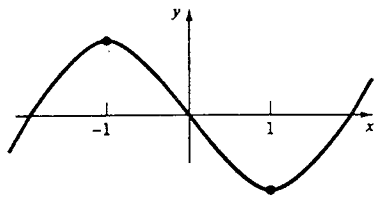
29.



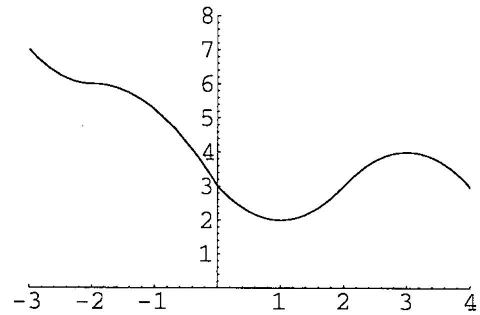
30.



31.



32.



33. Since $f''(x) = 6x - 2(a+b+c)$, set $d = \frac{1}{3}(a+b+c)$. Note that $f''(d) = 0$ and that f is concave down on $(-\infty, d)$ and concave up on (d, ∞); $(d, f(d))$ is a point of inflection.

34. $f'(x) = 2cx - 2x^{-3}$, $f''(x) = 2c + 6x^{-4}$. To have a point of inflection at 1 we need

$f''(1) = 0 \implies 2c + 6 = 0 \implies c = -3$

35. Since $(-1, 1)$ lies on the graph, $1 = -a + b$.

Since $f''(x)$ exists for all x and there is a pt of inflection at $x = \frac{1}{3}$, we must have $f''\left(\frac{1}{3}\right) = 0$.

Therefore

$$0 = 2a + 2b.$$

Solving these two equations, we find $a = -\frac{1}{2}$ and $b = \frac{1}{2}$.

Verification: the function

$$f(x) = -\frac{1}{2}x^3 + \frac{1}{2}x^2$$

has second derivative $f''(x) = -3x + 1$. This does change sign at $x = \frac{1}{3}$.

36. $f(x) = Ax^{1/2} + Bx^{-1/2}$; $f(1) = 4 \implies A + B = 4$.

$f'(x) = \frac{1}{2}Ax^{-1/2} - \frac{1}{2}Bx^{-3/2}$, $f''(x) = -\frac{1}{4}Ax^{-3/2} + \frac{3}{4}Bx^{-5/2}$.

To have a point of inflection at $(1, 4)$, we need $f''(1) = 0 \implies -\frac{1}{4}A + \frac{3}{4}B = 0$.

Solving the two equations gives $A = 3$, $B = 1$.

37. First, we require that $\left(\frac{1}{6}\pi, 5\right)$ lie on the curve:

$$5 = \tfrac{1}{2}A + B.$$

Next we require that $\dfrac{d^2y}{dx^2} = -4A\cos 2x - 9B\sin 3x$ be zero (and change sign) at $x = \dfrac{1}{6}\pi$:

$$0 = -2A - 9B.$$

Solving these two equations, we find $A = 18$, $B = -4$.

Verification: the function

$$f(x) = 18\cos 2x - 4\sin 3x$$

has second derivative $f''(x) = -72\cos 2x + 36\sin 3x$. This does change sign at $x = \frac{1}{6}\pi$.

38. $f(x) = Ax^2 + Bx + C;\quad f'(x) = 2Ax + B;\quad f''(x) = 2A.$

(a) Concave up $\implies f''(x) > 0 \implies A > 0$; to decrease between A and B we need

$$f'(x) < 0, \text{ for } x \text{ between } A \text{ and } B \implies B \le -2A^2.$$

(b) Concave down $\implies f''(x) < 0 \implies A < 0$; to have $f'(x) > 0$ for x between A and B we need $2A^2 + B \ge 0$ and $2AB + B \ge 0$, that is, $B \ge -2A^2$ and $B(2A+1) \ge 0$. If $A > -\frac{1}{2}$, then we need $B \ge 0$ (and automatically $B \ge -2A^2$). If $A \le -\frac{1}{2}$, then we need $B \le 0$ and $B \ge -2A^2$. The conditions are: $-\frac{1}{2} < A < 0,\ B \ge 0$, or $A \le -\frac{1}{2},\ -2A^2 \le B \le 0$.

39. Let $f'(x) = 3x^2 - 6x + 3$. Then we must have $f(x) = x^3 - 3x^2 + 3x + c$ for some constant c. Note that $f''(x) = 6x - 6$ and $f''(1) = 0$. Since $(1,-2)$ is a point of inflection of the graph of f, $(1,-2)$ must lie on the graph. Therefore,

$$1^3 - 3(1)^2 + 3(1) + c = -2 \quad \text{which implies} \quad c = -3$$

and so $f(x) = x^3 - 3x^2 + 3x - 3$.

40. $f'(x) = \cos x$ and $f''(x) = -\sin x = -f(x)$.

Thus f is concave down when $f''(x) < 0 \implies f(x) > 0$.

Similarly, f is concave up when $f''(x) > 0 \implies f(x) < 0$.

$g'(x) = -\sin x$ and $g''(x) = -\cos x = -g(x)$, so $g(x)$ has the same property.

41. (a) $p''(x) = 6x + 2a$ is negative for $x < -a/3$, and positive for $x > -a/3$. Therefore, the graph of p has a point of inflection at $x = -a/3$.

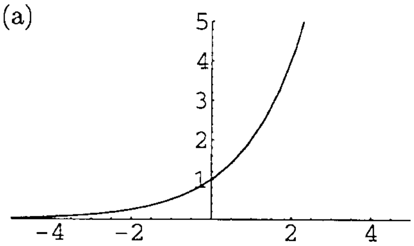
(b) $p'(x) = 3x^2 + 2ax + b$ has two real zeros iff $a^2 > 3b$.

42. It is sufficient to show that the x-coordinate of the point of inflection is the x-coordinate of the mid-point of the line segment connecting the local extrema. It is easy to show that the x-coordinate of the point of inflection is $x_0 = -\frac{1}{3}a$. Now suppose that p has local extrema at x_1 and x_2, $x_1 \ne x_2$. Then

$$p'(x_1) = p'(x_2) = 0 \quad\Rightarrow\quad 3x_1^2 + 2ax_1 + b - (3x_2^2 + 2ax_2 + b) = 0 \quad\Rightarrow\quad x_1 + x_2 = -\tfrac{2}{3}a.$$

Thus, $\dfrac{x_1 + x_2}{2} = -\frac{1}{3}a = x_0$.

43. (a)

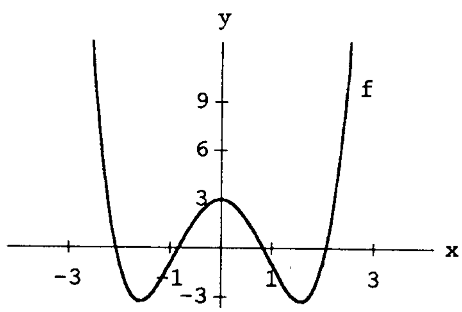


(b) No. If $f"(x) < 0$ and $f'(x) < 0$ for all x, then $f(x) < f'(0)x + f(0)$ on $(0, \infty)$, which implies that $f(x) \to -\infty$ as $x \to \infty$.

44. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$.

Then $f"(x) = n(n-1)a_n x^{n-2} + ... + 2a_2$ is a degree $n-2$ polynomial which can have at most $n-2$ roots. Hence f has at most $n-2$ points of inflection.

45.

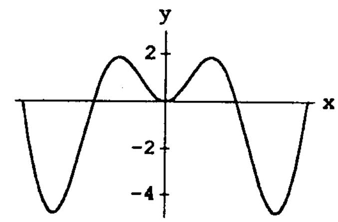


(a) concave up on $(-4, -0.913) \cup (0.913, 4)$

concave down on $(-0.913, 0.913)$

(b) pts of inflection at $x = -0.913,\ 0.913$

46.

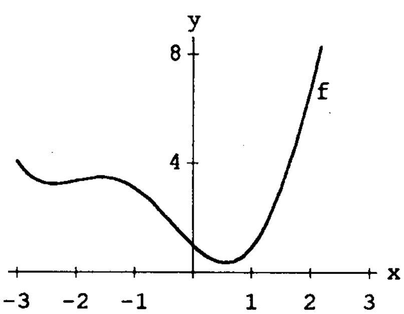


(a) concave up on $(-1.077, 1.077)$

concave down on $(-\pi, -1.077) \cup (1.077, \pi)$

(b) pts of inflection at $x = -1.077,\ 1.077$

47.



(a) concave up on:

$(-\pi, -1.996) \cup (-.0345, 2.550)$

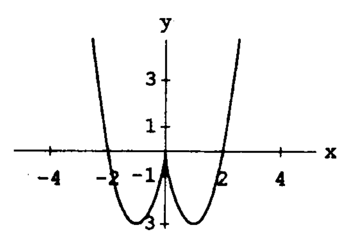
concave down on:

$(-1.996, -0.345) \cup (2.550, \pi)$

(b) pts of inflection at:

$x = -1.996,\ -0.345,\ 2,550$

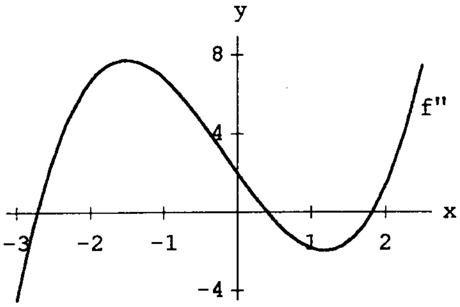
48.



(a) concave up on $(-5, 5)$

(b) no pts of inflection

49.



(a) concave up on:

$(-2.726, 0.402) \cup (1.823, 2.5)$

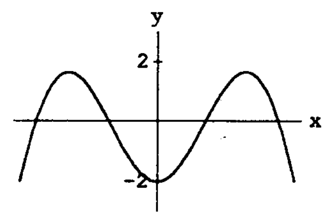
concave down on:

$(-3, -2.726) \cup (0.402, 1.823)$

(b) pts of inflection at:

$x = -2.726, 0.402, 1.823.$

50.



(a) concave up on:

$(-2.743, -1.091) \cup (1.091, 2.743)$

concave down on:

$(-\pi, -2.743) \cup (-1.091, 1.091) \cup (2.743, \pi)$

(b) pts of inflection at:

$x = -2.742, -1.091, 1.091, 2.743$

SECTION 4.7

1. (a) ∞ (b) $-\infty$ (c) ∞ (d) 1
(e) 0 (f) $x = -1,\ x = 1$ (g) $y = 0,\ y = 1$

2. (a) d (b) C (c) $x = a,\ x = b$
(d) $y = d$ (e) p (f) q

3. vertical: $x = \frac{1}{3}$; horizontal: $y = \frac{1}{3}$

4. vertical: $x = -2$; horizontal: none

5. vertical: $x = 2$; horizontal: none

6. vertical: none; horizontal: $y = 0$

7. vertical: $x = \pm 3$; horizontal: $y = 0$

8. vertical: $x = 16$; horizontal: $y = 0$

9. vertical: $x = -\frac{4}{3}$; horizontal: $y = \frac{4}{9}$

10. vertical: $x = \frac{1}{3}$; horizontal: $y = \frac{4}{9}$

11. vertical: $x = \frac{5}{2}$; horizontal: $y = 0$

12. vertical: $x = \frac{1}{2}$; horizontal: $y = -\frac{1}{8}$

13. vertical: none; horizontal: $y = \pm\frac{3}{2}$

14. vertical: $x = 8$; horizontal: $y = 0$

15. vertical: $x = 1$; horizontal: $y = 0$

16. vertical: $x = \pm 1$; horizontal: $y = \pm 2$

17. vertical: none; horizontal: $y = 0$

18. vertical: none; horizontal: $y = 0$

19. vertical: $x = \left(2n + \frac{1}{2}\right)\pi$; horizontal: none

20. vertical: $x = 2n\pi$; horizontal: none

21. $f'(x) = \frac{4}{3}(x+3)^{1/3}$; neither

22. $f'(x) = \frac{2}{5}x^{-3/5}$; cusp

23. $f'(x) = -\frac{4}{5}(2-x)^{-1/5}$; cusp

24. neither; $f(-1)$ is not defined

25. $f'(x) = \frac{6}{5}x^{-2/5}\left(1 - x^{3/5}\right)$; tangent

26. $f'(x) = \frac{7}{5}(x-5)^{2/5}$; neither

27. $f(-2)$ undefined; neither

28. $f'(x) = \frac{3}{7}(2-x)^{-4/7}$; tangent

29. $f'(x) = \begin{cases} \frac{1}{2}(x-1)^{-1/2}, & x > 1 \\ -\frac{1}{2}(1-x)^{-1/2}, & x < 1; \end{cases}$ cusp

30. $f'(x) = (4x-3)(x-1)^{-2/3}$; tangent

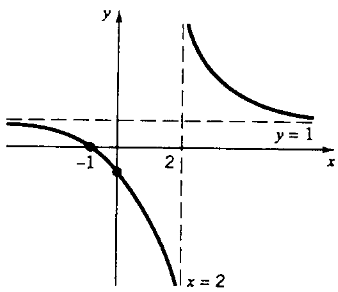
31. $f'(x) = \begin{cases} \frac{1}{3}(x+8)^{-23}, & x > -8 \\ -\frac{1}{3}(x+8)^{-2/3}, & x < -8; \end{cases}$ cusp

32. f is not defined for $x > 2$; neither

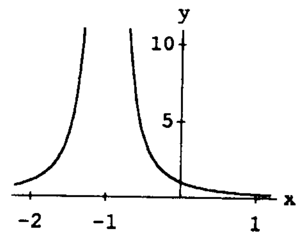
33. f not continuous at 0; neither

34. f is not continuous at 0; neither

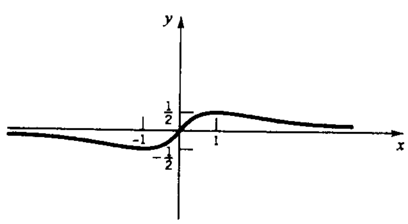
35.



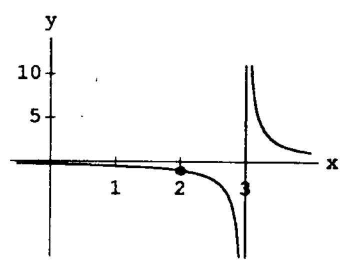
36.



37.



38.



39. $f(x) = x - 3x^{1/3}$

(a) $f'(x) = 1 - \dfrac{1}{x^{2/3}}$

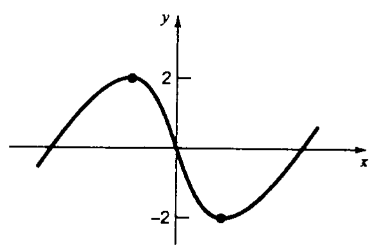
f is increasing on $(-\infty, -1] \cup [1, \infty)$

f is decreasing on $[-1, 1]$

(b) $f''(x) = \frac{2}{3}\,x^{-5/3}$

concave up on $(0, \infty)$; concave down on $(-\infty, 0)$

vertical tangent at $(0, 0)$

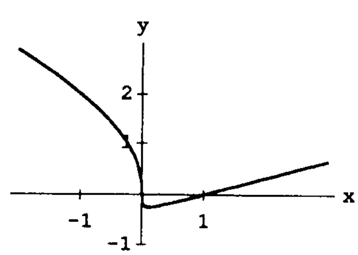


40. $f(x) = x^{2/3} - x^{1/3}$

(a) $f'(x) = \frac{2}{3}\,x^{-1/3} - \frac{1}{3}\,x^{-2/3} = \dfrac{2x^{1/3} - 1}{3x^{2/3}}$

f is increasing on $\left[\frac{1}{8}, \infty\right)$

f is decreasing on $\left(-\infty, \frac{1}{8}\right]$



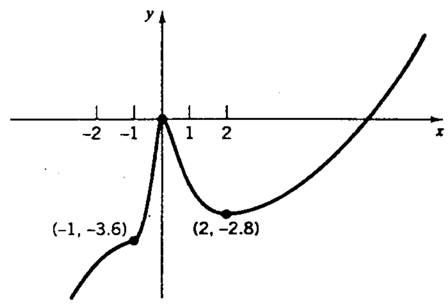
(b) $f''(x) = -\frac{2}{9}x^{-4/3} + \frac{2}{9}x^{-5/3} = \dfrac{2(1 - x^{1/3})}{9x^{5/3}}$

concave up on $(0, 1)$

concave down on $(-\infty, 0) \cup (1, \infty)$

vertical tangent at $(0, 0)$

41. $f(x) = \frac{3}{5}x^{5/3} - 3x^{2/3}$



(a) $f'(x) = x^{2/3} - 2x^{-1/3} = \dfrac{x-2}{x^{1/3}}$

f is increasing on $(-\infty, 0] \cup [2, \infty)$

f is decreasing on $[0, 2]$

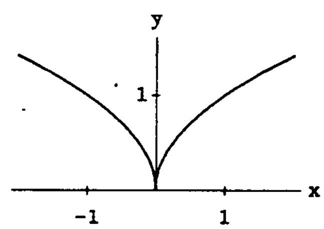
(b) $f''(x) = \frac{2}{3}x^{-1/3} + \frac{2}{3}x^{-4/3} = \dfrac{2x+2}{3x^{4/3}}$

concave up on $(-1, \infty)$; concave down on $(-\infty, -1)$

vertical cusp at $(0, 0)$

42. $f(x) = \sqrt{|x|}$

$$= \begin{cases} x^{1/2}, & x \geq 0 \\ (-x)^{1/2}, & x < 0. \end{cases}$$



(a) $f'(x) = \begin{cases} \frac{1}{2}x^{-1/2}, & x > 0 \\ -\frac{1}{2}(-x)^{-1/2}, & x < 0; \end{cases}$

f is increasing on $[0, \infty)$

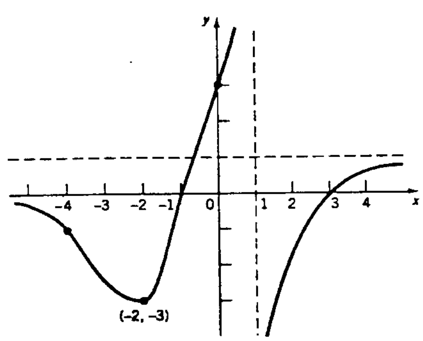
f is decreasing on $(-\infty, 0]$

(b) $f''(x) = \begin{cases} -\frac{1}{4}x^{-3/2}, & x > 0 \\ -\frac{1}{4}(-x)^{-3/2}, & x < 0; \end{cases}$

concave down on $(-\infty, 0) \cup (0, \infty)$

vertical cusp at $(0, 0)$

43.

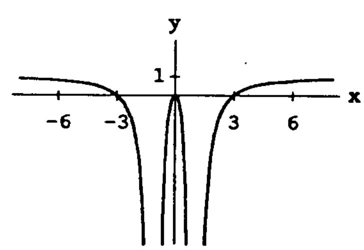


vertical asymptote: $x = 1$

horizontal asymptotes: $y = 0$, $y = 2$

no vertical tangents or cusps

44.

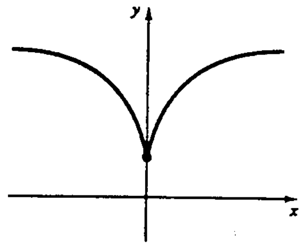


vertical asymptotes: $x = 1$, $x = -1$

horizontal asymptote: $y = 1$

no vertical tangents or cusps

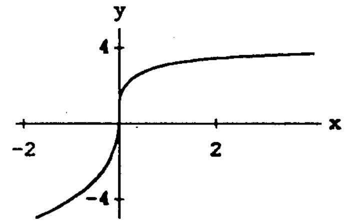
45.



no asymptotes

vertical cusp at $(0, 1)$

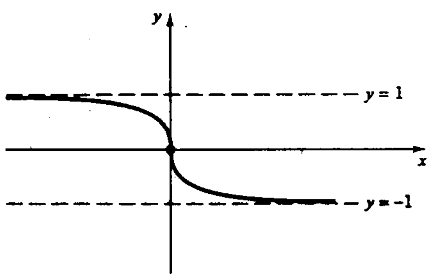
46.



horizontal asymptote: $y = 4$

vertical tangent at $(0, 1)$

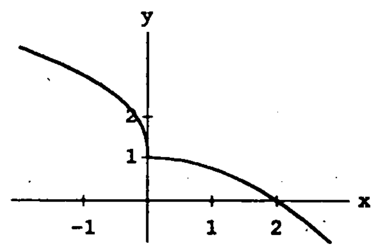
47.



horizontal asymptotes: $y = -1,\ y = 1$

vertical tangent at $(0, 0)$

48.



no asymptotes

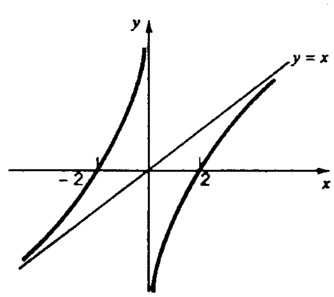
vertical tangent at $(0, 1)$

49. (a) p odd; (b) p even.

50. (a) Follow the hint.

(b) $[r(x) - (ax + b)] = \dfrac{Q(x)}{q(x)} \to 0$ as $x \to \pm\infty$, since $\deg[Q(x)] < \deg[q(x)]$.

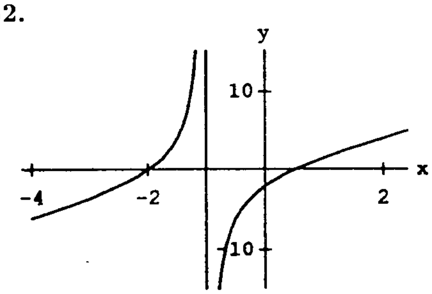
51.



vertical asymptote: $x = 0$

oblique asymptote: $y = x$

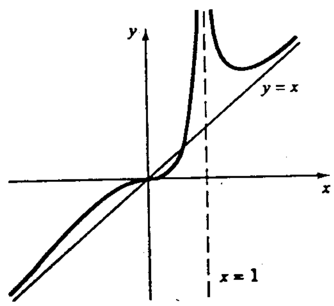
52.



vertical asymptote: $x = -1$

oblique asymptote: $y = 2x + 3$

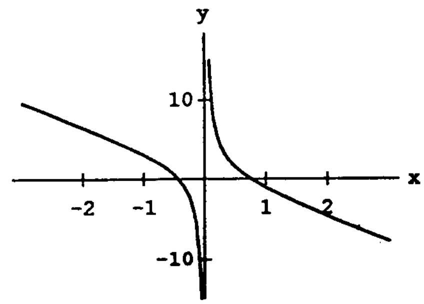
53.



vertical asymptote: $x = 1$

oblique asymptote: $y = x$

54.



vertical asymptote: $x = 0$

oblique asymptote: $y = -3x + 1$

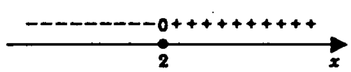
SECTION 4.8

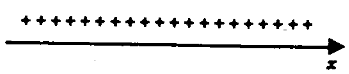
[Rough sketches; not scale drawings]

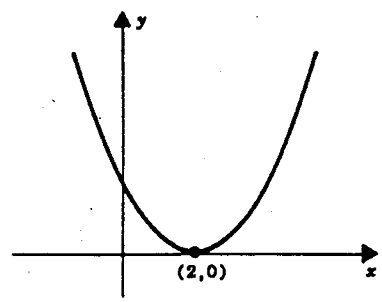
1. $f(x) = (x-2)^2$

$f'(x) = 2(x-2)$

$f''(x) = 2$

f' :


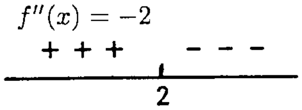
f'':


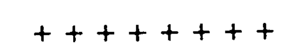


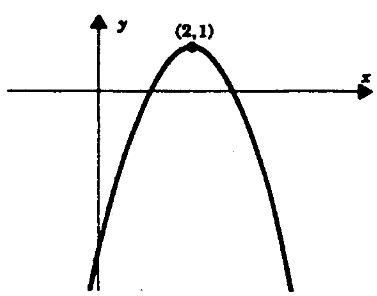
2. $f(x) = 1 - (x-2)^2$

$f'(x) = -2(x-2)$

$f''(x) = -2$

f' :


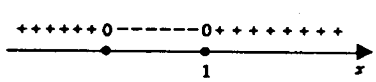
f'':


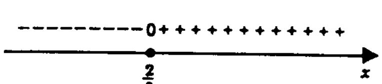


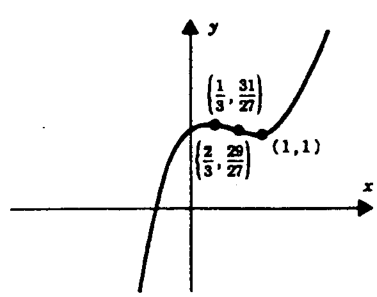
3. $f(x) = x^3 - 2x^2 + x + 1$

$f'(x) = (3x-1)(x-1)$

$f''(x) = 6x - 4$

f' :


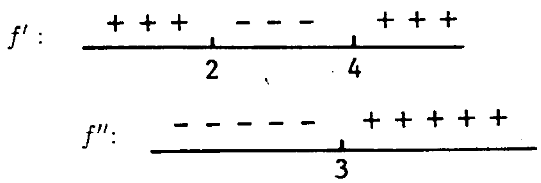
f'':


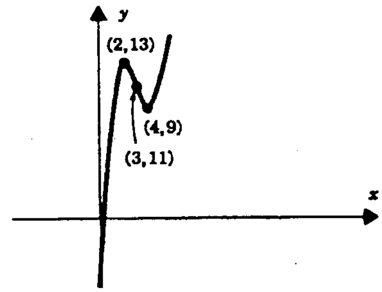


4. $f(x) = x^3 - 9x^2 + 24x - 7$

$f'(x) = 3x^2 - 18x + 24$

$f''(x) = 6x - 18$

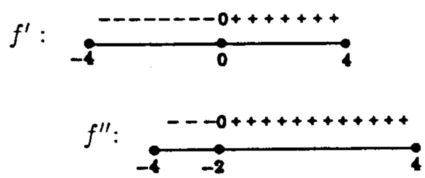


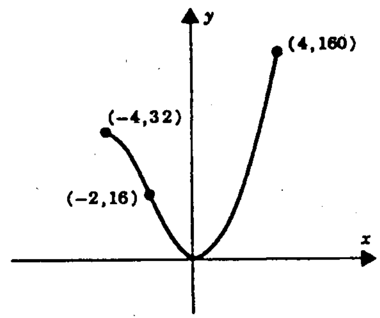


5. $f(x) = x^3 + 6x^2, \quad x \in [-4, 4]$

$f'(x) = 3x(x+4)$

$f''(x) = 6x + 12$

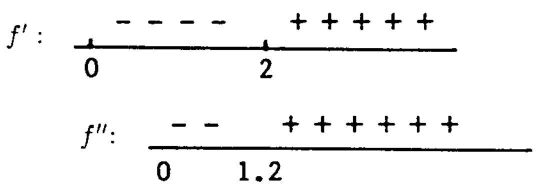


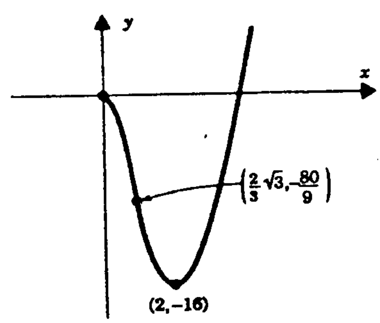


6. $f(x) = x^4 - 8x^2, \quad x \in [0, \infty)$

$f'(x) = 4x^3 - 16x$

$f''(x) = 12x^2 - 16$

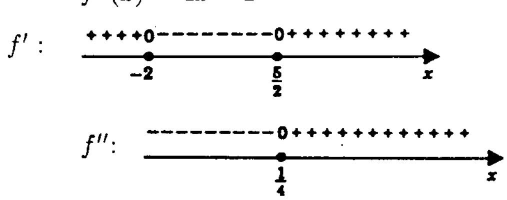


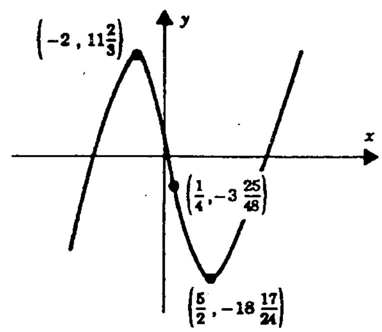


7. $f(x) = \frac{2}{3}x^3 - \frac{1}{2}x^2 - 10x - 1$

$f'(x) = (2x-5)(x+2)$

$f''(x) = 4x - 1$





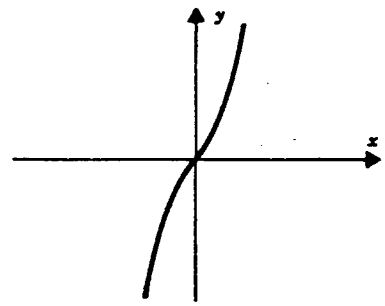
8. $f(x) = x(x^2+4)^2 = x^5 + 8x^3 + 16x$

$f'(x) = 5x^4 + 24x^2 + 16$

$f''(x) = 20x^3 + 48x = 4x(5x^2+12)$

f': + + + + + + + + + (0)

f'': − − − − (0) + + + + +



9. $f(x) = x^2 + 2x^{-1}$

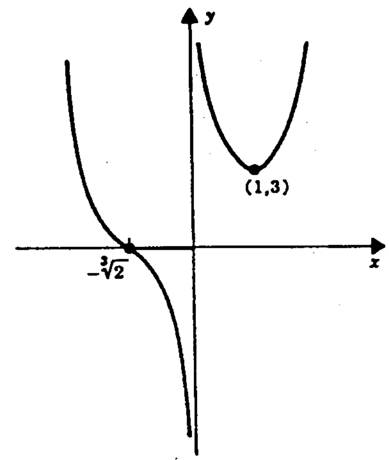
$f'(x) = 2x - 2x^{-2} = 2\left(x^3-1\right)/x^2$

$f''(x) = 2 + 4x^{-3}$

f': − − − − − − − − (0) − − − − 0 + + + + (1) x

f'': + + + 0 − − − − − ($-\sqrt[3]{2}$) (0) + + + + + + + + + x

asymptote: $x = 0$



10. $f(x) = x - x^{-1}$,

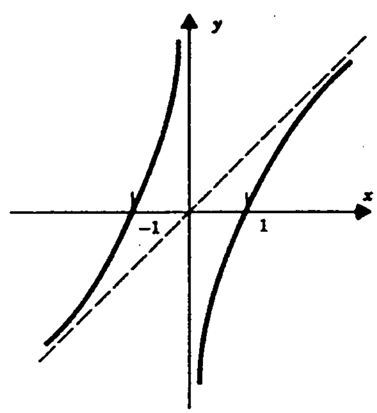
$f'(x) = 1 + x^{-2}$

$f''(x) = -x^{-3}$

f': + + + (0) + + + +

f'': + + + (0) − − − −

asymptote: $y = x$



11. $f(x) = (x-4)/x^2$

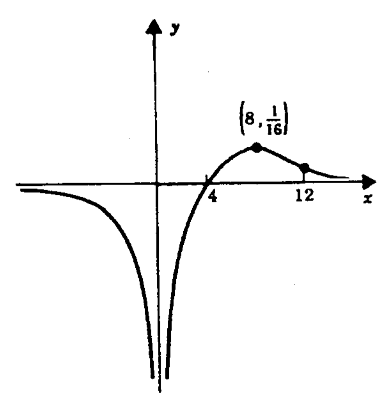
$f'(x) = (8-x)/x^3$

$f''(x) = (2x-24)/x^4$

f': − − − (0) + + + + + + + + 0 (8) − − − − − − x

f'': − − − (0) − − − − − − − − − − − − 0 + + + + (12) x

asymptotes: $x = 0,\ y = 0$



12. $f(x) = \dfrac{x+2}{x^3} = \dfrac{1}{x^2} + \dfrac{2}{x^3}$

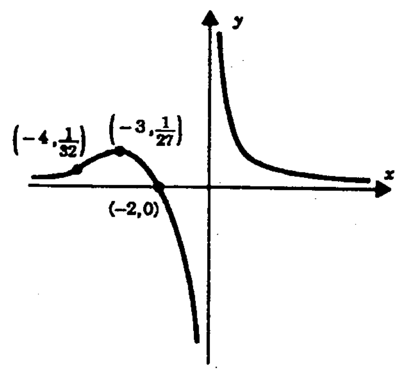
$f'(x) = -\dfrac{2}{x^3} - \dfrac{6}{x^4} = \dfrac{-2x-6}{x^4}$

$f''(x) = \dfrac{6}{x^4} + \dfrac{24}{x^5} = \dfrac{6(x+4)}{x^5}$

f' : + + + (−3) − − − (0) − −

f'': + + (−4) − − − − (0) + + +

asymptotes: $x = 0,\ y = 0$



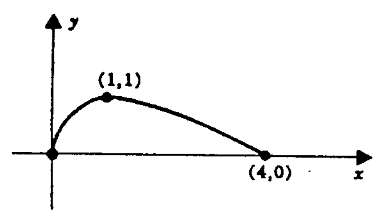
13. $f(x) = 2x^{1/2} - x, \quad x \in [0,4]$

$f'(x) = x^{-1/2}\left(1 - x^{1/2}\right)$

$f''(x) = -\frac{1}{2}x^{-3/2}$

f' : [0] + + + 0 (1) − − − − − − − − − − − − [4]

f'': [0] − − − − − − − − − − − − − − − − − [4]



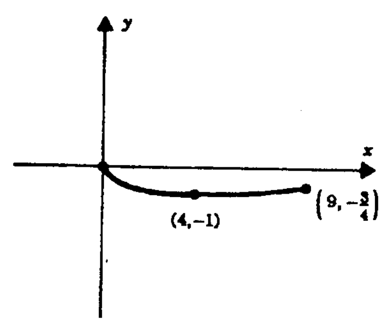
14. $f(x) = \frac{1}{4}x - \sqrt{x}, \quad x \in [0,9]$

$f'(x) = \frac{1}{4} - \frac{1}{2}x^{-1/2}$

$f''(x) = \frac{1}{4}x^{-3/2}$

f' : [0] − − − − (4) + + + + + + [9]

f'': [0] + + + + + + + + + [9]



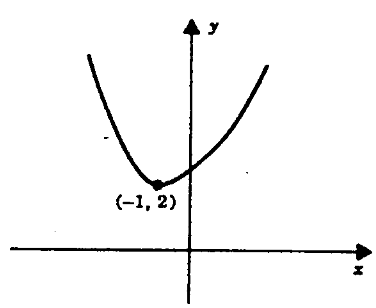
15. $f(x) = 2 + (x+1)^{6/5}$

$f'(x) = \frac{6}{5}(x+1)^{1/5}$

$f''(x) = \frac{6}{25}(x+1)^{-4/5}$

f' : − − − − − − − − − 0 (−1) + + + + + + + + + + + + x

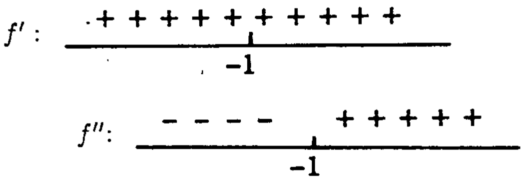
f'': + + + + + + + + dne (−1) + + + + + + + + + + + + x

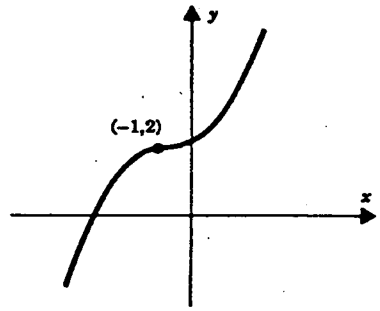


16. $f(x) = 2 + (x+1)^{7/5}$

$f'(x) = \frac{7}{5}(x+1)^{2/5}$

$f''(x) = \frac{14}{25}(x+1)^{-3/5}$

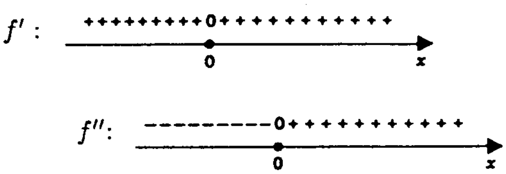


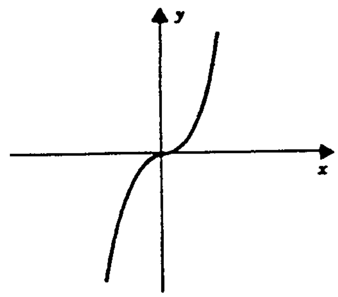


17. $f(x) = 3x^5 + 5x^3$

$f'(x) = 15x^2\left(x^2+1\right)$

$f''(x) = 30x\left(2x^2+1\right)$

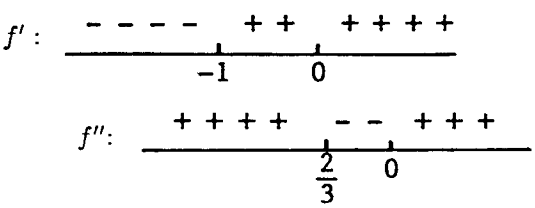


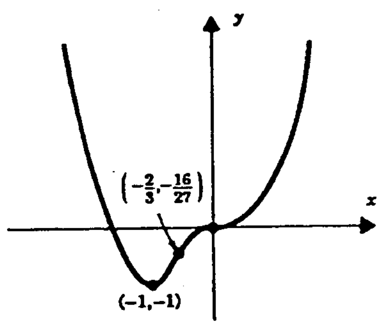


18. $f(x) = 3x^4 + 4x^3$

$f'(x) = 12x^3 + 12x^2 = 12x^2(x+1)$

$f''(x) = 36x^2 + 24x = 12x(3x+2)$

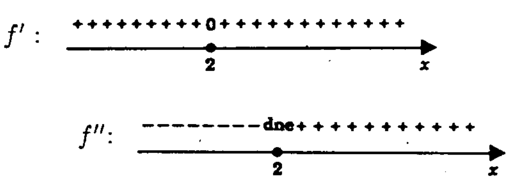


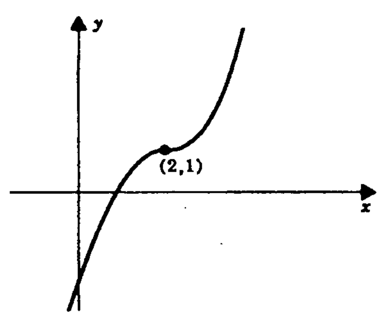


19. $f(x) = 1 + (x-2)^{5/3}$

$f'(x) = \frac{5}{3}(x-2)^{2/3}$

$f''(x) = \frac{10}{9}(x-2)^{-1/3}$





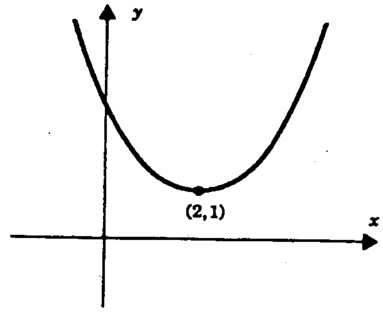
20. $f(x) = 1 + (x-2)^{4/3}$

$f'(x) = \frac{4}{3}(x-2)^{1/3}$

$f''(x) = \frac{4}{9}(x-2)^{-2/3}$

f' : $- - - -$ (2) $+ + + + +$

f'': $+ + + + + + + + + +$



21. $f(x) = \dfrac{2x}{4x-3}$

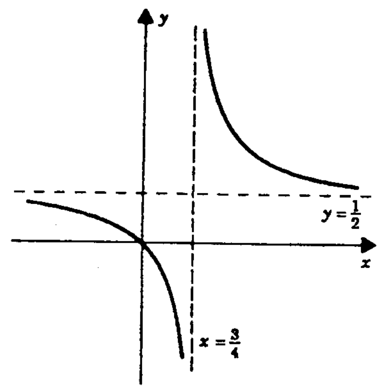
$f'(x) = -6(4x-3)^{-2}$

$f''(x) = 48(4x-3)^{-3}$

f' : $- - - - - - -$ ($\frac{3}{4}$) $- - - - - - -$ x

f'': $- - - - - - - -$ ($\frac{3}{4}$) $+ + + + + + + +$ x

asymptotes: $x = 3/4$, $y = 1/2$



22. $f(x) = \dfrac{2x^2}{x+1}$

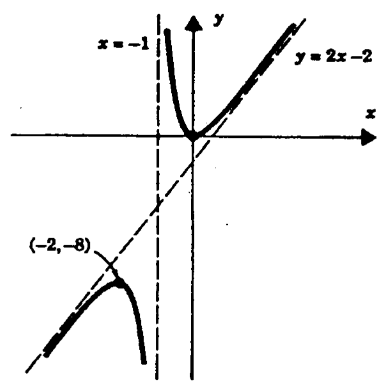
$f'(x) = \dfrac{2x(x+2)}{(x+1)^2}$

$f''(x) = \dfrac{4}{(x+1)^3}$

f' : $+ + +$ (−2) $- - -$ (0) $+ + +$

f'': $- - - -$ (−1) $+ + + +$

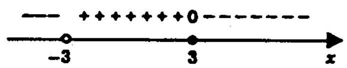
asymptote: $y = 2x - 2$

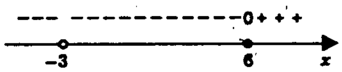


23. $f(x) = \dfrac{x}{(x+3)^2}$

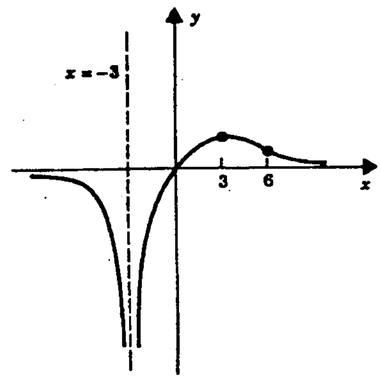
$f'(x) = \dfrac{3-x}{(x+3)^3}$

$f''(x) = \dfrac{2x-12}{(x+3)^4}$

f' : 

f'': 

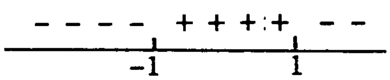
asymptotes: $x = -3,\ y = 0$

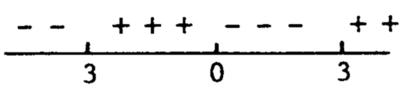


24. $f(x) = \dfrac{x}{x^2+1}$

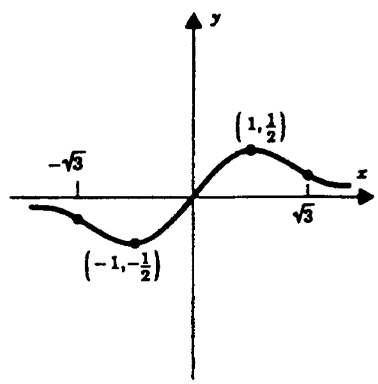
$f'(x) = \dfrac{1-x^2}{(x^2+1)^2}$

$f''(x) = \dfrac{2x(x^2-3)}{(x^2+1)^3}$

f' : 

f'': 

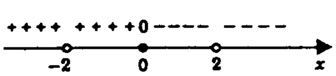
asymptote: $y = 0$

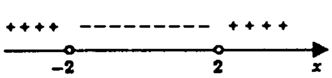


25. $f(x) = \dfrac{x^2}{x^2-4}$

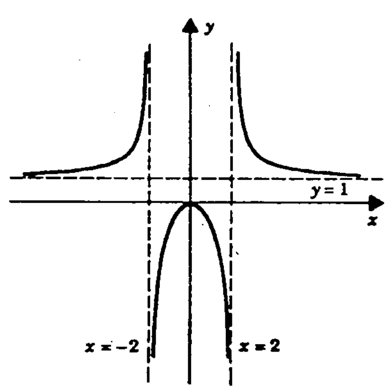
$f'(x) = \dfrac{-8x}{(x^2-4)^2}$

$f''(x) = \dfrac{8\left(3x^2+4\right)}{\left(x^2-4\right)^3}$

f' : 

f'': 

asymptotes: $x = -2,\ x = 2,\ y = 1$



26. $f(x) = \dfrac{2x}{x-4}$

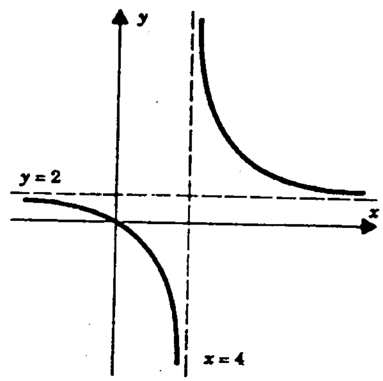
$f'(x) = \dfrac{-8}{(x-4)^2}$

$f''(x) = \dfrac{16}{(x-4)^3}$

f' : − − − − − − − − − − (4)

f'': − − − − | + + + + (4)

asymptotes: $x = 4,\ y = 2$



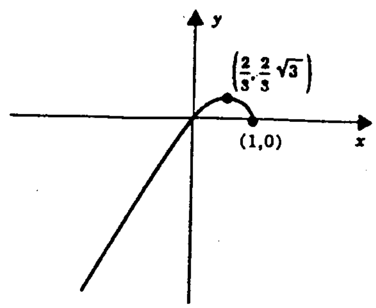
27. $f(x) = x(1-x)^{1/2}$

$f'(x) = \frac{1}{2}(1-x)^{-1/2}(2-3x)$

$f''(x) = \frac{1}{4}(1-x)^{-3/2}(3x-4)$

f' : ++++++++++0−−−−−− ($\frac{2}{3}$, 1)

f'': −−−−−−−−−−−−−−−− (1)



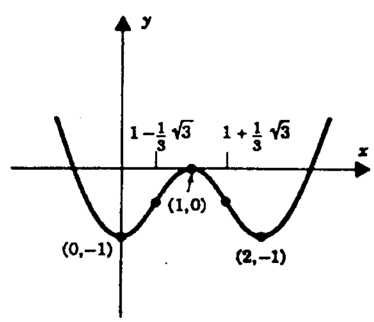
28. $f(x) = (x-1)^4 - 2(x-1)^2$

$f'(x) = 4(x-1)^3 - 4(x-1)$

$f''(x) = 12(x-1)^2 - 4$

f' : − − | + + + | − − | + + (0, 1, 2)

f'': + + + | − − − | + + + (.4, 1.6)



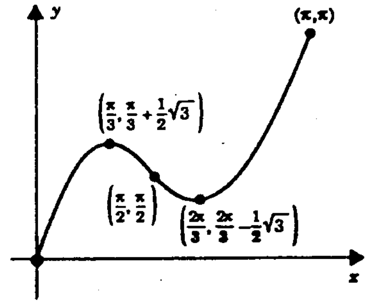
29. $f(x) = x + \sin 2x, \quad x \in [0, \pi]$

$f'(x) = 1 + 2\cos 2x$

$f''(x) = -4\sin 2x$

f': +++++0------0+++++ on 0, $\frac{\pi}{3}$, $\frac{2\pi}{3}$, π

f'': --------0++++++++++ on 0, $\frac{\pi}{2}$, π



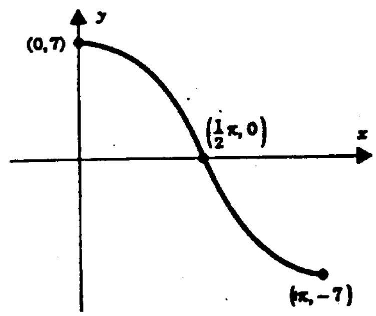
30. $f(x) = \cos^3 x + 6\cos x, \quad x \in [0, \pi]$

$f'(x) = -3\sin x(\cos^2 x + 2)$

$f''(x) = -9\cos^3 x$

f': - - - - - - - - - - - - on 0, π

f'': - - - - - + + + + on $\frac{\pi}{2}$



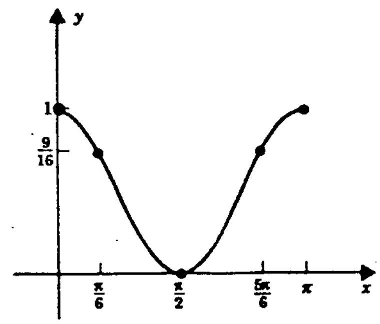
31. $f(x) = \cos^4 x, \quad x \in [0, \pi]$

$f'(x) = -4\cos^3 x \sin x$

$f''(x) = 4\cos^2 x\left(3\sin^2 x - \cos^2 x\right)$

f': ----------0++++++++++ on 0, $\frac{\pi}{2}$, π

f'': --0++++++0++++++0-- on 0, $\frac{\pi}{6}$, $\frac{\pi}{2}$, $\frac{5\pi}{6}$, π



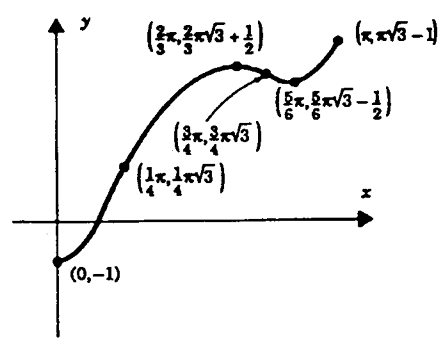
32. $f(x) = \sqrt{3}x - \cos 2x, \quad x \in [0, \pi]$

$f'(x) = \sqrt{3} + 2\sin 2x$

$f''(x) = 4\cos 2x$

f': + + + + + − − + + on 0, $\frac{2\pi}{3}$, $\frac{5\pi}{6}$, π

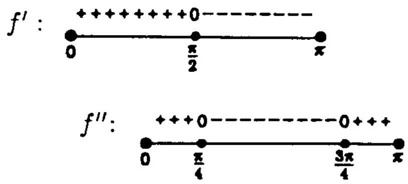
f'': + + − − − − + + + on 0, $\frac{\pi}{4}$, $\frac{3\pi}{4}$, π

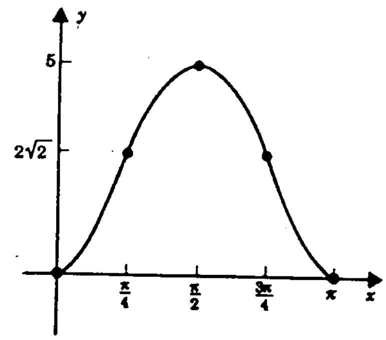


33. $f(x) = 2\sin^3 x + 3\sin x, \quad x \in [0, \pi]$

$f'(x) = 3\cos x\,(2\sin^2 x + 1)$

$f''(x) = 9\sin x\,(1 - 2\sin^2 x)$

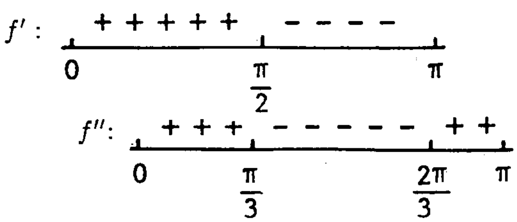


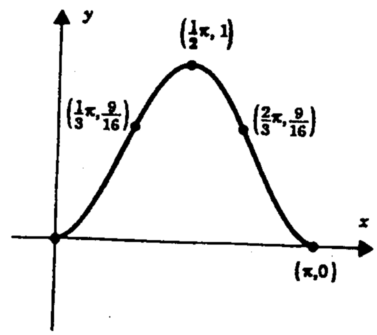


34. $f(x) = \sin^4 x, \quad x \in [0, \pi]$

$f'(x) = 4\sin^3 x\cos x$

$f''(x) = 4\sin^2 x\,(3\cos^2 x - \sin^2 x)$

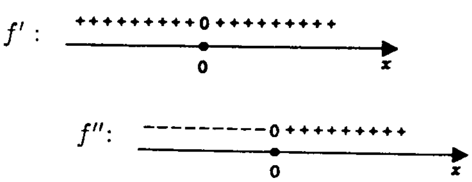


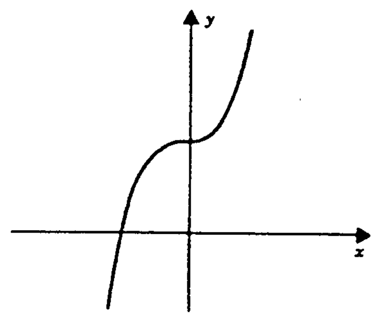


35. $f(x) = [(x+1) - 1]^3 + 1$

$f'(x) = 3x^2$

$f''(x) = 6x$

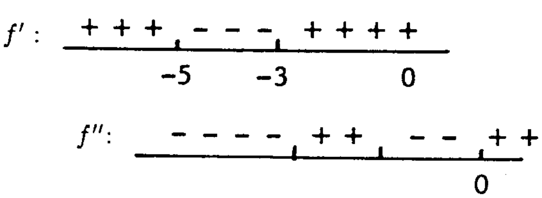


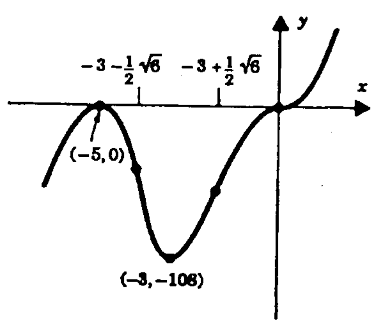


36. $f(x) = x^3(x+5)^2$

$f'(x) = 5x^2(x+3)(x+5)$

$f''(x) = 10x(2x^2 + 12x + 15)$



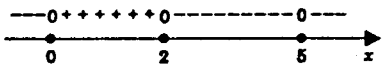


37. $f(x) = x^2(5-x)^3$

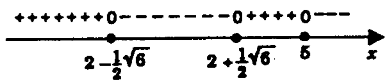
$f'(x) = 5x(2-x)(5-x)^2$

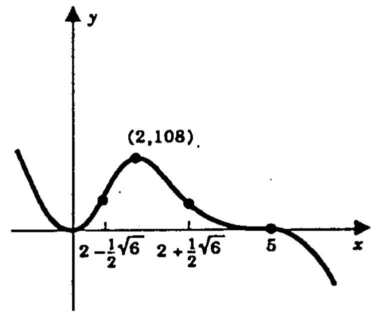
$f''(x) = 10(5-x)\left(2x^2 - 8x + 5\right)$

f' :



f'':



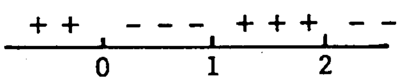


38. $f(x) = \begin{cases} 4 - 2x + x^2, & x \le 0,\ x \ge 2 \\ 4 + 2x - x^2, & 0 < x < 2 \end{cases}$

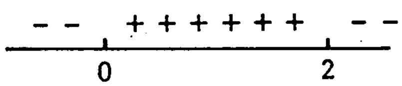
$f'(x) = \begin{cases} -2 + 2x, & x < 0,\ x > 2 \\ 2 - 2x, & 0 < x < 2 \end{cases}$

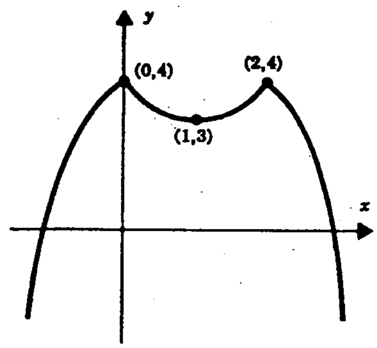
$f''(x) = \begin{cases} 2, & x < 0,\ x > 2 \\ -2, & 0 < x < 2 \end{cases}$

f' :



f'':



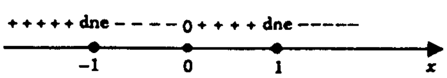


39. $f(x) = \begin{cases} 4 - x^2, & |x| > 1 \\ x^2 + 2, & -1 \le x \le 1 \end{cases}$

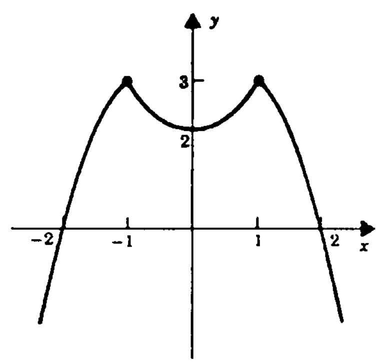
$f'(x) = \begin{cases} -2x, & |x| > 1 \\ 2x, & -1 < x < 1 \end{cases}$

$f''(x) = \begin{cases} -2, & |x| > 1 \\ 2, & -1 < x < 1 \end{cases}$

f' :



f'':



40. $f(x) = x - x^{1/3}$

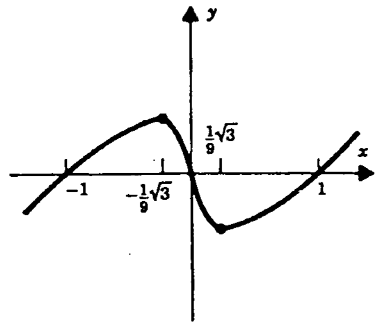
$f'(x) = 1 - \frac{1}{3}x^{-2/3}$

$f''(x) = \frac{2}{9}x^{-5/3}$

f' : + + + | − − − | + + +

f'': − − − − − | + + + +
0

vertical tangent at $(0, 0)$



41. $f(x) = x(x-1)^{1/5}$

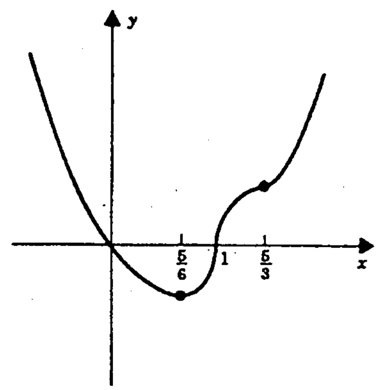
$f'(x) = \frac{1}{5}(x-1)^{-4/5}(6x-5)$

$f''(x) = \frac{2}{25}(x-1)^{-9/5}(3x-5)$

f' : − − − − − 0 + + + + dne + + + + + + + + + + + +
$\frac{5}{6}$ 1 x

f'': + + + + + + + + + + dne − − − − − − − − − 0 + + + +
1 $\frac{5}{3}$ x

vertical tangent at $(1, 0)$



42. $f(x) = x^2(x-7)^{1/3}$

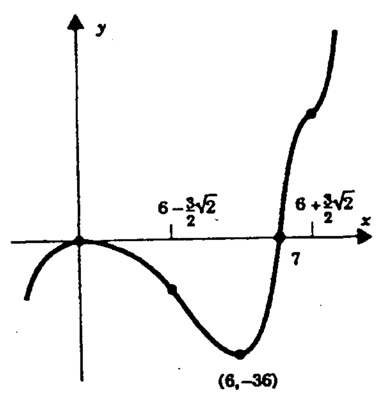
$$f'(x) = \frac{7x(x-6)}{3(x-7)^{2/3}}$$

$$f''(x) = \frac{14(2x^2 - 24x + 63)}{9(x-7)^{5/3}}$$

f' : + + + | − − − | + + +
0 6

f'': − − − − − | + + | − | +
7

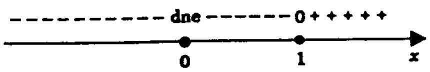
vertical tangent at $(7, 0)$

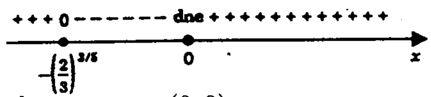


43. $f(x) = x^2 - 6x^{1/3}$

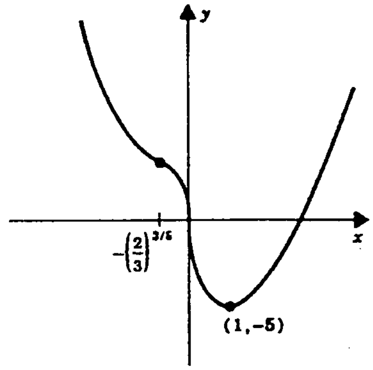
$f'(x) = 2x^{-2/3}\left(x^{5/3} - 1\right)$

$f''(x) = \frac{2}{3}x^{-5/3}\left(3x^{5/3} + 2\right)$

f' :


f'':


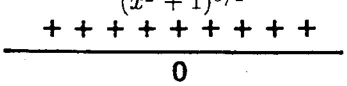
vertical tangent at $(0, 0)$

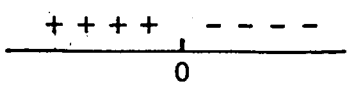


44. $f(x) = \dfrac{2x}{\sqrt{x^2+1}}$

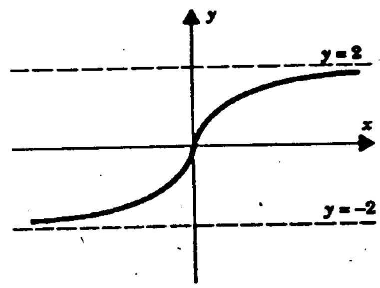
$f'(x) = \dfrac{2}{(x^2+1)^{3/2}}$

$f''(x) = \dfrac{-6x}{(x^2+1)^{5/2}}$

f' :


f'':


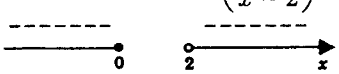
asymptotes: $y = 2$, $y = -2$

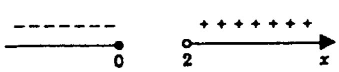


45. $f(x) = \left(\dfrac{x}{x-2}\right)^{1/2}$; $x \le 0$, $x > 2$

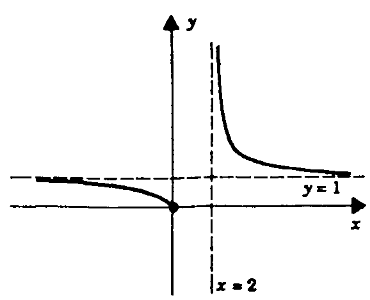
$f'(x) = -\left(\dfrac{x}{x-2}\right)^{-1/2}(x-2)^{-2}$

$f''(x) = (2x-1)\left(\dfrac{x}{x-2}\right)^{-3/2}(x-2)^{-4}$

f' :


f'':


asymptotes: $x = 2$, $y = 1$



46. $f(x) = \left(\dfrac{x}{x+4}\right)^{1/2}; \quad x < -4,\ x \geq 0$

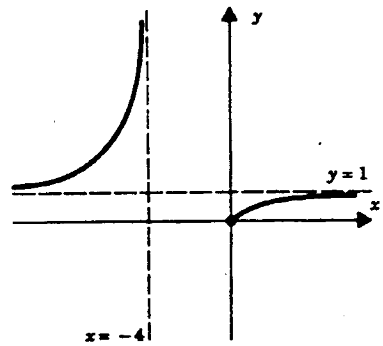
$f'(x) = 2\left(\dfrac{x}{x+4}\right)^{-1/2}(x+4)^{-2}$

$f''(x) = \dfrac{-4(x+1)}{(x+4)^{5/2}x^{3/2}}$

f': + + + (−4) (1) + + + +

f'': + + + (−4) (1) − − − −

asymptotes: $x = -4,\ y = 1$



47. $f(x) = x^2\left(x^2-2\right)^{-1/2}, \quad |x| > \sqrt{2}$

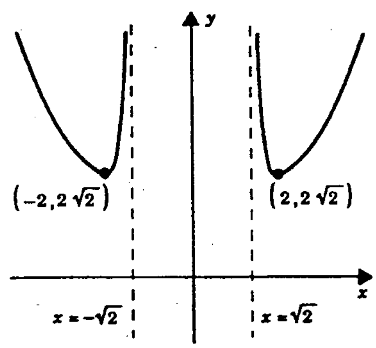
$f'(x) = x\left(x^2-4\right)\left(x^2-2\right)^{-3/2}$

$f''(x) = 2\left(x^2+4\right)\left(x^2-2\right)^{-5/2}$

f': − − 0 + + + (−2, $-\sqrt{2}$) — 0 + + + + ($\sqrt{2}$, 2) x

f'': + + + + + + ($-\sqrt{2}$) + + + + + + + + + ($\sqrt{2}$) x

asymptotes: $x = -\sqrt{2},\ x = \sqrt{2}$



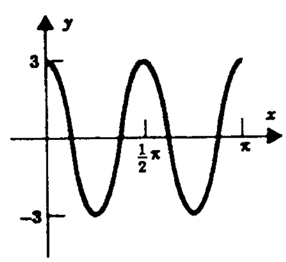
48. $f(x) = 3\cos 4x, \quad x \in [0, \pi]$

$f'(x) = -12\sin 4x$

$f''(x) = -48\cos 4x$

f': − − + + − − + + (0, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$, π)

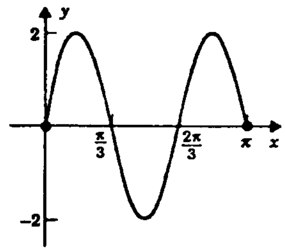
f'': − + + − − + + − − (0, $\frac{\pi}{8}$, $\frac{3\pi}{8}$, $\frac{5\pi}{8}$, $\frac{7\pi}{8}$, π)



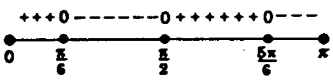
49. $f(x) = 2\sin 3x, \quad x \in [0, \pi]$

$f'(x) = 6\cos 3x$

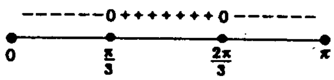
$f''(x) = -18\sin 3x$



f' :



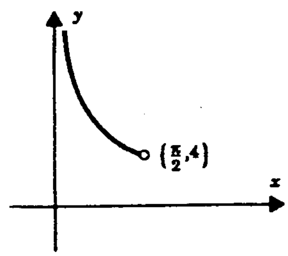
f'':



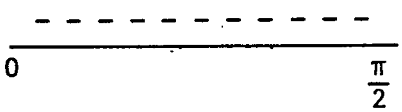
50. $f(x) = 3 + 2\cot x + \csc^2 x, \quad x \in (0, \pi/2)$

$f'(x) = -2\csc^2 x\,(1 + \cot x)$

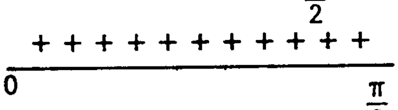
$f''(x) = 2\csc^2 x\,(3\cot^2 x + 2\cot x + 1)$



f' :



f'':



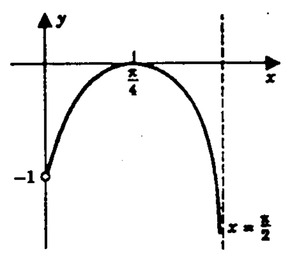
asymptote: $x = 0$

51. $f(x) = 2\tan x - \sec^2 x, \quad x \in (0, \pi/2)$

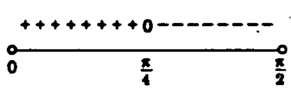
$= -(1 - \tan x)^2$

$f'(x) = 2\sec^2 x\,(1 - \tan x)$

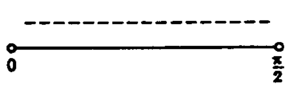
$f''(x) = -2\sec^2 x\,(3\tan^2 x - 2\tan x + 1)$



f' :



f'':

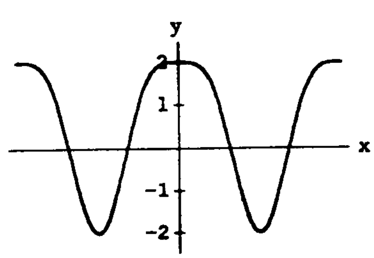


asymptote: $x = \frac{1}{2}\pi$

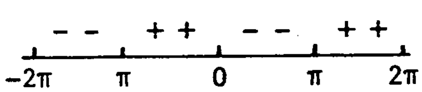
52. $f(x) = 2\cos x + \sin^2 x$

$f'(x) = 2\sin x\,(\cos x - 1)$

$f''(x) = 2(2\cos^2 x - \cos x - 1)$

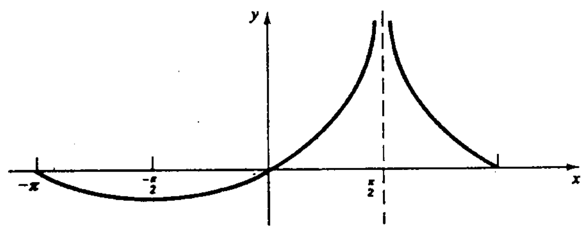


f' :



f'':

53. $f(x) = \dfrac{\sin x}{1 - \sin x}, \quad x \in (-\pi, \pi)$



$$f'(x) = \frac{\cos x}{(1-\sin x)^2}$$

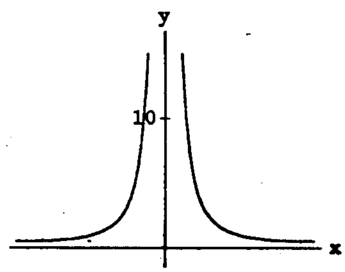
$$f''(x) = \frac{1 - \sin x + \cos^2 x}{(1-\sin x)^3}$$

f' : ---- 0 + + + + + dne ----
$-\pi \quad -\frac{\pi}{2} \quad 0 \quad \frac{\pi}{2} \quad \pi$

f'': + + + + + + + + + dne + + +
$-\pi \quad -\frac{\pi}{2} \quad 0 \quad \frac{\pi}{2} \quad \pi$

asymptote: $x = \frac{1}{2}\pi$

54. $f(x) = \dfrac{1}{1 - \cos x}, \quad x \in (-\pi, \pi)$



$$f'(x) = \frac{-\sin x}{(1-\cos x)^2}$$

$$f''(x) = \frac{1 - \cos x - \cos^2 x}{(1-\cos x)^3}$$

f' : + + + + – – – –
0

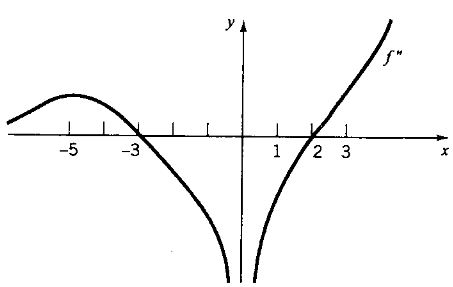
f'': + + + + + + + + + + +
0

asymptote: $x = 0$

55. (a) f increases on $(-\infty, -1] \cup (0, 1] \cup [3, \infty)$;

f decreases on $[-1, 0) \cup [1, 3]$; critical numbers: $x = -1, 0, 1, 3$.

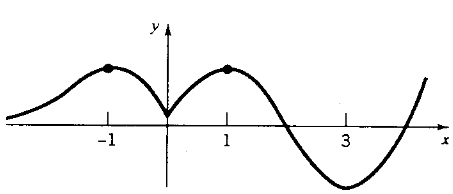
(b)



concave up on $(-\infty, -3) \cup (2, \infty)$

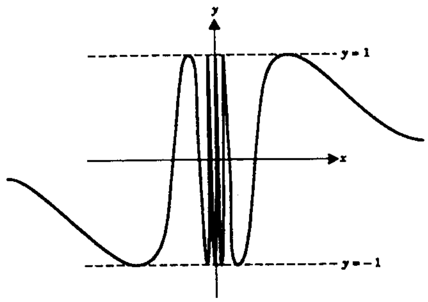
concave down on $(-3, 0) \cup (0, 2)$.

(c)

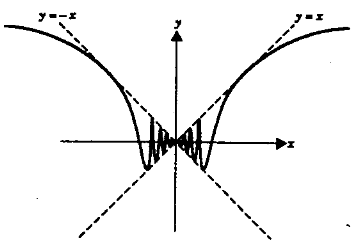


The graph does not necessarily have a horizontal asymptote.

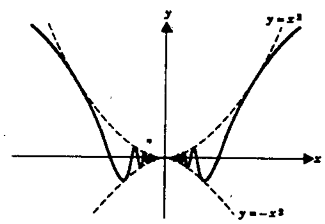
56. (a)



(b)



(c)



(d) F is discontinuous at 0; $\lim_{x\to 0} \sin(1/x)$ does not exist

G is continuous at 0: $|x\sin(1/x)| = |x|\,|\sin(1/x)| \le |x| \to 0,$

$$\text{so that } \lim_{x\to 0} G(x) = 0 = G(0).$$

H is continuous at 0: $\left|x^2\sin(1/x)\right| = |x|^2\,|\sin(1/x)| \le |x|^2 \to 0,$

$$\text{so that } \lim_{x\to 0} H(x) = 0 = H(0).$$

(e) F is not differentiable at 0 since it is not continuous at 0.

G is not differentiable at 0: $\displaystyle\lim_{h\to 0} \frac{G(h)-G(0)}{h} = \lim_{h\to 0} \sin(1/h)$ does not exist.

H is differentiable at 0: $\displaystyle H'(0) = \lim_{h\to 0} \frac{H(h)-H(0)}{h} = \lim_{h\to 0} h\sin(1/h) = 0.$

57. Solve the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

for y:

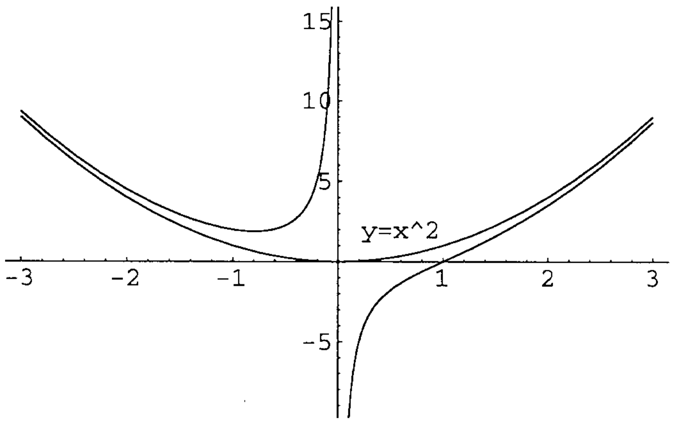
$$y^2 = \frac{b^2(x^2-a^2)}{a^2} \quad \text{and}$$

$$y = \pm\frac{b}{a}\sqrt{x^2-a^2} = \pm\frac{b}{a}\,x\sqrt{1-\frac{a^2}{x^2}}$$

Now, for $|x|$ large, $y \cong \pm\dfrac{b}{a}\,x.$

58. $f(x) - x^2 = \dfrac{x^3 - 1}{x} - x^2 = -\dfrac{1}{x}$.

Since $\lim\limits_{x \to \pm\infty} -\dfrac{1}{x} = 0$, so does $\lim\limits_{x \to \pm\infty} f(x) - x^2$.



CHAPTER 5

SECTION 5.1

1. $L_f(P)=0(\frac{1}{4})+\frac{1}{2}(\frac{1}{4})+1(\frac{1}{2})=\frac{5}{8}$, $\quad U_f(P)=\frac{1}{2}(\frac{1}{4})+1(\frac{1}{4})+2(\frac{1}{2})=\frac{11}{8}$

2. $L_f(P)=\frac{2}{3}(\frac{1}{3})+\frac{1}{4}(\frac{5}{12})+0(\frac{1}{4})+(-1)(1)=-\frac{97}{144}$,

$U_f(P)=1(\frac{1}{3})+\frac{2}{3}(\frac{5}{12})+\frac{1}{4}(\frac{1}{4})+0(1)=\frac{97}{144}$

3. $L_f(P)=\frac{1}{4}(\frac{1}{2})+\frac{1}{16}(\frac{1}{4})+0(\frac{1}{4})=\frac{9}{64}$, $\quad U_f(P)=1(\frac{1}{2})+\frac{1}{4}(\frac{1}{4})+\frac{1}{16}(\frac{1}{4})=\frac{37}{64}$

4. $L_f(P)=\frac{15}{16}(\frac{1}{4})+\frac{3}{4}(\frac{1}{4})+0(\frac{1}{2})=\frac{27}{64}$, $\quad U_f(P)=1(\frac{1}{4})+\frac{15}{16}(\frac{1}{4})+\frac{3}{4}(\frac{1}{2})=\frac{55}{64}$

5. $L_f(P)=1(\frac{1}{2})+\frac{9}{8}(\frac{1}{2})=\frac{17}{16}$, $\quad U_f(P)=\frac{9}{8}(\frac{1}{2})+2(\frac{1}{2})=\frac{25}{16}$

6. $L_f(P)=0(\frac{1}{25})+\frac{1}{5}(\frac{3}{25})+\frac{2}{5}(\frac{5}{25})+\frac{3}{5}(\frac{7}{25})+\frac{4}{5}(\frac{9}{25})=\frac{14}{25}$,

$U_f(P)=\frac{1}{5}(\frac{1}{25})+\frac{2}{5}(\frac{3}{25})+\frac{3}{5}(\frac{5}{25})+\frac{4}{5}(\frac{7}{25})+1(\frac{9}{25})=\frac{19}{25}$

7. $L_f(P)=\frac{1}{16}(\frac{3}{4})+0(\frac{1}{2})+\frac{1}{16}(\frac{1}{4})+\frac{1}{4}(\frac{1}{2})=\frac{3}{16}$, $\quad U_f(P)=1(\frac{3}{4})+\frac{1}{16}(\frac{1}{2})+\frac{1}{4}(\frac{1}{4})+1(\frac{1}{2})=\frac{43}{32}$

8. $L_f(P)=\frac{9}{16}(\frac{1}{4})+\frac{1}{16}(\frac{1}{2})+0(\frac{1}{2})+\frac{1}{16}(\frac{1}{4})+\frac{1}{4}(\frac{1}{2})=\frac{5}{16}$,

$U_f(P)=1(\frac{1}{4})+\frac{9}{16}(\frac{1}{2})+\frac{1}{16}(\frac{1}{2})+\frac{1}{4}(\frac{1}{4})+1(\frac{1}{2})=\frac{9}{8}$

9. $L_f(P)=0\left(\frac{\pi}{6}\right)+\frac{1}{2}\left(\frac{\pi}{3}\right)+0\left(\frac{\pi}{2}\right)=\frac{\pi}{6}$, $\quad U_f(P)=\frac{1}{2}\left(\frac{\pi}{6}\right)+1\left(\frac{\pi}{3}\right)+1\left(\frac{\pi}{2}\right)=\frac{11\pi}{12}$

10. $L_f(P)=\frac{1}{2}(\frac{\pi}{3})+0(\frac{\pi}{6})+(-1)(\frac{\pi}{2})=-\frac{\pi}{3}$, $\quad U_f(P)=1(\frac{\pi}{3})+\frac{1}{2}(\frac{\pi}{6})+0(\frac{\pi}{2})=\frac{5\pi}{12}$

11. (a) $L_f(P)\le U_f(P)$ but $3\not\le 2$.

(b) $L_f(P)\le \int_{-1}^{1} f(x)\,dx \le U_f(P)$ but $3\not\le 2\le 6$.

(c) $L_f(P)\le \int_{-1}^{1} f(x)\,dx \le U_f(P)$ but $3\le 10\not\le 6$.

12. (a) $L_f(P)=(x_0+3)(x_1-x_0)+(x_1+3)(x_2-x_1)+\cdots+(x_{n-1}+3)(x_n-x_{n-1})$,

$U_f(P)=(x_1+3)(x_1-x_0)+(x_2+3)(x_2-x_1)+\cdots+(x_n+3)(x_n-x_{n-1})$

(b) For each index i

$$x_{i-1}+3\le \frac{1}{2}(x_{i-1}+x_i)+3\le x_i+3$$

Multiplying by $\Delta x_i=x_i-x_{i-1}$ gives

$$(x_{i-1}+3)\,\Delta x_i\le \frac{1}{2}({x_i}^2-x_{i-1}^2)+3(x_i-x_{i-1})\le (x_i+3)\,\Delta x_i.$$

Summing from $i=1$ to $i=n$, we find that

$$L_f(P)\le \frac{1}{2}({x_1}^2-{x_0}^2)+3(x_1-x_0)\cdots+\frac{1}{2}({x_n}^2-{x_{n-1}}^2)+3(x_n-x_{n-1})\le U_f(P)$$

The middle sum collapses to

$$\frac{1}{2}({x_n}^2 - {x_0}^2) + 3(x_n - x_0) = \frac{1}{2}(b^2 - a^2) + 3(b - a)$$

Thus

$$\int_a^b (x+3)\,dx = \frac{1}{2}(b^2 - a^2) + 3(b-a)$$

13. (a) $L_f(P) = -3x_1(x_1 - x_0) - 3x_2(x_2 - x_1) - \cdots - 3x_n(x_n - x_{n-1}),$

$U_f(P) = -3x_0(x_1 - x_0) - 3x_1(x_2 - x_1) - \cdots - 3x_{n-1}(x_n - x_{n-1})$

(b) For each index i

$$-3x_i \le -\tfrac{3}{2}(x_i + x_{i-1}) \le -3x_{i-1}.$$

Multiplying by $\Delta x_i = x_i - x_{i-1}$ gives

$$-3x_i\,\Delta x_i \le -\tfrac{3}{2}({x_i}^2 - x_{i-1}^2) \le -3x_{i-1}\,\Delta x_i.$$

Summing from $i = 1$ to $i = n$, we find that

$$L_f(P) \le -\tfrac{3}{2}({x_1}^2 - {x_0}^2) - \cdots - \tfrac{3}{2}({x_n}^2 - x_{n-1}^2) \le U_f(P).$$

The middle sum collapses to

$$-\tfrac{3}{2}({x_n}^2 - {x_0}^2) = -\tfrac{3}{2}(b^2 - a^2).$$

Thus

$$L_f(P) \le -\frac{3}{2}(b^2 - a^2) \le U_f(P) \quad \text{so that} \quad \int_a^b -3x\,dx = -\frac{3}{2}(b^2 - a^2).$$

14. (a) $L_f(P) = (1 + 2x_0)(x_1 - x_0) + (1 + 2x_1)(x_2 - x_1) + \cdots + (1 + 2x_{n-1})(x_n - x_{n-1}),$

$U_f(P) = (1 + 2x_1)(x_1 - x_0) + (1 + 2x_2)(x_2 - x_1) + \cdots + (1 + 2x_n)(x_n - x_{n-1})$

(b) For each index i

$$1 + 2x_{i-1} \le 1 + (x_{i-1} + x_i) \le 1 + 2x_i$$

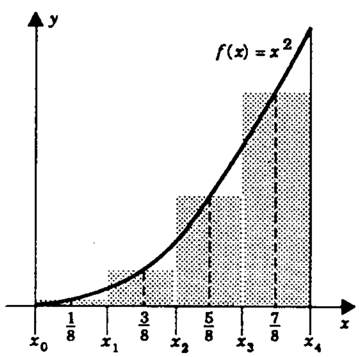
Multiplying by $\Delta x_i = x_i - x_{i-1}$ gives

$$(1 + 2x_{i-1})\,\Delta x_i \le (x_i - x_{i-1}) + ({x_i}^2 - x_{i-1}^2) \le (1 + 2x_i)\,\Delta x_i.$$

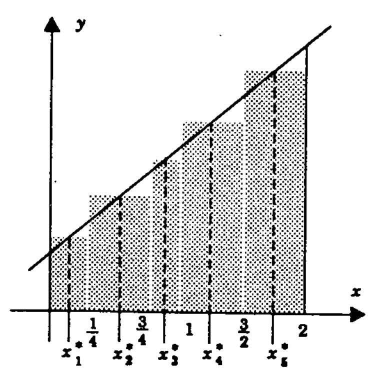
Proceeding as before, we get

$$\int_a^b (1 + 2x)\,dx = (b - a) + (b^2 - a^2)$$

15.



16.



17. (a) $\Delta x_1 = \Delta x_2 = \frac{1}{8}, \quad \Delta x_3 = \Delta x_4 = \Delta x_5 = \frac{1}{4}$

(b) $\|P\| = \frac{1}{4}$

(c) $m_1 = 0, \quad m_2 = \frac{1}{4}, \quad m_3 = \frac{1}{2}, \quad m_4 = 1, \quad m_5 = \frac{3}{2}$

(d) $f(x_1^*) = \frac{1}{8}, \quad f(x_2^*) = \frac{3}{8}, \quad f(x_3^*) = \frac{3}{4}, \quad f(x_4^*) = \frac{5}{4},$

$f(x_5^*) = \frac{3}{2}$

(e) $M_1 = \frac{1}{4}, \quad M_2 = \frac{1}{2}, \quad M_3 = 1, \quad M_4 = \frac{3}{2}, \quad M_5 = 2$

(f) $L_f(P) = \frac{25}{32}$ (g) $S^*(P) = \frac{15}{16}$

(h) $U_f(P) = \frac{39}{32}$ (i) $\displaystyle\int_a^b f(x)\,dx = 1$

18. Since m_i is the minimum of f on $[x_{i-1}, x_i]$, and M_i is the maximum on $[x_{i-1}, x_i]$, we have $m_i \le f(x_i^*) \le M_i$, where, by summing, $L_f(f) \le S^*(P) \le U_f(P)$.

19. $L_f(P) = x_0{}^3(x_1 - x_0) + x_1{}^3(x_2 - x_1) + \cdots + x_{n-1}^3(x_n - x_{n-1})$

$U_f(P) = x_1{}^3(x_1 - x_0) + x_2{}^3(x_2 - x_1) + \cdots + x_n{}^3(x_n - x_{n-1})$

For each index i

$$x_{i-1}^3 \le \tfrac{1}{4}\left(x_i{}^3 + x_i{}^2 x_{i-1} + x_i x_{i-1}^2 + x_{i-1}^3\right) \le x_i{}^3$$

and thus by the hint

$$x_{i-1}^3(x_i - x_{i-1}) \le \tfrac{1}{4}\left(x_i{}^4 - x_{i-1}^4\right) \le x_i{}^3(x_i - x_{i-1}).$$

Adding up these inequalities, we find that

$$L_f(P) \le \tfrac{1}{4}\left(x_n{}^4 - x_0{}^4\right) \le U_f(P).$$

Since $x_n = 1$ and $x_0 = 0$, the middle term is $\dfrac{1}{4}$: $\displaystyle\int_0^1 x^3\,dx = \frac{1}{4}$.

20. (a) $L_f(P) = x_0{}^4(x_1 - x_0) + x_1{}^4(x_2 - x_1) + \cdots + x_{n-1}{}^4(x_n - x_{n-1}),$

$U_f(P) = x_1{}^4(x_1 - x_0) + x_2{}^4(x_2 - x_1) + \cdots + x_n{}^4(x_n - x_{n-1})$

(b) For each index i

$$x_{i-1}{}^4 \le \frac{x_i{}^4 + x_i{}^3 x_{i-1} + x_i{}^2 x_{i-1}{}^2 + x_i x_{i-1}{}^3 + x_{i-1}{}^4}{5} \le x_i{}^4$$

Multiplying by $\Delta x_i = x_i - x_{i-1}$ gives

$$x_{i-1}{}^4\,\Delta x_i \le \frac{1}{5}(x_i{}^5 - x_{i-1}{}^5) \le x_i{}^4\,\Delta x_i.$$

Summing and collapsing the middle sum gives

$$L_f(P) \le \frac{1}{5}({x_n}^5 - {x_0}^5) \le U_f(P),$$

Thus

$$\int_0^1 x^4\,dx = \frac{1}{5}(1^5 - 0^5) = \frac{1}{5}.$$

21. Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a regular partition of $[a, b]$ and let $\Delta x = (b - a)/n$.

Since f is increasing on $[a, b]$,

$$L_f(P) = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x$$

and

$$U_f(P) = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

Now,

$$U_f(P) - L_f(P) = [f(x_n) - f(x_0)]\Delta x = [f(b) - f(a)]\Delta x.$$

22. Proceed as in Exercise 21.

23. Necessarily holds: $L_g(P) \le \int_a^b g(x)\,dx < \int_a^b f(x)\,dx \le U_f(P)$.

24. Need not hold. Consider the partition $\{0, 2, 3\}$ on $[0, 3]$ where $f(x) = x$ and $g(x) = 1$. Then $\int_a^b f(x)\,dx = 4\frac{1}{2}$ and $\int_a^b g(x)\,dx = 3$, but $L_g(P) = 3$ and $L_f(P) = 2$.

25. Necessarily holds: $L_g(P) \le \int_a^b g(x)\,dx < \int_a^b f(x)\,dx$

26. Need not hold. Consider the partition $\{0, 1, 3\}$ on $[0, 3]$ where $f(x) = 2$ and $g(x) = 3 - x$. Then $\int_a^b f(x)\,dx = 6$ and $\int_a^b g(x)\,dx = 4\frac{1}{2}$, but $U_g(P) = 7$ and $U_f(P) = 6$.

27. Necessarily holds: $U_f(P) \ge \int_a^b f(x)\,dx > \int_a^b g(x)\,dx$

28. Need not hold. Use the same counter example as Exercise 24.

29. (a) By Definition 5.1.5, $L_f(P) \le I \le U_f(P)$. Subtracting $L_f(P)$ from these inequalities gives

$$0 \le I - L_f(P) \le U_f(P) - L_f(P)$$

(b) From Definition 5.1.5,

$$L_f(P) - U_f(P) \le I - U_f(P) \le 0$$

Now multiply by -1 and the result follows.

30.

$$\begin{aligned} I - L_f(P) &\le U_f(P) - L_f(P) \quad \text{by Exercise 29} \\ &= |f(b) - f(a)|\Delta x \quad \text{by Exercise 21 and 22} \end{aligned}$$

similarly for $U_f(P) - I$

31. (a) $f'(x) = \dfrac{x}{\sqrt{1+x^2}} > 0$ for $x \in (0, 2)$. Thus, f is increasing on $[0, 2]$.

(b) Let $P = \{x_0, x_1, \ldots, x_n\}$ be a regular partition of $[0, 2]$ and let $\Delta x = 2/n$

By Exercise 30,

$$\int_0^2 f(x)\,dx - L_f(P) \le |f(2)-f(0)|\,\frac{2}{n} = \frac{2(\sqrt{5}-1)}{n} \cong \frac{2.47}{n}$$

It now follows that $\int_0^2 f(x)\,dx - L_f(P) < 0.1$ if $n > 25$.

(c) $\int_0^2 f(x)\,dx \cong 2.96$

32. (a) $f'(x) = \dfrac{-2x}{1+x^2} < 0$ on $(0,1)$ $\Rightarrow$ f is decreasing.

(b) $U_f(P) - \int_0^1 f(x)\,dx \le |f(1)-f(0)|\Delta x = |\frac{1}{2}-1|\frac{1}{n} = \dfrac{1}{2n}$.

so need $\dfrac{1}{2n} = 0.05$, or $n = 10$.

(c) Using $U_f(P)$ with $n = 10$, we have $\displaystyle\int_0^1 \frac{1}{1+x^2}\,ds \cong 0.76$

33. Let S be the set of positive integers for which the statement is true. Since $1 = \dfrac{1(2)}{2} = 1$, $1 \in S$. Assume that $k \in S$. Then

$$\begin{aligned} 1+2+\cdots+k+k+1 = (1+2+\cdots+k)+k+1 &= \frac{k(k+1)}{2}+k+1 \\ &= \frac{k+1)(k+2)}{2} \end{aligned}$$

Thus, $k+1 \in S$ and so S is the set of positive integers.

34. See Exercise 5 in section 1.8.

35. Let $f(x) = x$ and let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a regular partition of $[0,b]$. Then $\Delta x = b/n$ and $x_i = \dfrac{ib}{n}$, $i = 0, 1, 2, \ldots, n$.

(a) Since f is increasing on $[0,b]$,

$$\begin{aligned} L_f(P) &= \left[f(0) + f\left(\frac{b}{n}\right) + f\left(\frac{2b}{n}\right) + \cdots + f\left(\frac{(n-1)b}{n}\right)\right]\frac{b}{n} \\ &= \left[0 + \frac{b}{n} + \frac{2b}{n} + \cdots + \frac{(n-1)b}{n}\right]\frac{b}{n} \\ &= \frac{b^2}{n^2}\left[1+2+\cdots+(n-1)\right] \end{aligned}$$

(b)

$$\begin{aligned} U_f(P) &= \left[f\left(\frac{b}{n}\right) + f\left(\frac{2b}{n}\right) + \cdots + f\left(\frac{(n-1)b}{n}\right) + f(b)\right]\frac{b}{n} \\ &= \left[\frac{b}{n} + \frac{2b}{n} + \cdots + \frac{(n-1)b}{n} + b\right]\frac{b}{n} \\ &= \frac{b^2}{n^2}\left[1+2+\cdots+(n-1)+n\right] \end{aligned}$$

(c) By Exercise 33,

$$L_f(P) = \frac{b^2}{n^2}\cdot\frac{(n-1)n}{2} \quad \text{and} \quad U_f(P) = \frac{b^2}{n^2}\cdot\frac{n(n+1)}{2}.$$

As $n\to\infty$, $L_f(P)\to\dfrac{b^2}{2}$ and $U_f(P)\to\dfrac{b^2}{2}$. Therefore, $\int_0^2 x\,dx = \dfrac{b^2}{2}$.

36. (a) Note that $x_i = \dfrac{ih}{n}$ and $\Delta x = \dfrac{b}{n}$, so since $f(x)$ is increasing on $[0,b]$,

$$L_f(P) = \left(\frac{0b}{n}\right)^2\left(\frac{b}{n}\right) + \left(\frac{b}{n}\right)^2\left(\frac{b}{n}\right) + \left(\frac{2b}{n}\right)^2\left(\frac{b}{n}\right) + \cdots + \left(\frac{(n-1)b}{n}\right)^2\left(\frac{b}{n}\right)$$

$$= \frac{b^3}{n^3}[0^2+1^2+2^2+\cdots+(n-1)^2].$$

(b) Similar to (a).

(c) By Exercise 34, $L_f(P) = \dfrac{b^3}{n^3}\dfrac{(n-1)n(2n-1)}{6}$, $U_f(P) = \dfrac{b^3}{n^3}\dfrac{n(n+1)(2n+1)}{6}$.

As $n\to\infty$, $\dfrac{(n-1)n(2n-1)}{6n^3}\to\dfrac{2}{6}=\dfrac{1}{3}$, $\dfrac{n(n+1)(2n+1)}{6n^3}\to\dfrac{2}{6}=\dfrac{1}{3}$,

so $L_f(P), U_f(P)\to\dfrac{b^3}{3}$, and therefore $\displaystyle\int_0^b x^2\,dx = \frac{1}{3}b^3$.

37. (a) $\dfrac{1}{n^2}(1+2+\cdots+n) = \dfrac{1}{n^2}\left[\dfrac{n(n+1)}{2}\right] = \dfrac{1}{2}+\dfrac{1}{2n}$

(b) $S_n^* = \dfrac{1}{2}+\dfrac{1}{2n}$, $\displaystyle\int_0^1 x\,dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}$

$$\left|S_n^* - \int_0^1 x\,dx\right| = \frac{1}{2n} < \frac{1}{n} < \epsilon \quad \text{if} \quad n > \frac{1}{\epsilon}$$

38. (a) $\dfrac{1}{n^3}(1^2+2^2+\cdots+n^2) = \dfrac{1}{n^3}\left[\dfrac{n(n+1)(2n+1)}{6}\right] = \dfrac{1}{3}+\dfrac{1}{2n}+\dfrac{1}{6n^2}$

(b) $S_n^* = \dfrac{1}{3}+\dfrac{1}{2n}+\dfrac{1}{6n^2}$, $\displaystyle\int_0^1 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}$

$$\left|S_n^* - \int_0^1 x^2\,dx\right| = \frac{1}{2n}+\frac{1}{6n^2} < \frac{1}{n} < \epsilon \quad \text{if} \quad n > \frac{1}{\epsilon}$$

39. (a) $\dfrac{1}{n^4}\left(1^3+2^3+\cdots+n^3\right) = \dfrac{1}{n^4}\left[\dfrac{n^2(n+1)^2}{4}\right] = \dfrac{1}{4}+\dfrac{1}{2n}+\dfrac{1}{4n^2}$

(b) $S_n^* = \dfrac{1}{4}+\dfrac{1}{2n}+\dfrac{1}{4n^2}$, $\displaystyle\int_0^1 x^3dx = \left[\frac{1}{4}x^4\right]_0^1 = \frac{1}{4}$

$$\left|S_n^* - \int_0^1 x^3dx\right| = \frac{1}{2n}+\frac{1}{4n^2} < \frac{1}{n} < \epsilon \quad \text{if} \quad n > \frac{1}{\epsilon}$$

40. Choose each x_i^* so that $f(x_i^*) = m_i$. Then $S_i^*(P) = L_f(P)$.

Similarly, choosing each x_i^* so that $f(x_i^*) = M_i$ gives $S_i^*(P) = U_f(P)$.

Also, choosing each x_i^* so that $f(x_i^*) = \frac{1}{2}(m_i + M_i)$ (they exist by the intermediate value theorem) gives

$$\begin{aligned} S_i^*(P) &= \frac{1}{2}(m_1 + M_1)\Delta x_1 + \cdots + \frac{1}{2}(m_n + M_n)\Delta x_n \\ &= \frac{1}{2}[m_1\Delta x_1 + \cdots + m_n\Delta x_n + M_1\Delta x_1 + \cdots + M_n\Delta x_n] \\ &= \frac{1}{2}[L_f(P) + U_f(P)]. \end{aligned}$$

41. Let P be an arbitrary partition of [0, 4]. Since each $m_i = 2$ and each $M_i \geq 2$,

$$L_g(P) = 2\Delta x_1 + \cdots + 2\Delta x_n = 2(\Delta x_1 + \cdots + \Delta x_n) = 2 \cdot 4 = 8$$

and

$$U_g(P) \geq 2\Delta x_1 + \cdots + 2\Delta x_n = 2(\Delta x_1 + \cdots + \Delta x_n) = 2 \cdot 4 = 8.$$

Thus

$$L_g(P) \leq 8 \leq U_g(P) \quad \text{for all partitions } P \text{ of } [0, 4].$$

Uniqueness: Suppose that

(*) $$L_g(P) \leq I \leq U_g(P) \text{ for all partitions } P \text{ of } [0, 4].$$

Since $L_g(P) = 8$ for all P, I is at least 8. Suppose now that $I > 8$ and choose a partition P of $[0, 4]$ with max $\Delta x_i < \frac{1}{5}(I - 8)$ and

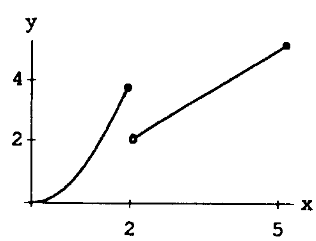
$$0 = x_1 < \cdots < x_{i-1} < 3 < x_i < \cdots < x_n = 4.$$

Then

$$\begin{aligned} U_g(P) &= 2\Delta x_1 + \cdots + 2\Delta x_{i-1} + 7\Delta x_i + 2\Delta x_{i+1} + \cdots + 2\Delta x_n \\ &= 2(\Delta x_1 + \cdots + \Delta x_n) + 5\Delta x_i \\ &= 8 + 5\Delta x_i < 8 + \tfrac{5}{5}(I - 8) = I \end{aligned}$$

and I does not satisfy (∗). This contradiction proves that I is not greater than 8 and therefore $I = 8$.

42. (a)



(b) $$\begin{aligned} \text{Area} &= \int_0^5 f(x)\,dx \\ &= \int_0^2 x^2\,dx + \int_2^5 x\,dx \\ &= \frac{1}{3}2^3 + \frac{1}{2}(5^2 - 2^2) \\ &= \frac{79}{6} \end{aligned}$$

43. Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be any partition of $[2, 10]$.

(a) Since each subinterval $[x_{i-1}, x_i]$ contains both rational and irrational numbers, $m_i = 4$

and $M_i = 7$. Thus,

$$L_f(P) = 4\Delta x_1 + 4\Delta x_2 + \cdots + 4\Delta x_n = 4(\Delta x_1 + \Delta x_2 + \cdots + \Delta x_n) = 4(10-2) = 32$$

and

$$U_f(P) = 7\Delta x_1 + 7\Delta x_2 + \cdots + 7\Delta x_n = 7(\Delta x_1 + \Delta x_2 + \cdots + \Delta x_n) = 7(10-2) = 56$$

Therefore, $L_f(P) \leq 40 \leq U_f(P)$.

(b) Every number $I \in [32, 56]$ satisfies the inequalities

$$L_f(P) \leq I \leq U_f(P) \quad \text{for all partitions } P$$

(c) See part (a).

44. (a) $L_f(P) \cong 5.1331$, $U_f(P) \cong 5.9331$

(b) $\frac{1}{2}[L_f(P) + U_f(P)] \cong 5.5331$ (c)$S^*(P) \cong 5.5371$

45. (a) $L_f(P) \cong 0.6105$, $U_f(P) \cong 0.7105$

(b) $\frac{1}{2}[L_f(P) + U_f(P)] \cong 0.6605$ (c) $S^*(P) \cong 0.6684$

46. (a) $L_f(P) \cong 3.9192$, $U_f(P) \cong 4.9192$

(b) $\frac{1}{2}[L_f(P) + U_f(P)] \cong 4.4192$ (c) $S^*(P) \cong 4.4094$

PROJECT 5.1

1. (a) If the object traveled at its minimum speed m_i on $[t_{i-1}, t_i]$, then it would travel a distance of $m_i\Delta t_i$ units; if the object traveled at its maximum speed M_i on $[t_{i-1}, t_i]$, then it would travel a distance of $M_i\Delta t_i$ units. Thus, the actual distance traveled, s_i, must be somewhere inbetween. i.e.

$$m_i\Delta t_i \leq s_i \leq M_i\Delta t_i, \quad i = 1, 2, ..., n$$

Adding these inequalities, we get

$$m_1\Delta t_1 + m_2\Delta t_2 + \cdots + m_n\Delta t_n \leq D \leq M_1\Delta t_1 + M_2\Delta t_2 + \cdots + M_n\Delta t_n,$$

where $D = s_1 + s_2 + \cdots + s_n$. Since the result holds for any partition P, we have

$$D = \int_a^b s(t)\, dt.$$

(b) Since $m_i \leq s(t_i^*0 \leq M_i$ for each i,

$$m_1\Delta t_1 + m_2\Delta t_2 + \cdots + m_n\Delta t_n \leq S^*(P) \leq M_1\Delta t_1 + M_2\Delta t_2 + \cdots + M_n\Delta t_n,$$

Thus $\lim\limits_{||P||\to 0} S*(P) = \int_a^b s(t)\, dt.$

2. Distance $= \int_0^5 s(t)\, dt = \int_0^5 (t+3)\, dt = \left[\frac{t^2}{2} + 3t\right]_0^5 = \frac{55}{2}$ units.

3. Distance $= \int_0^3 s(t)\, dt = \int_0^3 (t^2 + t + 1)\, dt = \left[\frac{t^3}{3} + \frac{t^2}{2} + t\right]_0^3 = \frac{33}{2}$ units.

4. Distance $= \int_0^1 s(t)\, dt = \int_0^1 (t^3 + 2t)\, dt = \left[\frac{t^4}{4} + t^2\right]_0^1 = \frac{5}{4}$ units.

SECTION 5.2

1. (a) $\int_0^5 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^5 f(x)\,dx = 4 + 1 = 5$

(b) $\int_1^2 f(x)\,dx = \int_0^2 f(x)\,dx - \int_0^1 f(x)\,dx = 4 - 6 = -2$

(c) $\int_1^5 f(x)\,dx = \int_0^5 f(x)\,dx - \int_0^1 f(x)\,dx = 5 - 6 = -1$

(d) 0 (e) $\int_2^0 f(x)\,dx = -\int_0^2 f(x)\,dx = -4$

(f) $\int_5^1 f(x)\,dx = -\int_1^5 f(x)\,dx = 1$

2. (a) $\int_4^8 f(x)\,dx = \int_1^8 f(x)\,dx - \int_1^4 f(x)\,dx = 11 - 5 = 6$

(b) $\int_4^3 f(x)\,dx = -\int_3^4 f(x)\,dx = -7$

(c) $\int_1^3 f(x)\,dx = \int_1^4 f(x)\,dx - \int_3^4 f(x)\,dx = 5 - 7 = -2$

(d) $\int_3^8 f(x)\,dx = \int_1^8 f(x)\,dx - \int_1^3 f(x)\,dx = 11 - (-2) = 13$

(e) $\int_8^4 f(x)\,dx = -\int_4^8 f(x)\,dx = -6$

(f) $\int_4^4 f(x)\,dx = 0$

3. With $P = \left\{1, \frac{3}{2}, 2\right\}$ and $f(x) = \frac{1}{x}$, we have

$$0.5 < \frac{7}{12} = L_f(P) \le \int_1^2 \frac{dx}{x} \le U_f(P) = \frac{5}{6} < 1.$$

4. Using $P = \{0, \frac{1}{2}, 1\}$, we have $0.6 < 0.65 = L_f(P) \le \int_0^1 \frac{1}{1+x^2}\,dx \le U_f P = 0.9 < 1.$

5. (a) $F(0) = 0$ (b) $F'(x) = x\sqrt{x+1}$ (c) $F'(2) = 2\sqrt{3}$

(d) $F(2) = \int_0^2 t\sqrt{t+1}\,dt$ (e) $-F(x) = \int_x^0 t\sqrt{t+1}\,dt$

6. (a) $F(\pi) = \int_\pi^\pi t\sin t\,dt = 0$ (b) By Theorem 5.2.5, $F'(x) = x\sin x$.

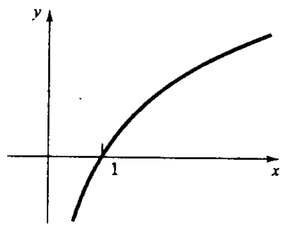
(c) $F'(\frac{\pi}{2}) = \frac{\pi}{2}\sin\frac{\pi}{2} = \frac{\pi}{2}$ (d) $F(2\pi) = \int_\pi^{2\pi} t\sin t\,dt$

(e) $-F(x) = \int_x^{\pi} t \sin t \, dt.$

7. (a) $F'(x) = \dfrac{1}{x} > 0$ for $x > 0$.
Thus, F is increasing on $(0, \infty)$;
there are no critical numbers.

(b) $F''(x) = -\dfrac{1}{x^2} < 0$ for $x > 0$.
The graph of F is concave down on $(0, \infty)$;
there are no points of inflection.

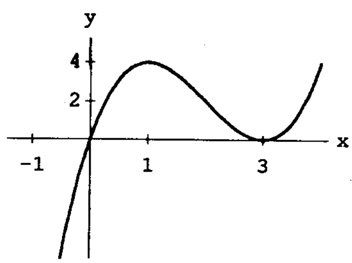
(c)



8. (a) $F'(x) = x(x-3)^2$,
F is increasing on $[0, \infty)$;
F is decreasing on $(-\infty, 0]$;
critical numbers $0, -3$.

(b) $F''(x) = (x-3)^2 + 2x(x-3) = 3(x-3)(x-1)$.
The graph of F is concave up on $(-\infty, 1) \cup (3, \infty)$,
The graph of F is concave down on $(1, 3)$,
Inflection points at $x = 1$, $x = 3$

(c)



9. $F'(x) = \dfrac{1}{x^2+9}$; (a) $\dfrac{1}{10}$ (b) $\dfrac{1}{9}$ (c) $\dfrac{4}{37}$ (d) $\dfrac{-2x}{(x^2+9)^2}$

10. $F'(x) = -\sqrt{x^2+1}$ (a) $-\sqrt{2}$ (b) -1 (c) $-\frac{1}{2}\sqrt{5}$ (d) $\dfrac{-x}{\sqrt{x^2+1}}$

11. $F'(x) = -x\sqrt{x^2+1}$; (a) $\sqrt{2}$ (b) 0 (c) $-\frac{1}{4}\sqrt{5}$ (d) $-(\sqrt{x^2+1} - \dfrac{x^2}{\sqrt{x^2+1}})$

12. $F'(x) = \sin \pi x$ (a) 0 (b) 0 (c) 1 (d) $\pi \cos \pi x$

13. $F'(x) = \cos \pi x$; (a) -1 (b) 1 (c) 0 (d) $-\pi \sin \pi x$

14. $F'(x) = (x+1)^3$ (a) 0 (b) 1 (c) $\dfrac{27}{8}$ (d) $3(x+1)^2$

15. (a) Since $P_1 \subseteq P_2$, $U_f(P_2) \leq U_f(P_1)$ but $5 \not\leq 4$.
(b) Since $P_1 \subseteq P_2$, $L_f(P_1) \leq L_f(P_2)$ but $5 \not\leq 4$.

16. We know this is true for $a < c < b$. Assume $a < b$. If $c = a$ or $c = b$, the equality becomes $\int_a^b f(x)\,dt = \int_a^b f(x)\,dt$, trivially true. If $c < a$, we get

$$\int_a^c f(t)\,dt + \int_c^b f(t)\,dt = -\int_c^a f(t)\,dt + \int_c^b f(t)\,dt = \int_a^b f(t)\,dt, \text{ as desired}$$

The cases $a > b$ and $a = b$ are proved in the same manner.

17. Let $u = x^3$. Then $F(u) = \int_1^u t\cos t\,dt$ and

$$\frac{dF}{dx} = \frac{dF}{du}\frac{du}{dx} = u\cos u\,(3x^2) = 3x^5\cos x^3.$$

18. Let $u = \cos x$. $\quad \frac{dF}{dx} = \frac{dF}{du}\frac{du}{dx} = \sqrt{1-u^2}\,(-\sin x) = \sqrt{1-\cos^2 x}\,(-\sin x) = -|\sin x|\sin x$

19. $F(x) = \int_{x^2}^1 (t - \sin^2 t)\,dt = -\int_1^{x^2} (t - \sin^2 t)\,dt$. Let $u = x^2$. Then

$$\frac{dF}{dx} = \frac{dF}{du}\frac{du}{dx} = -(u - \sin^2 u)(2x) = 2x\left[\sin^2(x^2) - x^2\right].$$

20. Let $u = \sqrt{x}$. $\quad \frac{dF}{dx} = \frac{dF}{du}\frac{du}{dx} = \frac{u^2}{1+u^4}\frac{1}{2\sqrt{x}} = \frac{x}{1+x^2}\frac{1}{2\sqrt{x}}$

21. (a) $F(0) = 0$

(b) $F'(0) = 2 + \frac{\sin 2(0)}{1+0^2} = 2$

(c) $F''(0) = \frac{(1+0)^2 2\cos 2(0) - \sin 2(0)(2)(0)}{(1+0)^2} = 2$

22. (a) $F(0) = 0$

(b) Let $u = x^2$. Then $f(u) = 2\sqrt{u} + \int_0^u \frac{\sin 2t}{1+t^2}\,dt$.

$$\frac{dF}{dx} = \frac{dF}{du}\frac{du}{dx} = 2 + \frac{\sin 2u}{1+u^2}(2x) = 2 + \frac{\sin 2x^2}{1+x^4}(2x)$$

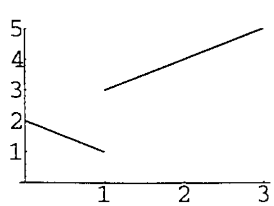
23. $F'(x) = \frac{x-1}{1+x^2} = 0 \implies x = 1$ is a critical number.

$F''(x) = \frac{(1+x^2) - 2x(x-1)}{(1+x^2)^2}$, so $F''(1) = \frac{1}{2} > 0$ means $x = 1$ is a local minimum.

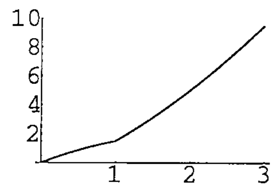
24. $F'(x) = \frac{x-4}{1+x^2} = 0 \implies x = 4$ is a critical number.

$F''(x) = \frac{(1+x^2) - 2x(x-4)}{(1+x^2)^2}$, so $F''(4) = \frac{1}{17} > 0$ means $x = 4$ is a local minimum.

25. (a)



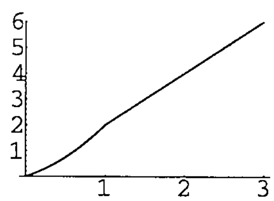
(b)



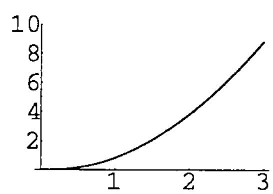
$$F(x) = \begin{cases} 2x - \frac{1}{2}x^2 & 0 \le x \le 1 \\ 2x + \frac{1}{2}x^2 - 1 & 1 < x \le 3 \end{cases}$$

(c) f is discontinuous at $x = 1$, F is continuous but not differentiable at $x = 1$.

26. (a)



(b)



$$F(x) = \begin{cases} \frac{x^3}{3} - \frac{x^2}{2} & 0 \le x \le 1 \\ x^2 - \frac{1}{6} & 1 < x \le 3 \end{cases}$$

(c) f is continuous at $x = 1$, but not differentiable. F is continuous and differentiable at $x = 1$.

27. $f(x) = \dfrac{d}{dx}\left(\dfrac{2x}{4+x^2}\right) = \dfrac{8-2x^2}{(4+x^2)^2}$

(a) $f(0) = \frac{1}{2}$ (b) $f(x) = 0$ at $x = -2,\ 2$

28. (a) $F(x) = \int_0^x t f(t)\,dt = \sin x - x\cos x.$

$F'(x) = xf(x) = \cos x - \cos x + x\sin x = x\sin x \implies f(\frac{\pi}{2}) = \sin\frac{\pi}{2} = 1$

(b) $f'(x) = \cos x$

29. By the hint $\dfrac{F(b) - F(a)}{b-a} = F'(c)$ for some c in $(a,\, b)$. The result follows by observing that

$$F(b) = \int_a^b f(t)\,dt, \quad F(a) = 0, \quad \text{and} \quad F'(c) = f(c).$$

30. Let $\epsilon > 0$. Since f is continuous at x, there exists $\delta > 0$ such that

(*) if $|k| < \delta$ then $|f(x+k) - f(x)| < \epsilon$.

suppose now that $|h| < \delta$. Obviously

$$m_h = f(x + k_h') \text{ for some number } k_h' \text{ which satisfies } |f(k_h')| < \delta.$$

and

$$M_h = f(x + k_h'') \text{ for some number } k_h'' \text{ which satisfies } |f(k_h'')| < \delta.$$

It follows from (*) that

$$|m_h - f(x)| < \epsilon \quad and \quad |M_h - f(x)| < \epsilon.$$

31. Set $G(x) = \int_a^x f(t)\,dt$. Then $F(x) = \int_c^a f(t)\,dt + G(x)$. First, note that $\int_c^a f(t)\,dt$ is a constant. By (5.2.5) G, and thus F, is continuous on $[a, b]$, is differentiable on (a, b), and $F'(x) = G'(x) = f(x)$ for all x in (a, b).

32. Mimic the argument given for the right-hand limit.

33. (a) $F(x) = -\int_0^c f(t)\,dt + \int_0^x f(t)\,dt$ and $G(x) = -\int_0^d f(t)\,dt + \int_0^x f(t)\,dt$.

Thus $F'(x) = f(x)dx + G'(x)$, so by theorem 4.2.4, F and G differ by a constant.

(b) $F(x) - G(x) = -\int_0^c f(t)\,dt + \int_0^d f(t)\,dt$

$= -\int_0^c f(t)\,dt + \int_0^c f(t)\,dt + \int_c^d f(t)\,dt \;= \int_c^d f(t)\,dt.$

34. (a) $F'(x) = x\int_1^x f(u)\,du$

(b) $F'(1) = 0$

(c) $F''(x) = xf(x) + \int_1^x f(u)\,du$

(d) $F''(1) = f(1)$

PROJECT 5.2

1. (a) Dom $(F) = (-\infty, \infty)$

(b) By Theorem 5.2.5, F is continuous on (∞, ∞).

(c) $$F(-x) = \int_0^{-x} \sin^2(t^2)\,dt$$

Set $u = -t$. Then $du = -dt$, and $u = 0$ at $t = 0$, $u = x$ at $t = -x$. Then

$$F(-x) = \int_0^x \sin^2(-u)^2\,(-du) = -\int_0^x \sin^2(u^2)\,du = -F(x).$$

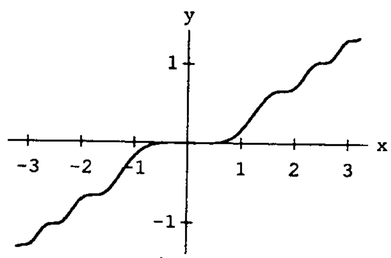
2. (a) F is differentiable for all x by Theorem 5.2.5. (b) $F'(x) = \sin^2(x^2)$

(c,d) $F'(x) \geq 0$ for all x, and $F'(x) = 0$ only when $x = \pm\sqrt{n\pi}$ for some integer n, $n \geq 0$.

Thus, F is increasing on (∞, ∞).

Thus, F is increasing on (∞, ∞).

3. $F''(x) = 2\sin(x^2)\cos(x^2)2x = 2x\sin(2x^2)$ $F''(x) = 0$ when $x^2 = \frac{n}{2}\pi$, $n = 0, 1, 2, \ldots$. That is, when $x = \pm\sqrt{n\pi/2}$. Since F'' changes signs at each of these values, the points $\left(\pm\sqrt{n\pi/2},\, F\left(\pm\sqrt{n\pi/2}\right)\right)$ are points of inflection. The graph of F is concave up on $\left(0, \sqrt{\pi/2}\right)$, concave down on $\left(\sqrt{\pi/2}, \sqrt{\pi}\right)$, and so on.

4. 

SECTION 5.3

1. $\displaystyle\int_0^1 (2x-3)\,dx = [x^2 - 3x]_0^1 = (-2) - (0) = -2$

2. $\displaystyle\int_0^1 (3x+2)\,dx = \left[\frac{3x^2}{2} + 2x\right]_0^1 = \frac{7}{2}$

3. $\displaystyle\int_{-1}^0 5x^4\,dx = [x^5]_{-1}^0 = (0) - (-1) = 1$

4. $\displaystyle\int_1^2 (2x + x^2)\,dx = \left[x^2 + \frac{1}{3}x^3\right]_1^2 = \frac{16}{3}$

5. $\displaystyle\int_1^4 2\sqrt{x}\,dx = 2\int_1^4 x^{1/2}\,dx = 2\left[\frac{2}{3}x^{3/2}\right]_1^4 = \frac{4}{3}\left[x^{3/2}\right]_1^4 = \frac{4}{3}(8-1) = \frac{28}{3}$

6. $\displaystyle\int_0^4 \sqrt[3]{x}\,dx = \int_0^4 x^{\frac{1}{3}}\,dx = \left[\frac{3}{4}x^{4/3}\right]_0^4 = \frac{3}{4}4^{4/3} = 3\sqrt[3]{4}$

7. $\displaystyle\int_1^5 2\sqrt{x-1}\,dx = \int_1^5 2(x-1)^{1/2}\,dx = \left[\frac{4}{3}(x-1)^{3/2}\right]_1^5 = \frac{4}{3}[4^{3/2} - 0] = \frac{32}{3}$

8. $\displaystyle\int_1^2 \left(\frac{3}{x^3} + 5x\right)dx = \left[-\frac{3}{2}x^{-2} + \frac{5}{2}x^2\right]_1^2 = \frac{69}{8}$

9. $\displaystyle\int_{-2}^0 (x+1)(x-2)\,dx = \int_{-2}^0 (x^2 - x - 2)\,dx = \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-2}^0 = \left[0 - \left(\frac{-8}{3} - 2 + 4\right)\right] = \frac{2}{3}$

10. $\displaystyle\int_1^0 (t^3+t^2)\,dt = \left[\frac{1}{4}t^4+\frac{1}{3}t^3\right]_1^0 = -\frac{7}{12}$

11. $\displaystyle\int_1^2 \left(3t+\frac{4}{t^2}\right)\,dt = \int_1^2 (3t+4t^{-2})\,dt = \left[\frac{3}{2}t^2-4t^{-1}\right]_1^2 = \left[(6-2)-\left(\frac{3}{2}-4\right)\right] = \frac{13}{2}$

12. $\displaystyle\int_{-1}^{-1} 7x^6\,dx = 0$

13. $\displaystyle\int_0^1 (x^{3/2}-x^{1/2})\,dx = \left[\frac{2}{5}x^{5/2}-\frac{2}{3}x^{3/2}\right]_0^1 = \left[\left(\frac{2}{5}-\frac{2}{3}\right)-0\right] = -\frac{4}{15}$

14. $\displaystyle\int_0^1 (x^{3/4}-2x^{1/2})\,dx = \left[\frac{4}{7}x^{7/4}-\frac{4}{3}x^{3/2}\right]_0^1 = -\frac{16}{21}$

15. $\displaystyle\int_0^1 (x+1)^{17}\,dx = \left[\frac{1}{18}(x+1)^{18}\right]_0^1 = \frac{1}{18}(2^{18}-1)$

16. $\displaystyle\int_0^a (a^2x-x^3)\,dx = \left[\frac{a^2x^2}{2}-\frac{x^4}{4}\right]_0^a = \frac{a^4}{4}$

17. $\displaystyle\int_0^a (\sqrt{a}-\sqrt{x})^2\,dx = \int_0^a (a-2\sqrt{a}\,x^{1/2}+x)\,dx = \left[ax-\frac{4}{3}\sqrt{a}\,x^{3/2}+\frac{x^2}{2}\right]_0^a = a^2-\frac{4}{3}a^2+\frac{a^2}{2} = \frac{1}{6}a^2$

18. $\displaystyle\int_{-1}^1 (x-2)^2\,dx = \left[\frac{1}{3}(x-2)^3\right]_{-1}^1 = \frac{26}{3}$

19. $\displaystyle\int_1^2 \frac{6-t}{t^3}\,dt = \int_1^2 (6t^{-3}-t^{-2})\,dt = \left[-3t^{-2}+t^{-1}\right]_1^2 = \left[\frac{-3}{4}+\frac{1}{2}\right]-[-3+1] = \frac{7}{4}$

20. $\displaystyle\int_1^2 \left(\frac{2-t}{t^3}\right)\,dt = \int_1^2 (2t^{-3}-t^{-2})\,dt = \left[-t^{-2}+t^{-1}\right]_1^2 = \frac{1}{4}$

21. $\displaystyle\int_0^1 x^2(x-1)\,dx = \int_0^1 (x^3-x^2)\,dx = \left[\frac{x^4}{4}-\frac{x^3}{3}\right]_0^1 = -\frac{1}{12}$

22. $\displaystyle\int_1^3 (x^2-\frac{1}{x^2})\,dx = \left[\frac{1}{3}x^3+\frac{1}{x}\right]_1^3 = 8$

23. $\displaystyle\int_1^2 2x(x^2+1)\,dx = \int_1^2 (2x^3+2x)\,dx = \left[\frac{x^4}{2}+x^2\right]_1^2 = 12-\frac{3}{2} = \frac{21}{2}$

24. $\displaystyle\int_0^1 3x^2(x^3+1)\,dx = \int_0^1 (3x^5+3x^2)\,dx = \left[\frac{1}{2}x^6+x^3\right]_0^1 = \frac{3}{2}$

25. $\displaystyle\int_0^{\pi/2} \cos x\,dx = [\sin x]_0^{\pi/2} = 1$

26. $\displaystyle\int_0^{\pi} 3\sin x\,dx = [-3\cos x]_0^{\pi} = 6$

27. $\displaystyle\int_0^{\pi/4} 2\sec^2 x\,dx = 2\,[\tan x]_0^{\pi/4} = 2$

28. $\displaystyle\int_{\pi/6}^{\pi/3} \sec x\ \tan x\, dx = [\sec x]_{\pi/6}^{\pi/3} = 2 - \frac{2\sqrt{3}}{3}$

29. $\displaystyle\int_{\pi/6}^{\pi/4} \csc x\ \cot x\, dx = [-\csc x]_{\pi/6}^{\pi/4} = -\sqrt{2} - (-2) = 2 - \sqrt{2}$

30. $\displaystyle\int_{\pi/4}^{\pi/3} -\csc^2\ x\, dx = [\cot x]_{\pi/4}^{\pi/3} = \frac{\sqrt{3}}{3} - 1$

31. $\displaystyle\int_0^{2\pi} \sin x\, dx = [-\cos x]_0^{2\pi} = -1 - (-1) = 0$

32. $\displaystyle\int_0^{\pi} \frac{1}{2} \cos x\, dx = \left[\frac{1}{2} \sin x\right]_0^{\pi} = 0$

33. $\displaystyle\int_0^{\pi/3} \left(\frac{2}{\pi}x - 2\sec^2 x\right)\ dx = \left[\frac{1}{\pi}x^2 - 2\tan x\right]_0^{\pi/3} = \frac{\pi}{9} - 2\sqrt{3}$

34. $\displaystyle\int_{\pi/4}^{\pi/2} \csc x(\cot x - 3\csc x)\, dx = \int_{\pi/4}^{\pi/2} (\csc x \cot x - 3\csc^2 x)\, dx = [-\csc x + 3\cot x]_{\pi/4}^{\pi/2} = \sqrt{2} - 4$

35. $\displaystyle\int_0^3 \left[\frac{d}{dx}\left(\sqrt{4+x^2}\right)\right]\ dx = \left[\sqrt{4+x^2}\right]_0^3 = \sqrt{13} - 2$

36. $\displaystyle\int_0^{\pi/2} \left[\frac{d}{dx}(\sin^3 x)\right] = [\sin^3 x]_0^{\pi/2} = 1$

37. (a) $\displaystyle F(x) = \int_1^x (t+2)^2\, dt \quad \Longrightarrow \quad F'(x) = (x+2)^2$

(b) $\displaystyle\int_1^x (t+2)^2\, dt = \left[\frac{t^3}{3} + 2t^2 + 4t\right]_1^x = \frac{x^3}{3} + 2x^2 + 4x - 6\frac{1}{3}$

$\Longrightarrow \quad F'(x) = x^2 + 4x + 4 = (x+2)^2$

38. (a) $\displaystyle F(x) = \int_0^x (\cos t - \sin t)\, dt \quad \Longrightarrow \quad F'(x) = \cos x - \sin x$

(b) $\displaystyle\int_0^x (\cos t - \sin t)\, dt = [\sin t + \cos t]_0^x = \sin x - \cos x - 1$

$\Longrightarrow \quad F'(x) = \cos x - \sin x$

39. (a) $\displaystyle F(x) = \int_1^{2x+1} \frac{1}{2} \sec u\ \tan u\, du \quad \Rightarrow \quad F'(x) = \sec(2x+1)\ \tan(2x+1)$

(b) $\displaystyle\int_1^{2x+1} \frac{1}{2} \sec u\ \tan u\, du = \left[\frac{1}{2}\sec u\right]_1^{2x+1} = \frac{1}{2}\sec(2x+1) - \frac{1}{2}\sec 1$

$$\Longrightarrow \quad F'(x) = \sec(2x+1)\,\tan(2x+1)$$

40. (a) $F(x) = \int_{x^2}^{2} t(t-1)\,dt \quad \Rightarrow \quad F'(x) = -x^2(x^2-1)2x$

(b) $\int_{x^2}^{2} t(t-1)\,dt = \left[\frac{t^3}{3} - \frac{t^2}{2}\right]_{x^2}^{2} = \frac{2}{3} - \frac{x^6}{3} + \frac{x^4}{2}$

$$\Longrightarrow \quad F'(x) = -2x^5 + 2x^3 = -2x^3(x^2-1)$$

41. (a) $F(x) = \int_2^x \frac{dt}{t}$ (b) $F(x) = -3 + \int_2^x \frac{dt}{t}$

42. (a) $F(x) = \int_3^x \sqrt{1+t^2}\,dt$ (b) $F(x) = 1 + \int_3^x \sqrt{1+t^2}\,dt$

43. $\text{Area} = \int_0^4 (4x - x^2)\,dx = \left[2x^2 - \frac{x^3}{3}\right]_0^4 = \frac{32}{3}$

44. $\text{Area} = \int_1^9 (x\sqrt{x}+1)\,dx = \int_1^9 (x^{3/2}+1)\,dx = \left[\frac{2}{5}x^{5/2} + x\right]_1^9 = \frac{524}{5}$

45. $\text{Area} = \int_{-\pi/2}^{\pi/4} 2\cos x\,dx = 2\left[\sin x\right]_{-\pi/2}^{\pi/4} = \sqrt{2} + 2$

46. $\text{Area} = \int_0^{\pi/3} (\sec x \tan x)\,dx = \left[\sec x\right]_0^{\pi/3} = \sqrt{2} - 1$

47. (a) $\int_2^5 (x-3)\,dx = \left[\frac{x^2}{2} - 3x\right]_2^5 = \frac{3}{2}$

(b) $\int_2^5 |x-3|\,dx = \int_2^3 (3-x)\,dx + \int_3^5 (x-3)\,dx$

$$= \left[3x - \frac{x^2}{2}\right]_2^3 + \left[\frac{x^2}{2} - 3x\right]_3^5 = \frac{5}{2}$$

48. (a) $\int_{-4}^{2} (2x+3)\,dx = \left[x^2+3x\right]_{-4}^{2} = 6$

(b) $\int_{-4}^{2} |2x+3|\,dx = \int_{-4}^{-3/2} (-2x-3)\,dx + \int_{-3/2}^{2} (2x+3)\,dx = \left[-x^2-3x\right]_{-4}^{-3/2} + \left[x^2+3x\right]_{-3/2}^{2} = \frac{37}{2}$

49. (a) $\int_{-2}^{2} (x^2-1)\,dx = \left[\frac{x^3}{3} - x\right]_{-2}^{2} = \frac{4}{3}$

(b)

$$\int_{-2}^{2} |x^2-1|\,dx = \int_{-2}^{-1} (x^2-1)\,dx + \int_{-1}^{1} (1-x^2)\,dx + \int_{1}^{2} (x^2-1)\,dx$$

$$= \left[\frac{x^3}{3} - x\right]_{-2}^{-1} + \left[x - \frac{x^3}{3}\right]_{-1}^{1} + \left[\frac{x^3}{3} - x\right]_{1}^{2} = 4$$

50. (a) $\int_{-\pi/2}^{\pi} \cos x\,dx = \left[\sin x\right]_{-\pi/2}^{\pi} = 1$

(b) $\int_{-\pi/2}^{\pi} |\cos x|\, dx = \int_{-\pi/2}^{\pi/2} \cos x\, dx + \int_{\pi/2}^{\pi} -\cos x\, dx = [\sin x]_{-\pi/2}^{\pi/2} + [-\sin x]_{\pi/2}^{\pi} = 3$

51. (a) $x(t) = \int_0^t (10u - u^2)\, du = \left[5u^2 - \frac{u^3}{3}\right]_0^t = 5t^2 - \frac{t^3}{3}, \quad 0 \le t \le 10$

(b) $v'(t) = 10 - 2t$; v has an absolute maximum at $t = 5$. The object's position at $t = 5$ is

$x(5) = \frac{250}{3}$.

52. (a) We need $x(t)$ such that $x'(t) = 3\sin t + 4\cos t$ and $x(0) = 1$

Then $x(t) = -3\cos t + 4\sin t + C$, $x(0) = -3 + C = 1 \implies C = 4$

$\implies x(t) = -3\cos t + 4\sin t + 4$.

(b) Maximum displacement when $v(t) = 0$: $3\sin t + 4\cos t = 0$

$\implies \tan t = -\frac{4}{3} \implies \sin t = \frac{4}{5}, \quad \cos t = -\frac{3}{5}$

So $x_{max} = -3(\frac{-3}{5}) + 4(\frac{4}{5}) + 4 = 9$

53. $\int_0^4 f(x)\, dx = \int_0^1 (2x+1)\, dx + \int_1^4 (4-x)\, dx = [x^2 + x]_0^1 + \left[4x - \frac{x^2}{2}\right]_1^4 = \frac{13}{2}$

54. $\int_{-2}^4 f(x)\, dx = \int_{-2}^0 x^2\, dx + \int_0^4 (\frac{1}{2}x + 2)\, dx = \left[\frac{x^3}{3}\right]_{-2}^0 + \left[\frac{x^2}{4} + 2x\right]_0^4 = \frac{44}{3}$

55. $\int_{-\pi/2}^{\pi} f(x)\, dx = \int_{-\pi/2}^{\pi/3} \cos x\, dx + \int_{\pi/3}^{\pi} \left[\frac{3}{\pi}x + 1\right] dx = [\sin x]_{-\pi/2}^{\pi/3} + \left[\frac{3x^2}{2\pi} + x\right]_{\pi/3}^{\pi}$

$= \frac{2+\sqrt{3}}{2} + 2\pi$

56. $\int_0^{3\pi/2} f(x)\, dx = \int_0^{\pi/2} 2\sin x\, dx + \int_{\pi/2}^{3\pi/2} \frac{1}{2}\cos x\, dx = [-2\cos x]_0^{\pi/2} + \left[\frac{1}{2}\sin x\right]_{\pi/2}^{3\pi/2} = 1$

57. (a) f is continuous on $[-2, 2]$.

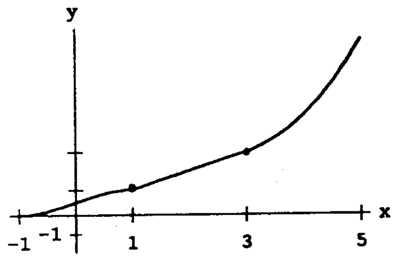
For $x \in [-2, 0]$, $g(x) = \int_{-2}^x (t+2)dt = \left[\frac{1}{2}t^2 + 2t\right]_{-2}^x = \frac{1}{2}x^2 + 2x + 2$.

For $x \in [0, 1]$, $g(x) = \int_{-2}^0 (t+2)\, dt + \int_0^x 2\, dt = 2 + [2t]_0^x = 2 + 2x$.

For $x \in [1, 2]$, $g(x) = \int_{-2}^0 (t+2)\, dt + \int_0^1 2\, dt + \int_1^x (4 - 2x)\, dx = 2 + 2 + [4t - t^2]_1^x = 1 + 4x - x^2$.

Thus $g(x) = \begin{cases} \frac{1}{2}x^2 + 2x + 2, & -2 \le x \le 0 \\ 2x + 2, & 0 \le x \le 1 \\ 1 + 4x - x^2, & 1 \le x \le 2 \end{cases}$

(b)



(c) f is continuous on $[-2, 2]$; f is differentiable on $(-2, 0)$, $(0, 1)$, and $(1, 2)$.

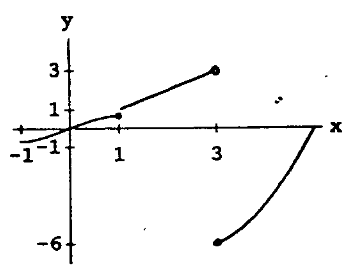
g is differentiable on $(-2, 2)$.

58. (a) $g(x) = \int_{-1}^{x} f(x)\,dt = \int_{-1}^{x} (1 - t^2)\,dt = \left[t - \frac{t^3}{3}\right]_{-1}^{x} = x - \frac{x^3}{3} + \frac{2}{3},$ for $-1 \le x \le 1$

$$g(x) = \int_{-1}^{1} (1 - t^2)\,dt + \int_{1}^{x} 1\,dt = \frac{4}{3} + [t]_1^x = \frac{1}{3} + x, \quad \text{for } 1 < x < 3$$

$$g(x) = \int_{-1}^{3} f(t)\,dt + \int_{3}^{x} (2t - 5)\,dt = \frac{10}{3} + \left[t^2 - 5t\right]_3^x = \frac{28}{3} + x^2 - 5x, \quad \text{for } 3 \le x \le 5$$

(b)



(c) f is continuous on $[-1, 1) \cup (1, 5]$, f is differentiable on $(-1, 1) \cup (1, 3) \cup (3, 5)$.

g is differentiable on $(-1, 1) \cup (1, 5)$.

59. Follows from Theorem 5.3.2 since $f(x)$ is an antiderivative of $f'(x)$.

60. Let $F(x) = f^2(x)$. Then $F'(x) = 2f(x)f'(x)$.

Thus $\int_a^b f(t)f'(t)\,dt = \frac{1}{2}\int_a^b F'(t)\,dt = \frac{1}{2}[F(b) - F(a)] = \frac{1}{2}[f^2(b) - f^2(a)].$

61. $\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x); \quad \int_a^x \frac{d}{dt}[f(t)]\,dt = f(x) - f(a)$

62. For $n = 2$ this is (5.3.6). For the induction step, write

$$\alpha_1 f_1(x) + \alpha_2 f_2(x) + \cdots + \alpha_{n+1} f_{n+1}(x),$$

as

$$[\alpha_1 f_1(x) + \alpha_2 f_2(x) + \cdots + \alpha_n f_n(x)] + \alpha_{n+1} f_{n+1}(x).$$

SECTION 5.4

1. $A=\int_0^1 (2+x^3)\,dx=\left[2x+\frac{x^4}{4}\right]_0^1=\frac{9}{4}$

2. $A=\int_0^2 (x+2)^{-2}\,dx=\left[\frac{-1}{x+2}\right]_0^2=\frac{1}{4}$

3. $A=\int_3^8 \sqrt{x+1}\,dx=\int_3^8 (x+1)^{1/2}\,dx=\left[\frac{2}{3}(x+1)^{3/2}\right]_3^8=\frac{2}{3}[27-8]=\frac{38}{3}$

4. $A=\int_0^8 (3x^2+x^3)\,dx=\left[x^3+\frac{1}{4}x^4\right]_0^8=1536$

5. $A=\int_0^1 (2x^2+1)^2\,dx=\int_0^1 (4x^4+4x^2+1)\,dx=\left[\frac{4}{5}x^5+\frac{4}{3}x^3+x\right]_0^1=\frac{47}{15}$

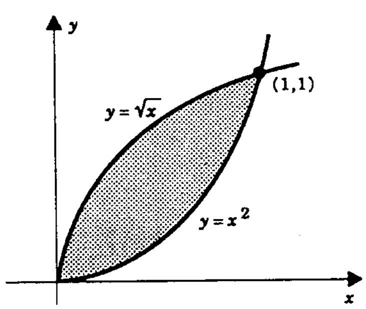
6. $A=\int_0^8 \frac{1}{2\sqrt{x+1}}\,dx=\left[\sqrt{x+1}\right]_0^8=2$

7. $A=\int_1^2 [0-(x^2-4)]\,dx=\int_1^2 (4-x^2)\,dx=\left[4x-\frac{x^3}{3}\right]_1^2=\left[8-\frac{8}{3}\right]-\left[4-\frac{1}{3}\right]=\frac{5}{3}$

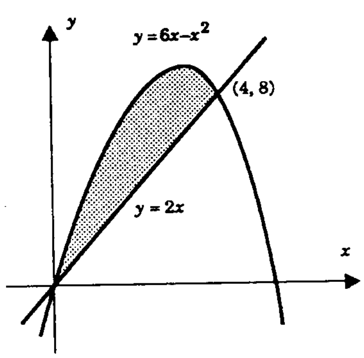
8. $A=\int_{\pi/6}^{\pi/3} \cos x\,dx=[\sin x]_{\pi/6}^{\pi/3}=\frac{\sqrt{3}-1}{2}$

9. $A=\int_{\pi/3}^{\pi/2} \sin x\,dx=[-\cos x]_{\pi/3}^{\pi/2}=(0)-\left(-\frac{1}{2}\right)=\frac{1}{2}$

10. $A=-\int_{-2}^{-1} (x^3+1)\,dx=-\left[\frac{x^4}{4}+x\right]_{-2}^{-1}=\frac{11}{4}$

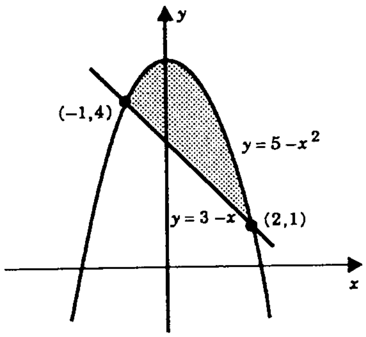
11. 

$$A=\int_0^1 [x^{1/2}-x^2]\,dx$$
$$=\left[\tfrac{2}{3}x^{3/2}-\tfrac{1}{3}x^3\right]_0^1=\tfrac{1}{3}$$

12. 

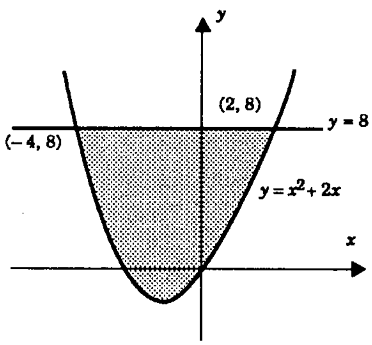
$$A=\int_0^4 (6-x^2-2x)\,dx=\frac{32}{3}$$

13.



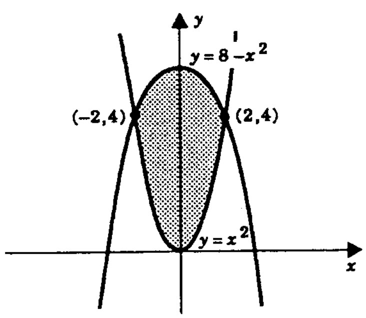
$$A = \int_{-1}^{2} [(5-x^2)-(3-x)]\,dx$$
$$= \int_{-1}^{2} (2+x-x^2)\,dx$$
$$= \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2}$$
$$= \left[4+2-\tfrac{8}{3}\right] - \left[-2+\tfrac{1}{2}+\tfrac{1}{3}\right] = \tfrac{9}{2}$$

14.



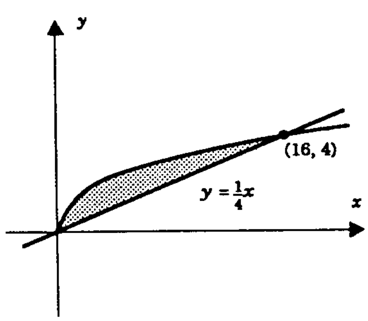
$$A = \int_{-4}^{2} (8-x^2-2x)\,dx = 36$$

15.



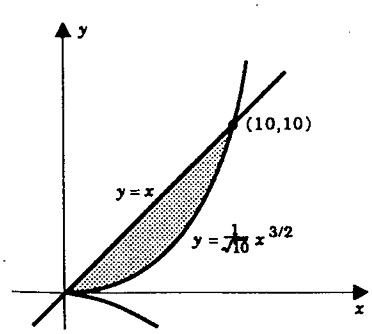
$$A = \int_{-2}^{2} [(8-x^2)-(x^2)]\,dx$$
$$= \int_{-2}^{2} (8-2x^2)\,dx$$
$$= \left[8x - \tfrac{2}{3}x^3\right]_{-2}^{2}$$
$$= \left[16-\tfrac{16}{3}\right] - \left[-16+\tfrac{16}{3}\right] = \tfrac{64}{3}$$

16.

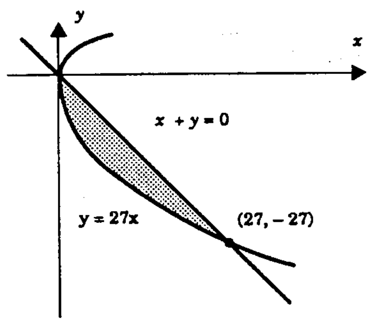


$$A = \int_{0}^{16} \left(\sqrt{x} - \frac{1}{4}x\right)\,dx = \frac{32}{3}$$

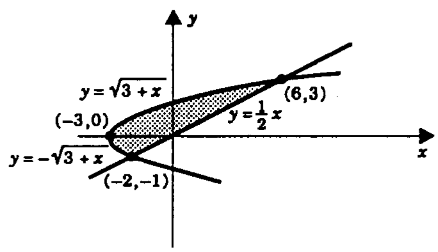
17.



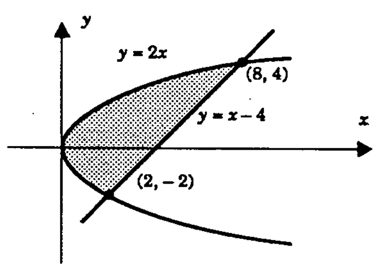
$$A = \int_{0}^{10} \left[x - \frac{1}{\sqrt{10}}\,x^{3/2}\right]\,dx$$
$$= \left[\frac{x^2}{2} - \frac{2\sqrt{10}}{50}\,x^{5/2}\right]_{0}^{10}$$
$$= 50 - \frac{2\sqrt{10}}{50}(10)^{5/2} = 10$$

18. 

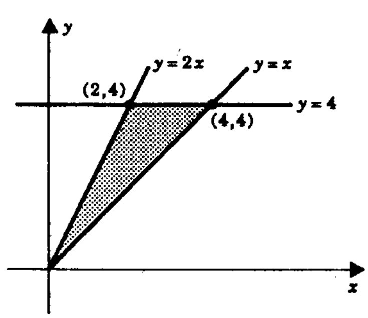
$$A = \int_0^{27} [-x - (-\sqrt{27x})]\,dx = \frac{243}{2}$$

19. 

$$\begin{aligned} A &= \int_{-3}^{-2} [(\sqrt{3+x}) - (-\sqrt{3+x})]\,dx + \int_{-2}^{6} \left[(\sqrt{3+x}) - \left(\frac{1}{2}x\right)\right]\,dx \\ &= \int_{-3}^{-2} 2(3+x)^{1/2}\,dx + \int_{-2}^{6} \left[(3+x)^{1/2} - \frac{1}{2}x\right]\,dx \\ &= \left[\frac{4}{3}(3+x)^{3/2}\right]_{-3}^{-2} + \left[\frac{2}{3}(3+x)^{3/2} - \frac{x^2}{4}\right]_{-2}^{6} = \left[\frac{4}{3} - 0\right] + \left[(18-9) - \left(\frac{2}{3} - 1\right)\right] = \frac{32}{3} \end{aligned}$$

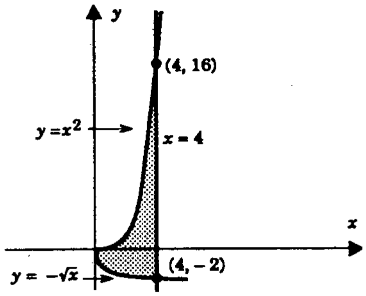
20. 

$$\begin{aligned} A &= \int_0^2 \left[\sqrt{2x} - (-\sqrt{2x})\right]\,dx + \int_2^8 \left[\sqrt{2x} - x + 4\right]\,dx \\ &= 18 \end{aligned}$$

21. 

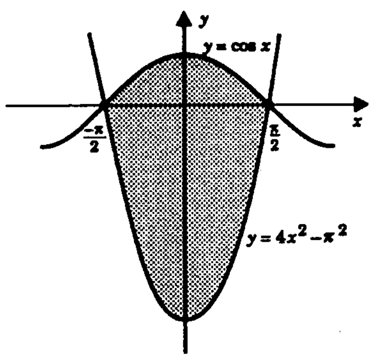
$$\begin{aligned} A &= \int_0^2 [2x - x]\,dx + \int_2^4 [4 - x]\,dx \\ &= \left[\tfrac{1}{2}x^2\right]_0^2 + \left[4x - \tfrac{1}{2}x^2\right]_2^4 \\ &= 2 + [8 - 6] = 4 \end{aligned}$$

22.



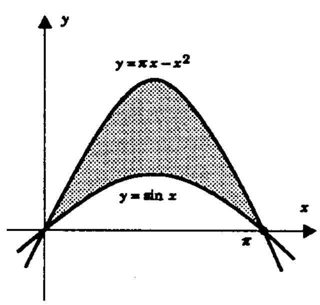
$$A = \int_0^4 \left(x^2 + \sqrt{x}\right)\,dx = \frac{80}{3}$$

23.



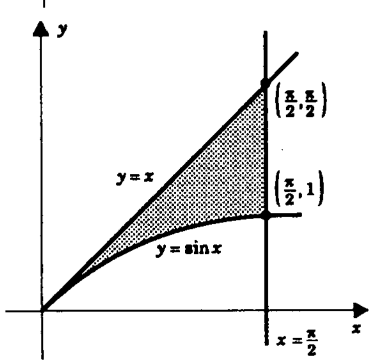
$$\begin{aligned} A &= \int_{-\pi/2}^{\pi/2} \left[\cos x - (4x^2 - \pi^2)\right] dx \\ &= \left[\sin x - \tfrac{4}{3}x^3 + \pi^2 x\right]_{-\pi/2}^{\pi/2} \\ &= \left[1 - \tfrac{1}{6}\pi^3 + \tfrac{1}{2}\pi^3\right] - \left[-1 + \tfrac{1}{6}\pi^3 - \tfrac{1}{2}\pi^3\right] \\ &= 2 + \tfrac{2}{3}\pi^3 \end{aligned}$$

24.



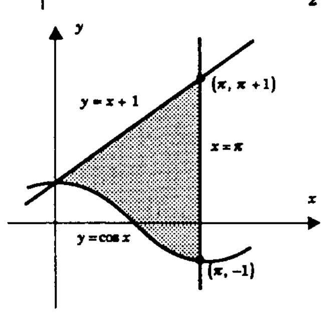
$$A = \int_0^{\pi} \left(\pi x - x^2 - \sin x\right)\,dx = \frac{\pi^3}{6} - 2$$

25.



$$\begin{aligned} A &= \int_0^{\pi/2} [x - \sin x]\,dx \\ &= \left[\frac{x^2}{2} + \cos x\right]_0^{\pi/2} \\ &= \frac{\pi^2}{8} - 1 \end{aligned}$$

26.



$$A = \int_0^{\pi} (x + 1 - \cos x)\,dx = \frac{\pi}{2}(\pi + 2)$$

27. (a) $\int_{-3}^{4}(x^2-x-6)\,dx=\left[\frac{1}{3}x^3-\frac{1}{2}x^2-6x\right]_{-3}^{4}=-\frac{91}{6};$

the area of the region bounded by the graph of f and the x-axis for $x\in[-3,-2]\cup[3,4]$ minus the area of the region bounded the graph of f and the x-axis for $x\in[-2,3]$.

(b) $A=\int_{-3}^{-2}(x^2-x-6)\,dx+\int_{-2}^{3}(-x^2+x+6)\,dx+\int_{3}^{4}(x^2-x-6)\,dx$

$=\left[\frac{1}{3}x^3-\frac{1}{2}x^2-6x\right]_{-3}^{-2}+\left[-\frac{1}{3}x^3+\frac{1}{2}x^2+6x\right]_{-2}^{3}+\left[\frac{1}{3}x^3-\frac{1}{2}x^2-6x\right]_{3}^{4}=\frac{17}{6}+\frac{125}{6}+\frac{17}{6}=\frac{53}{2}$

(c) $A=-\int_{-2}^{3}(x^2-x-6)\,dx=\frac{125}{6}$

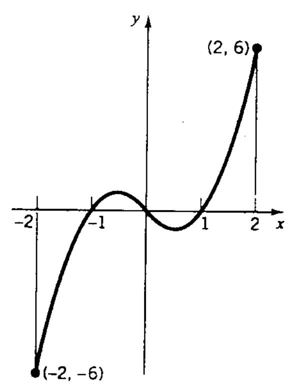
28. (a) $\int_{-\pi/2}^{3\pi/4}2\sin x\,dx=\left[-2\cos x\right]_{-\pi/2}^{3\pi/4}=\sqrt{2}=$ area above $-$ area below

(b) $A=\int_{-\pi/2}^{0}-2\sin x\,dx+\int_{0}^{3\pi/4}2\sin x\,dx=\left[2\cos x\right]_{-\pi/2}^{0}+\left[-2\cos x\right]_{0}^{3\pi/4}=\sqrt{2}+4$

(c) $A=\int_{-\pi/2}^{0}-2\sin x\,dx=\left[2\cos x\right]_{-\pi/2}^{0}=2$

29. (a) $\int_{-2}^{2}(x^3-x)\,dx=\left[\frac{1}{4}x^4-\frac{1}{2}x^2\right]_{-2}^{2}=0$

(b)



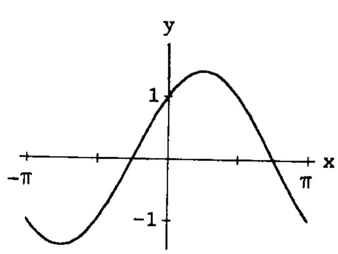
$A=2\left[-\int_{0}^{1}(x^3-x)\,dx+\int_{1}^{2}(x^3-x)\,dx\right]$

$=-2\left[\frac{1}{4}x^4-\frac{1}{2}x^2\right]_{0}^{1}+2\left[\frac{1}{4}x^4-\frac{1}{2}x^2\right]_{1}^{2}$

$=\frac{1}{2}+\frac{9}{2}=5$

30. (a) $\int_{-\pi}^{\pi}(\cos x+\sin x)\,dx=\left[\sin x-\cos x\right]_{-\pi}^{\pi}=0$

(b)

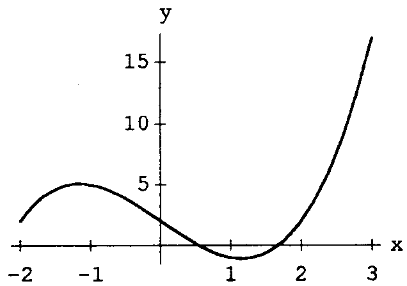


$A=-\int_{-\pi}^{-\pi/4}f(x)\,dx+\int_{-\pi/4}^{3\pi/4}f(x)\,dx-\int_{3\pi/4}^{\pi}f(x)\,dx$

$=4\sqrt{2}$

31. (a) $\int_{-2}^{3}(x^3-4x+2)\,dx=\left[\frac{1}{4}x^4-2x^2+2x\right]_{-2}^{3}=\frac{65}{4}$

(b)



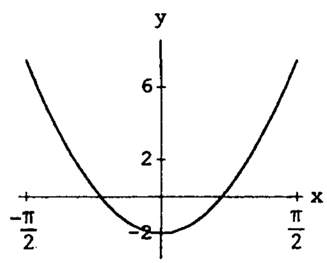
$$A\cong\int_{-2}^{0.54}(x^3-4x+2)\,dx-\int_{0.54}^{1.68}(x^3-4x+2)\,dx+\int_{1.68}^{3}(x^3-4x+2)\,dx$$

$$=\left[\tfrac{1}{4}x^4-2x^2+2x\right]_{-2}^{0.54}-\left[\tfrac{1}{4}x^4-2x^2+2x\right]_{0.54}^{1.68}+\left[\tfrac{1}{4}x^4-2x^2+2x\right]_{1.68}^{3}$$

$$=8.52+.81+8.54=17.87$$

32. (a) $\int_{-\pi/2}^{\pi/2}(3x^2-2\cos x)\,dx=\left[x^3-2\sin x\right]_{-\pi/2}^{\pi/2}=\frac{\pi^3}{4}-4$

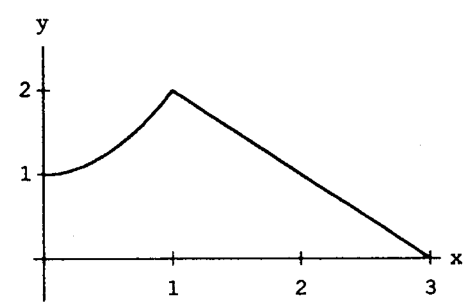
(b)



$$A\cong-\int_{-0.72}^{0.72}(3x^2-2\cos x)\,dx+2\int_{0.72}^{\pi/2}(3x^2-2\cos x)\,dx$$

$$\cong 7.45$$

33.

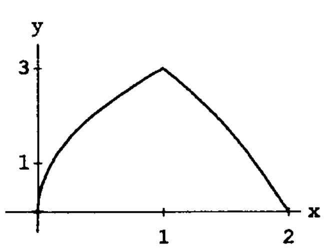


$$A=\int_0^1(x^2+1)\,dx+\int_1^3(3-x)\,dx$$

$$=\left[\tfrac{1}{3}x^3+x\right]_0^1+\left[3x-\tfrac{1}{2}x^2\right]_1^3$$

$$=\frac{4}{3}+2=\frac{10}{3}$$

34.

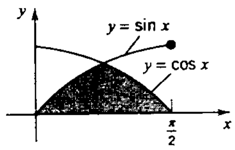


$$A=\int_0^1 3\sqrt{x}\,dx+\int_1^2(4-x^2)\,dx$$

$$=\left[2x^{3/2}\right]_0^1+\left[4x-\frac{x^3}{3}\right]_1^2$$

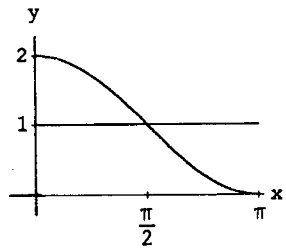
$$=2+\frac{5}{3}=\frac{11}{3}$$

35.



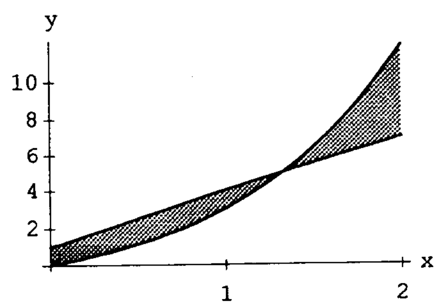
$$A = \int_0^{\pi/4} \sin x\, dx + \int_{\pi/4}^{\pi/2} \cos x\, dx$$
$$= [-\cos x]_0^{\pi/4} + [\sin x]_{\pi/4}^{\pi/2}$$
$$= 2 - \sqrt{2}$$

36.



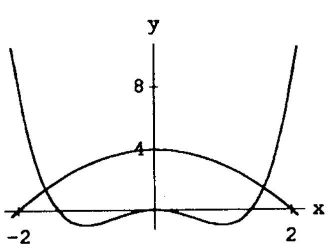
$$A = 2\int_0^{\pi/2} (1 + \cos x - 1)\, dx$$
$$= [2\sin x]_0^{\pi/2}$$
$$= 2$$

37.



$$A \cong \int_0^{1.32} [3x + 1 - (x^3 + 2x)]\, dx + \int_{1.32}^{2} [x^3 + 2x - (3x + 1)]\, dx$$
$$= \int_0^{1.32} (x + 1 - x^3)\, dx + \int_{1.32}^{2} (x^3 - x - 1)\, dx$$
$$= \left[\tfrac{1}{2}\, x^2 + x - \tfrac{1}{4}\, x^4\right]_0^{1.32} + \left[\tfrac{1}{4}\, x^4 - \tfrac{1}{2}\, x^2 - x\right]_{1.32}^{2} = 2.86$$

38.



$$A = \int_{-\sqrt{\frac{1+\sqrt{17}}{2}}}^{\sqrt{\frac{1+\sqrt{17}}{2}}} (4 - x^2 - x^4 + 2x^2)\, dx$$
$$\cong 11.34$$

SECTION 5.5

1. $\displaystyle\int \frac{dx}{x^4} = \int x^{-4}\,dx = -\frac{1}{3}x^{-3} + C$

2. $\displaystyle\int (x-1)^2\,dx = \int (x^2 - 2x + 1)\,dx = \frac{1}{3}x^3 - x^2 + x + C$

3. $\displaystyle\int (ax+b)\,dx = \frac{1}{2}\,ax^2 + bx + C$

4. $\displaystyle\int (ax^2+b)\,dx = \frac{1}{3}ax^3 + bx + C$

5. $\displaystyle\int \frac{dx}{\sqrt{1+x}} = \int (1+x)^{-1/2}\,dx = 2(1+x)^{1/2} + C$

6. $\displaystyle\int \frac{x^3+1}{x^5}\,dx = \int x^{-2} + x^{-5}\,dx = -x^{-1} - \frac{1}{4}x^{-4} + C$

7. $\displaystyle\int \left(\frac{x^3-1}{x^2}\right)\,dx = \int (x - x^{-2})\,dx = \frac{1}{2}\,x^2 + x^{-1} + C$

8. $\displaystyle\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)\,dx = \int \left(x^{1/2} - x^{-1/2}\right)\,dx = \frac{2}{3}x^{3/2} - 2x^{1/2} + C$

9. $\displaystyle\int (t-a)(t-b)\,dt = \int [t^2 - (a+b)t + ab]\,dt = \frac{1}{3}\,t^3 - \frac{a+b}{2}t^2 + abt + C$

10. $\displaystyle\int (t^2-a)(t^2-b)\,dt = \int \left(t^4 - (a+b)t^2 + ab\right)\,dt = \frac{1}{5}t^5 - \frac{1}{3}(a+b)t^3 + abt + C$

11. $\displaystyle\int \frac{(t^2-a)(t^2-b)}{\sqrt{t}}\,dt \quad = \int [t^{7/2} - (a+b)t^{3/2} + abt^{-1/2}]\,dt$

$\displaystyle\qquad = \tfrac{2}{9}t^{9/2} - \tfrac{2}{5}(a+b)t^{5/2} + 2abt^{1/2} + C$

12. $\displaystyle\int \left(2-\sqrt{x}\right)\left(2+\sqrt{x}\right)\,dx = \int (4-x)\,dx = 4x - \frac{1}{2}x^2 + C$

13. $\displaystyle\int g(x)g'(x)\,dx = \frac{1}{2}[g(x)]^2 + C$

14. $\displaystyle\int \sin x\cos x\,dx = \frac{1}{2}\sin^2 x + C$

15. $\displaystyle\int \tan x\sec^2 x\,dx = \int \sec x\,\frac{d}{dx}\,[\sec x]\,dx = \frac{1}{2}\sec^2 x + C$

$\displaystyle\int \tan x\sec^2 x\,dx = \int \tan x\,\frac{d}{dx}\,[\tan x]\,dx = \frac{1}{2}\tan^2 x + C$

16. $\displaystyle\int \frac{g'(x)}{[g(x)]^2}\,dx = -\frac{1}{g(x)} + C$

17. $\displaystyle\int \frac{4}{(4x+1)^2}\,dx = \int 4(4x+1)^{-2}\,dx = -(4x+1)^{-1} + C$

18. $\displaystyle\int \frac{3x^2}{(x^3+1)^2}\,dx = -\frac{1}{x^3+1} + C,$ (use Exercise 16)

19. $\displaystyle f(x) = \int f'(x)\,dx = \int (2x-1)\,dx = x^2 - x + C.$

Since $f(3) = 4$, we get $4 = 9 - 3 + C$ so that $C = -2$ and

$$f(x) = x^2 - x - 2.$$

20. $\displaystyle f(x) = \int (3-4x)\,dx = 3x - 2x^2 + C, \quad f(1) = 6 \implies f(x) = -2x^2 + 3x + 5$

21. $\displaystyle f(x) = \int f'(x)\,dx = \int (ax+b)\,dx = \tfrac{1}{2}\,ax^2 + bx + C.$

Since $f(2) = 0$, we get $0 = 2a + 2b + C$ so that $C = -2a - 2b$ and

$$f(x) = \tfrac{1}{2}ax^2 + bx - 2a - 2b.$$

22. $\displaystyle f(x) = \int (ax^2 + bx + c)\,dx = \frac{1}{3}ax^3 + \frac{1}{2}bx^2 + cx + K,$

$\displaystyle f(0) = 0 \implies f(x) = \frac{a}{3}x^3 + \frac{b}{2}x^2 + cx$

23. $\displaystyle f(x) = \int f'(x)\,dx = \int \sin x\,dx = -\cos x + C.$

Since $f(0) = 2$, we get $2 = -1 + C$ so that $C = 3$ and

$$f(x) = 3 - \cos x.$$

24. $\displaystyle f(x) = \int \cos x\,dx = \sin x + C, \quad f(\pi) = 3 \implies f(x) = 3 + \sin x$

25. First,

$$f'(x) = \int f''(x)\,dx = \int (6x-2)\,dx = 3x^2 - 2x + C.$$

Since $f'(0) = 1$, we get $1 = 0 + C$ so that $C = 1$ and

$$f'(x) = 3x^2 - 2x + 1.$$

Next,

$$f(x) = \int f'(x)\,dx = \int (3x^2 - 2x + 1)\,dx = x^3 - x^2 + x + K.$$

Since $f(0) = 2$, we get $2 = 0 + K$ so that $K = 2$ and

$$f(x) = x^3 - x^2 + x + 2.$$

26. $\displaystyle f'(x) = \int -12x^2\,dx = -4x^3 + C, \quad f'(0) = 1 \implies f'(x) = -4x^3 + 1$

$\displaystyle f(x) = \int (-4x^3 + 1)\,dx = -x^4 + x + K, \quad f(0) = 2 \implies f(x) = -x^4 + x + 2$

27. First,

$$f'(x) = \int f''(x)\,dx = \int (x^2 - x)\,dx = \frac{1}{3}x^3 - \frac{1}{2}x^2 + C.$$

Since $f'(1) = 0$, we get $0 = \frac{1}{3} - \frac{1}{2} + C$ so that $C = \frac{1}{6}$ and

$$f'(x) = \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{6}.$$

Next,

$$f(x) = \int f'(x)\,dx = \int \left(\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{6}\right)\,dx = \frac{1}{12}x^4 - \frac{1}{6}x^3 + \frac{1}{6}x + K.$$

Since $f(1) = 2$, we get $2 = \frac{1}{12} - \frac{1}{6} + \frac{1}{6} + K$ so that $K = \frac{23}{12}$ and

$$f(x) = \frac{x^4}{12} - \frac{x^3}{6} + \frac{x}{6} + \frac{23}{12} = \frac{1}{12}(x^4 - 2x^3 + 2x + 23).$$

28. $f'(x) = \int (1 - x)\,dx = x - \dfrac{x^2}{2} + C, \quad f'(2) = 1 \implies f'(x) = x - \dfrac{x^2}{2} + 1$

$f(x) = \int (x - \dfrac{x^2}{2} + 1)\,dx = \dfrac{x^2}{2} - \dfrac{x^3}{6} + x + K, \quad f(2) = 0 \implies f(x) = -\dfrac{x^3}{6} + \dfrac{x^2}{2} + x - \dfrac{8}{3}$

29. First,

$$f'(x) = \int f''(x)\,dx = \int \cos x\,dx = \sin x + C.$$

Since $f'(0) = 1$, we get $1 = 0 + C$ so that $C = 1$ and

$$f'(x) = \sin x + 1.$$

Next,

$$f(x) = \int f'(x)\,dx = \int (\sin x + 1)\,dx = -\cos x + x + K.$$

Since $f(0) = 2$, we get $2 = -1 + 0 + K$ so that $K = 3$ and

$$f(x) = -\cos x + x + 3.$$

30. $f'(x) = \int \sin x\,dx = -\cos x + C, \quad f'(0) = -2 \implies f'(x) = -\cos x - 1$

$f(x) = \int (-\cos x - 1)\,dx = -\sin x - x + K, \quad f(0) = 1 \implies f(x) = 1 - \sin x - x$

31. First,

$$f'(x) = \int f''(x)\,dx = \int (2x - 3)\,dx = x^2 - 3x + C.$$

Then,

$$f(x) = \int f'(x)\,dx = \int (x^2 - 3x + C)\,dx = \frac{1}{3}x^3 - \frac{3}{2}x^2 + Cx + K.$$

Since $f(2) = -1$, we get

(1) $$-1 = \tfrac{8}{3} - 6 + 2C + K;$$

and, from $f(0) = 3$, we conclude that

(2) $$3 = 0 + K.$$

Solving (1) and (2) simultaneously, we get $K = 3$ and $C = -\frac{1}{3}$ so that

$$f(x) = \tfrac{1}{3}x^3 - \tfrac{3}{2}x^2 - \tfrac{1}{3}x + 3.$$

32. $f'(x) = \int (5 - 4x)\,dx = 5x - 2x^2 + C,$

$f(x) = \int (5x - 2x^2 + C)\,dx = \dfrac{5}{2}x^2 - \dfrac{2}{3}x^3 + Cx + K$

$f(1) = \dfrac{5}{2} - \dfrac{2}{3} + C + K = 1, \quad f(0) = k = -2 \implies f(x) = -\dfrac{2}{3}x^3 + \dfrac{5}{2}x^2 + \dfrac{7}{6}x - 2$

33. $\dfrac{d}{dx}\left[\int f(x)\,dx\right] = f(x); \quad \int \dfrac{d}{dx}[f(x)]\,dx = f(x) + C$

34. $\int [f(x)g''(x) - g(x)f''(x)]\,dx = \int [f(x)g''(x) + f'(x)g'(x) - f'(x)g'(x) - g(x)f''(x)]\,dx$

$= \int \left(\dfrac{d}{dx}[f(x)g'(x)] - \dfrac{d}{dx}[f'(x)g(x)]\right) dx = f(x)g'(x) - g(x)f'(x) + C$

35. (a) $x(t) = \int v(t)\,dt = \int (6t^2 - 6)\,dt = 2t^3 - 6t + C.$

Since $x(0) = -2$, we get $-2 = 0 + C$ so that $C = -2$ and

$$x(t) = 2t^3 - 6t - 2. \quad \text{Therefore} \quad x(3) = 34.$$

Three seconds later the object is 34 units to the right of the origin .

(b) $s = \int_0^3 |v(t)|\,dt = \int_0^3 |6t^2 - 6|\,dt = \int_0^1 (6 - 6t^2)\,dt + \int_1^3 (6t^2 - 6)\,dt$

$= [6t - 2t^3]_0^1 + [2t^3 - 6t]_1^3 = 4 + [36 - (-4)] = 44.$

The object traveled 44 units.

36. (a) $v(t) = \int a(t)\,dt = \int (t + 2)^3\,dt = \dfrac{1}{4}(t + 2)^4 + C,$

$v(0) = 3 \implies v(t) = \dfrac{1}{4}(t + 2)^4 - 1$

(b) $x(t) = \int \left[\dfrac{(t + 2)^4}{4} - 1\right] dt = \dfrac{(t + 2)^5}{20} - t + K, \quad x(0) = 0 \implies x(t) = \dfrac{(t + 2)^5}{20} - t - \dfrac{8}{5}$

37. (a) $v(t) = \int a(t)\,dt = \int (t + 1)^{-1/2}\,dt = 2(t + 1)^{1/2} + C.$

Since $v(0) = 1$, we get $1 = 2 + C$ so that $C = -1$ and

$$v(t) = 2(t + 1)^{1/2} - 1.$$

(b) We know $v(t)$ by part (a). Therefore,

$$x(t) = \int v(t)\,dt = \int [2(t + 1)^{1/2} - 1]\,dt = \frac{4}{3}(t + 1)^{3/2} - t + C.$$

Since $x(0) = 0$, we get $0 = \frac{4}{3} - 0 + C$ so that

$$C = -\tfrac{4}{3} \quad \text{and} \quad x(t) = \tfrac{4}{3}(t + 1)^{3/2} - t - \tfrac{4}{3}.$$

38. (a) $x(t) = \int t(1 - t)\,dt = \dfrac{t^2}{2} - \dfrac{t^3}{3} + C, \quad x(0) = -2 \implies x(t) = \dfrac{t^2}{2} - \dfrac{t^3}{3} - 2$

$x(10) = -\dfrac{856}{3}$: $285\frac{1}{3}$ units to the left of the origin.

(b) $s = \int_0^{10} |v(t)|\,dt = \int_0^1 t(1-t)\,dt + \int_1^{10} -t(1-t)\,dt = \left[\dfrac{t^2}{2} - \dfrac{t^3}{3}\right]_0^1 - \left[\dfrac{t^2}{2} - \dfrac{t^3}{3}\right]_1^{10}$

$= \dfrac{851}{3} = 283\frac{2}{3}$ units.

39. (a) $v_0 = 60$ mph $= 88$ feet per second. In general, $v(t) = at + v_0$. Here, in feet and seconds, $v(t) = -20t + 88$. Thus $v(t) = 0$ at $t = 4.4$ seconds.

(b) In general, $x(t) = \frac{1}{2}at^2 + v_0 t + x_0$. Here we take $x_0 = 0$. In feet and seconds

$$x(t) = -10(4.4)^2 + 88(4.4) = 10(4.4)^2 = 193.6 \text{ ft.}$$

40. Let acceleration $= a$. Then $v(t) = \int a\,dt = at + v_0$.

$$x(t) = \int v(t)\,dt = \int (at + v_0)\,dt = \frac{1}{2}at^2 + v_0 t + x_0 = \frac{1}{2}\left[v(t) + v_0\right]t + x_0$$

41.

$$[v(t)]^2 = (at + v_0)^2 = a^2t^2 + 2av_0 t + {v_0}^2$$
$$= {v_0}^2 + a(at^2 + 2v_0 t)$$
$$= {v_0}^2 + 2a(\tfrac{1}{2}at^2 + v_0 t)$$

$x(t) = \frac{1}{2}at^2 + v_0 t + x_0$

$$= {v_0}^2 + 2a\left[x(t) - x_0\right]$$

42. (a) $v(t) = at + v_0$, and by Exercise 40 $x(t) = \dfrac{1}{2}\left[v(t) + v_0\right]t$, so

$$a = \frac{v(t) - v_0}{t} = \frac{v(t) - v_0}{2x(t)}(v(t) + v_0) = \frac{v(t)^2 - {v_0}^2}{2x(t)} = \frac{58.7^2 - 88^2}{2.264} = -8.15 \text{ ft/sec}^2$$

[Note 60 mph $= 88$ ft/sec, 40 mph $= 58\frac{2}{3}$ ft/sec¿]

(b) $t = \dfrac{2x(t)}{v(t) + v_0} = \dfrac{2.264}{58\frac{2}{3} + 88} = 3.6$ sec

(c) We don't know $x(t)$, so we will use $t = \dfrac{v(t) - v_0}{a} = \dfrac{0 - 88}{-8.15} = 10.8$ sec

(d) $x(t) = \dfrac{1}{2}\left[v(t) + v_0\right]t = \dfrac{1}{2}[0 + 88]\,10.8 = 475.2$ ft

43. The car can accelerate to 60 mph (88 ft/sec) in 20 seconds thereby covering a distance of 880 ft. It can decelerate from 88 ft/sec to 0 ft/sec in 4 seconds thereby covering a distance of 176 ft. At full speed, 88 ft/sec, it must cover a distance of

$$\frac{5280}{2} - 880 - 176 = 1584 \text{ ft.}$$

This takes $\dfrac{1584}{88} = 18$ seconds. The run takes at least $20 + 18 + 4 = 42$ seconds.

44. $v(t) = \int \sin t\,dt = -\cos t + C$, $v(0) = v_0 \implies v(t) = -\cos t + v_0 + 1$

$$x(t)=\int(-\cos t+v_0+1)\,dt=-\sin t+(v_0+1)t+K, \quad x(0)=x_0 \implies x(t)=x_0+(v_0+1)t-\sin t$$

45. $v(t)=\int a(t)\,dt=\int(2A+6Bt)\,dt=2At+3Bt^2+C.$

Since $v(0)=v_0$, we have $v_0=0+C$ so that $v(t)=2At+3Bt^2+v_0$.

$$x(t)=\int v(t)\,dt=\int(2At+3Bt^2+v_0)\,dt=At^2+Bt^3+v_0t+K.$$

Since $x(0)=x_0$, we have $x_0=0+K$ so that $K=x_0$ and

$$x(t)=x_0+v_0t+At^2+Bt^3.$$

46. $v(t)=\int\cos t\,dt=\sin t+C, \quad v(0)=v_0 \implies v(t)=\sin t+v_0$

$x(t)=\int(\sin t+v_0)\,dt=-\cos t+v_0t+K, \quad x(0)=x_0 \implies x(t)=x_0+1+v_0t-\cos t$

47.

$$x'(t)=t^2-5, \qquad y'(t)=3t,$$

$$x(t)=\tfrac{1}{3}t^3-5t+C. \qquad y(t)=\tfrac{3}{2}t^2+K.$$

When $t=2$, the particle is at (4, 2). Thus, $x(2)=4$ and $y(2)=2$.

$$4=\tfrac{8}{3}-10+C \implies C=\tfrac{34}{3}. \qquad 2=6+K \implies K=-4.$$

$$x(t)=\tfrac{1}{3}t^3-5t+\tfrac{34}{3}, \qquad y(t)=\tfrac{3}{2}t^2-4.$$

Four seconds later the particle is at $(x(6),\,y(6))=(\tfrac{160}{3},\,50)$.

48. $x(t)=\int(t-2)\,dt=\dfrac{t^2}{2}-2t+C, \quad x(4)=3 \implies x(t)=\dfrac{t^2}{2}-2t+3$

$y(t)=\int\sqrt{t}\,dt=\dfrac{2}{3}t^{3/2}+K, \quad y(4)=1 \implies y(t)=\dfrac{2}{3}t^{3/2}-\dfrac{13}{3}$

5 seconds later, $t=9$, so position is $(x(9),y(9))=(\tfrac{51}{2},\tfrac{41}{3})$.

49. Since $v(0)=2$, we have $2=A\cdot 0+B$ so that $B=2$. Therefore

$$x(t)=\int v(t)\,dt=\int(At+2)\,dt=\frac{1}{2}At^2+2t+C.$$

Since $x(2)=x(0)-1$, we have

$$2A+4+C=C-1 \quad \text{so that} \quad A=-\tfrac{5}{2}.$$

50. $x(t)=\int(At^2+1)\,dt=\dfrac{1}{3}At^3+t+C$

$x(1)-x(0)=(\dfrac{1}{3}A+1+C)-C=\dfrac{A}{3}+1=0 \implies A=-3$

Distance traveled $=\displaystyle\int_0^{1/\sqrt{3}}(1-3t^2)\,dt+\int_{1/\sqrt{3}}^1(3t^2-1)\,dt=\left[t-t^3\right]_0^{1/\sqrt{3}}+\left[t^3-t\right]_{1/\sqrt{3}}^1=\frac{4\sqrt{3}}{9}$

51. $x(t)=\int v(t)\,dt=\int\sin t\,dt=-\cos t+C$

Since $x(\pi/6)=0$, we have $\quad 0=-\dfrac{\sqrt{3}}{2}+C \quad$ so that $\quad C=\dfrac{\sqrt{3}}{2} \quad$ and $\quad x(t)=\dfrac{\sqrt{3}}{2}-\cos t.$

(a) At $t = 11\pi/6$ sec.

(b) We want to find the smallest $t_0 > \pi/6$ for which $x(t_0) = 0$ and $v(t_0) > 0$. We get $t_0 = 13\pi/6$ seconds.

52. $x(t) = \int \cos t\, dt = \sin t + C, \quad x\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} + C = 0 \implies x(t) = \sin t - \frac{1}{2}$

(a) $x(t) = 0$ at $t = \frac{5}{6}\pi$ sec.

(b) $x(t) = 0$ and $v(t) > 0 \implies t = \dfrac{13\pi}{6}$ sec

53. The mean-value theorem. With obvious notation

$$\frac{x(1/12) - x(0)}{1/12} = \frac{4}{1/12} = 48.$$

By the mean-value theorem there exists some time t_0 at which

$$x'(t_0) = \frac{x(1/12) - x(0)}{1/12}.$$

54. (Taking the direction of motion as positive, speed and velocity are the same.) Let v be the speed of the motorcycle at time 0, the time when the brakes are applied. The distance between the motorcycle and the haywagon t time units later is given by

$$d(t) = -\frac{1}{2}at^2 + (v_1 - v)t + s$$

[$v_1 t + s$ gives the position of the haywagon, $\frac{1}{2}at^2 + vt$ gives the position of the motorcycle]. Collision can be avoided only if the quadratic

$$d(t) = -\frac{1}{2}at^2 + (v_1 - v)t + s$$

remains positive. This can be true only if the discriminant of the quadratic,

$$B^2 - 4AC = (v_1 - v)^2 + 2as = (v - v_1)^2 + 2as$$

remains negative. Observe that

$$(v - v_1)^2 + 2as < 0 \quad \text{iff} \quad v < v_1 + \sqrt{2|a|s}$$

55. $$\frac{v'(t)}{[v(t)]^2} = 2 \implies -[v(t)]^{-1} = 2t - {v_0}^{-1}.$$

$$\implies [v(t)]^{-1} = {v_0}^{-1} - 2t \implies v(t) = \frac{1}{{v_0}^{-1} - 2t} = \frac{v_0}{1 - 2tv_0}.$$

SECTION 5.6

1. $\left\{\begin{array}{l} u = 2 - 3x \\ du = -3\,dx \end{array}\right\}; \quad \displaystyle\int \frac{dx}{(2-3x)^2} = \int (2-3x)^{-2}\,dx = -\frac{1}{3}\int u^{-2}\,du = \frac{1}{3}u^{-1} + C$

$$= \tfrac{1}{3}(2-3x)^{-1} + C$$

2. $\left\{\begin{array}{c} u = 2x+1 \\ du = 2\,dx \end{array}\right\}$; $\displaystyle\int \frac{dx}{\sqrt{2x+1}} = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \sqrt{u} + C = \sqrt{2x+1} + C$

3. $\left\{\begin{array}{c} u = 2x+1 \\ du = 2\,dx \end{array}\right\}$; $\displaystyle\int \sqrt{2x+1}\,dx = \int (2x+1)^{1/2}\,dx = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{3}u^{3/2} + C$

$$= \frac{1}{3}(2x+1)^{3/2} + C$$

4. $\left\{\begin{array}{c} u = ax+b \\ du = a\,dx \end{array}\right\}$; $\displaystyle\int \sqrt{ax+b} = \frac{1}{a}\int \sqrt{u}\,du = \frac{2}{3a}u^{3/2} + C$

$$= \frac{2}{3a}(ax+b)^{3/2} + C$$

5. $\left\{\begin{array}{c} u = ax+b \\ du = a\,dx \end{array}\right\}$; $\displaystyle\int (ax+b)^{3/4}\,dx = \frac{1}{a}\int u^{3/4}\,du = \frac{4}{7a}u^{7/4} + C$

$$= \frac{4}{7a}(ax+b)^{7/4} + C$$

6. $\left\{\begin{array}{c} u = ax^2+b \\ du = 2ax\,dx \end{array}\right\}$; $\displaystyle\int 2ax(ax^2+b)^4\,dx = \int u^4\,du = \frac{1}{5}u^5 + C$

$$= \frac{1}{5}(ax^2+b)^5 + C$$

7. $\left\{\begin{array}{c} u = 4t^2+9 \\ du = 8t\,dt \end{array}\right\}$; $\displaystyle\int \frac{t}{(4t^2+9)^2}\,dt = \frac{1}{8}\int \frac{du}{u^2} = -\frac{1}{8}u^{-1} + C = -\frac{1}{8}\left(4t^2+9\right)^{-1} + C$

8. $\left\{\begin{array}{c} u = t^2+1 \\ du = 2t\,dt \end{array}\right\}$; $\displaystyle\int \frac{3t}{(t^2-1)^2}\,dt = \frac{3}{2}\int \frac{du}{u^2} = -\frac{3}{2u} + C$

$$= \frac{-3}{2(t^2+1)} + C$$

9. $\left\{\begin{array}{c} u = 1+x^3 \\ du = 3x^2\,dx \end{array}\right\}$; $\displaystyle\int x^2\left(1+x^3\right)^{1/4}\,dx = \frac{1}{3}\int u^{1/4}\,du = \frac{4}{15}u^{5/4} + C$

$$= \frac{4}{15}(1+x^3)^{5/4} + C$$

10. $\left\{\begin{array}{l} u = a + bx^n \\ du = nbx^{n-1}\,dx \end{array}\right\};\quad \int x^{n-1}\sqrt{a+bx^n}\,dx = \frac{1}{bn}\int \sqrt{u}\,du = \frac{2}{3bn}u^{3/2} + C$

$$= \frac{2}{3bn}\left(a+bx^n\right)^{3/2} + C$$

11. $\left\{\begin{array}{l} u = 1 + s^2 \\ du = 2s\,ds \end{array}\right\};\quad \int \frac{s}{\left(1+s^2\right)^3}\,ds = \frac{1}{2}\int \frac{du}{u^3} = -\frac{1}{4}u^{-2} + C = -\frac{1}{4}(1+s^2)^{-2} + C$

12. $\left\{\begin{array}{l} u = 6 - 5s^2 \\ du = -10s\,ds \end{array}\right\};\quad \int \frac{2s}{\sqrt[3]{6-5s^2}}\,ds = -\frac{1}{5}\int \frac{1}{\sqrt[3]{u}}\,du = -\frac{3}{10}u^{2/3} + C$

$$= \frac{-3}{10}\left(6-5s^2\right)^{2/3} + C$$

13. $\left\{\begin{array}{l} u = x^2 + 1 \\ du = 2x\,dx \end{array}\right\};\quad \int \frac{x}{\sqrt{x^2+1}}\,dx = \int \left(x^2+1\right)^{-1/2}\,x\,dx = \frac{1}{2}\int u^{-1/2}\,du$

$$= u^{1/2} + C = \sqrt{x^2+1} + C$$

14. $\left\{\begin{array}{l} u = 1 - x^3 \\ du = -3x^2\,dx \end{array}\right\};\quad \int \frac{x^2}{\left(1-x^3\right)^{2/3}}\,dx = -\frac{1}{3}\int \frac{du}{u^{2/3}} = -u^{1/3} + C$

$$= -(1-x^3)^{1/3} + C$$

15. $\left\{\begin{array}{l} u = x^2 + 1 \\ du = 2x \end{array}\right\};\quad \int 5x\left(x^2+1\right)^{-3}\,dx = \frac{5}{2}\int u^{-3}\,du = -\frac{5}{4}u^{-2} + C$

$$= -\frac{5}{4}(x^2+1)^{-2} + C$$

16. $\left\{\begin{array}{l} u = 1 - x^4 \\ du = -4x^3\,dx \end{array}\right\};\quad \int 2x^3(1-x^4)^{-1/4}\,dx = -\frac{1}{2}\int u^{-1/4}\,du = -\frac{2}{3}u^{3/4} + C$

$$= -\frac{2}{3}(1-x^4)^{3/4} + C$$

17. $\left\{\begin{array}{l} u = x^{1/4} + 1 \\ du = \frac{1}{4}x^{-3/4}\,dx \end{array}\right\};\quad \int x^{-3/4}\left(x^{1/4}+1\right)^{-2}\,dx \quad = 4\int u^{-2}\,du = -4u^{-1} + C$

$$= -4(x^{1/4}+1)^{-1} + C$$

18. $\left\{\begin{array}{l} u = x^2 + 3x + 1 \\ du = (2x+3)\,dx \end{array}\right\};\quad \int \frac{4x+6}{\sqrt{x^3+3x+1}}\,dx = 2\int \frac{du}{\sqrt{u}} = 4\sqrt{u} + C$

$$= 4\sqrt{x^2+3x+1} + C$$

19. $\left\{\begin{array}{l} u = 1 - a^4x^4 \\ du = -4a^4x^3\,dx \end{array}\right\}$; $\displaystyle\int \frac{b^3x^3}{\sqrt{1-a^4x^4}}\,dx = -\frac{b^3}{4a^4}\int u^{-1/2}\,du = -\frac{b^3}{2a^4}u^{1/2} + C$

$$= -\frac{b^3}{2a^4}\sqrt{1-a^4x^4} + C$$

20. $\left\{\begin{array}{l} u = a + bx^n \\ du = bnx^{n-1}\,dx \end{array}\right\}$; $\displaystyle\int \frac{x^{n-1}}{\sqrt{a+bx^n}}\,dx = \frac{1}{bn}\int \frac{du}{\sqrt{u}} = \frac{2}{bn}\sqrt{u} + C$

$$= \frac{2}{bn}\sqrt{a+bx^n} + C$$

21 $\left\{\begin{array}{l|l} u = x^2+1 & x = 0 \implies u = 1 \\ du = 2x\,dx & x = 1 \implies u = 2 \end{array}\right\}$; $\displaystyle\int_0^1 x\left(x^2+1\right)^3\,dx = \frac{1}{2}\int_1^2 u^3\,du$

$$= \frac{1}{8}\left[u^4\right]_1^2 = \frac{15}{8}$$

22. $\left\{\begin{array}{l|l} u = 4+2x^3 & x = -1 \implies u = 2 \\ du = 6x^2\,dx & x = 0 \implies u = 4 \end{array}\right\}$; $\displaystyle\int_{-1}^0 3x^2\left(4+2x^3\right)^2\,dx = \frac{1}{2}\int_2^4 u^2\,du$

$$= \left[\frac{1}{6}u^3\right]_2^4 = \frac{28}{3}$$

23. 0; integrand is an odd function

24. $\left\{\begin{array}{l|l} u = r^2+16 & r = 0 \implies u = 16 \\ du = 2r\,dr & r = 3 \implies u = 25 \end{array}\right\}$; $\displaystyle\int_0^3 \frac{r}{\sqrt{r^2+16}}\,dr = \frac{1}{2}\int_{16}^{25}\frac{du}{\sqrt{u}}$

$$= \left[\sqrt{u}\right]_{16}^{25} = 1$$

25. $\left\{\begin{array}{l|l} u = a^2-y^2 & y = 0 \implies u = a^2 \\ du = -2y\,dy & y = a \implies u = 0 \end{array}\right\}$; $\displaystyle\int_0^a y\sqrt{a^2-y^2}\,dy = -\frac{1}{2}\int_{a^2}^0 u^{1/2}\,du$

$$= -\tfrac{1}{3}\left[u^{3/2}\right]_{a^2}^0 = \tfrac{1}{3}\left(a^2\right)^{3/2} = \tfrac{1}{3}|a|^3$$

26.

$\left\{\begin{array}{l|l} u = 1 - \dfrac{y^3}{a^3} & y = -a \implies u = 2 \\ du = -\dfrac{3y^2}{a^3}\,dy & y = 0 \implies u = 1 \end{array}\right\}$; $\displaystyle\int_{-a}^0 y^2\left(1-\frac{y^3}{a^3}\right)^{-2}\,dy = -\frac{a^3}{3}\int_2^1 u^{-2}\,du$

$$= -\frac{a^3}{3}\left[-u^{-1}\right]_2^1 = \frac{a^3}{6}$$

27.

$\left\{\begin{array}{l|l} u = 2x^2+1 & x = 0 \implies u = 1 \\ du = 4xdx & x = 2 \implies u = 9 \end{array}\right\}$; $\displaystyle\int_0^2 x\sqrt{2x^2+1}\,dx = \frac{1}{4}\int_1^9 \sqrt{u}\,du$

$$= \left[\frac{1}{6}u^{\frac{3}{2}}\right]_1^9 = \frac{13}{3}$$

28.

$$\left\{\begin{array}{l|l} u = 2x^2+1 & x=0 \implies u=1 \\ du = 4x\,dx & x=2 \implies u=9 \end{array}\right\}; \qquad \int_0^2 \frac{x}{(2x^2+1)^2}\,dx = \frac{1}{4}\int_1^9 u^{-2}\,du$$

$$= \left[-\frac{1}{4u}\right]_1^9 = \frac{2}{9}$$

29.

$$\left\{\begin{array}{l|l} u = 1+x^{-2} & x=1 \implies u=2 \\ du = -2x^{-3}dx & x=2 \implies u=\frac{5}{4} \end{array}\right\}; \qquad \int_1^2 x^{-3}(1+x^{-2})^{-3}\,dx = -\frac{1}{2}\int_2^{\frac{5}{4}} u^{-3}\,du$$

$$= \left[\frac{u^{-2}}{4}\right]_2^{\frac{5}{4}} = \frac{39}{400}$$

30.

$$\left\{\begin{array}{l|l} u = (x+2)(x+3) & x=0 \implies u=6 \\ du = (2x+5)dx & x=1 \implies u=12 \end{array}\right\}; \qquad \int_0^1 \frac{2x+5}{(x+2)^2(x+3)^2}\,dx = \int_6^{12} \frac{1}{u^2}\,du$$

$$= \left[-\frac{1}{u}\right]_6^{12} = \frac{1}{12}$$

31.

$$\left\{\begin{array}{l} u = x+1 \\ du = dx \end{array}\right\}; \qquad \int x\sqrt{x+1}\,dx = \int (u-1)\sqrt{u}\,du = \int \left(u^{3/2} - u^{1/2}\right)du$$

$$= \tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2} + C = \tfrac{2}{5}(x+1)^{5/2} - \tfrac{2}{3}(x+1)^{3/2} + C$$

32.

$$\left\{\begin{array}{l} u = x-1 \\ du = dx \end{array}\right\}; \qquad \int 2x\sqrt{x-1}\,dx = \int 2(u+1)\sqrt{u}\,du = 2\int \left(u^{3/2} + u^{1/2}\right)du$$

$$= \frac{4}{5}u^{5/2} + \frac{4}{3}u^{3/2} + C = \frac{4}{5}(x-1)^{5/2} + \frac{4}{3}(x-1)^{3/2} + C$$

33.

$$\left\{\begin{array}{l} u = 2x-1 \\ du = dx \end{array}\right\}; \qquad \int x\sqrt{2x-1}\,dx = \frac{1}{2}\int \frac{(u-1)}{2}\sqrt{u}\,du = \frac{1}{4}\int \left(u^{3/2} + u^{1/2}\right)du$$

$$= \frac{1}{10}u^{5/2} + \frac{1}{6}u^{3/2} + C = \frac{1}{10}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{3/2} + C$$

34.

$$\left\{\begin{array}{l} u = 2t+3 \\ du = 2\,dt \end{array}\right\}; \qquad \int t(2t+3)^8\,dt = \frac{1}{2}\int \frac{1}{2}(u-3)^8u^8\,du = \frac{1}{4}\int \left(u^9 - 3u^8\right)du$$

$$= \frac{1}{40}u^{10} - \frac{1}{12}u^9 + C = \frac{1}{40}(2t+3)^{10} - \frac{1}{12}(2t+3)^9 + C$$

35.

$$\left\{\begin{array}{l|l} u = x+1 & x=0 \implies u=1 \\ du = dx & x=1 \implies u=2 \end{array}\right\}; \qquad \int_0^1 \frac{x+3}{\sqrt{x+1}}\,dx = \int_1^2 \frac{u+2}{\sqrt{u}}\,du$$

$$= \int_1^2 \left(u^{1/2} + 2u^{-1/2}\right)du$$

$$= \left[\tfrac{2}{3}u^{3/2} + 4u^{1/2}\right]_1^2 = \tfrac{2}{3}\sqrt{8} + 4\sqrt{2} - \tfrac{2}{3} - 4 = \tfrac{16}{3}\sqrt{2} - \tfrac{14}{3}$$

36. $\left\{\begin{array}{l|l} u = x^2+1 & x=-1 \implies u=2 \\ du = 2x\,dx & x=0 \implies u=1 \end{array}\right\}$; $\displaystyle\int_{-1}^{0} x^3\left(x^2+1\right)^6 dx = \frac{1}{2}\int_2^1 (u-1)u^6\,du$

$$= \frac{1}{2}\int_2^1 \left(u^7-u^6\right) du = \left[\frac{1}{16}u^8 - \frac{1}{14}u^7\right]_2^1 = -\frac{255}{16}+\frac{127}{14} = -\frac{769}{112}$$

37.

$\left\{\begin{array}{c} u = x^2+1 \\ du = 2xdx \end{array}\right\}$; $\displaystyle\int x\sqrt{x^2+1}\,dx = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{3}u^{\frac{3}{2}} + C = \frac{1}{3}(x^2+1)^{\frac{3}{2}} + C.$

Also, $1 = \frac{1}{3}(0^2+1)+C \implies C = \frac{2}{3}$. Thus $y = \frac{1}{3}(x^2+1)^{\frac{3}{2}} + \frac{2}{3}$.

38.

$\left\{\begin{array}{c} u = 1+\sqrt{x} \\ du = \dfrac{1}{2\sqrt{x}}dx \end{array}\right\}$; $\displaystyle -\int \frac{1}{2\sqrt{x}(1+\sqrt{x})^2}\,dx = -\int u^{-2}\,du = \frac{1}{u}+C = \frac{1}{1+\sqrt{x}}+C.$

Also, $\frac{1}{3} = \frac{1}{1+\sqrt{4}} + C \implies C = 0$ Thus $y = \dfrac{1}{1+\sqrt{x}}$

39. $\displaystyle\int \cos(3x+1)\,dx = -frac13\sin(3x+1)+C$

40. $\displaystyle\int \sin 2\pi x\,dx = -\frac{1}{2\pi}\cos 2\pi x + C$

41. $\displaystyle\int \csc^2 \pi x\,dx = -\frac{1}{\pi}\cot \pi x + C$

42. $\displaystyle\int \sec 2x \tan 2x\,dx = \frac{1}{2}\sec 2x + C$

43. $\left\{\begin{array}{c} u = 3-2x \\ du = -2\,dx \end{array}\right\}$; $\displaystyle\int \sin(3-2x)\,dx = \int -\frac{1}{2}\sin u\,du = \frac{1}{2}\cos u + C = \frac{1}{2}\cos(3-2x)+C$

44. $\left\{\begin{array}{c} u = \sin x \\ du = \cos x\,dx \end{array}\right\}$; $\displaystyle\int \sin^2 x\cos x\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3 x + C$

45. $\left\{\begin{array}{c} u = \cos x \\ du = -\sin x\,dx \end{array}\right\}$; $\displaystyle\int \cos^4 x\sin x\,dx = \int -u^4\,du = -\frac{1}{5}u^5 + C = -\frac{1}{5}\cos^5 x + C$

46. $\left\{\begin{array}{c} u = x^2 \\ du = 2x\,dx \end{array}\right\}$; $\displaystyle\int x\sec^2 x^2\,dx = \frac{1}{2}\int \sec^2 u\,du = \frac{1}{2}\tan u + C = \frac{1}{2}\tan x^2 + C$

47. $\left\{\begin{array}{c} u = x^{1/2} \\ du = \frac{1}{2}x^{-1/2}\,dx \end{array}\right\}$; $\displaystyle\int x^{-1/2}\sin x^{1/2}\,dx = \int 2\sin u\,du = -2\cos u + C$

$$= -2\cos x^{1/2} + C$$

48. $\left\{\begin{array}{c} u = 1-2x \\ du = -2\,dx \end{array}\right\};\quad \int \csc(1-2x)\cot(1-2x)\,dx = -\frac{1}{2}\int \csc u \cot u\,du = \frac{1}{2}\csc u + C$

$$= \frac{1}{2}\csc(1-2x) + C$$

49. $\left\{\begin{array}{c} u = 1+\sin x \\ du = \cos x\,dx \end{array}\right\};\quad \int \sqrt{1+\sin x}\;\cos x\,dx = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C$

$$= \tfrac{2}{3}(1+\sin x)^{3/2} + C$$

50. $\left\{\begin{array}{c} u = 1+\cos x \\ du = -\sin x\,dx \end{array}\right\};\quad \int \frac{\sin x}{\sqrt{1+\cos x}}\,dx = -\int \frac{du}{\sqrt{u}} = -2\sqrt{u} + C = -2\sqrt{1+\cos x} + C$

51. $\int \frac{1}{\cos^2 x}\,dx = \int \sec^2 x\,dx = \tan x + C$

52. $\int (1+\tan^2 x)\sec^2 x\,dx = \int \sec^2 x\,dx + \int \tan^2 x \sec^2 x\,dx$

$= \tan x + \int u^2\,du = \tan x + \frac{1}{3}u^3 + C = \tan x + \frac{1}{3}\tan^3 x + C$

53. $\left\{\begin{array}{c} u = x^2 \\ du = 2x\,dx \end{array}\right\};\quad \int x\sin^3 x^2 \cos x^2\,dx = \int \frac{1}{2}\sin^3 u \cos u\,du = \frac{1}{8}\sin^4 u + C$

$$= \tfrac{1}{8}\sin^4 x^2 + C$$

54. $\int \frac{1}{\sin^2 x}\,dx = \int \csc^2 x\,dx = -\cot x + C$

55. $\left\{\begin{array}{c} u = 1+\tan x \\ du = \sec^2 x\,dx \end{array}\right\};\quad \int \frac{\sec^2 x}{\sqrt{1+\tan x}}\,dx = \int u^{-1/2}\,du = 2u^{1/2} + C$

$$= 2(1+\tan x)^{1/2} + C$$

56.

$\left\{\begin{array}{c} u = 4x^3+7 \\ du = 12x^2\,dx \end{array}\right\};\quad \int x^2 \sin\left(4x^3-7\right)\,dx = \frac{1}{12}\int \sin u\,du = -\frac{1}{12}\cos u + C$

$$= -\frac{1}{12}\cos(4x^3-7) + C$$

57. 0; the sine is an odd function

58. $\int_{-\pi/3}^{\pi/3} \sec x \tan x\,dx = \left[\sec x\right]_{-\pi/3}^{\pi/3} = 0$

59. $\int_{1/4}^{1/3} \sec^2 \pi x\,dx = \frac{1}{\pi}\left[\tan \pi x\right]_{1/4}^{1/3} = \frac{1}{\pi}(\sqrt{3}-1)$

60. $\int_0^1 \cos^2\left(\frac{\pi}{2}x\right)\sin\left(\pi 2\right)\,dx = \frac{-2}{3\pi}\left[\cos^3 \frac{\pi}{2}x\right]_1^0 = \frac{2}{3\pi}$

61. $\displaystyle\int_0^{\pi/2} \sin^3 x \cos x\, dx = \frac{1}{4}\left[\sin^4 x\right]_0^{\pi/2} = \frac{1}{4}$

62. $\displaystyle\int_0^{\pi} x \cos x^2\, dx = \frac{1}{2}\left[\sin x^2\right]_0^{\pi} = \frac{1}{2}\sin \pi^2$

63. $\displaystyle\int \sin^2 x\, dx = \int \frac{1-\cos 2x}{2}\, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

64. $\displaystyle\int \cos^2\, dx = \int \frac{1+\cos 2x}{2}\, dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$

65. $\displaystyle\int \cos^2 5x\, dx = \int \frac{1+\cos 10x}{2}\, dx = \frac{1}{2}x + \frac{1}{20}\sin 10x + C$

66. $\displaystyle\int \sin^2 3x\, dx = \int \frac{1-\cos 6x}{2}\, dx = \frac{1}{2}x - \frac{1}{12}\sin 6x + C$

67. $\displaystyle\int_0^{\pi/2} \cos^2 2x\, dx = \int_0^{\pi/2} \frac{1+\cos 4x}{2}\, dx = \left[\frac{1}{2}x + \frac{1}{8}\sin 4x\right]_0^{\pi/2} = \frac{\pi}{4}$

68. $\displaystyle\int_0^{2\pi} \sin^2 x\, dx = \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{2\pi} = \pi$

69. $\displaystyle A = \int_0^{\frac{\pi}{2}} [\cos x - (-\sin x)]\, dx = [\sin x - \cos x]_0^{\frac{\pi}{2}} = 2$

70. $\displaystyle A = \int_0^{1/4} (\cos \pi x - \sin \pi x)\, dx = \frac{1}{\pi}\left[\sin \pi x + \cos \pi x\right]_0^{1/4} = \frac{1}{\pi}(\sqrt{2}-1)$

71. $\displaystyle A = \int_0^{1/4} \left(\cos^2 \pi x - \sin^2 \pi x\right)\, dx = \int_0^{1/4} \cos 2\pi x\, dx == \frac{1}{2\pi}\left[\sin 2\pi x\right]_0^{1/4} = \frac{1}{2\pi}$

72. $\displaystyle\int_0^{1/4} (\cos^2 \pi x + \sin^2 \pi x)\, dx = \int_0^{1/4} 1\, dx = [x]_0^{1/4} = \frac{1}{4}$

73.

$$\begin{aligned} A = \int_{1/6}^{1/4} \left(\csc^2 \pi x - \sec^2 \pi x\right)\, dx &= \frac{1}{\pi}\left[-\cot \pi x - \tan \pi x\right]_{1/6}^{1/4} \\ &= \frac{1}{\pi}\left(-2 + \cot\frac{\pi}{6} + \tan\frac{\pi}{6}\right) \\ &= \frac{1}{\pi}\left(-2 + \sqrt{3} + \frac{1}{\sqrt{3}}\right) = \frac{1}{3\pi}(4\sqrt{3}-6) \end{aligned}$$

74. $\displaystyle\int \sin x \cos x = \int -u\, du = -\frac{1}{2}u^2 + K = -\frac{1}{2}\cos^2 x + K, \quad (u = \cos x)$

$\displaystyle\frac{1}{2}\sin^2 x + C = -\frac{1}{2}\cos^2 x + K \quad \text{where} \quad K = C + \tfrac{1}{2}$

75. (a) $\left\{\begin{array}{c} u = \sec x \\ du = \sec x \tan x\, dx \end{array}\right\};\quad \int \sec^2 x \tan x\, dx = \int u\, du = \frac{1}{2}u + C$

$$= \tfrac{1}{2}\sec^2 x + C$$

(b) $\left\{\begin{array}{c} u = \tan x \\ du = \sec^2 x\, dx \end{array}\right\};\quad \int \sec^2 x \tan x\, dx = \int u\, du = \frac{1}{2}u^2 + C'$

$$= \tfrac{1}{2}\tan^2 x + C'$$

(c) $C' = C + \frac{1}{2}$

76.
$$A = 4\int_0^r \sqrt{r^2 - x^2}\, dx = 4\int_0^{\pi/2} \sqrt{r^2 - r^2 \sin^2 u}(r\cos u)\, du \quad (x = r\sin u)$$
$$= 4\int_0^{\pi/2} r^2 \cos^2 u\, du = 4r^2\left[\frac{1}{2}u + \frac{1}{4}\sin 2u\right]_0^{\pi/2} = \pi r^2$$

77.
$$A = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\, dx = \frac{4b}{a}\left(\frac{\text{area of circle of radius } a}{4}\right)$$
$$= \frac{4b}{a}\left(\frac{\pi a^2}{4}\right) = \pi ab$$

78. (a)

$$\left\{\begin{array}{c|c} u = x - c & x = a + c \implies u = a \\ du = dx & x = b + c \implies u = b \end{array}\right\};\quad \int_{a+c}^{b+c} f(x - c)\, dx = \int_a^b f(u)\, du = \int_a^b f(x)\, dx$$

(b)

$$\left\{\begin{array}{c|c} u = \dfrac{x}{c} & x = ac \implies u = a \\ du = \dfrac{dx}{c} & x = bc \implies u = b \end{array}\right\};\quad \int_{\frac{x}{c}}^{\frac{b}{c}} f(\frac{x}{c})\, dx = \int_a^b f(u)\, du = \int_a^b f(x)\, dx$$

SECTION 5.7

1. Yes; $\int_a^b [f(x) - g(x)]\, dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx > 0.$

2. No; take, for example, the function $f(x) = x$ and $g(x) = 0$ on $[-\frac{1}{2}, 1]$.

3. Yes; otherwise we would have $f(x) \le g(x)$ for all $x \in [a,\, b]$ and it would follow that

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

4. No; take, for example, the function $f(x) = 0$ and $g(x) = -1$ on $[0, 1]$.

5. No; take $f(x) = 0$, $g(x) = -1$ on $[0, 1]$.

6. Yes; $\int_a^b |f(x)|\,dx \ge \int_a^b f(x)\,dx$ and we are assuming that $\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$.

7. No; take, for example, any odd function on an interval of the form $[-c, c]$.

8. Yes; if $f(x) \neq 0$ for each $x \in [a, b]$, then by continuity either $f(x) > 0$ for all $x \in [a, b]$, or $f(x) < 0$ for all $x \in [a, b]$. In either case

$$\int_a^b f(x)\,dx \neq 0$$

9. No; $\int_{-1}^1 x\,dx = 0$ but $\int_{-1}^1 |x|\,dx \neq 0$.

10. Yes; $\left|\int_a^b f(x)\,dx\right| = |0| = 0$

11. Yes; $U_f(P) \ge \int_a^b f(x)\,dx = 0.$

12. No; if $f(x) = 0$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\,dx = 0, \quad \text{and} \quad U_f(P) = 0 \text{ for all } P.$$

13. No; $L_f(P) \le \int_a^b f(x)\,dx = 0.$

14. No; take $f(x) = x$ on $[-1.1]$;

$$\int_{-1}^1 x\,dx = 0 \text{ but } \int_{-1}^1 x^2\,dx = \frac{2}{3}$$

15. Yes; $\int_a^b [f(x) + 1]\,dx = \int_a^b f(x)\,dx + \int_a^b 1\,dx = 0 + b - a = b - a.$

16. $\frac{d}{dx}\left[\int_u^b f(t)\,dt\right] = \frac{d}{du}\left[\int_u^b f(t)\,dt\right]\frac{du}{dx} = \frac{d}{du}\left[-\int_b^u f(t)\,dt\right]\frac{du}{dx} = -f(u)\frac{du}{dx}.$

17. $\frac{d}{dx}\left[\int_0^{1+x^2} \frac{dt}{\sqrt{2t+5}}\right] = \frac{1}{\sqrt{2(1+x^2)+5}}\,\frac{d}{dx}\left(1+x^2\right) = \frac{2x}{\sqrt{2x^2+7}}$

18. $\frac{d}{dx}\left[\int_1^{x^2} \frac{dt}{t}\right] = \frac{1}{x^2}\,2x = \frac{2}{x}.$

19. $\dfrac{d}{dx}\left[\int_x^a f(t)\,dt\right] = \dfrac{d}{dx}\left[-\int_a^x f(t)\,dt\right] = -f(x)$

20. $\dfrac{d}{dx}\left[\int_0^{x^3} \dfrac{dt}{\sqrt{1+t^2}}\right] = \dfrac{3x^2}{\sqrt{1+x^6}}.$

21. $\dfrac{d}{dx}\left[\int_{x^2}^3 \dfrac{\sin t}{t}\,dt\right] = -\dfrac{d}{dx}\left[\int_3^{x^2} \dfrac{\sin t}{t}\,dt\right] = -\dfrac{\sin(x^2)}{x^2}(2x) = -\dfrac{2\sin(x^2)}{x}$

22. $\dfrac{d}{dx}\left[\int_{\tan x}^4 \sin(t^2)\,dt\right] = -\sin\left(\tan^2 x\right)\sec^2 x.$

23. $\dfrac{d}{dx}\left[\int_1^{\sqrt{x}} \dfrac{t^2}{1+t^2}\,dt\right] = \dfrac{x}{1+x}\cdot\dfrac{1}{2\sqrt{x}} = \dfrac{\sqrt{x}}{2(1+x)}$

24. $\dfrac{d}{dx}\left[\int_u^v f(t)\,dt\right] = \dfrac{d}{dx}\left[\int_a^v f(t)\,dt - \int_a^u f(t)\,dt\right] = f(v)\dfrac{dv}{dx} - f(u)\dfrac{du}{dx}.$

25. $\dfrac{d}{dx}\left[\int_x^{x^2} \dfrac{dt}{t}\right] = \dfrac{1}{x^2}\dfrac{d}{dx}\left(x^2\right) - \dfrac{1}{x}\dfrac{d}{dx}(x) = \dfrac{2x}{x^2} - \dfrac{1}{x} = \dfrac{1}{x}$

26. $\dfrac{d}{dx}\left[\int_{\sqrt{x}}^{x^2+x} \dfrac{dt}{2+\sqrt{t}}\right] = \dfrac{1}{2+\sqrt{x^2+x}}(2x+1) - \dfrac{1}{2+\sqrt[4]{x}}\cdot\dfrac{1}{2\sqrt{x}}$

27. $\dfrac{d}{dx}\left[\int_{\tan x}^{2x} t\sqrt{1+t^2}\,dt\right] = 2x\sqrt{1+(2x)^2}\,(2) - \tan x\,\sqrt{1+\tan^2 x}\,(\sec^2 x)$

$$= 4x\sqrt{1+4x^2} - \tan x\ \sec^2 x\ |\sec x|$$

28. $\dfrac{d}{dx}\left[\int_{3x}^{1/x} \cos 2t\,dt\right] = \cos\left(\dfrac{2}{x}\right)\left(\dfrac{-1}{x^2}\right) - \cos 6x\,(3) = -\dfrac{\cos(2/x)}{x^2} - 3\cos 6x$

29. (a) With P a partition of $[a,\,b]$

$$L_f(P) \le \int_a^b f(x)\,dx.$$

If f is nonnegative on $[a,\,b]$, then $L_f(P)$ is nonnegative and, consequently, so is the integral. If f is positive on $[a,\,b]$, then $L_f(P)$ is positive and, consequently, so is the integral.

(b) Take F as an antiderivative of f on $[a,\,b]$. Observe that

$$F'(x) = f(x) \text{ on } (a,\,b) \quad\text{and}\quad \int_a^b f(x)\,dx = F(b) - F(a).$$

If $f(x) \ge 0$ on $[a,\,b]$, then F is nondecreasing on $[a,\,b]$ and $F(b) - F(a) \ge 0$.

If $f(x) > 0$ on $[a,\,b]$, then F is increasing on $[a,\,b]$ and $F(b) - F(a) > 0$.

30. Set $h(x) = g(x) - f(x)$ and apply Property I to h.

31. Consider the trivial partition P of $[a, b]$ into the single interval $[a, b]$.

Then $L_f(P) = m(b-a)$ and $U_f(P) = M(b-a)$.

Thus $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

32. Suppose $f(c) > 0$ for some $c \in (a, b)$. Then by Exercise 48, Section 2.4, there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in (c-\delta, c+\delta)$. Also, we can choose δ such that $(c-\delta, c+\delta) \subset (a, b)$.

Then $\displaystyle\int_a^b |f(x)|\,dx \geq \int_{c-\delta}^{c+\delta} |f(x)| dx > 0$, a contradiction. The same holds if $f(c) < 0$ for some c.

Thus $f(x) = 0$ for all $x \in (a, b)$. Then since f is continuous on $[a, b]$, we must have $f(a) = f(b) = 0$, so $f(x) = 0$ for all $x \in [a, b]$.

33.
$$H(x) = \int_{2x}^{x^3-4} \frac{x\,dt}{1+\sqrt{t}} = x\int_{2x}^{x^3-4} \frac{dt}{1+\sqrt{t}},$$

$$H'(x) = x \cdot \left[\frac{3x^2}{1+\sqrt{x^3-4}} - \frac{2}{1+\sqrt{2x}}\right] + 1 \cdot \int_{2x}^{x^3-4} \frac{dt}{1+\sqrt{t}},$$

$$H'(2) = 2\left[\frac{12}{3} - \frac{2}{3}\right] + \underbrace{\int_4^4 \frac{dt}{1+\sqrt{t}}}_{=0} = \frac{20}{3}$$

34.
$$H(x) = \frac{1}{x}\int_3^x [2t - 3H'(t)]\,dt,$$

$$H'(x) = \frac{-1}{x^2}\int_3^x [2t - 3H'(t)]\,dt + \frac{1}{x}[2x - 3H'(t)],$$

$$H'(2) = \frac{-1}{3^2}\int_3^3 [2t - 3H'(t)]\,dt + \frac{1}{3}[2 \cdot 3 - 3H'(3)]$$

$$= \frac{-1}{3^2} \cdot 0 + 2 - H'(3) \implies H'(3) = 1.$$

35. (a) Let $u = -x$. Then $du = -\,dx$; and $u = 0$ when $x = 0$, $u = a$ when $x = -a$.

$$\int_{-a}^0 f(x)\,dx = -\int_a^0 f(-u)\,du = \int_0^a f(-u)\,du = \int_0^a f(-x)\,dx$$

36. (a) $\displaystyle\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx$

In first integral, use $u = -x, du = -dx, u(-a) = a, u(0) = 0, x = -u$, and note that

$f(x) = f(-u) = -f(u)$ since f is odd. Then

$$\int_{-a}^0 f(x)\,dx = -\int_a^0 f(-u)\,du = \int_0^a f(-u)\,du = -\int_0^a f(u)\,du$$

So $\displaystyle\int_{-a}^a f(x)\,dx = -\int_0^a f(u)\,du + \int_0^a f(x)\,dx = 0$

(b) As above, but now $f(x) = f(-u) = f(u)$ since f is even, so

$$\int_{-a}^{0} f(x)\,dx = -\int_{a}^{0} f(-u)\,du = \int_{0}^{a} f(-u)\,du = \int_{0}^{a} f(u)\,du,$$

hence $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

37. $\int_{-\pi/4}^{\pi/4} (x + \sin 2x)\,dx = 0$ since $f(x) = x + \sin 2x$ is an odd function.

38. $\frac{t^3}{1+t^2}$ is an odd function, so $\int_{-3}^{3} \frac{t^3}{1+t^2}\,dt = 0$

39. $\int_{-\pi/3}^{\pi/3} (1+x^2-\cos x)\,dx = 2\int_{0}^{\pi/3} (1+x^2-\cos x)\,dx$ since $f(x) = 1+x^2-\cos x$ is an even function.

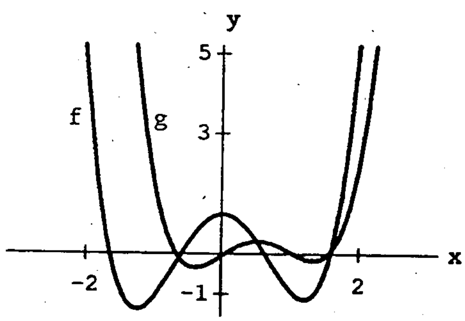
$$2\int_{0}^{\pi/3} (1+x^2-\cos x)\,dx = 2\left[x + \frac{1}{3}x^3 - \sin x\right]_0^{\pi/3} = \frac{2}{3}\pi + \frac{2}{81}\pi^3 - \sqrt{3}$$

40. $2x$ and $\sin x$ are odd, and x^2 and $\cos 2x$ are even, so

$$\int_{-\pi/4}^{\pi/4} (x^2 - 2x + \sin x + \cos 2x)\,dx = 2\int_{0}^{\pi/4} (x^2 + \cos 2x)\,dx = 2\left[\frac{x^3}{3} + \frac{1}{2}\sin 2x\right]_0^{\pi/4} = \frac{\pi^3}{96} + 1$$

PROJECT 5.7

1. (a) Let $f(x) = x^4 - 3x^2 + 0.1x + 1$.



(b) $\text{dom}(f) = \text{dom}(g)$. A comparson of ranges is:

range $(f) = [-1.37, \infty)$

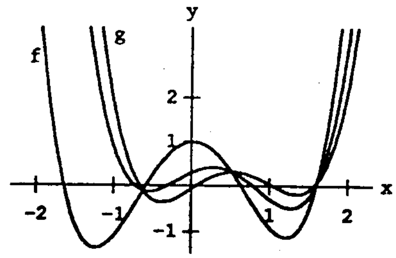
range $(g) = [-0.34, \infty)$

(c) Setting $g(x) = f(x)$ gives

$$\frac{f(x-1) + f(x)}{2} = f(x)$$

which implies $f(x-1) = f(x)$. This occurs at the points where $x \cong -0.6280, 0.5200, 1.6081$.

2. (a) $f(x) = x^4 - 3x^2 + 0.1x + 1$



(b) g and h have the same domain as f. A comparson of ranges is:

range $(f) = [-1.37, \infty)$

range $(h) = [-0.47, \infty)$

(c) The graphs of f, g, and h all intersect at the same points.

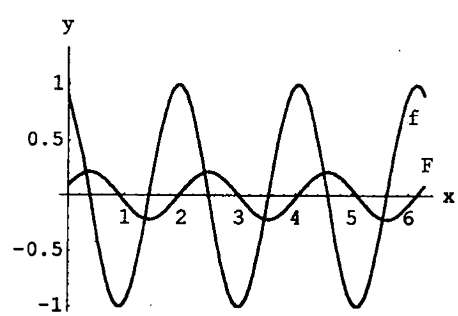
(d) They will all intersect at the same points. Proof:

Let $m(x) = \dfrac{f(x-1) + kf(x)}{k+1}$ be a moving average for f.

Then $f(x-1) + kf(x) = (k+1)f(x)$, so we get $f(x-1) = f(x)$.

i.e. We get the same points of intersection as in Project 1(c).

3. (a)



The moving average g smooths everything; the moving average F still oscillates, but it oscillates "slower" and with less amplitude.

(b) In general, $F'(x) = \dfrac{1}{a}[f(x) - f(x-a)]$ and

$$G(x) = \frac{1}{a}\int_{x-a}^{x} f'(t)dt = \frac{1}{a}[f(x) - f(x-a)] = F'(x)$$

SECTION 5.8

1. A.V. $= \frac{1}{c}\int_0^c (mx+b)\,dx = \frac{1}{c}\left[\frac{m}{2}x^2 + bx\right]_0^c = \frac{mc}{2} + b;$ at $x = c/2$

2. A.V. $= \frac{1}{2}\int_{-1}^{1} x^2\,dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3};$ at $x = \pm\frac{\sqrt{3}}{3}.$

3. A.V. $= \frac{1}{2}\int_{-1}^{1} x^3\,dx = 0$ since the integrand is odd; at $x = 0$

4. A.V. $= \frac{1}{3}\int_1^4 x^{-2}\,dx = \frac{1}{3}\left[-\frac{1}{x}\right]_1^4 = \frac{1}{3}\cdot\frac{3}{4} = \frac{1}{4};$ at $x = 2.$

5. A.V. $= \frac{1}{4}\int_{-2}^{2} |x|\,dx = \frac{1}{2}\int_0^2 |x|\,dx = \frac{1}{2}\int_0^2 x\,dx = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2 = 1;$ at $x = \pm 1$

6. A.V. $= \frac{1}{16}\int_{-8}^{8} x^{1/3}\,dx = 0$ (odd function); at; $x = 0$

7. A.V. $= \frac{1}{2}\int_0^2 \left(2x - x^2\right)\,dx = \frac{1}{2}\left[x^2 - \frac{x^3}{3}\right]_0^2 = \frac{2}{3};$ at $x = 1 \pm \frac{1}{3}\sqrt{3}$

8. A.V. $= \frac{1}{3}\int_0^3 (3-2x)\,dx = \frac{1}{3}\left[3x - x^2\right]_0^3 = 0;$ at $x = \frac{3}{2}.$

9. A.V. $= \frac{1}{9}\int_0^9 \sqrt{x}\,dx = \frac{1}{9}\left[\frac{2}{3}x^{3/2}\right]_0^9 = 2;$ at $x = 4$

10. A.V. $= \frac{1}{4}\int_{-2}^{2} (4-x^2)\,dx = \frac{1}{4}\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{8}{3};$ at $x = \pm\frac{2\sqrt{3}}{3}.$

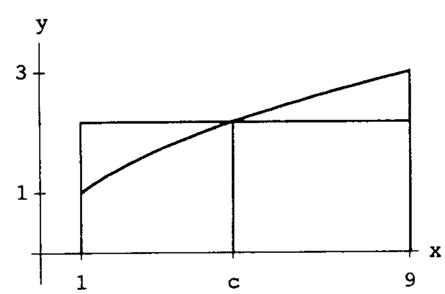
11. A.V. $= \frac{1}{2\pi}\int_0^{2\pi} \sin x\,dx = \frac{1}{2\pi}\left[-\cos x\right]_0^{2\pi} = 0;$ at $x = 0,\ \pi,\ 2\pi$

12. A.V. $= \frac{1}{\pi}\int_0^{\pi} \cos x\,dx = \frac{1}{\pi}\left[\sin x\right]_0^{\pi} = 0;$ at $x = \frac{\pi}{2}.$

13. (a) A.V.$= \frac{1}{8}\int_1^9 \sqrt{x}\,dx = \frac{1}{8}\left[\frac{2}{3}\,x^{3/2}\right]_1^9 = \frac{13}{6}$

(b) $\sqrt{x} = \frac{13}{6} \Longrightarrow x = 4.694$

(c)



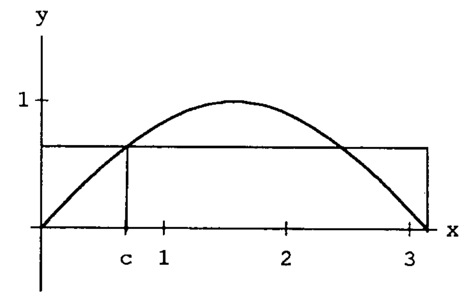
14. (a) A.V.$= \frac{1}{3}\int_{-1}^{2}(x^3 - x + 1)\,dx = \frac{1}{3}\left[\frac{x^4}{4} - \frac{x^2}{2} + x\right]_{-1}^{2} = \frac{7}{4}$

(b) $x^3 - x + 1 = \frac{7}{4};$ at $x \cong 1.263$

15. (a) A.V. $= \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = \frac{1}{\pi}[-\cos x]_0^{\pi} = \frac{2}{\pi}$

(b) $\sin x = \frac{2}{\pi} \Longrightarrow x = 0.691$

(c)



16. (a) A.V. $= \frac{12}{5\pi}\int_{-\pi/4}^{\pi/6} 2\cos 2x\,dx = \frac{12}{5\pi}[\sin 2x]_{-\pi/4}^{\pi/6} = \frac{12}{5\pi}\left[\frac{\sqrt{3}}{2} + 1\right] = \frac{6}{5\pi}(\sqrt{3} + 2) \cong 1.426$

(b) $2\cos 2x = 1.426$ at $x \cong \pm 0.389$

17. A.V. $= \frac{1}{b-a}\int_a^b x^n\,dx = \frac{1}{b-a}\left[\frac{x^{n+1}}{n+1}\right]_a^b = \frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)}.$

18. (a) for constant f, $f(b)(b-a) = \int_a^b f(x)\,dx$

(b) for increasing f, $f(b)(b-a) > \int_a^b f(x)\,dx$

(c) for decreasing f, $f(b)(b-a) < \int_a^b f(x)\,dx$

19. Average of f' on $[a, b] = \frac{1}{b-a}\int_a^b f'(x)\,dx = \frac{1}{b-a}\,[f(x)]_a^b = \frac{f(b) - f(a)}{b-a}.$

20. (a) True, because $\int_a^b (f + g)\,dx = \int_a^b f\,dx + \int_a^b g\,dx.$

(b) True, because $\int_a^b \alpha f\,dx = \alpha\int_a^b f\,dx.$

(c) False; take $f(x) = g(x) = x$ on $[0, 1]:$ $A.V.(f\,g) = \frac{1}{3},$ $(A.V.(f))(A.V.(g)) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}.$

21. (a) A.V. $= \frac{1}{\sqrt{3}}\int_0^{\sqrt{3}} y\,dx = \frac{1}{\sqrt{3}}\int_0^{\sqrt{3}} x^2\,dx = \frac{1}{\sqrt{3}}\left[\frac{x^3}{3}\right]_0^{\sqrt{3}} = 1$

(b) A.V. $= \frac{1}{3}\int_0^3 x\,dy = \frac{1}{3}\int_0^3 \sqrt{y}\,dy = \frac{1}{3}\left[\frac{2}{3}y^{3/2}\right]_0^3 = \frac{2}{3}\sqrt{3}$

(c)
$$A.V. = \frac{1}{\sqrt{3}}\int_0^{\sqrt{3}} \sqrt{(x-0)^2+(x^2-0)^2}\,dx = \frac{1}{\sqrt{3}}\int_0^{\sqrt{3}} x\sqrt{1+x^2}\,dx$$
$$= \frac{1}{\sqrt{3}}\left[\frac{1}{3}\left(1+x^2\right)^{3/2}\right]_0^{\sqrt{3}} = \frac{7}{9}\sqrt{3}$$

22. (a) Distance to x-axis is y, so $A.V. = \int_0^1 y\,dx = \int_0^1 mx\,dx = \frac{1}{2}m.$

(b) Distance to y-axis is x, so $A.V. = \int_0^1 x\,dx = \frac{1}{2}$

(c) Distance to origin is $\sqrt{x^2+y^2}$, so $A.V. = \int_0^1 \sqrt{x^2+m^2x^2}\,dx = \int_0^1 x\sqrt{1+m^2}\,dx = \frac{1}{2}\sqrt{1+m^2}$

23. The distance the stone has fallen after t seconds is given by $s(t) = 16t^2$.

(a) The terminal velocity after x seconds is $s'(x) = 32x$. The average velocity is $\dfrac{s(x)-s(0)}{x-0} = 16x$. Thus the terminal velocity is twice the average velocity.

(b) For the first $\frac{1}{2}x$ seconds, aver. vel. $= \dfrac{s\left(\frac{1}{2}x\right)-s(0)}{\frac{1}{2}x-0} = 8x.$

For the next $\frac{1}{2}x$ seconds, aver. vel. $= \dfrac{s(x)-s\left(\frac{1}{2}x\right)}{x-\frac{1}{2}x} = 24x.$

Thus, for the first $\frac{1}{2}x$ seconds the average velocity is one-third of the average velocity during the next $\frac{1}{2}x$ seconds.

24. Obviously since $\int_{-a}^{a} f(x)\,dx = 0$

25. Suppose $f(x) \neq 0$ for all x in (a,b). Then, since f is continuous, either

$f(x) > 0$ on (a,b) or $f(x) > 0$ on (a,b).

In either case, $\int_a^b f(x)\,dx \neq 0$.

26.

$$\frac{1}{b-a}\int_a^b (mx+k)\,dx = \frac{1}{b-a}\left[\frac{mx^2}{2}+kx\right]_a^b$$
$$= \frac{m(b+a)}{2}+k.$$

Thus, f takes on its average value at $x = \dfrac{b+a}{2}$.

27. (a) $v(t)-v(0) = \int_0^t a\,du;\quad v(0)=0$. Thus $v(t) = at$.

$x(t)-x(0) = \int_0^t v(u)\,du;\quad x(0)=x_0$. Thus $x(t) = \int_0^t au\,du + x_0 = \frac{1}{2}at^2 + x_0.$

(b) $v_{avg} = \dfrac{1}{t_2-t_1}\displaystyle\int_{t_1}^{t_2} at\,dt = \dfrac{1}{t_2-t_1}\left[\frac{1}{2}at^2\right]_{t_1}^{t_2}$

$$= \frac{at_2^2 - at_1^2}{2(t_2 - t_1)} = \frac{v(t_1) + v(t_2)}{2}$$

28. (a) Mass: $M = \int_a^b \lambda(x)\,dx = \int_0^6 \frac{12}{\sqrt{1+x}}\,dx = 24\left[\sqrt{1+x}\right]_0^6 = 24(\sqrt{7} - 1)$

Center of mass:

$$x_n = \frac{1}{M}\int_a^b x\lambda(x)\,dx = \frac{1}{M}\int_0^6 \frac{12x}{\sqrt{x+1}}\,dx = \frac{12}{M}\int_1^7 \frac{u-1}{\sqrt{u}}\,du \quad (u = x+1)$$

$$= \frac{12}{M}\int_1^7 \left(\sqrt{u} - \frac{1}{\sqrt{u}}\right)\,du = \frac{12}{M}\left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right]_1^7 = \frac{2}{3}\cdot\frac{2\sqrt{7}+1}{\sqrt{7}-1}.$$

(b) $\text{A.V.} = \frac{1}{6}\int_0^6 \frac{12}{\sqrt{x+1}}\,dx = \frac{M}{6} = 4(\sqrt{7} - 1).$

29. (a)

$$M = \int_0^L k\sqrt{x}\,dx = k\left[\frac{2}{3}x^{3/2}\right]_0^L = \frac{2}{3}kL^{3/2}$$

$$x_M M = \int_0^L x\left(k\sqrt{x}\,\right)\,dx = \int_0^L kx^{3/2}\,dx = \left[\frac{2}{5}kx^{5/2}\right]_0^L = \frac{2}{5}kL^{5/2}$$

$$x_M = \left(\tfrac{2}{5}kL^{5/2}\right) / \left(\tfrac{2}{3}kL^{3/2}\right) = \tfrac{3}{5}L$$

(b)

$$M = \int_0^L k\,(L-x)^2\,dx = \left[-\frac{1}{3}k\,(L-x)^3\right]_0^L = \frac{1}{3}kL^3$$

$$x_M M = \int_0^L x\left[k\,(L-x)^2\right]\,dx = \int_0^L k\left(L^2x - 2Lx^2 + x^3\right)\,dx$$

$$= k\left[\tfrac{1}{2}L^2x^2 - \tfrac{2}{3}Lx^3 + \tfrac{1}{4}x^4\right]_0^L = \tfrac{1}{12}kL^4$$

$$x_M = \left(\tfrac{1}{12}kL^4\right) / \left(\tfrac{1}{3}kL^3\right) = \tfrac{1}{4}L$$

30.

$$x_M\,M = \int_a^b x\lambda(x)\,dx$$

$$= \int_{x_0}^{x_1} x\lambda(x)\,dx + \int_{x_1}^{x_2} x\lambda(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} x\lambda(x)\,dx$$

$$= x_{M_1}\,M_1 + x_{M_2}\,M_2 + \cdots + x_{M_n}\,M_n$$

31.

$$\tfrac{1}{4}LM = \tfrac{1}{8}LM_1 + x_{M_2}M_2$$

$$x_{M_2} = \frac{1}{M_2}\left(\frac{1}{4}LM - \frac{1}{8}LM_1\right) = \frac{L}{8M_2}(2M - M_1)$$

32. By Exercise 30, $x_M\, M = x_{M_1}\, M_1 + x_{M_2}\, M_2$, so $\frac{2}{3}LM = \frac{1}{4}LM_1 + \frac{7}{8}LM_2.$

Also, $M_1 + M_2 = M$. Solving gives: $M_1 = \frac{1}{3}M, \quad M_2 = \frac{2}{3}M.$

33. Let $M = \int_a^{a+L} kx\,dx$, where a is the point of the first cut.

Thus $M = \left[\frac{kx^2}{2}\right]_a^{a+L} = \frac{k}{2}(2aL + L^2)$. Hence $a = \frac{2M - kL^2}{2kL}$, and $a + L = \frac{2M + kL^2}{2kL}$.

34. The average slope of f on $[a, b]$ is $\frac{1}{b-a}\int_a^b f'(x)\,dx = \frac{f(b) - f(a)}{b-a}.$

Geometrically, this is the slope of the line through $(a, f(a))$, and $(b, f(b))$.

35. If f is continuous on $[a,\, b]$, then, by Theorem 5.2.5, F satisfies the conditions of the mean-value theorem of differential calculus (Theorem 4.1.1). Therefore, by that theorem, there is at least one number c in $(a,\, b)$ for which

$$F'(c) = \frac{F(b) - F(a)}{b-a}.$$

Then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F'(c)(b-a) = f(c)(b-a).$$

36.

$$\begin{pmatrix} \text{min of } f \\ \\ \text{on } [c-h, c+h] \end{pmatrix} \leq \begin{pmatrix} \text{average of } f \\ \\ \text{on } [c-h, c+h] \end{pmatrix} \leq \begin{pmatrix} \text{max of } f \\ \\ \text{on } [c-h, c+h] \end{pmatrix}$$

By continuity, as $h \to 0^+$

$$\begin{pmatrix} \text{min of } f \\ \\ \text{on } [c-h, c+h] \end{pmatrix} \to f(c) \quad \text{and} \quad \begin{pmatrix} \text{max of } f \\ \\ \text{on } [c-h, c+h] \end{pmatrix} \to f(c).$$

By the pinching theorem the middle term must also tend to $f(c)$.

37. If f and g take on the same average value on every interval $[a,\, x]$, then

$$\frac{1}{x-a}\int_a^x f(t)\,dt = \frac{1}{x-a}\int_a^x g(t)\,dt.$$

Multiplication by $(x - a)$ gives

$$\int_a^x f(t)\,dt = \int_a^x g(t)\,dt.$$

Differentiation with respect to x gives $f(x) = g(x)$. This shows that, if the averages are everywhere the same, then the functions are everywhere the same.

38. Partition $[a, b]$ into n subintervals of equal length $\dfrac{b-a}{n}$, where $P = \{x_0, \ldots, x_n\}$ and x_i^* is a point from $[x_{i-1}, x_i]$. Then the average value of f on $[a, b]$ is:

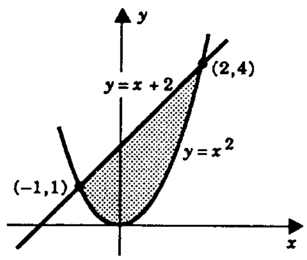
$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\,dx &= \frac{1}{b-a}\lim_{\|P\|\to 0}\left[f(x_1^*)\left(\frac{b-a}{n}\right)+\cdots+f(x_n^*)\left(\frac{b-a}{n}\right)\right]\\ &= \left(\frac{1}{b-a}\right)\lim_{n\to\infty}\frac{b-a}{n}[f(x_1^*)+\cdots+f(x_n^*)]\\ &= \lim_{n\to\infty}\frac{1}{n}[f(x_1^*)+\cdots+f(x_n^*)],\end{aligned}$$

which is the limit of arithmetic averages of values of f on $[a, b]$.

CHAPTER 6

SECTION 6.1

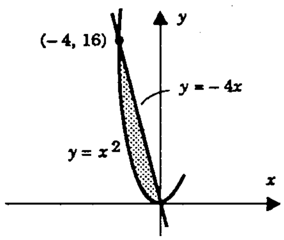
1.



(a) $\int_{-1}^{2} [(x+2) - x^2]\,dx$

(b) $\int_{0}^{1} [(\sqrt{y}) - (-\sqrt{y})]\,dy + \int_{1}^{4} [(\sqrt{y}\,) - (y-2)]\,dy$

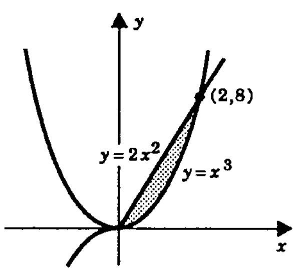
2.



(a) $\int_{-4}^{0} \left[(-4x) - x^2\right]\,dx$

(b) $\int_{0}^{16} \left[\left(-\frac{1}{4}y\right) - (-\sqrt{y})\right]\,dy$

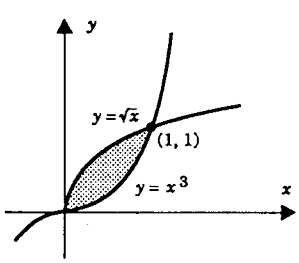
3.



(a) $\int_{0}^{2} \left[(2x^2) - (x^3)\right]\,dx$

(b) $\int_{0}^{8} \left[\left(y^{1/3}\right) - \left(\frac{1}{2}y\right)^{1/2}\right]\,dy$

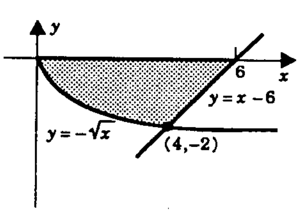
4.



(a) $\int_{0}^{1} \left[\sqrt{x} - x^3\right]\,dx$

(b) $\int_{0}^{1} \left[y^{1/3} - y^2\right]\,dy$

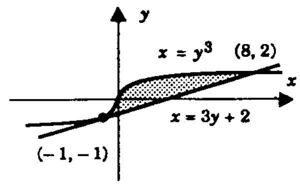
5.



(a) $\int_{0}^{4} \left[(0) - (-\sqrt{x}\,)\right]\,dx + \int_{4}^{6} [(0) - (x-6)]\,dx$

(b) $\int_{-2}^{0} \left[(y+6) - (y^2)\right]\,dy$

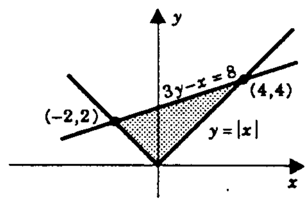
6.



(a) $\int_{-1}^{8}\left[x^{1/3}-\left(\frac{x-2}{3}\right)\right]\,dx$

(b) $\int_{-1}^{2}\left[(3y+2)-y^3\right]\,dy$

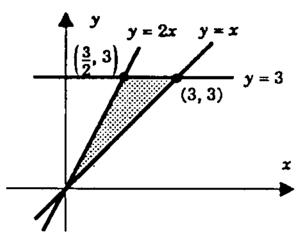
7.



(a) $\int_{-2}^{0}\left[\left(\frac{8+x}{3}\right)-(-x)\right]\,dx+\int_{0}^{4}\left[\left(\frac{8+x}{3}\right)-(x)\right]\,dx$

(b) $\int_{0}^{2}\left[(y)-(-y)\right]\,dy+\int_{2}^{4}\left[(y)-(3y-8)\right]\,dy$

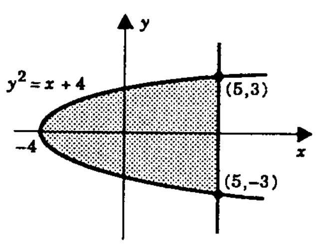
8.



(a) $\int_{0}^{3/2}\left[2x-x\right]\,dx+\int_{3/2}^{3}\left[3-x\right]dx$

(b) $\int_{0}^{3}\left[y-\frac{1}{2}y\right]\,dy$

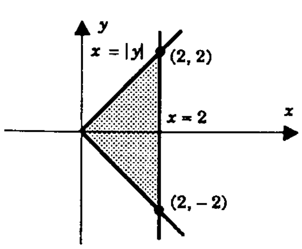
9.



(a) $\int_{-4}^{5}\left[(\sqrt{4+x}\,)-(-\sqrt{4+x}\,)\right]\,dx$

(b) $\int_{-3}^{3}\left[(5)-(y^2-4)\right]\,dy$

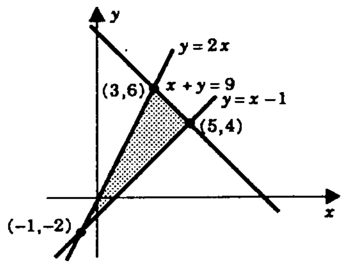
10.



(a) $\int_{0}^{2}\left[x-(-x)\right]\,dx$

(b) $\int_{-2}^{0}\left[2-(-y)\right]\,dy+\int_{0}^{2}\left[2-y\right]dy$

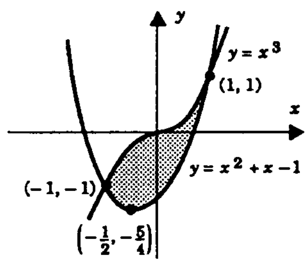
11.



(a) $\displaystyle\int_{-1}^{3}[(2x)-(x-1)]\,dx+\int_{3}^{5}[(9-x)-(x-1)]\,dx$

(b) $\displaystyle\int_{-2}^{4}\left[(y+1)-\left(\frac{1}{2}y\right)\right]\,dy+\int_{4}^{6}\left[(9-y)-\left(\frac{1}{2}y\right)\right]\,dy$

12.

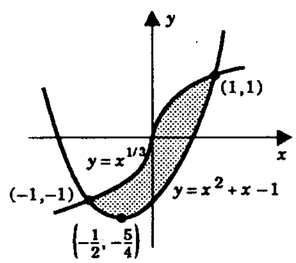


(a) $\displaystyle\int_{-1}^{1}\left[x^3-(x^2+x-1)\right]\,dx$

(b) $\displaystyle\int_{-5/4}^{-1}\left[\left(-\frac{1}{2}+\frac{1}{2}\sqrt{4y+5}\right)-\left(-\frac{1}{2}-\frac{1}{2}\sqrt{4y+5}\right)\right]\,dy$

$\displaystyle\qquad+\int_{-1}^{1}\left[\left(-\frac{1}{2}+\frac{1}{2}\sqrt{4y+5}\right)-y^{1/3}\right]\,dy$

13.

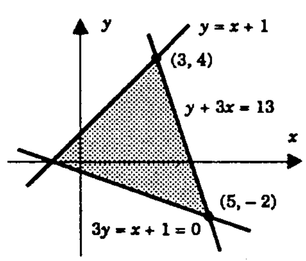


(a) $\displaystyle\int_{-1}^{1}\left[\left(x^{1/3}\right)-\left(x^2+x-1\right)\right]\,dx$

(b) $\displaystyle\int_{-5/4}^{-1}\left[\left(-\frac{1}{2}+\frac{1}{2}\sqrt{4y+5}\right)-\left(-\frac{1}{2}-\frac{1}{2}\sqrt{4y+5}\right)\right]\,dy$

$\displaystyle\qquad+\int_{-1}^{1}\left[\left(-\frac{1}{2}+\frac{1}{2}\sqrt{4y+5}\right)-\left(y^3\right)\right]\,dy$

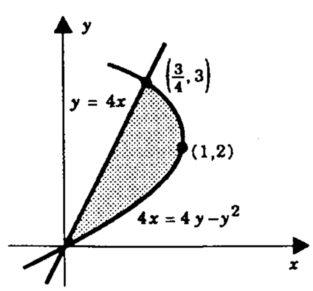
14.



(a) $\displaystyle\int_{-1}^{3}\left[(x+1)-\left(\frac{-x-1}{3}\right)\right]\,dx+\int_{3}^{5}\left[(13-3x)-\left(\frac{-x-1}{3}\right)\right]\,dx$

(b) $\displaystyle\int_{-2}^{0}\left[\left(\frac{13-y}{3}\right)-(-3y-1)\right]\,dy+\int_{0}^{4}\left[\left(\frac{13-y}{3}\right)-(y-1)\right]\,dy$

15.

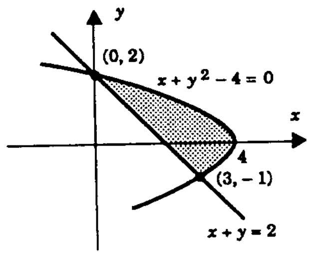


$$A=\int_{0}^{3}\left[\left(\frac{4y-y^2}{4}\right)-\left(\frac{y}{4}\right)\right]\,dy$$

$$=\int_{0}^{3}\left(\frac{3}{4}y-\frac{1}{4}y^2\right)\,dy$$

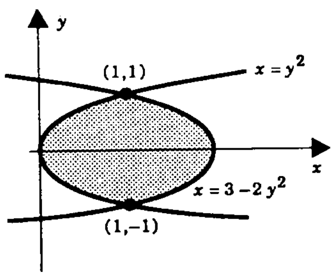
$$=\left[\tfrac{3}{8}y^2-\tfrac{1}{12}y^3\right]_0^3=\tfrac{9}{8}$$

16.



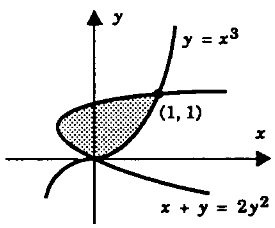
$$A = \int_{-1}^{2} \left[(4 - y^2) - (2 - y)\right]\, dy$$
$$= \int_{-1}^{2} (2 + y - y^2)\, dy$$
$$= \left[2y + \tfrac{y^2}{2} - \tfrac{y^3}{3}\right]_{-1}^{2} = \frac{9}{2}$$

17.



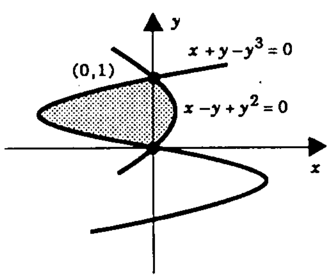
$$A = 2\int_{0}^{1} \left[(3 - 2y^2) - (y^2)\right]\, dy$$
$$= 2\int_{0}^{1} (3 - 3y^2)\, dy$$
$$= 2\left[3y - y^3\right]_{0}^{1} = 2\,(2) = 4$$

18.



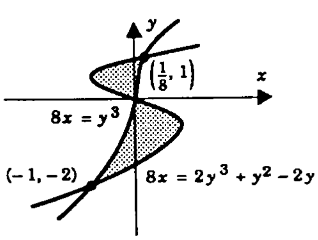
$$A = \int_{0}^{1} \left[y^{1/3} - (2y^2 - y)\right]\, dy$$
$$= \int_{0}^{1} (y^{1/3} + y - 2y^2)\, dy$$
$$= \left[\frac{3}{4}y^{4/3} + \frac{y^2}{2} - \frac{2}{3}y^3\right]_{0}^{1} = \frac{7}{12}$$

19.



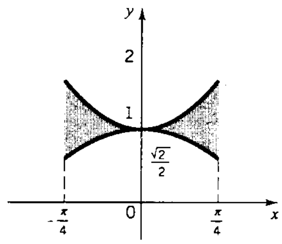
$$A = \int_{0}^{1} \left[(y - y^2) - (y^3 - y)\right]\, dy$$
$$= \int_{0}^{1} (2y - y^2 - y^3)\, dy$$
$$= \left[y^2 - \tfrac{1}{3}y^3 - \tfrac{1}{4}y^4\right]_{0}^{1} = \tfrac{5}{12}$$

20.



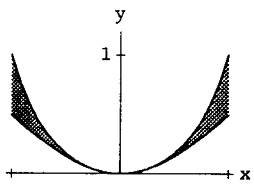
$$A = \int_{-2}^{0} \left[\frac{1}{8}(2y^3 + y^2 - 2y) - \frac{1}{8}y^3\right]\, dy$$
$$+ \int_{0}^{1} \left[\frac{1}{8}y^3 - \frac{1}{8}(2y^3 + y^2 - 2y)\right]\, dy$$
$$= \frac{1}{8}\left[\frac{y^4}{4} + \frac{y^3}{3} - y^2\right]_{-2}^{0} + \frac{1}{8}\left[-\frac{y^4}{4} - \frac{y^3}{3} + y^2\right]_{0}^{1} = \frac{37}{96}$$

21.



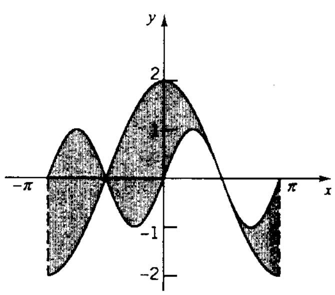
$$A = \int_{-\pi/4}^{\pi/4} \left[\sec^2 x - \cos x\right]\, dx$$
$$= 2\int_0^{\pi/4} \left[\sec^2 x - \cos x\right]\, dx$$
$$= 2\left[\tan x + \sin x\right]_0^{\pi/4} = 2\left[1 + \sqrt{2}/2\right] = 2 + \sqrt{2}$$

22.



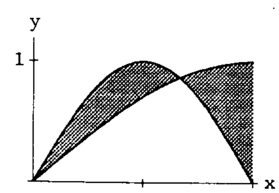
$$A = \int_{-\pi/4}^{\pi/4} (\tan^2 x - \sin^2 x)\, dx = \int_{-\pi/4}^{\pi/4} \left(\sec^2 x - \frac{3}{2} + \frac{\cos 2x}{2}\right) dx$$
$$= \left[\tan x - \frac{3}{2}x + \frac{\sin 2x}{4}\right]_{-\pi/4}^{\pi/4} = \frac{5}{2} - \frac{3\pi}{4}$$

23.



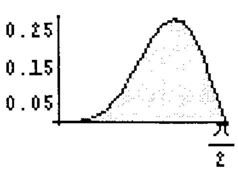
$$A = \int_{-\pi}^{-\pi/2} \left[\sin 2x - 2\cos x\right]\, dx + \int_{-\pi/2}^{\pi/2} \left[2\cos x - \sin 2x\right]\, dx$$
$$+ \int_{\pi/2}^{\pi} \left[\sin 2x - 2\cos x\right]\, dx$$
$$= \left[-\tfrac{1}{2}\cos 2x - 2\sin x\right]_{\pi}^{\pi/2} + \left[2\sin x + \tfrac{1}{2}\cos 2x\right]_{-\pi/2}^{\pi/2}$$
$$+ \left[-\tfrac{1}{2}\cos 2x - 2\cos x\right]_{\pi/2}^{\pi} = 8$$

24.



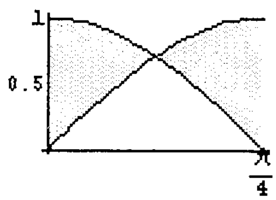
$$A = \int_0^{\pi/3} (\sin 2x - \sin x)\, dx + \int_{\pi/3}^{\pi/2} (\sin x - \sin 2x)\, dx$$
$$= \left[-\frac{\cos 2x}{2} + \cos x\right]_0^{\pi/3} + \left[-\cos x + \frac{\cos 2x}{2}\right]_{\pi/3}^{\pi/2} = \frac{1}{2}$$

25.



$$A = \int_0^{\pi/2} (\sin^4 x \cos x)\, dx$$
$$= \int_0^1 u^4\, du, \text{ where } u = \sin x$$
$$= \left[\frac{u^5}{5}\right]_0^1 = \frac{1}{5}$$

26.



$$A = \int_0^{\pi/8} (\cos 2x - \sin 2x)\, dx + \int_{\pi/8}^{\pi/4} (\sin 2x - \cos 2x)\, dx$$

$$= \left[\frac{\sin 2x}{2} + \frac{cos 2x}{2}\right]_0^{\pi/8} + \left[-\frac{\cos 2x}{2} - \frac{\sin 2x}{2}\right]_{\pi/8}^{\pi/4} = 2\sqrt{2} - 1$$

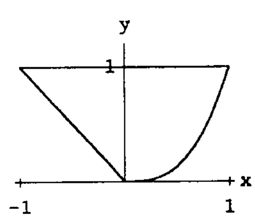
27. $$A = \int_0^1 \left[3x - \frac{1}{3}\, x\right] dx + \int_1^3 \left[-x + 4 - \frac{1}{3}\, x\right] dx = \left[\frac{4}{3}\, x^2\right]_0^1 + \left[-\frac{2}{3}\, x^2 + 4x\right]_1^3 = 4$$

28. $$A = \int_0^2 \left[(x+1) - \left(1 - \frac{x}{2}\right)\right] dx + \int_2^3 [(x+1) - (4x-8)]\, dx = \left[\frac{x^2}{4}\right]_0^2 + \left[-\frac{3x^2}{2} + 9x\right]_2^3 = \frac{5}{2}$$

29. $$A = \int_{-2}^1 [x - (-2)]\, dx + \int_1^5 [1 - (-2)]\, dx + \int_5^7 \left[-\frac{3}{2}\, x + \frac{17}{2} - (-2)\right] dx$$

$$= \left[\tfrac{1}{2}\, x^2 + 2x\right]_{-2}^1 + [3x]_1^5 \left[-\tfrac{3}{4}\, x^2 + \tfrac{21}{2} x\right]_5^7 = \frac{39}{2}$$

30.

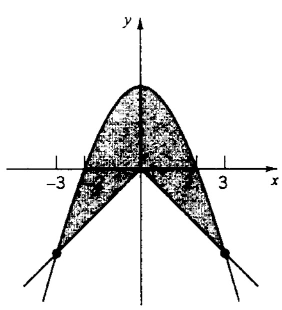


$$A = \int_0^1 \left[y^{1/3} - (-y)\right] dy$$

$$= \int_0^1 (y^{1/3} + y)\, dy$$

$$= \left[\frac{3}{4} y^{4/3} + \frac{y^2}{2}\right]_0^1 = \frac{5}{4}$$

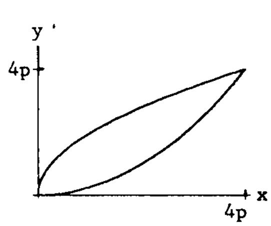
31.



$$A = \int_{-3}^0 \left[6 - x^2 - x\right] dx + \int_0^3 \left[6 - x^2 - (-x)\right] dx$$

$$= \left[6x - \frac{1}{3}\, x^3 - \frac{1}{2}\, x^2\right]_{-3}^0 + \left[6x - \frac{1}{3}\, x^3 + \frac{1}{2}\, x^2\right]_0^3 = 27$$

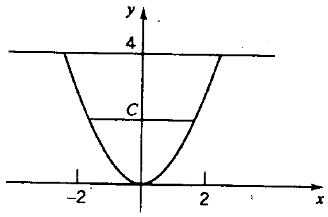
32.



$$A = \int_0^{4p} \left(\sqrt{4px} - \frac{x^2}{4p}\right) dx$$

$$= \left[\sqrt{4p}\, \frac{2}{3} x^{3/2} - \frac{x^3}{12p}\right]_0^{4p} = \frac{16}{3} p^2$$

33.



$C = 4^{2/3}$

$$\int_0^{\sqrt{c}} \left[c - x^2\right]\, dx = \frac{1}{2}\int_0^2 \left[4 - x^2\right]\, dx$$

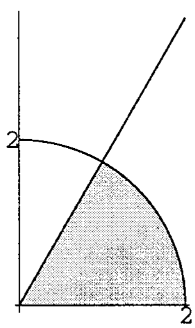
$$\left[cx - \tfrac{1}{3}x^3\right]_0^{\sqrt{3}} = \tfrac{1}{2}\left[4x - \tfrac{1}{3}x^3\right]_0^2$$

$$\tfrac{2}{3}c^{3/2} = \tfrac{8}{3} \quad \text{and} \quad c = 4^{2/3}$$

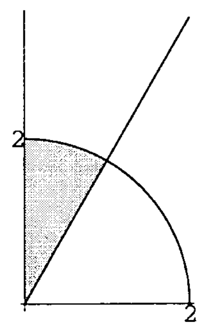
34. We want $\displaystyle\int_0^c \cos x\, dx = \frac{1}{2}\int_0^{\pi/2} \cos x\, dx \implies [\sin x]_0^c = \frac{1}{2}[\sin x]_0^{\pi/2} = \frac{1}{2}$

$$\implies \sin c = \frac{1}{2} \implies c = \frac{\pi}{6}$$

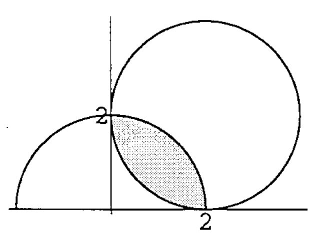
35. $\displaystyle A = \int_0^1 (\sqrt{4 - x^2} - \sqrt{3}x)\, dx$



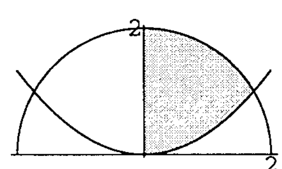
36. $\displaystyle A = \int_0^1 (\sqrt{3}x\, dx + \int_1^2 \sqrt{4 - x^2}\, dx$



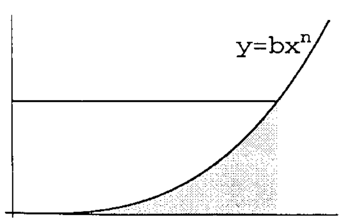
37. $\displaystyle A = \int_0^2 [\sqrt{4 - x^2} - (2 - \sqrt{4x - x^2})]\, dx$



38. $\displaystyle A = \int_0^{\sqrt{3}} (\sqrt{4 - x^2} - \frac{x^2}{3})\, dx$



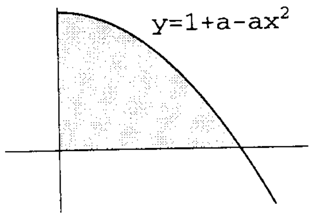
39.



The area under the curve is $\displaystyle A_c = \int_0^a bx^n\, dx = \frac{ba^{n+1}}{n+1}$.

For the rectangle, $A_r = ba^{n+1}$. Thus the ratio is $\displaystyle\frac{1}{n+1}$.

40.



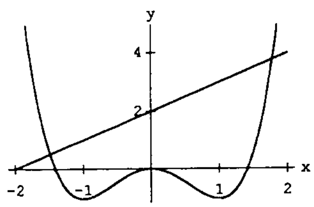
(a) $A = \int_0^{\sqrt{\frac{1+a}{a}}} (1 + a - ax^2)\, dx$

$$= \left[x + ax - \frac{ax^3}{3}\right]_0^{\sqrt{\frac{1+a}{a}}} = \frac{4(1+a)}{3}\sqrt{\frac{1+a}{a}}.$$

(b) $A' = \frac{4}{3}\sqrt{\frac{1+a}{a}} - 2\frac{1+a}{3a^2\sqrt{\frac{1+a}{a}}} = 0$

$$\Longrightarrow \quad a = \frac{1}{2}.$$

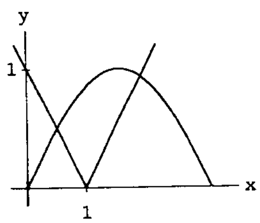
41.



$$A \cong \int_{-1.49}^{1.79} \left[x + 2 - (x^4 - 2x^2)\right]\, dx$$

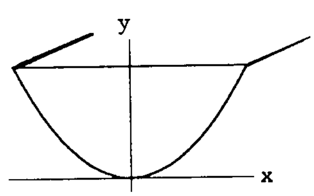
$$= \left[\tfrac{1}{2}x^2 - 2x - \tfrac{1}{5}x^5 + \tfrac{2}{3}x^3\right]_{-1.49}^{1.78} \cong 7.93$$

42.



$A \cong 0.67$

43.



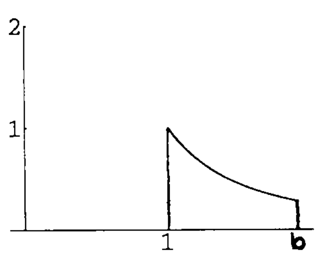
$$V = 8 \cdot 12 \int_{-3}^{3} \left[4 - \frac{4}{9}x^2\right]\, dx$$

$$= 96 \cdot 2 \int_0^3 \left[4 - \frac{4}{9}x^2\right]\, dx$$

$$= 192\, \left[4x - \tfrac{4}{27}x^3\right]_0^3$$

$$= 1536 \text{ cu. in.} \cong 0.89 \text{ cu. ft.}$$

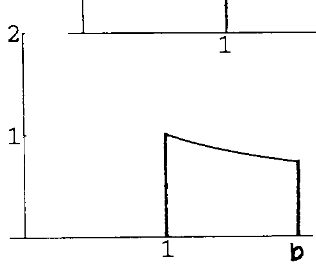
44.



(a) $A = \int_1^b \frac{1}{x^2}\, dx = \left[-\frac{1}{x}\right]_1^b = 1 - \frac{1}{b}.$

(b) $1 - \frac{1}{b} \to 1$ as $b \to \infty$, so it has finite area.

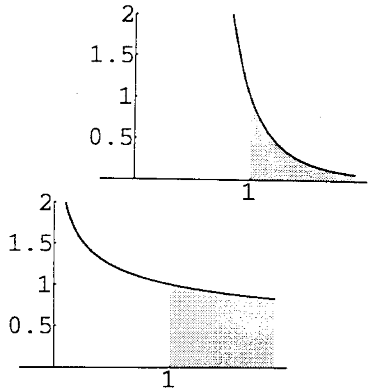
45.



(a) $A = \int_1^b \frac{1}{\sqrt{x}}\, dx = \left[2\sqrt{x}\right]_1^b = 2\sqrt{b} - 2.$

(b) $2\sqrt{b} - 2 \to \infty$ as $b \to \infty$, so it has "infinite" area.

46.



(a) $A = \int_1^b \frac{1}{x^p}\,dx = \left[\frac{x^{1-p}}{1-p}\right]_1^b = \frac{1}{p-1}\left[1 - b^{1-p}\right].$

$A \to \frac{1}{p-1}$ as $b \to \infty$

(b) $A \to -\infty$ as $b \to \infty$

PROJECT 6.1

1. (a) $C = \dfrac{\int_0^1 [x - L(x)]\,dx}{\int_0^1 x\,dx} = \dfrac{\int_0^1 [x - L(x)]\,dx}{\left[\frac{x^2}{2}\right]_0^1} = 2\int_0^1 [x - L(x)]\,dx$

(b) The area between $y = x$ and the Lorenz curve must be between 0 and $\frac{1}{2}$. The area under $y = x$ is equal to $\frac{1}{2}$. Thus their ratio must be between 0 and 1.

A coefficient close to 1 would mean the curve is close to the line $y = x$.

Thus the income distribution would be almost absolutely equal.

Similarly, a coefficient close to 0 would give a very unequal income distribution.

2. (a) $L(50) = \frac{7}{12}\left(\frac{1}{2}\right)^2 + \frac{5}{12}\left(\frac{1}{2}\right) = 35.4$

(b) $C = 2\int_0^1 \left[x - \frac{7}{12}x^2 - \frac{5}{12}x\right]dx = 2\int_0^1 \left[\frac{7}{12}x - \frac{7}{12}x^2\right]dx$

$= 2\left[\frac{7}{24}x^2 - \frac{7}{36}x^3\right]_0^1 = \frac{7}{36}$

3. 1935, $L(x) = x^{2.4}$.

Thus $C = 2\int_0^1 (x - x^{2.4})\,dx = 2\left[\frac{x^2}{2} - \frac{x^{3.4}}{3.4}\right]_0^1 \cong 0.412$

1947, $L(x) = x^{1.6}$.

Thus $C = 2\int_0^1 (x - x^{1.6})\,dx = 2\left[\frac{x^2}{2} - \frac{x^{2.6}}{2.6}\right]_0^1 \cong 0.231$

4. (a) L is defined and continuous on $[0,1]$, and $L(0) = 0$, $L(1) = 1$.

$L'(x) = \dfrac{5x^2(x^2+12)}{(4+x^2)^2} > 0$ on $(0,1)$ $\implies$ increasing.

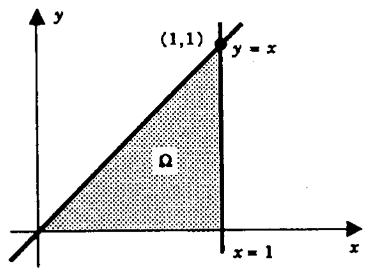
$$L''(x) = \frac{40x(12 - x^2)}{(4 + x^2)^3} \geq 0 \text{ on } (0, 1) \quad \Longrightarrow \text{ concave up}$$

(b) $C = 2\int_0^1 [x - L(x)]\, dx \cong 2(0.2)\,[f(0.1) + f(0.3) + \cdots + f(0.9)] \cong 0.472$

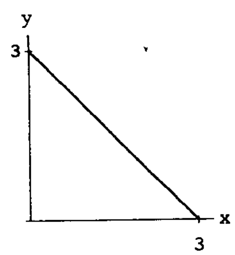
(c) $C \cong 0.463$

SECTION 6.2

1. $V = \int_0^1 \pi\left[(x)^2 - (0)^2\right] dx = \pi\left[\frac{x^3}{3}\right]_0^1 = \frac{\pi}{3}$

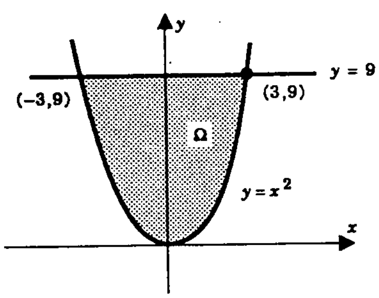


2. $V = \int_0^3 \pi\,(3 - x)^2\, dx = \left[-\frac{\pi(3 - x)^3}{3}\right]_0^3 = 9\pi$

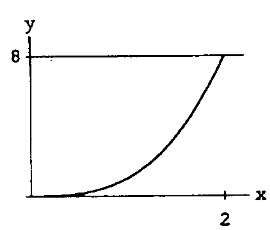


3. $V = \int_{-3}^3 \pi\left[(9)^2 - (x^2)^2\right] dx = 2\int_0^3 \pi\left(81 - x^4\right) dx$

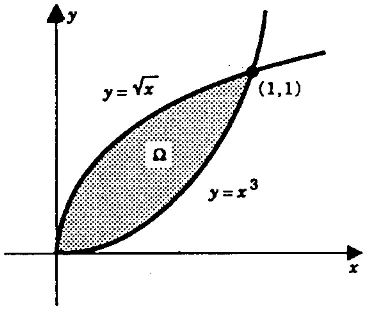
$= 2\pi\left[81x - \frac{x^5}{5}\right]_0^3 = \frac{1944\pi}{5}$



4. $V = \int_0^2 \pi\left[8^2 - x^6\right] dx = \pi\left[64x - \frac{x^7}{7}\right]_0^2 = \frac{768}{7}\pi$

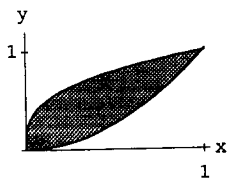


5.



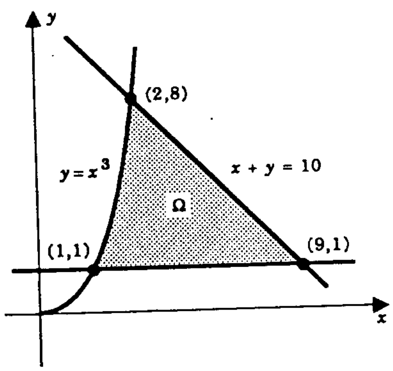
$$V=\int_0^1 \pi\left[(\sqrt{x}\,)^2-(x^3)^2\right]dx$$
$$=\int_0^1 \pi\left(x-x^6\right)dx$$
$$=\pi\left[\frac{1}{2}x^2-\frac{1}{7}x^7\right]_0^1=\frac{5\pi}{14}$$

6.



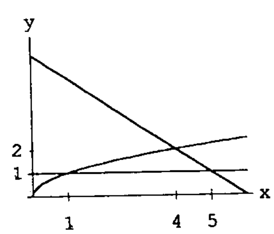
$$V=\int_0^1 \pi\left[x^{2/3}-x^4\right]dx=\pi\left[\frac{3}{5}x^{5/3}-\frac{x^5}{5}\right]_0^1=\frac{2\pi}{5}$$

7.



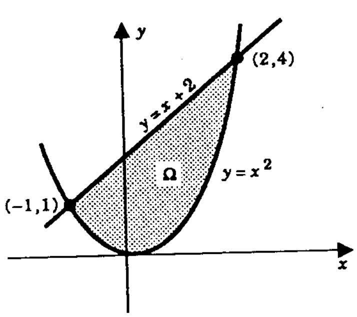
$$V=\int_1^2 \pi\left[(x^3)^2-(1)^2\right]dx+\int_2^9 \pi\left[(10-x)^2-(1)^2\right]dx$$
$$=\int_1^2 \pi\left(x^6-1\right)dx+\int_2^9 \pi\left(99-20x+x^2\right)dx$$
$$=\pi\left[\frac{1}{7}x^7-x\right]_1^2+\pi\left[99x-10x^2+\frac{1}{3}x^3\right]_2^9=\frac{3790\pi}{21}$$

8.



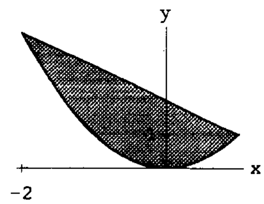
$$V=\int_1^4 \pi[x-1]\,dx+\int_4^5 \pi\left[(6-x)^2-1\right]dx$$
$$=\pi\left[\frac{x^2}{2}-x\right]_1^4+\pi\left[-\frac{(6-x)^3}{3}-x\right]_4^5=\frac{35}{6}\pi$$

9.



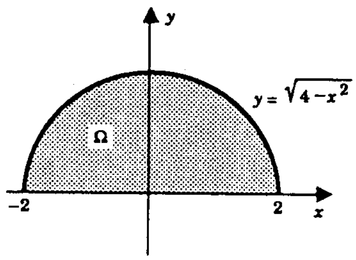
$$V=\int_{-1}^2 \pi\left[(x+2)^2-(x^2)^2\right]dx$$
$$=\int_{-1}^2 \pi\left(x^2+4x+4-x^4\right)dx$$
$$=\pi\left[\tfrac{1}{3}x^3+2x^2+4x-\tfrac{1}{5}x^5\right]_{-1}^2=\frac{72}{5}\pi$$

10.



$$V = \int_{-2}^{1} \pi \left[(2-x)^2 - x^4\right] dx = \pi \left[-\frac{(2-x)^3}{3} - \frac{x^5}{5}\right]_{-2}^{1} = \frac{72}{5}\pi$$

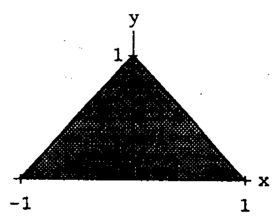
11.



$$V = \int_{-2}^{2} \pi \left[\sqrt{4-x^2}\,\right]^2 dx = 2\int_0^2 \pi\left(4-x^2\right) dx$$

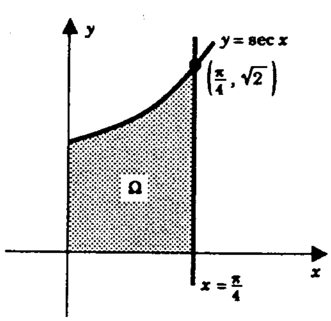
$$= 2\pi\left[4x - \frac{x^3}{3}\right]_0^2 = \frac{32}{3}\pi$$

12.



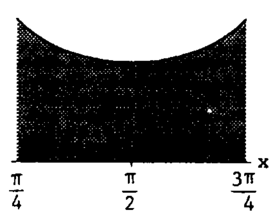
$V = 2$ (Volume of cone of radius 1, height 1)$=2 \cdot \frac{1}{3}\pi = \frac{2}{3}\pi$

13.



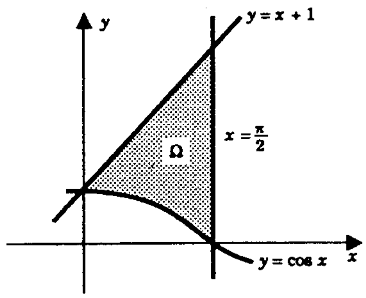
$$V = \int_0^{\pi/4} \pi \sec^2 x\, dx = \pi\left[\tan x\right]_0^{\pi/4} = \pi$$

14.



$$V = \int_{\pi/4}^{3\pi/4} \pi\left[\csc^2 x - 0\right] dx = \pi\left[-\cot x\right]_{\pi/4}^{3\pi/4} = 2\pi$$

15.

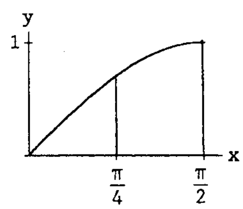


$$V = \int_0^{\pi/2} \pi\left[(x+1)^2 - (\cos x)^2\right]\,dx$$

$$= \int_0^{\pi/2} \pi\left[(x+1)^2 - \left(\frac{1}{2}+\frac{1}{2}\cos 2x\right)\right]\,dx$$

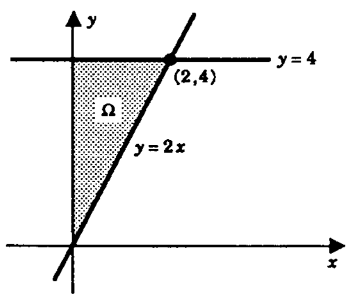
$$= \pi\left[\frac{1}{3}(x+1)^3 - \frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi/2} = \frac{\pi^2}{24}\left(\pi^2+6\pi+6\right)$$

16.



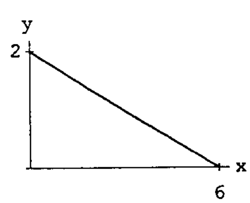
$$V = \int_{\pi/4}^{\pi/2} \pi\sin^2 x\,dx = \pi\left[\frac{x}{2} - \frac{1}{4}\sin 2x\right]_{\pi/4}^{\pi/2} = \frac{1}{8}\pi(\pi+2)$$

17.



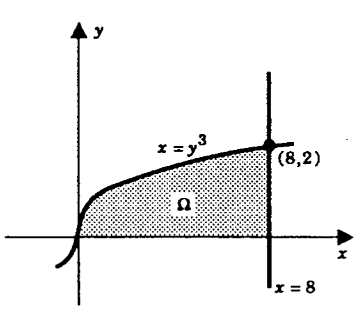
$$V = \int_0^4 \pi\left(\frac{y}{2}\right)^2\,dy = \frac{\pi}{12}\left[y^3\right]_0^4 = \frac{16\pi}{3}$$

18.



$V =$ (Volume of cone of radius 6, height 2) $= \frac{1}{3}\pi\, 6^2 \cdot 2 = 24\pi$

19.

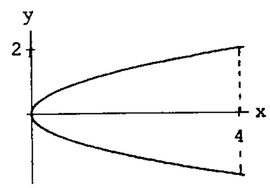


$$V = \int_0^2 \pi\left[(8)^2 - \left(y^3\right)^2\right]\,dy$$

$$= \int_0^2 \pi\left(64 - y^6\right)\,dy$$

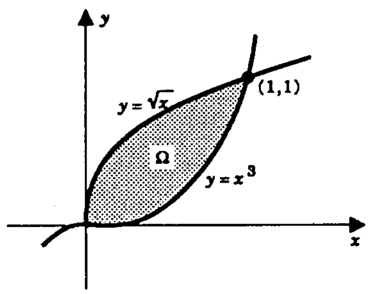
$$= \pi\left[64y - \tfrac{1}{7}y^7\right]_0^2 = \tfrac{768}{7}\pi$$

20.



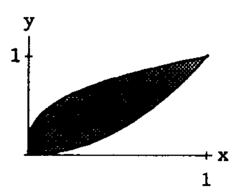
$$V=\int_{-2}^{2}\pi\left[4^2-y^4\right]dy=\pi\left[16y-\frac{y^5}{5}\right]_{-2}^{2}=\frac{256}{5}\pi$$

21.



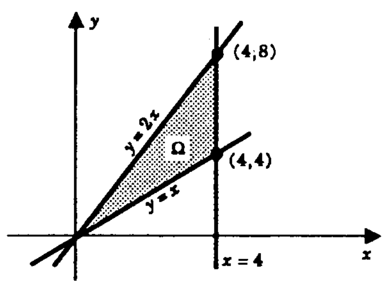
$$\begin{aligned}V&=\int_0^1\pi\left[\left(y^{1/3}\right)^2-\left(y^2\right)^2\right]dy\\&=\int_0^1\pi\left[y^{2/3}-y^4\right]dy\\&=\pi\left[\tfrac{3}{5}y^{5/3}-\tfrac{1}{5}y^5\right]_0^1=\tfrac{2}{5}\pi\end{aligned}$$

22.



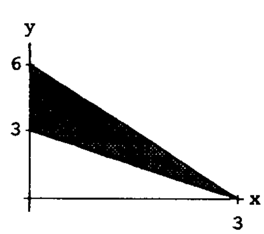
$$\begin{aligned}V&=\int_0^1\pi\left[(\sqrt{y})^2-(y^3)^2\right]dy=\int_0^1\pi(y-y^6)\,dy\\&=\pi\left[\frac{y^2}{2}-\frac{y^7}{7}\right]_0^1=\frac{5}{14}\pi\end{aligned}$$

23.



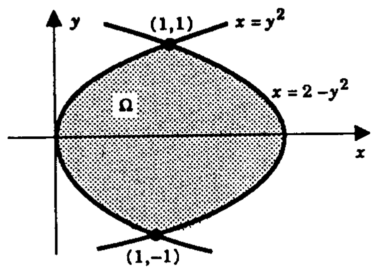
$$\begin{aligned}V&=\int_0^4\pi\left[y^2-\left(\frac{y}{2}\right)^2\right]dy+\int_4^8\pi\left[4^2-\left(\frac{y}{2}\right)^2\right]dy\\&=\int_0^4\pi\left[\frac{3}{4}y^2\right]dy+\int_4^8\pi\left[16-\frac{1}{4}y^2\right]dy\\&=\pi\left[\tfrac{1}{4}y^3\right]_0^4+\pi\left[16y-\tfrac{1}{12}y^3\right]_4^8=\tfrac{128}{3}\pi\end{aligned}$$

24.



$$\begin{aligned}V&=\int_0^3\pi\left[\left(3-\frac{y}{2}\right)^2-(3-y)^2\right]dy+\int_3^6\pi\left(3-\frac{y}{2}\right)^2dy\\&=\pi\left[-\frac{2}{3}\left(3-\frac{y}{2}\right)^3+\frac{(3-y)^3}{3}\right]_0^3+\pi\left[-\frac{2}{3}\left(3-\frac{y}{2}\right)^3\right]_3^6=9\pi\end{aligned}$$

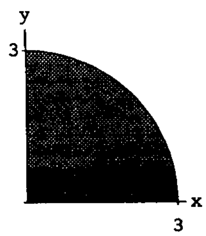
25.



$$V=\int_{-1}^{1}\pi\left[\left(2-y^2\right)^2-\left(y^2\right)^2\right]\,dy$$

$$=2\int_0^1\pi\left[4-4y^2\right]\,dy=2\pi\left[4y-\frac{4}{3}y^3\right]_0^1=\frac{16}{3}\pi$$

26.



$$V=\int_0^3\pi(9-y^2)\,dy=\pi\left[9y-\frac{y^3}{3}\right]_0^3=18\pi$$ (half sphere of radius 3.

27. (a) $V=\int_{-r}^{r}\left(2\sqrt{r^2-x^2}\right)^2\,dx=8\int_0^r\left(r^2-x^2\right)\,dx=8\left[r^2x-\frac{1}{3}x^3\right]_0^r=\frac{16}{3}r^3$

(b) $V=\int_{-r}^{r}\frac{\sqrt{3}}{4}\left(2\sqrt{r^2-x^2}\right)^2\,dx=2\sqrt{3}\int_0^r\left(r^2-x^2\right)\,dx=\frac{4\sqrt{3}}{3}r^3$

28. For each $x\in[-3,3]$, the length of the base of the cross-section at x is $2y=\frac{4}{3}\sqrt{9-x^2}$.

(a) The area of each truangle is $\frac{\sqrt{3}}{4}s^2$.

Thus $V=\int_{-3}^{3}\frac{\sqrt{3}}{4}\cdot\frac{16}{9}(9-x^2)\,dx=\frac{4\sqrt{3}}{9}\int_{-3}^{3}(9-x^2)\,dx$

$=\frac{4\sqrt{3}}{9}\left[9x-\frac{x^3}{3}\right]_{-3}^{3}=16\sqrt{3}.$

(b) The area of each square is s^2

Thus $V=\int_{-3}^{3}\frac{16}{9}(9-x^2)\,dx=\frac{16}{9}\int_{-3}^{3}(9-x^2)\,dx$

$=\frac{16}{9}\left[9x-\frac{x^3}{3}\right]_{-3}^{3}=64.$

29. (a) $V=\int_{-2}^{2}\left(4-x^2\right)^2\,dx=2\int_0^2\left(16-8x^2+x^4\right)\,dx=2\left[16x-\frac{8}{3}x^3+\frac{1}{5}x^5\right]_0^2=\frac{512}{15}$

(b) $V=\int_{-2}^{2}\frac{\pi}{2}\left(\frac{4-x^2}{2}\right)^2\,dx=\frac{\pi}{4}\int_0^2\left(4-x^2\right)^2\,dx=\frac{\pi}{4}\left(\frac{256}{15}\right)=\frac{64}{15}\pi$

(c) $V = \int_{-2}^{2} \frac{\sqrt{3}}{4}\left(4-x^2\right)^2 dx = \frac{\sqrt{3}}{2}\int_0^2 \left(4-x^2\right)^2 dx = \frac{\sqrt{3}}{2}\left(\frac{256}{15}\right) = \frac{128}{15}\sqrt{3}$

30. (a) $V = \int_0^1 2\sqrt{x}h\,dx + \int_1^3 2\cdot\frac{1}{\sqrt{2}}\sqrt{3-x}\,h\,dx = \frac{4h}{3}\left[x^{3/2}\right]_0^1 + \left[\frac{-2\sqrt{2}h}{3}(3-x)^{3/2}\right]_1^3 = 4h$

(b) $V = \int_0^1 \left(\frac{1}{2}\cdot 2\sqrt{x}\cdot\sqrt{3}\sqrt{x}\right) dx + \int_1^3 \left(\frac{1}{2}\cdot\sqrt{2}\sqrt{3-x}\cdot\frac{\sqrt{3}}{\sqrt{2}}\cdot\sqrt{3-x}\right) dx$

$= \sqrt{3}\int_0^1 x\,dx + \frac{\sqrt{3}}{2}\int_1^3 (3-x)\,dx = \frac{3\sqrt{3}}{2}$

(c) $V = \int_0^1 \left(\frac{1}{2}\cdot 2\sqrt{x}\cdot\sqrt{x}\right) dx + \int_1^3 \left(\frac{1}{2}\cdot\sqrt{2}\sqrt{3-x}\cdot\frac{\sqrt{2}}{2}\cdot\sqrt{3-x}\right) dx$

$= \int_0^1 x\,dx + \frac{1}{2}\int_1^3 (3-x)\,dx = \frac{3}{2}$

31. (a) $V = \int_0^4 [(\sqrt{y}\,) - (-\sqrt{y}\,)]^2\,dy = \int_0^4 4y\,dy = \left[2y^2\right]_0^4 = 32$

(b) $V = \int_0^4 \frac{\pi}{2}\left(\sqrt{y}\right)^2 dy = \frac{\pi}{2}\int_0^4 y\,dy = \frac{\pi}{2}\left[\frac{1}{2}y^2\right]_0^4 = 4\pi$

(c) $V = \int_0^4 \frac{\sqrt{3}}{4}[(\sqrt{y}\,) - (-\sqrt{y}\,)]^2\,dy = \sqrt{3}\int_0^4 y\,dy = 8\sqrt{3}$

32. (a) $V = \int_{-1}^1 (3-3y^2)h\,dy = h\left[3y - y^3\right]_{-1}^1 = 4h$

(b) $V = \int_{-1}^1 \frac{1}{2}(3-3y^2)\frac{\sqrt{3}}{2}(3-3y^2)\,dy = \frac{9\sqrt{3}}{4}\int_{-1}^1 (1-2y^2+y^4)\,dy$

$= \frac{9\sqrt{3}}{4}\left[y - \frac{2}{3}y^3 + \frac{y^5}{5}\right]_{-1}^1 = \frac{12}{5}\sqrt{3}$

(c) $V = \int_{-1}^1 \frac{1}{2}(3-3y^2)\frac{1}{2}(3-3y^2)\,dy = \frac{1}{\sqrt{3}}\cdot$[Volume in (b)] $= \frac{12}{5}$

33. (a) $V = \int_0^4 (4-x)^2\,dx = \left[16x - 4x^2 + \frac{x^3}{3}\right]_0^4 = \frac{64}{3}.$

(b) $V = \frac{1}{4}\int_0^4 (4-x)^2\,dx = \frac{1}{4}\left[16x - 4x^2 + \frac{x^3}{3}\right]_0^4 = \frac{16}{3}.$

34. (a) Area of each triangle $= y^2$, thus $V = 2\int_0^a \left(b^2 - \frac{b^2}{a^2}x^2\right) dx = 2\left[b^2x - \frac{b^2}{3a^2}x^3\right]_0^a$

$= \frac{4}{3}ab^2.$

(b) Area of each square $= 4y^2$, thus $V = 4\cdot$ the answer to part (a) $= \dfrac{16}{3}ab^2$.

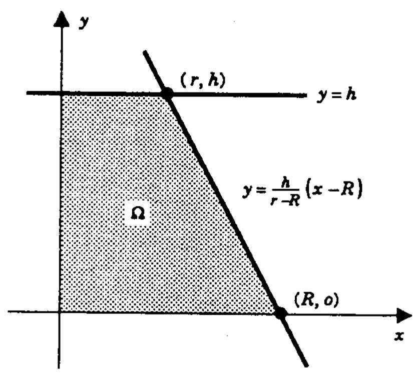
(c) Area of each triangle $= 2y$, thus $V = 2\displaystyle\int_0^a 2\sqrt{b^2 - \frac{b^2}{a^2}x^2}\,dx$

$$= \tfrac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx = \tfrac{4b}{a}\left[\tfrac{x}{2}\sqrt{a^2 - x^2} + \tfrac{a^2}{2}\sin^{-1}\tfrac{x}{a}\right]_0^a = ab\pi.$$

35. $$V = \int_{-a}^{a} \pi\left(b\sqrt{1 - \frac{x^2}{a^2}}\right)^2 dx = \frac{2b^2}{a^2}\int_0^a \pi\left(a^2 - x^2\right)^2 dx = \frac{2b^2}{a^2}\pi\left[a^2x - \frac{1}{3}x^3\right]_0^a$$
$$= \frac{2b^2}{a^2}\pi\left(\frac{2}{3}a^3\right) = \frac{4}{3}\pi ab^2$$

36. $$V = \int_{-b}^{b} \pi\left(\frac{a}{b}\sqrt{b^2 - y^2}\right)^2 dy = \frac{\pi a^2}{b^2}\int_{-b}^{b}(b^2 - y^2)\,dy$$
$$= \frac{\pi a^2}{b^2}\left[b^2y - \frac{y^3}{3}\right]_{-b}^{b} = \frac{4}{3}\pi a^2 b$$

37.



The specified frustum is generated by revolving the region Ω about the y-axis.

$$V = \int_0^h \pi\left[\frac{r-R}{h}y + R\right]^2 dy$$
$$= \pi\left[\frac{h}{3(r-R)}\left(\frac{r-R}{h}y + R\right)^3\right]_0^h$$
$$= \frac{\pi h}{3(r-R)}\left(r^3 - R^3\right) = \frac{\pi h}{3}\left(r^2 + rR + R^2\right)$$

38. $V = 2\displaystyle\int_0^{a/2} \pi\left(\sqrt{3}x\right)^2 dx = 6\pi\left[\frac{x^3}{3}\right]_0^{a/2} = \frac{1}{4}\pi a^3$ (twice the volume of cone of radius $\frac{\sqrt{3}}{2}a$, height $\frac{a}{2}$)

39. Capacity of basin $= \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = \frac{2}{3}\pi r^3$.

(a) $$\text{Volume of water } = \int_{r/2}^{r} \pi\left[\sqrt{r^2 - x^2}\,\right]^2 dx$$
$$= \pi\int_{r/2}^{r}\left(r^2 - x^2\right)dx = \pi\left[r^2x - \frac{1}{3}x^3\right]_{r/2}^{r} = \frac{5}{24}\pi r^3.$$

The basin is $\left(\frac{5}{24}\pi r^3\right)(100)\,/\,\left(\frac{2}{3}\pi r^3\right) = 31\frac{1}{4}\%$ full.

(b) Volume of water $= \displaystyle\int_{2r/3}^{r} \pi\left[\sqrt{r^2 - x^2}\,\right]^2 dx = \pi\int_{2r/3}^{r}\left(r^2 - x^2\right)dx = \frac{8}{81}\pi r^3.$

The basin is $\left(\frac{8}{81}\pi r^3\right)(100)\,/\,\left(\frac{2}{3}\pi r^3\right) = 14\frac{22}{27}\%$ full.

40. $V = \int_a^b \pi\left(\sqrt{r^2-x^2}\right)^2 dx = \pi\int_a^b (r^2-x^2)\,dx = \pi\left[r^2x - \frac{x^3}{3}\right]_a^b = \pi r^2(b-a) - \frac{1}{3}\pi(b^3-a^3).$

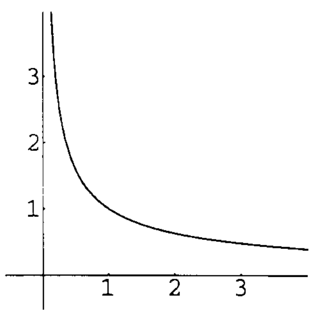
41. $V = \int_h^r \pi r^2 - y^2\,dy = \pi\left[r^2y - \frac{y^3}{3}\right]_h^r = \frac{\pi}{3}\left[2r^3 - 3r^2h + h^3\right].$

42. Imagine the punchbowl upside down on the x, y-plane centered over the origin.

(a) $V = \pi\int_1^{12}(144-y^2)\,dy = \pi\left[144y - \frac{y^3}{3}\right]_1^{12} = \pi(1584 - \frac{1727}{3})$ in^3 or about 13.7 gallons.

(b) $V = \pi\int_1^{10}(144-y^2)\,dy = \pi\left[144y - \frac{y^3}{3}\right]_1^{10} = 963\pi$ in^3 or about 13.1 gallons.

43. (a)



(b) $A(b) = \int_1^b x^{-\frac{2}{3}}\,dx = 3(b^{\frac{1}{3}} - 1)$

(c) $V(b) = \int_1^b \pi(x^{-\frac{2}{3}})^2\,dx = 3\pi(1 - b^{-\frac{1}{3}})$

(d) As $b \Rightarrow \infty, A(b) \Rightarrow \infty$ and $V(b) \Rightarrow 3\pi$.

44. (a) $A(c) = \int_c^1 x^{-\frac{2}{3}}\,dx = \left[3x^{\frac{1}{3}}\right]_c^1 = 3(1 - c^{\frac{1}{3}}).$

(c) $V(c) = \pi\int_c^1 (x^{-\frac{2}{3}})^2\,dx = \pi\left[-3x^{-\frac{1}{3}}\right]_c^1 = 3\pi(c^{-\frac{1}{3}} - 1).$

(d) As $c \to \infty$, $\quad c^{\frac{1}{3}} \to 0 \quad$ and $c^{-\frac{1}{3}} \to \infty$.

Thus $A(c) \to 3$, and $V(c) \to \infty$.

45. If the depth of the liquid in the container is h feet, then the volume of the liquid is:

$$V(h) = \int_0^h \pi\left(\sqrt{y+1}\right)^2 dy = \int_0^h [y+1]\,dy.$$

Differentiation with respect to t gives

$$\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt} = \pi(h+1)\frac{dh}{dt}.$$

Now, since $\frac{dV}{dt} = 2$, it follows that $\frac{dh}{dt} = \frac{2}{\pi(h+1)}$. Thus

$$\left.\frac{dh}{dt}\right|_{h=1} = \frac{2}{2\pi} = \frac{1}{\pi} \text{ ft/min} \quad \text{and} \quad \left.\frac{dh}{dt}\right|_{h=2} = \frac{2}{3\pi} \text{ ft/min.}$$

46. Outer radius $= f(x) + k$, inner radius $= k$, so $V = \int_a^b \pi \left([f(x)+k]^2 - k^2\right)\, dx$.

47. The cross section with coordinate x is a washer with outer radius k, inner radius $k - f(x)$, and area

$$A(x) = \pi k^2 - \pi[k - f(x)]^2 = \pi \left(2kf(x) - [f(x)]^2\right)$$

Thus

$$V = \int_a^b \pi \left(2kf(x) - [f(x)]^2\right)\, dx$$

48. (a) Outer radius $= f(x) - k$, inner radius $= g(x) - k$.

(b) $V = \int_a^b \pi \left([k - g(x)]^2 - [k - f(x)]^2\right)\, dx$

49. $V = \int_0^4 \pi \left[4\sqrt{x} - x\right]\, dx = \pi\left[-2\cos x\right]_0^\pi - \frac{\pi}{2}\int_0^\pi (1 - \cos 2x)\, dx$

50. $V = \int_0^3 \pi \left([(x+1)+1]^2 - \left[(x-1)^2+1\right]^2\right) dx = \int_0^3 \pi\left[(x+2)^2 - (x^2-2x+2)^2\right] dx$

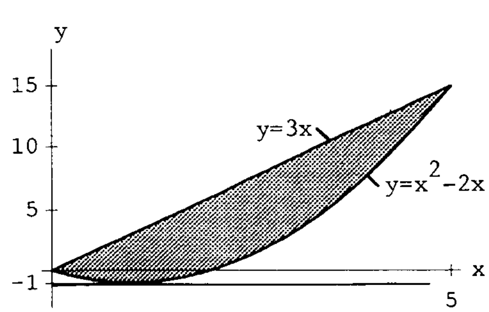
$= \int_0^3 \pi(10x - 7x^2 + 4x^3 - x^4)\, dx = \pi\left[5x^2 - \frac{7}{3}x^3 + x^4 - \frac{x^5}{5}\right]_0^3 = \frac{72}{5}\pi$

51. $V = \int_0^\pi \pi\left[2\sin x - \sin^2 x\right] dx = \pi\left[\frac{8}{3}x^{3/2} - \frac{1}{2}x^2\right]_0^4 = \frac{40\pi}{3} = 4\pi - \frac{\pi}{2}\left[1 - \frac{1}{2}\sin 2x\right]_0^\pi = 4\pi - \frac{1}{2}\pi^2$

52. $V = \int_{\pi/4}^\pi \pi\left[(1-\cos x)^2 - (1-\sin x)^2\right] dx = \pi\int_{\pi/4}^\pi (2\sin x - 2\cos x + \cos 2x)\, dx$

$= \pi\left[-2\cos x - 2\sin x + \frac{1}{2}\sin 2x\right]_{\pi/4}^\pi = \frac{3}{2} + 2\sqrt{2}$

53.



$V = \int_0^5 \pi\left([3x-) - 1)]^2 - \left[x^2 - 2x - (-1)\right]^2\right) dx$

$= \pi \int_0^5 [3x+1]^2\, dx - \int_0^5 [x-1]^4\, dx$

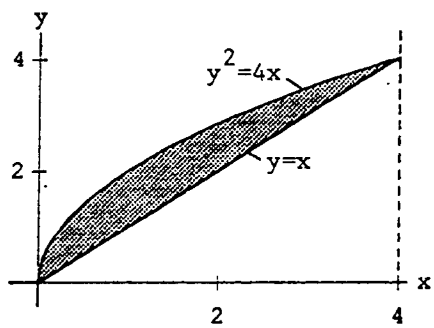
$= \pi\left[\frac{1}{9}(3x+1)^3\right]_0^5 - \frac{1}{5}\left[x-1\right]_0^5$

$= 250\pi$

54. (a) $V = \int_0^1 2\pi(2+x)\left(\sqrt{x} - x^2\right)\,dx = 2\pi\int_0^1 \left(2x^{1/2} - 2x^2 + x^{3/2} - x^3\right)\,dx = \frac{49}{30}\pi$

(b) $V = \int_0^1 2\pi(3-x)\left(\sqrt{x} - x^2\right)\,dx = 2\pi\int_0^1 \left(3x^{1/2} - 3x^2 - x^{3/2} + x^3\right)\,dx = \frac{17}{10}\pi$

55.



(a) $V = \int_0^4 \pi\left[\left(\sqrt{4x}\right)^2 - x^2\right]\,dx$

$= \pi\int_0^4 \left[4x - x^2\right]\,dx$

$= \pi\left[2x^2 - \frac{1}{3}x^3\right]_0^4 = \frac{32\pi}{3}$

(b) $V = \int_0^4 \pi\left[\left(\frac{1}{4}y^2 - 4\right)^2 - (y-4)^2\right]\,dy$

$= \pi\int_0^4 \left[\frac{1}{16}y^4 - 3y^2 + 8y\right]\,dy$

$= \pi\left[\frac{1}{80}y^5 - y^3 + 4y^2\right]_0^4 = \frac{64\pi}{5}$

56. (a) $V = \int_0^{16} \pi\left[\left(5 - \frac{y}{4}\right)^4 - (5 - \sqrt{y})^2\right]\,dy = \pi\int_0^{16}\left(10\sqrt{y} - \frac{7}{2}y + \frac{y^2}{16}\right)\,dy$

$= \pi\left[\frac{20}{3}y^{3/2} - \frac{7}{4}y^2 + \frac{y^3}{48}\right]_0^{16} = 64\pi$

(b) $V = \int_0^{16} \pi\left[(\sqrt{y}+1)^2 - \left(\frac{y}{4}+1\right)^2\right]\,dy = \pi\int_0^{16}\left(\frac{y}{2} + 2\sqrt{y} - \frac{y^2}{16}\right)\,dy$

$= \pi\left[\frac{y^2}{4} + \frac{4}{3}y^{3/2} - \frac{y^3}{48}\right]_0^{16} = 64\pi$

57. (a) $V = \int_0^4 \pi\left(x^{3/2}\right)^2\,dx = \pi\int_0^4 x^3\,dx = \pi\left[\frac{1}{4}x^4\right]_0^4 = 64\pi$

(b) $V = \int_0^8 \pi\left(4 - y^{3/2}\right)^2\,dy = \pi\int_0^8\left(16 - 8y^{2/3} + y^{4/3}\right)\,dy$

$= \pi\left[16y - \frac{24}{5}y^{5/3} + \frac{3}{7}y^{7/3}\right]_0^8 = \frac{1024}{35}\pi$

(c) $V = \int_0^4 \pi\left[(8)^2 - \left(8 - x^{3/2}\right)^2\right]\,dx = \pi\int_0^4\left(16x^{3/2} - x^3\right)\,dx$

$= \pi\left[\frac{32}{5}x^{5/2} - \frac{1}{4}x^4\right]_0^4 = \frac{704}{5}\pi$

(d) $V = \int_0^8 \pi\left[(4)^2 - \left(y^{2/3}\right)^2\right]\,dy = \pi\int_0^8\left(16 - y^{4/3}\right)\,dy = \pi\left[16y - \frac{3}{7}y^{7/3}\right]_0^8 = \frac{512}{7}\pi$

58. (a) $V = \int_0^8 \pi y^{4/3}\,dy = \pi\left[\frac{3}{7}y^{7/3}\right]_0^8 = \frac{384}{7}\pi$

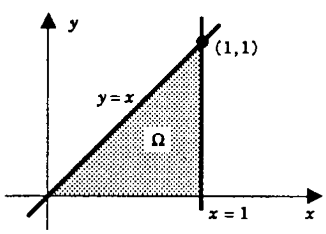
(b) $V = \int_0^4 \pi(8 - x^{3/2})^2\,dx = \pi\int_0^4 \left(64 - 16x^{3/2} + x^3\right)\,dx = \pi\left[64x - \frac{32}{5}x^{5/2} + \frac{x^4}{4}\right]_0^4 = \frac{576}{5}\pi$

(c) $V = \int_0^8 \pi\left[4^2 - (4 - y^{2/3})^2\right]\,dy = \pi\int_0^8 (8y^{2/3} - y^{4/3})\,dy = \pi\left[\frac{24}{5}y^{5/3} - \frac{3}{7}y^{7/3}\right]_0^8 = \frac{3456}{35}\pi$

(d) $V = \int_0^4 \pi\left[8^2 - x^3\right]\,dx = \pi\left[64x - \frac{x^4}{4}\right]_0^4 = 192\pi$

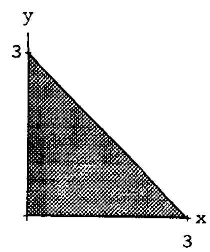
SECTION 6.3

1.



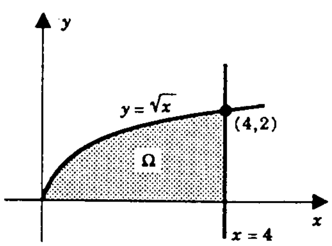
$$V = \int_0^1 2\pi x\,[x - 0]\,dx = 2\pi\int_0^1 x^2\,dx$$
$$= 2\pi\left[\frac{1}{3}x^3\right]_0^1 = \frac{2\pi}{3}$$

2.



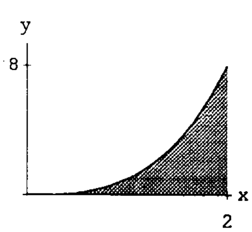
$$V = \int_0^3 2\pi x(3 - x)\,dx = 2\pi\int_0^3 (3x - x^2)\,dx$$
$$= 2\pi\left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = 9\pi$$

3.



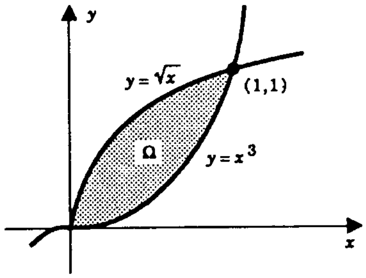
$$V = \int_0^4 2\pi x\left[\sqrt{x} - 0\right]\,dx = 2\pi\int_0^4 x^{3/2}\,dx$$
$$= 2\pi\left[\tfrac{2}{5}x^{5/2}\right]_0^4 = \frac{128}{5}\pi$$

4.



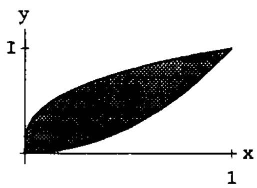
$$V = \int_0^2 2\pi x x^3\,dx = 2\pi\left[\frac{x^5}{5}\right]_0^2 = \frac{64}{5}\pi$$

5.



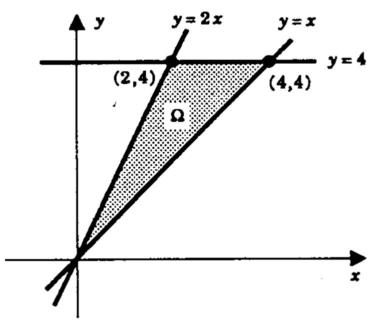
$$V = \int_0^1 2\pi x\left[\sqrt{x} - x^3\right]\,dx$$
$$= 2\pi \int_0^1 \left(x^{3/2} - x^4\right)\,dx$$
$$= 2\pi\left[\frac{2}{5}x^{5/2} - \frac{1}{5}x^5\right]_0^1 = \frac{2\pi}{5}$$

6.



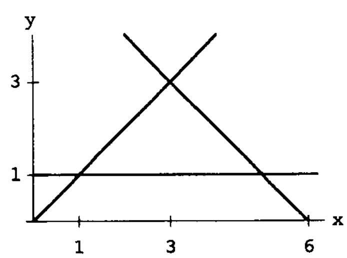
$$V = \int_0^1 2\pi x\left[x^{1/3} - x^2\right]\,dx = 2\pi\int_0^1 \left(x^{4/3} - x^3\right)\,dx$$
$$= 2\pi\left[\frac{3}{7}x^{7/3} - \frac{x^4}{4}\right]_0^1 = \frac{5\pi}{14}$$

7.



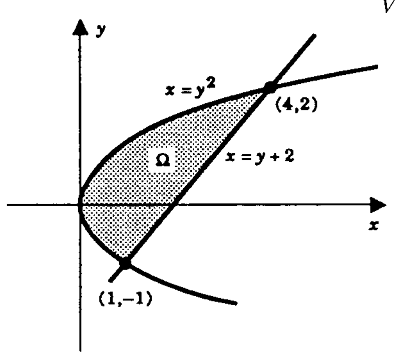
$$V = \int_0^2 2\pi x\,[2x - x]\,dx + \int_2^4 2\pi x\,[4 - x]\,dx$$
$$= 2\pi\int_0^2 x^2\,dx + 2\pi\int_2^4 \left(4x - x^2\right)\,dx$$
$$= 2\pi\left[\tfrac{1}{3}x^3\right]_0^2 + 2\pi\left[2x^2 - \tfrac{1}{3}x^3\right]_2^4 = 16\pi$$

8.



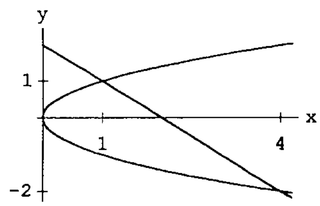
$$V = \int_1^3 2\pi x(x-1)\,dx + \int_3^5 2\pi x(6 - x - 1)\,dx$$
$$= 2\pi\int_1^3 (x^2 - x)\,dx + 2\pi\int_3^5 (5x - x^2)\,dx$$
$$= 2\pi\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^3 + 2\pi\left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_3^5 = 24\pi$$

9.



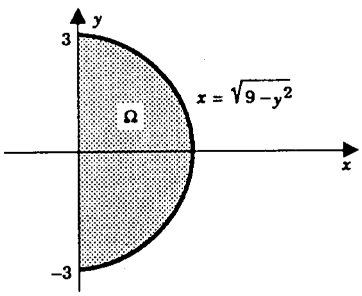
$$V = \int_0^1 2\pi x\left[(\sqrt{x}\,) - (-\sqrt{x}\,)\right]\,dx + \int_1^4 2\pi x\left[(\sqrt{x}\,) - (x-2)\right]\,dx$$
$$= 4\pi\int_0^1 x^{3/2}\,dx + 2\pi\int_1^4 \left(x^{3/2} - x^2 + 2x\right)\,dx$$
$$= 4\pi\left[\tfrac{2}{5}x^{5/2}\right]_0^1 + 2\pi\left[\tfrac{2}{5}x^{5/2} - \tfrac{1}{3}x^3 + x^2\right]_1^4 = \tfrac{72}{5}\pi$$

10.



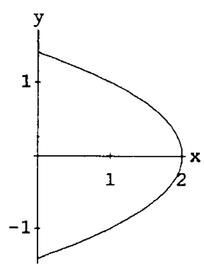
$$V = \int_0^1 2\pi x \cdot 2\sqrt{x}\,dx + \int_1^4 2\pi x(2 - x + \sqrt{x})\,dx$$
$$= 4\pi \int_0^1 x^{3/2}\,dx + 2\pi \int_1^4 (2x - x^2 + x^{3/2})\,dx$$
$$= 4\pi \left[\frac{2}{5}x^{5/2}\right]_0^1 + 2\pi \left[x^2 - \frac{x^3}{3} + \frac{2}{5}x^{5/2}\right]_1^4 = \frac{72}{5}\pi$$

11.



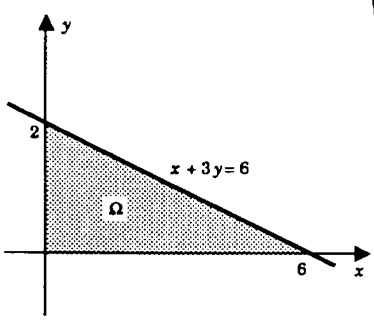
$$V = \int_0^3 2\pi x \left[\sqrt{9 - x^2} - \left(-\sqrt{9 - x^2}\right)\right] dx$$
$$= 4\pi \int_0^3 x\left(9 - x^2\right)^{1/2}\,dx$$
$$= 4\pi \left[-\tfrac{1}{3}\left(9 - x^2\right)^{3/2}\right]_0^3 = 36\pi$$

12.



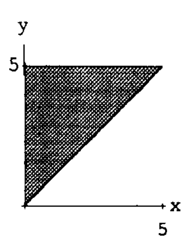
$$V = \int_0^1 2\pi x \cdot 2x\,dx + \int_1^2 2\pi x \cdot 2\sqrt{2 - x}\,dx$$
$$= 4\pi \int_0^1 x^2\,dx - 4\pi \int_1^0 (2 - u)\sqrt{u}\,du \qquad (u = 2 - x)$$
$$= 4\pi \left[\frac{x^3}{3}\right]_0^1 - 4\pi \left[\frac{4}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_1^0 = \frac{76}{15}\pi$$

13.



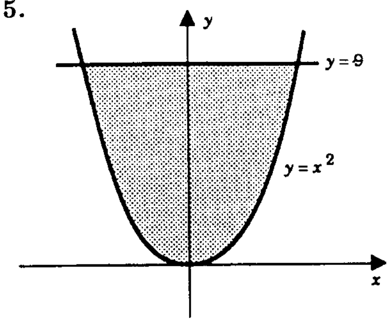
$$V = \int_0^2 2\pi y\,[6 - 3y]\,dy$$
$$= 6\pi \int_0^2 \left(2y - y^2\right)\,dy$$
$$= 6\pi \left[y^2 - \tfrac{1}{3}y^3\right]_0^2 = 8\pi$$

14.



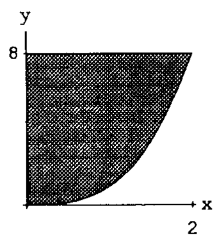
$$V = \int_0^5 2\pi y \cdot y\,dy = 2\pi \left[\frac{y^3}{3}\right]_0^5 = \frac{250}{3}\pi$$

15.



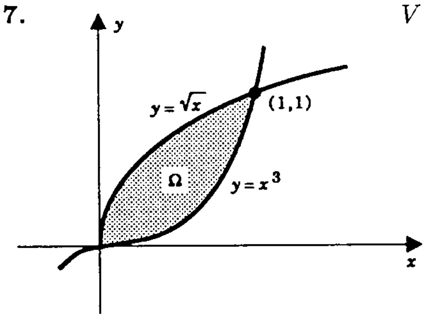
$$V=\int_0^9 2\pi y\,[(\sqrt{y}\,)-(-\sqrt{y}\,)]\,dy$$
$$=4\pi\int_0^9 y^{3/2}\,dy$$
$$=4\pi\left[\tfrac{2}{5}y^{5/2}\right]_0^9=\tfrac{1944}{5}\pi$$

16.



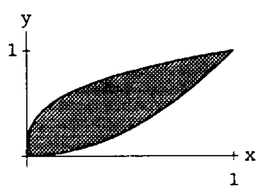
$$V=\int_0^8 2\pi y y^{1/3}\,dy=2\pi\int_0^8 y^{4/3}\,dy$$
$$=2\pi\left[\frac{3}{7}y^{7/3}\right]_0^8=\frac{768}{7}\pi$$

17.



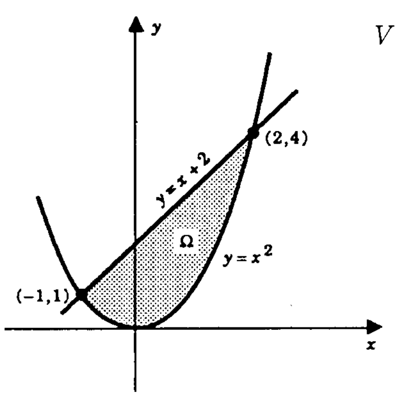
$$V=\int_0^1 2\pi y\left[y^{1/3}-y^2\right]\,dy$$
$$=2\pi\int_0^1\left(y^{4/3}-y^3\right)\,dy$$
$$=2\pi\left[\tfrac{3}{7}y^{7/3}-\tfrac{1}{4}y^4\right]_0^1=\tfrac{5}{14}\pi$$

18.



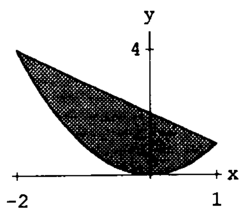
$$V=\int_0^1 2\pi y(\sqrt{y}-y^3)\,dy=2\pi\int_0^1(y^{3/2}-y^4)\,dy$$
$$=2\pi\left[\frac{2}{5}y^{5/2}-\frac{y^5}{5}\right]_0^1=\frac{2}{5}\pi$$

19.



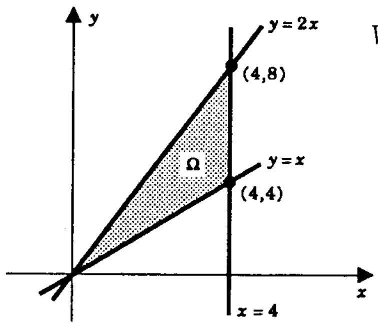
$$V=\int_0^1 2\pi y\,[(\sqrt{y}\,)-(-\sqrt{y}\,)]\,dy+\int_1^4 2\pi y\,[(\sqrt{y}\,)-(y-2)]\,dy$$
$$=4\pi\int_0^1 y^{3/2}\,dy+2\pi\int_1^4\left(y^{3/2}-y^2+2y\right)\,dy$$
$$=4\pi\left[\tfrac{2}{5}y^{5/2}\right]_0^1+2\pi\left[\tfrac{2}{5}y^{5/2}-\tfrac{1}{3}y^3+y^2\right]_1^4=\tfrac{72}{5}\pi$$

20.



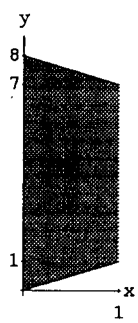
$$V = \int_0^1 2\pi y 2\sqrt{y}\,dy + \int_1^4 2\pi y(2 - y + \sqrt{y})\,dy$$
$$= 4\pi \int_0^1 y^{3/2}\,dy + 2\pi \int_1^4 (2y - y^2 + y^{3/2})\,dy$$
$$= 4\pi \left[\frac{2}{5}y^{5/2}\right]_0^1 + 2\pi \left[y^2 - \frac{y^3}{3} + \frac{2}{5}y^{5/2}\right]_1^4 = \frac{72}{5}\pi$$

21.



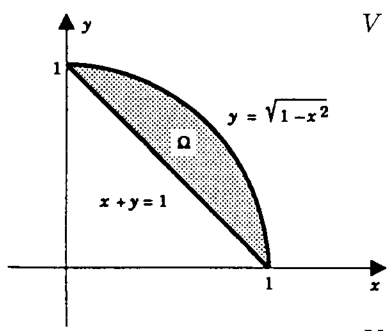
$$V = \int_0^4 2\pi y\left[y - \frac{y}{2}\right]\,dy + \int_4^8 2\pi y\left[4 - \frac{y}{2}\right]\,dy$$
$$= \pi \int_0^4 y^2\,dy + \pi \int_4^8 \left(8y - y^2\right)\,dy$$
$$= \pi\left[\tfrac{1}{3}y^3\right]_0^4 + \pi\left[4y^2 - \tfrac{1}{3}y^3\right]_4^8 = 64\pi$$

22.



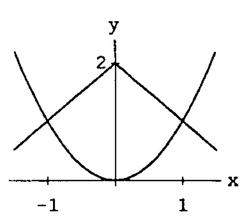
$$V = \int_1^4 2\pi y(y-1)\,dy + \int_4^7 2\pi y(8 - y - 1)\,dy$$
$$= 2\pi \int_1^4 (y^2 - y)\,dy + 2\pi \int_4^7 (7y - y^2)\,dy$$
$$= 2\pi \left[\frac{y^3}{3} - \frac{y^2}{2}\right]_1^4 + 2\pi \left[\frac{7y^2}{2} - \frac{y^3}{3}\right]_4^7 = 72\pi$$

23.



$$V = \int_0^1 2\pi y\left[\sqrt{1-y^2} - (1-y)\right]\,dy$$
$$= 2\pi \int_0^1 \left[y\left(1-y^2\right)^{1/2} - y + y^2\right]\,dy$$
$$= 2\pi \left[-\frac{1}{3}\left(1-y^2\right)^{3/2} - \frac{1}{2}y^2 + \frac{1}{3}y^3\right]_0^1 = \frac{\pi}{3}$$

24.



$$V = \int_0^1 2\pi y 2\sqrt{y}\,dy + \int_1^2 2\pi y 2(2-y)\,dy$$
$$= 4\pi \int_0^1 y^{3/2}\,dy + 4\pi \int_1^2 (2y - y^2)\,dy$$
$$= 4\pi \left[\frac{2}{5}y^{5/2}\right]_0^1 + 4\pi \left[y^2 - \frac{y^3}{3}\right]_1^2 = \frac{64}{15}\pi$$

25. (a) $V = \int_0^1 2\pi x\left[1-\sqrt{x}\right]\,dx$ (b) $V = \int_0^1 \pi\, y^4\,dy$

$= \pi\left[\frac{1}{5}\,y^5\right]_0^1 = \frac{1}{5}\pi$

26. (a) $V = \int_0^1 \pi\left[(2-\sqrt{x})^2 - 1^2\right]\,dx$

(b) $V = \int_0^1 2\pi(2-y)y^2\,dy = 2\pi\int_0^1 (2y^2-y^3)\,dy = 2\pi\left[\frac{2}{3}y^3 - \frac{y^4}{4}\right]_0^1 = \frac{5}{6}\pi$

27. (a) $V = \int_0^1 \pi\left(x - x^4\right)\,dx$ (b) $V = \int_0^1 2\pi y\left(\sqrt{y}-y^2\right)\,dy$

$= \pi\left[\frac{1}{2}\,x^2 - \frac{1}{5}\,x^5\right]_0^1 = \frac{3\pi}{10}$

28. (a) $V = \int_0^1 2\pi(x+2)(\sqrt{x}-x^2)\,dx = 2\pi\int_0^1\left(x^{3/2}+2\sqrt{x}-2x^2-x^3\right)\,dx$

$= 2\pi\left[\frac{2}{5}x^{5/2} + \frac{4}{3}x^{3/2} - \frac{2}{3}x^3 - \frac{x^4}{4}\right]_0^1 = \frac{49}{30}\pi$

(b) $V = \int_0^1 \pi\left[(\sqrt{y}+2)^2 - (y^2+2)^2\right]\,dy$

29. (a) $V = \int_0^1 2\pi x\cdot x^2\,dx$ (b) $V = \int_0^1 \pi(1-y)\,dy$

$= 2\pi\ \left[\frac{1}{4}x^4\right]_0^1 = \frac{\pi}{2}$

30. (a) $V = \int_0^1 \pi\left[(\sqrt{x}+1)^2 - (x^2+1)^2\right]\,dx$

(b) $V = \int_0^1 2\pi(y+1)(\sqrt{y}-y^2)\,dy = 2\pi\int_0^1 (y^{3/2}+\sqrt{y}-y^2-y^3)\,dy$

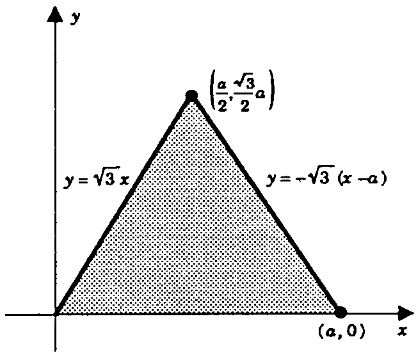
$= 2\pi\left[\frac{2}{5}y^{5/2} + \frac{2}{3}y^{3/2} - \frac{y^3}{3} - \frac{y^4}{4}\right]_0^1 = \frac{29}{30}\pi$

31. $V = \int_0^a 2\pi x\left[2b\sqrt{1-\frac{x^2}{a^2}}\,\right]\,dx = \frac{4\pi b}{a}\int_0^a x\left(a^2-x^2\right)^{1/2}\,dx = \frac{4\pi b}{a}\left[-\frac{1}{3}\left(a^2-x^2\right)^{3/2}\right]_0^a = \frac{4}{3}\pi a^2 b$

32. $V = \int_0^b 2\pi y\frac{2a}{b}\sqrt{b^2-y^2}\,dy = \frac{4\pi a}{b}\int_0^b y(b^2-y^2)^{1/2}\,dy$

$= \frac{4\pi a}{b}\left[-\frac{1}{3}(b^2-y^2)^{3/2}\right]_0^b = \frac{4}{3}\pi a b^2$

33.



By the shell method

$$V = \int_0^{a/2} 2\pi x \left(\sqrt{3}x\right)\,dx + \int_{a/2}^{a} 2\pi x \left[\sqrt{3}\,(a - x)\right]\,dx$$
$$= 2\pi\sqrt{3}\int_0^{a/2} x^2\,dx + 2\pi\sqrt{3}\int_{a/2}^{a} \left(ax - x^2\right)\,dx$$
$$= 2\pi\sqrt{3}\left[\frac{1}{3}x^3\right]_0^{a/2} + 2\pi\sqrt{3}\left[\frac{a}{2}x^2 - \frac{1}{3}x^3\right]_{a/2}^{a} = \frac{\sqrt{3}}{4}a^3\pi$$

34. $$V = \int_0^{\sqrt{r^2-a^2}} 2\pi x\left(\sqrt{r^2 - x^2} - a\right)\,dx = 2\pi\int_0^{\sqrt{r^2-a^2}}\left[x(r^2 - x^2)^{1/2} - ax\right]\,dx$$
$$= 2\pi\left[-\frac{1}{3}(r^2 - x^2)^{3/2} - \frac{a}{2}x^2\right]_0^{\sqrt{r^2-a^2}} = \frac{1}{3}\pi(2r^3 + a^3 - 3ar^2)$$

35. (a) $$V = \int_0^8 2\pi y\left[4 - y^{2/3}\right]\,dy = 2\pi\int_0^8\left(4y - y^{5/3}\right)\,dy = 2\pi\left[2y^2 - \frac{3}{8}y^{8/3}\right]_0^8 = 64\pi$$

(b) $$V = \int_0^4 2\pi\,(4 - x)\left[x^{3/2}\right]\,dx = 2\pi\int_0^4\left(4x^{3/2} - x^{5/2}\right)\,dx$$
$$= 2\pi\left[\tfrac{8}{5}x^{5/2} - \tfrac{2}{7}x^{7/2}\right]_0^4 = \tfrac{1024}{35}\pi$$

(c) $$V = \int_0^8 2\pi\,(8 - y)\left[4 - y^{2/3}\right]\,dy = 2\pi\int_0^8\left(32 - 4y - 8y^{2/3} + y^{5/3}\right)\,dy$$
$$= 2\pi\left[32y - 2y^2 - \tfrac{24}{5}y^{5/3} + \tfrac{3}{8}y^{8/3}\right]_0^8 = \tfrac{704}{5}\pi$$

(d) $$V = \int_0^4 2\pi x\left[x^{3/2}\right]\,dx = 2\pi\int_0^4 x^{5/2}\,dx = 2\pi\left[\frac{2}{7}x^{7/2}\right]_0^4 = \frac{512}{7}\pi$$

36. (a) $$V = \int_0^4 2\pi x(8 - x^{3/2})\,dx = 2\pi\int_0^4(8x - x^{5/2})\,dx = 2\pi\left[4x^2 - \frac{2}{7}x^{7/2}\right]_0^4 = \frac{384}{7}\pi$$

(b) $$V = \int_0^8 2\pi(8 - y)y^{2/3}\,dy = 2\pi\int_0^8(8y^{2/3} - y^{5/3})\,dy = 2\pi\left[\frac{24}{5}y^{5/3} - \frac{3}{8}y^{8/3}\right]_0^8 = \frac{576}{5}\pi$$

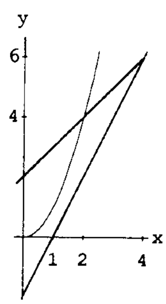
(c) $$V = \int_0^4 2\pi(4 - x)(8 - x^{3/2}\,dx = 2\pi\int_0^8(8y^{2/3} - y^{5/3})\,dy$$
$$= 2\pi\left[32x - 4x^2 - \frac{8}{5}x^{5/2} + \frac{2}{7}x^{7/2}\right]_0^4 = \frac{3456}{35}\pi$$

(d) $$V = \int_0^8 2\pi y\,y^{2/3}\,dy = 2\pi\int_0^8 y^{5/3}\,dy = 2\pi\left[\frac{3}{8}y^{8/3}\right]_0^8 = 192\pi$$

37. (a) $F'(x) = \sin x + x\cos x - \sin x = x\cos x = f(x)$.

(b) $V = \int_0^{\pi/2} 2\pi x \cdot \cos x\, dx = 2\pi\, [x \sin x + cos x]_0^{\pi/2} = \pi^2 - 2\pi$

38. (a)



By the shell method

$$V = \int_0^2 2\pi x(x^2 - 2x + 2)\, dx + \int_2^4 2\pi x\,(x + 2 - 2x + 2)\; dx$$
$$= 2\pi \int_0^2 (x^3 - 2x^2 + 2x)\, dx + 2\pi \int_2^4 (4x - x^2)\, dx$$
$$= 2\pi \left[\frac{x^4}{4} - \frac{2}{3}x^3 + x^2\right]_0^2 + 2\pi \left[2x^2 - \frac{x^3}{3}\right]_2^4 = 16\pi$$

39. (a) $V = \int_0^1 2\sqrt{3}\pi x^2\, dx + \int_1^2 2\pi x\sqrt{4 - x^2}\, dx$ (b) $V = \int_0^{\sqrt{3}} \pi \left[4 - \frac{4}{3}y^2\right]\, dy$

(c) $V = \int_0^{\sqrt{3}} \pi \left[4 - \frac{4}{3}y^2\right]\, dy = \pi\left[4y - \frac{4}{9}y^3\right]_0^{\sqrt{3}} = \frac{8\sqrt{3}\pi}{3}$

40. (a) $V = \int_0^1 \pi(\sqrt{3}x)^2\, dx + \int_1^2 \pi\left(\sqrt{4 - x^2}\right)^2\, dx$

(b) $V = \int_0^{\sqrt{3}} 2\pi y\left[\sqrt{4 - y^2} - \frac{y}{\sqrt{3}}\right]\, dy$

(c) use (a): $V = 3\pi \int_0^1 x^2\, dx + \pi \int_1^2 (4 - x^2)\, dx = \pi\, [x^3]_0^1 + \pi\left[4x - \frac{x^3}{3}\right]_1^2 = \frac{8}{3}\pi$

41. (a) $V = \int_0^4 2\sqrt{3}\,\pi x(2 - x)\, dx + \int_1^2 2\pi(2 - x)\sqrt{4 - x^2}\, dx$

(b) $V = \int_0^{\sqrt{3}} \pi\left[\left(2 - \frac{y}{\sqrt{3}}\right)^2 - \left(2 - \sqrt{4 - y^2}\right)^2\right]\, dy$

42. (a) $V = \int_0^1 \pi\left[(\sqrt{3}\,x + 1)^2 - 1^2\right]\, dx + \int_1^2 \pi\left[\left(\sqrt{4 - x^2} + 1\right)^2 - 1^2\right]\, dx$

(b) $V = \int_0^{\sqrt{3}} 2\pi(y + 1)\left(\sqrt{4 - y^2} - \frac{y}{\sqrt{3}}\right)\, dy$

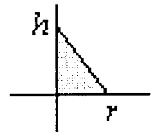
43. (a) $V = \int_{b-a}^{b+a} 2\pi x\sqrt{a^2 - (x - b)^2}\, dx$

(b) $V = \int_a^a \pi\left[\left(b + \sqrt{a^2 - y^2}\right)^2 - \left(b - \sqrt{a^2 - y^2}\right)^2\right]\, dy$

44. $V = \int_{-a}^{a} 2\pi(a - x)2\sqrt{a^2 - x^2}\, dx = 4\pi a \int_{-a}^{a} \sqrt{a^2 - x^2}\, dx - 4\pi \int_{-a}^{a} x\sqrt{a^2 - x^2}\, dx$

$= 4\pi a(\text{ Area of half circle }) - 4\pi\left[-\frac{1}{3}(a^2 - x^2)^{3/2}\right]_{-a}^{a} = 4\pi a \cdot \pi a^2 - 0 = 4\pi^2 a^3$

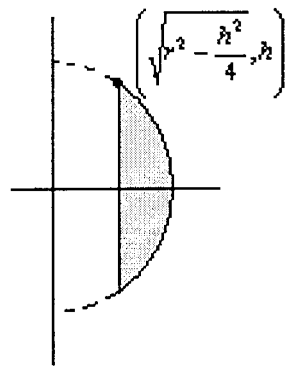
45. $V = \int_0^r 2\pi x(h - \frac{h}{r}x)\,dx = 2\pi h\left[\frac{x^2}{2} - \frac{x^3}{3r}\right]_0^r = \frac{\pi r^2 h}{3}.$



46. $V = 2\int_{\sqrt{r^2-h^2/4}}^{r} 2\pi x\sqrt{r^2-x^2}\,dx$

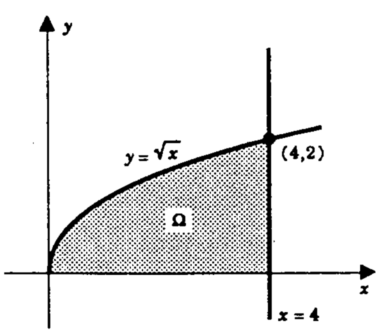
$= -2\pi\int_{h^2/4}^{0} u^{\frac{1}{2}}\,du$ (where $u = r^2 - x^2$)

$= 2\pi\left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^{h^2/4} = \frac{\pi h^3}{6}.$



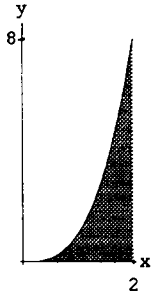
SECTION 6.4

1.



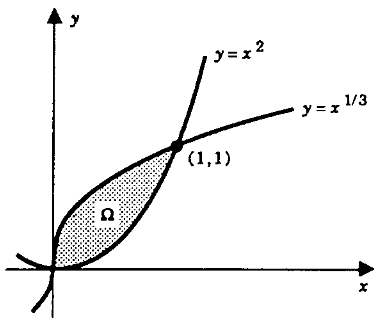
$$\begin{aligned}
A &= \int_0^4 \sqrt{x}\,dx = \frac{16}{3}\\
\overline{x}A &= \int_0^4 x\sqrt{x}\,dx = \frac{64}{5}, \quad \overline{x} = \frac{12}{5}\\
\overline{y}A &= \int_0^4 \frac{1}{2}\left(\sqrt{x}\right)^2\,dx = 4, \quad \overline{y} = \frac{3}{4}\\
V_x &= 2\pi\overline{y}A = 8\pi, \quad V_y = 2\pi\overline{x}A = \frac{128}{5}\pi
\end{aligned}$$

2.



$$\begin{aligned}
A &= \int_0^2 x^3\,dx = 4\\
\overline{x}A &= \int_0^2 x\,x^3\,dx = \frac{32}{5}, \quad \overline{x} = \frac{8}{5}\\
\overline{y}A &= \int_0^2 \frac{1}{2}(x^3)^2\,dx = \frac{64}{7}, \quad \overline{y} = \frac{16}{7}\\
V_x &= 2\pi\overline{y}A = \frac{128}{7}\pi, \quad V_y = 2\pi\overline{x}A = \frac{64}{5}\pi
\end{aligned}$$

3.



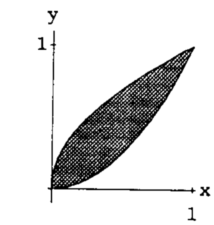
$$A = \int_0^1 \left(x^{1/3} - x^2\right)\,dx = \frac{5}{12}$$

$$\overline{x}A = \int_0^1 x\left(x^{1/3} - x^2\right)\,dx = \frac{5}{28}, \quad \overline{x} = \frac{3}{7}$$

$$\overline{y}A = \int_0^1 \frac{1}{2}\left[\left(x^{1/3}\right)^2 - \left(x^2\right)^2\right]\,dx = \frac{1}{5}, \quad \overline{y} = \frac{12}{25}$$

$$V_x = 2\pi\overline{y}A = \tfrac{2}{5}\pi, \quad V_y = 2\pi\overline{x}A = \tfrac{5}{14}\pi$$

4.



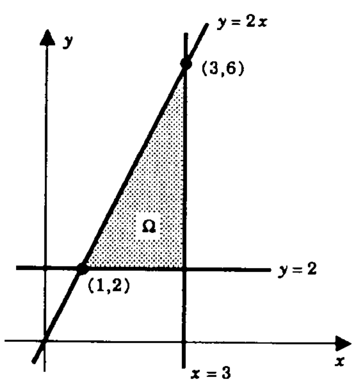
$$A = \int_0^1 (\sqrt{x} - x^3)\,dx = \frac{5}{12}$$

$$\overline{x}A = \int_0^1 x(\sqrt{x} - x^3)\,dx = \frac{1}{5}, \quad \overline{x} = \frac{12}{25}$$

$$\overline{y}A = \int_0^1 \frac{1}{2}(x - x^6)\,dx = \frac{5}{28}, \quad \overline{y} = \frac{3}{7}$$

$$V_x = 2\pi\overline{y}A = \frac{5}{14}\pi, \quad V_y = 2\pi\overline{x}A = \frac{2}{5}\pi$$

5.



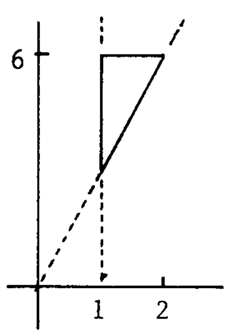
$$A = \int_1^3 (2x - 2)\,dx = 4$$

$$\overline{x}A = \int_1^3 x\,(2x - 2)\,dx = \frac{28}{3}, \quad \overline{x} = \frac{7}{3}$$

$$\overline{y}A = \int_1^3 \frac{1}{2}\left[(2x)^2 - (2)^2\right]\,dx = \frac{40}{3}, \quad \overline{y} = \frac{10}{3}$$

$$V_x = 2\pi\overline{y}A = \tfrac{80}{3}\pi, \quad V_y = 2\pi\overline{x}A = \tfrac{56}{3}\pi$$

6.



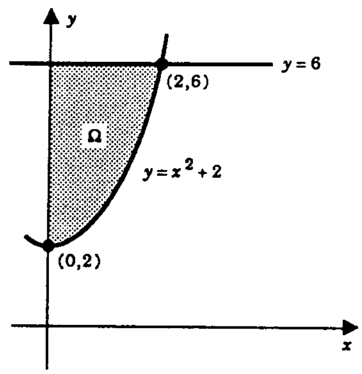
$$A = \frac{1}{2}\cdot 3\cdot 1 = \frac{3}{2}$$

$$\overline{x}A = \int_1^2 x(6 - 3x)\,dx = 2, \quad \overline{x} = \frac{4}{3}$$

$$\overline{y}A = \int_1^2 \frac{1}{2}\left(6^2 - (3x)^2\right)\,dx = \frac{15}{2}, \quad \overline{y} = 5$$

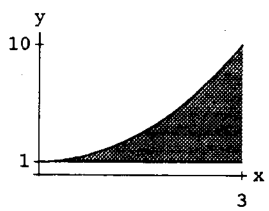
$$V_x = 2\pi\overline{y}A = 15\pi, \quad V_y = 2\pi\overline{x}A = 4\pi$$

7.



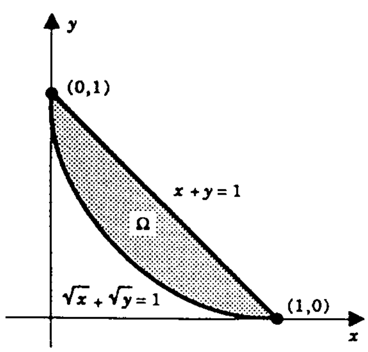
$$\begin{aligned}
A &= \int_0^2 \left[6 - \left(x^2 + 2x\right)\right]\,dx = \frac{16}{3}\\
\overline{x}A &= \int_0^2 x\left[6 - \left(x^2 + 2\right)\right]\,dx = 4, \quad \overline{x} = \frac{3}{4}\\
\overline{y}A &= \int_0^2 \frac{1}{2}\left[(6)^2 - \left(x^2 + 2\right)^2\right]\,dx = \frac{352}{15}, \quad \overline{y} = \frac{22}{5}\\
V_x &= 2\pi\overline{y}A = \tfrac{704}{15}\pi, \quad V_y = 2\pi\overline{x}A = 8\pi
\end{aligned}$$

8.



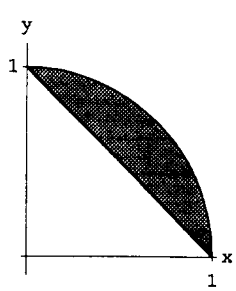
$$\begin{aligned}
A &= \int_0^3 \left[(x^2 + 1) - 1\right]\,dx = 9\\
\overline{x}A &= \int_0^3 x\left[(x^2 + 1) - 1\right]\,dx = \frac{81}{4}, \quad \overline{x} = \frac{9}{4}\\
\overline{y}A &= \int_0^3 \frac{1}{2}\left[(x^2 + 1)^2 - 1^2\right]\,dx = \frac{999}{30}, \quad \overline{y} = \frac{111}{30}\\
V_x &= 2\pi\overline{y}A = \frac{333}{5}\pi, \quad V_y = 2\pi\overline{x}A = \frac{81}{2}\pi
\end{aligned}$$

9.



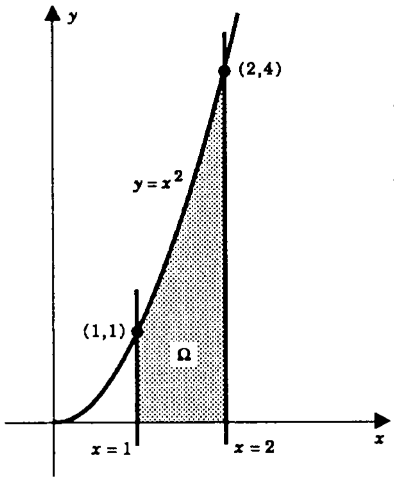
$$\begin{aligned}
A &= \int_0^1 \left[(1 - x) - \left(1 - \sqrt{x}\right)^2\right]\,dx = \frac{1}{3}\\
\overline{x}A &= \int_0^1 x\left[(1 - x) - \left(1 - \sqrt{x}\,\right)^2\right]\,dx = \frac{2}{15}, \quad \overline{x} = \frac{2}{5}\\
\overline{y} &= \tfrac{2}{5} \qquad \text{by symmetry}\\
V_x &= 2\pi\overline{y}A = \tfrac{4}{15}\pi, \quad V_y = \tfrac{4}{15}\pi \qquad \text{by symmetry}
\end{aligned}$$

10.



$$\begin{aligned}
A &= \frac{\pi}{4} - \frac{1}{2} = \frac{\pi - 2}{4}\\
\overline{x}A &= \int_0^1 x\left(\sqrt{1 - x^2} + x - 1\right)\,dx = \frac{1}{6}, \quad \overline{x} = \frac{2}{3(\pi - 2)}\\
\overline{y} &= \frac{2}{3(\pi - 2)} \qquad \text{by symmetry}\\
V_x &= 2\pi\overline{y}A = \frac{\pi}{3}, \quad V_y = 2\pi\overline{x}A = \frac{\pi}{3}
\end{aligned}$$

11.



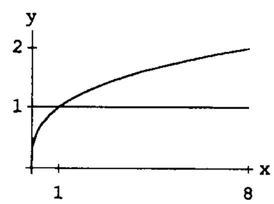
$$A = \int_1^2 x^2\,dx = \frac{7}{3}$$

$$\overline{x}A = \int_1^2 x\left(x^2\right)\,dx = \frac{15}{4}, \quad \overline{x} = \frac{45}{28}$$

$$\overline{y}A = \int_1^2 \frac{1}{2}\left(x^2\right)^2\,dx = \frac{31}{10}, \quad \overline{y} = \frac{93}{70}$$

$$V_x = 2\pi\overline{y}A = \tfrac{31}{5}\pi, \quad V_y = 2\pi\overline{x}A = \tfrac{15}{2}\pi$$

12.



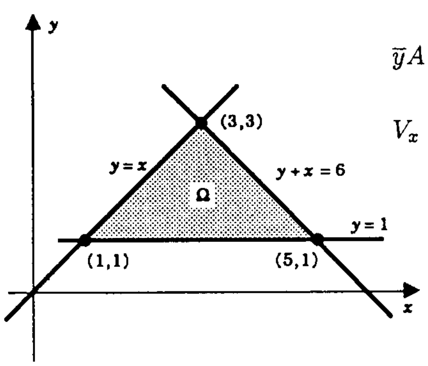
$$A = \int_1^8 (x^{1/3} - 1)\,dx = \frac{17}{4}$$

$$\overline{x}A = \int_1^8 x(x^{1/3} - 1)\,dx = \frac{321}{14}, \quad \overline{x} = \frac{642}{119}$$

$$\overline{y}A = \int_1^8 \frac{1}{2}(x^{2/3} - 1^2)\,dx = \frac{58}{10}, \quad \overline{y} = \frac{116}{85}$$

$$V_x = 2\pi\overline{y}A = \frac{58}{5}\pi, \quad V_y = 2\pi\overline{x}A = \frac{321}{7}\pi$$

13.

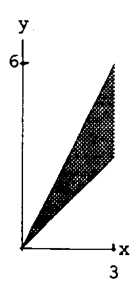


$A = \tfrac{1}{2}bh = 4;$ by symmetry, $\overline{x} = 3$

$$\overline{y}A = \int_1^3 y\left[(6-y) - y\right]\,dy = \frac{20}{3}, \quad \overline{y} = \frac{5}{3}$$

$$V_x = 2\pi\overline{y}A = \tfrac{40}{3}\pi, \quad V_y = 2\pi\overline{x}A = 24\pi$$

14.



$$A = \frac{9}{2}$$

$$\overline{x}A = \int_0^3 x(2x - x)\,dx = 9, \quad \overline{x} = 2$$

$$\overline{y}A = \int_0^3 \frac{1}{2}\left(4x^2 - x^2\right)\,dx = \frac{27}{2}, \quad \overline{y} = 3$$

$$V_x = 2\pi\overline{y}A = 27\pi, \quad V_y = 2\pi\overline{x}A = 18\pi$$

15. $\left(\frac{5}{2}, 5\right)$

16. $A = \int_{-1}^{3} (4x - x^2 - 2x + 3)\, dx = \dfrac{32}{3}$

$\overline{x}A = \int_{-1}^{3} x(2x - x^2 + 3)\, dx = \dfrac{32}{3} \implies \overline{x} = 1$

$\overline{y}A = \int_{-1}^{3} \dfrac{1}{2}\left[(4x - x^2)^2 - (2x-3)^2\right]\, dx = \dfrac{32}{5} \implies \overline{y} = \dfrac{3}{5}$

17. $\left(1, \frac{8}{5}\right)$

18. $A = \int_{-1}^{3} (2x + 3 - x^2)\, dx = \dfrac{32}{3}$

$\overline{x}A = \int_{-1}^{3} x(2x + 3 - x^2)\, dx = \dfrac{32}{3} \implies \overline{x} = 1$

$\overline{y}A = \int_{-1}^{3} \dfrac{1}{2}\left[(2x+3)^2 - x^4\right]\, dx = \dfrac{544}{15} \implies \overline{y} = \dfrac{17}{5}$

19. $\left(\frac{10}{3}, \frac{40}{21}\right)$

20. $A = \int_{0}^{2} (x - x^2 + \sqrt{2x})\, dx = 2$

$\overline{x}A = \int_{0}^{2} x(x - x^2 + \sqrt{2x})\, dx = \dfrac{28}{15} \implies \overline{x} = \dfrac{14}{15}$

$\overline{y}A = \int_{0}^{2} \dfrac{1}{2}\left[(x - x^2)^2 - 2x\right]\, dx = -\dfrac{22}{15} \implies \overline{y} = -\dfrac{11}{15}$

21. $(2, 4)$

22. $A = \int_{1}^{6} (6x - x^2 - 6 + x)\, dx; \quad \overline{x}A = \int_{1}^{6} x(6x - x^2 - 6 + x)\, dx;$

$\overline{y}A = \int_{1}^{6} \dfrac{1}{2}\left[(6x - x^2)^2 - (6-x)^2\right]\, dx. \implies \overline{x} = \dfrac{7}{2}, \quad \overline{y} = 5$

23. $\left(-\frac{3}{5}, 0\right)$

24. $A = \int_{0}^{a} (\sqrt{a} - \sqrt{x})^2\, dx; \quad \overline{x}A = \int_{0}^{a} x(\sqrt{a} - \sqrt{x})^2\, dx;$

$\overline{y}A = \int_{0}^{a} \dfrac{1}{2}(\sqrt{a} - \sqrt{x})^4\, dx. \implies \overline{x} = \dfrac{a}{5}, \quad \overline{y} = \dfrac{a}{5}$

25. (a) $(0,0)$ by symmetry

(b) Ω_1 smaller quarter disc, Ω_2 the larger quarter disc

$$A_1 = \frac{1}{16}\pi, \quad A_2 = \pi; \quad \overline{x}_1 = \overline{y}_1 = \frac{2}{3\pi}, \quad \overline{x}_2 = \overline{y}_2 = \frac{8}{3\pi} \quad \text{(Problem 1)}$$

$$\overline{x}A = \left(\frac{8}{3\pi}\right)(\pi) - \frac{2}{3\pi}\left(\frac{1}{16}\pi\right)\frac{63}{24}, \quad A = \frac{15}{16}\pi$$

$$\overline{x} = \left(\frac{63}{24}\right) \bigg/ \left(\frac{15\pi}{16}\right) = \frac{14}{5\pi}, \quad \overline{y} = \overline{x} = \frac{14}{5\pi} \quad \text{(symmetry)}$$

(c) $\overline{x} = 0, \quad \overline{y} = \dfrac{14}{5\pi}$

26. $A = \dfrac{1}{2}\pi ab; \quad \overline{x} = 0$ by symmetry.

$$\overline{y}A = \int_{-a}^{a} \frac{1}{2}\left(\frac{b}{a}\sqrt{a^2 - x^2}\right) dx = \frac{2}{3}ab^2 \implies \overline{y} = \frac{4b}{3\pi}.$$

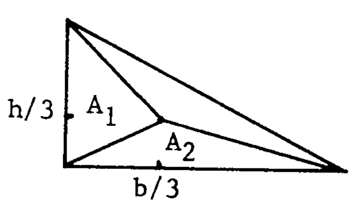
27. Use theorem of Pappus. Centroid of rectangle is located

$$c + \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2} \text{ units}$$

from line l. The area of the rectangle is ab. Thus,

$$\text{volume } = 2\pi\left[c + \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}\right](ab) = \pi ab\left(2c + \sqrt{a^2 + b^2}\right).$$

28. (a)



$$A_1 = \tfrac{1}{2}h \cdot \tfrac{b}{3} = \tfrac{1}{6}hb; \qquad A_2 = \tfrac{1}{2}b \cdot \tfrac{h}{3} = \tfrac{1}{6}hb.$$

$$A_3 = \tfrac{1}{2}bh - (A_1 + A_2) = \tfrac{1}{2}bh - \tfrac{1}{3}bh = \tfrac{1}{6}bh.$$

(b) Hypotenuse has equation $hx + by - bh = 0$, so distance from $(\frac{b}{3}, \frac{h}{3})$ to hypotenuse is

$$d = \frac{\left|\frac{hb}{3} + \frac{bh}{3} - bh\right|}{\sqrt{h^2 + b^2}} = \frac{bh}{3\sqrt{h^2 + b^2}}$$

(c) $V = 2\pi dA = 2\pi \dfrac{bh}{3\sqrt{h^2 + b^2}} \cdot \dfrac{1}{2}bh = \dfrac{\pi b^2 h^2}{3\sqrt{h^2 + b^2}}$

29. (a) $\left(\frac{2}{3}a, \frac{1}{3}h\right)$ (b) $\left(\frac{2}{3}a + \frac{1}{3}b, \frac{1}{3}h\right)$ (c) $\left(\frac{1}{3}a + \frac{1}{3}b, \frac{1}{3}h\right)$

30. (a) $V = 2\pi\overline{y}A = 2\pi\dfrac{1}{3}h\dfrac{1}{2}bh = \dfrac{1}{3}\pi bh^2$ [using Exercise 29(c)]

(b) $V = 2\pi \overline{x} A = 2\pi \, \frac{1}{3}(a+b)\, \frac{1}{2}bh = \frac{1}{3}\,\pi(a+b)bh.$

31. (a) $V = \frac{2}{3}\pi R^3 \sin^3\theta + \frac{1}{3}\pi R^3 \sin^2\theta\cos\theta = \frac{1}{3}\pi R^3 \sin^2\theta\,(2\sin\theta + \cos\theta)$

(b) $$\overline{x} = \frac{V}{2\pi A} = \frac{\frac{1}{3}\pi R^3 \sin^2\theta\,(2\sin\theta+\cos\theta)}{2\pi\left(\frac{1}{2}R^2\sin\theta\cos\theta + \frac{1}{4}\pi R^2\sin^2\theta\right)} = \frac{2R\,\sin\theta\,(2\sin\theta+\cos\theta)}{3\,(\pi\sin\theta + 2\cos\theta)}$$

32. (a) $\overline{x} = 0, \overline{y} = 0$ (by symmetry).

(b) $\overline{x} = 0$ (by symmetry about y-axis), $\overline{y} = r + \dfrac{4r}{3\pi}$ by Example 6.

(c) $\overline{x} = 0$ (by symmetry about y-axis).

$$\overline{y} = \frac{1}{A}\int_{-r}^{r}\frac{1}{2}\left[\left(r+\sqrt{r^2-x^2}\right)^2 - (-r)^2\right]dx = \frac{1}{A}\left[r\int_{-r}^{r}\sqrt{r^2-x^2}\,dx + \frac{1}{2}\int_{-r}^{r}(r^2-x^2)\,dx\right]$$

$$= \frac{1}{\frac{\pi r^2}{2} + 4r^2}\left[r\frac{\pi r^2}{2} + \frac{2}{3}r^3\right] = \frac{r}{3}\left(\frac{3\pi+4}{\pi+8}\right)$$

(d) as in (c): $\overline{x} = 0$, (by symmetry about y-axis), $\overline{y} = -\frac{r}{3}\left(\frac{3\pi+4}{\pi+8}\right)$

(e) $\overline{x} = 0$, $\overline{y} = 0$ (by symmetry).

(f) $A = \pi r^2 + 4r^2$

$$\overline{x}A = \int_{-r}^{r} x\left(2r + \sqrt{r^2-x^2}\right)dx + \int_{r}^{2r} x\,2\sqrt{r^2-(x-r)^2}\,dx = 0 + \overline{x}_{\Omega_2}A_{\Omega_2}$$

$$\Longrightarrow\ \overline{x} = \overline{x}_{\Omega_2}\frac{A_{\Omega_2}}{A} = \left(r + \frac{4r}{3\pi}\right)\frac{\pi r^2/2}{\pi r^2 + 4r^2} = \frac{r}{6}\left(\frac{3\pi+4}{\pi+4}\right)$$

By symmetry about the line $y = x$, $\overline{y} = \overline{x} = \dfrac{r}{6}\left(\dfrac{3\pi+4}{\pi+4}\right)$

(g) $A = \frac{3}{2}\pi r^2 + 4r^2$

$$\overline{x}A = \overline{x}_{S\cup\Omega_1\cup\Omega_2}\,A_{S\cup\Omega_1\cup\Omega_2} \Longrightarrow \overline{x} = \frac{r}{6}\left(\frac{3\pi+4}{\pi+4}\right)\frac{\pi r^2+4r^2}{\frac{3}{2}\pi r^2 + 4r^2} = \frac{r}{3}\left(\frac{3\pi+4}{\pi+8}\right)$$

$\overline{y} = 0$, (by symmetry about x-axis).

33. (a) The mass contributed by $[x_{i-1}, x_i]$ is approximately $\lambda(x_i^*)\,\Delta x_i$ where x_i^* is the midpoint of $[x_{i-1}, x_i]$. The sum of these contributions,

$$\lambda(x_1^*)\,\Delta x_1 + \cdots + \lambda(x_n^*)\,\Delta x_n,$$

is a Riemann sum, which as $\|P\| \to 0$, tends to the given integral.

(b) Take M_i as the mass contributed by $[x_{i-1}, x_i]$. Then $x_{M_i}M_i \cong x_i^*\lambda(x_i^*)\,\Delta x_i$ where x_i^* is the midpoint of $[x_{i-1}, x_i]$. Therefore

$$x_M M = x_{M_1} M_1 + \cdots + x_{M_n} M_n \cong x_1^* \lambda (x_1^*) \, \Delta x_1 + \cdots + x_n^* \lambda (x_n^*) \, \Delta x_n.$$

As $\|P\| \to 0$, the sum on the right converges to the given integral.

PROJECT 6.4

1. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. P breaks up $[a, b]$ into n subintervals $[x_{i-1}, x_i]$. Choose x_i^* as the midpoint of $[x_{i-1}, x_i]$. By revolving the ith midpoint rectangle about x-axis, we obtain a solid cylinder of volume $V_i = \pi \left[f(x_i^*) \right]^2 \Delta x_i$ and centroid (center) on the x-axis at $x = x_i^*$. The union of all these cylinders has centroid at $x = \overline{x}_P$ where

$$x_p V_p = \pi x_1^* \left[f(x_1^*) \right]^2 \Delta x_1 + \cdots + \pi x_n^* \left[f(x_n^*) \right]^2 \Delta x_n$$

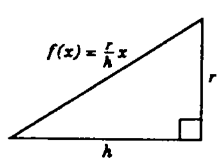
(Here V_P represents the union of the n cylinders.) As $\|P\| \to 0$, the union of the cylinders tends to the shape of S and the equation just derived tends to Formula 6.4.5.

2. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. P breaks up $[a, b]$ into n subintervals $[x_{i-1}, x_i]$. Choose x_i^* as the midpoint of $[x_{i-1}, x_i]$. By revolving the ith midpoint rectangle about y-axis, we obtain a solid cylinder of volume $V_i = 2\pi x_i^* f(x_i^*) \, \Delta x_i$ and centroid (center) on the y-axis at $y = \frac{1}{2} f(x_i^*)$. The union of all these cylindrical shells has centroid at $y = \overline{y}_P$ where

$$\overline{y}_p V_p = \pi x_1^* \left[f(x_1^*) \right]^2 \Delta x_1 + \cdots + \pi x_n^* \left[f(x_n^*) \right]^2 \Delta x_n$$

(Here V_P represents volume of the union of the n cylindrical shells.) As $\|P\| \to 0$, the union of the cylinders tends to the shape of S and the equation just derived tends to Formula 6.4.6.

3. (a)



$$\overline{x} \left(\frac{1}{3} \pi r^2 h \right) = \int_0^h \pi x \left(\frac{r}{h} x \right)^2 \, dx = \frac{1}{4} \pi r^2 h^2$$

$$\overline{x} = \left(\tfrac{1}{4} \pi r^2 h^2 \right) / \left(\tfrac{1}{3} \pi r^2 h \right) = \tfrac{3}{4} h.$$

The centroid of the cone lies on the axis of the cone at a distance $\frac{3}{4} h$ from the vertex.

(b) The hemisphere is obtained by rotating $f(x) = \sqrt{r^2 - x^2}, x \in [0, r]$, around the x-axis.

$$V_x = \frac{2}{3} \pi r^3; \quad \overline{x} V_x = \int_0^r \pi x (r^2 - x^2) \, dx = \pi \left[r^2 \frac{x^2}{2} - \frac{x^4}{4} \right]_0^r = \pi \frac{r^4}{4}$$

$$\Longrightarrow \overline{x} = \frac{3r}{8} \quad \left(\tfrac{3r}{8} \text{ units from the center of the base along the axis} \right).$$

(c) $$V_x = \int_0^a \pi \frac{b^2}{a^2} \left(a^2 - x^2 \right) \, dx = \frac{2}{3} \pi a b^2, \quad \overline{x} V_x = \int_0^a \pi x \frac{b^2}{a^2} \left(a^2 - x^2 \right) \, dx = \frac{1}{4} \pi a^2 b^2$$

$$\overline{x} = \left(\tfrac{1}{4} \pi a^2 b^2 \right) / \left(\tfrac{2}{3} \pi a b^2 \right) = \tfrac{3}{8} a; \quad \text{centroid } \left(\tfrac{3}{8} a, 0 \right)$$

(d) (i) $V_x = \int_0^1 \pi\left(\sqrt{x}\,\right)^2\,dx = \frac{1}{2}\pi, \quad \overline{x}V_x = \int_0^1 \pi x\left(\sqrt{x}\,\right)^2\,dx = \frac{1}{3}\pi$

$\overline{x} = \left(\frac{1}{3}\pi\right)/\left(\frac{1}{2}\pi\right) = \frac{2}{3};$ centroid $\left(\frac{2}{3}, 0\right)$

(ii) $V_y = \int_0^1 2\pi x\sqrt{x}\,\,dx = \frac{4}{5}\pi, \quad \overline{y}V_y = \int_0^1 \pi x\left(\sqrt{x}\,\right)^2\,dx = \frac{1}{3}\pi$

$\overline{y} = \left(\frac{1}{3}\pi\right)/\left(\frac{4}{5}\pi\right) = \frac{5}{12};$ centroid $\left(0, \frac{5}{12}\right)$

(e) (i) $V_x = \int_0^2 \pi(4-x^2)^2\,dx = \frac{256}{15}\pi$

$$\overline{x}V_x = \int_0^2 \pi x(4-x^2)^2\,dx = \pi\left[-\frac{(4-x^2)^3}{6}\right]_0^2 = \frac{32\pi}{3} \implies \overline{x} = \frac{5}{8}; \quad \overline{y} = 0$$

(ii) $V_y = \int_0^2 2\pi x(4-x^2)\,dx = 8\pi$

$$\overline{y}V_y = \int_0^2 \pi x(4-x^2)^2\,dx = \frac{32}{3}\pi \implies \overline{y} = \frac{4}{3}; \quad \overline{x} = 0$$

SECTION 6.5

1. $W = \int_1^4 x\left(x^2+1\right)^2\,dx = \frac{1}{6}\left[\left(x^2+1\right)^3\right]_1^4 = 817.5$ ft-lb

2. $W = \int_3^8 2x\sqrt{x+1}\,dx = \int_4^9 2(u-1)\sqrt{u}\,du = 2\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_4^9 = \frac{2152}{15}$ ft-lb

3. $W = \int_1^3 x\sqrt{x^2+7}\,dx = \frac{1}{3}\left[(x^2+7)^{\frac{3}{2}}\right]_0^3 = \frac{1}{3}(64 - 7^{\frac{3}{2}})$ft-lb

4. $W = \int_0^{\frac{\pi}{4}} (x^2 + \cos 2x\,dx = \left[\frac{x^3}{3} + \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}} = (\frac{\pi^3}{192} + \frac{1}{2})$ft-lb

5. $W = \int_{\pi/6}^{pi} (x + \sin 2x)\,dx = \left[\frac{1}{2}\,x^2 - \frac{1}{2}\cos 2x\right]_{\pi/6}^{\pi} = \frac{35}{72}\pi^2 - \frac{1}{4}$ Newton-meters

6. $W = \int_0^{\pi/2} \frac{\cos 2x}{\sqrt{2+\sin 2x}}\,dx = \left[(2+\sin 2x)^{1/2}\right]_0^{\pi/2} = 0$

7. By Hooke's law, we have $600 = -k(-1)$. Therefore $k = 600$.

   The work required to compress the spring to 5 inches is given by

$$W = \int_{10}^{5} 600(x-10)\,dx = 600\left[\frac{1}{2}x^2 - 10x\right]_{10}^{5}$$

$$= 7500 \text{ in-lb, or } 625 \text{ ft-lb}$$

8. Work done by spring $= -5 = \displaystyle\int_1^3 -kx\,dx = -k\left[\frac{x^2}{2}\right]_1^3 = -4k \implies k = \frac{5}{4}.$

We want s such that $-6 = \displaystyle\int_0^s -\frac{5}{4}x\,dx = -\frac{5}{4}\frac{s^2}{2} \implies s = \frac{4\sqrt{3}}{\sqrt{5}}$ feet.

9. To counteract the restoring force of the spring we must apply a force $F(x) = kx$.

Since $F(4) = 200$, we see that $k = 50$ and therefore $F(x) = 50x$.

(a) $W = \displaystyle\int_0^1 50x\,dx = 25$ ft-lb (b) $W = \displaystyle\int_0^{3/2} 50x\,dx = \frac{225}{4}$ ft-lb

10. (a) $W = \displaystyle\int_0^a -kx\,dx = -\frac{k}{2}a^2 \implies \int_0^{2a} -kx\,dx = -\frac{k}{2}(2a)^2 = 4\left(-\frac{k}{2}a^2\right) = 4W$

(b) $\displaystyle\int_0^{na} -kx\,dx = -\frac{k}{2}n^2a^2 = n^2W$

(c) $\displaystyle\int_a^{2a} -kx\,dx = -\frac{k}{2}(4a^2 - a^2) = 3\left(-\frac{k}{2}a^2\right) = 3W$

(d) $\displaystyle\int_a^{na} -kx\,dx = \int_0^{na} -kx\,dx - \int_0^a -kx\,dx = n^2W - W = (n^2-1)W$

11. Let L be the natural length of the spring.

$$\int_{2-L}^{2.1-L} kx\,dx = \frac{1}{2}\int_{2.1-L}^{2.2-L} kx\,dx$$

$$\left[\tfrac{1}{2}kx^2\right]_{2-L}^{2.1-L} = \tfrac{1}{2}\left[\tfrac{1}{2}kx^2\right]_{2.1-L}^{2.2-L}$$

$$(2.1-L)^2 - (2-L)^2 = \tfrac{1}{2}\left[(2.2-L)^2 - (2.1-L)^2\right].$$

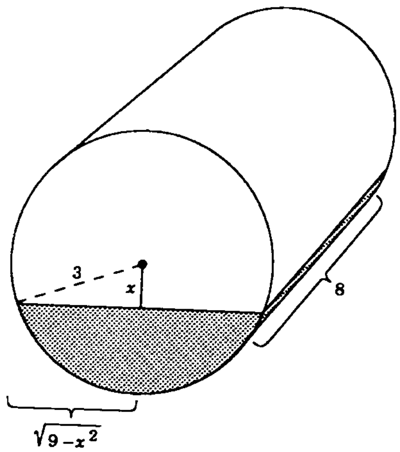
Solve this equation for L and you will find that $L = 1.95$.

Answer: 1.95 ft

12. (a) $W = \displaystyle\int_a^b \sigma s(x)A(x)\,dx = \int_0^6 62.5\,x\,4\pi\,dx = 4,500\pi$ ft-lb

(b) $W = \displaystyle\int_0^6 62.5\,(x+5)\cdot 4\pi\,dx = 12,000\pi$ ft-lb.

13.



(a) $W = \int_0^3 (x+3)(60)(8)\left(2\sqrt{9-x^2}\right)dx$

$= 960\int_0^3 x\left(9-x^2\right)^{1/2}dx$

$+2880\underbrace{\int_0^3 \sqrt{9-x^2}\,dx}_{\text{area of quarter circle of radius 3}}$

$= 960\left[-\frac{1}{3}\left(9-x^2\right)^{3/2}\right]_0^3 + 2880\left[\frac{9}{4}\pi\right]$

$= (8640 + 6480\pi)$ ft-lb

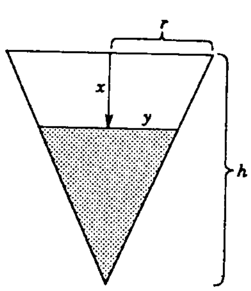
(b) $W = \int_0^3 (x+7)(60)(8)\left(2\sqrt{9-x^2}\right)dx = 960\int_0^3 x\left(9-x^2\right)^{1/2}dx + 6720\int_0^3 \sqrt{9-x^2}\,dx$

$= (8640 + 15120\pi)$ ft-lb

14. $W = \int_0^3 60(3-x)16\sqrt{9-x^2}\,dx = 720\cdot 4\int_0^3 \sqrt{9-x^2}\,dx - 960\int_0^3 x\sqrt{9-x^2}\,dx$

$= 720(9\pi) - 960\left[-\frac{1}{3}(9-x^2)^{3/2}\right]_0^3 = (6480\pi - 8640)$ ft-lb

15.



By similar triangles

$$\frac{h}{r} = \frac{h-x}{y} \quad \text{so that} \quad y = \frac{r}{h}(h-x).$$

Thus, the area of a cross section of the fluid at a depth of x feet is

$$\pi y^2 = \pi\frac{r^2}{h^2}(h-x)^2.$$

(a) $W = \int_0^{h/2} x\sigma\left[\pi\frac{r^2}{h^2}(h-x)^2\right]dx = \frac{\sigma\pi r^2}{h^2}\int_0^{h/2}\left(h^2x - 2hx^2 + x^3\right)dx = \frac{11}{192}\sigma\pi r^2h^2$ ft-lb

(b) $W = \int_0^{h/2} (x+k)\sigma\left[\pi\frac{r^2}{h^2}(h-x)^2\right]dx = \frac{11}{192}\pi r^2h^2\sigma + \frac{7}{24}\pi r^2hk\sigma$ ft-lb

16. $W = \int_0^h \sigma(h-x)\pi \frac{r^2}{h^2}(h-x)^2\,dx = \sigma\pi\frac{r^2}{h^2}\left[-\frac{(h-x)^4}{4}\right]_0^h = \frac{1}{4}\sigma\pi r^2 h^2$ ft-lb.

17. $y = \frac{3}{4}x^2,\ 0 \le x \le 4$

(a) $W = \int_0^{12} \sigma(12-y)\pi x^2\,dy = \frac{4}{3}\pi\sigma\int_0^{12}(12y - y^2)\,dy$

$= \frac{4}{3}\pi\sigma\left[6y^2 - \frac{y^3}{3}\right]_0^{12} = \frac{4}{3}\pi\sigma(288) = 384\pi\sigma$ newton-meters.

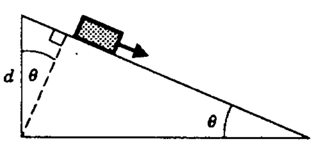
(b) $W = \int_0^{12} \sigma(13-y)\pi x^2\,dy = \frac{4}{3}\pi\sigma\int_0^{12}(13y - y^2)\,dy$

$= \frac{4}{3}\pi\sigma\left[\frac{13y^2}{2} - \frac{y^3}{3}\right]_0^{12} = \frac{4}{3}\pi\sigma(360) = 480\pi\sigma$ newton-meters.

18. $W = \int_{r_1}^{r_2} F\,dr = \int_{r_1}^{r_2} -\frac{GmM}{r^2}\,dr = \left[\frac{GmM}{r}\right]_{r_1}^{r_2} = GmM\left[\frac{1}{r_2} - \frac{1}{r_1}\right]$

19. $W = \int_0^{80}(80-x)15\,dx = 15\left[80x - \frac{1}{2}x^2\right]_0^{80} = 48{,}000$ ft-lb

20. (a) $W = wd$ ft-lb



(b) component of force along the inclined plane$=w\sin\theta$ lb

distance traveled$=\frac{d}{\sin\theta}$ ft; $W = (w\sin\theta)\frac{d}{\sin\theta} = wd$ ft-lb.

21. (a) $W = 200 \cdot 100 = 20{,}000$ ft-lb

(b) $W = \int_0^{100}[(100-x)+200]\,dx$

$= \int_0^{100}(400-2x)\,dx$

$= \left[400x - x^2\right]_0^{100} = 30{,}000$ ft-lb

22. (a) Loses 50 pounds over 100 feet, so weight x feet from top is

$$W(x) = 200 - \frac{50}{100}(100-x) = 150 + \frac{x}{2}$$

Thus, work $= \int_0^{100}\left(150 + \frac{x}{2}\right)dx = 17{,}500$ ft-lb

(b) Just add the work done lifting the chain, namely $\int_0^{100} 2x\,dx = 10{,}000$ ft-lb.

Total work $=27{,}500$ ft-lb

23. The bag is raised 8 feet and loses a total of 3 pounds at a constant rate. Thus, the bag loses sand at the rate of 3/8 lb/ft. After the bag has been raised x feet it weighs $100 - \dfrac{3x}{8}$ pounds.

$$W = \int_0^8 \left(100 - \frac{3x}{8}\right) dx = \left[100 - \frac{3x^2}{16}\right]_0^8 = 788 \text{ ft-lb.}$$

24. Wieght at depth x is $40 - \dfrac{1}{20}(8.3)(40 - x) = 0.415x + 23.4$.

Thus $W = \displaystyle\int_0^{40} (0.415x + 23.4)\, dx = \left[0.415\frac{x^2}{2} + 23.4x\right]_0^{40} = 1268$ ft-lb.

25. (a) $W = \displaystyle\int_0^l x\sigma\, dx = \frac{1}{2}\sigma l^2$ ft-lb (b) $W = \displaystyle\int_0^l (x + l)\, \sigma\, dx = \frac{3}{2}\sigma l^2$ ft-lb

26. Work $= wh + \displaystyle\int_0^h \sigma x\, dx = \left(wh + \frac{1}{2}\sigma h^2\right)$ ft-lb.

27. Thirty feet of cable and the steel beam weighing a total of

$$800 + 30(6) = 980 \text{ lb}$$

are raised 20 feet. The work requires (20)(980) ft-lb.

Next, the remaining 20 feet of cable is raised a varying distance and wound onto the steel drum. Thus the total work is given by

$$W = (20)\,(980) + \int_0^{20} 6x\, dx = 19,600 + 1,200 = 20,800 \text{ ft-lb.}$$

28. Reaches height x at time $t = x/n$, and at that time weighs $w - 8.3pt = w - 8.3px/n$ pounds,

Therefore, work$= \displaystyle\int_0^m \left(w - 8.3p\frac{x}{n}\right) dx = \left(wm - \frac{4.15pm^2}{n}\right)$ ft-lb.

29. By the hint

$$W = \int_a^b F(x)\, dx = \int_a^b ma\, dx = \int_a^b mv\frac{dv}{dx}\, dx = \int_{v_a}^{v_b} mv\, dv = \left[\frac{1}{2}mv^2\right]_{v_a}^{v_b} = \frac{1}{2}v_b^2 - \frac{1}{2}v_a^2$$

30. (a) Acceleration: $a = \dfrac{88}{15}$ feet/sec^2, Force: $F = ma = \dfrac{w}{g}a = \dfrac{3000}{32} \cdot \dfrac{88}{15} = 550$ lbs

$v(t) = at$, so $p = \dfrac{dW}{dt} = F\,(x(t))\, v(t) = 550 \cdot \dfrac{88}{15}t$

The engine must be able to sustain this until $t = 15$,

so need $p = 550.88$ ft-lb/s $= 88$ horse power.

(b) Now there is a $\dfrac{4}{\sqrt{10016}} \cdot 3000$ component of gravity acting against the motion, so the total force needed is $550 + \dfrac{4 \cdot 3000}{\sqrt{10016}} \cong 670$ lbs, so the required power is:

$$p = 670.88 \text{ ft-lb/sec} = 107.2 \text{ horse power.}$$

31. (a) The work required to pump the water out of the tank is given by

$$W=\int_5^{10}(62.5)\pi\,5^2x\,dx=1562.5\pi\left[\frac{1}{2}x^2\right]_5^{10}\cong 184,078\text{ ft-lb}$$

A $\frac{1}{2}$-horsepower pump can do 275 ft-lb of work per second. Therefore it will take

$$\frac{184,078}{275}\cong 669\text{ seconds}\cong 11\text{ min, 10 sec, to empty the tank.}$$

(b) The work required to pump the water to a point 5 feet above the top of the tank is given by

$$W=\int_5^{10}(62.5)\pi\,5^2(x+5)\,dx=\int_5^{10}(62.5)\pi\,5^2x\,dx+\int_5^{10}(62.5)\pi\,5^3\,dx\cong 306796\text{ ft-lb}$$

It will take a $\frac{1}{2}$-horsepower pump approximately 1,116 sec, or 18 min, 36 sec, in this case.

32. (a)

$$\begin{aligned}W&=\int_0^8 60x16\pi\,dx+\int_0^4 60(x+8)\pi(16-x^2)\,dx\\&=960\pi\left[\frac{x^2}{2}\right]_0^8+60\pi\int_0^4(128+16x-8x^2-x^3)\,dx\\&=30,720\pi+60\pi\left[128x+8x^2-\frac{8x^3}{3}-\frac{x^4}{4}\right]_0^4\\&=30,720\pi+24,320\pi\\&=55,040\pi\simeq 172,913\text{ft-lbs.}\end{aligned}$$

(b) Pump does $\frac{1}{2}(550)=275$ ft-lbs/sec.

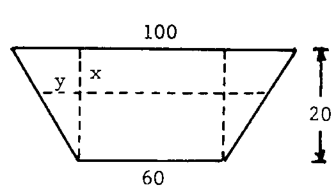
Therefore, it will take $\frac{172,913}{275}\simeq 629$ seconds, or 10.5 minutes.

SECTION 6.6

1. $F=\int_0^6(62.5)\cdot x\cdot 8\,dx=250\left[x^2\right]_0^6=9000\text{ lb}$

2. $F=\int_4^{10}62.5\cdot x\cdot 6\,dx=(62.5)(3)\left[x^2\right]_4^{10}=15,750\text{ lb}$

3. The width of the plate x meters below the surface is given by $w(x)=60+2(20-x)=100-2x$ (see the figure). The force against the dam is



$$\begin{aligned}F&=\int_0^{20}9800x(100-2x)\,dx\\&=9800\int_0^{20}(100x-2x^2)\,dx\end{aligned}$$

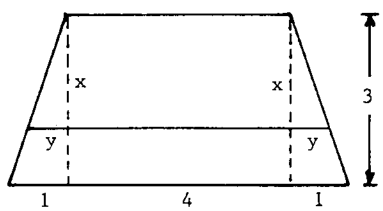
$$= 9800\left[50x^2 - \tfrac{2}{3}x^3\right]_0^{20}$$

$$\cong 1.437 \times 10^8 \text{ Newtons}$$

4. $F = \displaystyle\int_{15}^{20} 9800 \cdot x \cdot 5\, dx = (4900)(5)\left[x^2\right]_{15}^{20} = 4,287,500 \text{ N}$

5. The width of the gate x meters below its top is given by $w(x) = 4 + \frac{2}{3}x$ (see the figure).

The force of the water against the gate is



$$F = \int_0^3 9800(10+x)\left(4 + \frac{2}{3}x\right) dx$$

$$= 9800 \int_0^3 \left[\frac{2}{3}x^2 + \frac{32}{3} + 40\right] dx$$

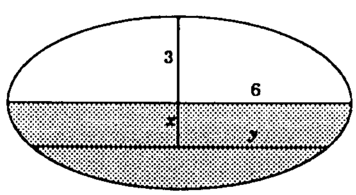
$$= 9800\left[\tfrac{2}{9}x^3 + \tfrac{16}{3}x^2 + 40x\right]_0^3$$

$$\cong 1.7052 \times 10^6 \text{ Newtons}$$

6. (a) $F = \displaystyle\int_0^{75} 62.5x \cdot 1000\, dx = (62.5)(1000)\left(\frac{75^2}{2}\right) = 175,781,250 \text{ lbs.}$

(b) $F = \displaystyle\int_0^{50} 62.5x \cdot 1000\, dx = (62.5)(1000)\left(\frac{50^2}{2}\right) = 78,125,000 \text{ lbs.}$

7.



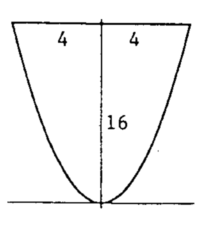
ellipse: $\dfrac{x^2}{3^2} + \dfrac{y^2}{6^2} = 1$

$$F = \int_0^3 (60)\,x\left[12\sqrt{1 - \frac{x^2}{9}}\right] dx$$

$$= 240 \int_0^3 x\left(9 - x^2\right)^{1/2} dx$$

$$= 240\left[-\tfrac{1}{3}\left(9 - x^2\right)^{3/2}\right]_0^3 = 2160 \text{ lb}$$

8.

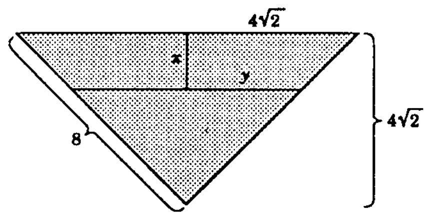


$$F = \int_0^{16} \sigma x w(x)\, dx = \int_0^{16} 70x2\sqrt{16-x}\, dx = 140\int_0^{16} x\sqrt{16-x}\, dx$$

$$= -140\int_{16}^{0} (16-u)\sqrt{u}\, du = -40\left[\frac{32}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_{16}^{0}$$

$$= \frac{114,688}{3} \text{ lb}$$

9.



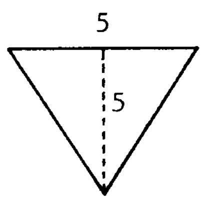
By similar triangles

$$\frac{4\sqrt{2}}{4\sqrt{2}} = \frac{y}{4\sqrt{2} - x} \quad \text{so} \quad y = 4\sqrt{2} - x.$$

$$F = \int_0^{4\sqrt{2}} (62.5)\,x\left[2\left(4\sqrt{2} - x\right)\right]\,dx$$

$$= 125\int_0^{4\sqrt{2}} \left(4\sqrt{2}\,x - x^2\right)\,dx = \frac{8000}{3}\sqrt{2}\ \text{lb}$$

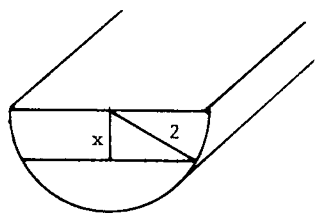
10.



By similar triangles, $\frac{W(x)}{5} = \frac{5 \cdot x}{5} \implies W(x) = 5 - x.$

$$F = \int_0^5 62.5x(5-x)\,dx = 62.5\left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_0^5 = \frac{15,625}{12}\ \text{lb}.$$

11.



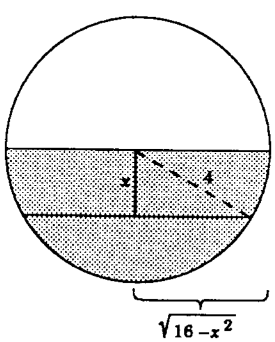
$$F = \int_0^2 (62.5)\cdot x \cdot 2\sqrt{4 - x^2}\,dx$$

$$= 125\int_0^2 x\sqrt{4 - x^2}\,dx$$

$$= -\tfrac{125}{3}\left[\left(4 - x^2\right)^{3/2}\right]_0^2 = 333.33\ \text{lb}$$

12. $F = \int_0^4 62.5y\,2\sqrt{4 - y}\,dy$

13.



$$F = \int_0^4 60x\left(2\sqrt{16 - x^2}\right)\,dx$$

$$= 120\int_0^4 x\left(16 - x^2\right)^{1/2}\,dx$$

$$= 120\left[-\tfrac{1}{3}\left(16 - x^2\right)^{3/2}\right]_0^4 = 2560\ \text{lb}$$

14. $F = \int_0^8 60x\,2\sqrt{16 - (x-4)^2}\,dx = 120\int_{-4}^4 (u+4)\sqrt{16 - u^2}\,du$

$$= 120\int_{-4}^4 u\sqrt{16 - u^2}\,du + 120\cdot 4\int_{-4}^4 \sqrt{16 - u^2}\,du$$

$$= 0 + 120\cdot 4\,[\text{Area of half circle}] = 480\cdot\frac{1}{2}\cdot 16\pi = 3840\pi\ \text{lb}.$$

15. (a) The width of the plate is 10 feet and the depth of the plate ranges from 8 feet to 14 feet. Thus

$$F = \int_8^{14} 62.5x\,(10)\,dx = 41,250 \text{ lb.}$$

(b) The width of the plate is 6 feet and the depth of the plate ranges from 6 feet to 16 feet. Thus

$$F = \int_6^{16} 62.5x\,(6)\;dx = 41,250 \text{ lb.}$$

16. $W(x) = 2\pi r = 30\pi.$ $\quad F = \displaystyle\int_0^{50} 60x30\pi\,dx = 900\pi 50^2 = 2,250,000\pi$ lb.

17. (a) Force on the sides:

$$F = \int_0^1 (9800)\,x\,14\,dx + \int_0^2 (9800)(1+x)\,7(2-x)\,dx$$

$$= 68,600\left[x^2\right]_0^1 + 68,600\int_0^2 \left[2 + x - x^2\right]\,dx$$

$$= 68,600 + 68,600\left[2x + \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^2$$

$$\cong 297,267 \text{ Newtons}$$

(b) Force at the shallow end:

$$F = \int_0^1 (9800)\cdot x\cdot 8\,dx = 39,200\left[x^2\right]_0^1 = 39,200 \text{ Newtons}$$

Force at the deep end:

$$F = \int_0^3 (9800)\cdot x\cdot 8\,dx = 39,200\left[x^2\right]_0^3 = 352,800 \text{ Newtons}$$

18. Following the argument given for a vertical plate, the approximate force on the i th strip is

$$\sigma x_i^* w(x_i)\sec\theta\Delta x_i$$

Therefore, the force F on the plate is given by

$$F = \int_a^b \sigma x w(x)\sec\theta\,dx$$

19. $F = \displaystyle\int_0^{14} (9800)\left(1 + \frac{1}{7}x\right)\cdot 8\frac{5\sqrt{2}}{7}\,dx = 392,000\frac{\sqrt{2}}{7}\left[x + \frac{1}{14}x^2\right]_0^1 4 \cong 2.217\times 10^6$ Newtons

20. (a) $F = \int_0^{100} 62.5x(1000)\sec(\pi/6)\,dx$

$$= \frac{125,000}{\sqrt{3}}\left[\frac{x^2}{2}\right]_0^{100} \cong 361,000,000 \text{ lbs}$$

(b) $F = \int_0^{75} 62.5x(1000)\sec(\pi/6)\,dx$

$$= \frac{125,000}{\sqrt{3}}\left[\frac{x^2}{2}\right]_0^{75} \cong 203,000,000 \text{ lbs}$$

21. $F = \displaystyle\int_a^b \sigma x w(x)\,dx = \sigma \int_a^b x w(x)\,dx = \sigma \overline{x} A$

where A is the area of the submerged surface and $\overline{x}$ is the depth of the centroid.

22. From Exercise 21, $F_1 = \sigma \overline{x}_1 A_1 = \sigma h_1 A_1$, $F_2 = \sigma h_2 A_2$. Since $A_1 = A_2$, $F_2 = \dfrac{h_2}{h_1} F_1$

CHAPTER 7

SECTION 7.1

1. Suppose $f(x_1) = f(x_2)$ $x_1 \neq x_2$. Then

$5x_1 + 3 = 5x_2 + 3 \ \Rightarrow \ x_1 = x_2$; v

f is one-to-one

$$\begin{aligned} f(t) &= x \\ 5t + 3 &= x \\ 5t &= x - 3 \\ t &= \tfrac{1}{5}(x - 3) \\ f^{-1}(x) &= \tfrac{1}{5}(x - 3) \end{aligned}$$

2. $f^{-1}(x) = \dfrac{1}{3}(x - 5)$

3. Suppose $f(x_1) = f(x_2)$ $x_1 \neq x_2$. Then

$4x_1 - 7 = 4x_2 - 7 \ \Rightarrow \ x_1 = x_2$;

f is one-to-one

$$\begin{aligned} f(t) &= x \\ 4t - 7 &= x \\ 4t &= x + 7 \\ t &= \tfrac{1}{4}(x + 7) \\ f^{-1}(x) &= \tfrac{1}{4}(x + 7) \end{aligned}$$

4. $f^{-1}(x) = \dfrac{1}{7}(x - 4)$

5. f is not one-to-one; e.g. $f(1) = f(-1)$

6. $f^{-1}(x) = x^{1/5}$.

7. $f'(x) = 5x^4 \geq 0$ on $(-\infty, \infty)$ and

$f'(x) = 0$ only at $x = 0$; f is increasing.

Therefore, f is one-to-one.

$$\begin{aligned} f(t) &= x \\ t^5 + 1 &= x \\ t^5 &= x - 1 \\ t &= (x - 1)^{1/5} \\ f^{-1}(x) &= (x - 1)^{1/5} \end{aligned}$$

8. not one-to-one; e.g. $f(0) = f(3)$

9. $f'(x) = 9x^2 \geq 0$ on $(-\infty, \infty)$ and

$f'(x) = 0$ only at $x = 0$; f is increasing.

Therefore, f is one-to-one.

10. $f^{-1}(x) = (x + 1)^{1/3}$

$$f(t) = x$$
$$1 + 3t^3 = x$$
$$t^3 = \tfrac{1}{3}(x-1)$$
$$t = \left[\tfrac{1}{3}(x-1)\right]^{1/3}$$
$$f^{-1}(x) = \left[\tfrac{1}{3}(x-1)\right]^{1/3}$$

11. $f'(x) = 3(1-x)^2 \geq 0$ on $(-\infty, \infty)$ and $f'(x) = 0$ only at $x = 1$; f is increasing. Therefore, f is one-to-one.

$$f(t) = x$$
$$(1-t)^3 = x$$
$$1 - t = x^{1/3}$$
$$t = 1 - x^{1/3}$$
$$f^{-1}(x) = 1 - x^{1/3}$$

12. not one-to-one; e.g. $f(0) = f(2)$.

13. $f'(x) = 3(x+1)^2 \geq 0$ on $(-\infty, \infty)$ and $f'(x) = 0$ only at $x = -1$; f is increasing. Therefore, f is one-to-one.

$$f(t) = x$$
$$(t+1)^3 + 2 = x$$
$$(t+1)^3 = x - 2$$
$$t + 1 = (x-2)^{1/3}$$
$$t = (x-2)^{1/3} - 1$$
$$f^{-1}(x) = (x-2)^{1/3} - 1$$

14. $f^{-1}(x) = \dfrac{1}{4}(x^{1/3} + 1)$

15. $f'(x) = \dfrac{3}{5x^{2/5}} > 0$ for all $x \neq 0$;

f is increasing on $(-\infty, \infty)$

$$f(t) = x$$
$$t^{3/5} = x$$
$$t = x^{5/3}$$
$$f^{-1}(x) = x^{5/3}$$

16. $f^{-1}(x) = (1-x)^3 + 2$

17. $f'(x) = 3(2-3x)^2 \geq 0$ for all x and $f'(x) = 0$ only at $x = 2/3$; f is increasing

$$\begin{aligned} f(t) &= x \\ (2-3t)^3 &= x \\ 2-3t &= x^{1/3} \\ 3t &= 2 - x^{1/3} \\ t &= \tfrac{1}{3}(2-x^{1/3}) \\ f^{-1}(x) &= \tfrac{1}{3}(2-x^{1/3}) \end{aligned}$$

18. not one-to-one; e.g. $f(1) = f(-1)$

19. $f'(x) = -\dfrac{1}{x^2} < 0$ for all $x \neq 0$; f is decreasing on $(-\infty, 0) \cup (0, \infty)$

$$\begin{aligned} f(t) &= x \\ \frac{1}{t} &= x \\ t &= \frac{1}{x} \\ f^{-1}(x) &= \frac{1}{x} \end{aligned}$$

20. $f^{-1}(x) = 1 - \dfrac{1}{x}$

21. f is not one-to-one; e.g. $f\left(\frac{1}{2}\right) = f(2)$

22. not one-to-one; e.g. $f(1) = f(2)$

23. $f'(x) = -\dfrac{3x^2}{(x^3+1)^2} \leq 0$ for all $x \neq -1$; f is decreasing on $(-\infty, -1) \cup (-1, \infty)$

$$\begin{aligned} f(t) &= x \\ \frac{1}{t^3+1} &= x \\ t^3 + 1 &= \frac{1}{x} \\ t^3 &= \frac{1}{x} - 1 \\ t &= \left(\frac{1}{x} - 1\right)^{1/3} \\ f^{-1}(x) &= \left(\frac{1}{x} - 1\right)^{1/3} \end{aligned}$$

24. $f^{-1}(x) = \dfrac{x}{1+x}$

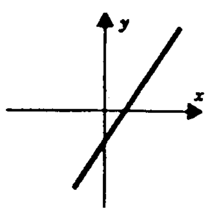
25. $f'(x) = \dfrac{1}{(x+2)^2} > 0$ for all $x \neq -2$;
f is increasing on $(-\infty, -2) \cup (-2, \infty)$

$$\begin{aligned} f(t) &= x \\ \frac{t+2}{t+1} &= x \\ t+2 &= xt + x \\ t(1-x) &= x - 2 \\ t &= \frac{x-2}{1-x} \\ f^{-1}(x) &= \frac{x-2}{1-x} \end{aligned}$$

26. not one-to-one; e.g. $f(1) = f(-3)$

27. they are equal.

28.



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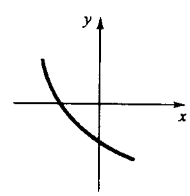

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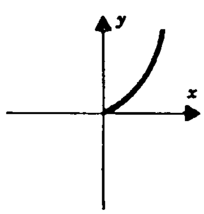
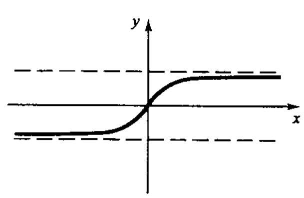


31.



32. (a) Suppose f and g are one-to-one, and that $f(g(x_1)) = f(g(x_2))$. Then since f is one-to-one, $g(x_1) = g(x_2)$, and since g is one-to-one this implies $x_1 = x_2$.

(b) Since $g^{-1}\left(f^{-1}\left(f\left(g(x)\right)\right)\right) = g^{-1}\left(g(x)\right) = x$, we have $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

33. $f'(x) = 3x^2 \geq 0$ on $I = (-\infty, \infty)$ and $f'(x) = 0$ only at $x = 0$; f is increasing on I and so it has an inverse.

$$f(2) = 9 \text{ and } f'(2) = 12; \quad \left(f^{-1}\right)'(9) = \frac{1}{f'(2)} = \frac{1}{12}$$

34. $f'(x) = -2 - 3x^2$; $\left(f^{-1}\right)'(4) = \dfrac{1}{f'\left(f^{-1}(4)\right)} = \dfrac{1}{f'(-1)} = -\dfrac{1}{5}$

35. $f'(x) = 1 + \dfrac{1}{\sqrt{x}} > 0$ on $I = (0, \infty)$; f is increasing on I and so it has an inverse.

$$f(4) = 8 \text{ and } f'(4) = 1 + \frac{1}{2} = \frac{3}{2}; \quad \left(f^{-1}\right)'(8) = \frac{1}{f'(4)} = \frac{1}{3/2} = \frac{2}{3}$$

36. $f'(x) = 1 + \cos x; \quad \left(f^{-1}\right)'(0) = \dfrac{1}{f'\left(f^{-1}(0)\right)} = \dfrac{1}{f'(0)} = \dfrac{1}{2}$

37. $f'(x) = 2 - \sin x > 0$ on $I = (-\infty, \infty)$; f is increasing on I and so it has an inverse.

$$f(\pi/2) = \pi \text{ and } f'(\pi/2) = 1; \quad \left(f^{-1}\right)'(\pi) = \frac{1}{f'(\pi/2)} = 1$$

38. $f'(x) = \dfrac{2}{(x-1)^2}; \quad \left(f^{-1}\right)'(3) = \dfrac{1}{f'\left(f^{-1}(3)\right)} = \dfrac{1}{f'(6)} = \dfrac{25}{2}$

39. $f'(x) = \sec^2 x > 0$ on $I = (-\pi/2, \pi/2)$; f is increasing on I and so it has an inverse.

$$f(\pi/3) = \sqrt{3} \text{ and } f'(\pi/3) = 4; \quad \left(f^{-1}\right)'(\sqrt{3}) = \frac{1}{f'(\pi/3)} = \frac{1}{4}$$

40. $f'(x) = 5x^4 + 6x^2 + 2; \quad \left(f^{-1}\right)'(-5) = \dfrac{1}{f'\left(f^{-1}(-5)\right)} = \dfrac{1}{f'(-1)} = \dfrac{1}{13}$

41. $f'(x) = 3x^2 + \dfrac{3}{x^4} > 0$ on $I = (0, \infty)$; f is increasing on I and so it has an inverse.

$f(1) = 2$ and $f'(1) = 6; \quad {f^{-1}}'(2) = \dfrac{1}{f'(1)} = \dfrac{1}{6}.$

42. $f'(x) = 1 - \sin x \geq 0$ on $I = [0, \pi]$, with $f(x) = 0$ for only one value on I and so it has an inverse.

$f(\pi) = -1$ and $f'(\pi) = 1; \quad {f^{-1}}'(-1) = \dfrac{1}{f'(\pi)} = 1.$

43. Let $x \in \text{dom}\left(f^{-1}\right)$ and let $f(z) = x$. Then

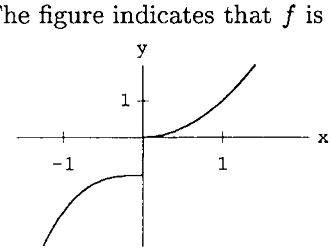
$$\left(f^{-1}\right)'(x) = \frac{1}{f'(z)} = \frac{1}{f(z)} = \frac{1}{x}$$

44. $\left(f^{-1}\right)'(x) = \dfrac{1}{f'\left(f^{-1}(x)\right)} = \dfrac{1}{1 + \left[f\left(f^{-1}(x)\right)\right]^2} = \dfrac{1}{1 + x^2}$

45. Let $x \in \text{dom}\left(f^{-1}\right)$ and let $f(z) = x$. Then

$$\left(f^{-1}\right)'(x) = \frac{1}{f'(z)} = \frac{1}{\sqrt{1 - [f(z)]^2}} = \frac{1}{\sqrt{1 - x^2}}$$

46. (a) The figure indicates that f is one-to-one. (b)



$$f^{-1}(x) = \begin{cases} (x+1)^{1/3}, & x < -1 \\ \sqrt{x}, & x \geq 0 \end{cases}$$

47. (a) $$f'(x) = \frac{(cx+d)a - (ax+b)c}{(cx+d)^2} = \frac{ad-bc}{(cx+d)^2}, \quad x \neq -d/c$$

Thus, $f'(x) \neq 0$ iff $ad - bc \neq 0$.

(b) $$\frac{at+b}{ct+d} = x$$
$$at + b = ctx + dx$$
$$(a - cx)t = dx - b$$
$$t = \frac{dx-b}{a-cx}; \quad f^{-1}(x) = \frac{dx-b}{a-cx}$$

48. $f = f^{-1} \implies \dfrac{ax+b}{cx+d} = \dfrac{dx-b}{a-cx}$

$\implies a^2x + ab - acx^2 = cdx^2 + d^2x - bd$

$\implies a = -d$ as long as either b or $c \neq 0$.

If $b = c = 0$, then $a = \pm d$.

49. (a) $f'(x) = \sqrt{1+x^2} > 0$, so f is always increasing, hence one-to-one.

(b) $\left(f^{-1}\right)'(0) = \dfrac{1}{f'\left(f^{-1}(0)\right)} = \dfrac{1}{f'(2)} = \dfrac{1}{\sqrt{5}}$.

50. (a) $f'(x) = \sqrt{16 + (2x)^4}\,(2) = 8\sqrt{1+x^4} > 0$ for all x.

(b) Since $f(1/2) = 0$, we have
$$\left(f^{-1}\right)'(0) = \frac{1}{f'(1/2)} = \frac{1}{2\sqrt{17}} = \frac{\sqrt{17}}{34}$$

51. (a) $g(t) \geq 0$ or $g(t) \leq 0$ on $[a, b]$, with $g(t) = 0$ only at "isolated" points.

(b) $g(x) \neq 0$

(c) $(f^{-1})'(x) = \dfrac{1}{g(x)}$

52. (a) $g(x_1) = g(x_2) \implies f(x_1 + c) = f(x_2 + c) \implies x_1 + c = x_2 + c \implies x_1 = x_2$, so g is one-to-one.

$g(t) = x \implies f(t + c) = x \implies t + c = f^{-1}(x) \implies t = f^{-1}(x) - c$

$\implies g^{-1}(x) = f^{-1}(x) - c$

(b) $h(x_1) = h(x_2) \implies f(cx_1) = f(cx_2) \implies cx_1 = cx_2 \implies x_1 = x_2$, so h is one-to-one.

$h(t) = x \implies f(ct) = x \implies ct = f^{-1}(x) \implies t = \dfrac{1}{c}f^{-1}(x)$

$\implies g^{-1}(x) = \dfrac{1}{c}f^{-1}(x)$

53. (a)

$$g'(x) = \frac{1}{f'[g(x)]}$$

$$g''(x) = -\frac{1}{(f'[g(x)])^2} f''[g(x)]g'(x)$$

$$= -\frac{f''[g(x)]}{(f'[g(x)])^3}$$

(b) If f is increasing and its graph is concave up (down), then the graph of g is concave down (up). On the other hand, if f is decreasing then the graphs of f and g have the the same concavity.

54. (a) No. If p is a polynomial of even degree, then $\lim_{x\to\pm\infty} p(x) = \infty$ or $\lim_{x\to\pm\infty} p(x) = -\infty$.

(b) Yes, for instance $P(x) = x^3$ has an inverse. $P(x) = x^3 - x$ does not have an inverse.

55. Let $f(x) = \sin x$ and let $y = f^{-1}(x)$. Then

$$\sin y = x$$

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y} \quad (y \neq \pm\pi/2)$$

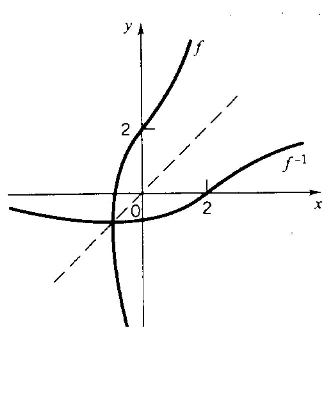
$$= \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}} \quad (x \neq \pm 1)$$

56. Let $y = f^{-1}(x)$. Then $\tan y = x$, so $\sec^2 y \dfrac{dy}{dx} = 1$.

Thus $\dfrac{dy}{dx} = \dfrac{1}{\sec^2 y} = \cos^2 y = \dfrac{1}{1 + x^2}$ (see figure).

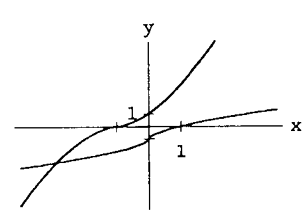
57. $f'(x) = 3x^2 + 3 > 0$ for all x;

f is increasing on $(-\infty, \infty)$



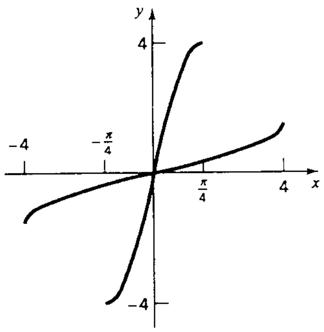
58. $f'(x) = \dfrac{3}{5} x^{-2/5} > 0 \ (x \neq 0)$

f is increasing on $(-\infty, \infty)$



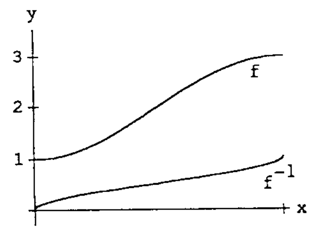
59. $f'(x) = 8\cos 2x > 0,\ x \in (-\pi/4, \pi/4);$

f is increasing on $[-\pi/4, \pi/4]$



60. $f'(x) = \sin 3x > 0,\ x \in (0, \pi/3)$

f is increasing on $[0, \pi/3]$



SECTION 7.2

1. $\ln 20 = \ln 2 + \ln 10 \cong 2.99$

2. $\ln 16 = \ln 2^4 = 4\ln 2 \cong 2.78$

3. $\ln 1.6 = \ln \frac{16}{10} = 2\ln 4 - \ln 10 \cong 0.48$

4. $\ln 3^4 = 4\ln 3 \cong 4.40$

5. $\ln 0.1 = \ln \frac{1}{10} = \ln 1 - \ln 10 \cong -2.30$

6. $\ln 2.5 = \ln \frac{10}{4} = \ln 10 - \ln 4 \cong 0.91$

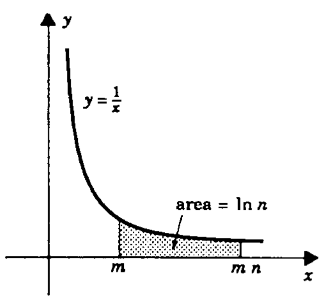
7. $\ln 7.2 = \ln \frac{72}{10} = \ln 8 + \ln 9 - \ln 10 \cong 1.98$

8. $\ln\sqrt{630} = \frac{1}{2}\ln(9\cdot 7\cdot 10) = \frac{1}{2}(\ln 9 + \ln 7 + \ln 10) \cong 3.23$

9. $\ln\sqrt{2} = \frac{1}{2}\ln 2 \cong 0.35$

10. $\ln 0.4 = \ln \frac{4}{10} = \ln 4 - \ln 10 \cong -0.91$

11.



12. $A = \int_x^1 \frac{1}{t}\,dt = -\int_1^x \frac{1}{t}\,dt$

$= -\ln x = \ln(1/x)$

13. $\frac{1}{2}\left[L_f(P) + U_f(P)\right] = \frac{1}{2}\left[\frac{763}{1980} + \frac{1691}{3960}\right] \cong 0.406$

14. $\ln 2.5 \cong \frac{1}{2}\left[L_f(P) + U_f(P)\right] \cong 0.921$

15. (a) $\ln 5.2 \cong \ln 5 + \frac{1}{5}(0.2) \cong 1.65$

(b) $\ln 4.8 \cong \ln 5 - \frac{1}{5}(0.2) \cong 1.57$

(c) $\ln 5.5 \cong \ln 5 + \frac{1}{5}(0.5) \cong 1.71$

16. (a) $\ln 10.3 \cong \ln 10 + \frac{1}{10}(0.3) \cong 2.33$

(b) $\ln 9.6 \cong \ln 10 + \frac{1}{10}(-0.4) \cong 2.26$

(c) $\ln 11 \cong \ln 10 + \frac{1}{10}(1) \cong 2.40$

17. $x = e^2$

18. $x = \dfrac{1}{e}$

19. $2 - \ln x = 0$ or $\ln x = 0$. Thus $x = e^2$ or $x = 1$.

20. $\ln x^{1/2} - \ln(2x-1) = 0 \Longrightarrow \ln\dfrac{\sqrt{x}}{2x-1} = 0 \Longrightarrow \dfrac{\sqrt{x}}{2x-1} = 1 \Longrightarrow x = 1$

21.
$$\ln\left[(2x+1)(x+2)\right] = 2\ln(x+2)$$
$$\ln\left[(2x+1)(x+2)\right] = \ln\left[(x+2)^2\right]$$
$$(2x+1)(x+2) = (x+2)^2$$
$$x^2 + x - 2 = 0$$
$$(x+2)(x-1) = 0$$
$$x = -2,\ 1$$

We disregard the solution $x = -2$ since it does not satisfy the initial equation.
Thus, the only solution is $x = 1$.

22. $2\ln(x+2) - \frac{1}{2}\ln x^4 = \ln\frac{(x+2)^2}{x^2} = 1 = \ln e \implies \frac{(x+2)^2}{x^2} = e \implies x = \frac{-2}{(1 \pm \sqrt{e})}$

23. See Exercises 3.1, Definition (3.1.5).
$$\lim_{x\to 1}\frac{\ln x}{x-1} = \frac{d}{dx}(\ln x)\bigg|_{x=1} = \frac{1}{x}\bigg|_{x=1} = 1$$

24. (a) By the mean-value theorem with $f(x) = \ln x$, there exists c between 1 and x such that
$$\frac{\ln x - \ln 1}{x-1} = f'(c) = \frac{1}{c}, \quad \text{so} \quad \ln x = \frac{x-1}{c}.$$
If $x > 1$, then $\frac{1}{x} < \frac{1}{c} < 1$ and $x - 1 > 0$ so $\frac{x-1}{x} < \ln x < x - 1$

If $0 < x < 1$, then $1 < \frac{1}{c} < \frac{1}{x}$ and $x - 1 < 0$ so $\frac{x-1}{x} < \ln x < x - 1$

(b) $\frac{1}{x} < \frac{\ln x}{x-1} < 1$ for $x > 1$, $\quad 1 < \frac{\ln x}{x-1} < \frac{1}{x}$ for $x < 1$,

so by the pinching theorem $\lim_{x\to 1}\frac{\ln x}{x-1} = \lim_{x\to 1} 1 = \lim_{x\to 1}\frac{1}{x} = 1$

25. (a) Let $P = \{1, 2, \ldots, n\}$ be a regular partition of $[1, n]$. Then
$$L_f(P) = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \int_1^n \frac{1}{t}\,dt < 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} = U_f(P)$$

(b) The sum of the shaded areas is give by
$$U_f(P) - \int_1^n \frac{1}{t}\,dt = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \ln n.$$

(c) Connect the points $(1,1), \left(2, \frac{1}{2}\right), \ldots, \left(n, \frac{1}{n}\right)$ by straight line segments.

The sum of the areas of the triangles that are formed is:
$$\frac{1}{2}\cdot 1\left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right)\right] = \frac{1}{2}\left(1 - \frac{1}{n}\right)$$
so
$$\frac{1}{2}\left(1 - \frac{1}{n}\right) < \gamma$$

The sum of the areas of the indicated rectangles is:

$$1\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n-1}-\frac{1}{n}\right)\right]=1-\frac{1}{n}$$

so

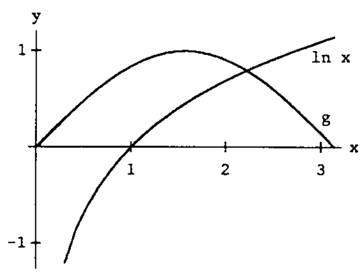
$$\gamma < 1-\frac{1}{n}$$

Letting $n \to \infty$ we have $\frac{1}{2} < \gamma < 1$.

26. (a) $\ln 1 - g(1) = 1 > 0,\quad \ln 2 - g(2) = \ln 2 - 2 < 0,$ so by the intermediate-value theorem $\ln r - g(r) = 0$ for some $r \in [1,2]$.

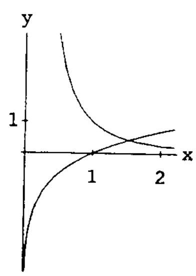
(b) $r \cong 1.7915$

27. (a) Let $G(x) = \sin x - \ln x$. Then $G(3) = \sin 3 - \ln 3 \cong 0.96 > 0$ and $G(2) = \sin 2 - \ln 2 \cong -0.22 < 0$. Thus, G has at least one zero on $[2,3]$ which implies that there is at least one number $r \in [2,3]$ such that $\sin r = \ln r$.

(b) 

$r \cong 2.2191$

28. (a) $\ln 1 - \dfrac{1}{1^2} = -1 < 0,\quad \ln 2 - \dfrac{1}{2^2} \cong 0.69 - \dfrac{1}{4} > 0,$ so by the intermediate-value theorem $\ln r - \dfrac{1}{r^2} = 0$ for some $r \in [1,2]$.

(b) 

$r \cong 1.5316$

29. $L = 1$

30. $\lim_{x\to 0} x \ln|x| = 1$

31. $L = 0$

SECTION 7.3

1. $\text{dom}(f) = (0,\infty),\quad f'(x) = \dfrac{1}{4x}(4) = \dfrac{1}{x}$

2. $\text{dom}(f) = \left(-\dfrac{1}{2},\infty\right),\quad f'(x) = \dfrac{2}{2x+1}$

3. $\text{dom}(f) = (-1, \infty), \quad f'(x) = \dfrac{1}{x^3+1}\dfrac{d}{dx}\left(x^3+1\right) = \dfrac{3x^2}{x^3+1}$

4. $\text{dom}(f) = (-1, \infty), \quad f'(x) = \dfrac{3}{x+1}$

5. $\text{dom}(f) = (-\infty, \infty), \quad f(x) = \dfrac{1}{2}\ln\left(1+x^2\right)$ so $f'(x) = \dfrac{1}{2}\left[\dfrac{1}{1+x^2}(2x)\right] = \dfrac{x}{1+x^2}$

6. $\text{dom}(f) = (0, \infty), \quad f'(x) = \dfrac{3(\ln x)^2}{x}$

7. $\text{dom}(f) = \{x \mid x \neq \pm 1\}, \quad f'(x) = \dfrac{1}{x^4-1}\dfrac{d}{dx}\left(x^4-1\right) = \dfrac{4x^3}{x^4-1}$

8. $\text{dom}(f) = (1, \infty), \quad f'(x) = \dfrac{1}{x\ln x}$

9. $\text{dom}(f) = (0, \infty), \quad f'(x) = x^2\dfrac{d}{dx}(\ln x) + 2x(\ln x) = x + 2x\ln x$

10. $\text{dom}(f) = (-\infty, -2) \cup (-2, 1) \cup (1, \infty), \quad f'(x) = \dfrac{1}{x+2} - \dfrac{3x^2}{x^3-1}$ (rewrite $f(x)$ as $\ln|x+2| - \ln|x^3-1|$)

11. $\text{dom}(f) = (0, 1) \cup (1, \infty), \quad f(x) = (\ln x)^{-1}$ so $f'(x) = -(\ln x)^{-2}\dfrac{d}{dx}(\ln x) = -\dfrac{1}{x(\ln x)^2}$

12. $\text{dom}(f) = (-\infty, \infty), \quad f'(x) = \dfrac{x}{2(x^2+1)}$ (rewrite $f(x)$ as $\dfrac{1}{4}\ln(x^2+1)$.)

13. $\text{dom}(f) = (0, \infty), \quad f'(x) = \cos(\ln x)\left(\dfrac{1}{x}\right) = \dfrac{\cos(\ln x)}{x}$

14. $\text{dom}(f) = (0, \infty), \quad f'(x) = -\dfrac{\sin(\ln x)}{x}$

15. $\displaystyle\int \frac{dx}{x+1} = \ln|x+1| + C$

16. $\displaystyle\int \frac{dx}{3-x} = -\int \frac{dx}{x-3} = -\ln|x-3| + C$

17. $\left\{\begin{array}{c} u = 3-x^2 \\ du = -2x\,dx \end{array}\right\}; \quad \displaystyle\int \frac{x}{3-x^2}\,dx = -\frac{1}{2}\int\frac{du}{u} = -\frac{1}{2}\ln|u| + C = -\frac{1}{2}\ln|3-x^2| + C$

18. $\displaystyle\int \frac{x+1}{x^2}\,dx = \int\left(\frac{1}{x} + \frac{1}{x^2}\right)dx = \ln|x| - \frac{1}{x} + C$

19. $\left\{\begin{array}{c} u = 3x \\ du = 3dx \end{array}\right\}; \quad \displaystyle\int \tan 3x\,dx = \frac{1}{3}\int \tan u\,du = \frac{1}{3}\ln|\sec u| + C = \frac{1}{3}\ln|\sec 3x| + C$

20. $\displaystyle\int \sec\frac{\pi}{2}x\,dx = \frac{2}{\pi}\ln\left|\sec\frac{\pi}{2}x + \tan\frac{\pi}{2}x\right| + C$

21. $\left\{\begin{array}{c} u = x^2 \\ du = 2x\,dx \end{array}\right\}$; $\displaystyle\int x\sec x^2\,dx = \frac{1}{2}\int \sec u\,du = \frac{1}{2}\ln|\sec u + \tan u| + C$

$= \frac{1}{2}\ln|\sec x^2 + \tan x^2| + C$

22. $\left\{\begin{array}{c} u = 2 + \cot x \\ du = -\csc^2 x\,dx \end{array}\right\}$; $\displaystyle\int \frac{\csc^2 x}{2 + \cot x}\,dx = -\int \frac{du}{u} = -\ln|u| + C = -\ln|2 + \cot x| + C.$

23. $\left\{\begin{array}{c} u = 3 - x^2 \\ du = -2x\,dx \end{array}\right\}$; $\displaystyle\int \frac{x}{(3 - x^2)^2}\,dx = -\frac{1}{2}\int \frac{du}{u^2} = \frac{1}{2u} + C = \frac{1}{2(3 - x^2)} + C$

24. $\displaystyle\int \frac{\ln(x + a)}{x + a}\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{1}{2}[\ln(x + a)]^2 + C$

25. $\left\{\begin{array}{c} u = 2 + \cos x \\ du = -\sin x\,dx \end{array}\right\}$; $\displaystyle\int \frac{\sin x}{2 + \cos x}\,dx = -\int \frac{1}{u}\,du = -\ln|u| + C = -\ln|2 + \cos x| + C$

26. $\left\{\begin{array}{c} u = 4 - \tan 2x \\ du = -2\sec^2 2x\,dx \end{array}\right\}$; $\displaystyle\int \frac{\sec^2 2x}{4 - \tan 2x}\,dx = -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln|u| + C$

$= -\frac{1}{2}\ln|4 - \tan 2x| + C$

27. $\displaystyle\int \left(\frac{1}{x + 2} - \frac{1}{x - 2}\right)dx = \ln|x + 2| - \ln|x - 2| + C = \ln\left|\frac{x + 2}{x - 2}\right| + C$

28. $\left\{\begin{array}{c} u = 2x^3 - 1 \\ du = 6x^2\,dx \end{array}\right\}$; $\displaystyle\int \frac{x^2}{2x^3 - 1}\,dx = \frac{1}{6}\int \frac{du}{u} = \frac{1}{6}\ln|u| + C = \frac{1}{6}\ln|2x^3 - 1| + C$

29. $\left\{u = \ln x,\ du = \dfrac{dx}{x}\right\}$; $\displaystyle\int \frac{dx}{x(\ln x)^2} = \int \frac{du}{u^2} = -\frac{1}{u} + C = -\frac{1}{\ln x} + C$

30. $\displaystyle\int \frac{\sec 2x\tan 2x}{1 + \sec 2x}\,dx = \frac{1}{2}\int \frac{1}{1 + u}\,du = \frac{1}{2}\ln|1 + u| + C = \frac{1}{2}\ln|1 + \sec 2x| + C$

31. $\left\{\begin{array}{c} u = \sin x + \cos x \\ du = (\cos x - \sin x)\,dx \end{array}\right\}$; $\displaystyle\int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx = -\int \frac{1}{u}\,du = -\ln|u| + C$

$= -\ln|\sin x + \cos x| + C$

32. $\left\{\begin{array}{c} u = \sqrt{x} \\ du = \dfrac{1}{2\sqrt{x}}\,dx \end{array}\right\}$; $\displaystyle\int \frac{1}{\sqrt{x}(1 + \sqrt{x})}\,dx = 2\int \frac{du}{1 + u} = 2\ln|1 + u| + C = 2\ln|1 + \sqrt{x}| + C$

33. $\left\{u = 1 + x\sqrt{x}\,,\ du = \frac{3}{2}x^{1/2}\,dx\right\};\quad \int \frac{\sqrt{x}}{1+x\sqrt{x}}\,dx = \frac{2}{3}\int \frac{du}{u} = \frac{2}{3}\ln|u| + C$
$= \frac{2}{3}\ln\left|1 + x\sqrt{x}\,\right| + C$

34. $\int \frac{\tan(\ln x)}{x}\,dx = \int \tan u\,du = \ln|\sec u| + C = \ln|\sec(\ln x)| + C$

35.

$$\left\{\begin{array}{l} u = \ln|\sin x| \\ \\ du = \cot x\,dx \end{array}\right\};\quad \int \cot x \ln|\sin x|\,dx = \int u\,du = \frac{1}{2}u^2 + C$$
$$= \tfrac{1}{2}\left(\ln|\sin x|\right)^2 + C$$

36. $\left\{\begin{array}{l} u = \ln \sec x \\ \\ du = \tan x\,dx \end{array}\right\};\quad \int u\,du = \frac{u^2}{2} + C = \frac{[\ln(\sec x)]^2}{2} + C$

37. $\int (1+\sec x)^2\,dx = \int \left(1 + 2\sec x + \sec^2 x\right)\,dx = x + 2\ln|\sec x + \tan x| + \tan x + C$

38. $\int x\left(\frac{1}{x^2-a^2} - \frac{1}{x^2-b^2}\right)\,dx = \frac{1}{2}\ln|x^2-a^2| - \frac{1}{2}\ln|x^2-b^2| + C = \frac{1}{2}\ln\left|\frac{x^2-a^2}{x^2-b^2}\right| + C$

39.

$$\left\{\begin{array}{l} u = \ln\,|\sec x + \tan x| \\ \\ du = \sec x\,dx \end{array}\right\};$$

$$\int \frac{\sec x}{\sqrt{\ln\,|\sec x + \tan x|}}\,dx = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{\ln\,|\sec x + \tan x|} + C$$

40. $\int (3 - \csc x)^2\,dx = \int (9 - 6\csc x + \csc^2 x)\,dx = 9x - 6\ln|\csc x - \cot x| - \cot x + C$

41. $\int_1^e \frac{dx}{x} = [\ln x]_1^e = \ln e - \ln 1 = 1 - 0 = 1$

42. $\displaystyle\int_1^{e^2} \frac{dx}{x} = [\,\ln|x|\,]_1^{e^2} = \ln e^2 - \ln 1 = 2$

43. $\displaystyle\int_e^{e^2} \frac{dx}{x} = [\ln x]_e^{e^2} = \ln e^2 - \ln e = 2 - 1 = 1$

44. $\displaystyle\int_0^1 \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx = \left[\ln\left|\frac{x+1}{x+2}\right|\right]_0^1 = \ln\frac{2}{3} - \ln\frac{1}{2} = \ln\frac{4}{3}$

45. $\displaystyle\int_4^5 \frac{x}{x^2-1}\,dx = \left[\frac{1}{2}\ln|x^2-1|\right]_4^5 = \frac{1}{2}(\ln 24 - \ln 15) = \frac{1}{2}\ln\frac{8}{5}$

46. $\displaystyle\int_{1/4}^{1/3} \tan\pi x\,dx = \frac{1}{\pi}[\ln|\sec\pi x|]_{1/4}^{1/3} = \frac{1}{\pi}\left(\ln 2 - \ln\sqrt{2}\right) = \frac{\ln 2}{2\pi}.$

47. $\left\{\begin{array}{l|lcl} u = 1+\sin x & x = \pi/6 & \Longrightarrow & u = 3/2 \\ & & & \\ du = \cos x\,dx & x = \pi/2 & \Longrightarrow & u = 2 \end{array}\right\};\quad \displaystyle\int_{\pi/6}^{\pi/2} \frac{\cos x}{1+\sin x}\,dx = \int_{3/2}^{2} \frac{du}{u} = [\ln u]_{3/2}^{2} = \ln\frac{4}{3}$

48. $\displaystyle\int_{\pi/4}^{\pi/2} (1+\csc x)^2\,dx = \int_{\pi/4}^{\pi/2} (1 + 2\csc x + \csc^2 x)\,dx = [x + 2\ln|\csc x - \cot x| - \cot x]_{\pi/4}^{\pi/2}$

$$= \frac{\pi}{4} + 1 - 2\ln(\sqrt{2}-1)$$

49. $\displaystyle\int_{\pi/4}^{\pi/2} \cot x\,dx = [\,\ln|\sin x|\,]_{\pi/4}^{\pi/2} = \ln 1 - \ln\frac{\sqrt{2}}{2} = \ln\sqrt{2} = \frac{1}{2}\ln 2$

50. $\displaystyle\int_1^e \frac{\ln x}{x}\,dx = \left[\frac{(\ln x)^2}{2}\right]_1^e = \frac{1}{2}$

51.

$$\ln|g(x)| = 2\ln\left(x^2+1\right) + 5\ln|x-1| + 3\ln x$$

$$\frac{g'(x)}{g(x)} = 2\left(\frac{2x}{x^2+1}\right) + \frac{5}{x-1} + \frac{3}{x}$$

$$g'(x) = \left(x^2+1\right)^2(x-1)^5 x^3\left(\frac{4x}{x^2+1} + \frac{5}{x-1} + \frac{3}{x}\right)$$

52.
$$\ln|g(x)| = \ln|x| + \ln|x+a| + \ln|x+b| + \ln|x+c|$$
$$\frac{g'(x)}{g(x)} = \frac{1}{x} + \frac{1}{x+a} + \frac{1}{x+b} + \frac{1}{x+c}$$
$$g'(x) = x(x+a)(x+b)(x+c)\left(\frac{1}{x} + \frac{1}{x+a} + \frac{1}{x+b} + \frac{1}{x+c}\right)$$

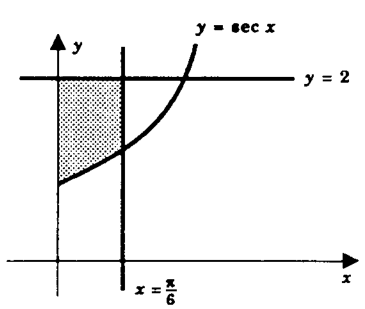
53.
$$\ln|g(x)| = 4\ln|x| + \ln|x-1| - \ln|x+2| - \ln\left(x^2+1\right)$$
$$\frac{g'(x)}{g(x)} = \frac{4}{x} + \frac{1}{x-1} - \frac{1}{x+2} - \frac{2x}{x^2+1}$$
$$g'(x) = \frac{x^4(x-1)}{(x+2)\left(x^2+1\right)}\left(\frac{4}{x} + \frac{1}{x-1} - \frac{1}{x+2} - \frac{2x}{x^2+1}\right)$$

54.
$$\ln|g(x)| = \ln|1+x| + \ln|2+x| + \ln|x| - \ln|4+x| - \ln|2-x|$$
$$\frac{g'(x)}{g(x)} = \frac{1}{1+x} + \frac{1}{2+x} + \frac{1}{x} - \frac{1}{4+x} - \frac{-1}{2-x}$$
$$g'(x) = \frac{(1+x)(2+x)x}{(4+x)(2-x)}\left(\frac{1}{1+x} + \frac{1}{2+x} + \frac{1}{x} - \frac{1}{4+x} + \frac{1}{2-x}\right)$$

55.
$$\ln|g(x)| = \tfrac{1}{2}\left(\ln|x-1| + \ln|x-2| - \ln|x-3| - \ln|x-4|\right)$$
$$\frac{g'(x)}{g(x)} = \frac{1}{2}\left(\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4}\right)$$
$$g'(x) = \frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)}}\left(\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4}\right)$$

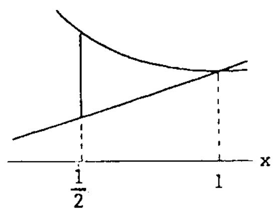
56.
$$\ln|g(x)| = \ln x^2 + \ln(x^2+1) + \ln(x^2+2) - \ln|x^2-1| - \ln|x^2-5|$$
$$\frac{g'(x)}{g(x)} = \frac{2}{x} + \frac{2x}{x^2+1} + \frac{2x}{x^2+2} - \frac{2x}{x^2-1} - \frac{2x}{x^2-5}$$
$$g'(x) = \frac{x^2(x^2+1)(x^2+2)}{(x^2-1)(x^2-5)}\left(\frac{2}{x} + \frac{2x}{x^2+1} + \frac{2x}{x^2+2} - \frac{2x}{x^2-1} - \frac{2x}{x^2-5}\right)$$

57.



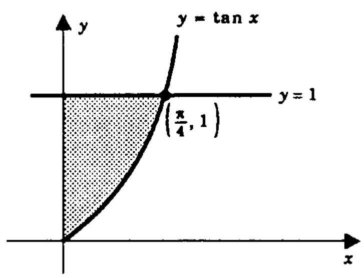
$$A = \int_0^{\pi/6} (2 - \sec x)\, dx$$
$$= [2x - \ln|\sec x + \tan x|]_0^{\pi/6}$$
$$= \frac{\pi}{3} - \ln\left|\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right| = \frac{\pi}{3} - \frac{1}{2}\ln 3$$

58.



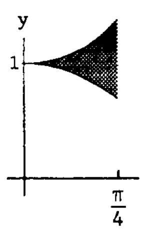
$$A=\int_{1/2}^{1}\left(\csc\frac{\pi}{2}x-x\right)dx$$
$$=\left[\frac{2}{\pi}\ln\left|\csc\frac{\pi}{2}x-\cot\frac{\pi}{2}x\right|-\frac{x^2}{2}\right]_{1/2}^{1}$$
$$=-\frac{2}{\pi}\ln(\sqrt{2}-1)-\frac{3}{8}=\frac{2}{\pi}\ln(1+\sqrt{2})-\frac{3}{8}$$

59.



$$A=\int_0^{\pi/4}(1-\tan x)\,dx$$
$$=[x-\ln|\sec x|]_0^{\pi/4}$$
$$=\frac{\pi}{4}-\ln\sqrt{2}=\frac{\pi}{4}-\frac{1}{2}\ln 2$$

60.



$$A=\int_0^{\pi/4}(\sec x-\cos x)\,dx$$
$$=[\ln|\sec x+\tan x|-\sin x]_0^{\pi/4}$$
$$=\ln(1+\sqrt{2})-\frac{\sqrt{2}}{2}$$

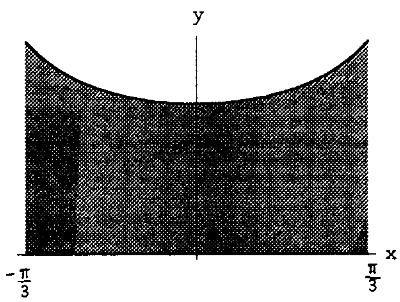
61. $A=\int_1^4\left[\frac{5-x}{4}-\frac{1}{x}\right]dx=\left[\frac{5}{4}x-\frac{1}{8}x^2-\ln x\right]_1^4=\frac{15}{8}-\ln 4$

62. $A=\int_1^2\left(3-x-\frac{2}{x}\right)dx=\left[3x-\frac{x^2}{2}-2\ln x\right]_1^2=\frac{3}{2}-2\ln 2$

63. $V=\int_0^8\pi\left(\frac{1}{\sqrt{1+x}}\right)^2dx=\pi\int_0^8\frac{1}{1+x}\,dx=\pi\,[\ln|1+x|]_0^8=\pi\ln 9$

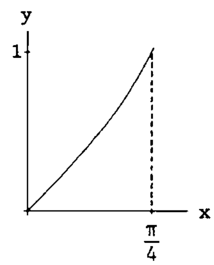
64. By shells: $V=\int_0^3 2\pi x\cdot\frac{3}{1+x^2}\,dx=\left[3\pi\ln(1+x^2)\right]_0^3=3\pi\ln 10$

65.



$$V = \int_{-\pi/3}^{\pi/3} \pi \left(\sqrt{\sec x}\right)^2 dx$$
$$= 2\pi \int_0^{\pi/3} \sec x \, dx$$
$$= 2\pi \left[\ln|\sec x + \tan x|\right]_0^{\pi/3} = 2\pi \ln(2+\sqrt{3})$$

66.



$$V = \int_0^{\pi/4} \pi \tan^2 x \, dx$$
$$= \pi \int_0^{\pi/4} (\sec^2 x - 1)\, dx$$
$$= \pi \left[\tan x - x\right]_0^{\pi/4}$$
$$= \pi\left(1 - \frac{\pi}{4}\right)$$

67. $v(t) = \int a(t)\, dt = \int -(t+1)^{-2}\, dt = \dfrac{1}{t+1} + C.$

Since $v(0) = 1$, we get $1 = 1 + C$ so that $C = 0$. Then

$$s = \int_0^4 |v(t)|\, dt = \int_0^4 \frac{dt}{t+1} = \left[\ln(t+1)\right]_0^4 = \ln 5.$$

The particle traveled $\ln 5$ ft.

68. $V(t) = \int a(t)\, dt = \int -(t+1)^{-2}\, dt = \dfrac{1}{t+1} + C, \quad v(0) = 2 \implies v(t) = \dfrac{1}{t+1} + 1$

Then $\quad s = \displaystyle\int_0^4 v(t)\, dt = \int_0^4 \left(\frac{1}{1+t} + 1\right) dt = \left[\ln(t+1) + t\right]_0^4 = 4 + \ln 5$ ft

69. $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$

$\dfrac{d^2}{dx^2}(\ln x) = -\dfrac{1}{x^2}$

$\dfrac{d^3}{dx^3}(\ln x) = \dfrac{2}{x^3}$

70. $\dfrac{d}{dx}(\ln(1-x)) = \dfrac{-1}{1-x}$

$\dfrac{d^2}{dx^2}(\ln(1-x)) = \dfrac{-1}{(1-x)^2}$

$\dfrac{d^3}{dx^3}(\ln(1-x)) = \dfrac{-2}{(1-x)^3}$

$$\frac{d^4}{dx^4}(\ln x) = -\frac{2\cdot 3}{x^4} \qquad\qquad \frac{d^4}{dx^4}(\ln(1-x)) = \frac{-2\cdot 3}{(1-x)^4}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$\frac{d^n}{dx^n}(\ln x) = (-1)^{n-1}\frac{(n-1)!}{x^n} \qquad\qquad \frac{d^n}{dx^n}[\ln(1-x)] = -\frac{(n-1)!}{(1-x)^n}$$

71. $\dfrac{d^n}{dx^n}(\ln 2x) = \dfrac{d^n}{dx^n}[\ln 2 + \ln x] = 0 + \dfrac{d^n}{dx^n}(\ln x) = (-1)^{n-1}\dfrac{(n-1)!}{x^n}$ (See Exercise 69)

72. $\dfrac{d^n}{dx^n}\left(\ln\dfrac{1}{x}\right) = -\dfrac{d^n}{dx^n}(\ln x) = (-1)^n\dfrac{(n-1)!}{x^n}$ (See Exercise 69)

73. $\displaystyle\int \csc x\,dx = \int \frac{\csc x(\csc x - \cot x)}{\csc x - cotx}\,dx = \int \frac{\csc^2 - \csc x\,\cot x}{\csc x - \cot x}\,dx$

$$\left\{\begin{array}{l} u = \csc x - \cot x \\ \\ du = (-\csc x\,\cot x + \csc^2 x)\,dx \end{array}\right\}; \quad \int \csc x\,dx = \int \frac{du}{u} = \ln|u| + C$$

$$= \ln|\csc x - \cot x| + C$$

74. (a) If $g(x) = g_1(x)g_2(x)$, (7.3.7) gives $g'(x) = g(x)\left[\dfrac{g_1'(x)}{g_1(x)} + \dfrac{g_2'(x)}{g_2(x)}\right]$

$$\Longrightarrow\ g'(x) = g_1(x)g_2(x)\left[\frac{g_1'(x)}{g_1(x)} + \frac{g_2'(x)}{g_2(x)}\right] = g_1'(x)\,g_2(x) + g_1(x)\,g_2'(x).$$

(b) If $g(x) = \dfrac{g_1(x)}{g_2(x)} = g_1(x)\left(\dfrac{1}{g_2(x)}\right)$, then

$$\Longrightarrow\ g'(x) = g_1(x)\left(\frac{1}{g_2(x)}\right)\left(\frac{g_1'(x)}{g_1(x)} + \frac{\frac{d}{dx}(1/g_2(x))}{1/g_2(x)}\right) = \frac{g_1'(x)g_2(x) - g_1(x)g_2'(x)}{[g_2(x)]^2}$$

75. $f(x) = \ln(4-x),\quad x < 4$

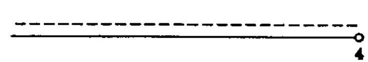
$$f'(x) = \frac{1}{x-4}$$

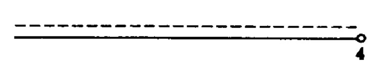
$$f''(x) = \frac{-1}{(x-4)^2}$$

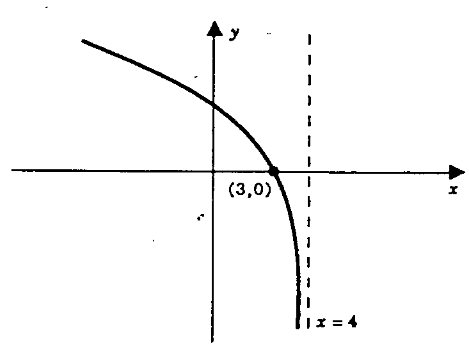
(i) domain $(-\infty, 4)$

(ii) decreases throughout

(iii) no extreme values

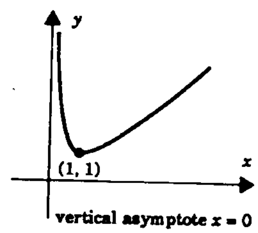
f' :


f'' :




(iv) concave down throughout:
no pts of inflection

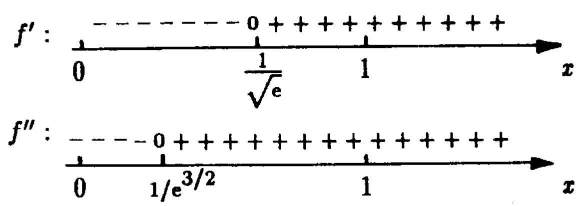
76. $f'(x) = 1 - \dfrac{1}{x}$



$f''(x) = -\dfrac{1}{x^2}$

(i) domain $(0, \infty)$
(ii) decreases on $(0, 1]$, increases on $[1, \infty)$
(iii) $f(1) = 1$ local and absolute min
(iv) concave up on $(0, \infty)$;
no pts of inflection

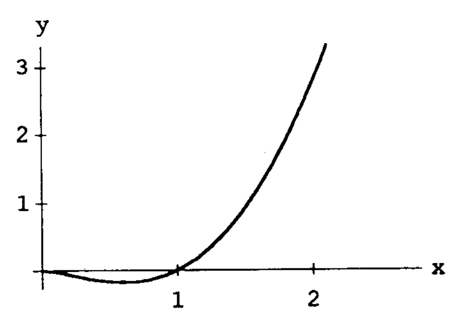
77. $f(x) = x^2 \ln x,\ x > 0$



$f'(x) = 2x \ln x + x$

$f''(x) = 2 \ln x + 3$

(i) domain $(0,\ \infty)$
(ii) decreases on $(0,\ 1/\sqrt{e}]$, increases on $[1/\sqrt{e},\ \infty)$
(iii) $f(1/\sqrt{e}) = -1/2e$ local and absolute min
(iv) concave down on $\left(0,\ 1/e^{3/2}\right)$,
concave up on $\left(1/e^{3/2},\ \infty\right)$;
pt of inflection at $\left(1/e^{3/2},\ -3/2e^3\right)$



78. $f'(x) = -\dfrac{2x}{4 - x^2}$

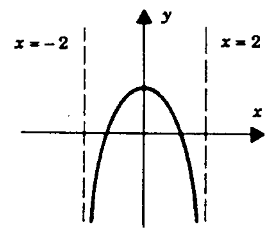
$f''(x) = -\dfrac{4(2 + x^2)}{(4 - x^2)^2}$

(i) domain $(-2, 2)$

(ii) increases on $(-2, 0]$, decreases on $[0, 2)$

(iii) $f(0) = \ln 4$ local and absolute max

(iv) concave down on $(-2, 2)$;
no pts of inflection



79. Let $f(x) = x^2 \ln \dfrac{1}{x}$. Then $f'(x) = -x + 2x \ln \dfrac{1}{x} = 0 \implies ln\dfrac{1}{x} = \dfrac{1}{2} \Rightarrow x = \dfrac{1}{\sqrt{e}}$.

80. (a) Because $\sqrt{1 - \sin^2 x} > 0$ for all $x \in [0, \pi]$, $x \neq \frac{\pi}{2}$.

(b) $\displaystyle\int_0^{\pi} \sqrt{1 - \sin^2 x}\, dx = \int_0^{\pi} \sqrt{\cos^2 x}\, dx = \int_0^{\pi} |\cos x|\, dx = 2 \int_0^{\pi/2} \cos x\, dx = 2 [\sin x]_0^{\pi/2} = 2$

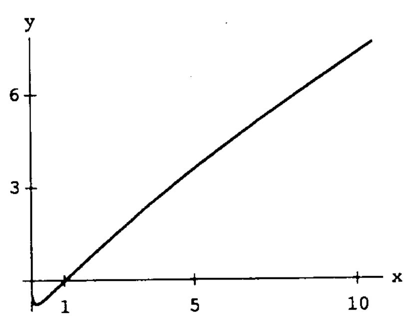
81. Average slope $= \displaystyle\frac{1}{b-a} \int_a^b \frac{1}{x}\, dx = \frac{1}{(b-a)} \ln \frac{b}{a}$

82. (a) $f'(x) = \dfrac{1}{2x}(2) = \dfrac{1}{x}$; $g'(x) = \dfrac{1}{3x}(3) = \dfrac{1}{x}$

(b) $F'(x) = \dfrac{1}{kx}(k) = \dfrac{1}{x}$

(c) $F(x) = \ln kx = \ln k + \ln x$, so $F'(x) = 0 + \dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$.

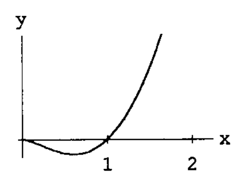
83.



x-intercept: 1; abs min at $x = 1/e^2$;

abs max at $x = 10$

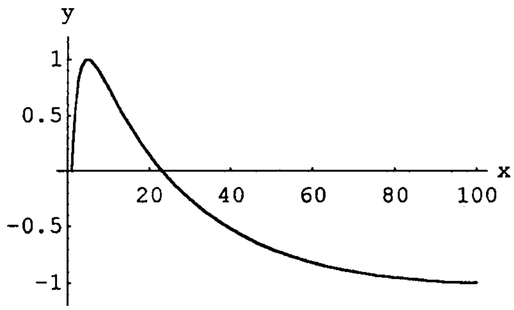
84.



x-intercept at $x = 1$; abs min at $x \cong 0.6065$;

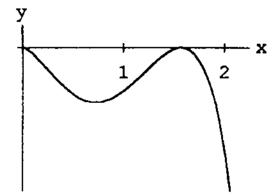
abs max at $x = 2$

85.



x-intercepts: $1, 23.1407$; abs min at $x = 100$; abs max at $x = 4.8105$

86.

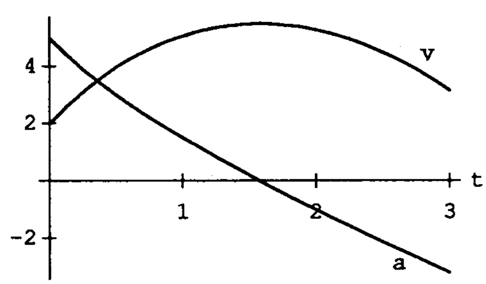


x-intercept at $x = \pi/2$; abs max at $x = \pi/2$; local min at $x \cong 0.7269$; abs min at $x = 2$;

87. (a)

$$\begin{aligned} v(t) - v(0) &= \int_0^t a(u)du, \quad 0 \leq t \leq 3 \\ v(t) &= \int_0^t \left[4 - 2(u+1) + \frac{3}{u+1}\right] du + 2 \\ &= \left[2u - u^2 + 3\ln|u+1|\right]_0^t = 2 + 2t - t^2 + 3\ln(t+1) \end{aligned}$$

(b)

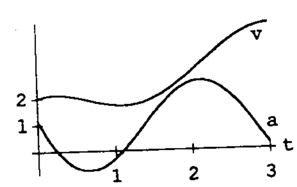


(c) max velocity at $t = 1.5811$; min velocity at $t = 0$

88. (a) $v(t) = \int a(t)\,dt = \int \left[2\cos 2(t+1) + \frac{2}{t+1}\right] dt = \sin 2(t+1) + 2\ln(t+1) + C$

$v(0) = 2 \implies v(t) = \sin 2(t+1) + 2\ln(t+1) + 2 - \sin 2$

(b)



(c) max at $t \cong 6.1389$; min at $t \cong 1.1092$

SECTION 7.4

1. $\frac{dy}{dx} = e^{-2x}\frac{d}{dx}(-2x) = -2e^{-2x}$

2. $\frac{dy}{dx} = 3e^{2x+1}\cdot 2 = 6e^{2x+1}$

3. $\frac{dy}{dx} = e^{x^2-1}\frac{d}{dx}\left(x^2-1\right) = 2xe^{x^2-1}$

4. $\frac{dy}{dx} = 2e^{-4x}(-4) = -8e^{-4x}$

5. $\frac{dy}{dx} = e^x\frac{d}{dx}(\ln x) + \ln x\frac{d}{dx}(e^x) = e^x\left(\frac{1}{x}+\ln x\right)$

6. $\frac{dy}{dx} = 2xe^x + x^2e^x$

7. $\frac{dy}{dx} = x^{-1}\frac{d}{dx}\left(e^{-x}\right) + e^{-x}\frac{d}{dx}\left(x^{-1}\right) = -x^{-1}e^{-x} - e^{-x}x^{-2} = -\left(x^{-1}+x^{-2}\right)e^{-x}$

8. $\frac{dy}{dx} = e^{\sqrt{x}+1}\left(\frac{1}{2\sqrt{x}}\right)$

9. $\frac{dy}{dx} = \frac{1}{2}\left(e^x - e^{-x}\right)$

10. $\frac{dy}{dx} = \frac{1}{2}\left(e^x - e^{-x}(-1)\right) = \frac{1}{2}\left(e^x + e^{-x}\right)$

11. $\frac{dy}{dx} = e^{\sqrt{x}}\frac{d}{dx}\left(\ln\sqrt{x}\,\right) + \ln\sqrt{x}\,\frac{d}{dx}\left(e^{\sqrt{x}}\right) = e^{\sqrt{x}}\left(\frac{1}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}\right) + \ln\sqrt{x}\frac{e^{\sqrt{x}}}{2\sqrt{x}} = \frac{1}{2}e^{\sqrt{x}}\left(\frac{1}{x}+\frac{\ln\sqrt{x}}{\sqrt{x}}\right)$

12. $\frac{dy}{dx} = 3(3-2e^{-x})^2\left(-2e^{-x}(-1)\right) = 6e^{-x}\left(3-2e^{-x}\right)^2$

13. $\frac{dy}{dx} = 2\left(e^{x^2}+1\right)\frac{d}{dx}\left(e^{x^2}+1\right) = 2\left(e^{x^2}+1\right)e^{x^2}\frac{d}{dx}\left(x^2\right) = 4xe^{x^2}\left(e^{x^2}+1\right)$

14. $\frac{dy}{dx} = 2\left(e^{2x}-e^{-2x}\right)\cdot\left(2e^{2x}+2e^{-2x}\right) = 4\left(e^{4x}-e^{-4x}\right)$

15. $\frac{dy}{dx} = \left(x^2-2x+2\right)\frac{d}{dx}\left(e^x\right) + e^x\left(2x-2\right) = x^2e^x$

16. $\frac{dy}{dx} = 2xe^x + x^2e^x - e^{x^2} - xe^{x^2}2x = (x^2+2x)e^x - (2x^2+1)e^{x^2}$

17. $\frac{dy}{dx} = \frac{\left(e^x+1\right)e^x - \left(e^x-1\right)e^x}{\left(e^x+1\right)^2} = \frac{2e^x}{\left(e^x+1\right)^2}$

18. $\frac{dy}{dx} = \frac{2e^{2x}(e^{2x}+1) - (e^{2x}-1)2e^{2x}}{(e^{2x}+1)^2} = \frac{4e^{2x}}{(e^{2x}+1)^2}$

19. $y = e^{4\ln x} = \left(e^{\ln x}\right)^4 = x^4$ so $\frac{dy}{dx} = 4x^3.$

20. $y = \ln e^{3x} = 3x \implies \frac{dy}{dx} = 3$

21. $f'(x) = \cos\left(e^{2x}\right)\, e^2x\,2 = 2e^{2x}\cos\left(e^{2x}\right)$

22. $f'(x) = e^{\sin 2x}\cdot 2\cos 2x$

23. $f'(x) = e^{-2x}(-\sin x) + e^{-2x}(-2)\cos x = -\,e^{-2x}(2\cos x + \sin x)$

24. $f'(x) = -\dfrac{1}{\cos e^{2x}} \cdot \sin e^{2x}(2e^{2x}) = -2e^{2x}\ \tan(e^{2x})$

25. $\displaystyle\int e^{2x}\,dx = \frac{1}{2}e^{2x} + C$

26. $\displaystyle\int e^{-2x}\,dx = -\frac{1}{2}e^{-2x} + C$

27. $\displaystyle\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$

28. $\displaystyle\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$

29. $\{u = x^2, \quad du = 2x\,dx\}$; $\displaystyle\int xe^{x^2}\,dx = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C$

30. $\displaystyle\int xe^{-x^2}\,dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C$

31. $\left\{u = \dfrac{1}{x}, \quad du = -\dfrac{1}{x^2}\,dx\right\}$; $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx = -\int e^u\,du = -e^u + C = -e^{1/x} + C$

32. $\displaystyle\int \frac{e^{2\sqrt{x}}}{\sqrt{x}}\,dx = e^{2\sqrt{x}} + C$

33. $\displaystyle\int \ln e^x\,dx = \int x\,dx = \frac{1}{2}x^2 + C$

34. $\displaystyle\int e^{\ln x}\,dx = \int x\,dx = \frac{x^2}{2} + C$

35. $\displaystyle\int \frac{4}{\sqrt{e^x}}\,dx = \int 4e^{-x/2}\,dx = -8e^{-x/2} + C$

36. $\displaystyle\int \frac{e^x}{e^x+1}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln(e^x+1) + C$

37. $\left\{\begin{array}{c} u = e^x + 1 \\ du = e^x\,dx \end{array}\right\}$; $\displaystyle\int \frac{e^x}{\sqrt{e^x+1}}\,dx = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{e^x+1} + C$

38. $\left\{\begin{array}{c} u = e^x + 1 \\ du = e^x\,dx \end{array}\right\}$; $\displaystyle\int 2\frac{e^x}{\sqrt[3]{e^x+1}}\,dx = \int 2u^{-1/3}\,du = 3u^{2/3} + C = 3(e^x+1)^{2/3} + C$

39. $\left\{\begin{array}{c} u = 2e^{2x} + 3 \\ du = 4e^{2x}\,dx \end{array}\right\}$; $\displaystyle\int \frac{e^{2x}}{2e^{2x}+3}\,dx = \frac{1}{4}\int \frac{du}{u} = \frac{1}{4}\ln u + C = \frac{1}{4}\ln\left(2e^{2x}+3\right) + C$

40. $\left\{\begin{array}{c} u = e^{ax^2}+1 \\ du = 2axe^{ax^2}\,dx \end{array}\right\}$; $\displaystyle\int \frac{xe^{ax^2}}{e^{ax^2}+1}\,dx = \frac{1}{2a}\int \frac{du}{u} = \frac{1}{2a}\ln|u| + C = \frac{1}{2a}\ln\left(e^{ax^2}+1\right) + C$

41. $\{u = \sin x, \quad du = \cos x\,dx\}$; $\displaystyle\int \cos x\, e^{\sin x}\,dx = \int e^u\,du = e^u + C = e^{\sin x} + C$

42. $\left\{\begin{array}{c} u = e^{-2x} \\ du = -2e^{-2x}\,dx \end{array}\right\}$; $\displaystyle\int \frac{\sin\left(e^{-2x}\right)}{e^{2x}}\,dx = -\frac{1}{2}\int \sin u\,du = \frac{1}{2}\cos u + C = \frac{1}{2}\cos\left(e^{-2x}\right) + C$

43. $\{u = e^{-x}, \quad du = -\,e^{-x}\,dx\}$;

$\int e^{-x}\left[1+\cos\left(e^{-x}\right)\right]dx = -\int (1+\cos u)\,du = -u - \sin u + C = -e^{-x} - \sin\left(e^{-x}\right) + C$

44. $\left\{\begin{array}{c} u = \tan 2x \\ du = 2\sec^2 2x\,dx \end{array}\right\}$; $\displaystyle\int \sec^2 2xe^{\tan 2x}\,dx = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{\tan 2x} + C$

45. $\displaystyle\int_0^1 e^x\,dx = \left[e^x\right]_0^1 = e - 1$

46. $\displaystyle\int_0^1 e^{-kx}\,dx = -\frac{1}{k}\left[e^{-kx}\right]_0^1 = \frac{1}{k}\left(1 - e^{-k}\right)$

47. $\displaystyle\int_0^{\ln \pi} e^{-6x}\,dx = \left[-\frac{1}{6}e^{-6x}\right]_0^{\ln \pi} = -\frac{1}{6}e^{-6\ln \pi} + \frac{1}{6}e^0 = \frac{1}{6}\left(1 - \pi^{-6}\right)$

48. $\displaystyle\int_0^1 xe^{-x^2}\,dx = -\frac{1}{2}\left[e^{-x^2}\right]_0^1 = \frac{1}{2}\left(1 - \frac{1}{e}\right)$

49. $\displaystyle\int_0^1 \frac{e^x+1}{e^x}\,dx = \int_0^1 \left(1 + e^{-x}\right)\,dx = \left[x - e^{-x}\right]_0^1 = \left(1 - e^{-1}\right) - (0-1) = 2 - \frac{1}{e}$

50. $\displaystyle\int_0^1 \frac{4-e^x}{e^x}\,dx = \int_0^1 (4e^{-x} - 1)\,dx = \left[-4e^{-x} - x\right]_0^1 = 3 - 4e^{-1}$

51. $\displaystyle\int_0^{\ln 2} \frac{e^x}{e^x+1}\,dx = \left[\ln\left(e^x+1\right)\right]_0^{\ln 2} = \ln\left(e^{\ln 2}+1\right) - \ln\left(e^0+1\right) = \ln 3 - \ln 2 = \ln\frac{3}{2}$

52. $\displaystyle\int_0^1 \frac{e^x}{4-e^x}\,dx = \left[-\ln\left|e^x - 4\right|\right]_0^1 = \ln\left(\frac{3}{4-e}\right)$

53. $\displaystyle\int_0^1 x\left(e^{x^2}+2\right)\,dx = \int_0^1 \left(xe^{x^2}+2x\right)\,dx = \left[\frac{1}{2}e^{x^2} + x^2\right]_0^1 = \left(\tfrac{1}{2}e + 1\right) - \left(\tfrac{1}{2} + 0\right) = \tfrac{1}{2}(e+1)$

54. $\displaystyle\int_0^{\ln\frac{\pi}{4}} e^x \sec e^x\,dx = [\ln|\sec e^x + \tan e^x|]_0^{\ln\frac{\pi}{4}} = \ln\left(\frac{\sqrt{2}+1}{2}\right).$

55. $y' = ake^{kx} = bke^{-kx} = 0 \quad \Longrightarrow \quad x = \dfrac{1}{2k}\ln\dfrac{b}{a}.$

At this point, $y = 2\sqrt{ab}$.

56. $y' = -2xe^{-x^2}$

$y" = -2x(-2xe^{-x^2}) - 2e^{-x^2} = e^{-x^2}(4x^2-2) = 0 \quad \Longrightarrow \quad x = \pm\sqrt{\dfrac{1}{2}}$

57. $A = 2xe^{-x^2}$

$A' = 2x(-2xe^{-x^2}) + 2e^{-x^2} = 2e^{-x^2}(1-2x^2) = 0 \quad \Longrightarrow \quad x = \pm\sqrt{\dfrac{1}{2}}$ and $y = \dfrac{1}{\sqrt{e}}$.

Put the vertices at $\left(\pm\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{e}}\right)$.

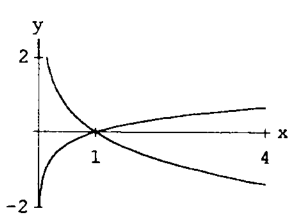
58. $A = y\ln y$, and $\dfrac{dA}{dt} = (1+\ln y)\dfrac{dy}{dt}$.

At $y = 3$, $\quad \dfrac{dy}{dt} = \dfrac{1}{2}$. $\quad$ Thus $\dfrac{dA}{dt} = \dfrac{1}{2}(1+\ln 3)$ square units per minute.

59.

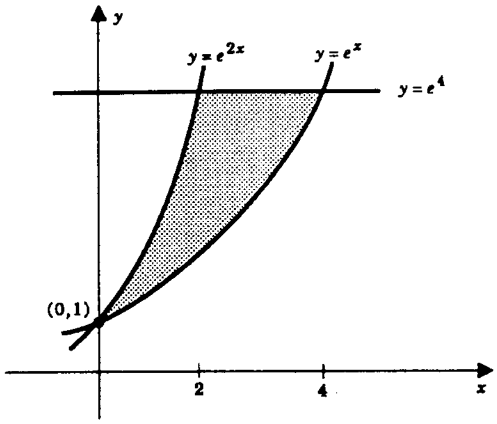
$$\begin{aligned}
x(t) &= Ae^{ct} + Be^{-ct} \\
x'(t) &= Ace^{ct} - Bce^{-ct} \\
x''(t) &= Ac^2e^{ct} + Bc^2e^{-ct} \\
&= c^2\left(Ae^{ct} + Be^{-ct}\right) \\
&= c^2x(t)
\end{aligned}$$

60.



$$\begin{aligned}
A &= \int_{-\ln 4}^{0} (4 - e^{-y})\,dy + \int_0^{\frac{1}{2}\ln 4} (4 - e^{2y})\,dy \\
&= \left[4y + e^{-y}\right]_{-\ln 4}^{0} + \left[4y - \frac{1}{2}e^{2y}\right]_0^{\frac{1}{2}\ln 4} \\
&= 12\ln 2 - \frac{9}{2}
\end{aligned}$$

61.



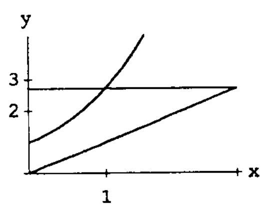
$$A = \int_0^2 \left(e^{2x} - e^x\right)\,dx + \int_2^4 \left(e^4 - e^x\right)\,dx$$

$$= \left[\tfrac{1}{2}e^{2x} - e^x\right]_0^2 + \left[e^4 x - e^x\right]_2^4$$

$$= \left(\tfrac{1}{2}e^4 - e^2 - \tfrac{1}{2} + 1\right) + \left(4e^4 - e^4 - 2e^4 + e^2\right)$$

$$= \tfrac{1}{2}\left(3e^4 + 1\right)$$

62.

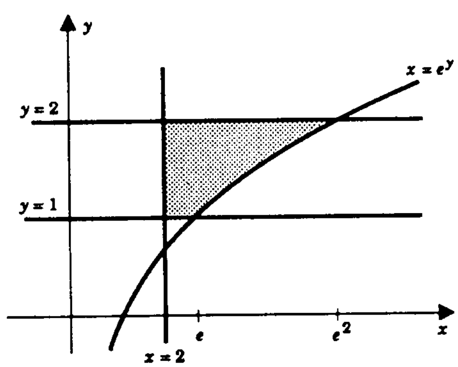


$$A = \text{triangle} \; - \; \text{upper left corner}$$

$$= \frac{1}{2}e^2 - \int_0^1 (e - e^x)\,dx = \frac{1}{2}e^2 - [ex - e^x]_0^1$$

$$= \frac{1}{2}e^2 - 1$$

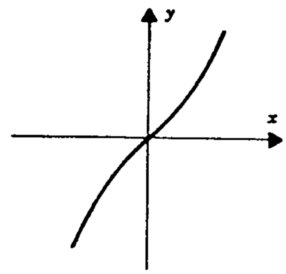
63.



$$A = \int_1^2 \left(e^y - 2\right)\,dy$$

$$= [e^y - 2y]_1^2 = e^2 - e - 2$$

64. $f'(x) = \frac{1}{2}(e^x + e^{-x})$



$f''(x) = \frac{1}{2}(e^x - e^{-x})$

(i) domain $(-\infty, \infty)$

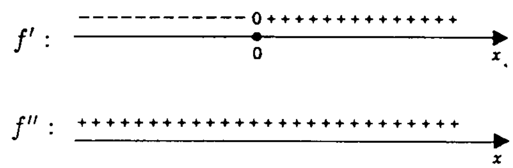
(ii) increases on $(-\infty, \infty)$

(iii) no extreme values

(iv) concave down on $(-\infty, 0)$, concave up on $(0, \infty)$

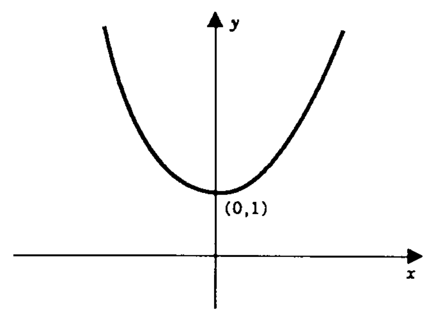
pt of inflection $(0, 0)$

65. $f(x) = \frac{1}{2}(e^x + e^{-x})$



$f'(x) = \frac{1}{2}(e^x - e^{-x})$

$f''(x) = \frac{1}{2}(e^x + e^{-x})$



(i) domain $(-\infty, \infty)$

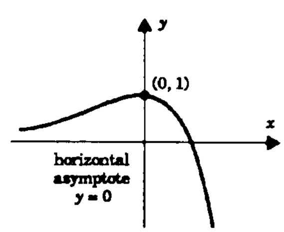
(ii) decreases on $(-\infty, 0]$, increases on $[0, \infty)$

(iii) $f(0) = 1$ local and absolute min

(iv) concave up everywhere

66. $f'(x) = -xe^x$

$f''(x) = -e^x - xe^x = -e^x(1 + x)$



(i) domain $(-\infty, \infty)$

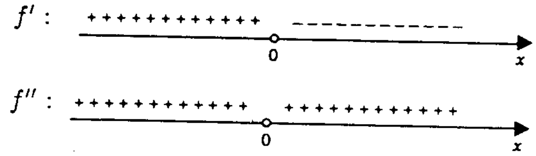
(ii) increases on $(-\infty, 0]$, decreases on $[0, \infty)$

(iii) $f(0) = 1$ local and absolute max

(iv) concave up on $(-\infty, -1)$, concave down on $(-1, \infty)$

pt of inflection $(-1, 2/e)$

67. $f(x) = e^{(1/x)^2}$



$f'(x) = \dfrac{-2}{x^3} e^{(1/x)^2}$

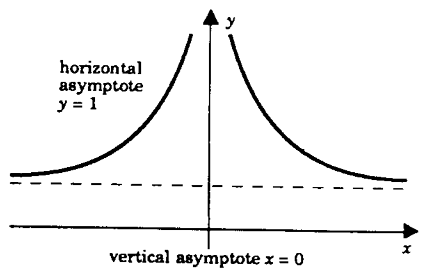
$f''(x) = \dfrac{6x^2 + 4}{x^6} e^{(1/x)^2}$

(i) domain $(-\infty, 0) \cup (0, \infty)$

(ii) increases on $(-\infty, 0)$, decreases on $(0, \infty)$

(iii) no extreme values

(iv) concave up on $(-\infty, 0)$ and on $(0, \infty)$



68. $f'(x) = (2x - x^2)e^{-x}$

$f''(x) = e^{-x}(2 - 4x + x^2)$

(i) domain $(-\infty, \infty)$

(ii) decreases on $(-\infty, 0]$, and on $[2, \infty)$, increases on $[0, 2]$

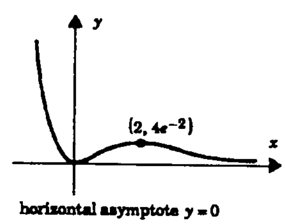
(iii) $f(0) = 0$ local and absolute min,

$f(2) = 4e^{-2}$ local max

(iv) concave up on $(-\infty, 2 - \sqrt{2})$ and $(2 + \sqrt{2}, \infty)$,

concave down on $(2 - \sqrt{2}, 2 + \sqrt{2})$;

points of inflection at $x = 2 \pm \sqrt{2}$.



69. (a) For $y = e^{ax}$ we have $dy/dx = ae^{ax}$. Therefore the line tangent to the curve $y = e^{ax}$ at an arbitrary point $(x_0,\, e^{ax_0})$ has equation

$$y - e^{ax_0} = ae^{ax_0}\,(x - x_0)\,.$$

The line passes through the origin iff $e^{ax_0} = (ae^{ax_0})\,x_0$ iff $x_0 = 1/a$. The point of tangency is $(1/a,\, e)$. This is point B. By symmetry, point A is $(-1/a,\, e)$.

(b) The tangent line at B has equation $y = aex$. By symmetry

$$A_{\mathrm{I}} = 2\int_0^{1/a} (e^{ax} - aex)\; dx = 2\left[\frac{1}{a}e^{ax} - \frac{1}{2}aex^2\right]_0^{1/a} = \frac{1}{a}\,(e-2)\,.$$

(c) The normal at B has equation

$$y - e = -\frac{1}{ae}\left(x - \frac{1}{a}\right).$$

This can be written

$$y = -\frac{1}{ae}x + \frac{a^2e^2+1}{a^2e}.$$

Therefore
$$A_{\text{II}} = 2\int_0^{1/a}\left(-\frac{1}{ae}x + \frac{a^2e^2+1}{a^2e} - e^{ax}\right)dx = \frac{1+2a^2e}{a^3e}.$$

70. By induction. True for $n=0: e^x > 1$ for $x>0$.

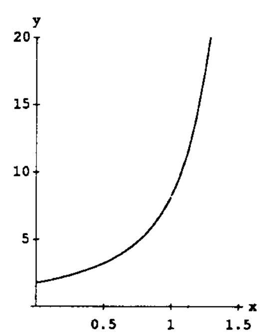
Assume true for n. Then

$$\begin{aligned} e^x &= 1 + \int_0^x e^t\,dt > 1 + \int_0^x \left(1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!}\right)dt \\ &= 1 + \left[t + \frac{t^2}{2} + \frac{t^3}{3!} + \cdots + \frac{t^{n+1}}{(n+1)!}\right]_0^x \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n+1}}{(n+1)!} \end{aligned}$$

So the result is true for $n+1$

71. For $x > (n+1)!$
$$e^x > 1 + x + \cdots + \frac{x^{n+1}}{(n+1)!} > \frac{x^{n+1}}{(n+1)!} = x^n\left[\frac{x}{(n+1)!}\right] > x^n.$$

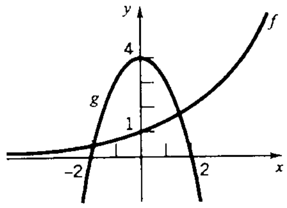
72. Numerically, $L \cong 10$;
$$\lim_{x\to 0}\frac{e^{10x}-1}{x} = \lim_{x\to 0}\frac{e^{10x}-e^{10(0)}}{x-0}$$
is the derivative of $f(x)=e^{10x}$ at $x=0$. Note that
$$f'(x) = 10e^{10x} \quad\text{and}\quad f'(0)=10$$



73. Numerically, $8.15 \le L \le 8.16$;
$$\lim_{x\to 1}\frac{e^{x^3}-e}{x-1} = \lim_{x\to 1}\frac{e^{x^3}-e^1}{x-1}$$
is the derivative of $f(x) = e^{x^3}$ at $x=1$.
Note that $f'(x) = 3x^2e^{x^3}$; $f'(1) = 3e \cong 8.15485$.

74. Numerically, $L \cong 2.72$;
$$\lim_{x\to 1}\frac{e^x-e}{\ln x} = \left(\lim_{x\to 1}\frac{e^x-e^1}{x-1}\right)\left(\lim_{x\to 1}\frac{x-1}{\ln x - \ln 1}\right) = (e)(1) = e \cong 2.72$$
(The first limit is the derivative of $f(x)=e^x$ at $x=1$. This is e. The second limit is the reciprocal of the derivative of $g(x)=\ln x$ at $x=1$. This is 1.)

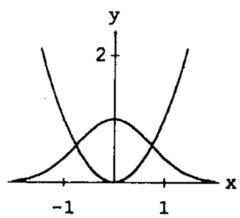
75. (a)



(b) $x_1 \cong -1.9646,\ x_2 \cong 1.0580$

(c) $A \cong \int_{-1.9646}^{1.0580} \left[4 - x^2 - e^x\right] dx = \left[4x - \frac{1}{3}x^3 - e^x\right]_{-1.9646}^{1.0580} \cong 6.4240$

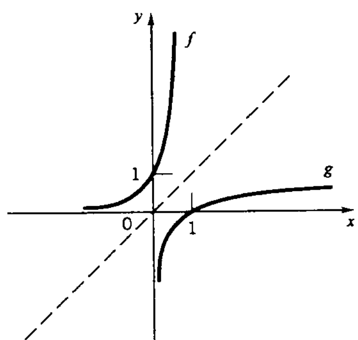
76. (a)



(b) Intersect at $x \cong \pm 0.7531$

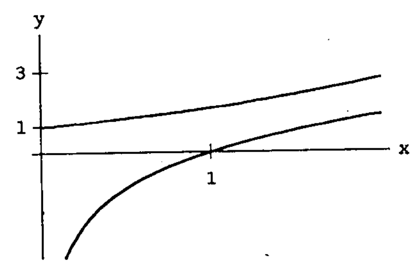
(c) Area $\cong 0.98$

77.



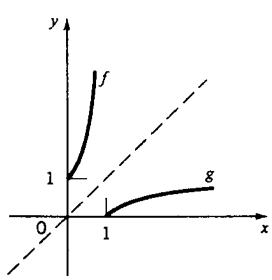
$f\left(g(x)\right) = e^{2\ln\sqrt{x}} = e^{2(1/2)\ln x} = e^{\ln x} = x$

78.



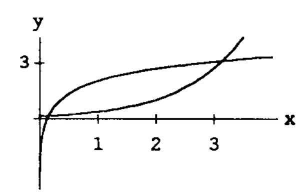
$f\left(g(x)\right) = e^{\frac{\ln x^2}{2}} = e^{\frac{2\ln x}{2}} = e^{\ln x} = x$

79.



$f\left(g(x)\right) = e^{\left(\sqrt{\ln x}\right)^2} = e^{\ln x} = x$

80.



$f\left(g(x)\right) = e^{2+\ln x-2} = e^{\ln x} = x$

PROJECT 7.4

1. (a) $\ln a + \frac{1}{n} = \int_1^{1+\frac{1}{n}} \frac{dt}{t} \qquad \leq \int_1^{1+\frac{1}{n}} 1\, dt$

(since $\frac{1}{t} \leq 1$ throughout the interval of integration) $= 1$.

(b) $\ln a + \frac{1}{n} = \int_1^{1+\frac{1}{n}} \frac{dt}{t} \geq \int_1^{1+\frac{1}{n}} \frac{dt}{1+\frac{1}{n}}$

(since $\frac{1}{t} \geq \frac{1}{1+\frac{1}{n}}$ throughout the interval of integration) $= \frac{1}{1+\frac{1}{n}}\,\frac{1}{n} = \frac{1}{n+1}$.

2. From 1 (a), we get $1 + \frac{1}{n} \leq e^{\frac{1}{n}}$, and thus

$$\left(1+\frac{1}{n}\right)^n \leq e$$

From 1 (b), we get $e^{\frac{1}{n+1}} \leq 1 + \frac{1}{n}$, and thus

$$e \leq \left(1+\frac{1}{n}\right)^{n+1}$$

3. For $n = 1000$, we get $e \cong 2.72$.

For $n = 10,000$, we get $e \cong 2.718$.

For $n = 100,000$, we get $e \cong 2.7183$.

For $n = 1,000,000$, we get $e \cong 2.71828$.

4. Define g on $(-1, \infty)$ by setting

$$g(h) = \left\{ \begin{array}{ll} \frac{1}{h}\ln(1+h), & h \in (-1,0) \cup (0,\infty) \\ 1, & h = 0 \end{array} \right]$$

By the hint the function g is continuous at 0. The composition $e^{g(h)}$ is therefore continuous at 0; that is, $\lim_{h\to 0}(1+h)^{1/h} = e$.

SECTION 7.5

1. $\log_2 64 = \log_2 (2^6) = 6$

2. $\log_2 \frac{1}{64} = \log_2 2^{-6} = -6$

3. $\log_{64}(1/2) = \frac{\ln(1/2)}{\ln 64} = \frac{-\ln 2}{6\ln 2} = -\frac{1}{6}$

4. $\log_{10} 0.01 = \log_{10} 10^{-2} = -2$

5. $\log_5 1 = \log_5 (5^0) = 0$

6. $\log_5 0.2 = \log_5 5^{-1} = -1$

7. $\log_5 (125) = \log_5 (5^3) = 3$

8. $\log_2 4^3 = \log_2 2^6 = 6$

9. $\log_p xy = \frac{\ln xy}{\ln p} = \frac{\ln x + \ln y}{\ln p} = \frac{\ln x}{\ln p} + \frac{\ln y}{\ln p} = \log_p x + \log_p y$

10. $\log_p \frac{1}{x} = \frac{\ln \frac{1}{x}}{\ln p} = -\frac{\ln x}{\ln p} = -\log_p x.$

11. $\log_p x^y = \frac{\ln x^y}{\ln p} = y\frac{\ln x}{\ln p} = y \log_p x$

12. $\log_p \frac{x}{y} = \frac{\ln \frac{x}{y}}{\ln p} = \frac{\ln x - \ln y}{\ln p} = \log_p x - \log_p y$

13. $10^x = e^x \implies \left(e^{\ln 10}\right) = e^x \implies e^{x \ln 10} = e^x$

$\implies x \ln 10 = x \implies x(\ln 10 - 1) = 0 \implies$ Thus, $x = 0.$

14. $\log_5 x = 0.04 \implies x = 5^{0.04}$

15. $\log_x 10 = \log_4 100 \implies \frac{\ln 10}{\ln x} = \frac{\ln 100}{\ln 4} \implies \frac{\ln 10}{\ln x} = \frac{2 \ln 10}{2 \ln 2}$

$\implies \ln x = \ln 2$ Thus, $x = 2.$

16. $\log_x 2 = \log_3 x \implies \frac{\ln 2}{\ln x} = \frac{\ln x}{\ln 3} \implies \ln x = \pm\sqrt{(\ln 2)(\ln 3)} \implies x = e^{\pm\sqrt{(\ln 2)(\ln 3)}}$

17. The logarithm function is increasing. Thus,

$$e^{t_1} < a < e^{t_2} \quad \implies \quad t_1 = \ln e^{t_1} < \ln a < \ln e^{t_2} = t_2.$$

18. Since the exponential function is increasing, $e^{\ln x_1} < e^b < e^{\ln x_2}$, so $x_1 < e^b < x_2$

19. $f'(x) = 3^{2x}(\ln 3)(2) = 2(\ln 3)3^{2x}$

20. $g'(x) = 4^{3x^2}(\ln 4)\, 6x$

21. $f'(x) = 2^{5x}(\ln 2)(5)3^{\ln x} + 2^{5x}3^{\ln x}(\ln 3)\frac{1}{x} = 2^{5x}3^{\ln x}\left(5 \ln 2 + \frac{\ln 3}{x}\right)$

22. $F'(x) = 5^{-2x^2+x}(\ln 5)(-4x + 1)$

23. $g'(x) = \frac{1}{2}\,(\log_3 x)^{-1/2}\left(\frac{1}{\ln 3}\right)\frac{1}{x} = \frac{1}{2(\ln 3)x\sqrt{\log_3 x}}$

24. $h'(x) = 7^{\sin x^2}(\ln 7)(\cos x^2)\, 2x$

25. $f'(x) = \sec^2(\log_5 x)\,(1 \ln 5)\,\frac{1}{x} = \frac{\sec^2(\log_5 x)}{x \ln 5}$

26. $g'(x) = \dfrac{1}{\ln 10}\dfrac{\frac{1}{x}x^2 - \ln x(2x)}{x^4} = \dfrac{x - 2x\ln x}{x^4 \ln 10}.$

27. $F'(x) = -\sin(2^x + 2^{-x})\,[2^x \ln 2 - 2^{-x}\ln 2] = \ln 2\,(2^{-x} - 2^x)\sin(2^x + 2^{-x})$

28. $h'(x) = a^{-x}\ln a(-1)\cos bx + a^{-x}(-\sin bx)b = -(\ln a)a^{-x}\cos bx - ba^{-x}\sin bx$

29. $\displaystyle\int 3^x\,dx = \frac{3^x}{\ln 3} + C$

30. $\displaystyle\int 2^{-x}\,dx = -\frac{2^{-x}}{\ln 2} + C$

31. $\displaystyle\int \left(x^3 + 3^{-x}\right)\,dx = \frac{1}{4}x^4 - \frac{3^{-x}}{\ln 3} + C$

32. $\displaystyle\int x10^{-x^2}\,dx = -\frac{1}{2}\int 10^u\,du = -\frac{10^u}{2\ln 10} + C = -\frac{10^{-x^2}}{2\ln 10} + C$

33. $\displaystyle\int \frac{dx}{x\ln 5} = \frac{1}{\ln 5}\int\frac{dx}{x} = \frac{\ln|x|}{\ln 5} + C = \log_5|x| + C$

34. $\displaystyle\int \frac{\log_5 x}{x}\,dx = \frac{1}{\ln 5}\int\frac{\ln x}{x}\,dx = \frac{1}{\ln 5}\frac{1}{2}(\ln x)^2 + C = \frac{(\ln x)^2}{2\ln 5} + C$

35.
$$\begin{aligned}\int \frac{\log_2 x^3}{x}\,dx &= \frac{1}{\ln 2}\int\frac{\ln x^3}{x}\,dx = \frac{3}{\ln 2}\int\frac{\ln x}{x}\,dx\\ &= \frac{3}{\ln 2}\left[\frac{1}{2}(\ln x)^2\right] + C = \frac{3}{\ln 4}(\ln x)^2 + C\end{aligned}$$

36. Write $c = b^{\log_b c}$ Then $\log_a c = \log_a\left(b^{\log_b c}\right) = (\log_b c)(\log_a b)$.

37. $f'(x) = \dfrac{1}{x\ln 3}$ so $f'(e) = \dfrac{1}{e\ln 3}$

38. $f(x) = x\log_3 x;\quad f'(x) = \log_3 x + x\cdot\dfrac{1}{x\ln 3} = \dfrac{\ln x + 1}{\ln 3};\quad f'(e) = \dfrac{2}{\ln 3}$

39. $f'(x) = \dfrac{1}{x\ln x}$ so $f'(e) = \dfrac{1}{e\ln e} = \dfrac{1}{e}$

40. $f(x) = \log_3(\log_2 x) = \dfrac{\ln\left(\frac{\ln x}{\ln 2}\right)}{\ln 3} = \dfrac{\ln(\ln x) - \ln(\ln 2)}{\ln 3};\quad f'(x) = \dfrac{1}{\ln 3}\cdot\dfrac{1}{\ln x}\cdot\dfrac{1}{x} \implies f'(e) = \dfrac{1}{e\ln 3}$

41.
$$f(x) = p^x$$
$$\ln f(x) = x \ln p$$
$$\frac{f'(x)}{f(x)} = \ln p$$
$$f'(x) = f(x) \ln p$$
$$f'(x) = p^x \ln p$$

42.
$$f(x) = p^{g(x)}$$
$$\ln f(x) = g(x) \ln p$$
$$\frac{f'(x)}{f(x)} = g'(x) \ln p$$
$$f'(x) = f(x)g'(x) \ln p = p^{g(x)} g'(x) \ln p$$

43.
$$y = (x+1)^x$$
$$\ln y = x \ln (x+1)$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{x}{x+1} + \ln (x+1)$$
$$\frac{dy}{dx} = (x+1)^x \left[\frac{x}{x+1} + \ln (x+1)\right]$$

44.
$$y = (\ln x)^x$$
$$\ln y = x \ln(\ln x)$$
$$\frac{1}{y}\frac{dy}{dx} = \ln(\ln x) + x\frac{1}{\ln x} \cdot \frac{1}{x}$$
$$\frac{dy}{dx} = (\ln x)^x \left[\ln(\ln x) + \frac{1}{\ln x}\right]$$

45.
$$y = (\ln x)^{\ln x}$$
$$\ln y = \ln x [\ln (\ln x)]$$
$$\frac{1}{y}\frac{dy}{dx} = \ln x \left[\frac{1}{x \ln x}\right] + \frac{1}{x}[\ln (\ln x)]$$
$$\frac{dy}{dx} = (\ln x)^{\ln x} \left[\frac{1 + \ln (\ln x)}{x}\right]$$

46.
$$y = \left(\frac{1}{x}\right)^x$$
$$\ln y = x \ln \frac{1}{x} = -x \ln x$$
$$\frac{1}{y}\frac{dy}{dx} = -\ln x - x \cdot \frac{1}{x} = -\ln x - 1$$
$$\frac{dy}{dx} = -\left(\frac{1}{x}\right)^x [1 + \ln x]$$

47.
$$y = x^{\sin x}$$
$$\ln y = (\sin x)(\ln x)$$
$$\frac{1}{y}\frac{dy}{dx} = (\cos x)(\ln x) + \sin x \left(\frac{1}{x}\right)$$
$$\frac{dy}{dx} = x^{\sin x}\left[(\cos x)(\ln x) + \frac{\sin x}{x}\right]$$

48.
$$y = (\cos x)^{x^2+1}$$
$$\ln y = (x^2+1)\ln(\cos x)$$
$$\frac{1}{y}\frac{dy}{dx} = 2x \ln(\cos x) + (x^2+1)\left(-\frac{\sin x}{\cos x}\right)$$
$$\frac{dy}{dx} = (\cos x)^{x^2+1}\left[2x \ln(\cos x) - (x^2+1)\tan x\right]$$

49.
$$y = (\sin x)^{\cos x}$$
$$\ln y = (\cos x)(\ln[\sin x])$$
$$\frac{1}{y}\frac{dy}{dx} = (-\sin x)(\ln[\sin x]) + (\cos x)\left(\frac{1}{\sin x}\right)(\cos x)$$
$$\frac{dy}{dx} = (\sin x)^{\cos x}\left[\frac{\cos^2 x}{\sin x} - (\sin x)(\ln[\sin x])\right]$$

50.
$$y = x^{x^2}$$
$$\ln y = x^2 \ln x$$
$$\frac{1}{y}\frac{dy}{dx} = 2x \ln x + x^2 \cdot \frac{1}{x}$$
$$\frac{dy}{dx} = x^{x^2+1}(2\ln x + 1)$$

51.

$$y = x^{2^x}$$

$$\ln y = 2^x \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = 2^x \ln 2 \ln x + 2^x\left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} = x^{2^x}\left[2^x \ln 2 \ln x + \frac{2^x}{x}\right]$$

52

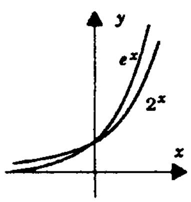
$$y = (\tan x)^{\sec x}$$

$$\ln y = \sec x \ln(\tan x)$$

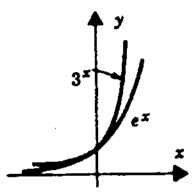
$$\frac{1}{y}\frac{dy}{dx} = \sec x \tan x \ln(\tan x) + \sec x \cdot \frac{\sec^2 x}{\tan x}$$

$$\frac{dy}{dx} = (\tan x)^{\sec x}\left[\sec x \tan x \ln(\tan x) + \sec^3 x \cot x\right]$$

53.



54.



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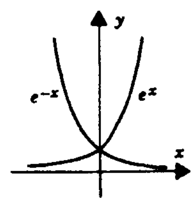

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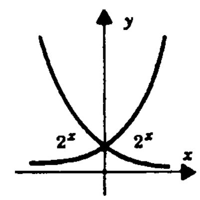

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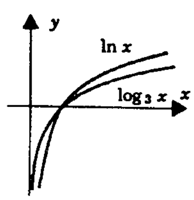

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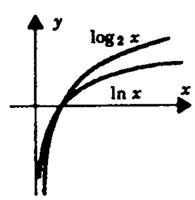

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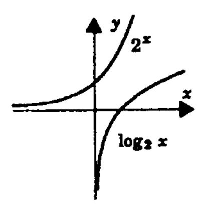
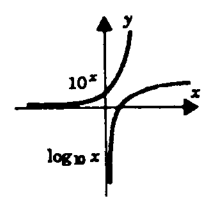


60.



61. $\displaystyle\int_1^2 2^{-x}\,dx = \left[-\frac{2^{-x}}{\ln 2}\right]_1^2 = \frac{1}{4\ln 2}$

62. $\displaystyle\int_0^1 4^x\,dx = \left[\frac{4^x}{\ln 4}\right]_0^1 = \frac{3}{\ln 4}$

63. $\displaystyle\int_1^4 \frac{dx}{x\ln 2} = [\log_2 x]_1^4 = \log_2 4 - 0 = 2$

64. $\displaystyle\int_0^2 p^{x/2}\,dx = \left[\frac{2p^{x/2}}{\ln p}\right]_0^2 = \frac{2(p-1)}{\ln p}$

65. $\displaystyle\int_0^1 x10^{1+x^2}\,dx = \left[\frac{1}{2\ln 10}10^{1+x^2}\right]_0^1 = \frac{1}{2\ln 10}(100-10) = \frac{45}{\ln 10}$

66. $\displaystyle\int_0^1 \frac{5p^{\sqrt{x+1}}}{\sqrt{x+1}}\,dx = \left[\frac{10p^{\sqrt{x+1}}}{\ln p}\right]_0^1 = \frac{10}{\ln p}\left(p^{\sqrt{2}} - p\right)$

67. $\displaystyle\int_0^1 \left(2^x + x^2\right)dx = \left[\frac{2^x}{\ln 2} + \frac{x^3}{3}\right]_0^1 = \frac{1}{3} + \frac{1}{\ln 2}$

68. $7^{1/\ln 7} \cong 2.71828.$ $\quad 7^{1/\ln 7} = \left(e^{\ln 7}\right)^{1/\ln 7} = e^1 \cong 2.71828.$

69. approx 16.99999; $5^{\ln 17/\ln 5} = \left(e^{\ln 5}\right)^{\ln 17/\ln 5} = e^{\ln 17} = 17$

70. approx 54.59815; $16^{1/\ln 2} = \left(e^{\ln 16}\right)^{1/\ln 2} = e^{\ln 16/\ln 2} = e^{4\ln 2/\ln 2} = e^4 \cong 54.59815$

SECTION 7.6

1. We begin with $A(t) = A_0 e^{rt}$

and take $A_0 = \$500$ and $t = 10$. The interest earned is given by

$$A(10) - A_0 = 500\left(e^{10r} - 1\right).$$

Thus, (a) $500\left(e^{0.6} - 1\right) \cong \$411.06$ (b) $500\left(e^{0.8} - 1\right) \cong \$612.77$

(c) $500\,(e - 1) \cong \$859.14$.

2. We want $A(t) = A_0 e^{rt} = 2A_0$, so $e^{rt} = 2 \implies rt = \ln 2 \implies t = \dfrac{\ln 2}{r}$

(a) $t = \dfrac{\ln 2}{0.06} \cong 11.55$ years. (b) $t = \dfrac{\ln 2}{0.08} \cong 8.66$ years. (c) $t = \dfrac{\ln 2}{0.1} \cong 6.93$ years.

3. In general

$$A(t) = A_0 e^{rt}.$$

We set

$$3A_0 = A_0 e^{20r}$$

and solve for r:

$$3 = e^{20r}, \quad \ln 3 = 20r, \quad r = \frac{\ln 3}{20} \cong 5\tfrac{1}{2}\%.$$

4. We want $A(10) = A_0 e^{10r} = 2A_0$, so $r = \dfrac{\ln 2}{10} \cong 6.9\%$

5. Let $P = P(t)$ denote the bacteria population at time t. Then $P'(t) = kP(t)$ (k constant) $P(0) = 1000$, and $P(t) = 1000\,e^{kt}$. Since $P(1/2) = 2P(0) = P(0)e^{k/2}$, it follows that $e^{k/2} = 2$ or $k = 2\ln 2$. Now

$$P(t) = 1000e^{(2\ln 2)t} \quad \text{and} \quad P(2) = 1000\,e^{(2\ln 2)2} = 16{,}000$$

6. $P(t) = P_0 e^{kt}$. $P(4) = P_0 e^{k4} = 3p_0 \implies k = \dfrac{\ln 3}{4}$

(a) $P(12) = P_0 e^{\frac{\ln 3}{4}12} = P_0 e^{3\ln 3} = 27P_0 = 1{,}000{,}000 \implies P_0 \cong 37{,}037$ bacteria

(b) $P(t) = P_0 e^{\frac{\ln 3}{4}t} = 2P_0 \implies \dfrac{\ln 3}{4}t = \ln 2 \implies t = \dfrac{4\ln 2}{\ln 3} \cong 2.52$ hours.

7. (a) $P(t) = 10{,}000e^{t\ln 2} = 10{,}000(2)^t$

(b) $P(26) = 10,000(2)^{26}$, $P(52) = 10,000(2)^{52}$

8. $qC = Ce^{kp} \implies q = e^{kp} \implies p = \frac{1}{k}\ln q$

9. (a) $P(10) = P(0)e^{0.035(10)t} = P(0)e^{0.35t}$. Thus it increases by $e^{0.35}$.

(b) $2P(0) = P(0)e^{15k} \implies k = \frac{\ln 2}{15}$.

10. Let $P(t)$ be the world population t years after 1970.

Then $P(10) = P(0)e^{10k} = 227$ and $P(20) = P(0)e^{20k} = 249 \implies e^{10k} = \frac{249}{227}$.

Thus $P(0) \simeq 206.94$ million.

11. $P(40) = 206.94e^{40k} = 206.94(\frac{249}{227})^4 = 299.6$ million.

$P(28) = 206.94e^{28k} = 206.94(\frac{249}{227})^{2.8} = 268.1$ million.

12. $Pe^{kt} = 2P \Rightarrow kt = \ln 2$.

Since $k = \frac{1}{10}\ln\left(\frac{249}{227}\right)$, we get $t = \dfrac{10\ln 2}{\ln\left(\frac{249}{227}\right)} \simeq 74.9$ years.

13. $4.5e^{0.0164t} = 30 \implies 0.0164t = \ln\frac{30}{4.5} \implies t \simeq 115.7$ years.

Thus maximum population will be reached in 2095.

14. $\frac{ds}{dx} = -\frac{s}{V} \implies s(x) = Ce^{-x/V} = s_0e^{-x/V}$. We want $s(x) = \frac{1}{2}s_0$, so $e^{-x/V} = \frac{1}{2}$,

hence $x = V\ln\frac{1}{2} = -V\ln 2$; $x = V\ln 2 = 10,000\ln 2 \cong 6931$ gallons

15.

$$V'(t) = ktV(t)$$

$$V'(t) - ktV(t) = 0$$

$$e^{-kt^2/2}V'(t) - kte^{-kt^2/2}V(t) = 0$$

$$\frac{d}{dt}\left[e^{-kt^2/2}V(t)\right] = 0$$

$$e^{-kt^2/2}V(t) = C$$

$$V(t) = Ce^{kt^2/2}.$$

Since $V(0) = C = 200$, $V(t) = 200e^{kt^2/2}$.

Since $V(5) = 160$,

$$200e^{k(25/2)} = 160, \quad e^{k(25/2)} = \tfrac{4}{5}, \quad e^k = \left(\tfrac{4}{5}\right)^{2/25}$$

and therefore

$$V(t) = 200\left(\tfrac{4}{5}\right)^{t^2/25} \text{ liters.}$$

16. $A(t) = A_0e^{kt}$. $A(5) = A_0e^{5k} = \frac{2}{3}A_0 \Longrightarrow k = \frac{\ln(2/3)}{5} \Longrightarrow A(t) = A_0e^{\frac{\ln(2/3)}{5}t}$

$A(t) = \frac{1}{2}A_0 \Longrightarrow \frac{\ln(2/3)}{5}t = \ln\frac{1}{2} \Longrightarrow t = \frac{5\ln(1/2)}{\ln(2/3)} \cong 8.55$ years.

17. Take two years ago as time $t = 0$. In general

$$(*) \qquad A(t) = A_0e^{kt}.$$

We are given that

$$A_0 = 5 \quad \text{and} \quad A(2) = 4.$$

Thus,

$$4 = 5e^{2k} \quad \text{so that} \quad \tfrac{4}{5} = e^{2k} \quad \text{or} \quad e^k = \left(\tfrac{4}{5}\right)^{1/2}.$$

We can write

$$A(t) = 5\left(\tfrac{4}{5}\right)^{t/2}$$

and compute $A(5)$ as follows:

$$A(5) = 5\left(\tfrac{4}{5}\right)^{5/2} = 5e^{\frac{5}{2}\ln(4/5)} \cong 5e^{-0.56} \cong 2.86.$$

About 2.86 gm will remain 3 years from now.

18. Let $t = 0$ correspond to a year ago. Then $A(t) = 4e^{kt}$, and $A(1) = 3 \Longrightarrow 4e^k = 3 \Longrightarrow k = \ln(3/4)$
Therefore, $A(t) = 4e^{\ln(3/4)t} = 4\left(\frac{3}{4}\right)^t$. Ten years ago, $t = -9$; $A(-9) = 4\left(\frac{3}{4}\right)^{-9} \cong 53.27$ grams.

19. A fundamental property of radioactive decay is that the percentage of substance that decays during any year is constant:

$$100\left[\frac{A(t) - A(t+1)}{A(t)}\right] = 100\left[\frac{A_0e^{kt} - A_0e^{k(t+1)}}{A_0e^{kt}}\right] = 100(1 - e^k)$$

If the half-life is n years, then

$$\tfrac{1}{2}A_0 = A_0e^{kn} \quad \text{so that} \quad e^k = \left(\tfrac{1}{2}\right)^{1/n}.$$

Thus, $100\left[1 - \left(\frac{1}{2}\right)^{1/n}\right]$ % of the material decays during any one year.

20. $A(t) = ne^{kt}$. $A(5) = ne^{5k} = m \Longrightarrow 5k = \ln(m/n) \Longrightarrow k = \frac{1}{5}\ln(m/n)$ and $A(t) = ne^{\frac{\ln(m/n)}{5}t}$.

$A(10) = ne^{2\ln(m/n)} = n\left(\frac{m}{n}\right)^2 = \frac{m^2}{n}$ grams.

21. (a) $A(1620) = A_0e^{1620k} = \frac{1}{2}A_0 \Longrightarrow k = \frac{\ln\frac{1}{2}}{1620} \simeq -0.00043.$

Thus $A(500) = A_0e^{500k} = 0.807A_0$. Hence 80.7% will remain.

(b) $0.25A_0 = A_0e^{kt} \Longrightarrow t = 3240$ years.

22. $Ae^{5.3k} = \frac{1}{2}A \Longrightarrow k \simeq -0.1308.$

(a) $A(8) = A_0e^{8k} \simeq 0.351A_0$. Thus 35.1% will remain.

(b) $100 = A_0e^{3k} \implies A_0 \simeq 148$ grams.

23. (a) $x_1(t) = 10^6 t, \quad x_2(t) = e^t - 1$

(b) $\frac{d}{dt}[x_1(t) - x_2(t)] = \frac{d}{dt}[10^6 t - (e^t - 1)] = 10^6 - e^t$

This derivative is zero at $t = 6 \ln 10 \cong 13.8$. After that the derivative is negative.

(c) $x_2(15) < e^{15} = (e^3)^5 \cong 20^5 = 2^5(10^5) = 3.2(10^6) < 15(10^6) = x_1(15)$

$x_2(18) = e^{18} - 1 = (e^3)^6 - 1 \cong 20^6 - 1 = 64(10^6) - 1 > 18(10^6) = x_1(18)$

$x_2(18) - x_1(18) \cong 64(10^6) - 1 - 18(10^6) \cong 46(10^6)$

(d) If by time t_1 EXP has passed LIN, then $t_1 > 6 \ln 10$. For all $t \geq t_1$ the speed of EXP is greater than the speed of LIN:

$$\text{for} \quad t \geq t_1 > 6 \ln 10, \quad v_2(t) = e^t > 10^6 = v_1(t).$$

24. (a) $x_1(t) = t, \quad x_3(t) = 10^6 \ln(t+1)$

(b) $\frac{d}{dt}[x_3(t) - x_1(t)] = \frac{d}{dt}\left[10^6 \ln(t+1) - t\right] = \frac{10^6}{t+1} - 1$

This derivative is 0 at $t = 10^6 - 1$. After that the derivative is negative.

(c) $x_1(10^7 - 1) = 10^7 - 1 < 7(\ln 10)10^6 = 10^6 \ln 10^7 = x_3(10^7 - 1)$

$x_3(10^8 - 1) = 10^6 \ln 10^8 = (10^6)8 \ln 10 < (10^6)24 < 10^8 - 1 = x_1(10^8 - 1)$

(d) If by time t_1 LIN had passed LOG, then $t_1 > 10^6 - 1$. For all $t \geq t_1$ the speed of LIN is greater than the speed of LOG:

$$\text{for} \quad t \geq t_1 > 10^6 - 1, \quad v_1(t) = t > \frac{10^6}{t+1} = v_3(t).$$

25. Let $p(h)$ denote the pressure at altitude h. The equation $\frac{dp}{dh} = kp$ gives

$$(*) \qquad p(h) = p_0 e^{kh}$$

where p_0 is the pressure at altitude zero (sea level).

Since $p_0 = 15$ and $p(10000) = 10$,

$$10 = 15e^{10000k}, \quad \tfrac{2}{3} = e^{10000k}, \quad \tfrac{1}{10000} \ln \tfrac{2}{3} = k.$$

Thus, $(*)$ can be written

$$p(h) = 15\left(\tfrac{2}{3}\right)^{h/10000}.$$

(a) $p(5000) = 15\left(\tfrac{2}{3}\right)^{1/2} \cong 12.25 \text{ lb/in.}^2$.

(b) $p\,(15000) = 15\left(\frac{2}{3}\right)^{3/2} \cong 8.16$ lb/in.^{2}.

26. $P = 20,000\, e^{-(0.06)(4)} \cong \$15,732.56.$

27. From Exercise 26, we have $6000 = 10,000e^{-8r}$. Thus

$$e^{-8r} = \frac{6000}{10,000} = \frac{3}{5} \Rightarrow -8r = \ln(3/5) \quad \text{and} \quad r \cong 0.064 \quad \text{or} \quad r = 6.4\%$$

28. (a) $P = 50,000\, e^{-(0.04)(20)} \cong \$22,466.45$

(b) $P = 50,000\, e^{-(0.06)(20)} \cong \$15,059.71$

(c) $P = 50,000\, e^{-(0.08)(20)} \cong \$10,094.83$

29. The future value of \$16,000 at an interest rate r, t years from now is given by $Q(t) = 16,000\, e^{rt}$. Thus

(a) For $r = 0.05$: $\quad P(3) = 16,000\, e^{(0.05)3} \cong 18,589.35$ or \$18,589.35.

(b) For $r = 0.08$: $\quad P(3) = 16,000\, e^{(0.08)3} \cong 20,339.99$ or \$20,339.99.

(c) For $r = 0.12$: $\quad P(3) = 16,000\, e^{(0.12)3} \cong 22,933.27$ or \$22,933.27.

30. $\dfrac{dv}{dt} = -kv \Longrightarrow v = ce^{-kt}, \quad v(0) = ce^0 = c,$ so c is velocity when power is shut off.

31. By Exercise 30

$(*)$
$$v\,(t) = Ce^{-kt}, \quad t \text{ in seconds.}$$

We use the initial conditions

$$v\,(0) = C = 4 \text{ mph } = \tfrac{1}{900} \text{ mi/sec} \qquad \text{and} \qquad v\,(60) = 2 = \tfrac{1}{1800} \text{ mi/sec}$$

to determine e^{-k}:

$$\tfrac{1}{1800} = \tfrac{1}{900}e^{-60k}, \qquad e^{60k} = 2, \qquad e^k = 2^{1/60}.$$

Thus, $(*)$ can be written

$$v\,(t) = \tfrac{1}{900}\, 2^{-t/60}.$$

The distance traveled by the boat is

$$s = \int_0^{60} \frac{1}{900} 2^{-t/60}\, dt = \frac{1}{900}\left[\frac{-60}{\ln 2} 2^{-t/60}\right]_0^{60} = \frac{1}{30\ln 2} \text{ mi } = \frac{176}{\ln 2} \text{ ft} \quad \text{(about 254 ft).}$$

32. Since the amount $A(t)$ of raw sugar present after t hours decreases at a rate proportional to A, we have

$$A(t) = A_0 e^{kt}.$$

We are given $\quad A_0 = 1000 \quad$ and $\quad A(10) = 800. \quad$ Thus,

$$800 = 1000e^{10k}, \quad \tfrac{4}{5} = e^{10k}, \quad e^k = \left(\tfrac{4}{5}\right)^{1/10}$$

so that

$$A(t) = 1000\left(\tfrac{4}{5}\right)^{t/10}.$$

Now,

$$A(20) = 1000\left(\frac{4}{5}\right)^{20/10} = 640;$$

after 10 more hours of inversion there will remain 640 pounds.

33. Let $A(t)$ denote the amount of ^{14}C remaining t years after the organism dies. Then $A(t) = A(0)e^{kt}$ for some constant k. Since the half-life of ^{14}C is 5700 years, we have

$$\frac{1}{2} = e^{5700k} \quad \Rightarrow k = -\frac{\ln 2}{5700} \cong 0.000122 \text{ and } A(t) = A(0)e^{-0.000122t}$$

If 25% of the original amount of ^{14}C remains after t years, then

$$0.25A(0) = A(0)e^{-0.000122t} \quad \Rightarrow \quad t = \frac{\ln 0.25}{-0.000122} \cong 11,400 \text{ (years)}$$

34. $A(t) = A_0 e^{-\frac{\ln 2}{5700}t}$; $A(2000) = A_0\, e^{-\frac{\ln 2}{5700}\cdot 2000} \cong 0.78\, A_0$; 78% remains

SECTION 7.7

1.	0	**2.**	$-\frac{\pi}{3}$	**3.**	$\pi/3$	**4.**	$\frac{\pi}{3}$	**5.**	$2\pi/3$
6.	$\frac{3\pi}{4}$	**7.**	$-\pi/4$	**8.**	$-\frac{\pi}{4}$	**9.**	$-2/\sqrt{3}$	**10.**	$\frac{\sqrt{3}}{2}$
11.	$1/2$	**12.**	$\frac{\pi}{4}$	**13.**	1.1630	**14.**	−0.9190	**15.**	−0.4580
16.	1.2598	**17.**	1.2002	**18.**	1.7133				

19. $\dfrac{dy}{dx} = \dfrac{1}{1+(x+1)^2} = \dfrac{1}{x^2+2x+2}$ **20.** $\dfrac{dy}{dx} = \dfrac{1}{1+(\sqrt{x})^2}\cdot\dfrac{1}{2\sqrt{x}} = \dfrac{1}{2\sqrt{x}(1+x)}$

21. $f'(x) = \dfrac{1}{|2x^2|\sqrt{(2x^2)^2-1}}\dfrac{d}{dx}\left(2x^2\right) = \dfrac{2x}{x\sqrt{4x^4-1}}$

22. $f'(x) = e^x \sin^{-1} x + e^x \dfrac{1}{\sqrt{1-x^2}} = e^x\left[\sin^{-1} x + \dfrac{1}{\sqrt{1-x^2}}\right]$

23. $f'(x) = \sin^{-1} 2x + x\dfrac{1}{\sqrt{1-(2x)^2}}\dfrac{d}{dx}(2x) = \sin^{-1} 2x + \dfrac{2x}{\sqrt{1-4x^2}}$

24. $f'(x) = e^{\tan^{-1} x} \cdot \dfrac{1}{1+x^2}$

25. $\dfrac{du}{dx} = 2\left(\sin^{-1} x\right) \dfrac{d}{dx}\left(\sin^{-1} x\right) = \dfrac{2\sin^{-1} x}{\sqrt{1-x^2}}$

26. $\dfrac{dy}{dx} = \dfrac{1}{1+(e^x)^2} \cdot e^x = \dfrac{e^x}{1+e^{2x}}$

27. $\dfrac{dy}{dx} = \dfrac{x\left(\dfrac{1}{1+x^2}\right) - (1)\tan^{-1} x}{x^2} = \dfrac{x - (1+x^2)\tan^{-1} x}{x^2(1+x^2)}$

28. $\dfrac{dy}{dx} = \dfrac{1}{|\sqrt{x^2+2}|\sqrt{\left(\sqrt{x^2+2}\right)^2 - 1}} \cdot \dfrac{x}{\sqrt{x^2+2}} = \dfrac{x}{(x^2+2)\sqrt{x^2+1}}$

29. $f'(x) = \dfrac{1}{2}\left(\tan^{-1} 2x\right)^{-1/2} \dfrac{d}{dx}\left(\tan^{-1} 2x\right) = \dfrac{1}{2}\left(\tan^{-1} 2x\right)^{-1/2} \dfrac{2}{1+(2x)^2} = \dfrac{1}{(1+4x^2)\sqrt{\tan^{-1} 2x}}$

30. $f'(x) = \dfrac{1}{\tan^{-1} x} \cdot \dfrac{1}{1+x^2} = \dfrac{1}{(1+x^2)\tan^{-1} x}$

31. $\dfrac{dy}{dx} = \dfrac{1}{1+(\ln x)^2} \dfrac{d}{dx}(\ln x) = \dfrac{1}{x\left[1+(\ln x)^2\right]}$

32. $g'(x) = \dfrac{-\sin x}{|\cos x + 2|\sqrt{(\cos x + 2)^2 - 1}}$

33. $\dfrac{d\theta}{dr} = \dfrac{1}{\sqrt{1-\left(\sqrt{1-r^2}\,\right)^2}} \dfrac{d}{dr}\left(\sqrt{1-r^2}\,\right) = \dfrac{1}{\sqrt{r^2}} \cdot \dfrac{-r}{\sqrt{1-r^2}} = -\dfrac{r}{|r|\sqrt{1-r^2}}$

34. $\dfrac{d\theta}{dr} = \dfrac{1}{\sqrt{1-[r/(r+1)]^2}} \cdot \dfrac{1}{(r+1)^2} = \dfrac{1}{(r+1)\sqrt{2r+1}}$

35. $g'(x) = 2x\,\sec^{-1}\left(\dfrac{1}{x}\right) + x^2 \cdot \dfrac{1}{\left|\dfrac{1}{x}\right|\sqrt{\dfrac{1}{x^2} - 1}} \cdot \left(-\dfrac{1}{x^2}\right) = 2x\,\sec^{-1}\left(\dfrac{1}{x}\right) - \dfrac{x^2}{\sqrt{1-x^2}}$

36. $\dfrac{d\theta}{dr} = \dfrac{1}{1+[1/(1+r^2)]^2} \cdot \dfrac{-2r}{(1+r^2)^2} = \dfrac{-2r}{r^4+2r^2+2}$

37. $\dfrac{dy}{dx} = \cos\left[\sec^{-1}(\ln x)\right] \cdot \dfrac{1}{|\ln x|\sqrt{(\ln x)^2 - 1}} \cdot \dfrac{1}{x} = \dfrac{\cos\left[\sec^{-1}(\ln x)\right]}{x\,|\ln x|\sqrt{(\ln x)^2 - 1}}$

38. $f'(x) = e^{\sec^{-1} x} \cdot \dfrac{1}{|x|\sqrt{x^2-1}} = \dfrac{e^{\sec^{-1} x}}{|x|\sqrt{x^2-1}}$

39. $f'(x) = \dfrac{-x}{\sqrt{c^2-x^2}} + \dfrac{c}{\sqrt{1-(x/c)^2}} \cdot \left(\dfrac{1}{c}\right) = \dfrac{c-x}{\sqrt{c^2-x^2}} = \sqrt{\dfrac{c-x}{c+x}}$

40. $f'(x) = \dfrac{1}{3\sqrt{1-(3x-4x^2)^2}} \cdot (3-8x) = \dfrac{(3-8x)}{3\sqrt{1-(3x-4x^2)^2}}$

41. $$\frac{dy}{dx} = \frac{\sqrt{c^2-x^2}\ (1) - x\left(\dfrac{-x}{\sqrt{c^2-x^2}}\right)}{\left(\sqrt{c^2-x^2}\ \right)^2} - \frac{1}{\sqrt{1-(x/c)^2}}\left(\frac{1}{c}\right)$$
$$= \frac{c^2}{(c^2-x^2)^{3/2}} - \frac{1}{(c^2-x^2)^{1/2}} = \frac{x^2}{(c^2-x^2)^{3/2}}$$

42. $f'(x) = \sqrt{c^2-x^2} - \dfrac{x^2}{\sqrt{c^2-x^2}} + c^2\dfrac{1}{\sqrt{1-(x/c)^2}} \cdot \dfrac{1}{c} = \dfrac{2c^2-2x^2}{\sqrt{c^2-x^2}} = 2\sqrt{c^2-x^2}$

43. $\left\{\begin{array}{c} au = x+b \\ a\,du = dx \end{array}\right\}$; $\displaystyle\int \frac{dx}{\sqrt{a^2-(x+b)^2}} = \int \frac{a\,du}{\sqrt{a^2-a^2u^2}} = \int \frac{du}{\sqrt{1-u^2}}$
$$= \sin^{-1} u + C = \sin^{-1}\left(\frac{x+b}{a}\right) + C$$

44. $\left\{\begin{array}{c} au = x+b \\ a\,du = dx \end{array}\right\}$; $\displaystyle\int \frac{dx}{a^2+(x+b)^2} = \frac{1}{a^2}\int \frac{a\,du}{1+u^2} = \frac{1}{a}\tan^{-1} u + C = \frac{1}{a}\tan^{-1}\left(\frac{x+b}{a}\right) + C$

45. $\left\{\begin{array}{c} au = x+b \\ a\,du = dx \end{array}\right\}$; $\displaystyle\int \frac{dx}{(x+b)\sqrt{(x+b)^2-a^2}} = \int \frac{a\,du}{au\sqrt{a^2u^2-a^2}} = \frac{1}{a}\int \frac{du}{u\sqrt{u^2-1}}$
$$= \frac{1}{a}\sec^{-1}|u| + C = \frac{1}{a}\sec^{-1}\left(\frac{|x+b|}{a}\right) + C$$

46. $\text{dom}(f) = [-1,1]$, $\quad \text{range}(f) = [0,\pi]$. $\quad y = \cos^{-1} x \implies \cos y = x \implies -\sin y\dfrac{dy}{dx} = 1$
$\implies \dfrac{dy}{dx} = -\dfrac{1}{\sin y} = -\dfrac{1}{\sqrt{1-\cos^2 y}} = -\dfrac{1}{\sqrt{1-x^2}}.$

47. $\text{dom}(f) = (-\infty,\infty)$, $\quad \text{range}\,(f) = (0,\pi)$
$$y = \cot^{-1} x$$
$$\cot y = x$$
$$-\csc^2 y\,\frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = -\frac{1}{\csc^2 x} = -\frac{1}{1+x^2}$$

48. $\text{dom}(f) = (-\infty,-1] \cup [1,\infty)$, $\quad \text{range}(f) = \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
$y = \csc^{-1} x \implies \csc y = x \implies -\csc y\cot y\dfrac{dy}{dx} = 1 \implies \dfrac{dy}{dx} = \dfrac{-1}{\csc y\cot y} = \dfrac{-1}{|x|\sqrt{x^2-1}}$

49. $\displaystyle\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \frac{\pi}{4}$

50. $\displaystyle\int_{-1}^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_{-1}^1 = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$

51. $\displaystyle\int_0^{1/\sqrt{2}} \frac{dx}{\sqrt{1-x^2}} = \left[\sin^{-1} x\right]_0^{1/\sqrt{2}} = \frac{\pi}{4}$

52. $\displaystyle\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1} \frac{x}{2}\right]_0^1 = \frac{\pi}{6}$

53. $\displaystyle\int_0^5 \frac{dx}{25+x^2} \underset{(7.7.8)}{=} \left[\frac{1}{5}\tan^{-1} \frac{x}{5}\right]_0^5 = \frac{\pi}{20}$

54. $\displaystyle\int_5^8 \frac{dx}{x\sqrt{x^2-16}} = \frac{1}{4}\left[\sec^{-1} \frac{x}{4}\right]_5^8 = \frac{1}{4}\left(\sec^{-1} 2 - \sec^{-1} \frac{5}{4}\right) = \frac{\pi}{12} - \frac{1}{4}\sec^{-1}\frac{5}{4}$

55. $\left\{\begin{array}{c|cc} 3u = 2x & x=0 & \Longrightarrow \quad u=0 \\ 3\,du = 2\,dx & x=3/2 & \Longrightarrow \quad u=1 \end{array}\right\};\quad \displaystyle\int_0^{3/2} \frac{dx}{9+4x^2} = \frac{1}{6}\int_0^1 \frac{du}{1+u^2} = \frac{1}{6}\left[\tan^{-1} u\right]_0^1 = \frac{\pi}{24}$

56. $\displaystyle\int_2^5 \frac{dx}{9+(x-2)^2} = \frac{1}{3}\left[\tan^{-1}\left(\frac{x-2}{3}\right)\right]_2^5 = \frac{1}{3}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{12}$

57. $\left\{\begin{array}{c|cc} u = 4x & x=3/4 & \Longrightarrow \quad u=3 \\ du = 4\,dx & x=3 & \Longrightarrow \quad u=12 \end{array}\right\};$

$$\int_{3/4}^3 \frac{dx}{x\sqrt{16x^2-9}} = \int_3^{12} \frac{du/4}{(u/4)\sqrt{u^2-9}} = \frac{1}{3}\left[\sec^{-1}\left(\frac{|u|}{3}\right)\right]_{3/4}^3 = \frac{1}{3}\sec^{-1} 4 \cong 0.4391$$

58. $\displaystyle\int_4^6 \frac{dx}{(x-3)\sqrt{x^2-6x+8}} = \int_4^6 \frac{dx}{(x-3)\sqrt{(x-3)^2-1}} = \left[\sec^{-1}(x-3)\right]_4^6 = \sec^{-1} 3 - \sec^{-1} 1 = \sec^{-1} 3$

59. $\displaystyle\int_{-3}^{-2} \frac{dx}{\sqrt{4-(x+3)^2}} \underset{(7.7.13)}{=} \left[\sin^{-1}\left(\frac{x+3}{2}\right)\right]_{-3}^{-2} = \frac{\pi}{6}$

60. $\displaystyle\int_{\ln 2}^{\ln 3} \frac{e^{-x}}{\sqrt{1-e^{-2x}}}\,dx = \left[-\sin^{-1} e^{-x}\right]_{\ln 2}^{\ln 3} = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{6} - \sin^{-1}\left(\frac{1}{3}\right)$

61. $\left\{\begin{array}{c|cc} u = e^x & x=0 & \Longrightarrow \quad u=1 \\ du = e^x\,dx & x=\ln 2 & \Longrightarrow \quad u=2 \end{array}\right\};$

$$\int_0^{\ln 2} \frac{e^x}{1+e^{2x}}\,dx = \int_1^2 \frac{du}{1+u^2} = \left[\tan^{-1} u\right]_1^2 = \tan^{-1} 2 - \frac{\pi}{4} \cong 0.322$$

62. $\displaystyle\int_0^{1/2} \frac{dx}{\sqrt{3-4x^2}} = \frac{1}{\sqrt{3}}\int_0^{1/2} \frac{dx}{\sqrt{1-\frac{4}{3}x^2}} = \frac{1}{2}\left[\sin^{-1}\frac{2x}{\sqrt{3}}\right]_0^{1/2} = \frac{1}{2}\sin^{-1}\left(\frac{\sqrt{3}}{3}\right)$

63. $\left\{\begin{array}{c} u = x^2 \\ du = 2x\,dx \end{array}\right\}$; $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1} u + C = \frac{1}{2}\sin^{-1} x^2 + C$

64. $\displaystyle\int \frac{\sec^2 x}{\sqrt{9-\tan^2 x}}\,dx = \int \frac{du}{\sqrt{9-u^2}} = \sin^{-1}\left(\frac{u}{3}\right) + C = \sin^{-1}\left(\frac{\tan x}{3}\right) + C$

65. $\left\{\begin{array}{c} u = x^2 \\ du = 2x\,dx \end{array}\right\}$; $\displaystyle\int \frac{x}{1+x^4}\,dx = \frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\tan^{-1} x^2 + C$

66. $\displaystyle\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{dx}{\sqrt{4-(x-2)^2}} = \sin^{-1}\left(\frac{x-2}{2}\right) + C$

67. $\left\{\begin{array}{c} u = \tan x \\ du = \sec^2 x\,dx \end{array}\right\}$; $\displaystyle\int \frac{\sec^2 x}{9+\tan^2 x}\,dx = \int \frac{du}{9+u^2} = \tan^{-1}\left(\frac{u}{3}\right) + C = \tan^{-1}\left(\frac{\tan x}{3}\right) + C$

68. $\displaystyle\int \frac{\cos x}{3+\sin^2 x}\,dx = \int \frac{du}{3+u^2} = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + C = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\sin x}{\sqrt{3}}\right) + C$

69. $\left\{\begin{array}{c} u = \sin^{-1} x \\ du = \dfrac{1}{\sqrt{1-x^2}}\,dx \end{array}\right\}$; $\displaystyle\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\left(\sin^{-1} x\right)^2 + C$

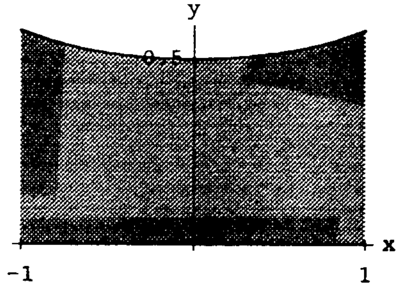
70. $\displaystyle\int \frac{\tan^{-1} x}{1+x^2}\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{1}{2}\left(\tan^{-1} x\right)^2 + C$

71. $\left\{\begin{array}{c} u = \ln x \\ du = \dfrac{1}{x}\,dx \end{array}\right\}$; $\displaystyle\int \frac{dx}{x\sqrt{1-(\ln x)^2}} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}(\ln x) + C$

72. $\displaystyle\int \frac{1}{x}\cdot\frac{1}{1+(\ln x)^2}\,dx = \int \frac{du}{1+u^2} = \tan^{-1} u + C = \tan^{-1}(\ln x) + C$

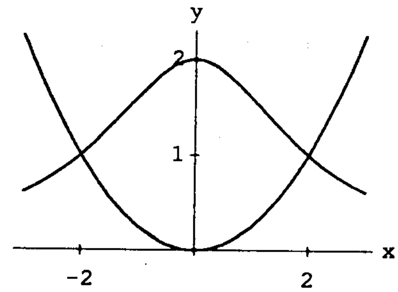
73. $\displaystyle A = \int_{-1}^{1} \frac{1}{\sqrt{4-x^2}}\,dx = 2\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$

$\displaystyle = 2\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1 = \frac{\pi}{3}$



74. $A=\int_{-3}^{3}\frac{3}{9+x^2}\,dx=3\left[\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]_{-3}^{3}=\frac{\pi}{4}+\frac{\pi}{4}=\frac{\pi}{2}$

75. $\frac{8}{x^2+4}=\frac{1}{4}x^2 \;\Rightarrow\; x=\pm 2$



$$A=\int_{-2}^{2}\left(\frac{8}{x^2+4}-\frac{1}{4}x^2\right)dx=2\int_{0}^{2}\left(\frac{8}{x^2+4}-\frac{1}{4}x^2\right)dx$$

$$=2\left[8\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)-\frac{1}{12}x^3\right]_0^2=2\pi-\frac{4}{3}$$

76. $V=\int_{2\sqrt{3}}^{6}2\pi x\cdot\frac{1}{x^2\sqrt{x^2-9}}\,dx=2\pi\int_{2\sqrt{3}}^{6}\frac{dx}{x\sqrt{x^2-9}}=2\pi\left[\frac{1}{3}\sec^{-1}\left(\frac{x}{3}\right)\right]_{2\sqrt{3}}^{6}=\frac{\pi^2}{9}$

77. Let x be the distance between the motorist and the point on the road where the line determined by the sign intersects the road. Then, from the given figure,

$$\theta=\tan^{-1}\left(\frac{s+k}{x}\right)-\tan^{-1}\frac{s}{x},\quad 0<x<\infty$$

and

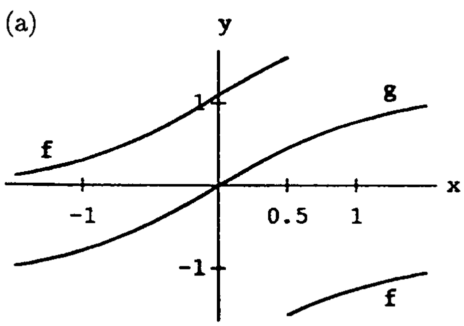
$$\frac{d\theta}{dx}=\frac{1}{1+\frac{(s+k)^2}{x^2}}\left(-\frac{s+k}{x^2}\right)-\frac{1}{1+\frac{s^2}{x^2}}\left(-\frac{s}{x^2}\right)$$
$$=\frac{-(s+k)}{x^2+(s+k)^2}+\frac{s}{x^2+s^2}=\frac{s^2k+sk^2-kx^2}{[x^2+(s+k)^2]\,[x^2+s^2]}$$

Setting $d\theta/dx=0$ we get $x=\sqrt{s^2+sk}$. Since θ is essentially 0 when x is close to 0 and when x is "large," we can conclude that θ is a maximum when $x=\sqrt{s^2+sk}$.

78. $y=\tan^{-1}\frac{x}{30};\quad \frac{dy}{dt}=\frac{1}{1+(x/30)^2}\cdot\frac{1}{30}\cdot\frac{dx}{dt}$

If $\frac{dy}{dt}=6$ and $x=50$ then $\frac{dy}{dt}=\frac{30}{900+(50)^2}\cdot 6=\frac{9}{170}$ rad/sec

79. (a)



(b) $\lim_{x\to(1/2)^+}f(x)=-\frac{\pi}{2};\quad \lim_{x\to(1/2)^-}f(x)=\frac{\pi}{2}$

(c) $f'(x)=\frac{1}{1+\left(\frac{2+x}{1-2x}\right)^2}\,\frac{(1-2x)-(2+x)2}{(1-2x)^2}$

$$=\frac{(1-2x)^2}{(1-2x)^2+(2+x)^2}\,\frac{5}{(1-2x)^2}=\frac{1}{1+x^2}$$

(d) This is clear from the graphs in part (a).

(e) Evaluating at $x=0$ gives $C_1=\tan^{-1}2$;

evaluating at $x=1$ gives $C_2=\tan^{-1}3-\pi/4$

80. (a) $f'(x) = \dfrac{1}{1+\left(\frac{a+x}{1-ax}\right)^2} \cdot \dfrac{1+a^2}{(1-ax)^2} = \dfrac{1+a^2}{(1+a^2)(1+x^2)} = \dfrac{1}{1+x^2}$

(b) Like Exercise 79(d)

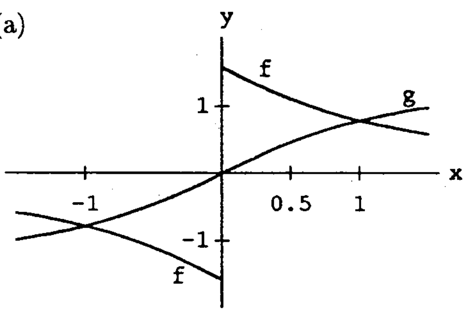
(c) Let $g(x) = \tan^{-1} x$. For $x = 0 < \dfrac{1}{a}$, $g(0) = 0$, $f(0) = \tan^{-1}(a)$, so

$$f(x) = g(x) + \tan^{-1} a \quad \text{for} \quad x < \frac{1}{a}, \quad \text{i.e.,} \quad C_1 = \tan^{-1} a$$

For $x > \dfrac{1}{a}$, note that $\lim\limits_{x\to\infty} \tan^{-1} x = \dfrac{\pi}{2}$, $\lim\limits_{x\to\infty} \tan^{-1}\left(\dfrac{a+x}{1-ax}\right) = \tan^{-1}(-1/a)$ so

$$f(x) = g(x) + \tan^{-1}(-1/a) - \frac{\pi}{2} \quad \text{for} \quad x > \frac{1}{a}, \quad \text{i.e.,} \quad C_2 = \tan^{-1}(-1/a) - \frac{\pi}{2}$$

81. (a)



(b) $\lim\limits_{x\to 0+} f(x) = \dfrac{\pi}{2}$; $\lim\limits_{x\to 0-} f(x) = -\dfrac{\pi}{2}$

(c) $f'(x) = \dfrac{1}{1+\left(\dfrac{1}{x}\right)^2} \dfrac{-1}{x^2} = \dfrac{-1}{1+x^2}$

(d) This is clear from the graphs in part (a).

(e) Evaluating at $x = 1$ gives $C_1 = \pi/2$;

evaluating at $x = -1$ gives $C_2 = -\pi/2$

82. Numerical work suggests limit $\cong 1$. One way to see this is to note that the limit is the derivative of $f(x) = \sin^{-1} x$ at $x = 0$ and this derivative is 1:

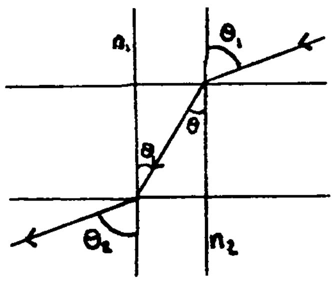
$$f'(x) = \frac{1}{\sqrt{1-x^2}}, \quad f'(0) = 1$$

83. $I = \displaystyle\int_0^{0.5} \frac{1}{\sqrt{1-x^2}}\, dx \cong \frac{1}{10}\left[f(0.05) + f(0.15) + f(0.25) + f(0.35) + f(0.45)\right] \cong 0.523;$

and $\sin(0.523) \cong 0.499$. Explanation: $I = \sin^{-1}(0.5)$ and $\sin\left[\sin^{-1}(0.5)\right] = 0.5$.

PROJECT 7.7

1. (a)



$n_1 \sin\theta_1 = n\sin\theta = n_2\sin\theta_2$

2. (a) Think of n and θ as functions of altitude y. Then

$$n\sin\theta = C$$

Differentiation with respect to y gives

$$n\cos\theta\frac{d\theta}{dy} + \frac{dn}{dy}\sin\theta = 0, \quad \cot\theta\frac{d\theta}{dy} + \frac{1}{n}\frac{dn}{dy} = 0$$

and so when $\alpha = \frac{\pi}{2} - \theta$,

$$\frac{1}{n}\frac{dn}{dy} = -\cot\theta\frac{d\theta}{dy} = -\frac{dy}{dx}\left(-\frac{d\alpha}{dy}\right) = \frac{d\alpha}{dx}$$

Now, $\alpha = \tan^{-1}\left(\frac{dy}{dx}\right)$ and $\frac{d\alpha}{dx} = \dfrac{\frac{d^2y}{d^2x}}{1+\left(\frac{dy}{dx}\right)^2}$.

(b) $1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2\alpha = 1 + \cot^2\theta = \csc^2\theta = \frac{n^2}{C^2} = (\text{a constant})\cdot[n(y)]^2$.

(c) $n(y) = \frac{k}{|y+b|}$, with b, k constants, $k > 0$.

SECTION 7.8

1. $\dfrac{dy}{dx} = \cosh x^2 \dfrac{d}{dx}\left(x^2\right) = 2x \cosh x^2$

2. $\dfrac{dy}{dx} = \sinh(x + a)$

3. $\dfrac{dy}{dx} = \dfrac{1}{2}\left(\cosh ax\right)^{-1/2}(a \sinh ax) = \dfrac{a \sinh ax}{2\sqrt{\cosh ax}}$

4. $\dfrac{dy}{dx} = a \cosh^2 ax + a \sinh^2 ax = a(\cosh^2 ax + \sinh^2 ax)$

5. $\dfrac{dy}{dx} = \dfrac{(\cosh x - 1)(\cosh x) - \sinh x(\sinh x)}{(\cosh x - 1)^2} = \dfrac{1}{1 - \cosh x}$

6. $\dfrac{dy}{dx} = \dfrac{x \cosh x - \sinh x}{x^2}$

7. $\dfrac{dy}{dx} = ab \cosh bx - ab \sinh ax = ab\,(\cosh bx - \sinh ax)$

8. $\dfrac{dy}{dx} = e^x(\cosh x + \sinh x) + e^x(\sinh x + \cosh x) = 2e^x(\cosh x + \sinh x)$

9. $\dfrac{dy}{dx} = \dfrac{1}{\sinh ax}(a \cosh ax) = \dfrac{a \cosh ax}{\sinh ax}$

10. $\dfrac{dy}{dx} = \dfrac{1}{1 - \cosh ax}(-\sinh ax)a = \dfrac{-a \sinh ax}{1 - \cosh ax}$

11. $\dfrac{dy}{dx} = \cosh\left(e^{2x}\right)e^{2x}(2) = 2e^{2x} \cosh\left(e^{2x}\right)$

12. $\dfrac{dy}{dx} = \sinh\left(\ln x^3\right) \cdot \dfrac{1}{x^3} \cdot 3x^2 = \dfrac{3 \sinh\left(\ln x^3\right)}{x}$

13. $\dfrac{dy}{dx} = -e^{-x} \cosh 2x + 2e^{-x} \sinh 2x$

14. $\dfrac{dy}{dx} = \dfrac{1}{1 + \sinh^2 x} \cdot \cosh x = \dfrac{1}{\cosh x}$

15. $\dfrac{dy}{dx} = \dfrac{1}{\cosh x}(\sinh x) = \tanh x$

16. $\dfrac{dy}{dx} = \dfrac{1}{\sinh x} \cdot \cosh x = \dfrac{\cosh x}{\sinh x}$

17. $\ln y = x \ln \sinh x;\quad \dfrac{1}{y}\dfrac{dy}{dx} = \ln \sinh x + x\,\dfrac{\cosh x}{\sinh x}$ and $\dfrac{dy}{dx} = (\sinh x)^x\,[\ln \sinh x + x \coth x]$

18. $y = x^{\cosh x} \implies \ln y = \cosh x \ln x \implies \dfrac{1}{y}\dfrac{dy}{dx} = \sinh x \ln x + \cosh x \dfrac{1}{x}$

$$\implies \frac{dy}{dx} = x^{\cosh x}\left(\sinh x \ln x + \frac{\cosh x}{x}\right)$$

19.

$$\begin{aligned}\cosh^2 t - \sinh^2 t &= \left(\frac{e^t + e^{-t}}{2}\right)^2 - \left(\frac{e^t + e^{-t}}{2}\right)^2 \\ &= \frac{1}{4}\left\{\left(e^{2t} + 2 + e^{-2t}\right) - \left(e^{2t} - 2 + e^{-2t}\right)\right\} = \frac{4}{4} = 1\end{aligned}$$

20.

$$\begin{aligned}\sinh t \cosh s + \cosh t \sinh s &= \frac{1}{2}(e^t - e^{-t})\,\frac{1}{2}(e^s + e^{-s}) + \frac{1}{2}(e^t + e^{-t})\,\frac{1}{2}(e^s - e^{-s}) \\ &= \frac{1}{4}\left(e^{s+t} - e^{s-t} + e^{t-s} - e^{-(s+t)}\right) + \frac{1}{4}\left(e^{s+t} + e^{s-t} - e^{t-s} - e^{-(s+t)}\right) \\ &= \frac{1}{2}\left(e^{s+t} - e^{-(s+t)}\right) = \sinh(s+t)\end{aligned}$$

21.

$$\begin{aligned}\cosh t \cosh s + \sinh t \sinh s &= \left(\frac{e^t + e^{-t}}{2}\right)\left(\frac{e^s + e^{-s}}{2}\right) + \left(\frac{e^t - e^{-t}}{2}\right)\left(\frac{e^s - e^{-s}}{2}\right) \\ &= \tfrac{1}{4}\left\{(e^{t+s} + e^{t-s} + e^{s-t} + e^{-t-s}) + (e^{t+s} - e^{t-s} - e^{s-t} + e^{-s-t})\right\} \\ &= \frac{1}{4}\left\{2e^{t+s} + 2e^{-(t+s)}\right\} = \frac{e^{t+s} + e^{-(t+s)}}{2} = \cosh(t+s)\end{aligned}$$

22. Follows from Exercise 20, with $s = t$.

23. Set $s = t$ in $\cosh(t+s) = \cosh t \cosh s + \sinh t \sinh s$ to get $\cosh(2t) = \cosh^2 t + \sinh^2 t$. Then use Exercise 19 to obtain the other two identities.

24. $\cosh(-t) = \dfrac{1}{2}\left(e^{-t} + e^{-(-t)}\right) = \dfrac{1}{2}\left(e^{-t} + e^{t}\right) = \cosh t$

25. $\sinh(-t) = \dfrac{e^{(-t)} - e^{-(-t)}}{2} = -\dfrac{e^t - e^{-t}}{2} = -\sinh t$

26. $y = 5\cosh x + 4\sinh x = \frac{5}{2}(e^x + e^{-x}) + \frac{4}{2}(e^x - e^{-x}) = \frac{9}{2}e^x + \frac{1}{2}e^{-x}$

$\frac{dy}{dx} = \frac{9}{2}e^x - \frac{1}{2}e^{-x} = \frac{e^{-x}}{2}(9e^{2x} - 1);\quad \frac{dy}{dx} = 0 \implies e^{2x} = \frac{1}{9} \implies x = -\ln 3.$

$\frac{d^2y}{dx^2} = \frac{9}{2}e^x + \frac{1}{2}e^{-x} > 0$ for all x, so abs min occurs at $x = -\ln 3$.

At $x = -\ln 3$, $y = \frac{9}{2}(\frac{1}{3}) + \frac{1}{2}(3) = 3.$

27. $y = -5\cosh x + 4\sinh x = -\frac{5}{2}(e^x + e^{-x}) + \frac{4}{2}(e^x - e^{-x}) = -\frac{1}{2}e^x - \frac{9}{2}e^{-x}$

$\frac{dy}{dx} = -\frac{1}{2}e^x + \frac{9}{2}e^{-x} = \frac{e^{-x}}{2}\left(9 - e^{2x}\right);\quad \frac{dy}{dx} = 0 \implies e^x = 3 \text{ or } x = \ln 3$

$\frac{d^2y}{dx^2} = -\frac{1}{2}e^x - \frac{9}{2}e^{-x} < 0$ for all x so abs max occurs at $x = \ln 3$.

The abs max is $y = -\frac{1}{2}e^{\ln 3} - \frac{9}{2}e^{-\ln 3} = -\frac{1}{2}(3) - \frac{9}{2}\left(\frac{1}{3}\right) = -3.$

28. $y = 4\cosh x + 5\sinh x = \frac{4}{2}(e^x + e^{-x}) + \frac{5}{2}(e^x - e^{-x}) = \frac{9}{2}e^x - \frac{1}{2}e^{-x}$

$\frac{dy}{dx} = \frac{9}{2}e^x + \frac{1}{2}e^{-x} > 0$ always increasing, so no extreme values.

29. $[\cosh x + \sinh x]^n = \left[\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\right]^n$

$= [e^x]^n = e^{nx} = \frac{e^{nx} + e^{-nx}}{2} + \frac{e^{nx} - e^{-nx}}{2} = \cosh nx + \sinh nx$

30. $y = A\cosh cx + B\sinh cx,\quad y' = Ac\sinh cx + Bc\cosh cx,\quad y'' = Ac^2\cosh cx + Bc^2\sinh cx$

$\implies y'' = c^2 y$

31.

$y = A\cosh cx + B\sinh cx;\qquad y(0) = 2 \implies 2 = A.$

$y' = Ac\sinh cx + Bc\cosh cx;\qquad y'(0) = 1 \implies 1 = Bc.$

$y'' = Ac^2\cosh cx + Bc^2\sinh cx = c^2 y;\qquad y'' - 9y = 0 \implies (c^2 - 9)\,y = 0.$

Thus, $c = 3$, $B = \frac{1}{3}$, and $A = 2$.

32. From Exercise 30, $y'' = c^2 y$, so $c = \frac{1}{2}$.

$1 = y(0) = A\cosh 0 + B\sinh 0 = A \implies A = 1$

$2 = y'(0) = Ac\sinh 0 + Bc\cosh 0 = Bc \implies B = \frac{2}{c} = 4$

33. $\frac{1}{a}\sinh ax + C$

34. $\frac{1}{a}\cosh ax + C$

35. $\frac{1}{3a}\sinh^3 ax + C$

36. $\frac{1}{3a}\cosh^3 ax + C$

37. $\frac{1}{a}\ln(\cosh ax) + C$

38. $\frac{1}{a}\ln|\sinh ax| + C$

39. $-\frac{1}{a\cosh ax} + C$

40.

$$\int \sinh^2 x\,dx = \int \frac{1}{4}(e^{2x} - 2 + e^{-2x})\,dx = \frac{1}{4}\left(\frac{1}{2}e^{2x} - 2x - \frac{1}{2}e^{-2x}\right) + C$$
$$= \frac{1}{4}\sinh 2x - \frac{1}{2}x + C = \frac{1}{2}\sinh x\cosh x - \frac{1}{2}x + C$$

41. From the identity $\cosh 2t = 2\cosh^2 t - 1$ (Exercise 23), we get $\cosh^2 t = \frac{1}{2}(1 + \cosh 2t)$. Thus,

$$\int \cosh^2 x\,dx = \frac{1}{2}\int (1 + \cosh 2x)\,dx$$
$$= \frac{1}{2}\left(x + \frac{1}{2}\sinh 2x\right) + C$$
$$= \frac{1}{2}(x + \sinh x\cosh x) + C$$

42. $\int \sinh 2x e^{\cosh 2x}\,dx = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{\cosh 2x} + C$

43. $\left\{\begin{array}{l} u = \sqrt{x} \\ du = dx/2\sqrt{x} \end{array}\right\}$; $\int \frac{\sinh\sqrt{x}}{\sqrt{x}}\,dx = 2\int \sinh u\,du = 2\cosh u + C = 2\cosh\sqrt{x} + C$

44. $\int \frac{\sinh x}{1 + \cosh x}\,dx = \int \frac{du}{1+u} = \ln|1+u| + C = \ln(1 + \cosh x) + C$

45. $A.V. = \frac{1}{1-(-1)}\int_{-1}^{1}\cosh x\,dx = \frac{1}{2}[\sinh x]_{-1}^{1} = \frac{e^2 - 1}{2e} \cong 1.175$

46. $A.V. = \frac{1}{4}\int_0^4 \sinh 2x\,dx = \frac{1}{8}[\cosh 2x]_0^4 = \frac{1}{8}\left[\frac{e^8 + e^{-8}}{2} - 1\right]$

47. $V = \int_0^1 \pi\left(\cosh^2 x - \sinh^2 x\right)\,dx = \int_0^1 \pi\,dx = \pi$

48. (a) $\lim_{x\to\infty}\frac{\sinh x}{e^x} = \lim_{x\to\infty}\frac{e^x - e^{-x}}{2e^x} = \lim_{x\to\infty}\left(\frac{1}{2} - \frac{e^{-2x}}{2}\right) = \frac{1}{2}$

(b) $\lim_{x\to\infty}\frac{\cosh x}{e^{ax}} = \lim_{x\to\infty}\frac{e^x + e^{-x}}{2e^{ax}} = \lim_{x\to\infty}\frac{1}{2}\left(e^{x-ax} + e^{-x-ax}\right)$

For $0 < a < 1$, limit=∞. For $a > 1$, limit=0.

SECTION 7.9

1. $\dfrac{dy}{dx} = 2\tanh x \,\text{sech}^2 x$

2. $\dfrac{dy}{dx} = 2\tanh 3x \,\text{sech}^2 3x \cdot 3 = 6\tanh 3x \,\text{sech}\, 3x$

3. $\dfrac{dy}{dx} = \dfrac{1}{\tanh x}\,\text{sech}^2 x = \text{sech}\, x \,\text{csch}\, x$

4. $\dfrac{dy}{dx} = \text{sech}^2(\ln x) \cdot \dfrac{1}{x}$

5. $\dfrac{dy}{dx} = \cosh\left(\tan^{-1} e^{2x}\right) \dfrac{d}{dx}\left(\tan^{-1} e^{2x}\right) = \dfrac{2e^{2x}\cosh\left(\tan^{-1} e^{2x}\right)}{1+e^{4x}}$

6. $\dfrac{dy}{dx} = -\,\text{sech}\,(3x^2+1)\tanh(3x^2+1)(6x) = -6x\,\text{sech}\,(3x^2+1)\tanh(3x^2+1)$

7. $\dfrac{dy}{dx} = -\,\text{csch}^2\left(\sqrt{x^2+1}\right)\dfrac{d}{dx}\left(\sqrt{x^2+1}\right) = -\dfrac{x}{\sqrt{x^2+1}}\,\text{csch}^2\left(\sqrt{x^2+1}\right)$

8. $\dfrac{dy}{dx} = \dfrac{1}{\text{sech}\, x}\cdot(-\,\text{sech}\, x)(\tanh x) = -\tanh x$

9. $\dfrac{dy}{dx} = \dfrac{(1+\cosh x)\,(-\,\text{sech}\, x\tanh x) - \text{sech}\, x\,(\sinh x)}{(1+\cosh x)^2}$

$$= \frac{-\,\text{sech}\, x\,(\tanh x + \cosh x\tanh x + \sinh x)}{(1+\cosh x)^2} = \frac{-\,\text{sech}\, x\,(\tanh x + 2\sinh x)}{(1+\cosh x)^2}$$

10. $\dfrac{dy}{dx} = \dfrac{\sinh x(1+\,\text{sech}\, x) - \cosh x(-\,\text{sech}\, x)\tanh x}{(1+\,\text{sech}\, x)^2} = \dfrac{\sinh x + 2\tanh x}{(1+\,\text{sech}\, x)^2}$

11. $\dfrac{d}{dx}(\coth x) = \dfrac{d}{dx}\left[\dfrac{\cosh x}{\sinh x}\right] = \dfrac{\sinh x\,(\sinh x) - \cosh x\,(\cosh x)}{\sinh^2 x}$

$$= -\frac{\cosh^2 x - \sinh^2 x}{\sinh^2 x} = \frac{-1}{\sinh^2 x} = -\,\text{csch}^2 x$$

12. $\dfrac{d}{dx}(\,\text{sech}\, x) = \dfrac{d}{dx}\left(\dfrac{1}{\cosh x}\right) = \dfrac{-1}{(\cosh x)^2}\cdot\sinh x = -\dfrac{1}{\cosh x}\cdot\dfrac{\sinh x}{\cosh x} = -\,\text{sech}\, x\tanh x$

13. $\dfrac{d}{dx}(\,\text{csch}\, x) = \dfrac{d}{dx}\left[\dfrac{1}{\sinh x}\right] = -\dfrac{\cosh x}{\sinh^2 x} = -\,\text{csch}\, x\coth x$

14. $\tanh(t+s) = \dfrac{\sinh(t+s)}{\cosh(t+s)} = \dfrac{\sinh t\cosh s + \cosh t\sinh s}{\cosh t\cosh s + \sinh t\sinh s} = \dfrac{\tanh t + \tanh s}{1+\tanh t\tanh s}$

15. (a) By the hint $\operatorname{sech}^2 x_0 = \dfrac{9}{25}$. Take $\operatorname{sech} x_0 = \dfrac{3}{5}$ since $\operatorname{sech} x = \dfrac{1}{\cosh x} > 0$ for all x.

(b) $\cosh x_0 = \dfrac{1}{\operatorname{sech} x_0} = \dfrac{5}{3}$

(c) $\sinh x_0 = \cosh x_0 \tanh x_0 = \left(\dfrac{5}{3}\right)\left(\dfrac{4}{5}\right) = \dfrac{4}{3}$

(d) $\coth x_0 = \dfrac{\cosh x_0}{\sinh x_0} = \dfrac{5/3}{4/3} = \dfrac{5}{4}$

(e) $\operatorname{csch} x_0 = \dfrac{1}{\sinh x_0} = \dfrac{3}{4}$

16. $\operatorname{sech}^2 t_0 = 1 - \tanh^2 t_0 = 1 - \dfrac{25}{144} = \dfrac{119}{144} \implies \operatorname{sech}^2 t_0 = \dfrac{\sqrt{119}}{12}$; $\cosh t_0 = \dfrac{1}{\operatorname{sech} t_0} = \dfrac{12}{\sqrt{119}}$;

$\sinh t_0 = \cosh t_0 \tanh t_0 = \dfrac{12}{\sqrt{119}} \cdot \dfrac{-5}{12} = \dfrac{-5}{\sqrt{119}}$; $\coth t_0 = \dfrac{1}{\tanh t_0} = \dfrac{-12}{5}$;

$\operatorname{csch} t_0 = \dfrac{1}{\sinh t_0} = -\dfrac{\sqrt{119}}{5}$.

17. If $x \le 0$, the result is obvious. Suppose then that $x > 0$. Since $x^2 \ge 1$, we have $x \ge 1$. Consequently

$$x - 1 = \sqrt{x-1}\,\sqrt{x-1} \le \sqrt{x-1}\,\sqrt{x+1} = \sqrt{x^2-1}$$

and therefore $$x - \sqrt{x^2-1} \le 1.$$

18. We will show that

$$\tanh\left[\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right] = x \quad \text{for all} \quad x \in [-1,1].$$

First we observe that

$$\tanh s = \frac{e^s - e^{-s}}{e^s + e^{-s}} \quad \text{and therefore} \quad \tanh(\ln t) = \frac{t - 1/t}{t + 1/t} = \frac{t^2-1}{t^2+1}$$

It follows that

$$\tanh\left[\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right] = \tanh\left(\ln\sqrt{\frac{1+x}{1-x}}\right) = \frac{\frac{1+x}{1-x} - 1}{\frac{1+x}{1-x} + 1} = \frac{2x}{2} = x.$$

19. By Theorem 7.9.2,

$$\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{d}{dx}\left[\ln\left(x + \sqrt{x^2+1}\right)\right] = \frac{1}{x + \sqrt{x^2+1}}\left(1 + \frac{x}{\sqrt{x^2+1}}\right) = \frac{1}{\sqrt{x^2+1}}.$$

20. $$\frac{d}{dx}(\cos^{-1} x) = \frac{d}{dx}\left[\ln(x + \sqrt{x^2-1})\right] = \frac{1}{x + \sqrt{x^2-1}}\left(1 + \frac{x}{\sqrt{x^2-1}}\right) = \frac{1}{\sqrt{x^2-1}}.$$

21. By Theorem 7.9.2

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{d}{dx}\left[\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right] = \frac{1}{2}\frac{1}{\left(\dfrac{1+x}{1-x}\right)}\left(\frac{(1-x)(1) - (1+x)(-1)}{(1-x)^2}\right)$$

$$= \frac{1}{\left(\dfrac{1+x}{1-x}\right)(1-x)^2} = \frac{1}{1-x^2}.$$

22. $y = \text{sech}^{-1}x \implies \text{sech}\, y = x \implies \cosh y = \dfrac{1}{x} \implies y = \cosh^{-1}\left(\dfrac{1}{x}\right),$

so $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{(1/x)^2 - 1}} \cdot \left(\dfrac{-1}{x^2}\right) = \dfrac{-1}{x\sqrt{1-x^2}}.$

23. Let $y = \text{csch}^{-1}x$. Then $\text{csch}\, y = x$ and $\sinh y = \dfrac{1}{x}$.

$$\sinh y = \frac{1}{x}$$

$$\cosh y \frac{dy}{dx} = -\frac{1}{x^2}$$

$$\frac{dy}{dx} = -\frac{1}{x^2 \cosh y} = -\frac{1}{x^2\sqrt{1+(1/x)^2}} = -\frac{1}{|x|\sqrt{1+x^2}}$$

24. $y = \coth^{-1}x \implies \coth y = x \implies \tanh y = \dfrac{1}{x} \implies y = \tanh^{-1}\left(\dfrac{1}{x}\right),$ so

$\dfrac{dy}{dx} = \dfrac{1}{1-(1/x)^2} \cdot \left(-\dfrac{1}{x^2}\right) = \dfrac{-1}{x^2-1} = \dfrac{1}{1-x^2}$

25. (a) $\dfrac{dy}{dx} = -\,\text{sech}\, x \tanh x = -\dfrac{\sinh x}{\cosh^2 x}$

$\dfrac{dy}{dx} = 0$ at $x = 0$; $\dfrac{dy}{dx} > 0$ if $x < 0$; $\dfrac{dy}{dx} < 0$ if $x > 0$

f is increasing on $(-\infty, 0]$ and decreasing on $[0, \infty)$; $f(0) = 1$ is the absolute maximum of f.

(b) $\dfrac{d^2y}{dx^2} = -\dfrac{\cosh^2 x - 2\sinh^2 x}{\cosh^3 x} = \dfrac{\sinh^2 x - 1}{\cosh^3 x}$

$\dfrac{d^2y}{dx^2} = 0 \Rightarrow \sinh x = \pm 1$

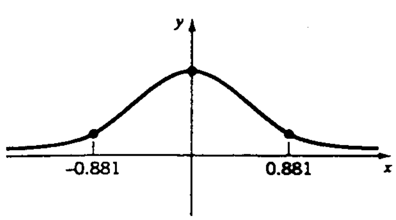
$\sinh x = 1 \Rightarrow \dfrac{e^x - e^{-x}}{2} = 1 \Rightarrow e^{2x} - 2e^x - 1 = 0 \Rightarrow x = \ln(1+\sqrt{2}) \cong 0.881;$

$\sinh x = -1 \Rightarrow \dfrac{e^x - e^{-x}}{2} = -1 \Rightarrow e^{2x} + 2e^x - 1 = 0 \Rightarrow x = -\ln(1+\sqrt{2}) = -0.881$

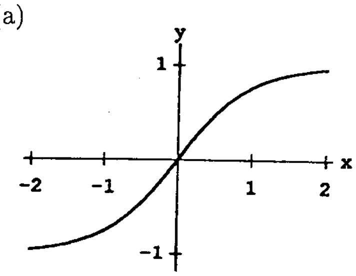
(c) The graph of f is concave up on $(-\infty, -0.881) \cup (0.881, \infty)$ and concave down on $(-0.881, 0.881)$;

points of inflection at $x = \pm 0.881$

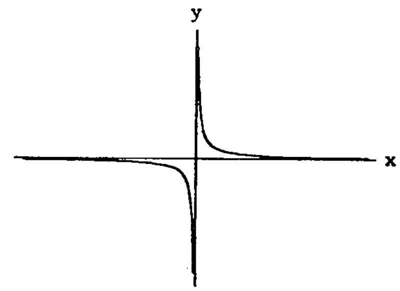
(d)



26. (a)



(b)



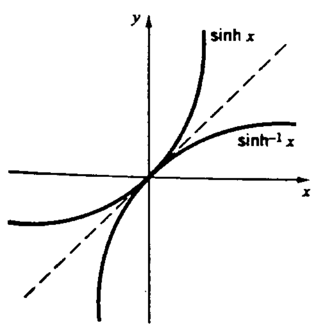
27. $y = \sinh x; \quad \dfrac{dy}{dx} = \cosh x; \quad \dfrac{d^2y}{dx^2} = \sinh x.$

$\dfrac{d^2y}{dx^2} = 0 \;\Rightarrow\; \sinh x = 0 \;\Rightarrow\; x = 0.$

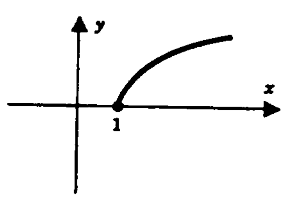
$y = \sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right); \quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{x^2+1}}; \quad \dfrac{d^2y}{dx^2} = -\dfrac{x}{(x^2+1)^{3/2}}.$

$\dfrac{d^2y}{dx^2} = 0 \;\Rightarrow\; -\dfrac{x}{(x^2+1)^{3/2}} = 0 \;\Rightarrow\; x = 0.$

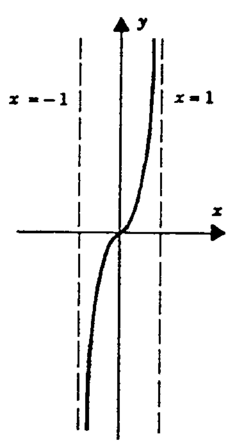
It is easy to verify that $(0,0)$ is a point of inflection for both graphs.



28. (a)



(b)



29. (a) $\tan\phi = \sinh x$

$$\phi = \tan^{-1}(\sinh x)$$

$$\frac{d\phi}{dx} = \frac{\cosh x}{1+\sinh^2 x} = \frac{\cosh x}{\cosh^2 x} = \frac{1}{\cosh x} = \text{sech}\, x$$

(b) $\sinh x = \tan\phi$

$$x = \sinh^{-1}(\tan\phi) = \ln\left(\tan\phi + \sqrt{\tan^2\phi + 1}\right) = \ln(\tan\phi + \sec\phi) = \ln(\sec\phi + \tan\phi)$$

(c) $x = \ln(\sec\phi + \tan\phi)$

$$\frac{dx}{d\phi} = \frac{\sec\phi\tan\phi + \sec^2\phi}{\tan\phi + \sec\phi} = \sec\phi$$

30. $$V = \int_{-1}^{1} \pi\,\text{sech}^2 x\,dx = [\pi\tanh x]_{-1}^{1} = \pi\left(\frac{e-e^{-1}}{e+e^{-1}} - \frac{e^{-1}-e}{e^{-1}+e}\right) = 2\pi\left(\frac{e^2-1}{e^2+1}\right)$$

31. $$\int \tanh x\,dx = \int \frac{\sinh x}{\cosh x}\,dx$$

$$\left\{\begin{array}{c} u = \cosh x \\ du = \sinh x\,dx \end{array}\right\}; \quad \int \frac{\sinh x}{\cosh x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln\cosh x + C$$

32. $$\int \coth x\,dx = \int \frac{\cosh x}{\sinh x}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sinh x| + C$$

33. $$\int \text{sech}\, x\,dx = \int \frac{1}{\cosh x}\,dx = \int \frac{2}{e^x + e^{-x}}\,dx = \int \frac{2e^x}{e^{2x}+1}\,dx$$

$$\left\{\begin{array}{c} u = e^x \\ du = e^x\,dx \end{array}\right\}; \quad \int \frac{2e^x}{e^{2x}+1}\,dx = 2\int \frac{1}{u^2+1}\,du = 2\tan^{-1} u + C = 2\tan^{-1}(e^x) + C$$

34. $$\int \text{csch}\, x\,dx = \int \frac{2}{e^x - e^{-x}}\,dx = 2\int \frac{e^x}{e^{2x}-1}\,dx = -2\int \frac{du}{1-u^2} \quad (u = e^x)$$

$$= \begin{cases} -2\tanh^{-1} e^x + C, & e^x < 1 \\ -2\coth^{-1} e^x + C, & e^x > 1. \end{cases}$$

35. $$\left\{\begin{array}{c} u = \text{sech}\, x \\ du = -\,\text{sech}\, s\,\tanh x\,dx \end{array}\right\}; \quad \int \text{sech}^3 x\,\tanh x\,dx = -\int u^2\,du = -\frac{1}{3}u^3 + C = -\frac{1}{3}\,\text{sech}^3 x + C$$

36. $\displaystyle\int x\,\text{sech}^2 x^2\,dx = \frac{1}{2}\int \text{sech}^2 u\,du = \frac{1}{2}\tanh u + C = \frac{1}{2}\tanh x^2 + C$

37. $\left\{\begin{array}{c} u = \ln(\cosh x) \\ du = \tanh x\,dx \end{array}\right\};\quad \displaystyle\int \tanh x\,\ln(\cosh x)\,dx = \int u\,du = \frac{1}{2}u^2 + C$

$$= \frac{1}{2}\left[\ln(\cosh x)\right]^2 + C$$

38. $\displaystyle\int \frac{1+\tanh x}{\cosh^2 x}\,dx = \int\left(\text{sech}^2 x + \frac{\sinh x}{\cosh^3 x}\right)dx = \tanh x - \frac{1}{2\cosh^2 x} + C = \tanh x - \frac{1}{2}\text{sech}^2 x + C$

39. $\left\{\begin{array}{c} u = 1+\tanh x \\ du = \text{sech}^2 x\,dx \end{array}\right\};\quad \displaystyle\int \frac{\text{sech}^2 x}{1+\tanh x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C$

$$= \ln|1+\tanh x| + C$$

40. $\displaystyle\int \tanh^5 x\,\text{sech}^2 x\,dx = \frac{1}{6}\tanh^6 x + C$

41. $\left\{\begin{array}{c} x = a\sinh u \\ dx = a\cosh u\,du \end{array}\right\};\quad \displaystyle\int \frac{dx}{\sqrt{a^2+x^2}}\,dx = \int \frac{a\cosh u}{\sqrt{a^2+a^2\sinh^2 u}}\,du$

$$= \int du = u + C = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

42. $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\int \frac{1}{\sqrt{(x/a)^2-1}}\,dx = \int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1}(u) + C = \cosh^{-1}\left(\frac{x}{a}\right) + C$

43. Suppose $|x| < a$.

$\left\{\begin{array}{c} x = a\tanh u \\ dx = a\,\text{sech}^2 u\,du \end{array}\right\};\quad \displaystyle\int \frac{dx}{a^2-x^2}\,dx = \int \frac{a\,\text{sech}^2 u}{a^2-a^2\tanh^2 u}\,du$

$$= \frac{1}{a}\int du = \frac{u}{a} + C = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + C$$

The other case is done in the same way.

44. (a) $v(0)=\sqrt{\dfrac{mg}{k}}\tanh 0=0$

$$v'(t)=\sqrt{\frac{mg}{k}}\operatorname{sech}^2\left(\sqrt{\frac{gk}{m}}\,t\right)\left(\sqrt{\frac{gk}{m}}\right)=g\operatorname{sech}^2\left(\sqrt{\frac{gk}{m}}\,t\right)$$

$$mg-kv^2=mg-k\frac{mg}{k}\tanh^2\left(\sqrt{\frac{gk}{m}}\,t\right)=mg\left[1-\tanh^2\left(\sqrt{\frac{gk}{m}}\,t\right)\right]$$

$$=mg\operatorname{sech}^2\left(\sqrt{\frac{gk}{m}}\,t\right)=m\frac{dv}{dt}$$

(b) $\lim\limits_{t\to\infty}v(t)=\lim\limits_{t\to\infty}\sqrt{\dfrac{mg}{k}}\tanh\left(\sqrt{\dfrac{gk}{m}}\,t\right)=\sqrt{\dfrac{mg}{k}}.$

CHAPTER 8

SECTION 8.1

1. $\displaystyle\int e^{2-x}dx = -e^{2-x} + C$

2. $\displaystyle\int \cos\frac{2}{3}x\,dx = \frac{3}{2}\sin\frac{2}{3}x + C$

3. $\displaystyle\int_0^1 \sin\pi x\,dx = \left[-\frac{1}{\pi}\cos\pi x\right]_0^1 = \frac{2}{\pi}$

4. $\displaystyle\int_0^t \sec\pi x\tan\pi x\,dx = \frac{1}{\pi}\left[\sec\pi x\right]_0^t$

$\displaystyle = \tfrac{1}{\pi}(\sec\pi t - 1)$

5. $\displaystyle\int \sec^2(1-x)\,dx = -\tan(1-x) + C$

6. $\displaystyle\int \frac{dx}{5^x} = \int 5^{-x}\,dx = \frac{-1}{\ln 5}5^{-x} + C = -\frac{1}{5^x\ln 5} + C$

7. $\displaystyle\int_{\pi/6}^{\pi/3} \cot x\,dx = \left[\ln(\sin x)\right]_{\pi/6}^{\pi/3} = \ln\frac{\sqrt{3}}{2} - \ln\frac{1}{2} = \frac{1}{2}\ln 3$

8. $\displaystyle\int_0^1 \frac{x^3}{1+x^4}\,dx = \frac{1}{4}\left[\ln(1+x^4)\right]_0^1 = \frac{1}{4}\ln 2$

9. $\displaystyle\left\{\begin{array}{l} u = 1-x^2 \\ du = -2x\,dx \end{array}\right\};\quad \int \frac{x\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int u^{-1/2}\,du = -u^{1/2} + C = -\sqrt{1-x^2} + C$

10. $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{dx}{\cos^2 x} = \int_{-\pi/4}^{\pi/4} \sec^2 x\,dx = \left[\tan x\right]_{-\pi/4}^{\pi/4} = 2$

11. $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{\sin x}{\cos^2 x}\,dx = \int_{-\pi/4}^{\pi/4} \sec x\tan x\,dx = \left[\sec x\right]_{-\pi/4}^{\pi/4} = 0$

12. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2e^{\sqrt{x}} + C$

13. $\displaystyle\left\{\begin{array}{lll} u = 1/x \mid x = 1 & \Longrightarrow & u = 1 \\ du = -\frac{dx}{x^2} \mid x = 2 & \Longrightarrow & u = 1/2 \end{array}\right\};$

$$\int_1^2 \frac{e^{1/x}}{x^2}\,dx = \int_1^{1/2} -e^u\,du = \left[-e^u\right]_1^{1/2} = e - \sqrt{e}$$

14. $\displaystyle\int \frac{x^3}{\sqrt{1-x^4}}\,dx = -\frac{1}{4}\int \frac{du}{\sqrt{u}} = -\frac{1}{2}\sqrt{u} + C = -\frac{1}{2}\sqrt{1-x^4} + C$

15. $\displaystyle\int_0^c \frac{dx}{x^2+c^2} = \left[\frac{1}{c}\tan^{-1}\left(\frac{x}{c}\right)\right]_0^c = \frac{\pi}{4c}$

16. $\displaystyle\int a^x e^x\,dx = \int (ae)^x\,dx = \frac{(ae)^x}{\ln(ae)} + C = \frac{a^x e^x}{1+\ln a} + C$

17. $\left\{\begin{array}{l} u = 3\tan\theta + 1 \\ du = 3\sec^2\theta\, d\theta \end{array}\right\};$

$$\int \frac{\sec^2\theta}{\sqrt{3\tan\theta+1}}\,d\theta = \frac{1}{3}\int u^{-1/2}\,du = \frac{2}{3}u^{1/2} + C = \frac{2}{3}\sqrt{3\tan\theta+1} + C$$

18. $\displaystyle\int \frac{\sin\phi}{3-2\cos\phi}\,d\phi = \frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(3-2\cos\phi) + C$

19. $\displaystyle\int \frac{e^x}{ae^x - b}\,dx = \frac{1}{a}\ln|ae^x - b| + C$

20. $\displaystyle\int \frac{dx}{x^2-4x+13} = \int\frac{dx}{(x-2)^2+9} = \frac{1}{3}\tan^{-1}\left(\frac{x-2}{3}\right) + C$

21. $\left\{\begin{array}{l} u = x + 1 \\ du = dx \end{array}\right\};$

$$\begin{aligned}\int \frac{x}{(x+1)^2+4}\,dx &= \int\frac{u-1}{u^2+4}\,du = \int\frac{u}{u^2+4}\,du - \int\frac{du}{u^2+4} \\ &= \frac{1}{2}\ln|u^2+4| - \frac{1}{2}\tan^{-1}\frac{u}{2} + C \\ &= \frac{1}{2}\ln|(x+1)^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C\end{aligned}$$

22. $\displaystyle\int \frac{\ln x}{x}\,dx = \frac{1}{2}(\ln x)^2 + C$

23. $\left\{\begin{array}{l} u = x^2 \\ du = 2x\,dx \end{array}\right\};\quad \displaystyle\int\frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int\frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1}u + C$

$$= \tfrac{1}{2}\sin^{-1}(x^2) + C$$

24. $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx = \int\frac{du}{1+u^2} = \tan^{-1}u + C = \tan^{-1}e^x + C$

25. $\left\{\begin{array}{l} u = x + 3 \\ du = dx \end{array}\right\};\quad \displaystyle\int\frac{dx}{x^2+6x+10} = \int\frac{dx}{(x+3)^2+1} = \int\frac{du}{u^2+1}$

$$= \tan^{-1}u + C = \tan^{-1}(x+3) + C$$

26. $\displaystyle\int e^x\tan e^x\,dx = \int\tan u\,du = \ln|\sec u| + C = \ln|\sec e^x| + C$

27. $\displaystyle\int x\sin x^2\,dx = -\frac{1}{2}\cos x^2 + C$

28. $\displaystyle\int \frac{x+1}{\sqrt{1-x^2}}\,dx = \int\frac{x}{\sqrt{1-x^2}}\,dx + \int\frac{dx}{\sqrt{1-x^2}} = -\sqrt{1-x^2} + \sin^{-1}x + C$

29. $\displaystyle\int \tan^2 x\,dx = \int(\sec^2 x - 1)\,dx = \tan x - x + C$

30. $\displaystyle\int \cosh 2x \sinh^3 2x\,dx = \frac{1}{8}\sinh^4 2x + C$

31. $\left\{\begin{array}{lll} u = \ln x | \; x = 1 & \Longrightarrow & u = 0 \\ du = -\dfrac{dx}{x} | \; x = e & \Longrightarrow & u = 1 \end{array}\right\};$

$$\int_1^e \frac{\ln x^3}{x}\,dx = \int_1^e \frac{3\ln x}{x}\,dx = 3\int_0^1 u\,du = 3\left[\frac{u^2}{2}\right]_0^1 = \frac{3}{2}$$

32. $\displaystyle\int_0^{\pi/4} \frac{\tan^{-1} x}{1+x^2}\,dx = \frac{1}{2}\left[(\tan^{-1} x)^2\right]_0^{\pi/4} = \frac{1}{2}$

33. $\left\{\begin{array}{l} u = \sin^{-1} x \\ du = \dfrac{dx}{\sqrt{1-x^2}} \end{array}\right\}; \quad \displaystyle\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\left(\sin^{-1} x\right)^2 + C$

34. $\displaystyle\int e^x \cosh(2-e^x)\,dx = -\int \cosh u\,du = -\sinh u + C = -\sinh(2-e^x) + C$

35. $\left\{\begin{array}{l} u = \ln x \\ du = \frac{1}{x}\,dx \end{array}\right\}; \quad \displaystyle\int \frac{1}{x\ln x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\ln x| + C$

36. $\displaystyle\int_{-1}^1 \frac{x^2}{x^2+1}\,dx = \int_{-1}^1 \frac{x^2+1-1}{x^2+1}\,dx = \int_{-1}^1 \left(1 - \frac{1}{x^2+1}\right)dx = \left[x - \tan^{-1} x\right]_{-1}^1 = 2 - \frac{\pi}{2}$

37. $\left\{\begin{array}{lll} u = \cos x | \; x = 0 & \Longrightarrow & u = 1 \\ du = -\sin x\,dx | \; x = \pi/4 & \Longrightarrow & u = \sqrt{2}/2 \end{array}\right\};$

$$\begin{aligned} \int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x}\,dx &= \int_0^{\pi/4} \sec^2 x\,dx + \int_0^{\pi/4} \frac{\sin x}{\cos^2 x}\,dx \\ &= [\tan x]_0^{\pi/4} - \int_1^{\sqrt{2}/2} \frac{du}{u^2} \\ &= 1 + \left[\frac{1}{u}\right]_1^{\sqrt{2}/2} = \sqrt{2} \end{aligned}$$

38. $\displaystyle\int_0^{1/2} \frac{1+x}{\sqrt{1-x^2}}\,dx = \left[\sin^{-1} x - \sqrt{1-x^2}\right]_0^{1/2} = \frac{\pi}{6} + 1 - \frac{\sqrt{3}}{2}$

39. (formula 99) $\displaystyle\int \sqrt{x^2-4}\,dx = \frac{x}{2}\sqrt{x^2-4} - 2\ln|x + \sqrt{x^2-4}| + C$

40. (formula 87) $\displaystyle\int \sqrt{4-x^2}\,dx = \frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2} + C$

41. (formula 18) $\displaystyle\int \cos^2 2t\,dt = \frac{1}{2}[\sin 2t - \frac{1}{3}\sin^3 2t] + C$

42. (formula 38) $\displaystyle\int \sec^4 t\,dt = \frac{1}{3}\sec^3 t\sin t + \frac{2}{3}\int \sec^2 t\,dt = \frac{1}{3}\sec^3 t\sin t + \frac{2}{3}\tan t + C$

43. (formula 108) $\displaystyle\int \frac{1}{x(2x+3)}\,dx = \frac{1}{3}\ln\left|\frac{x}{2x+3}\right| + C$

44. (formula 106) $\displaystyle\int \frac{x}{2+3x}\,dx = \frac{1}{9}(2+3x-2\ln|2+3x|)+C$

45. (formula 81) $\displaystyle\int \frac{\sqrt{x^2+9}}{x^2}\,dx = -\frac{\sqrt{x^2+9}}{x} + \ln|x+\sqrt{x^2+9}|+C$

46. (formula 103) $\displaystyle\int \frac{1}{x^2\sqrt{x^2-2}}\,dx = \frac{\sqrt{x^2-2}}{2x}+C$

47. (formula 11) $\displaystyle\int x^3\ln x\,dx = x^4\left(\frac{\ln x}{4}-\frac{1}{16}\right)+C$

48. (formulas 23,24) $\displaystyle\int x^3\sin x\,dx = -x^3\cos x + 3\int x^2\cos x\,dx$

$= -x^3\cos x + 3x^2\cos x - 6\int x\sin x\,dx = -x^3\cos x + 3x^2\cos x + 6x\cos x - 6\sin x + C$

49.

$$\begin{aligned}\int_0^{\pi}\sqrt{1+\cos x}\,dx &= \int_0^{\pi}\sqrt{2\cos^2\left(\frac{x}{2}\right)}\,dx \\ &= \sqrt{2}\int_0^{\pi}\cos\left(\frac{x}{2}\right)\,dx \qquad \left[\cos\left(\frac{x}{2}\right)\geq 0 \text{ on } [0,\pi]\right] \\ &= 2\sqrt{2}\left[\sin\left(\frac{x}{2}\right)\right]_0^{\pi} = 2\sqrt{2}\end{aligned}$$

50. (a) Clear since $du = \sec^2 x\,dx$. (b) Clear since $du = \sec x\tan x\,dx$

(c) Since $\sec^2 x = 1+\tan^2 x$, $\frac{1}{2}\tan^2 x + C_1 = \frac{1}{2}\sec^2 x + C_2$ with $C_2 = C_1 - \frac{1}{2}$

51. (a) $\displaystyle\int_0^{\pi}\sin^2 nx\,dx = \int_0^{\pi}\left[\frac{1}{2}-\frac{\cos 2nx}{2}\right]dx = \left[\frac{x}{2}-\frac{\sin 2nx}{4n}\right]_0^{\pi} = \frac{\pi}{2}$

(b) $\displaystyle\int_0^{\pi}\sin nx\,\cos nx\,dx = \frac{1}{2}\int_0^{\pi}\sin 2nx\,dx = -\left[\frac{\cos 2nx}{4n}\right]_0^{\pi} = 0$

(c) $\displaystyle\int_0^{\pi/n}\sin nx\,\cos nx\,dx = \frac{1}{2}\int_0^{\pi/n}\sin 2nx\,dx = -\left[\frac{\cos 2nx}{4n}\right]_0^{\pi/n} = 0$

52. (a) $\displaystyle\int \sin^3 x\,dx = \int(1-\cos^2 x)\sin x\,dx = -\cos x + \frac{1}{3}\cos^3 x + C$

(b) $\displaystyle\int \sin^5 x\,dx = \int(1-\cos^2 x)^2\sin x\,dx = \int(1-2\cos^2 x+\cos^4 x)\sin x\,dx$

$\displaystyle = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$

(c) Write $\sin^{2k+1} x$ as $\sin^{2k} x\sin x = (1-\cos^2 x)^k\sin x$, expand $(1-\cos^2 x)^k$ and then integrate.

53. (a) $\int \tan^3 x\,dx = \int \tan^2 x \tan x\,dx = \int (\sec^2 x - 1)\tan x\,dx$

$$= \int \sec^2 x \tan x\,dx - \int \tan x\,dx$$

$$= \int u\,du - \int \tan x\,dx \quad (u = \tan x, \quad du = \sec^2 x\,dx)$$

$$= \frac{1}{2}u^2 - \ln|\sec x| + C = \frac{1}{2}\tan^2 x - \ln|\sec x| + C$$

(b) $\int \tan^5 x\,dx = \int \tan^3 x \tan^2 x\,dx = \int \tan^3 x(\sec^2 x - 1)\,dx$

$$= \int \tan^3 x \sec^2 x\,dx - \int \tan^3 x\,dx$$

$$= \int u^3\,du - \int \tan^3 x\,dx \quad (u = \tan x \quad du = \sec^2 x\,dx)$$

$$= \frac{1}{4}u^4 - \frac{1}{2}\tan^2 x + \ln|\sec x| + C$$

$$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \ln|\sec x| + C$$

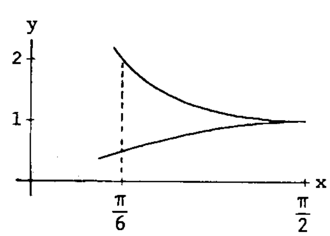
(c) $$\int \tan^7 x\,dx = \int \tan^5 x \sec^2 x\,dx - \int \tan^5 x\,dx$$

$$= \frac{1}{6}\tan^6 x - \frac{1}{4}\tan^4 x + \frac{1}{2}\tan^2 x - \ln|\sec x| + C$$

(d) $$\int \tan^{2k+1} x\,dx = \int \tan^{2k-1} x \tan^2 x\,dx = \int \tan^{2k-1} x \sec^2 x\,dx - \int \tan^{2k-1} x\,dx$$

$$= \frac{1}{2k}\tan^{2k} - \int \tan^{2k-1} x\,dx$$

54. (a)



(b) $$A = \int_{\pi/6}^{\pi/2} (\csc x - \sin x)\,dx$$

$$= [\ln|\csc x - \cot x| + \cos x]_{\pi/6}^{\pi/2}$$

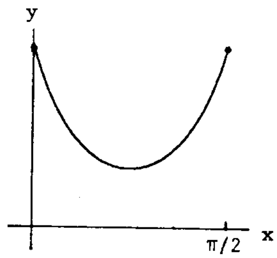
$$= -\ln(2 - \sqrt{3}) - \frac{\sqrt{3}}{2}$$

(c) $$V = \int_{\pi/6}^{\pi/2} \pi(\csc^2 x - \sin^2 x)\,dx$$

$$= \pi\left[-\cot x - \frac{x}{2} + \frac{1}{4}\sin 2x\right]_{\pi/6}^{\pi/2}$$

$$= \frac{7\pi\sqrt{3}}{8} - \frac{\pi^2}{6}$$

55. (a)



(b) $\sin x + \cos x = \sqrt{2}\,[\sin x\,\cos(\pi/4) + \cos x\,\sin(\pi/4)]$

$$= \sqrt{2}\,\sin\left(x + \frac{\pi}{4}\right);$$

$A = \sqrt{2}$, $B = \pi/4$

(c) $$\text{Area} = \int_0^{\pi/2} \frac{1}{\sin x + \cos x}\,dx = \frac{1}{\sqrt{2}}\int_0^{\pi/2} \frac{1}{\sin\left(x + \frac{\pi}{4}\right)}\,dx$$

$$\left\{\begin{array}{ll} u = x + \pi/4 | & x = 0 \implies u = \pi/4 \\ du = dx | & x = \pi/2 \implies u = 3\pi/4 \end{array}\right\};$$

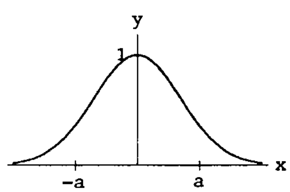
$$\frac{1}{\sqrt{2}}\int_0^{\pi/2} \frac{1}{\sin\left(x + \frac{\pi}{4}\right)}\,dx = \frac{\sqrt{2}}{2}\int_{\pi/4}^{3\pi/4} \frac{1}{\sin u}\,du$$

$$= \frac{\sqrt{2}}{2}\int_{\pi/4}^{3\pi/4} \csc u\,du$$

$$= \frac{\sqrt{2}}{2}\left[\ln|\csc u - \cot u|\right]_{\pi/4}^{3\pi/4}$$

$$= \frac{\sqrt{2}}{2}\ln\left[\frac{\sqrt{2}+1}{\sqrt{2}-1}\right]$$

56. (a)



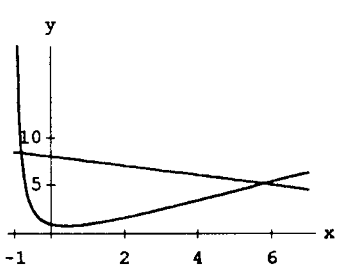
(b) $$V = \int_0^a 2\pi x e^{-x^2}\,dx$$

$$= \left[-\pi e^{-x^2}\right]_0^a = \pi(1 - e^{-a^2})$$

(c) $\pi(1 - e^{-a^2}) = 2$

$$a = \sqrt{-\ln(1 - 2/\pi)} \cong 1.0061$$

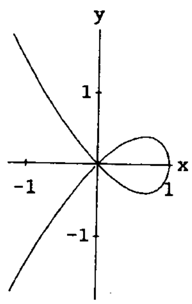
57. (a)



(b) $x_1 \cong -0.80$, $x_2 \cong 5.80$

(c)
$$\text{Area} \cong \int_{-0.80}^{5.80}\left[8-\frac{1}{2}x-\frac{x^2+1}{x+1}\right]dx=\int_{-0.80}^{5.80}\left[8-\frac{1}{2}x-x+1-\frac{2}{x+1}\right]dx$$
$$=\int_{-0.80}^{5.80}\left[9-\frac{3}{2}x-\frac{2}{x+1}\right]dx$$
$$=\left[9x-\frac{3}{4}x^2-2\ln|x+1|\right]_{-0.80}^{5.80}\cong 27.6$$

58. (a)



(b) Let $u=1-x,\ du=-dx,\ u(0)=1,\ u(1)=0$

$$A=2\int_0^1 x\sqrt{1-x}\,dx$$
$$A=-2\int_1^0(1-u)\sqrt{u}\,du$$
$$=-2\left[\tfrac{2}{3}u^{3/2}-\tfrac{2}{5}u^{5/2}\right]_1^0=\tfrac{8}{15}$$

SECTION 8.2

1.

$u=x,$	$dv=e^{-x}\,dx$
$du=dx,$	$v=-e^{-x}$

$$\int xe^{-x}\,dx=-xe^{-x}-\int -e^{-x}\,dx=-xe^{-x}-e^{-x}+C$$

2.

$$\int_0^2 x2^x\,dx=\left[\frac{x2^x}{\ln 2}\right]_0^2-\int_0^2\frac{2^x}{\ln 2}\,dx$$

$u=x,$	$dv=2^x\,dx$
$du=dx,$	$v=\dfrac{2^x}{\ln 2}$

$$=\frac{8}{\ln 2}-\left[\frac{2^x}{(\ln 2)^2}\right]_0^2$$
$$=\frac{8}{\ln 2}-\frac{3}{(\ln 2)^2}$$

3. $\left\{\begin{array}{c} t=-x^3 \\ dt=-3x^2\,dx\end{array}\right\};\quad \int x^2e^{-x^3}\,dx=-\frac{1}{3}\int e^t\,dt=-\frac{1}{3}e^t+C=-\frac{1}{3}e^{-x^3}+C$

4. $\int x\ln x^2\,dx=\frac{1}{2}\int \ln u\,du=\frac{1}{2}u\ln u-\frac{1}{2}u+C=\frac{1}{2}x^2\ln x^2-\frac{x^2}{2}+C=x^2\ln x-\frac{x^2}{2}+C$

5.

$$\int x^2e^{-x}\,dx = -x^2e^{-x} - \int -2xe^{-x}\,dx = -x^2e^{-x} + 2\int xe^{-x}\,dx$$

$$\boxed{\begin{array}{ll} u = x^2, & dv = e^{-x}\,dx \\ du = 2x\,dx, & v = -e^{-x} \end{array}} \qquad = -x^2e^{-x} + 2\left[-xe^{-x} - \int -e^{-x}\,dx\right]$$

$$\boxed{\begin{array}{ll} u = x, & dv = e^{-x}\,dx \\ du = dx, & v = -e^{-x} \end{array}} \qquad = -x^2e^{-x} + 2\left(-xe^{-x} - e^{-x}\right) + C$$

$$= -e^{-x}\left(x^2 + 2x + 2\right) + C$$

$$\int_0^1 x^2e^{-x}\,dx = \left[-e^{-x}\left(x^2 + 2x + 2\right)\right]_0^1 = 2 - 5e^{-1}$$

6.

$$\int x^3e^{-x^2}\,dx = -\frac{x^2}{2}e^{-x^2} - \int -\frac{1}{2}\cdot 2xe^{-x^2}\,dx$$

$$\boxed{\begin{array}{ll} u = x^2, & dv = xe^{-x^2}\,dx \\ du = 2x\,dx, & v = -\frac{1}{2}e^{-x^2} \end{array}} \qquad = -\frac{x^2e^{-x^2}}{2} + \int xe^{-x^2}\,dx$$

$$= -\frac{x^2e^{-x^2}}{2} - \frac{1}{2}e^{-x^2} + C$$

7.

$$\int x^2(1-x)^{-1/2}\,dx = -2x^2(1-x)^{1/2} + 4\int x(1-x)^{1/2}\,dx$$

$$\boxed{\begin{array}{ll} u = x^2, & dv = (1-x)^{-1/2}\,dx \\ du = 2x\,dx, & v = -2(1-x)^{1/2} \end{array}} \qquad = -2x^2(1-x)^{1/2} + 4\left[-\frac{2x}{3}(1-x)^{3/2} + \int \frac{2}{3}(1-x)^{3/2}\,dx\right]$$

$$\boxed{\begin{array}{ll} u = x, & dv = (1-x)^{1/2}\,dx \\ du = dx, & v = -\frac{2}{3}(1-x)^{3/2} \end{array}} \qquad = -2x^2(1-x)^{1/2} - \frac{8x}{3}(1-x)^{3/2} - \frac{16}{15}(1-x)^{5/2} + C$$

Or, use the substitution $t = 1 - x$ (no integration by parts needed) to obtain:

$$-2(1-x)^{1/2} + \tfrac{4}{3}(1-x)^{3/2} - \tfrac{2}{5}(1-x)^{5/2} + C.$$

8. $\displaystyle\int \frac{dx}{x(\ln x)^3} = \int \frac{du}{u^3} = -\frac{1}{2u^2} + C = \frac{-1}{2(\ln x)^2} + C$

9.

$$\int x\ln\sqrt{x}\,dx = \frac{1}{2}\int x\ln x\,dx = \frac{1}{2}\left[\frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx\right]$$

$$\boxed{\begin{array}{ll} u = \ln x, & dv = x\,dx \\ du = \dfrac{dx}{x}, & v = \dfrac{1}{2}x^2 \end{array}} \qquad = \tfrac{1}{4}x^2\ln x - \tfrac{1}{8}x^2 + C$$

$$\int_1^{e^2} x\ln\sqrt{x}\,dx = \left[\tfrac{1}{4}x^2\ln x - \tfrac{1}{8}x^2\right]_1^{e^2} = \tfrac{3}{8}e^4 + \tfrac{1}{8}$$

10.

$$\left\{\begin{array}{c|c} u = x+1 & u(0)=1 \\ du = dx & u(3)=4 \end{array}\right\} \quad \int_0^3 x\sqrt{x+1}\,dx = \int_1^4 (u-1)\sqrt{u}\,du = \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^4 = \frac{116}{15}$$

11.

$$\int \frac{\ln(x+1)}{\sqrt{x+1}}\,dx = 2\sqrt{x+1}\ln(x+1) - \int \frac{2\,dx}{\sqrt{x+1}}$$

$$\boxed{\begin{array}{ll} u = \ln(x+1), & dv = \dfrac{dx}{\sqrt{x+1}} \\ du = \dfrac{dx}{x+1}, & v = 2\sqrt{x+1} \end{array}}$$

$$= 2\sqrt{x+1}\,\ln(x+1) - 4\sqrt{x+1} + C$$

12.

$$\int x^2(e^x-1)\,dx = \int (x^2e^x - x^2)\,dx = \int x^2e^x\,dx - \frac{x^3}{3} = x^2e^x - \int 2xe^x\,dx - \frac{x^3}{3}$$

$$= x^2e^x - 2xe^x + \int 2e^x\,dx - \frac{x^3}{3} = e^x(x^2-2x+2) - \frac{x^3}{3} + C$$

13.

$$\int (\ln x)^2\,dx = x(\ln x)^2 - 2\int \ln x\,dx$$

$$\boxed{\begin{array}{ll} u = (\ln x)^2, & dv = dx \\ du = \dfrac{2\ln x}{x}\,dx, & v = x \end{array}}$$

$$= x(\ln x)^2 - 2\left[x\ln x - \int dx\right]$$

$$\boxed{\begin{array}{ll} u = \ln x, & dv = dx \\ du = \dfrac{dx}{x}, & v = x \end{array}}$$

$$= x(\ln x)^2 - 2x\ln x + 2x + C$$

14.

$$\left\{\begin{array}{c} u = x+5 \\ du = dx \end{array}\right\} \quad \int x(x+5)^{-14}\,dx = \int (u-5)u^{-14}\,du = \int (u^{-13} - 5u^{-14})\,du$$

$$= -\frac{1}{12}u^{-12} + \frac{5}{13}u^{-13} + C$$

$$= -\frac{1}{12}(x+5)^{-12} + \frac{5}{13}(x+5)^{-13} + C$$

15.

$$\int x^3\,3^x\,dx = \frac{x^3\,3^x}{\ln 3} - \frac{3}{\ln 3}\int x^2\,3^x\,dx$$

$u = x^3,$	$dv = 3^x\,dx$
$du = 3x^2\,dx,$	$v = \dfrac{3^x}{\ln 3}$

$$= \frac{x^3\,3^x}{\ln 3} - \frac{3}{\ln 3}\left[\frac{x^2\,3^x}{\ln 3} - \frac{2}{\ln 3}\int x\,3^x\,dx\right]$$

$u = x^2,$	$dv = 3^x\,dx$
$du = 2x\,dx$	$v = \dfrac{3^x}{\ln 3}$

$$= \frac{x^3\,3^x}{\ln 3} - \frac{3x^2\,3^x}{(\ln 3)^2} + \frac{6}{(\ln 3)^2}\int x\,3^x\,dx$$

$$= \frac{x^3\,3^x}{\ln 3} - \frac{3x^2\,3^x}{(\ln 3)^2} + \frac{6}{(\ln 3)^2}\left[\frac{x3^x}{\ln 3} - \frac{1}{\ln 3}\int 3^x\,dx\right]$$

$u = x,$	$dv = 3^x\,dx$
$du = dx,$	$v = \dfrac{3^x}{\ln 3}$

$$= 3^x\left[\frac{x^3}{\ln 3} - \frac{3x^2}{(\ln 3)^2} + \frac{6x}{(\ln 3)^3} - \frac{6}{(\ln 3)^4}\right] + C$$

16.

$u = \ln x,$	$dv = \sqrt{x}\,dx$
$du = \dfrac{1}{x}\,dx,$	$v = \dfrac{2}{3}x^{3/2}$

$$\int \sqrt{x}\ln x\,dx = \frac{2}{3}x^{3/2}\ln x - \int \frac{2}{3}x^{3/2}\cdot\frac{1}{x}\,dx = \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C$$

17.

$$\int x\,(x+5)^{14}\,dx = \frac{x}{15}\,(x+5)^{15} - \frac{1}{15}\int (x+5)^{15}\,dx$$

$u = x,$	$dv = (x+5)^{14}\,dx$
$du = dx,$	$v = \frac{1}{15}\,(x+5)^{15}$

$$= \tfrac{1}{15}x\,(x+5)^{15} - \tfrac{1}{240}\,(x+5)^{16} + C$$

Or, use the substitution $t = x + 5$ (integration by parts not needed) to obtain:

$$\tfrac{1}{16}\,(x+5)^{16} - \tfrac{1}{3}\,(x+5)^{15} + C.$$

18.

$$\int (2^x + x^2)^2\,dx = \int (2^{2x} + 2x^2 2^x + x^n)\,dx = \frac{2^{2x}}{2\ln 2} + \frac{x^5}{5} + \int 2x^2 2^x\,dx$$

$$= \frac{4^x}{\ln 4} + \frac{x^5}{5} + \frac{2x^2 2^x}{\ln 2} - \int \frac{4x2^x}{\ln 2}\,dx$$

$$= \frac{4^x}{\ln 4} + \frac{x^5}{5} + \frac{2x^2 2^x}{\ln 2} - \frac{4x2^x}{(\ln 2)^2} + \int \frac{4\cdot 2^x}{(\ln 2)^2}\,dx$$

$$= \frac{4^x}{\ln 4} + \frac{x^5}{5} + 2^x\left[\frac{2x^2}{\ln 2} - \frac{4x}{(\ln 2)^2} + \frac{4}{(\ln 2)^3}\right] + C$$

19.

$$\int x\cos\pi x\,dx=\frac{1}{\pi}x\sin x-\frac{1}{\pi}\int\sin\pi x\,dx$$

$$\boxed{\begin{array}{ll} u=x, & dv=\cos\pi x\,dx \\ du=dx, & v=\dfrac{1}{\pi}\sin\pi x\end{array}}$$

$$=\frac{1}{\pi}x\sin\pi x+\frac{1}{\pi^2}\cos\pi x+C$$

$$\int_0^{1/2}x\,\cos\pi x\,dx=\left[\frac{1}{\pi}x\sin\pi x+\frac{1}{\pi^2}\cos\pi x\right]_0^{1/2}=\frac{1}{2\pi}-\frac{1}{\pi^2}$$

20.

$$\int_0^{\pi/2}x^2\sin x\,dx=\left[-x^2\cos x\right]_0^{\pi/2}+\int_0^{\pi/2}2x\cos x\,dx=\left[-x^2\cos x\right]_0^{\pi/2}+\left[2x\sin x\right]_0^{\pi/2}-\int_0^{\pi/2}2\sin x\,dx$$

$$=\left[-x^2\cos x+2x\sin x+2\cos x\right]_0^{\pi/2}=\pi-2$$

21.

$$\int x^2\,(x+1)^9\,dx=\frac{x^2}{10}\,(x+1)^{10}-\frac{1}{5}\int x\,(x+1)^{10}\,dx$$

$$\boxed{\begin{array}{ll} u=x^2, & dv=(x+1)^9\,dx \\ du=2x\,dx, & v=\frac{1}{10}\,(x+1)^{10}\end{array}}$$

$$=\frac{x^2}{10}\,(x+1)^{10}-\frac{1}{5}\left[\frac{x}{11}\,(x+1)^{11}-\frac{1}{11}\int(x+1)^{11}\,dx\right]$$

$$\boxed{\begin{array}{ll} u=x, & dv=(x+1)^{10}\,dx \\ du=dx, & v=\frac{1}{11}\,(x+1)^{11}\end{array}}$$

$$=\frac{x^2}{10}\,(x+1)^{10}-\frac{x}{55}\,(x+1)^{11}+\frac{1}{660}\,(x+1)^{12}+C$$

22.

$$\int x^2(2x-1)^{-7}\,dx=-\frac{x^2}{12}(2x-1)^{-6}+\int\frac{x}{6}(2x-1)^{-6}\,dx$$

$$\boxed{\begin{array}{ll} u=x^2 & dv=(2x-1)^{-7}\,dx \\ du=2x\,dx & v=\dfrac{(2x-1)^{-6}}{-12}\end{array}}$$

$$=-\frac{x^2}{12}(2x-1)^{-6}-\frac{x}{60}(2x-1)^{-5}+\int\frac{1}{60}(2x-1)^{-5}\,dx$$

$$=-\frac{x^2}{12}(2x-1)^{-6}-\frac{x}{60}(2x-1)^{-5}-\frac{1}{480}(2x-1)^{-4}+C$$

23.

$$\int e^x\sin x\,dx=-e^x\cos x+\int e^x\cos x\,dx$$

$$\boxed{\begin{array}{ll} u=e^x, & dv=\sin x\,dx \\ du=e^x\,dx, & v=-\cos x\end{array}}$$

$$=-e^x\cos x+e^x\sin x-\int e^x\sin x\,dx$$

$$\boxed{\begin{array}{ll} u=e^x, & dv=\cos x\,dx \\ du=e^x\,dx, & v=\sin x\end{array}}$$

Adding $\int e^x\sin x\,dx$ to both sides, we get

$$2\int e^x \sin x\,dx = -e^x\cos x + e^x \sin x$$

so that

$$\int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + C.$$

24.
$$\int (e^x+2x)^2\,dx = \int (e^{2x}+4xe^x+4x^2)\,dx = \frac{1}{2}e^{2x}+\frac{4}{3}x^3+\int 4xe^x\,dx$$
$$= \frac{1}{2}e^{2x}+\frac{4}{3}x^3+4xe^x-\int 4e^x\,dx = \frac{1}{2}e^{2x}+\frac{4}{3}x^3+4xe^x-4e^x+C$$

25.
$$\int \ln\left(1+x^2\right)\,dx = x\ln\left(1+x^2\right) - 2\int \frac{x^2}{1+x^2}\,dx$$

$u=\ln\left(1+x^2\right)$,	$dv = dx$
$du = \dfrac{2x}{1+x^2}\,dx$,	$v = x$

$$= x\ln\left(1+x^2\right) - 2\int \frac{x^2+1-1}{1+x^2}\,dx$$
$$= x\ln\left(1+x^2\right) - 2\int \left(1-\frac{1}{1+x^2}\right)\,dx$$
$$= x\ln\left(1+x^2\right) - 2x + 2\tan^{-1}x + C$$

$$\int_0^1 \ln(1+x^2)\,dx = \left[x\ln(1+x^2)-2x+2\tan^{-1}x\right]_0^1 = \ln 2 - 2 + \frac{\pi}{2}$$

26.
$$\int x\ln(x+1)\,dx = \int (x+1)\ln(x+1)\,dx - \int \ln(x+1)\,dx$$

(from Example 3)
$$= \frac{1}{2}(x+1)^2\ln(x+1) - \frac{1}{4}(x+1)^2 - (x+1)\ln(x+1) + (x+1) + C_1$$
$$= \frac{1}{2}(x^2-1)\ln(x+1) - \frac{1}{4}x^2 + \frac{1}{2}x + C$$

27.
$$\int x^n \ln x\,dx = \frac{x^{n+1}\ln x}{n+1} - \frac{1}{n+1}\int x^n\,dx$$

$u=\ln x$,	$dv = x^n\,dx$
$du = \dfrac{dx}{x}$,	$v = \dfrac{x^{n+1}}{n+1}$

$$= \frac{x^{n+1}\ln x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C$$

28.
$$\int e^{3x}\cos 2x\,dx = \frac{1}{2}e^{3x}\sin 2x - \int \frac{3}{2}e^{3x}\sin 2x\,dx$$

$u = e^{3x}$	$dv = \cos 2x\,dx$
$du = 3e^{3x}\,dx$	$v = \dfrac{1}{2}\sin 2x$

$$= \frac{1}{2}e^{3x}\sin 2x + \frac{3}{4}e^{3x}\cos 2x - \int \frac{9}{4}e^{3x}\cos 2x\,dx$$

$$\Longrightarrow\ \frac{13}{4}\int e^{3x}\cos 2x\,dx = \frac{1}{2}e^{3x}\sin 2x + \frac{3}{4}e^{3x}\cos 2x$$
$$\Longrightarrow\ \int e^{3x}\cos 2x\,dx = \frac{2}{13}e^{3x}\sin 2x + \frac{3}{13}e^{3x}\cos 2x + C$$

29. $\{t = x^2, \quad dt = 2x\,dx\}$; $\displaystyle\int x^3 \sin x^2\,dx = \frac{1}{2}\int t \sin t\,dt$

$u = t,$	$dv = \sin t\,dt$
$du = dt,$	$v = -\cos t$

$$= \frac{1}{2}\left[-t\cos t + \int \cos t\,dt\right]$$
$$= \tfrac{1}{2}(-t\cos t + \sin t) + C$$
$$= -\tfrac{1}{2}x^2\cos x^2 + \tfrac{1}{2}\sin x^2 + C$$

30. $$\int x^3 \sin x\,dx = -x^3\cos x + \int 3x^2\cos x\,dx$$

$u = x^3$	$dv = \sin x\,dx$
$du = 3x^2\,dx$	$v = -\cos x$

$$= -x^3\cos x + 3x^2\sin x - \int 6x\sin x\,dx$$
$$= -x^3\cos x + 3x^2\sin x + 6x\cos x - \int 6\cos x\,dx$$
$$= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$$

31. $\left\{\begin{array}{l|ll} u = 2x & x = 0 \implies & u = 0 \\ du = 2\,dx & x = 1/4 \implies & u = 1/2 \end{array}\right\}$;

$$\int_0^{1/4} \sin^{-1} 2x\,dx = \frac{1}{2}\int_0^{1/2} \sin^{-1} u\,du$$
$$= \frac{1}{2}\left[u\sin^{-1} u + \sqrt{1-u^2}\right]_0^{1/2} \qquad \text{[by(8.2.5)]}$$
$$= \frac{1}{2}\left[\frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1\right] = \frac{\pi}{24} + \frac{\sqrt{3}-2}{4}$$

32. $\left\{\begin{array}{l} u = \sin^{-1} 2x \\ du = \dfrac{2}{\sqrt{1-4x^2}}\,dx \end{array}\right\}$ $\displaystyle\int \frac{\sin^{-1} 2x}{\sqrt{1-4x^2}}\,dx = \frac{1}{2}\int u\,du = \frac{1}{4}u^2 + C = \frac{1}{4}\left(\sin^{-1} 2x\right)^2 + C$

33. $\left\{\begin{array}{l|ll} u = x^2 & x = 0 \implies & u = 0 \\ du = 2x\,dx & x = 1 \implies & u = 1 \end{array}\right\}$;

$$\int_0^1 x\tan^{-1} x\,dx = \frac{1}{2}\int_0^1 \tan^{-1} u\,du$$
$$= \frac{1}{2}\left[u\tan^{-1} u - \tfrac{1}{2}(1+x^2)\right]_0^1 \qquad \text{[by 8.2.6]}$$
$$= \frac{\pi}{8} - \frac{1}{2}\ln 2$$

34. $$\int \cos\sqrt{x}\,dx = \int \sqrt{x}\frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$$

$u = \sqrt{x}$	$dv = \dfrac{\cos\sqrt{x}}{\sqrt{x}}\,dx$
$du = \dfrac{1}{2\sqrt{x}}\,dx$	$v = 2\sin\sqrt{x}$

$$= 2\sqrt{x}\sin\sqrt{x} - \int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$$
$$= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$$

35.

$$\int x^2 \cosh 2x\,dx = \tfrac{1}{2}x^2 \sinh 2x - \int x \sinh 2x\,dx$$

$u = x^2,$	$dv = \cosh 2x\,dx$
$du = 2x\,dx,$	$v = \frac{1}{2}\sinh 2x$

$$= \tfrac{1}{2}x^2 \sinh 2x - \tfrac{1}{2}x\cosh 2x + \tfrac{1}{2}\int \cosh 2x\,dx$$

$u = x,$	$dv = \sinh 2x\,dx$
$du = dx,$	$v = \frac{1}{2}\cosh 2x$

$$= \tfrac{1}{2}x^2 \sinh 2x - \tfrac{1}{2}x\cosh 2x + \tfrac{1}{4}\sinh 2x + C$$

36. $\displaystyle\int_{-1}^{1} x\sinh(2x^2)\,dx = \frac{1}{4}\left[\cosh(2x^2)\right]_{-1}^{1} = 0$

37. Let $u = \ln x$, $du = \frac{1}{x}\,dx$. Then

$$\int \frac{1}{x}\sin^{-1}(\ln x)\,dx = \int \sin^{-1} u\,du = u\,\sin^{-1} u + \sqrt{(1-u^2)} + C$$
$$= (\ln x)\sin^{-1}(\ln x) + \sqrt{1-(\ln x)^2} + C$$

38.

$u = \cos(\ln x)$	$dv = dx$
$du = -\dfrac{\sin(\ln x)}{x}\,dx$	$v = x$

$$\int \cos(\ln x)\,dx = x\cos(\ln x) + \int \sin(\ln x)\,dx$$

$u = \sin(\ln x)$	$dv = dx$
$du = \dfrac{\cos(\ln x)}{x}\,dx$	$v = x$

$$= x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\,dx$$

$$\Longrightarrow \int \cos(\ln x)\,dx = \frac{1}{2}x\left[\cos(\ln x) + \sin(\ln x)\right] + C$$

39.

$$\int \sin(\ln x)\,dx = x\,\sin(\ln x) - \int \cos(\ln x)\,dx$$

$u = \sin(\ln x),$	$dv = dx$
$du = \cos(\ln x)\dfrac{1}{x}\,dx,$	$v = x$

$$= x\,\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\,dx$$

$u = \cos(\ln x),$	$dv = dx$
$du = -\,\sin(\ln x)\dfrac{1}{x}\,dx,$	$v = x$

Adding $\int \sin(\ln x)\,dx$ to both sides, we get

$$2\int \sin(\ln x)\,dx = x\,\sin(\ln x) - x\cos(\ln x)$$

so that

$$\int \sin(\ln x)\,dx = \tfrac{1}{2}\left[x\,\sin(\ln x) - x\cos(\ln x)\right] + C$$

40.

$$\boxed{\begin{array}{ll} u=(\ln x)^2 & dv=x^2dx \\ du=2\dfrac{\ln x}{x}\,dx & v=\dfrac{x^3}{3}\end{array}}\qquad \int_1^{2e} x^2(\ln x)^2\,dx=\left[\frac{x^3}{3}(\ln x)^2\right]_1^{2e}-\int_1^{2e}\frac{2x^2}{3}\ln x\,dx$$

$$\boxed{\begin{array}{ll} u=\ln x & dv=\dfrac{2x^2}{3}\,dx \\ du=\dfrac{1}{x}\,dx & v=\dfrac{2x^3}{9}\end{array}}\qquad =\left[\frac{x^3}{3}(\ln x)^2-\frac{2x^3}{9}\ln x\right]_1^{2e}+\int_1^{2e}\frac{2x^2}{9}\,dx$$

$$=\left[\frac{x^3}{3}(\ln x)^2-\frac{2x^3}{9}\ln x+\frac{2x^3}{27}\right]_1^{2e}$$

$$=\frac{8e^3}{3}\left[(\ln 2e)^2-\frac{2}{3}\ln 2e+\frac{2}{9}\right]-\frac{2}{27}$$

41.

$$\boxed{\begin{array}{ll} u=\ln x & dv=dx \\ du=\dfrac{1}{x}\,dx & v=x\end{array}}\qquad \int \ln x\,dx=x\ln x-\int dx=x\ln x-x+C$$

42.

$$\boxed{\begin{array}{ll} u=\tan^{-1}x & dv=dx \\ du=\dfrac{1}{1+x^2}\,dx & v=x\end{array}}\qquad \int \tan^{-1}x\,dx=x\tan^{-1}x-\int\frac{x}{1+x^2}\,dx$$

$$=x\tan^{-1}x-\frac{1}{2}\ln(1+x^2)+C$$

43. Cannot be solved

44.

$$\boxed{\begin{array}{ll} u=\ln x & dv=x^k dx \\ du=\dfrac{1}{x}\,dx & v=\dfrac{1}{k+1}x^{k+1}\end{array}}\qquad \int x^k\ln x\,dx=\frac{1}{k+1}x^{k+1}\ln x-\int\frac{1}{k+1}x^k\,dx$$

$$=\frac{1}{k+1}x^{k+1}\ln x-\frac{x^{k+1}}{(k+1)^2}+C$$

45.

$$\boxed{\begin{array}{ll} u=e^{ax} & dv=\sin bx dx \\ du=ae^{ax}\,dx & v=-\dfrac{1}{b}\cos bx\end{array}}\qquad \int e^{ax}\sin bx\,dx=-\frac{1}{b}e^{ax}\cos bx+\frac{a}{b}\int e^{ax}\cos bx\,dx$$

$$\boxed{\begin{array}{ll} u=e^{ax} & dv=\cos bx dx \\ du=ae^{ax}\,dx & v=\dfrac{1}{b}\sin bx\end{array}}\qquad =-\frac{1}{b}e^{ax}\cos bx+\frac{a}{b^2}e^{ax}\sin bx-\frac{a^2}{b^2}\int e^{ax}\sin bx\,dx.$$

$$\Longrightarrow\quad \int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+C$$

46.

$$\boxed{\begin{array}{ll} u=e^{ax} & dv=\cos bx dx \\ du=ae^{ax}\,dx & v=\dfrac{1}{b}\sin bx\end{array}}\qquad \int e^{ax}\cos bx\,dx=\frac{1}{b}e^{ax}\sin bx-\frac{a}{b}\int e^{ax}\sin bx\,dx$$

$u = e^{ax}$	$dv = \sin bx dx$
$du = ae^{ax}\,dx$	$v = -\frac{1}{b}\cos bx$

$$= \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx\,dx.$$

$$\Longrightarrow \quad \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C$$

47. The integrals cancel each other if you integrate by parts.

$\int e^{ax}\cosh bx\,dx = \frac{1}{2a}e^{ax} - \frac{x}{2} + C$

49. $A = \int_0^{1/2} \sin -1x\,dx = \left[x\,\sin^{-1}x + \sqrt{1-x^2}\right]_0^{1/2} = \frac{\pi}{12} + \frac{\sqrt{3}-2}{2}$

50.

$$A = \int_0^2 xe^{-2x}\,dx = \left[-\frac{xe^{-2x}}{2}\right]_0^2 + \int_0^2 \frac{e^{-2x}}{2}\,dx$$
$$= \left[-\frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4}\right]_0^2 = \frac{1}{4} - \frac{5}{4}e^{-4}$$

51. (a) $A = \int_1^e \ln x\,dx = [x\ln x - x]_1^e = 1$

(b) $\overline{x}A = \int_1^e x\ln x\,dx = \left[\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right]_1^e = \frac{1}{4}\left(e^2+1\right), \quad \overline{x} = \frac{1}{4}\left(e^2+1\right)$

$\overline{y}A = \int_1^e \frac{1}{2}(\ln x)^2\,dx = \frac{1}{2}\left[x(\ln x)^2 - 2x\ln x + 2x\right]_1^e = \frac{1}{2}e - 1, \quad \overline{y} = \frac{1}{2}e - 1$

(c) $V_x = 2\pi\overline{y}A = \pi(e-2), \quad V_y = 2\pi\overline{x}A = \frac{1}{2}\pi\left(e^2+1\right)$

52. (a) $A = \int_1^{2e} \frac{\ln x}{x}\,dx = \left[\frac{1}{2}(\ln x)^2\right]_1^{2e} = \frac{\ln 2e}{2} = \frac{1+\ln 2}{2}$

$$V = \int_1^{2e} \pi\left(\frac{\ln x}{x}\right)^2 dx$$

(b)

$u = (\ln x)^2,$	$dv = \frac{1}{x^2}\,dx$
$du = \frac{2\ln x}{x},\ dx$	$v = -\frac{1}{x}$

$$= \pi\left[-\frac{(\ln x)^2}{x}\right]_1^{2e} - \pi\int_1^{2e} -\frac{2\ln x}{x^2}\,dx$$
$$= \pi\left[-\frac{(\ln 2e)^2}{2e}\right] + 2\pi\left[-\frac{\ln x}{x}\right]_1^{2e} - 2\pi\int_1^{2e} -\frac{1}{x^2}\,dx$$
$$= \pi\left[-\frac{(\ln 2e)^2}{2e} - \frac{\ln 2e}{e} - \frac{1}{2e} + 1\right]$$

53. $\overline{x} = \frac{1}{e-1}, \quad \overline{y} = \frac{1}{4}(e+1)$

54. $\overline{x} = \frac{e-2}{e-1}, \quad \overline{y} = \frac{1}{4}\left(\frac{e+1}{e}\right)$

55. $\overline{x} = \frac{1}{2}\pi, \quad \overline{y} = \frac{1}{8}\pi$

56. $\overline{x} = \frac{\pi}{2} - 1, \quad \overline{y} = \frac{\pi}{8}$

57. (a) $M = \int_0^1 e^{kx}\,dx = \frac{1}{k}\left(e^k - 1\right)$

(b) $x_M M = \int_0^1 xe^{kx}\,dx = \frac{(k-1)\,e^k + 1}{k^2}, \qquad x_M = \frac{(k-1)\,e^k + 1}{k\,(e^k - 1)}$

58. (a) $M = \int_2^3 \ln x\,dx = [x\ln x - x]_2^3 = 3\ln 3 - 2\ln 2 - 1$

(b) $x_M\, M = \int_2^3 x\ln x\,dx = \left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_2^3 = \frac{9}{2}\ln 3 - 2\ln 2 - \frac{5}{4}$

$$\Longrightarrow \ x_M = \frac{18\ln 3 - 8\ln 2 - 5}{4(3\ln 3 - 2\ln 2 - 1)}$$

59. $V_y = \int_0^1 2\pi x\cos\frac{1}{2}\pi x\,dx = \left[4x\sin\frac{1}{2}\pi x + \frac{8}{\pi}\cos\frac{1}{2}\pi x\right]_0^1 = 4 - \frac{8}{\pi}$

60. $V_y = \int_0^\pi 2\pi x^2\sin x\,dx = 2\pi\left[-x^2\cos x + 2x\sin x + 2\cos x\right]_0^\pi = 2\pi(\pi^2 - 4)$

61. $V_y = \int_0^1 2\pi x^2 e^x\,dx = 2\pi\,(e - 2)$ (see Example 6)

62. $V_y = \int_0^{\pi/2} 2\pi x^2\cos x\,dx = 2\pi\left[x^2\sin x + 2x\cos x + 2\sin x\right]_0^{\pi/2} = \frac{\pi}{2}(\pi^2 - 8)$

63.

$$V_x = \int_0^1 \pi e^{2x}\,dx = \pi\left[\frac{1}{2}e^{2x}\right]_0^1 = \frac{1}{2}\pi\left(e^2 - 1\right)$$

$$\overline{x}V_x = \int_0^1 \pi x e^{2x}\,dx = \pi\left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right]_0^1 = \frac{1}{4}\pi\left(e^2 + 1\right)$$

$$\overline{x} = \frac{e^2 + 1}{2\,(e^2 - 1)}$$

64.

$$\overline{x}V_x = \int_0^{\pi/2} \pi x\sin^2 x\,dx = \frac{1}{8}\pi\left[2x^2 - 2x\sin 2x - \cos 2x\right]_0^{\pi/2} = \frac{1}{16}\pi(\pi^2 + 4)$$

$$V_x = \int_0^{\pi/2} \pi\sin^2 x\,dx = \pi\left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi/2} = \frac{1}{4}\pi^2$$

$$\overline{x} = \frac{\pi^2 + 4}{4\pi}$$

65.

$$A=\int_0^1 \cosh x\,dx=[\sinh x]_0^1=\sinh 1=\frac{e-e^{-1}}{2}=\frac{e^2-1}{2e}$$

$$\overline{x}A=\int_0^1 x\cosh x\,dx=[x\sinh x-\cosh x]_0^1=\sinh 1-\cosh 1+1=\frac{2\,(e-1)}{2e}$$

$$\overline{y}A=\int_0^1 \frac{1}{2}\cosh^2 x\,dx=\frac{1}{4}\,[\sinh x\cosh x+x]_0^1=\frac{1}{4}\,(\sinh 1\cosh 1+1)=\frac{e^4+4e^2-1}{16e^2}$$

Therefore $\overline{x}=\dfrac{2}{e+1}$ and $\overline{y}=\dfrac{e^4+4e^2-1}{8e\,(e^2-1)}$.

66. (a) $\overline{x}V_x=\displaystyle\int_0^1 \pi x\cosh^2 x\,dx=\frac{\pi}{8}\left[2x^2+2x\sinh 2x-\cosh 2x\right]_0^1=\frac{\pi}{8}(1+2\sinh 2-\cosh 2)(a)$

$$V_x=\int_0^1 \pi\cosh^2 x\,dx=\pi\left[\frac{x}{2}+\frac{1}{4}\sinh 2x\right]_0^1=\frac{\pi}{4}(2+\sinh 2)$$

$$\overline{x}=\frac{1+2\sinh 2-\cosh 2}{2(2+\sinh 2)}$$

(b) $\overline{y}V_y=\displaystyle\int_0^1 2\pi x\cosh^2 x\,dx=\frac{\pi}{8}(1+2\sinh 2-\cosh 2)$

$$V_y=\int_0^1 2\pi x\cosh x\,dx=2\pi\,[x\sinh x-\cosh x]_0^1=2\pi(\sinh 1-\cosh 1+1)$$

$$\overline{y}=\frac{1+2\sinh 2-\cosh 2}{16(\sinh 1-\cosh 1+1)}$$

67.

$u=x^2,$	$dv=e^{-x}\,dx$
$du=2x\,dx,$	$v=-e^{-x}$

$$\int x^n e^{ax}\,dx=\frac{x^n e^{ax}}{a}-\frac{n}{a}\int x^{n-1}e^{ax}\,dx$$

68.

$$\int(\ln x)^n\,dx=x(\ln x)^n-n\int\frac{x(\ln x)^{n-1}}{x}\,dx$$

$u=(\ln x)^n$	$dv=dx$
$du=\dfrac{n(\ln x)^{n-1}}{x}\,dx$	$v=x$

$$=x(\ln x)^n-n\int(\ln x)^{n-1}\,dx$$

69.

$$\begin{aligned}\int x^3e^{2x}\,dx&=\frac{1}{2}x^3e^{2x}-\frac{3}{2}\int x^2e^{2x}\,dx=\frac{1}{2}x^3e^{2x}-\frac{3}{2}\left[\frac{1}{2}x^2e^{2x}-\int xe^{2x}\,dx\right]\\&=\frac{1}{2}x^3e^{2x}-\frac{3}{4}x^2e^{2x}-\frac{3}{4}xe^{2x}-\frac{3}{4}\int e^{2x}\,dx\\&=\frac{1}{2}x^3e^{2x}-\frac{3}{4}x^2e^{2x}-\frac{3}{4}xe^{2x}-\frac{3}{8}e^{2x}+C\end{aligned}$$

70.

$$\begin{aligned}\int x^2e^{-x}\,dx&=-x^2e^{-x}+2\int xe^{-x}\,dx\\&=-x^2e^{-x}+2\left[-xe^{-x}+\int e^{-x}\,dx\right]\\&=-x^2e^{-x}-2xe^{-x}-2e^{-x}+C\end{aligned}$$

71.

$$\int (\ln x)^3\,dx = x(\ln x)^3 - 3\int (\ln x)^2\,dx = x(\ln x)^3 - 3x(\ln x)^2 + 6\int \ln x\,dx$$
$$= x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6\int dx$$
$$= x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C$$

72.

$$\int (\ln x)^4\,dx = x(\ln x)^4 - 4\int (\ln x)^3\,dx$$
$$= x(\ln x)^4 - 4\left[x(\ln x)^3 - 3\int (\ln x)^2\,dx\right]$$
$$= x(\ln x)^4 - 4x(\ln x)^3 + 12\int (\ln x)^2\,dx$$
$$= x(\ln x)^4 - 4x(\ln x)^3 + 12\left[x(\ln x)^2 - 2\int \ln x\,dx\right]$$
$$= x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24x\ln x + 24x + C$$

73. (a) Differentiating, $x^3e^x = Ax^3e^x + 3Ax^2e^x + 2Bxe^x + Bx^2e^x + Ce^x + Cxe^x + De^x$
$\implies \quad A = 1,\ B = -3,\ C = 6,\ \text{and } D = -6$

(b)

$$\int x^3e^x\,dx = x^3e^x - 3\int x^2e^x\,dx$$
$$= x^3e^x - 3x^2e^x + 6\int xe^x\,dx$$
$$= x^3e^x - 3x^2e^x + 6xe^x - 6\int e^x\,dx$$
$$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$$

74. Use induction on k. For $k = 0, P(x)$ is a constant and the result follows immediately.
Now suppose the statement is true for $k = n$ and let $P(x)$ have dgree $n + 1$.
Then $P(x) = ax^{n+1} + Q(x)$ for $Q(x)$ of degree$n$.

$$\int P(x)e^x\,dx = a\int x^{n+1}e^x\,dx + \int Q(x)e^x\,dx$$
$$= ax^{n+1}e^x - a(n+1)\int x^ne^x\,dx + \int Q(x)e^x\,dx$$
$$= ax^{n+1}e^x - a(n+1)(x^n - n(n-1)x^{n-1} + \cdots \pm 1)e^x + (Q(x) - Q'(x) + \cdots \pm Q^{(n)}(x))e^x + C$$
$$= [ax^{n+1} + Q(x) - [a(n+1)x^n + Q'(x)] + \cdots \pm Q^{(n)}(x)]]e^x + C$$
$$= [P(x) - P'(x) + \cdots \pm P^{(n)}(x)]e^x + C$$

75. Let $u = f(x)$, $dv = g''(x)\,dx$. Then $du = f'(x)\,dx$, $v = g'(x)$, and

$$\int_a^b f(x)g''(x)\,dx = [f(x)g'(x)]_a^b - \int_a^b f'(x)g'(x)\,dx = -\int_a^b f'(x)g'(x)\,dx \quad \text{since } f(a) = f(b) = 0$$

Now let $u = f'(x)$, $dv = g'(x)\,dx$. Then $du = f''(x)\,dx$, $v = g(x)$, and

$$-\int_a^b f'(x)g'(x)\,dx = [-f'(x)g(x)]_a^b + \int f''(x)g(x)\,dx = \int g(x)f''(x)\,dx \quad \text{since } g(a) = g(b) = 0$$

Therefore, if f and g have continuous second derivatives, and if $f(a) = g(a) = f(b) = g(b) = 0$, then

$$\int_a^b f(x)g''(x)\,dx = \int_a^b g(x)f''(x)\,dx$$

76. (a)

$$f(b) - f(a) = \int_a^b f'(x)\,dx = [f'(x)(x-b)]_a^b - \int_a^b f''(x)(x-b)\,dx \ (a)$$

$u = f'(x)$	$dv = dx$
$du = f''(x)\,dx$	$v = (x-b)$

$$= f'(a)(b-a) - \int_a^b f''(x)(x-b)\,dx$$

(b)

$$f(b) - f(a) = f'(a)(b-a) - \left[f''(x)\frac{(x-b)^2}{2}\right]_a^b + \int_a^b \frac{f'''(x)}{2}(x-b)^2\,dx$$

$u = f''(x)$	$dv = (x-b)\,dx$
$du = f'''(x)\,dx$	$v = \dfrac{(x-b)^2}{2}$

$$= f'(a)(b-a) + \frac{f''(a)}{2}(b-a)^2 + \int_a^b \frac{f'''(x)}{2}(x-b)^2\,dx$$

PROJECT 8.2

1. Let P be a regular partition of the interval $[0, n]$. The present value of R dollars continuously compounded at the rate r on the interval $[t_{i-1}, t_i]$ is approximately $Re^{-rt}\,\Delta t$. Therefore, it follows that the present value of the revenue stream over the time interval $[0, n]$ is given by the definite integral

$$P.V. = \int_0^n Re^{-rt}\,dt$$

2. (a) $P.V. = \int_0^4 1000e^{-0.04t}\,dt = -25,000\left[e^{-0.04t}\right]_0^4 = 25,000(1 - e^{-0.16}) \simeq \$3696.41.$

(b) $P.V. = \int_0^4 1000e^{-0.08t}\,dt = -12,500\left[e^{-0.08t}\right]_0^4 = 12,500(1 - e^{-0.32}) \simeq \$3423.14.$

3. (a)

$$\begin{aligned} \text{P.V.} &= \int_0^2 (1000 + 60t)\,e^{-0.05t}\,dt \\ &= 1000\int_0^2 e^{-t/20}\,dt + 60\int_0^2 te^{-t/20}\,dt \\ &= 1000\left[-20e^{-t/20}\right]_0^2 + 60\left[-20te^{-t/20} - 400e^{-t/20}\right]_0^2 \end{aligned}$$

(by parts)

$$= \left[-(44000+1200t)\,e^{-t/20}\right]_0^2 = 44000 - 46400\,e^{-0.10} \cong \$2016$$

(b) $$\text{P.V.} = \int_0^2 (1000+60t)\,e^{-0.1t}\,dt$$
$$= 1000\left[-10e^{-t/10}\right]_0^2 + 60\left[-10te^{-t/10} - 100e^{-t/10}\right]_0^2$$

(by parts)

$$= \left[-(16000+600t)\,e^{-t/10}\right]_0^2 = 16000 - 17200e^{-0.20} \cong \$1918$$

4. (a) $$P.V. = \int_3^4 (1000+60t)e^{-0.05t}\,dt = 1000\int_3^4 e^{-0.05t}\,dt + 60\int_3^4 te^{-0.05t}\,dt$$
$$= -20{,}000\left[e^{-0.05t}\right]_3^4 + 60\left[-20te^{-0.05t} - 400e^{-0.05t}\right]_3^4 \simeq \$1{,}015.64.$$

(b) $$P.V. = \int_3^4 (1000+60t)e^{-0.1t}\,dt = 1000\int_3^4 e^{-0.1t}\,dt + 60\int_3^4 te^{-0.1t}\,dt$$
$$= -10{,}000\left[e^{-0.1t}\right]_3^4 + 60\left[-10te^{-0.1t} - 100e^{-0.1t}\right]_3^4 \simeq \$852.67.$$

5. $r(t) = a\sin\omega t$, with a, ω positive constants.
$$P.V. = \int_0^n a\sin\omega t e^{-rt}\,dt = a\int_0^n e^{-rt}\sin\omega t\,dt = \frac{ae^{-rt}}{r^2+\omega^2}\left[-r\sin\omega t - \omega\cos\omega t\right]_0^n$$
$$= \frac{a\omega}{r^2+\omega^2} - \frac{ae^{-rn}}{r^2+\omega^2}(r\sin\omega n - \omega\cos\omega n).$$

SECTION 8.3

1. $$\int \sin^3 x\,dx = \int (1-\cos^2 x)\sin x\,dx = \frac{1}{3}\cos^3 x - \cos x + C$$

2. $$\int_0^{\pi/8} \cos^2 4x\,dx = \left[\frac{x}{2} + \frac{1}{16}\sin 8x\right]_0^{\pi/8} = \frac{\pi}{16}$$

3. $$\int_0^{\pi/6} \sin^2 3x\,dx = \int_0^{\pi/6} \frac{1-\cos 6x}{2}\,dx = \left[\frac{1}{2}x - \frac{1}{12}\sin 6x\right]_0^{\pi/6} = \frac{\pi}{12}$$

4. $$\int \cos^3 x\,dx = \int (1-\sin^2 x)\cos x\,dx = \sin x - \frac{1}{3}\sin^3 x + C$$

5.

$$\begin{aligned}\int \cos^4 x \sin^3 x\,dx &= \int \cos^4 x\,(1-\cos^2 x)\sin x\,dx\\ &= \int (\cos^4 x - \cos^6 x)\sin x\,dx\\ &= -\tfrac{1}{5}\cos^5 x + \tfrac{1}{7}\cos^7 x + C\end{aligned}$$

6. $\displaystyle\int \sin^3 x\cos^2 x\,dx = \int \cos^2(1-\cos^2 x)\sin x\,dx = -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$

7.

$$\begin{aligned}\int \sin^3 x\cos^3 x\,dx &= \int \sin^3 x\,(1-\sin^2 x)\cos x\,dx\\ &= \int (\sin^3 x - \sin^5 x)\cos x\,dx\\ &= \tfrac{1}{4}\sin^4 x - \tfrac{1}{6}\sin^6 x + C\end{aligned}$$

8.

$$\begin{aligned}\int \sin^2 x\cos^4 x\,dx &= \int (\sin x\cos x)^2\cos^2 x\,dx = \int \frac{1}{4}\sin^2 2x(\frac{1}{2}+\frac{1}{2}\cos 2x)\,dx\\ &= \frac{1}{8}\int \sin^2 2x\,dx + \frac{1}{8}\int \sin^2 2x\cos 2x\,dx = \frac{1}{8}\left(\frac{1}{2}x - \frac{1}{8}\sin 4x\right) + \frac{1}{48}\sin^3 2x + C\end{aligned}$$

9. $\displaystyle\int \sec^2 \pi x\,dx = \frac{1}{\pi}\tan \pi x + C$

10. $\displaystyle\int \csc^2 2x\,dx = -\frac{1}{2}\cot 2x + C$

11.

$$\begin{aligned}\int \tan^3 x\,dx &= \int (\sec^2 x - 1)\tan x\,dx\\ &= \int \tan x\sec^2 x\,dx - \int \tan x\,dx\\ &= \tfrac{1}{2}\tan^2 x + \ln|\cos x| + C\end{aligned}$$

12. $\displaystyle\int \cot^3 x\,dx = \int \cot x(\csc^2 x - 1)\,dx = -\frac{1}{2}\cot^2 x - \ln|\sin x| + C$

13.

$$\int \sin^4 x\,dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \cos^2 2x\right)\,dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right)\,dx$$

$$= \int \left(\frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\right)\,dx$$

$$= \tfrac{3}{8}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C$$

$$\int_0^{\pi} \sin^4 x\,dx = \left[\tfrac{3}{8}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x\right]_0^{\pi} = \tfrac{3}{8}\pi$$

14.

$$\int \cos^3 x \cos 2x\,dx = \int (1-\sin^2 x)(1-2\sin^2 x)\cos x\,dx = \int (1 - 3\sin^2 x + 2\sin^4 x)\cos x\,dx$$

$$= \sin x - \sin^3 x + \frac{2}{5}\sin^5 x + C$$

15.

$$\int \sin 2x \cos 3x\,dx = \int \frac{1}{2}\left[\sin(-x) + \sin 5x\right]\,dx$$

$$= \int \frac{1}{2}\left(-\sin x + \sin 5x\right)\,dx$$

$$= \tfrac{1}{2}\cos x - \tfrac{1}{10}\cos 5x + C$$

16.

$$\int_0^{\pi/2} \cos 2x \sin 3x\,dx = \int_0^{\pi/2} \frac{1}{2}\left[\sin(3x-2x) + \sin(3x+2x)\right]\,dx = \frac{1}{2}\int_0^{\pi/2} (\sin x + \sin 5x)\,dx$$

$$= \left[-\frac{1}{2}\cos x - \frac{1}{10}\cos 5x\right]_0^{\pi/2} = \frac{3}{5}$$

17. $\displaystyle\int \tan^2 x \sec^2 x\,dx = \frac{1}{3}\tan^3 x + C$

18. $\displaystyle\int \cot^2 x \csc^2 x\,dx = -\frac{1}{3}\cot^3 x + C$

19.

$$\int \csc^3 x\,dx = -\csc x \cot x - \int \csc x \cot^2 x\,dx$$

$u = \csc x,$	$dv = \csc^2 x\,dx$
$du = -\csc x \cot x\,dx,$	$v = -\cot x$

$$= -\csc x \cot x - \int \csc x\,(\csc^2 x - 1)\,dx$$

Thus

$$2\int \csc^3 x\,dx = -\csc x \cot x + \int \csc x\,dx$$

so that

$$\int \csc^3 x\,dx = -\frac{1}{2}\csc x \cot x + \frac{1}{2}\ln|\csc x - \cot x| + C.$$

20.

$$\int \sec^3 \pi x\,dx = \frac{1}{\pi}\sec \pi x \tan \pi x - \int \tan^2 \pi x \sec \pi x\,dx$$

$u = \sec \pi x$	$dv = \sec^2 \pi x\,dx$
$du = \pi \sec \pi x \tan \pi x\,dx$	$v = \frac{1}{\pi}\tan \pi x$

$$= \frac{1}{\pi}\sec \pi x \tan \pi x - \int \sec^3 \pi x\,dx + \int \sec \pi x\,dx$$

$$\Longrightarrow 2\int \sec^3 \pi x\,dx = \frac{1}{\pi}\sec \pi x \tan \pi x + \frac{1}{\pi}\ln|\sec \pi x + \tan \pi x|$$

$$\int \sec^3 \pi x\,dx = \frac{1}{2\pi}\sec \pi x \tan \pi x + \frac{1}{2\pi}\ln|\sec \pi x + \tan \pi x| + C$$

21.

$$\begin{aligned}\int \sin^2 x \sin 2x\,dx &= \int \sin^2 x\,(2\sin x \cos x)\,dx \\ &= 2\int \sin^3 x \cos x\,dx \\ &= \tfrac{1}{2}\sin^4 x + C\end{aligned}$$

22.

$$\begin{aligned}\int_0^{\pi/2} \cos^4 x\,dx &= \left[\frac{1}{4}\cos^3 x \sin x\right]_0^{\pi/2} + \frac{3}{4}\int_0^{\pi/2} \cos^2 x\,dx \\ &= \left[\frac{1}{4}\cos^3 x \sin x + \frac{3}{8}x + \frac{3}{16}\sin 2x\right]_0^{\pi/2} = \frac{3\pi}{16}\end{aligned}$$

23.

$$\begin{aligned}\int \sin^6 x\,dx &= \int \left(\frac{1-\cos 2x}{2}\right)^3 dx \\ &= \frac{1}{8}\int (1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x)\,dx \\ &= \frac{1}{8}\int \left[1 - 3\cos 2x + 3\left(\frac{1+\cos 4x}{2}\right) - \cos 2x\,(1 - \sin^2 2x)\right] dx \\ &= \frac{1}{8}\int \left(\frac{5}{2} - 4\cos 2x + \frac{3}{2}\cos 4x + \sin^2 2x \cos 2x\right) dx \\ &= \tfrac{5}{16}x - \tfrac{1}{4}\sin 2x + \tfrac{3}{64}\sin 4x + \tfrac{1}{48}\sin^3 2x + C\end{aligned}$$

24. $$\int \cos^5 x \sin^5 x\,dx = \int \cos^5 x \sin^4 x \sin x\,dx = \int \cos^5 x (1-\cos^2 x)^2 \sin x\,dx$$
$$= \int \cos^5 x (1 - 2\cos^2 x + \cos^4 x)\sin x\,dx = -\frac{\cos^6 x}{6} + \frac{1}{4}\cos^8 x - \frac{1}{10}\cos^{10} x + C$$

25. $$\int_{\pi/6}^{\pi/2} \cot^2 x\,dx = \int_{\pi/6}^{\pi/2} (\csc^2 x - 1)\,dx = [-\cot x - x]_{\pi/6}^{\pi/2} = \sqrt{3} - \frac{\pi}{3}$$

26. $$\int \tan^4 x\,dx = \int \tan^2 x(\sec^2 x - 1)\,dx = \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx$$
$$= \frac{1}{3}\tan^3 x - \int (\sec^2 x - 1)\,dx = \frac{1}{3}\tan^3 x - \tan x + x + C$$

27. $$\int \cot^3 x \csc^3 x\,dx = \int (\csc^2 x - 1)\csc^3 x \cot x\,dx$$
$$= \int (\csc^4 x - \csc^2 x)\csc x \cot x\,dx$$
$$= -\frac{1}{5}\csc^5 x + \frac{1}{3}\csc^3 x + C$$

28. $$\int \tan^3 x \sec^3 x\,dx = \int (\sec^2 x - 1)\sec^2 x \sec x \tan x\,dx$$
$$= \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$$

29. $$\int \sin 5x \sin 2x\,dx = \int \frac{1}{2}(\cos 3x - \cos 7x)\,dx$$
$$= \tfrac{1}{6}\sin 3x - \tfrac{1}{14}\sin 7x + C$$

30. $$\int_0^{\pi/4} \sin 5x \cos 2x\,dx = \int_0^{\pi/4} \frac{1}{2}\left[\sin(5x-2x) + \sin(5x+2x)\right]\,dx$$
$$= \frac{1}{2}\int_0^{\pi/4} (\sin 3x + \sin 7x)\,dx = -\left[\frac{\cos 3x}{6} + \frac{\cos 7x}{14}\right]_0^{\pi/4} = \frac{5+\sqrt{2}}{21}$$

31. $$\int \csc^4 2x\,dx = \int (1+\cot^2 2x)\csc^2 2x\,dx = -\frac{1}{2}\cot 2x - \frac{1}{6}\cot^3 2x + C$$

32. $$\int \sec^4 3x\,dx = \int (1+\tan^2 3x)\sec^2 3x\,dx = \frac{1}{9}\tan^3 3x + \frac{1}{3}\tan 3x + C$$

33. $$\int \tan^5 3x\,dx = \int \tan^3 3x\,(\sec^2 3x - 1)\,dx$$
$$= \int \tan^3 3x \sec^2 3x\,dx - \int \tan^3 3x\,dx$$
$$= \int \tan^3 3x \sec^2 3x\,dx - \int (\tan 3x \sec^2 3x - \tan 3x)\,dx$$
$$= \tfrac{1}{12}\tan^4 3x - \tfrac{1}{6}\tan^2 3x + \tfrac{1}{3}\ln|\sec 3x| + C$$

34.

$$\begin{aligned}\int \cot^5 2x\,dx &= \int (\cot^3 2x \csc^2 2x - \cot^3 2x)\,dx\\ &= \int (\cot^3 2x \csc^2 2x - \cot 2x \csc^2 2x + \cot 2x)\,dx\\ &= -\frac{1}{8}\cot^4 2x + \frac{1}{4}\cot^2 2x + \frac{1}{2}\ln|\sin 2x| + C\end{aligned}$$

35.

$$\begin{aligned}\int_{-1/6}^{1/3} \sin^4 3\pi x\,\cos^3 3\pi x\,dx &= \int_{-1/6}^{1/3} \sin^4 3\pi x\,\cos^2 3\pi x\,\cos 3\pi x\,dx\\ &= \int_{-1/6}^{1/3} \sin^4 3\pi x\,(1 - \sin^2 3\pi x\,\cos 3\pi x\,dx\\ &= \int_{-1}^{0} u^4(1-u^2)\frac{1}{3\pi}\,du \quad [u = \sin 3\pi x,\quad du = 3\pi\,\cos 3\pi x\,dx]\\ &= \frac{1}{3\pi}\left[\tfrac{1}{5}u^5 - \tfrac{1}{7}u^7\right]_{-1}^{0} = \frac{2}{105\pi}\end{aligned}$$

36.

$$\begin{aligned}\int_0^{1/2} \cos\pi x \cos\frac{\pi}{2}x\,dx &= \frac{1}{2}\int_0^{1/2}\left[\cos\left(\pi x - \frac{\pi}{2}x\right) + \cos\left(\pi x + \frac{\pi}{2}x\right)\right]dx\\ &= \frac{1}{2}\int_0^{1/2}\left(\cos\frac{\pi}{2}x + \cos\frac{3\pi}{2}x\right)dx = \frac{1}{2}\left[\frac{2}{\pi}\sin\frac{\pi}{2}x + \frac{2}{3\pi}\sin\frac{3\pi}{2}x\right]_0^{1/2}\\ &= \frac{2\sqrt{2}}{3\pi}\end{aligned}$$

37.

$$\begin{aligned}\int_0^{\pi/4} \cos 4x \sin 2x\,dx &= \int_0^{\pi/4} \frac{1}{2}(\sin 6x - \sin 2x)\,dx\\ &= \left[-\tfrac{1}{12}\cos 6x + \tfrac{1}{4}\cos 2x\right]_0^{\pi/4} = -\tfrac{1}{6}\end{aligned}$$

38.

$$\begin{aligned}\int (\sin 3x - \sin x)^2\,dx &= \int (\sin^2 3x - 2\sin 3x \sin x + \sin^2 x)\,dx\\ &= \int (\frac{1}{2} - \frac{1}{2}\cos 6x - \cos 2x + \cos 4x + \frac{1}{2} - \frac{1}{2}\cos 2x)\,dx\\ &= x - \frac{1}{12}\sin 6x + \frac{1}{4}\sin 4x - \frac{3}{4}\sin 2x + C\end{aligned}$$

39.

$$\int \sec^5 x\,dx = \tan x \sec^3 x - 3\int \sec^3 x \tan^2 x\,dx$$

$u = \sec^3 x,$	$dv = \sec^2 x\,dx$
$du = 3\sec^3 x \tan x\,dx,$	$v = \tan x$

$$= \tan x \sec^3 x - 3\int \sec^5 x\,dx + 3\int \sec^3 x\,dx.$$

Rearranging terms, we get

$$4\int \sec^5 x\,dx = \tan x \sec^3 x + 3\int \sec^3 x\,dx.$$

We have already seen that

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

Therefore

$$\int \sec^5 x\,dx = \frac{1}{4}\tan x \sec^3 x + \frac{3}{8}\sec x \tan x + \frac{3}{8}\ln|\sec x + \tan x| + C.$$

40.

$$\int \csc^5 x\,dx = -\csc^3 x \cot x - \int 3\csc^3 x \cot^2 x\,dx$$

$u = \csc^3 x$	$dv = \csc^2 x, dx$
$du = -3\csc^3 x \cot x\,dx$	$v = -\cot x$

$$= -\csc^3 x \cot x - 3\int \csc^5 x\,dx + 3\int \csc^3 x\,dx$$

$$4\int \csc^5 x\,dx = -\csc^3 x \cot x + 3\int \csc^3 x\,dx = -\csc^3 x \cot x - \frac{3}{2}\csc x \cot x + \frac{3}{2}\ln|\csc x - \cot x|$$

$$\int \csc^5 x\,dx = -\frac{1}{4}\csc^3 x \cot x - \frac{3}{8}\csc x \cot x + \frac{3}{8}\ln|\csc x - \cot x| + C$$

41.

$$\int \tan^4 x \sec^4 x\,dx = \int \tan^4 x\,(\tan^2 x + 1)\sec^2 x\,dx$$
$$= \int (\tan^6 x + \tan^4 x)\sec^2 x\,dx$$
$$= \frac{1}{7}\tan^7 x + \frac{1}{5}\tan^5 x + C$$

42. $\int \cot^4 x \csc^4 x\,dx = \int \cot^4 x(\cot^2 x + 1)\csc^2 x\,dx = -\frac{1}{7}\cot^7 x - \frac{1}{5}\cot^5 x + C$

43.

$$\int \sin(x/2)\cos 2x\,dx = \int \tfrac{1}{2}(\sin\left(\tfrac{5}{2}x\right) - \sin\left(\tfrac{3}{2}x\right)\,dx$$
$$= \tfrac{1}{3}\cos\left(\tfrac{3}{2}x\right) - \tfrac{1}{5}\cos\left(\tfrac{5}{2}x\right) + C$$

44. $\int_0^{2\pi} \sin^2 ax\,dx = \int_0^{2\pi}\left(\frac{1}{2} - \frac{1}{2}\cos 2ax\right)dx = \left[\frac{x}{2} - \frac{1}{4a}\sin 2ax\right]_0^{2\pi} = \pi - \frac{\sin 4\pi a}{4a}$

45. $\left\{\begin{array}{ll} u = \tan x \mid x = 0 & \Longrightarrow \quad u = 0 \\ du = \sec^2 x\,dx \mid x = \pi/4 & \Longrightarrow \quad u = 1 \end{array}\right\};$

$$\int_0^{\pi/4} \tan^3 x\,\sec^2 x\,dx = \int_0^1 u^3\,du = \left[\tfrac{1}{4}u^4\right]_0^1 = \tfrac{1}{4}$$

46. $\displaystyle\int_{\pi/4}^{\pi/2} \csc^3 x \cot x\,dx = \int_{\pi/4}^{\pi/2} \csc^2 x \csc x \cot x\,dx = \left[-\frac{1}{3}\csc^3 x\right]_{\pi/4}^{\pi/2} = \frac{2\sqrt{2}-1}{3}$

47. $\displaystyle\int_0^{\pi/6} \tan^2 2x\,dx = \int_0^{\pi/6} \left(\sec^2 2x - 1\right)\,dx = \left[\tfrac{1}{2}\tan 2x - x\right]_0^{\pi/6} = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$

48. $\displaystyle\int_0^{\pi/3} \tan x \sec^{3/2} x\,dx = \int_0^{\pi/3} \sec^{1/2} x \sec x \tan x\,dx = \left[\frac{2}{3}\sec^{3/2} x\right]_0^{\pi/3} = \frac{4\sqrt{2}-2}{3}$

49. $\displaystyle A = \int_0^{\pi} \sin^2 x\,dx = \int_0^{\pi} \tfrac{1}{2}(1-\cos 2x)\,dx = \tfrac{1}{2}\left[x - \tfrac{1}{2}\sin 2x\right]_0^{\pi} = \frac{\pi}{2}$

50. $\displaystyle V = \int_{-\pi/2}^{\pi/2} \pi\cos^2 x\,dx = \pi\left[\frac{x}{2} + \frac{\sin 2x}{4}\right]_{-\pi/2}^{\pi/2} = \frac{\pi^2}{2}$

51.
$$\begin{aligned}
V = \int_0^{\pi} \pi\left(\sin^2 x\right)^2\,dx = \pi\int_0^{\pi} \sin^4 x\,dx &= \pi\int_0^{\pi} \left(\tfrac{1}{2}(1-\cos 2x)\right)^2\,dx \\
&= \frac{\pi}{4}\int_0^{\pi} \left(1 - 2\cos 2x + \cos^2 2x\right)\,dx \\
&= \frac{\pi}{4}\left[x - \sin 2x\right]_0^{\pi} + \frac{\pi}{8}\int_0^{\pi} (1+\cos 4x)\,dx \\
&= \frac{\pi^2}{4} + \frac{\pi}{8}\left[x + \tfrac{1}{4}\sin 4x\right]_0^{\pi} = \frac{3\pi^2}{8}
\end{aligned}$$

52. $\displaystyle V = \int_0^{\pi/4} \pi(\cos^2 x - \sin^2 x)\,dx = \pi\int_0^{\pi/4} \cos 2x\,dx = \frac{\pi}{2}\left[\sin 2x\right]_0^{\pi/4} = \frac{\pi}{2}$

53. $\displaystyle V = \int_0^{\pi/4} \pi\left[1^2 - \tan^2 x\right]\,dx = \pi\int_0^{\pi/4} \left[2 - \sec^2 x\right]\,dx = \pi\left[2x - \tan x\right]_0^{\pi/4} = \frac{\pi^2}{2} - \pi$

54. $\displaystyle V = \int_0^{\pi/4} \pi\tan^4 x\,dx = \pi\left[\frac{1}{3}\tan^3 x - \tan x + x\right]_0^{\pi/4} = \pi\left(\frac{\pi}{4} - \frac{2}{3}\right)$

55.
$$\begin{aligned}
V = \int_0^{\pi/4} \pi\left[(\tan x + 1)^2 - 1^2\right]\,dx &= \pi\int_0^{\pi/4} \left[\tan^2 x + 2\tan x\right]\,dx \\
&= \pi\int_0^{\pi/4} \left(\sec^2 x + 2\tan x - 1\right)\,dx \\
&= \pi\left[\tan x + 2\ln|\sec x| - x\right]_0^{\pi/4} = \pi\left[\ln 2 + 1 - \frac{\pi}{4}\right]
\end{aligned}$$

56. $\displaystyle V = \int_0^{\pi/4} \pi\sec^4 x\,dx = \pi\left[\frac{1}{3}\tan^3 x + \tan x\right]_0^{\pi/4} = \frac{4\pi}{3}$

57. Suppose $m \neq n$:

$$\begin{aligned}
\int \sin mx \sin nx\,dx &= \int \tfrac{1}{2}[\cos(m-n)x - \cos(m+n)x]\,dx \\
&= \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} + C
\end{aligned}$$

Now suppose that $m = n$:

$$\int \sin mx \sin nx\, dx = \int \sin^2 mx\, dx = \int \tfrac{1}{2}(1 - \cos 2mx)\, dx$$
$$= \frac{1}{2}\left(x - \frac{1}{2m}\sin 2mx\right) + C = \frac{x}{2} - \frac{\sin 2mx}{4m} + C$$

58. (a)

$$\int \sin mx \cos nx\, dx = \frac{1}{2}\int [\sin(mx - nx) + \sin(mx + nx)]\, dx$$
$$= \begin{cases} \dfrac{-\cos(m-n)x}{2(m-n)} - \dfrac{\cos(m+n)x}{2(m+n)} + C, & \text{if } m \neq n \\ -\dfrac{\cos 2mx}{4m} + C, & \text{if } m = n \end{cases}$$

(b)

$$\int \cos mx \cos nx\, dx = \frac{1}{2}\int [\cos(m-n)x + \cos(m+n)x]\, dx$$
$$= \begin{cases} \dfrac{\sin(m-n)x}{2(m-n)} + \dfrac{\sin(m+n)x}{2(m+n)} + C, & \text{if } m \neq n \\ \dfrac{x}{2} + \dfrac{\sin 2mx}{4m} + C, & \text{if } m = n \end{cases}$$

59. Suppose $m \neq n$:

$$\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = \left[\frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}\right]_{-\pi}^{\pi} = 0$$

Suppose $m = n$:

$$\int_{-\pi}^{\pi} \sin^2 mx\, dx = \left[\frac{x}{2} - \frac{\sin 2mx}{4m}\right]_{-\pi}^{\pi} = \pi$$

60. (a) Using Exercise 58 and the fact that $\cos x$ is an even function, $\int_{-\pi}^{\pi} \sin mx \cos nx\, dx = 0$

(b)

$$\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \begin{cases} 0, & \text{if } m \neq n \\ \pi, & \text{if } m = n \end{cases}$$

61. Let $u = \cos^{n-1} x$, $dv = \cos x\, dx$. Then $du = (n-1)\cos^{n-2} x(-\sin x)\, dx$, $v = \sin x$ and

$$\int \cos^n x\, dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x\, dx$$
$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x(1 - \cos^2 x)\, dx$$
$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx - (n-1)\int \cos^n x\, dx$$

Adding $(n-1)\int \cos^n x\,dx$ to both sides, we get

$$n\int \cos^n x\,dx = \cos^{n-1} x\,\sin x + (n-1)\int \cos^{n-2} x\,dx$$

so that

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\,\sin x + \frac{(n-1)}{n}\int \cos^{n-2} x\,dx$$

62. (a) $\displaystyle\int_0^{\pi/2} \sin^n x\,dx = \left[-\frac{1}{n}\sin^{n-1} x\cos x\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx = 0 + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$

(b) For $n=2$, $\displaystyle\int_0^{\pi/2} \sin^2 x\,dx = \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\pi/2} = \frac{\pi}{4} = \frac{2-1}{2}\cdot\frac{\pi}{2}$

For $n=3$, $\displaystyle\int_0^{\pi/2} \sin^3 x\,dx = \frac{2}{3}\int_0^{\pi/2} \sin x\,dx = \frac{2}{3}\left[-\cos x\right]_0^{\pi/2} = \frac{2}{3}$

The result then follows from (a) by induction.

(c) $\displaystyle\int_0^{\pi/2} \cos^n x\,dx = \int_0^{\pi/2} \sin^n\left(\frac{\pi}{2} - x\right)\,dx = -\int_{\pi/2}^{0} \sin^n u\,du = \int_0^{\pi/2} \sin^n u\,du$

$(u = \pi/2 - x)$

63. $\displaystyle\int_0^{\pi/2} \sin^7 x\,dx = \frac{6\cdot 4\cdot 2}{7\cdot 5\cdot 3} = \frac{16}{35}$

64. $\displaystyle\int_0^{\pi/2} \cos^6 x\,dx = \left(\frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\right)\frac{\pi}{2} = \frac{5/pi}{32}$

65.

$$\begin{aligned}\int \cot^n x\,dx &= \int \cot^{n-2} x\,\cot^2 x\,dx\\ &= \int \cot^{n-2} x\left(\csc^2 x - 1\right)\,dx\\ &= -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\,dx\end{aligned}$$

66. (a) $\displaystyle\int \cot^3 x\,dx = -\frac{\cot^2 x}{2} - \int \cot x\,dx = -\frac{\cot^2 x}{2} - \ln|\sin x| + C$

(b) $\displaystyle\int \cot^4 x\,dx = -\frac{\cot^3 x}{3} - \int \cot^2 x\,dx = -\frac{\cot^3 x}{3} + \cot x + x + C$

(c) $\displaystyle\int \cot^5 2x\,dx = -\frac{\cot^4 2x}{8} - \int \cot^3 2x\,dx = -\frac{\cot^4 2x}{8} + \frac{\cot^2 2x}{4} + \frac{1}{2}\ln|\sin x| + C$

67. $\displaystyle\int \csc^n x\,dx = \int \csc^{n-2} x\,\csc^2 x\,dx$

Let $u = csc^{n-2}x$, $dv = \csc^2 x\,dx$. Then $du = -(n-2)\csc^{n-2} x\,\cot x\,dx$, $v = -\cot x$ and

$$\begin{aligned}\int \csc^n x\,dx &= -(n-2)\csc^{n-2} x\,\cot x - (n-2)\int \csc^{n-2} x\,\cot^2 x\,dx\\ &= -\csc^{n-2} x\,\cot x - (n-2)\int \csc^{n-2} x\left(\csc^2 x - 1\right)\,dx\\ &= -\csc^{n-2} x\,\cot x - (n-2)\int \left(\csc^n x - \csc^{n-2} x\right)\,dx\end{aligned}$$

Adding $(n-2)\int \csc^n x\,dx$ to both sides, we get

$$(n-1)\int \csc^n x\,dx = -\csc^{n-2} x\,\cot x + (n-2)\int \csc^{n-2} x\,dx.$$

Thus,

$$\int \csc^n x\,dx = \frac{-\csc^{n-2} x\,\cot x}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2} x\,dx.$$

68. (a) $\displaystyle\int \csc^3 x\,dx = -\frac{\csc x\cot x}{2} + \frac{1}{2}\int \csc x\,dx = -\frac{\csc x\cot x}{2} + \frac{1}{2}\ln|\csc x - \cot x| + C$

(b) $\displaystyle\int \csc^4 x\,dx = -\frac{\csc^2 x\cot x}{3} + \frac{2}{3}\int \csc^2 x\,dx = -\frac{\csc^2 x\cot x}{3} - \frac{2}{3}\cot x + C$

(c) $\displaystyle\int \csc^5 3x\,dx = -\frac{\csc^3 3x\cot 3x}{12} + \frac{3}{4}\int \csc^3 3x\,dx$

$$= -\frac{\csc^3 3x\cot 3x}{12} - -\frac{3\csc 3x\cot 3x}{8} + \frac{3}{8}\ln|\csc x - \cot x| + C$$

SECTION 8.4

1. $\left\{\begin{array}{l} x = a\sin u \\ dx = a\cos u\,du \end{array}\right\}$;

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos u\,du}{a\cos u} = \int du = u + C = \sin^{-1}\left(\frac{x}{a}\right) + C$$

2. $\left\{\begin{array}{l} x = \sqrt{2}\tan u \\ dx = \sqrt{2}\sec^2 u\,du \end{array}\right\}$;

$$\int \frac{dx}{(x^2+2)^{3/2}} = \int \frac{\sqrt{2}\sec^2 u\,du}{(2\tan^2 u+2)^{3/2}} = \int \frac{\sqrt{2}\sec^2 u}{2\sqrt{2}\sec^3 u}\,du = \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + C = \frac{1}{2}\frac{x}{\sqrt{x^2+2}} + C$$

3. $\left\{\begin{array}{l} x = \sqrt{5}\sin u \\ dx = \sqrt{5}\cos u\,du \end{array}\right\}$;

$$\int \frac{dx}{(5-x^2)^{3/2}} = \int \frac{\sqrt{5}\cos u\,du}{(5\cos^2 u)^{3/2}} = \frac{1}{5}\int \sec^2 u\,du = \frac{1}{5}\tan u + C = \frac{x}{5\sqrt{5-x^2}} + C$$

$$\int_0^1 \frac{dx}{(5-x^2)^{3/2}} = \left[\frac{x}{5\sqrt{5-x^2}}\right]_0^1 = \frac{1}{10}$$

4. $\left\{\begin{array}{l} u = x^2-4 \\ du = 2x\,dx \end{array}\right\}$;

$$\int_{5/2}^4 \frac{x}{\sqrt{x^2-4}}\,dx = \frac{1}{2}\int_{9/4}^{12} u^{-1/2}\,du = \left[u^{1/2}\right]_{9/4}^{12} = \sqrt{12} - \frac{3}{2} = \frac{4\sqrt{3}-3}{2}$$

5. $\left\{\begin{aligned} x &= \sec u \\ dx &= \sec u \tan u\, du \end{aligned}\right\};$

$$\begin{aligned}\int \sqrt{x^2-1}\, dx &= \int \tan^2 u \sec u\, du \\ &= \int (\sec^3 u - \sec u)\, du \\ &= \frac{1}{2}\sec u \tan u - \frac{1}{2}\ln|\sec u + \tan u| + C\end{aligned}$$

Section 8.4

$$= \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln|x + \sqrt{x^2-1}| + C$$

6. $$\int \frac{x}{\sqrt{4-x^2}}\, dx = -\int \frac{-2x}{2\sqrt{4-x^2}}\, dx = -\int \frac{du}{2\sqrt{u}} = -\sqrt{u} + C = -\sqrt{4-x^2} + C$$

7. $\left\{\begin{aligned} x &= 2\sin u \\ dx &= 2\cos u\, du \end{aligned}\right\};$

$$\begin{aligned}\int \frac{x^2}{\sqrt{4-x^2}}\, dx &= \int \frac{4\sin^2 u}{2\cos u} 2\cos u\, du \\ &= 2\int (1-\cos 2u)\, du \\ &= 2u - \sin 2u + C \\ &= 2u - 2\sin u \cos u + C \\ &= 2\sin^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}x\sqrt{4-x^2} + C\end{aligned}$$

8. $\left\{\begin{aligned} x &= 2\sec u \\ dx &= 2\sec u \tan u\, du \end{aligned}\right\};$

$$\begin{aligned}\int \frac{x^2}{\sqrt{x^2-4}}\, dx &= \int \frac{8\sec^3 u \tan u}{\sqrt{4(\sec^2 u - 1)}}\, du \\ &= 4\int \sec^3 u\, du \\ &= 2\sec u \tan u + 2\ln|\sec u + \tan u| + C_1 \\ &= \frac{1}{2}x\sqrt{x^2-4} + 2\ln\left|\frac{x+\sqrt{x^2-4}}{2}\right| + C_1 \\ &= \frac{1}{2}x\sqrt{x^2-4} + 2\ln|x + \sqrt{x^2-4}| + C \quad (C = C_1 - 2\ln 2)\end{aligned}$$

9. $\left\{\begin{aligned} u &= 1-x^2 \\ du &= -2x\, dx \end{aligned}\right\};$ $$\int \frac{x}{(1-x^2)^{3/2}}\, dx = -\frac{1}{2}\int \frac{du}{u^{3/2}} = u^{-1/2} + C = \frac{1}{\sqrt{1-x^2}} + C$$

10. $\left\{\begin{aligned} x &= 2\tan u \\ dx &= 2\sec^2 u\, du \end{aligned}\right\};$

$$\begin{aligned}\int \frac{x^2}{\sqrt{4+x^2}}\, dx &= \int \frac{8\tan^2 u \sec^2 u}{\sqrt{4(1+\tan^2 u)}}\, du = 4\int \tan^2 u \sec u\, du \\ &= 4\int (\sec^3 u - \sec u)\, du \\ &= 4\left(\frac{1}{2}\sec u \tan u - \frac{1}{2}\ln|\sec u + \tan u|\right) + C \\ &= \frac{1}{2}x\sqrt{x^2+4} - 2\ln\left(x + \sqrt{x^2+4}\right) + C\end{aligned}$$

(absorbing $2\ln 2$ into C)

11. $\left\{\begin{aligned} x &= \sin u \\ dx &= \cos u\, du \end{aligned}\right\};$

$$\begin{aligned}\int \frac{x^2}{(1-x^2)^{3/2}}\, dx &= \int \frac{\sin^2 u}{\cos^3 u}\cos u\, du = \int \tan^2 u\, du \\ &= \int (\sec^2 u - 1)\, du = \tan u - u + C\end{aligned}$$

$$= \frac{x}{\sqrt{1-x^2}} - \sin^{-1} x + C$$

$$\int_0^{1/2} \frac{x^2}{(1-x^2)^{3/2}}\,dx = \left[\frac{x}{\sqrt{1-x^2}} - \sin^{-1} x\right]_0^{1/2} = \frac{2\sqrt{3}-\pi}{6}$$

12. $\left\{\begin{aligned} u &= a^2+x^2 \\ dx &= 2x\,dx \end{aligned}\right\}$; $\displaystyle\int \frac{x}{a^2+x^2}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C$

$$= \frac{1}{2}\ln(a^2+x^2) + C$$

13. $\left\{\begin{aligned} u &= 4-x^2 \\ du &= -2x\,dx \end{aligned}\right\}$; $\displaystyle\int x\sqrt{4-x^2}\,dx = -\frac{1}{2}\int u^{1/2}\,du = -\frac{1}{3}u^{3/2} + C$

$$= -\frac{1}{3}(4-x^2)^{3/2} + C$$

14. $\left\{\begin{aligned} x &= 4\sin u \\ dx &= 4\cos u\,du \end{aligned}\right\}$; $\displaystyle\int \frac{x^3}{\sqrt{16-x^2}}\,dx = \frac{256\sin^3 u\cos u}{\sqrt{16-16\sin^2 u}}\,du = 64\int \sin^3 u\,du$

$$= 64\int (1-\cos^2 u)\sin u\,du = 64\left(-\cos u + \frac{\cos^3 u}{3}\right) + C$$

$$= \frac{1}{3}\left(\sqrt{16-x^2}\right)^3 - 16\sqrt{16-x^2} + C$$

$$\int_0^2 \frac{x^3}{\sqrt{16-x^2}}\,dx = \left[\frac{1}{3}(16-x^2)^{3/2} + 64\sqrt{16-x^2}\right]_0^2 = \frac{128}{3} - 12\sqrt{12}$$

15. $\left\{\begin{aligned} x &= 5\sin u \mid x = 0 \implies u = 0 \\ dx &= 5\cos u\,du \mid x = 5 \implies u = \pi/2 \end{aligned}\right\}$;

$$\int_0^5 x^2\sqrt{25-x^2}\,dx = \int_0^{\pi/2} (5\sin u)^2(5\cos u)^2\,du$$

$$= 625\int_0^{\pi/2} \left(\sin^2 u - \sin^4 u\right)\,du$$

$$= 625\left[\frac{1}{2}\cdot\frac{\pi}{2} - \frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2}\right] = \frac{625\pi}{16}$$ [see Exercise 62, Section 8.3]

16. $\left\{\begin{aligned} u &= e^x \\ du &= e^x\,dx \end{aligned}\right\}$; $\displaystyle\int \frac{e^x}{\sqrt{9-e^{2x}}}\,dx = \int \frac{du}{\sqrt{9-u^2}}$

$$= \sin^{-1}\left(\frac{u}{3}\right) + C = \sin^{-1}\left(\frac{e^x}{3}\right) + C$$

17. $\left\{\begin{aligned} x &= \sqrt{8}\tan u \\ dx &= \sqrt{8}\sec^2 u\,du \end{aligned}\right\}$; $\displaystyle\int \frac{x^2}{(x^2+8)^{3/2}}\,dx = \int \frac{8\tan^2 u}{(8\sec^2 u)^{3/2}}\sqrt{8}\sec^2 u\,du$

$$= \int \frac{\tan^2 u}{\sec u}\,du = \int \frac{\sec^2 u - 1}{\sec u}\,du$$

$$= \int (\sec u - \cos u)\, du$$

$$= \ln|\sec u + \tan u| - \sin u + C$$

$$= \ln\left(\frac{\sqrt{x^2+8}+x}{\sqrt{8}}\right) - \frac{x}{\sqrt{x^2+8}} + C$$

(absorb $-\ln\sqrt{8}$ in C)

$$= \ln\left(\sqrt{x^2+8}+x\right) - \frac{x}{\sqrt{x^2+8}} + C$$

18. $\left\{\begin{array}{l} x = \sin u \\ dx = \cos u\, du \end{array}\right\}$; $\displaystyle\int \frac{\sqrt{1-x^2}}{x^4}\, dx = \int \frac{\cos^2 u}{\sin^4 u}\, du = \int \cot^2 u \csc^2 u\, du$

$$= -\frac{\cot^3 u}{3} + C = -\frac{(1-x^2)^{3/2}}{3x^3} + C$$

19. $\left\{\begin{array}{l} x = a\sin u \\ dx = a\cos u\, du \end{array}\right\}$; $\displaystyle\int \frac{dx}{x\sqrt{a^2-x^2}} = \int \frac{a\cos u\, du}{a\sin u\,(a\cos u)} = \frac{1}{a}\int \csc u\, du$

$$= \frac{1}{a}\ln|\csc u - \cot u| + C$$

$$= \frac{1}{a}\ln\left|\frac{a-\sqrt{a^2-x^2}}{x}\right| + C$$

20. $\left\{\begin{array}{l} x = a\tan u \\ dx = a\sec^2 u\, du \end{array}\right\}$; $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\sec^2 u\, du}{\sqrt{a^2(\tan^2 u + 1)}}$

$$= \int \sec u\, du$$

$$= \ln|\sec u + \tan u| + C$$

$$= \ln\left|x + \sqrt{x^2+a^2}\right| + C$$

21. $\left\{\begin{array}{l} x = a\sec u \\ dx = a\sec u \tan u\, du \end{array}\right\}$; $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sec u\tan u\, du}{a\tan u} = \int \sec u\, du$

$$= \ln|\sec u + \tan u| + C$$

$$= \ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + C$$

(absorb $-\ln a$ in C)

$$= \ln\left|x + \sqrt{x^2-a^2}\right| + C$$

22. $\displaystyle\int_{-a}^{a} \sqrt{a^2-x^2}\, dx = \frac{1}{2}(\text{Area of circle of radius } a) = \frac{\pi a^2}{2}$

23. $\left\{\begin{array}{l} x = 3\tan u \\ dx = 3\sec^2 u\,du \end{array}\right\};$ $\quad \displaystyle\int \frac{x^3}{\sqrt{9+x^2}}\,dx = \int \frac{27\tan^3 u}{3\sec u}\cdot 3\sec^2 u\,du$

$$= 27\int \tan^3 u\,\sec u\,du$$

$$= 27\int \left(\sec^2 u - 1\right)\sec u\,\tan u\,du$$

$$= 27\left[\tfrac{1}{3}\sec^3 u - \sec u\right] + C$$

$$= \tfrac{1}{3}\left(9+x^2\right)^{3/2} - 9\left(9+x^2\right)^{1/2} + C$$

$$\int_0^3 \frac{x^3}{\sqrt{9+x^2}}\,dx = \left[\tfrac{1}{3}\left(9+x^2\right)^{3/2} - 9\left(9+x^2\right)^{1/2}\right]_0^3 = 18 - 9\sqrt{2}$$

24. $\left\{\begin{array}{l} x = \sec u \\ dx = \sec u\tan u\,du \end{array}\right\};$ $\quad \displaystyle\int \frac{\sqrt{x^2-1}}{x}\,dx = \int \frac{\tan u}{\sec u}\sec u\tan u\,du$

$$= \int \tan^2 u\,du = \int \left(\sec^2 u - 1\right)du$$

$$= \tan u - u + C = \sqrt{x^2-1} - \tan^{-1}\sqrt{x^2-1} + C$$

25. $\left\{\begin{array}{l} x = a\tan u \\ dx = a\sec^2 u\,du \end{array}\right\};$ $\quad \displaystyle\int \frac{dx}{x^2\sqrt{a^2+x^2}} = \int \frac{a\sec^2 u\,du}{a^2\tan^2 u\,(a\sec u)}$

$$= \frac{1}{a^2}\int \frac{\sec u}{\tan^2 u}\,du$$

$$= \frac{1}{a^2}\int \cot u\csc u\,du$$

$$= -\frac{1}{a^2}\cos u + C = -\frac{1}{a^2x}\sqrt{a^2+x^2} + C$$

26. $\left\{\begin{array}{l} x = a\sin u \\ dx = a\cos u\,du \end{array}\right\};$ $\quad \displaystyle\int \frac{dx}{x^2\sqrt{a^2-x^2}} = \int \frac{a\cos u}{a^2\sin^2 u\;a\cos u}\,du$

$$= \frac{1}{a^2}\int \csc^2 u\,du = -\frac{1}{a^2}\cot u + C$$

$$= -\frac{\sqrt{a^2-x^2}}{a^2x^2} + C$$

27. $\left\{\begin{array}{l} x = a\sec u \\ dx = a\sec u\tan u\,du \end{array}\right\};$ $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \int \frac{a\sec u\tan u\,du}{a^2\sec^2 u\,(a\tan u)}$

$$= \frac{1}{a^2}\int \cos u\,du$$

$$= \frac{1}{a^2}\sin u + C$$

$$= \frac{1}{a^2 x}\sqrt{x^2 - a^2} + C$$

28. $\left\{\begin{aligned} e^x &= 2\tan u \\ e^x\,dx &= 2\sec^2 u\,du \end{aligned}\right\}$; $\displaystyle\int \frac{dx}{e^x\sqrt{4+e^{2x}}} = \int \frac{e^x}{e^{2x}\sqrt{4+e^{2x}}}\,dx = \int \frac{2\sec^2 u}{4\tan^2 u 2\sec u}\,du$

$$= \frac{1}{4}\int \frac{\cos u}{\sin^2 u}\,du = -\frac{1}{4}\cdot\frac{1}{\sin u} + C$$

$$= -\frac{1}{4}\frac{\sqrt{4+e^{2x}}}{e^x} + C$$

29. $\left\{\begin{aligned} e^x &= 3\sec u \\ e^x dx &= 3\sec u\tan u\,du \end{aligned}\right\}$; $\displaystyle\int \frac{dx}{e^x\sqrt{e^{2x}-9}} = \int \frac{\tan u\,du}{3\sec u\,(3\tan u)}$

$$= \frac{1}{9}\int \cos u\,du$$

$$= \tfrac{1}{9}\sin u + C$$

$$= \tfrac{1}{9}e^{-x}\sqrt{e^{2x}-9} + C$$

30. $\left\{\begin{aligned} x - 1 &= 2\sec u \\ dx &= 2\sec u\tan u\,du \end{aligned}\right\}$; $\displaystyle\int \frac{dx}{\sqrt{x^2-2x-3}} = \int \frac{dx}{\sqrt{(x-1)^2-4}}$

$$= \int \frac{2\sec u\tan u}{2\tan u}\,du$$

$$= \int \sec u\,du = \ln|\sec u + \tan u| + C$$

$$= \ln\left|x - 1 + \sqrt{(x-1)^2-4}\right| + C$$

31. $(x^2 - 4x + 4)^{\frac{3}{2}} = \begin{cases} (x-2)^3, & x > 2 \\ (2-x)^3, & x < 2 \end{cases}$

$$\int \frac{dx}{(x^2-4x+4)^{3/2}} = \begin{cases} -\dfrac{1}{2(x-2)^2} + C\,, & x > 2 \\ \dfrac{1}{2(2-x)^2} + C\,, & x < 2 \end{cases}$$

32. $\left\{\begin{aligned} x - 3 &= 3\sin u \\ dx &= 3\cos u\,du \end{aligned}\right\}$; $\displaystyle\int \frac{x}{\sqrt{6x-x^2}}\,dx = \int \frac{x}{\sqrt{9-(x-3)^2}}\,dx = \int \frac{3+3\sin u}{3\cos u}\cdot 3\cos u\,du$

$$= 3\int (1+\sin u)\,du = 3u - 3\cos u + C$$

$$= 3\sin^{-1}\left(\frac{x-3}{3}\right) - \sqrt{6x-x^2} + C$$

33. $\left\{\begin{aligned} x-3 &= \sin u \\ dx &= \cos u\, du \end{aligned}\right\}$; $\displaystyle\int x\sqrt{6x-x^2-8}\, dx = \int x\sqrt{1-(x-3)^2}\, dx$

$$= \int (3+\sin u)(\cos u)\cos u\, du$$

$$= \int (3\cos^2 u + \cos^2 u \sin u)\, du$$

$$= \int \left[3\left(\frac{1+\cos 2u}{2}\right) + \cos^2 u \sin u\right] du$$

$$= \frac{3u}{2} + \frac{3}{4}\sin 2u - \frac{1}{3}\cos^3 u + C$$

$$= \frac{3}{2}\sin^{-1}(x-3) + \frac{3}{2}(x-3)\sqrt{6x-x^2-8} - \frac{1}{3}(6x-x^2-8)^{3/2} + C$$

34. $\left\{\begin{aligned} x+2 &= 3\tan u \\ dx &= 3\sec^2 u\, du \end{aligned}\right\}$; $\displaystyle\int \frac{x+2}{\sqrt{x^2+4x+13}}\, dx = \int \frac{x+2}{\sqrt{(x+2)^2+9}}\, dx$

$$= \int \frac{3\tan u}{3\sec u}\cdot 3\sec^2 u\, du = 3\int \sec u \tan u\, du$$

$$= 3\sec u + C = \sqrt{x^2+4x+13} + C$$

35. $\left\{\begin{aligned} x+1 &= 2\tan u \\ dx &= 2\sec^2 u\, du \end{aligned}\right\}$; $\displaystyle\int \frac{x}{(x^2+2x+5)^2}\, dx = \int \frac{x}{[(x+1)^2+4]^2}\, dx$

$$= \int \frac{2\tan u - 1}{(4\sec^2 u)^2} 2\sec^2 u\, du$$

$$= \frac{1}{8}\int \frac{2\tan u - 1}{\sec^2 u}\, du$$

$$= \frac{1}{8}\int (2\sin u\cos u - \cos^2 u)\, du$$

$$= \frac{1}{8}\int \left(2\sin u \cos u - \frac{1+\cos 2u}{2}\right) du$$

$$= \frac{1}{8}\left(\sin^2 u - \frac{u}{2} - \frac{\sin 2u}{4}\right) + C$$

$$= \frac{1}{8}\left[\left(\frac{x+1}{\sqrt{x^2+2x+5}}\right)^2 - \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) - \frac{1}{4}\overbrace{(2)\left(\frac{x+1}{\sqrt{x^2+2x+5}}\right)\left(\frac{2}{\sqrt{x^2+2x+5}}\right)}^{\sin 2u = 2\sin u\cos u}\right] + C$$

$$= \frac{x^2+x}{8(x^2+2x+5)} - \frac{1}{16}\tan^{-1}\left(\frac{x+1}{2}\right) + C$$

36. $\left\{\begin{aligned} x-1 &= 2\sec u \\ dx &= 2\sec u\tan u\,du \end{aligned}\right\}$; $\displaystyle\int \frac{x}{\sqrt{x^2-2x-3}}\,dx = \int \frac{x}{\sqrt{(x-1)^2-4}}\,dx$

$$= \int \frac{1+2\sec u}{2\tan u}\cdot 2\sec u\tan u\,du = \int (\sec u + 2\sec^2 u)\,du$$

$$= \ln|\sec u + \tan u| + 2\tan u + C$$

$$= \ln\left|x-1+\sqrt{x^2-2x-3}\right| + \sqrt{x^2-2x-3} + C$$

37. $\left\{\begin{aligned} x-3 &= \sin u \\ dx &= \cos u\,du \end{aligned}\right\}$; $\displaystyle\int \sqrt{6x-x^2-8}\,dx = \int \sqrt{1-(x-3)^2}\,dx$

$$= \int \cos^2 u\,du$$

$$= \int \frac{1+\cos 2u}{2}\,du$$

$$= \frac{u}{2} + \frac{\sin 2u}{4} + C$$

$$= \tfrac{1}{2}\left[\sin^{-1}(x-3) + (x-3)\sqrt{6x-x^2-8}\right] + C$$

38. $\left\{\begin{aligned} x+3 &= 3\sec u \\ dx &= 3\sec u\tan u\,du \end{aligned}\right\}$; $\displaystyle\int x\sqrt{x^2+6x}\,dx = \int x\sqrt{(x+3)^2-9}\,dx$

$$= \int (-3+3\sec u)3\tan u\,3\sec u\tan u\,du$$

$$= 27\int (-\tan^2 u\sec u + \tan^2 u\sec^2 u)\,du$$

$$= 27\left[-\frac{1}{2}\sec u\tan u + \frac{1}{2}\ln|\sec u+\tan u| + \frac{1}{3}\tan^3 u\right] + C$$

$$= -\frac{3}{2}(x+3)\sqrt{x^2+6x} + \frac{27}{2}\ln\left|x+3+\sqrt{x^2+6x}\right| + \frac{1}{3}(x^2+6x)^{3/2} + C$$

39. Let $x = a\tan u$. Then $dx = a\sec^2 u\,du$, $\sqrt{x^2+a^2} = a\sec u$, and

$$\int \frac{dx}{(x^2+a^2)^n} = \int \frac{a\sec^2 u}{a^{2n}\sec^{2n} u}\,du = \frac{1}{a^{2n-1}}\int \cos^{2n-2} u\,du$$

40. $\displaystyle\int \frac{1}{(x^2+4)^4}\,dx = \frac{1}{2^7}\int \cos^6 u\,du = \frac{1}{2^7}\left[\frac{5}{16}u + \frac{1}{u}\sin 2u + \frac{3}{64}\sin 4u - \frac{1}{48}\sin^2 2u\right] + C$

$$= \frac{1}{2^7}\left[\frac{5}{16}u + \frac{1}{2}\sin u\cos u + \frac{3}{16}\sin u\cos u(\cos^2 u - \sin^2 u) - \frac{1}{12}\sin^2 u\cos^2 u\right] + C$$

$$= \frac{1}{27}\left[\frac{5}{16}\tan^{-1}\left(\frac{x}{2}\right) + \frac{x}{x^2+4} + \frac{3}{8}\cdot\frac{x(4-x^2)}{(x^2+4)^2} - \frac{1}{3}\cdot\frac{x^2}{(x^2+4)^2}\right] + C$$

41. $\displaystyle\int \frac{1}{(x^2+1)^3}\,dx = \int \cos^4 u\,du$

$$= \frac{1}{4}\cos^3 u\,\sin u + \frac{3}{8}\cos u\,\sin u + \frac{3}{8}u + C \quad \text{[by the reduction formula (8.3.2)]}$$

$$= \frac{3}{8}\tan^{-1} x + \frac{3x}{8(x^2+1)} + \frac{x}{4(x^2+1)^2} + C$$

42.

$u = \tan^{-1} x$	$dv = x\,dx$
$du = \dfrac{1}{1+x^2}\,dx$	$v = \dfrac{x^2}{2}$

$$\int x\tan^{-1} x\,dx = \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$$

$x = \tan u \quad dx = \sec^2 u\,du$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \frac{\tan^2 u}{1+\tan^2 u}\sec^2 u\,du$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \tan^2 u\,du$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\tan u + \frac{1}{2}u + C$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x + C$$

$$= \tan^{-1} x\left(\frac{x^2}{2} + \frac{1}{2}\right) - \frac{1}{2}x + C$$

43.

$u = \sin^{-1} x$	$dv = x\,dx$
$du = \dfrac{1}{\sqrt{1-x^2}}\,dx$	$v = \dfrac{x^2}{2}$

$$\int x\sin^{-1} x\,dx = \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx$$

$x = \sin u \quad dx = \cos u\,du$

$$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\int \sin^2 u\,du$$

$$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\left[\frac{1}{2}u - \frac{1}{4}\sin 2u\right] + C$$

$$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{4}\sin^{-1} x + \frac{1}{8}(2x\sqrt{1-x^2}) + C$$

$$= \sin^{-1} x\left(\frac{x^2}{2} - \frac{1}{4}\right) + \frac{1}{4}x\sqrt{1-x^2} + C$$

44.

$$\boxed{x = 3\sec u \quad dx = 3\sec u \tan u\, du} \qquad \int_3^5 \frac{\sqrt{x^2-9}}{x}\,dx = 3\int_0^b \tan^2 u\,du \quad (\text{where } b = \sec^{-1}\frac{5}{3})$$

$$= 3\left[\tan u - u\right]_0^b$$

$$= 4 - 3\sec^{-1}\frac{5}{3}$$

45.

$$V = \int_0^1 \pi\left(\frac{1}{1+x^2}\right)^2 dx = \pi\int_0^1 \frac{1}{(1+x^2)^2}\,dx$$

$$= \pi\int_0^{\pi/4} \cos^2 u\,du \qquad [x = \tan u, \quad \text{see Ex.45}]$$

$$= \frac{\pi}{2}\int_0^{\pi/4} (1+\cos 2u)\,du$$

$$= \frac{\pi}{2}\left[u + \tfrac{1}{2}\sin 2u\right]_0^{\pi/4} = \frac{\pi^2}{8} + \frac{\pi}{4}$$

46. $A = 2\int_0^{\sqrt{r^2-h^2}} \left(\sqrt{r^2-x^2} - h\right) dx = 2\left[\frac{1}{2}x\sqrt{r^2-x^2} + \frac{1}{2}r^2\sin^{-1}\left(\frac{x}{r}\right) - hx\right]_0^{\sqrt{r^2-h^2}}$

$= r^2\sin^{-1}\left(\frac{\sqrt{r^2-h^2}}{r}\right) - h\sqrt{r^2-h^2}$

47. We need only consider angles θ between 0 and π. Assume first that $0 \le \theta \le \frac{\pi}{2}$.

The area of the triangle is: $\frac{1}{2}r^2\sin\theta\,\cos\theta$.

The area of the other region is given by:

$$\int_{r\cos\theta}^{r} \sqrt{r^2-x^2}\,dx = \left[\frac{x}{2}\sqrt{r^2-x^2} + \frac{r^2}{2}\sin^{-1}\frac{x}{r}\right]_{r\cos\theta}^{r}$$

$$= \frac{\pi r^2}{4} - \frac{r^2}{2}\sin\theta\,\cos\theta - \frac{r^2}{2}\sin^{-1}(\cos\theta)$$

$$= \frac{r^2\theta}{2} - \frac{r^2}{2}\sin\theta\,\cos\theta$$

Thus, the area of the sector is $A = \frac{1}{2}r^2\theta$.

Now suppose that $\frac{\pi}{2} < \theta \le \pi$. Then $A = \frac{1}{2}\pi r^2 - \frac{1}{2}r^2(\pi - \theta) = \frac{1}{2}r^2\theta$.

48. $A = 4\int_0^b \frac{a}{b}\sqrt{b^2+y^2}\,dy = \frac{4a}{b}\left[\frac{y}{2}\sqrt{b^2+y^2} + \frac{b^2}{2}\ln\left(y+\sqrt{b^2+y^2}\right)\right]_0^b = 2ab\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]$

49. $A = 2\int_3^5 4\sqrt{\frac{x^2}{9}-1}\,dx = 24\int_1^{\frac{5}{3}} \sqrt{u^2-1}\,du \quad (\text{where } u = \tfrac{x}{3})$

$= 24\left[\frac{u}{2}\sqrt{u^2-1} - \frac{1}{2}\ln\left|u+\sqrt{u^2-1}\right|\right]_1^{\frac{5}{3}} = \frac{80}{3} - 12\ln 3.$

50. $V = 2\int_{b-a}^{b+a} 2\pi x\sqrt{a^2-(x-b)^2}\,dx = 4\pi\int_{-a}^{a}(u+b)\sqrt{a^2-u^2}\,du \quad (\text{where } u = x-b)$

$= 2\pi^2 a^2 b.$

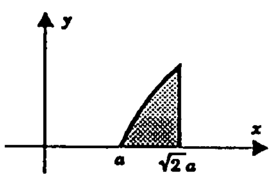
51. $M = \int_0^a \frac{dx}{\sqrt{x^2+a^2}} = \left[\ln\left(x+\sqrt{x^2+a^2}\right)\right]_0^a = \ln(1+\sqrt{2})$

$x_M M = \int_0^a \frac{x}{\sqrt{x^2+a^2}}\,dx = \left[\sqrt{x^2+a^2}\right]_0^a = (\sqrt{2}-1)a \qquad x_M = \frac{(\sqrt{2}-1)a}{\ln(1+\sqrt{2})}$

52. $M = \int_0^a (x^2+a^2)^{-3/2}\,dx = \left[\frac{1}{a^2}\sin u\right]_0^{\pi/4} = \frac{\sqrt{2}}{2a^2}$

$x_M\, M = \int_0^a \frac{x}{(x^2+a^2)^{3/2}}\,dx = \left[-(x^2+a^2)^{-1/2}\right]_0^a = \frac{2-\sqrt{2}}{2a} \implies x_M = (\sqrt{2}-1)a$

53.



$A = \int_a^{\sqrt{2}a} \sqrt{x^2-a^2}\,dx = \left[\frac{1}{2}x\sqrt{x^2-a^2} - \frac{1}{2}a^2\ln\left|x+\sqrt{x^2-a^2}\right|\right]_a^{\sqrt{2}a}$

$= \tfrac{1}{2}a^2[\sqrt{2} - \ln(\sqrt{2}+1)]$

$\overline{x}A = \int_a^{\sqrt{2}a} x\sqrt{x^2-a^2}\,dx = \frac{1}{3}a^3, \qquad \overline{y}A = \int_a^{\sqrt{2}a}\left[\frac{1}{2}(x^2-a^2)\right]dx = \frac{1}{6}a^3(2-\sqrt{2})$

$\overline{x} = \frac{2a}{3[\sqrt{2}-\ln(\sqrt{2}+1)]}, \qquad \overline{y} = \frac{(2-\sqrt{2})a}{3[\sqrt{2}-\ln(\sqrt{2}+1)]}$

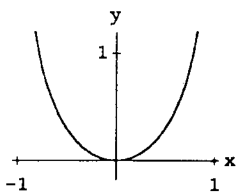
54. Using $\overline{y}$ and A found in Exercise 53,

$V_x = 2\pi\overline{y}A = 2\pi\frac{(2-\sqrt{2})a}{3\left[\sqrt{2}-\ln(\sqrt{2}+1)\right]}\cdot\frac{1}{2}a^2\left[\sqrt{2}-\ln(\sqrt{2}+1)\right] = \frac{1}{3}\pi a^3(2-\sqrt{2})$

$\overline{x}V_x = \int_a^{\sqrt{2}a} \pi x(x^2-a^2)\,dx = \frac{1}{4}\pi a^4 \implies \overline{x} = \frac{3a}{4(2-\sqrt{2})} = \frac{3}{8}a(2+\sqrt{2})$

55. $V_y = 2\pi\overline{R}A = \frac{2}{3}\pi a^3 \qquad \overline{y}V_y = \int_a^{\sqrt{2}a} \pi x(x^2-a^2)\,dx = \frac{1}{4}\pi a^4, \quad \overline{y} = \frac{3}{8}a$

56. (a)



(b) $A = \int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}\,dx$

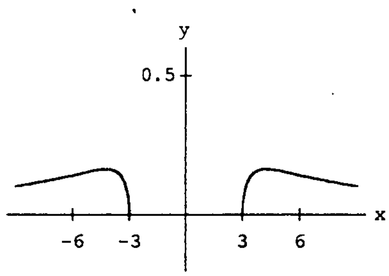
$= \int_0^{\pi/6} \sin^2 u\,du = \left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_0^{\pi/6} = \frac{2\pi - 3\sqrt{3}}{24}$

(c) $V = \int_0^{1/2} \frac{\pi x^4}{1-x^2}\,dx = \int_0^{\pi/6} \frac{\pi \sin^4 u}{\cos^2 u} \cos u\,du$

$= \pi \int_0^{\pi/6} (\sec u - \cos u - \sin^2 u \cos u)\,du$

$= \pi \left[\ln|\sec u + \tan u| - \sin u - \frac{1}{3}\sin^3 u\right]_0^{\pi/6} = \pi\left(\ln\sqrt{3} - \frac{13}{24}\right)$

57. (a)



(b) $A = \int_3^6 \frac{\sqrt{x^2-9}}{x^2}\,dx$

$= \int_0^{\pi/3} \frac{3\tan u}{9\sec^2 u} \cdot 3\sec u \tan u\,du \quad [x = 3\sec u]$

$= \int_0^{\pi/3} (\sec u - \cos u)\,du$

$= [\ln|\sec u + \tan u| - \sin u]_0^{\pi/3}$

$= \ln(2+\sqrt{3}) - \frac{\sqrt{3}}{2}$

(c) $\bar{x}A = \int_3^6 x \cdot \frac{\sqrt{x^2-9}}{x^2}\,dx = \int_0^{\pi/3} \left(3\sec^2 u - 3\right)\,du = [3\tan u - 3u]_0^{\pi/3} = 3\sqrt{3} - \pi$

Thus, $\bar{x} = \frac{2\left(3\sqrt{3} - \pi\right)}{2\ln\left(2+\sqrt{3}\right) - \sqrt{3}}$.

$\bar{y}A = \int \frac{1}{2}\left(\frac{\sqrt{x^2-9}}{x^2}\right)^2 dx = \frac{1}{2}\int_3^6 \left(\frac{1}{x^2} - \frac{9}{x^4}\right) dx = \frac{1}{2}\left[-\frac{1}{x} + \frac{3}{x^3}\right]_3^6 = \frac{5}{144}$

Thus, $\bar{y} = \frac{5}{72\left[2\ln\left(2+\sqrt{3}\right) - \sqrt{3}\right]}$.

SECTION 8.5

1. $\frac{1}{x^2+7x+6} = \frac{1}{(x+1)(x+6)} = \frac{A}{x+1} + \frac{B}{x+6}$

$1 = A(x+6) + B(x+1)$

$x = -6{:}\quad 1 = -5B \implies B = -1/5$

$x = -1{:}\quad 1 = 5A \implies A = 1/5$

$\frac{1}{x^2+7x+6} = \frac{1/5}{x+1} - \frac{1/5}{x+6}$

2. $$\frac{x^2}{(x-1)(x^2+4x+5)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4x+5}$$

$x^2 = A(x^2+4x+5) + (Bx+C)(x-1)$

$x = 1: \quad 1 = 10A \implies A = 1/10$

$x = 0: \quad 0 = 5A - C \implies C = 5A = 1/2$

Coefficient of x^2 is $\quad A + B = 1 \implies B = 1 - A = 9/10$

$$R(x) = \frac{\frac{1}{10}}{x-1} + \frac{\frac{9}{10}x + \frac{1}{2}}{x^2+4x+5}$$

3. $$\frac{x}{x^4-1} = \frac{x}{(x^2+1)(x+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1}$$

$x = (Ax+B)(x^2-1) + C(x-1)(x^2+1) + D(x+1)(x^2+1)$

$x = 1: \quad 1 = 4D \implies D = 1/4$

$x = -1: \quad -1 = -4C \implies C = 1/4$

$x = 0: \quad -B - C + D = 0 \implies B = 0$

$x = 2: \quad 6A + 5C + 15D = 2 \implies A = -1/2$

$$\frac{x}{x^4-1} = \frac{1/4}{x-1} + \frac{1/4}{x+1} - \frac{x/2}{x^2+1}$$

4. $$\frac{x^4}{(x-1)^3} = \frac{[(x-1)+1]^4}{(x-1)^3} = \frac{(x-1)^4 + 4(x-1)^3 + 6(x-1)^2 + 4(x-1) + 1}{(x-1)^3}$$

$$= x + 3 + \frac{6}{x-1} + \frac{4}{(x-1)^2} + \frac{1}{(x-1)^3}$$

5. $$\frac{x^2-3x-1}{x^3+x^2-2x} = \frac{x^2-3x-1}{x(x+2)(x+1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-1}$$

$x^2 - 3x - 1 = A(x+2)(x-1) + Bx(x-1) + Cx(x+2)$

$x = 0: \quad -1 = -2A \implies A = 1/2$

$x = -2: \quad 9 = 6B \implies B = 3/2$

$x = 1: \quad -3 = 3C \implies C = -1$

$$\frac{x^2-3x-1}{x^3+x^2-2x} = \frac{1/2}{x} + \frac{3/2}{x+2} - \frac{1}{x-1}$$

6. $$\frac{x^3+x^2+x+2}{x^4+3x^2+2} = \frac{x^3+x^2+x+2}{(x^2+1)(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$$

$x^3+x^2+x+2 = (Ax+B)(x^2+2) + (Cx+D)(x^2+1) = (A+C)x^3 + (B+D)x^2 + (2A+C)x + 2B + D$

$A + C = 1 = 2A + 1 \implies A = 0, \quad C = 1; \quad B + D = 1, \quad 2B + D = 2 \implies B = 1, \quad D = 0$

$$R(x) = \frac{1}{x^2+1} + \frac{x}{x^2+2}$$

7. $$\frac{2x^2+1}{x^3-6x^2+11x-6} = \frac{2x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

$2x^2 + 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$

$x = 1: \quad 3 = 2A \implies A = 3/2$

$x = 2: \quad 9 = -B \implies B = -9$

$x = 3: \quad 19 = 2C \implies C = 19/2$

$$\frac{2x^2+1}{x^3-6x^2+11x-6} = \frac{3/2}{x-1} - \frac{9}{x-2} + \frac{19/2}{x-3}$$

8. $$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x$

Constant term: A=1, $\quad x^4$ term: $0 = A + B \implies B = -1$

x^3 term: $0 = C$, $\quad x^2$ term: $0 = 2A + B + D \implies D = -1$

x term: $0 = E$

$$R(x) = \frac{1}{x} - \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}$$

9. $$\frac{7}{(x-2)(x+5)} = \frac{A}{x-2} + \frac{B}{x+5}$$

$$7 = A(x+5) + B(x-2)$$

$x = -5: \quad 7 = -7B \implies B = -1$

$x = 2: \quad 7 = 7A \implies A = 1$

$$\int \frac{7}{(x-2)(x+5)}\,dx = \int\left(\frac{1}{x-2} - \frac{1}{x+5}\right) = \ln|x-2| - \ln|x+5| + C = \ln\left|\frac{x-2}{x+5}\right| + C$$

10. $$\int \frac{x}{(x+1)(x+2)(x+3)}\,dx = \int\left(\frac{-1/2}{x+1} + \frac{2}{x+2} - \frac{3/2}{x+3}\right)dx$$
$$= -\frac{1}{2}\ln|x+1| + 2\ln|x+2| - \frac{3}{2}\ln|x+3| + C$$

11. $$\frac{2x^2+3}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$$2x^2 + 3 = Ax(x-1) + B(x-1) + Cx^2$$

$x = 0: \quad 3 = -B \implies B = -3$

$x = 1: \quad 5 = C \implies C = 5$

$x = -1: \quad 5 = 2A - 2B + C \implies A = -3$

$$\int \frac{2x^2+3}{x^2(x-1)}\,dx = \int\left(-\frac{3}{x} - \frac{3}{x^2} + \frac{5}{x-1}\right)dx = -3\ln|x| + \frac{3}{x} + 5\ln|x-1| + C$$

12. $$\int \frac{x^2+1}{x(x^2-1)}\,dx = \int\left(\frac{-1}{x} + \frac{1}{x-1} + \frac{1}{x+1}\right)dx = -\ln|x| + \ln|x-1| + \ln|x+1| + C = \ln\left|\frac{x^2-1}{x}\right| + C$$

13. We carry out the division until the numerator has degree smaller than the denominator:

$$\frac{x^5}{(x-2)^2} = \frac{x^5}{x^2-4x+4} = x^3+4x^2+12x+32+\frac{80x-128}{(x-2)^2}.$$

Then,

$$\frac{80x-128}{(x-2)^2} = \frac{80x-160+32}{(x-2)^2} = \frac{80}{x-2}+\frac{32}{(x-2)^2}$$

$$\begin{aligned}\int \frac{x^5}{(x-2)^2}\,dx &= \int\left(x^3+4x^2+12x+32+\frac{80}{x-2}+\frac{32}{(x-2)^2}\right)dx\\ &= \frac{1}{4}x^4+\frac{4}{3}x^3+6x^2+32x+80\ln|x-2|-\frac{32}{x-2}+C.\end{aligned}$$

14. $$\begin{aligned}\int \frac{x^5}{x-2}\,dx &= \int\left(x^4+2x^3+4x^2+8x+16+\frac{32}{x-2}\right)dx\\ &= \frac{1}{5}x^5+\frac{1}{2}x^4+\frac{4}{3}x^3+4x^2+16x+32\ln|x-2|+C\end{aligned}$$

15. $$\frac{x+3}{x^2-3x+2} = \frac{A}{x-1}+\frac{B}{x-2}$$

$$x+3 = A(x-2)+B(x-1)$$

$x=1$: $4=-A \implies A=-4$
$x=2$: $5=B \implies B=5$

$$\int \frac{x+3}{x^2-3x+2}\,dx = \int\left(\frac{-4}{x-1}+\frac{5}{x-2}\right)dx = -4\ln|x-1|+5\ln|x-2|+C$$

16. $$\int \frac{x^2+3}{x^2-3x+2}\,dx = \int\left(1+\frac{7}{x-2}-\frac{4}{x-1}\right)dx = x-7\ln|x-2|-4\ln|x-1|+C$$

17. $$\int \frac{dx}{(x-1)^3} = \int (x-1)^{-3}\,dx = -\frac{1}{2}(x-1)^{-2}+C = -\frac{1}{2(x-1)^2}+C$$

18. $$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x+1)^2+1} = \tan^{-1}(x+1)+C$$

19. $$\frac{x^2}{(x-1)^2(x+1)} = \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}$$

$$x^2 = A(x-1)(x+1)+B(x+1)+C(x-1)^2$$

$x=1$: $1=2B \implies B=1/2$
$x=-1$: $1=4C \implies C=1/4$

$x=0:\quad 0=-A+B+C \implies A=3/4$

$$\begin{aligned}\int \frac{x^2}{(x-1)^2(x+1)}\,dx &= \int\left(\frac{3/4}{x-1}+\frac{1/2}{(x-1)^2}+\frac{1/4}{x+1}\right)dx\\ &= \frac{3}{4}\ln|x-1|-\frac{1}{2(x-1)}+\frac{1}{4}\ln|x+1|+C\end{aligned}$$

20. $\displaystyle\int \frac{2x-1}{(x+1)^2(x-2)^2}\,dx = \int\left[\frac{-1/3}{(x+1)^2}+\frac{1/3}{(x-2)^2}\right]dx = \frac{1}{3}\left(\frac{1}{x+1}-\frac{1}{x-2}\right)+C$

21.

$$x^4-16=(x^2-4)(x^2+4)=(x-2)(x+2)(x^2+4)$$

$$\frac{1}{x^4-16}=\frac{A}{x-2}+\frac{B}{x+2}+\frac{Cx+D}{x^2+4}$$

$$1=A(x+2)(x^2+4)+B(x-2)(x^2+4)+(Cx+D)(x^2-4)$$

$x=2:\quad 1=32A \implies A=1/32$
$x=-2:\quad 1=-32B \implies B=-1/32$
$x=0:\quad 1=8A-8B-4D \implies D=-1/8$
$x=1:\quad 1=15A-5B-3C-3D \implies C=0$

$$\begin{aligned}\int\frac{dx}{x^4-16} &= \int\left(\frac{1/32}{x-2}-\frac{1/32}{x+2}-\frac{1/8}{x^2+4}\right)dx\\ &= \frac{1}{32}\ln|x-2|-\frac{1}{32}\ln|x+2|-\frac{1}{8}\left(\frac{1}{2}\tan^{-1}\frac{x}{2}\right)+C\\ &= \frac{1}{32}\ln\left|\frac{x-2}{x+2}\right|-\frac{1}{16}\tan^{-1}\frac{x}{2}+C\end{aligned}$$

22.

$$\begin{aligned}\int\frac{x}{x^3-1}\,dx &= \frac{1}{3}\int\left(\frac{1}{x-1}+\frac{1-x}{x^2+x+1}\right)dx\\ &= \frac{1}{3}\int\frac{1}{x-1}\,dx-\frac{1}{6}\int\frac{2x+1}{x^2+x+1}\,dx+\frac{1}{2}\int\frac{1}{x^2+x+1}\,dx\\ &= \frac{1}{3}\ln|x-1|-\frac{1}{6}\ln|(x^2+x+1)+\frac{1}{2}\int\frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx\\ &= \frac{1}{6}\ln\left(\frac{x^2-2x+1}{x^2+x+1}\right)+\frac{1}{\sqrt{3}}\tan^{-1}\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]+C\end{aligned}$$

23.

$$\frac{x^3+4x^2-4x-1}{(x^2+1)^2}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}$$

$$x^3+4x^2-4x-1=(Ax+B)(x^2+1)+(Cx+D)$$

$$\begin{array}{llll}
x=0: & -1=B+D & \implies D=-B-1 & \\
x=1: & 0=2A+2B+C+D & \implies 6=4B+2D & \implies B=4,\ D=-5 \\
x=-1: & 6=-2A+2B-C+D & 6=-2A+8-C-5 & \\
x=2: & 15=10A+5B+2C+D & 15=10A+20+2C-5 & \implies A=1,\ C=-5
\end{array}$$

$$\int \frac{x^3+4x^2-4x-1}{(x^2+1)^2}\,dx = \int\left(\frac{x}{x^2+1}+\frac{4}{x^2+1}-\frac{5x}{(x^2+1)^2}-\frac{5}{(x^2+1)^2}\right)dx$$

$$(*) \qquad = \frac{1}{2}\ln(x^2+1)+4\tan^{-1}x+\frac{5}{2(x^2+1)}-5\int\frac{dx}{(x^2+1)^2}$$

For this last integral we set

$$\left\{\begin{array}{l} x=\tan u \\ dx=\sec^2 u\,du\end{array}\right\};\quad \int\frac{dx}{(x^2+1)^2}=\int\frac{\sec^2 u\,du}{(1+\tan^2 u)^2}=\int\cos^2 u\,du$$

$$=\frac{1}{2}\int(1+\cos 2u)\,du$$

$$=\tfrac{1}{2}\left(u+\tfrac{1}{2}\sin 2u\right)+C=\tfrac{1}{2}(u+\sin u\cos u)+C$$

$$=\frac{1}{2}\left(\tan^{-1}x+\frac{x}{1+x^2}\right)+C.$$

Substituting this result in (*) and rearranging the terms, we get

$$\int\frac{x^3+4x^2-4x+1}{(x^2+1)^2}\,dx=\frac{1}{2}\ln(x^2+1)+\frac{3}{2}\tan^{-1}x+\frac{5(1-x)}{2(1+x^2)}+C.$$

24.

$$\int\frac{dx}{(x^2+16)^2}=\frac{1}{64}\int\cos^2 u\,du=\frac{1}{128}\left[u+\frac{\sin 2u}{2}\right]+C$$

$$=\frac{1}{128}\tan^{-1}\left(\frac{x}{4}\right)+\frac{1}{32}\cdot\frac{x}{(x^2+16)}+C$$

25.

$$\frac{1}{x^4+4}=\frac{Ax+B}{x^2+2x+2}+\frac{Cx+D}{x^2-2x+2}\qquad\text{(using the hint)}$$

$$1=(Ax+B)(x^2-2x+2)+(Cx+D)(x^2+2x+2)$$

$$\left.\begin{array}{ll}
x=0: & 1=2B+2D \\
x=1: & 1=A+B+5C+5D \\
x=-1: & 1=-5A+5B-C+D \\
x=2: & 1=4A+2B+20C+10D
\end{array}\right\}\implies \begin{array}{l} A=1/8 \\ B=1/4 \\ C=-1/8 \\ D=1/4\end{array}$$

$$\int \frac{dx}{x^4+4} = \frac{1}{8}\int \frac{x+2}{x^2+2x+2}\,dx - \frac{1}{8}\int \frac{x-2}{x^2-2x+2}\,dx$$

$$= \frac{1}{8}\int \frac{x+1}{x^2+2x+2}\,dx + \frac{1}{8}\int \frac{dx}{(x+1)^2+1} - \frac{1}{8}\int \frac{x-1}{x^2-2x+2}\,dx + \frac{1}{8}\int \frac{dx}{(x-1)^2+1}$$

$$= \tfrac{1}{16}\ln(x^2+2x+2) + \tfrac{1}{8}\tan^{-1}(x+1) - \tfrac{1}{16}\ln(x^2-2x+2) + \tfrac{1}{8}\tan^{-1}(x-1) + C$$

$$= \frac{1}{16}\ln\left(\frac{x^2+2x+2}{x^2-2x+2}\right) + \frac{1}{8}\tan^{-1}(x+1) + \frac{1}{8}\tan^{-1}(x-1) + C$$

26.
$$\int \frac{dx}{x^4+6} = \frac{\sqrt{2}}{64}\int\left(\frac{2x+4\sqrt{2}}{x^2+2\sqrt{2}x+4} - \frac{2x-4\sqrt{2}}{x^2-2\sqrt{2}x+4}\right)dx$$

$$= \frac{\sqrt{2}}{64}\int\left(\frac{2x+2\sqrt{2}}{x^2+2\sqrt{2}x+4} + \frac{2\sqrt{2}}{x^2+2\sqrt{2}x+4}\right)dx$$

$$- \frac{\sqrt{2}}{64}\int\left(\frac{2x-2\sqrt{2}}{x^2-2\sqrt{2}x+4} - \frac{2\sqrt{2}}{x^2-2\sqrt{2}x+4}\right)dx$$

$$= \frac{\sqrt{2}}{64}\ln|x^2+2\sqrt{2}x+4| + \frac{1}{16}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+\sqrt{2}}{\sqrt{2}}\right)$$

$$- \frac{\sqrt{2}}{64}\ln|x^2-2\sqrt{2}x+4| + \frac{1}{16}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-\sqrt{2}}{\sqrt{2}}\right) + C$$

$$= \frac{\sqrt{2}}{64}\ln\left(\frac{x^2+2\sqrt{2}x+4}{x^2-2\sqrt{2}x+4}\right) + \frac{\sqrt{2}}{32}\tan^{-1}\left(\frac{x+\sqrt{2}}{\sqrt{2}}\right) + \frac{\sqrt{2}}{32}\tan^{-1}\left(\frac{x-\sqrt{2}}{\sqrt{2}}\right) + C$$

27.
$$\frac{x-3}{x^3+x^2} = \frac{x-3}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$x-3 = Ax(x+1) + B(x+1) + Cx^2$$

$x=0$: $\quad -3 = B$

$x=-1$: $\quad -4 = C$

$x=1$: $\quad -2 = 2A + 2B + C \quad \Longrightarrow \quad A = 4$

$$\int \frac{x-3}{x^3+x^2}\,dx = 4\int \frac{1}{x}\,dx - 3\int \frac{1}{x^2}\,dx - 4\int \frac{1}{x+1}\,dx$$

$$= 4\ln|x| + \frac{3}{x} - 4\ln|x+1| + C = \frac{3}{x} + 4\ln\left|\frac{x}{x+1}\right| + C$$

28. $$\int \frac{1}{(x-1)(x^2+1)^2}\,dx$$
$$= \frac{1}{4}\int\left[\frac{1}{x-1}-\frac{x+1}{x^2+1}-\frac{2x+2}{(x^2+1)^2}\right]dx$$
$$= \frac{1}{4}\int\left[\frac{1}{x-1}-\frac{x}{x^2+1}-\frac{1}{x^2+1}-\frac{2x}{(x^2+1)^2}-\frac{2}{(x^2+1)^2}\right]dx$$
$$= \frac{1}{4}\ln|x-1|-\frac{1}{8}\ln(x^2+1)-\frac{1}{4}\tan^{-1}x+\frac{1}{4}\cdot\frac{1}{(x^2+1)}+\frac{1}{4}\left(\tan^{-1}x+\frac{x}{x^2+1}\right)+C$$
$$= \frac{1}{8}\ln\left(\frac{x^2-2x-1}{x^2+1}\right)+\frac{x+1}{4(x^2+1)}+C$$

29. $$\frac{x+1}{x^3+x^2-6x}=\frac{x+1}{x(x-2)(x+3)}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+3}$$
$$x+1=A(x-2)(x+3)+Bx(x+3)+Cx(x-2)$$

$x=0$: $1=-6A \implies A=-1/6$

$x=2$: $3=10B \implies B=3/10$

$x=-3$: $-2=15C \implies C=-2/15$

$$\int\frac{x+1}{x^3+x^2-6x}\,dx=-\tfrac{1}{6}\int\tfrac{1}{x}\,dx+\tfrac{3}{10}\int\tfrac{1}{x-2}\,dx-\tfrac{2}{15}\int\tfrac{1}{x+3}\,dx$$
$$=-\tfrac{1}{6}\ln|x|+\tfrac{3}{10}\ln|x-2|-\tfrac{2}{15}\ln|x+3|+C$$

30. $$\int\frac{x^3+x^2+x+3}{(x^2+1)(x^2+3)}\,dx=\int\left(\frac{1}{x^2+1}+\frac{x}{x^2+3}\right)dx=\tan^{-1}x+\frac{1}{2}\ln(x^2+3)+C$$

31. $$\int_0^2\frac{x}{x^2+5x+6}\,dx=\int_0^2\frac{x}{(x+2)(x+3)}\,dx=\int_0^2\left(\frac{3}{x+3}-\frac{2}{x+2}\right)dx$$
$$=[3\ln|x+3|-2\ln|x+2|]_0^2=\ln\left(\frac{125}{108}\right)$$

32. $$\int_1^3\frac{1}{x^3+x}\,dx=\int_1^3\left(\frac{1}{x}-\frac{x}{x^2+1}\right)dx=\left[\ln|x|-\frac{1}{2}\ln(x^2+1)\right]_1^3=\ln\left(\frac{3}{\sqrt{5}}\right)$$

33. $$\int_1^3\frac{x^2-4x+3}{x^3+2x^2+x}\,dx=\int_1^3\frac{x^2-4x+3}{x(x+1)^2}\,dx=\int_1^3\left(\frac{3}{x}-\frac{2}{x+1}-\frac{8}{(x+1)^2}\right)dx$$
$$=\left[3\ln|x|-2\ln|x+1|+\frac{8}{x+1}\right]_1^3=\ln\left(\frac{27}{4}\right)-2$$

34. $$\int_0^2\frac{x^3}{(x^2+2)^2}\,dx=\int_0^2\left(\frac{x}{x^2+2}-\frac{2x}{(x^2+2)^2}\right)dx=\left[\frac{1}{2}\ln(x^2+2)+\frac{1}{x^2+2}\right]_0^2=\ln\sqrt{3}-\frac{1}{3}$$

35.

$$\begin{aligned}\int \frac{\cos\theta}{sin^2\theta - 2\sin\theta - 8}\,d\theta &= \frac{1}{6}\int \frac{cos\theta}{\sin\theta - 4}\,d\theta - \frac{1}{6}\int \frac{cos\theta}{\sin\theta + 2}\,d\theta \\ &= \frac{1}{6}\ln|\sin\theta - 4| - \frac{1}{6}\ln|\sin\theta + 2| + C \\ &= \frac{1}{6}\ln\left|\frac{\sin\theta - 4}{\sin\theta + 2}\right| + C\end{aligned}$$

36.

$$\begin{aligned}\int \frac{e^t}{e^{2t} + 5e^t + 6}\,dt &= \int \frac{e^t}{e^t + 2}\,dt - \int \frac{e^t}{e^t + 3}\,dt \\ &= \ln|e^t + 2| - \ln|e^t + 3| + C \\ &= \ln\left|\frac{e^t + 2}{e^t + 3}\right| + C\end{aligned}$$

37.

$$\begin{aligned}\int \frac{1}{t[(\ln t)^2 - 4]}\,dt &= \frac{1}{4}\int \frac{1}{t(\ln t - 2}\,dt - \frac{1}{4}\int \frac{1}{t(\ln t + 2}\,dt \\ &= \frac{1}{4}\ln|\ln t - 2| - \frac{1}{4}\ln|\ln t + 2| + C \\ &= \frac{1}{4}\ln\left|\frac{\ln t - 2}{\ln t + 2}\right| + C\end{aligned}$$

38.

$$\begin{aligned}\int \frac{\sec^2\theta}{\tan^3\theta - \tan^2\theta}\,d\theta &= \int \frac{\sec^2\theta}{\tan\theta - 1}\,d\theta - \int \frac{\sec^2\theta}{\tan\theta}\,d\theta - \int \frac{\sec^2\theta}{\tan^2\theta}\,d\theta \\ &= \ln|\tan\theta - 1| - \ln|\tan\theta| + \cot\theta + C \\ &= \ln\left|\frac{\tan\theta - 1}{\tan\theta}\right| + \cot\theta + C\end{aligned}$$

39.

$$\begin{aligned}\int \frac{1}{a^2 - u^2}\,du &= \frac{1}{2a}\int \frac{1}{u + a}\,du - \frac{1}{2a}\int \frac{1}{u - a}\,du \\ &= \frac{1}{2a}(\ln|u + a| - \ln|u - a|) + C) \\ &= \frac{1}{2a}\ln\left|\frac{u + a}{u - a}\right| + C\end{aligned}$$

40.

$$\begin{aligned}\int \frac{1}{u(a + bu)}\,du &= \frac{1}{a}\int \frac{1}{u}\,du - \frac{1}{a}\int \frac{b}{a + bu}\,du \\ &= \frac{1}{a}(\ln|u| - \ln|a + bu|) + C \\ &= \frac{1}{a}\ln\left|\frac{u}{a + bu}\right| + C\end{aligned}$$

41.

$$\begin{aligned}\int \frac{1}{u^2(a+bu)}\,du &= -\frac{b}{a^2}\int \frac{1}{u}\,du + \frac{1}{a}\int \frac{1}{u^2}\,du + \frac{b}{a^2}\int \frac{b}{a+bu}\,du \\ &= \frac{b}{a^2}(\ln|a+bu| - \ln|u|) - \frac{1}{au} + C \\ &= \frac{b}{a^2}\ln\left|\frac{u}{a+bu}\right| - \frac{1}{au} + C\end{aligned}$$

42.

$$\begin{aligned}\int \frac{1}{u(a+bu)^2}\,du &= \frac{1}{a^2}\int \frac{1}{u}\,du - \frac{1}{a^2}\int \frac{b}{a+bu}\,du - \frac{b}{ab}\int \frac{b}{(a+bu)^2}\,du \\ &= \frac{1}{a^2}(\ln|u| - \ln|a+bu|) + \frac{b}{ab(a+bu)} + C \\ &= \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C\end{aligned}$$

43.

$$\begin{aligned}\int \frac{1}{(a+bu)(c+du)}\,du &= -\frac{1}{ad-bc}\int \frac{b}{a+bu}\,du + \frac{1}{ad-bc}\int \frac{d}{c+du}\,du \\ &= \frac{1}{ad-bc}\left(\ln\left|\frac{c+du}{a+bu}\right|\right) + C\end{aligned}$$

44.

$$\begin{aligned}\int \frac{1}{a^2-u^2}\,du &= \frac{1}{2a}\int \frac{1}{a+u}\,du + \frac{1}{2a}\int \frac{1}{a-u}\,du \\ &= \frac{1}{2a}\left(\ln\left|\frac{a+u}{a-u}\right|\right) + C\end{aligned}$$

45.

$$\begin{aligned}\int \frac{u}{a^2-u^2}\,du &= -\frac{1}{2}\int \frac{1}{v}\,dv, \quad \text{where} \quad v = a^2-u^2 \\ &= -\frac{1}{2}\ln v + C \\ &= -\frac{1}{2}\ln|a^2-u^2| + C\end{aligned}$$

46.

$$\begin{aligned}\int \frac{u^2}{a^2-u^2}\,du &= \int \frac{a^2}{a^2-u^2}\,du - \int du \\ &= a^2\left(\frac{1}{2a}\right)\left(\ln\left|\frac{a+u}{a-u}\right|\right) - u + C \quad \text{by Exercise 44} \\ &= -u + \frac{a}{2}\left(\ln\left|\frac{a+u}{a-u}\right|\right) + C\end{aligned}$$

47. Note that $$y = \frac{1}{x^2-1} = \frac{1}{2}\left[\frac{1}{x-1} - \frac{1}{x+1}\right]$$
and thus $$\frac{d^0y}{dx^0} = \left(\frac{1}{2}\right)(-1)^0\,0!\left[\frac{1}{(x-1)^{0+1}} - \frac{1}{(x+1)^{0+1}}\right].$$
The rest is a routine induction.

48. $$\int x^3 \tan^{-1} x\,dx = \frac{x^4}{4}\tan^{-1} x - \frac{1}{4}\int \frac{x^4}{x^2+1}\,dx$$

$u = \tan^{-1} x$	$dv = x^3\,dx$
$du = \frac{1}{x^2+1}\,dx$	$v = \frac{x^4}{4}$

$$= \frac{x^4}{4}\tan^{-1} x - \frac{1}{4}\int\left(x^2 - 1 + \frac{1}{x^2+1}\right)dx$$
$$= \frac{x^4}{4}\tan^{-1} x - \frac{x^3}{12} + \frac{x}{4} - \frac{1}{4}\tan^{-1} x + C = \frac{x^4-1}{4}\tan^{-1} x - \frac{x^3}{12} + \frac{x}{4} + C$$

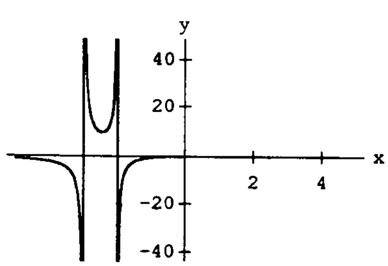
49. $$A = \int_0^1 \frac{dx}{x^2+1} = \left[\tan^{-1} x\right]_0^1 = \frac{1}{4}\pi$$
$$\overline{x}A = \int_0^1 \frac{x}{x^2+1}\,dx = \frac{1}{2}\left[\ln(x^2+1)\right]_0^1 = \frac{1}{2}\ln 2$$
$$\overline{y}A = \int_0^1 \frac{dx}{2(x^2+1)^2} = \frac{1}{2}\left[\tan^{-1} x + \frac{x}{x^2+1}\right]_0^1 = \frac{1}{8}(\pi+2)$$
$$\overline{x} = \frac{2\ln 2}{\pi};\quad \overline{y} = \frac{\pi+2}{2\pi}$$

50. $$\overline{x}V_x = \overline{y}V_y = \int_0^1 \frac{\pi x}{(x^2+1)^2}\,dx = \frac{1}{2}\left[-\frac{1}{x^2+1}\right]_0^1 = \frac{1}{4}\pi$$

(a) $$V_x = \int_0^1 \frac{\pi}{(x^2+1)^2}\,dx = \pi\left[\tan^{-1} x + \frac{x}{x^2+1}\right]_0^1 = \frac{1}{4}\pi(\pi+2);\quad \overline{x} = \frac{1}{\pi+2}$$

(b) $$V_y = \int_0^1 \frac{2\pi x}{x^2+1}\,dx = \left[\ln(x^2+1)\right]_0^1 = \pi\ln 2;\quad \overline{y} = \frac{1}{4\ln 2}$$

51. (a)



(b) $$A = \int_0^4 \frac{x}{x^2+5x+6}\,dx$$
$$= \int_0^4 \frac{x}{(x+2)(x+3)}\,dx$$
$$= \int_0^4 \left(\frac{3}{x+3} - \frac{2}{x+2}\right)dx$$
$$= \left[3\ln|x+3| - 2\ln|x+2|\right]_0^4 = 3\ln 7 - 5\ln 3$$

52. (a) $$V_y = \int_0^4 \frac{2\pi x^2}{x^2+5x+6}\,dx = 2\pi\int_0^4\left(1 + \frac{4}{x+2} - \frac{9}{x+3}\right)dx$$
$$= 2\pi\left[x + 4\ln|x+2| - 9\ln|x+3|\right]_0^4 = 2\pi(4 + 13\ln 3 - 9\ln 7)$$

(b)

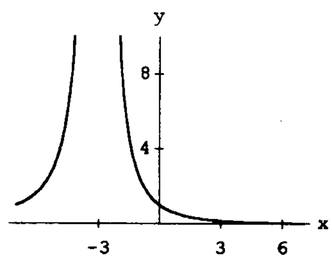
$$\overline{y}V_y = \int_0^4 \frac{\pi x^3}{(x^2+5x+6)^2}\,dx = \pi\int_0^4 \left[\frac{28}{x+2} - \frac{8}{(x+2)^2} - \frac{27}{x+3} - \frac{27}{(x+3)^2}\right] dx$$

$$= \pi\left[28\ln|x+2| + \frac{8}{x+2} - 27\ln|x+3| + \frac{27}{x+3}\right]_0^4$$

$$= \pi\left[\frac{4}{3} + \frac{27}{7} - 13 + 55\ln 3 - 27\ln 7\right]$$

$$\overline{y} = \frac{-\frac{164}{21} + 55\ln 3 - 27\ln 7}{2(4 + 13\ln 3 - 9\ln 7)}$$

53. (a)



(b) $$A = \int_{-2}^{9} \frac{9-x}{(x+3)^2}\,dx = \int_{-2}^{9}\left(\frac{12}{(x+3)^2} - \frac{1}{x+3}\right) dx = \left[-\ln|x+3| - \frac{12}{x+3}\right]_{-2}^{9} = 11 - \ln 12$$

54. **(a)**

$$V_x = \int_{-2}^{9} \pi\frac{(9-x)^2}{(x+3)^4}\,dx = \pi\int_{-2}^{9}\left[\frac{1}{(x+3)^2} - \frac{24}{(x+3)^3} + \frac{144}{(x+3)^4}\right] dx$$

$$= \pi\left[\frac{-1}{x+3} + \frac{12}{(x+3)^2} - \frac{48}{(x+3)^3}\right]_{-2}^{9} = \frac{1331}{36}\pi$$

(b)

$$\overline{x}V_x = \int_{-2}^{9} \pi x\frac{(9-x)^2}{(x+3)^4}\,dx = \pi\int_{-2}^{9}\left[\frac{1}{x+3} - \frac{27}{(x+3)^2} + \frac{216}{(x+3)^2} - \frac{432}{(x+3)^4}\right] dx$$

$$= \pi\left[\ln|x+3| + \frac{27}{x+3} - \frac{108}{(x+3)^2} + \frac{144}{(x+3)^3}\right]_{-2}^{9} = \pi\left(\ln 12 - \frac{737}{12}\right)$$

$$\overline{x} = \frac{36}{1331}\left(\ln 12 - \frac{737}{12}\right)$$

SECTION 8.6

1. $\left\{\begin{array}{c} x = u^2 \\ dx = 2u\,du \end{array}\right\}$; $\displaystyle\int \frac{dx}{1-\sqrt{x}} = \int \frac{2u\,du}{1-u} = 2\int\left(\frac{1}{1-u} - 1\right) du$

$$= -2(\ln|1-u| + u) + C = -2(\sqrt{x} + \ln|1-\sqrt{x}|) + C$$

2. $\left\{\begin{array}{c} u=\sqrt{x} \\ du=\frac{1}{2\sqrt{x}}\,dx \end{array}\right\};$

$$\int \frac{\sqrt{x}}{1+x}\,dx=\int \frac{2u^2}{1+u^2}\,du=\int\left(2-\frac{2}{1+u^2}\right)du$$
$$=2u-2\tan^{-1}u+C$$
$$=2\sqrt{x}-2\tan^{-1}\sqrt{x}+C$$

3. $\left\{\begin{array}{c} u^2=1+e^x \\ 2u\,du=e^x\,dx \end{array}\right\};$

$$\int \sqrt{1+e^x}\,dx=\int u\cdot\frac{2u\,du}{u^2-1}=2\int\left(1+\frac{1}{u^2-1}\right)du$$
$$=2\int\left(1+\frac{1}{2}\left[\frac{1}{u-1}-\frac{1}{u+1}\right]\right)du$$
$$=2u+\ln|u-1|-\ln|u+1|+C$$
$$=2\sqrt{1+e^x}+\ln\left[\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right]+C$$
$$=2\sqrt{1+e^x}+\ln\left[\frac{\left(\sqrt{1+e^x}-1\right)^2}{e^x}\right]+C$$
$$=2\sqrt{1+e^x}+2\ln\left(\sqrt{1+e^x}-1\right)-x+C$$

4. $\left\{\begin{array}{c} u^3=x \\ 3u^2\,du=dx \end{array}\right\};$

$$\int \frac{dx}{x(x^{1/3}-1)}=\int \frac{3\,du}{u(u-1)}=3\int\left(\frac{1}{u-1}-\frac{1}{u}\right)du$$
$$=3\ln|u-1|-3\ln|u|+C=3\ln|u-1|-\ln|u^3|+C$$
$$=3\ln|x^{1/3}-1|-\ln|x|+C$$

5. (a) $\left\{\begin{array}{c} u^2=1+x \\ 2u\,du=dx \end{array}\right\};$

$$\int x\sqrt{1+x}\,dx=\int(u^2-1)(u)2u\,du$$
$$=\int(2u^4-2u^2)\,du$$
$$=\tfrac{2}{5}u^5-\tfrac{2}{3}u^3+C$$
$$=\tfrac{2}{5}(x+1)^{5/2}-\tfrac{2}{3}(x+1)^{3/2}+C$$

(b) $\left\{\begin{array}{c} u=1+x \\ du=dx \end{array}\right\};$

$$\int x\sqrt{1+x}\,dx=\int(u-1)\sqrt{u}\,du=\int\left(u^{3/2}-u^{1/2}\right)du$$
$$=\tfrac{2}{5}u^{5/2}-\tfrac{2}{3}u^{3/2}+C$$
$$=\tfrac{2}{5}(1+x)^{5/2}-\tfrac{2}{3}(1+x)^{3/2}+C$$

6. (a)

$\left\{\begin{array}{c} u^2=1+x \\ 2u\,du=dx \end{array}\right\};$

$$\int x^2\sqrt{1+x}\,dx=\int(u^2-1)^2u\cdot 2u\,du$$
$$=\int(2u^6-4u^4+2u^2)\,du$$
$$=\frac{2}{7}u^7-\frac{4}{5}u^5+\frac{2}{3}u^3+C$$
$$=\frac{2}{7}(1+x)^{7/2}-\frac{4}{5}(1+x)^{5/2}+\frac{2}{3}(1+x)^{3/2}+C$$

(b)

$$\left\{\begin{array}{l} u = 1+x \\ du = dx \end{array}\right\}; \quad \int x^2\sqrt{1+x}\,dx = \int (u-1)^2\sqrt{u}\,du = \int (u^2-2u+1)\sqrt{u}\,du$$
$$= \frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C$$
$$= \frac{2}{7}(1+x)^{7/2} - \frac{4}{5}(1+x)^{5/2} + \frac{2}{3}(1+x)^{3/2} + C$$

7. $$\left\{\begin{array}{l} u^2 = x-1 \\ 2u\,du = dx \end{array}\right\}; \quad \int (x+2)\sqrt{x-1}\,dx = \int (u+3)(u)2u\,du$$
$$= \int (2u^4 + 6u^2)\,du$$
$$= \tfrac{2}{5}u^5 + 2u^3 + C$$
$$= \tfrac{2}{5}(x-1)^{5/2} + 2(x-1)^{3/2} + C$$

8. $$\left\{\begin{array}{l} u = x+2 \\ du = dx \end{array}\right\}; \quad \int (x-1)\sqrt{x+2}\,dx = \int (u-3)\sqrt{u}\,du$$
$$= \frac{2}{5}u^{5/2} - 2u^{3/2} + C$$
$$= \frac{2}{5}(x+2)^{5/2} - 2(x+2)^{3/2} + C$$

9. $$\left\{\begin{array}{l} u^2 = 1+x^2 \\ 2u\,du = 2x\,dx \end{array}\right\}; \quad \int \frac{x^3}{(1+x^2)^3}\,dx = \int \frac{x^2}{(1+x^2)^3}x\,dx = \int \frac{u^2-1}{u^6}u\,du$$
$$= \int (u^{-3} - u^{-5})\,du = \frac{1}{2}u^{-2} + \frac{1}{4}u^{-4} + C$$
$$= \frac{1}{4(1+x^2)^2} - \frac{1}{2(1+x^2)} + C$$
$$= -\frac{1+2x^2}{4(1+x^2)^2} + C$$

10. $$\left\{\begin{array}{l} u = 1+x \\ du = dx \end{array}\right\}; \quad \int x(1+x)^{1/3}\,dx = \int (u-1)u^{1/3}\,du$$
$$= \frac{3}{7}u^{7/3} - \frac{3}{4}u^{4/3} + C$$
$$= \frac{3}{7}(1+x)^{7/3} - \frac{3}{4}(1+x)^{4/3} + C$$

11. $$\left\{\begin{array}{l} u^2 = x \\ 2u\,du = dx \end{array}\right\}; \quad \int \frac{\sqrt{x}}{\sqrt{x}-1}\,dx = \int \left(\frac{u}{u-1}\right)2u\,du = 2\int \left(u+1+\frac{1}{u-1}\right)\,du$$
$$= u^2 + 2u + 2\ln|u-1| + C$$
$$= x + 2\sqrt{x} + 2\ln|\sqrt{x}-1| + C$$

12. $\left\{\begin{array}{c} u = x+1 \\ du = dx \end{array}\right\}$;

$$\int \frac{x}{\sqrt{1+x}}\,dx = \int \frac{u-1}{\sqrt{u}}\,du = \frac{2}{3}u^{3/2} - 2u^{1/2} + C = \frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + C$$

13. $\left\{\begin{array}{c} u^2 = x-1 \\ 2u\,du = dx \end{array}\right\}$;

$$\int \frac{\sqrt{x-1}+1}{\sqrt{x-1}-1}\,dx = \int \frac{u+1}{u-1}2u\,du = \int \left(2u+4+\frac{4}{u-1}\right)du$$
$$= u^2 + 4u + 4\ln|u-1| + C$$
$$= x - 1 + 4\sqrt{x-1} + 4\ln|\sqrt{x-1}-1| + C$$

(absorb -1 in C)

$$= x + 4\sqrt{x-1} + 4\ln|\sqrt{x-1}-1| + C$$

14. $\left\{\begin{array}{c} u = e^x \\ du = e^x\,dx \end{array}\right\}$;

$$\int \frac{1-e^x}{1+e^x}\,dx = \int \frac{1-u}{u(1+u)}\,du = \int\left(\frac{1}{u} - \frac{2}{1+u}\right)du = \ln|u| - 2\ln|1+u| + C = x - 2\ln(1+e^x) + C$$

15. $\left\{\begin{array}{c} u^2 = 1+e^x \\ 2u\,du = e^x\,dx \end{array}\right\}$;

$$\int \frac{dx}{\sqrt{1+e^x}} = \int \left(\frac{1}{u}\right)\frac{2u\,du}{u^2-1} = \int \left[\frac{1}{u-1} - \frac{1}{u+1}\right]du$$
$$= \ln|u-1| - \ln|u+1| + C$$
$$= \ln\left[\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right] + C$$
$$= \ln\left[\frac{(\sqrt{1+e^x}-1)^2}{e^x}\right] + C$$
$$= 2\ln(\sqrt{1+e^x}-1) - x + C$$

16. $\left\{\begin{array}{c} u = e^{-x} \\ du = -e^{-x}\,dx \end{array}\right\}$;

$$\int \frac{dx}{1+e^{-x}} = -\int \frac{du}{u(1+u)} = -\int\left(\frac{1}{u} - \frac{1}{1+u}\right)du = -\ln|u| + \ln|1+u| + C$$
$$= \ln\left|\frac{1+u}{u}\right| + C = \ln\left|\frac{1+e^x}{e^{-x}}\right| + C = \ln(1+e^x) + C$$

17. $\left\{\begin{array}{r} u^2 = x+4 \\ 2u\,du = dx \end{array}\right\}$;

$$\int \frac{x}{\sqrt{x+4}}\,dx = \int \frac{u^2-4}{u} 2u\,du = \int (2u^2-8)\,du$$
$$= \tfrac{2}{3}u^3 - 8u + C$$
$$= \tfrac{2}{3}(x+4)^{3/2} - 8(x+4)^{1/2} + C$$
$$= \tfrac{2}{3}(x-8)\sqrt{x+4} + C$$

18. $\left\{\begin{array}{r} u^2 = x-2 \\ 2u\,du = dx \end{array}\right\}$;

$$\int \frac{x+1}{x\sqrt{x-2}}\,dx = 2\int \frac{u^2+3}{u^2+2}\,du = 2\int \left(1 + \frac{1}{u^2+2}\right)\,du$$
$$= 2u + \sqrt{2}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$
$$= 2\sqrt{x-2} + \sqrt{2}\tan^{-1}\left(\sqrt{\frac{x-2}{2}}\right) + C$$

19. $\left\{\begin{array}{r} u^2 = 4x+1 \\ 2u\,du = 4\,dx \end{array}\right\}$;

$$\int 2x^2(4x+1)^{-5/2}\,dx = \int 2\left(\frac{u^2-1}{4}\right)^2 (u^{-5})\frac{u}{2}\,du$$
$$= \frac{1}{16}\int (1 - 2u^{-2} + u^{-4})\,du = \frac{1}{16}u + \frac{1}{8}u^{-1} - \frac{1}{48}u^{-3} + C$$
$$= \tfrac{1}{16}(4x+1)^{1/2} + \tfrac{1}{8}(4x+1)^{-1/2} - \tfrac{1}{48}(4x+1)^{-3/2} + C$$

20. $\left\{\begin{array}{r} u = x-1 \\ du = dx \end{array}\right\}$;

$$\int x^2\sqrt{x-1}\,dx = \int (u+1)^2\sqrt{u}\,du$$
$$= \frac{2}{7}u^{7/2} + \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C$$
$$= \frac{2}{7}(x-1)^{7/2} + \frac{4}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C$$

21. $\left\{\begin{array}{r} u^2 = ax+b \\ 2u\,du = a\,dx \end{array}\right\}$;

$$\int \frac{x}{(ax+b)^{3/2}}\,dx = \int \frac{\frac{u^2-b}{a}}{u^3}\,\frac{2u}{a}\,du$$
$$= \frac{2}{a^2}\int (1 - bu^{-2})\,du$$
$$= \frac{2}{a^2}(u + bu^{-1}) + C = \frac{2u^2+2b}{a^2u} + C$$
$$= \frac{4b+2ax}{a^2\sqrt{ax+b}} + C$$

22. $\left\{\begin{array}{r} u = ax+b \\ du = a\,dx \end{array}\right\}$

$$\int \frac{x}{\sqrt{ax+b}}\,dx = \frac{1}{a^2}\int \frac{u-b}{\sqrt{u}}\,du$$
$$= \frac{1}{a^2}\left[\frac{2}{3}u^{3/2} - 2bu^{1/2}\right] + C$$
$$= \frac{2}{3a^2}(ax-2b)\sqrt{ax+b} + C$$

23. $\left\{\begin{array}{ll} u = \tan(x/2), & dx = \dfrac{2}{1+u^2}\,du \\ \sin x = \dfrac{2u}{1+u^2}, & \cos x = \dfrac{1-u^2}{1+u^2} \end{array}\right\};$

$$\begin{aligned}\int \frac{1}{1+\cos x-\sin x}\,dx &= \int \frac{1}{1+\dfrac{1-u^2}{1+u^2}-\dfrac{2u}{1+u^2}}\cdot\frac{2}{1+u^2}\,du\\ &= \int \frac{1}{1-u}\,du\\ &= -\ln|1-u|+C = -\ln\left|1-\tan\left(\frac{x}{2}\right)\right|+C\end{aligned}$$

24. $\left\{\begin{array}{ll} u = \tan(x/2), & dx = \dfrac{2}{1+u^2}\,du \\ \cos x = \dfrac{1-u^2}{1+u^2} & \end{array}\right\};$

$$\begin{aligned}\int \frac{1}{2+\cos x}\,dx &= \int \frac{1}{2+\frac{1-u^2}{1+u^2}}\cdot\frac{2}{1+u^2}\,du = \int \frac{2}{2+2u^2+1-u^2}\,du = \int \frac{2}{3+u^2}\,du\\ &= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)+C = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\tan(x/2)}{\sqrt{3}}\right)+C\end{aligned}$$

25. $\left\{\begin{array}{ll} u = \tan(x/2), & dx = \dfrac{2}{1+u^2}\,du \\ \sin x = \dfrac{2u}{1+u^2} & \end{array}\right\};$

$$\begin{aligned}\int \frac{1}{2+\sin x}\,dx &= \int \frac{1}{2+\dfrac{2u}{1+u^2}}\cdot\frac{2}{1+u^2}\,du\\ &= \int \frac{1}{u^2+u+1}\,du = \int \frac{1}{\left(u+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,du\\ &= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{u+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+C\\ &= \frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{1}{\sqrt{3}}\left(2\tan(x/2)+1\right)\right]+C\end{aligned}$$

26. $\left\{\begin{array}{ll} u = \tan(x/2), & dx = \dfrac{2}{1+u^2}\,du \\ \sin x = \dfrac{2u}{1+u^2} & \end{array}\right\};$

$$\int \frac{\sin x}{1+\sin^2 x}\,dx = \int \frac{\frac{2u}{1+u^2}}{1+\frac{4u^2}{(1+u^2)^2}} \cdot \frac{2}{1+u^2}\,du$$

$$\left\{\begin{array}{c} v = u^2 \\ dv = 2u\,du \end{array}\right\} \quad = \int \frac{4u}{1+6u^2+u^4}\,du$$

$$= \int \frac{2}{1+6v+v^2}\,dv = \int \frac{2}{(v+3)^2-8}\,dv$$

$$= \frac{2}{\sqrt{8}} \ln\left|\frac{v+3-\sqrt{8}}{\sqrt{(v+3)^2-8}}\right| + C$$

$$= \frac{2}{\sqrt{8}} \ln\left|\frac{\tan^2(x/2)+3-\sqrt{8}}{\sqrt{\left[\tan^2(x/2)+3\right]^2-8}}\right| + C$$

27.

$$\left\{\begin{array}{cc} u = \tan(x/2), & dx = \dfrac{2}{1+u^2}\,du \\ \sin x = \dfrac{2u}{1+u^2}, & \tan x = \dfrac{2u}{1-u^2} \end{array}\right\};$$

$$\int \frac{1}{\sin x + \tan x}\,dx = \int \frac{1}{\dfrac{2u}{1+u^2} + \dfrac{2u}{1-u^2}} \cdot \frac{2}{1+u^2}\,du$$

$$= \int \frac{1-u^2}{2u}\,du = \frac{1}{2}\int \left(\frac{1}{u} - u\right)\,du$$

$$= \tfrac{1}{2}\left(\ln|u| - \tfrac{1}{2}u^2\right) + C = \tfrac{1}{2}\ln\,\|\tan(x/2)\| - \tfrac{1}{4}\left[\tan(x/2)\right]^2 + C$$

28.

$$\int \frac{1}{1+\sin x+\cos x}\,dx = \int \frac{2}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2}\,du$$

$$= \int \frac{2}{2+2u}\,du = \int \frac{du}{1+u}$$

$$= \int \ln|1+u| + C = \ln\left|1+\tan\left(\frac{x}{2}\right)\right| + C$$

29.

$$\left\{\begin{array}{cc} u = \tan(x/2), & dx = \dfrac{2}{1+u^2}\,du \\ \sin x = \dfrac{2u}{1+u^2}, & \cos x = \dfrac{1-u^2}{1+u^2} \end{array}\right\};$$

$$\int \frac{1-\cos x}{1+\sin x}\,dx = \int \frac{1-\dfrac{1-u^2}{1+u^2}}{1+\dfrac{2u}{1+u^2}} \cdot \frac{2}{1+u^2}\,du$$
$$= \int \frac{4u^2}{(1+u^2)(u+1)^2}\,du$$
$$= \int \left[\frac{2u}{1+u^2} - \frac{2}{u+1} + \frac{2}{(u+1)^2}\right] du$$
$$= \ln(u^2+1) - 2\ln|u+1| - \frac{2}{u+1} + C$$
$$= \ln\left[\frac{u^2+1}{(u+1)^2} - \frac{2}{u+1}\right] + C$$
$$= \ln\left[\frac{\tan^2(x/2)+1}{(\tan(x/2)+1)^2}\right] - \frac{2}{\tan(x/2)+1} + C = \ln\left|\frac{1}{1+\sin x}\right| - \frac{2}{\tan(x/2)+1} + C$$

30.
$$\int \frac{1}{5+3\sin x}\,dx = \int \frac{1}{5+\frac{6u}{1+u^2}} \cdot \frac{2}{1+u^2}\,du$$
$$= \int \frac{2}{5+5u^2+6u}\,du = \frac{2}{5}\int \frac{du}{(u-\frac{3}{5})^2+\frac{16}{25}}$$
$$= \frac{2}{5}\cdot\frac{5}{4}\tan^{-1}\left(\frac{u-3/5}{4/5}\right) + C$$
$$= \frac{1}{2}\tan^{-1}\left[\frac{5}{4}\left(\tan\left(\frac{x}{2}\right) - \frac{3}{5}\right)\right] + C$$

31. $\left\{\begin{array}{c} u^2 = x \\ 2u\,du = dx \end{array}\right\}$;
$$\int \frac{x^{3/2}}{x+1}\,dx = \int \frac{u^3}{u^2+1}\,2u\,du$$
$$= \int \frac{2u^4}{u^2+1}\,du = \int \left[2u^2 - 2 + \frac{2}{u^2+1}\right] du$$
$$= \frac{2}{3}u^3 - 2u + 2\tan^{-1}u + C = \frac{2}{3}x^{3/2} - 2x^{1/2} + 2\tan^{-1}x^{1/2} + C$$

$$\int_0^4 \frac{x^{3/2}}{x+1}\,dx = \left[\frac{2}{3}x^{3/2} - 2x^{1/2} + 2\tan^{-1}x^{1/2}\right]_0^4 = \frac{4}{3} + 2\tan^{-1}2$$

32. $\left\{\begin{array}{c} u^2 = x \\ 2u\,du = dx \end{array}\right\}$
$$\int_0^4 \frac{1}{1+\sqrt{x}}\,dx = \int_0^2 \frac{2u}{1+u}\,du$$
$$= \int_0^2 \left(2 - \frac{2}{1+u}\right) du = [2u - 2\ln|1+u|]_0^2$$
$$= 4 - 2\ln 3$$

33.

$$\begin{aligned}\int_0^{\pi/2} \frac{\sin 2x}{2+\cos x}\,dx &= \int_0^{\pi/2} \frac{2\sin x\,\cos x}{2+\cos x}\,dx \\ &= \int_0^1 \frac{2u}{2+u}\,du \qquad [u=\cos x,\quad du=-\sin x\,dx] \\ &= \int_0^1 \left(2-\frac{4}{2+u}\right)du \\ &= \left[2u-4\ln|2+u|\right]_0^1 = 2+4\ln\left(\tfrac{2}{3}\right)\end{aligned}$$

34. $\left\{\begin{array}{l} u=\tan(x/2), \quad dx=\dfrac{2}{1+u^2}\,du \\ \sin x = \frac{2u}{1+u^2} \end{array}\right\};$

$$\begin{aligned}\int_0^{\pi/2} \frac{1}{1+\sin x}\,dx &= \int_0^1 \frac{1}{1+\frac{2u}{1+u^2}}\cdot\frac{2u}{1+u^2}\,du \\ &= \int_0^1 \frac{2u}{1+2u+u^2}\,du = \int_0^1 \left[\frac{2}{u+1}-\frac{2}{(u+1)^2}\right]du \\ &= \left[2\ln|u+1|+\frac{2}{u+1}\right]_0^1 = 2\ln 2-1\end{aligned}$$

35. $\left\{\begin{array}{l} u=\tan(x/2), \quad dx=\dfrac{2}{1+u^2}\,du \\ \sin x=\dfrac{2u}{1+u^2}, \quad \cos x=\dfrac{1-u^2}{1+u^2} \end{array}\right\};$

$$\begin{aligned}\int \frac{1}{\sin x-\cos x-1}\,dx &= \int \frac{1}{\dfrac{2u}{1+u^2}-\dfrac{1-u^2}{1+u^2}-1}\cdot\frac{2}{1+u^2}\,du \\ &= \int \frac{1}{u-1}\,du \\ &= \ln|u-1|+C = \ln|\tan(x/2)-1|+C\end{aligned}$$

$$\int_0^{\pi/3} \frac{1}{\sin x-\cos x-1}\,dx = \left[\ln|\tan(x/2)-1|\right]_0^{\pi/3} = \ln\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right)$$

36. $\left\{\begin{array}{c} u^2=x \\ 2u\,du=dx \end{array}\right\}$

$$\begin{aligned}\int_0^1 \frac{\sqrt{x}}{1+\sqrt{x}}\,dx &= \int_0^1 \frac{2u^2}{1+u}\,du \\ &= \int_0^1 \left(2u-2+\frac{2}{1+u}\right)du \\ &= \left[u^2-2u+2\ln|1+u|\right]_0^1 = 2\ln 2-1\end{aligned}$$

37. $\left\{\begin{array}{l} u=\tan(x2), \quad dx=\dfrac{2}{1+u^2}\,du \\ \sin x=\dfrac{2u}{1+u^2}, \quad \cos x=\dfrac{1-u^2}{1+u^2} \end{array}\right\};$

$$\int \sec x\,dx = \int \frac{1}{\cos x}\,dx = \int \frac{1}{\frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2}\,du$$
$$= 2\int \frac{1}{1-u^2}\,du = 2\int \left[\frac{1/2}{1-u} + \frac{1/2}{1+u}\right]\,du$$
$$= \int \left[\frac{1}{1-u} + \frac{1}{1+u}\right]\,du = -\ln|1-u| + \ln|1+u| + C$$
$$= \ln\left|\frac{1+\tan(x/2)}{1-\tan(x/2)}\right| + C$$

38. (a)

$$\left\{\begin{array}{c} u = \sin x \\ du = \cos x\,dx \end{array}\right\} \quad \int \sec x\,dx = \int \frac{du}{1-u^2} = \frac{1}{2}\int\left(\frac{1}{1-u} + \frac{1}{1+u}\right)du$$
$$= -\frac{1}{2}\ln|1-u| + \frac{1}{2}\ln|1+u| + C$$
$$= \ln\sqrt{\frac{1+\sin x}{1-\sin x}} + C$$

(b) $\sqrt{\dfrac{1+\sin x}{1-\sin x}} = \sqrt{\dfrac{(1+\sin x)^2}{1-\sin^2 x}} = \left|\dfrac{1+\sin x}{\cos x}\right| = |\sec x + \tan x|$

39.

$$\int \csc x\,dx = \int \frac{\sin x}{\sin^2 x}\,dx = \int \frac{\sin x}{1-\cos^2 x}\,dx$$
$$= -\int \frac{1}{1-u^2}\,du \qquad [u = \cos x, \quad du = -\sin x\,dx]$$
$$= \frac{1}{2}\int\left[\frac{1}{u-1} - \frac{1}{u+1}\right]du$$
$$= \frac{1}{2}\left[\ln|u-1| - \ln|u+1|\right] + C = \ln\sqrt{\frac{1-\cos x}{1+\cos x}} + C$$

40. $\cosh\left(\frac{x}{2}\right) = \dfrac{1}{sech\left(\frac{x}{2}\right)} = \dfrac{1}{\sqrt{1-\tanh^2(\frac{x}{2})}} = \dfrac{1}{\sqrt{1-u^2}}$

$\sinh\left(\frac{x}{2}\right) = \sqrt{\cosh^2\left(\frac{x}{2}\right) - 1} = \dfrac{u}{\sqrt{1-u^2}}$

$\sinh x = 2\sinh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{2}\right) = \dfrac{2u}{1-u^2}, \quad \cosh x = \cosh^2\left(\frac{x}{2}\right) + \sinh^2\left(\frac{x}{2}\right) = \dfrac{1+u^2}{1-u^2}$

$du = \frac{1}{2}sech^2\left(\frac{x}{2}\right)dx \Longrightarrow dx = 2\cosh^2\left(\frac{x}{2}\right)du = \dfrac{2}{1-u^2}\,du$

41. $\left\{\begin{array}{c} u = \tanh(x/2), \quad dx = \dfrac{2}{1-u^2}\,du \\ \cosh x = \dfrac{1+u^2}{1-u^2}, \quad \text{sech}\, x = \dfrac{1-u^2}{1+u^2} \end{array}\right\};$

$$\int \operatorname{sech} x\,dx = \int \frac{1-u^2}{1+u^2}\cdot\frac{2}{1-u^2}\,du$$
$$= \int \frac{2}{1+u^2}\,du = 2\tan^{-1}u + C = 2\tan^{-1}(\tanh(x/2)) + C$$

42. $\displaystyle\int \frac{1}{1+\cosh x}\,dx = \int \frac{1}{1+\frac{1+u^2}{1-u^2}}\cdot\frac{2}{1-u^2}\,du = \int du = u + C = \tanh\left(\frac{x}{2}\right) + C$

43. $\left\{\begin{array}{cc} u = \tanh(x/2), & dx = \dfrac{2}{1-u^2}\,du \\ \sinh x = \dfrac{2u}{1-u^2}, & \cosh x = \dfrac{1+u^2}{1-u^2} \end{array}\right\};$

$$\int \frac{1}{\sinh x + \cosh x}\,dx = \int \frac{1}{\dfrac{2u}{1-u^2}+\dfrac{1+u^2}{1-u^2}}\cdot\frac{2}{1-u^2}\,du$$
$$= \int \frac{2}{(1+u)^2}\,du = \frac{-2}{u+1} + C = \frac{-2}{\tanh(x/2)+1} + C$$

44.
$$\int \frac{1-e^x}{1+e^x}\,dx = \int \frac{1-\cosh x - \sinh x}{1+\cosh x + \sinh x}\,dx$$
$$= \int \frac{1-\frac{1+u^2}{1-u^2}-\frac{2u}{1-u^2}}{1+\frac{1+u^2}{1-u^2}+\frac{2u}{1-u^2}}\cdot\frac{2}{1-u^2}\,du$$
$$= \int -\frac{2u}{1-u^2}\,du = \ln|1-u^2| + C = \ln\left|1-\tanh^2\left(\frac{x}{2}\right)\right| + C$$

SECTION 8.7

1. (a) $L_{12} = \frac{12}{12}[0+1+4+9+16+25+36+49+64+81+100+121] = 506$

(b) $R_{12} = \frac{12}{12}[1+4+9+16+25+36+49+64+81+100+121+144] = 650$

(c) $M_6 = \frac{12}{6}[1+9+25+49+81+121] = 572$

(d) $T_{12} = \frac{12}{24}[0+2(1+4+9+16+25+36+49+64+81+100+121)+144] = 578$

(e) $S_6 = \frac{12}{36}[0+144+2(4+16+36+64+100)+4(1+9+25+49+81+121)] = 576$

$$\int_0^{12} x^2\,dx = \left[\frac{1}{3}x^3\right]_0^{12} = 576$$

2. (a) 0.500000 (b) 0.500000 (c) 0.500000

$$\int_0^1 \sin^2 \pi x\,dx = \frac{1}{\pi}\left[\frac{\pi x}{2} - \frac{\sin 2\pi x}{4}\right]_0^1 = \frac{1}{2}$$

3. (a) $L_6 = \frac{3}{6}\left[\frac{1}{1+0}+\frac{1}{1+1/8}+\frac{1}{1+1}+\frac{1}{1+27/8}+\frac{1}{1+8}+\frac{1}{1+125/8}\right]$

$= \frac{1}{2}\left[1+\frac{8}{9}+\frac{1}{2}+\frac{8}{35}+\frac{1}{9}+\frac{8}{133}\right] \cong 1.394$

(b) $R_6 = \frac{3}{6}\left[\frac{1}{1+1/8}+\frac{1}{1+1}+\frac{1}{1+27/8}+\frac{1}{1+8}+\frac{1}{1+125/8}+\frac{1}{1+27}\right]$

$= \frac{1}{2}\left[\frac{8}{9}+\frac{1}{2}+\frac{8}{35}+\frac{1}{9}+\frac{8}{133}+\frac{1}{28}\right] \cong 0.9122$

(c) $M_3 = \frac{3}{3}\left[\frac{1}{1+1/8}+\frac{1}{1+27/8}+\frac{1}{1+125/8}\right] = \frac{8}{9}+\frac{8}{35}+\frac{8}{133} \cong 1.1852$

(d) $T_6 = \frac{3}{12}\left[1+2\left(\frac{8}{9}+\frac{1}{2}+\frac{8}{35}+\frac{1}{9}+\frac{8}{133}\right)+\frac{1}{28}\right] \cong 1.1533$

(e) $S_3 = \frac{3}{18}\left\{1+\frac{1}{28}+2\left[\frac{1}{2}+\frac{1}{9}\right]+4\left[\frac{8}{9}+\frac{8}{35}+\frac{8}{133}\right]\right\} \cong 1.1614$

4. (a) 0.4229 (b) 0.4339

5. (a) $\frac{1}{4}\pi \cong T_4 = \frac{1}{8}\left[1+2\left(\frac{1}{1+1/16}+\frac{1}{1+1/4}+\frac{1}{1+9/16}\right)+\frac{1}{1+1}\right]$

$= \frac{1}{8}\left[1+2\left(\frac{16}{17}+\frac{4}{5}+\frac{16}{25}\right)+\frac{1}{2}\right] \cong 0.7828$

(b) $\frac{1}{4}\pi \cong S_4 = \frac{1}{24}\left[1+\frac{1}{2}+2\left(\frac{16}{17}+\frac{4}{5}+\frac{16}{25}\right)+4\left(\frac{64}{65}+\frac{64}{73}+\frac{64}{89}+\frac{64}{113}\right)\right] \cong 0.7854$

6. (a) 0.8511 (b) 0.8542

7. (a) $M_4 = \frac{2}{4}\left[\cos\left(\frac{-3}{4}\right)^2+\cos\left(\frac{-1}{4}\right)^2+\cos\left(\frac{1}{4}\right)^2+\cos\left(\frac{3}{4}\right)^2\right] \cong 1.8440$

(b) $T_8 = \frac{2}{16}\Big[\cos(-1)^2+2\cos\left(\frac{-3}{4}\right)^2+2\cos\left(\frac{-1}{2}\right)^2+2\cos\left(\frac{-1}{4}\right)^2+$

$2\cos(0)^2+2\cos\left(\frac{1}{4}\right)^2+2\cos\left(\frac{1}{2}\right)^2+2\cos\left(\frac{3}{4}\right)^2+\cos(1)^2\Big] \cong 1.7915$

(c) $S_4 = \frac{2}{24}\Big\{\cos(-1)^2+\cos(1)^2+2\left[\cos\left(\frac{-1}{2}\right)^2+\cos(0)^2+\cos\left(\frac{1}{2}\right)^2\right]+$

$4\left[\cos\left(\frac{-3}{4}\right)^2+\cos\left(\frac{-1}{4}\right)^2+\cos\left(\frac{1}{4}\right)^2+\cos\left(\frac{3}{4}\right)^2\right]\Big\} \cong 1.8090$

8. (a) 3.0543 (b) 3.0615 (c) 3.0591

9. (a) $T_{10} = \frac{2}{20}\Big[e^{-0^2}+2e^{-(1/5)^2}+2e^{-(2/5)^2}+2e^{-(3/5)^2}+2e^{-(4/5)^2}+2e^{-1^2}+2e^{-(6/5)^2}+$

$$2e^{-(7/5)^2} + 2e^{-(8/5)^2} + 2e{-}(9/5)^2 + e{-}2^2\Big] \cong 0.8818$$

(b) $$S_5 = \frac{2}{30}\left\{e^{-0^2} + e^{-2^2} + 2\left[e^{-(2/5)^2} + e^{-(4/5)^2} + e^{-(6/5)^2} + e^{-(8/5)^2}\right] + 4\left[e^{-(1/5)^2} + e^{-(3/5)^2} + e^{-1^2} + e^{-(7/5)^2} + e^{-(9/5)^2}\right]\right\} \cong 0.8821$$

10. (a) 1.9133 (b) 1.9271 (c) 1.9225

11. Such a curve passes through the three points

$$(a_1, b_1), \quad (a_2, b_2), \quad (a_3, b_3)$$

iff

$$b_1 = a_1{}^2A + a_1B + C, \quad b_2 = a_2{}^2A + a_2B + C, \quad b_3 = a_3{}^2A + a_3B + C,$$

which happens iff

$$A = \frac{b_1(a_2 - a_3) - b_2(a_1 - a_3) + b_3(a_1 - a_2)}{(a_1 - a_3)(a_1 - a_2)(a_2 - a_3)},$$

$$B = -\frac{b_1(a_2{}^2 - a_3{}^2) - b_2(a_1{}^2 - a_3{}^2) + b_3(a_1{}^2 - a_2{}^2)}{(a_1 - a_3)(a_1 - a_2)(a_2 - a_3)},$$

$$C = \frac{a_1{}^2(a_2b_3 - a_3b_2) - a_2{}^2(a_1b_3 - a_3b_1) + a_3{}^2(a_1b_2 - a_2b_1)}{(a_1 - a_3)(a_1 - a_2)(a_2 - a_3)}.$$

12.

$$\begin{aligned}
&\frac{b-a}{6}\left[g(a) + 4g\left(\frac{a+b}{2}\right) + g(b)\right] \\
&= \frac{b-a}{6}\left\{(Aa^2 + Ba + C) + 4\left[A\left(\frac{a+b}{2}\right)^2 + B\left(\frac{a+b}{2}\right) + C\right] + (Ab^2 + Bb + C)\right\} \\
&= \frac{b-a}{6}\left\{A(b^2 + a^2) + B(b + a) + 2C + A(a^2 + 2ab + b^2) + 2B(a + b) + 4C\right\} \\
&= \frac{b-a}{6}\left\{2A(b^2 + ab + a^2) + 3B(b + a) + 6C\right\} \\
&= \frac{1}{3}A(b^3 - a^3) + \frac{1}{2}B(b^2 - a^2) + C(b - a) \\
&\int_a^b Ax^2\,dx + \int_a^b Bx\,dx + \int_a^b C\,dx = \int_a^b g(x)\,dx
\end{aligned}$$

13. (a) $$\left|\frac{(b-a)^3}{12n^2}f''(c)\right| = \frac{27}{12n^2}\,\frac{1}{4c^{3/2}} \leq \frac{9}{16n^2} < 0.01 \implies n^2 > \left(\frac{15}{2}\right)^2 \implies n \geq 8$$

(b) $$\left|\frac{(b-a)^5}{180n^4}f^{(4)}(c)\right| = \frac{243}{180n^4}\,\frac{15}{16c^{7/2}} \leq \frac{81}{64n^4} < 0.01 \implies n > \frac{3}{2}\sqrt{5} \implies n \geq 4$$

14. (a) $$\left|\frac{(b-a)^3}{12n^2}f''(c)\right| = \frac{8}{12n^2}20c^3 \leq \frac{8 \cdot 20 \cdot 27}{12n^2} < 0.01 \implies n \geq 190$$

(b) $\left|\dfrac{(b-a)^5}{180n^4}f^{(4)}(c)\right| = \dfrac{32}{180n^4}\cdot 120c \le \dfrac{32\cdot 120\cdot 3}{180n^4} < 0.01 \implies n \ge 9$

15. (a) $\left|\dfrac{(b-a)^3}{12n^2}f''(c)\right| = \dfrac{27}{12n^2}\,\dfrac{1}{4c^{3/2}} \le \dfrac{9}{16n^2} < 0.00001 \implies n > 75\sqrt{10} \implies n \ge 238$

(b) $\left|\dfrac{(b-a)^5}{180n^4}f^{(4)}(c)\right| = \dfrac{243}{180n^4}\,\dfrac{15}{16c^{7/2}} \le \dfrac{81}{64n^4} < 0.00001 \implies n > 15\left(\dfrac{5}{2}\right)^{1/4} \implies n \ge 19$

16. (a) $\dfrac{8\cdot 20\cdot 27}{12n^2} < 0.00001 \implies n \ge 6000$

(b) $\dfrac{32\cdot 120\cdot 3}{180n^4} < 0.00001 \implies n \ge 51$

17. (a) $\left|\dfrac{(b-a)^3}{12n^2}f''(c)\right| = \dfrac{\pi^3}{12n^2}\sin c \le \dfrac{\pi^3}{12n^2} < 0.001 \implies n > 5\pi\sqrt{\dfrac{10\pi}{3}} \implies n \ge 51$

(b) $\left|\dfrac{(b-a)^5}{180n^4}f^{(4)}(c)\right| = \dfrac{\pi^5}{180n^4}\sin c \le \dfrac{\pi^5}{180n^4} < 0.001 \implies n > \pi\left(\dfrac{50\pi}{9}\right)^{1/4} \implies n \ge 7$

18. (a) $\left|\dfrac{(b-a)^3}{12n^2}f''(c)\right| = \dfrac{\pi^3}{12n^2}\cos c \le \dfrac{\pi^3}{12n^2} < 0.001 \implies n \ge 51$

(b) $\left|\dfrac{(b-a)^5}{180n^4}f^{(4)}(c)\right| = \dfrac{\pi^5}{180n^4}\cos c \le \dfrac{\pi^5}{180n^4} < 0.001 \implies n \ge 7$

19. (a) $\left|\dfrac{(b-a)^3}{12n^2}f''(c)\right| = \dfrac{8}{12n^2}e^c \le \dfrac{8}{12n^2}e^3 < 0.01 \implies n > 10e\sqrt{\dfrac{2e}{3}} \implies n \ge 37$

(b) $\left|\dfrac{(b-a)^5}{180n^4}f^{(4)}(c)\right| = \dfrac{32}{180n^4}e^c \le \dfrac{8}{45n^4}e^3 < 0.01 \implies n > 2\left(\dfrac{10e^3}{9}\right)^{1/4} \implies n \ge 5$

20. (a) $\left|\dfrac{(b-a)^3}{12n^2}f''(c)\right| = \dfrac{(e-1)^3}{12n^2}\cdot\dfrac{1}{c^2} \le \dfrac{(e-1)^3}{12n^2} < 0.01 \implies n \ge 7$

(b) $\left|\dfrac{(b-a)^5}{180n^4}f^{(4)}(c)\right| = \dfrac{(e-1)^5}{180n^4}\cdot\dfrac{6}{c^4} \le \dfrac{6(e-1)^5}{180n^4} < 0.01 \implies n \ge 3$

21. (a) $\left|\dfrac{(b-a)^3}{12n^2}f''(c)\right| = \left|\dfrac{8}{12n^2}2e^{-c^2}(2c^2-1)\right| \le \dfrac{8}{3n^2}e^{-3/2} < 0.0001$

$\implies n > 100\sqrt{\tfrac{8}{3}e^{-3/2}} \implies n \ge 78$

(b) $\left|\dfrac{(b-a)^5}{2880n^4} f^{(4)}(c)\right| = \left|\dfrac{32}{2880n^4} 4e^{-c^2}\left(4c^4 - 12c^2 + 3\right)\right| \le \dfrac{32}{2880n^4} 12 < 0.0001$

$$\implies \quad n > 10\left[\frac{32\cdot 12}{2880}\right]^{1/4} \quad \implies \quad n \ge 7$$

22. (a) $\left|\dfrac{(b-a)^3}{12n^2} f''(c)\right| = \dfrac{8}{12n^2}\cdot e^c \le \dfrac{8}{12n^2}\cdot e^2 < 0.00001 \implies n \ge 702$

(b) $\left|\dfrac{(b-a)^5}{180n^4} f^{(4)}(c)\right| = \dfrac{32}{180n^4}\cdot e^c \le \dfrac{32}{180n^4}\cdot e^2 < 0.00001 \implies n \ge 20$

23. $f^{(4)}(x) = 0$ for all x; therefore by (8.7.3) the theoretical error is zero

24. If f is linear, $f''(x) = 0$ for all x, so the theoretical error is zero

25. (a) $\left|T_2 - \displaystyle\int_0^1 x^2\,dx\right| = \dfrac{3}{8} - \dfrac{1}{3} = \dfrac{1}{24} = E_2^T$

(b) $\left|S_1 - \displaystyle\int_0^1 x^4\,dx\right| = \dfrac{5}{24} - \dfrac{1}{5} = \dfrac{1}{120} = E_1^S$

26. Since $m_i \le f(x_{i-1}) \le M_i$, $m_i \le f(x_i) \le M_i$, and $m_i \le f\left(\frac{x_{i-1}+x_i}{2}\right) \le M_i$, we get

$$m_i \le \frac{1}{2}\left[f(x_{i-1}) + f(x_i)\right] \le M_i \quad \text{and} \quad m_i \le \frac{1}{6}\left[f(x_{i-1}) + 4f\left(\frac{x_{i-1}+x_i}{2}\right) + f(x_i)\right] \le M_i$$

So by the intermediate value theorem, we can find $a_i,\ b_i \in [x_{i-1}, x_i]$ such that

$$f(a_i) = \frac{1}{2}\left[f(x_{i-1}) + f(x_i)\right], \quad f(b_i) = \frac{1}{6}\left[f(x_{i-1}) + 4f\left(\frac{x_{i-1}+x_i}{2}\right) + f(x_i)\right]$$

So using a_i or b_i as x_i^*, we can write T_n and S_n as Riemann sums.

27. Let f be twice differentiable on $[a,b]$ with $f(x) > 0$ and $f''(x) > 0$, and let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a regular partition of $[a,b]$. Figure A shows a typical subinterval with the approximating trapezoid ABCD. Since the area under the curve is less than the area of the trapezoid, we can conclude that

$$\int_a^b f(x)\,dx \le T_n.$$

Figure A

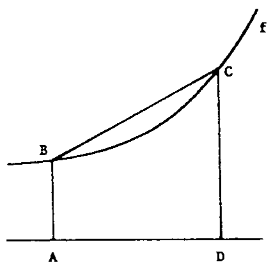
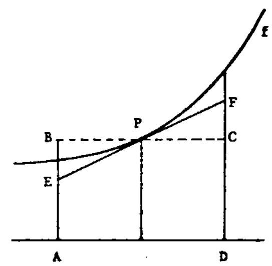


Figure B



Now consider Figure B. Since the triangles EBP and PFC are congruent, the area of the rectangle ABCD equals the area of the trapezoid AEFD, and since the area under the curve is greater than the area of AEFD it follows that

$$M_n \leq \int_a^b f(x)\,dx.$$

28.

$$\frac{1}{3}T_n + \frac{2}{3}M_n$$
$$= \frac{1}{3}\frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right] + \frac{2}{3}\cdot\frac{b-a}{n}\left[f\left(\frac{x_0+x_1}{2}\right) + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right)\right]$$
$$= \frac{b-a}{6n}\left\{f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n) + 4\left[f\left(\frac{x_0+x_1}{2}\right) + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right)\right]\right\}$$
$$= S_n$$

PROJECT 8.7

1. (a) Since $f(t) = e^{-t^2}$ is an even function,

$$\int_{-x}^0 e^{-t^2}\,dt = \int_0^x e^{-t^2}\,dt \quad \text{for all } x.$$

Now, $B(-x) = \int_0^{-x} e^{-t^2}\,dt = -\int_{-x}^0 e^{-t^2}\,dt = -\int_0^x e^{-t^2}\,dt = -B(x).$

Thus B is an odd function.

(b) Since $f(t) = e^{-t^2}$ is continuous on $(-\infty, \infty)$, $B(x) = \int_0^x e^{-t^2}\,dt$ is differentiable (Theorem 5.2.5)

(c) By Theorem 5.2.5, $B'(x) = e^{-x^2}$. Since $B'(x) > 0$, B is increasing on $(-\infty, \infty)$.

(d) $B''(x) = -2xe^{-x^2}$; the graph of B has a point of inflection at $(0, 1)$.

2. (a) $B(1) \simeq 0.746824$

$B(2) \simeq 0.882081$

$B(3) \simeq 0.886207$

$B(4) \simeq 0.886227$

$B(10) \simeq 0.886226$

(b) $\lim_{n\to\infty} B(x) \simeq 0.88623$ (5 decimal places).

$\lim_{n\to-\infty} B(x) \simeq -0.88623$ (by symmetry).

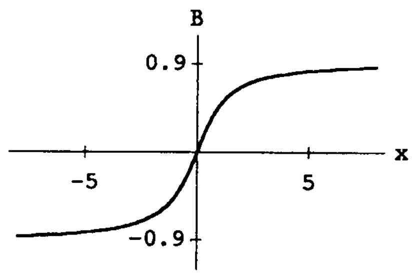
(c) $\lim_{n\to\infty} erf(x) = 1 \quad \left(\implies \quad \lim_{x\to\infty} B(x) = \frac{\sqrt{\pi}}{2}\right)$

(d) $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{t^2}{2}}\,dt = \frac{1}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du$ (where $t = \sqrt{u}$)

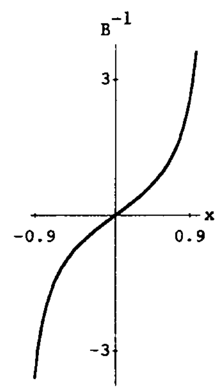
$= \frac{1}{2}erf(x)$. Thus $\lim_{x\to\infty} \Phi(x) = \frac{1}{2}$.

(e) $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\,dt = 1$

3. (a)



(b)



Since B is increasing on $(-\infty, \infty)$, B^{-1} exists

4. $2\Phi(\sqrt{2}x) = \frac{2}{\sqrt{2\pi}}\int_0^{\sqrt{2}x} e^{-\frac{t^2}{2}}\,dt$

$= \frac{2\sqrt{2}}{\sqrt{2\pi}}\left[\frac{1}{\sqrt{2}}\int_0^{\sqrt{2}x} e^{-\left(\frac{t}{\sqrt{2}}\right)^2}\,dt\right] = \frac{2}{\pi}\int_0^x e^{-t^2}\,dt = erf(x)$

SECTION 8.8

1. $y_1'(x) = \frac{1}{2}e^{x/2}; \qquad 2y_1' - y_1 = 2\left(\frac{1}{2}\right)e^{x/2} - e^{x/2} = 0;$ $\quad y_1$ is a solution.

$y_2'(x) = 2x + e^{x/2}; \qquad 2y_2' - y_2 = 2\left(2x + e^{x/2}\right) - \left(x^2 + 2e^{x/2}\right) = 4x - x^2 \neq 0;$

y_2 is not a solution.

2. $y_1' + xy_1 = -xe^{-x^2/2} + xe^{-x^2/2} = 0;$ not a solution

$y_2' + xy_2 = -Cxe^{-x^2/2} + x + Cxe^{-x^2/2} = x;$ $\quad y_2$ is a solution.

3. $y_1'(x) = \frac{-e^x}{(e^x+1)^2}; \qquad y_1' + y_1 = \frac{-e^x}{(e^x+1)^2} + \frac{1}{e^x+1} = \frac{1}{(e^x+1)^2} = y_1^2;$ $\quad y_1$ is a solution.

$y_2'(x) = \frac{-Ce^x}{(Ce^x+1)^2}; \qquad y_2' + y_2 = \frac{-Ce^x}{(Ce^x+1)^2} + \frac{1}{Ce^x+1} = \frac{1}{(Ce^x+1)^2} = y_2^2;$

y_2 is a solution.

4. $y_1'' + 4y_1 = -8\sin 2x + 8\sin 2x = 0;\quad y_1$ is a solution.

$y_2'' + 4y_2 = -2\cos x + 8\cos x = 6\cos x;$ not a solution.

5. $y_1'(x) = 2e^{2x},\quad y_1'' = 4e^{2x};\qquad y_1'' - 4y_1 = 4e^{2x} - 4e^{2x} = 0;\quad y_1$ is a solution.

$y_2'(x) = 2C\,\cosh 2x,\quad y_2'' = 4C\,\sinh 2x;\qquad y_2'' - 4y_2 = 4C\,\sinh 2x - 4C\,\sinh 2x = 0;$

y_2 is a solution.

6. $y_1'' - 2y_1' - 3y_1 = e^{-x} + 18e^{3x} - 2(-e^{-x} + 6e^{3x}) - 3(e^{-x} + 2e^{3x}) = 0;$ not a solution

$y_2'' - 2y_2' - 3y_2 = \frac{7}{4}\left[(6+9x)e^{3x} - 2(1+3x)e^{3x} - 3xe^{3x}\right] = 7e^{3x};\quad y_2$ is a solution.

7. $y' - 2y = 1;\qquad H(x) = \int (-2)\,dx = -2x,$ integrating factor: e^{-2x}

$$e^{-2x}y' - 2e^{-2x}y = e^{-2x}$$
$$\frac{d}{dx}\left[e^{-2x}y\right] = e^{-2x}$$
$$e^{-2x}y = -\frac{1}{2}e^{-2x} + C$$
$$y = -\frac{1}{2} + Ce^{2x}$$

8. $y' - \frac{2}{x}y = -1;\qquad H(x) = \int -\frac{2}{x}\,dx,$ integrating factor: x^{-2}

$$x^{-2}y' - \frac{2}{x^3}y = -x^{-2}$$
$$\frac{d}{dx}(x^{-2}y) = -x^{-2}$$
$$x^{-2}y = \frac{1}{x} + C$$
$$y = x + Cx^2$$

9. $y' + \frac{5}{2}y = 1;\qquad H(x) = \int \left(\frac{5}{2}\right)\,dx = \frac{5}{2}x,$ integrating factor: $e^{5x/2}$

$$e^{5x/2}y' + \frac{5}{2}e^{5x/2}y = e^{5x/2}$$
$$\frac{d}{dx}\left[e^{5x/2}y\right] = e^{5x/2}$$
$$e^{5x/2}y = \frac{2}{5}e^{5x/2} + C$$
$$y = \frac{2}{5} + Ce^{-5x/2}$$

10. $y' - y = -2e^{-x};\qquad H(x) = \int -\,dx,$ integrating factor: e^{-x}

$$e^{-x}y' - e^{-x}y = -2e^{-2x}$$
$$\frac{d}{dx}\left(e^{-x}y\right) = -2e^{-2x}$$
$$e^{-x}y = e^{-2x} + C$$
$$y = e^{-x} + Ce^{x}$$

11. $y' - 2y = 1 - 2x;\qquad H(x) = \int (-2)\,dx = -2x,\quad$ integrating factor: e^{-2x}

$$e{-}2xy' - 2e^{-2x}y = e^{-2x} - 2xe^{-2x}$$
$$\frac{d}{dx}\left[e^{-2x}y\right] = e^{-2x} - 2xe^{-2x}$$
$$e^{-2x}y = -\frac{1}{2}\,e^{-2x} + \frac{1}{2}\,x\,e^{-2x} + \frac{1}{2}\,e^{-2x} + C = x\,e^{-2x} + C$$
$$y = x + Ce^{2x}$$

12. $y' + \frac{2}{x}y = \frac{\cos x}{x^2};\qquad H(x) = \int \frac{2}{x}\,dx = 2\ln|x|,\quad$ integrating factor: x^2

$$x^2y' + 2xy = \cos x$$
$$\frac{d}{dx}[x^2y] = \cos x$$
$$x^2y = \sin x + C$$
$$y = \frac{\sin x}{x^2} + \frac{C}{x^2}$$

13. $y' - \frac{4}{x}\,y = -2n;\qquad H(x) = \int \left(-\frac{4}{x}\right)\,dx = -4\,\ln x = \ln\,x^{-4},\quad$ integrating factor: $e^{\ln\,x^{-4}} = x^{-4}$

$$x^{-4}y' - \frac{4}{x}\,x^{-4}y = -2nx^{-4}$$
$$\frac{d}{dx}\left[x^{-4}y\right] = -2nx^{-4}$$
$$x^{-4}y = \frac{2}{3}\,nx^{-3} + C$$
$$y = \frac{2}{3}\,nx + Cx^4$$

14. $y' + y = 2 + 2x;\qquad H(x) = \int dx,\quad$ integrating factor: e^x

$$e^xy' + e^xy = (2 + 2x)e^x$$
$$\frac{d}{dx}(e^xy) = 2(1 + x)e^x$$
$$e^xy = 2xe^x + C$$
$$y = 2x + Ce^{-x}$$

15. $y' - e^x\,y = 0;\qquad H(x) = \int -e^x\,dx = -e^x,\quad$ integrating factor: e^{-e^x}

$$e^{-e^x}y' - e^x\,e^{-e^x}\,y = 0$$
$$\frac{d}{dx}\left[e^{-e^x}\,y\right] = 0$$
$$e^{-e^x}\,y = C$$
$$y = Ce^{e^x}$$

16. $y' - y = e^x;\qquad H(x) = \int -\,dx,$ integrating factor: e^{-x}

$$\begin{aligned} e^{-x}y' - e^{-x}y &= 1 \\ \frac{d}{dx}(e^{-x}y) &= 1 \\ e^{-x}y &= x + C \\ y &= xe^x + Ce^x \end{aligned}$$

17. $y' + \dfrac{1}{1+e^x}\,y = \dfrac{1}{1+e^x};\qquad H(x) = \int \dfrac{1}{1+e^x}\,dx = \ln \dfrac{e^x}{1+e^x},$
integrating factor: $e^{H(x)} = \dfrac{e^x}{1+e^x}$

$$\begin{aligned} \frac{e^x}{1+e^x}\,y' + \frac{1}{1+e^x}\cdot\frac{e^x}{1+e^x}\,y &= \frac{1}{1+e^x}\cdot\frac{e^x}{1+e^x} \\ \frac{d}{dx}\left[\frac{e^x}{1+e^x}\,y\right] &= \frac{e^x}{(1+e^x)^2} \\ \frac{e^x}{1+e^x}\,y &= -\frac{1}{1+e^x} + C \\ y &= -e^{-x} + C\left(1+e^{-x}\right) \end{aligned}$$

This solution can also be written: $y = 1 + K\,(e^{-x}+1)$, where K is an arbitrary constant.

18. $y' + \dfrac{1}{x}y = \dfrac{1+x}{x}e^x;\qquad H(x) = \int \dfrac{1}{x}\,dx,$ integrating factor: x

$$\begin{aligned} xy' + y &= (1+x)e^x \\ \frac{d}{dx}(xy) &= (1+x)e^x \\ xy &= xe^x + C \\ y &= e^x + \frac{C}{x} \end{aligned}$$

19. $y' + 2xy = xe^{-x^2};\qquad H(x) = \int 2x\,dx = x^2,$ integrating factor: e^{x^2}

$$\begin{aligned} e^{x^2}y' + 2xe^{x^2}y &= x \\ \frac{d}{dx}\left[e^{x^2}y\right] &= x \\ e^{x^2}y &= \frac{1}{2}x^2 + C \\ y &= e^{-x^2}\left(\tfrac{1}{2}x^2 + C\right) \end{aligned}$$

20. $y' - \frac{1}{x}y = 2\ln x;$ $\quad H(x) = \int -\frac{1}{x}\,dx,$ $\quad$ integrating factor: $\frac{1}{x}$

$$\frac{1}{x}y' - \frac{1}{x^2}y = \frac{2}{x}\ln x$$

$$\frac{d}{dx}\left(\frac{1}{x}y\right) = \frac{2}{x}\ln x$$

$$\frac{1}{x}y = (\ln x)^2 + C$$

$$y = x(\ln x)^2 + Cx$$

21. $y' + \frac{2}{x+1}\,y = 0;$ $\quad H(x) = \int \frac{2}{x+1}\,dx = 2\,\ln(x+1) = \ln(x+1)^2,$

integrating factor: $e^{\ln(x+1)^2} = (x+1)^2$

$$(x+1)^2\,y' + 2(x+1)\,y = 0$$

$$\frac{d}{dx}\left[(x+1)^2\,y\right] = 0$$

$$(x+1)^2\,y = C$$

$$y = \frac{C}{(x+1)^2}$$

22. $y' + \frac{2}{x+1}y = (x+1)^{5/2};$ $\quad H(x) = \int \frac{2}{x+1}\,dx,$ $\quad$ integrating factor: $(x+1)^2$

$$(x+1)^2y' + 2(x+1)y = (x+1)^{9/2}$$

$$\frac{d}{dx}\left[(x+1)^2y\right] = (x+1)^{9/2}$$

$$(x+1)^2y = \frac{2}{11}(x+1)^{11/2} + C$$

$$y = \frac{2}{11}(x+1)^{7/2} + C(x+1)^{-2}$$

23. $y' + y = x;$ $\quad H(x) = \int 1\,dx = x,$ $\quad$ integrating factor : e^x

$$e^x\,y' + e^x\,y = xe^x$$

$$\frac{d}{dx}\left[e^x\,y\right] = xe^x$$

$$e^x\,y = xe^x - e^x + C$$

$$y = (x-1) + Ce^{-x}$$

$y(0) = -1 + C = 1 \implies C = 2.$ Therefore, $y = 2e^{-x} + x - 1$ is the solution which satisfies the side condition.

24. $y' - y = e^{2x}; \qquad H(x) = \int -\,dx, \quad$ integrating factor: e^{-x}

$$\frac{d}{dx}(e^{-x}y) = e^{x}$$

$$e^{-x}y = e^{x} + C$$

$$y = e^{2x} + Ce^{x}$$

$1 = y(1) = e^2 + Ce \implies C = \dfrac{1-e^2}{e}$ and $y = e^{2x} + \dfrac{1-e^2}{e}e^{x}$ is the solution which satisfies the side condition.

25. $y' + y = \dfrac{1}{1+e^x}; \qquad H(x) = \int 1\,dx = x, \quad$ integrating factor : e^x

$$e^x\,y' + e^x\,y = \frac{e^x}{1+e^x}$$

$$\frac{d}{dx}\left[e^x\,y\right] = \frac{e^x}{1+e^x}$$

$$e^x\,y = \ln\left(1+e^x\right) + C$$

$$y = e^{-x}\left[\ln\left(1+e^x\right) + C\right]$$

$y(0) = \ln 2 + C = e \quad \implies \quad C = e - \ln 2.$ Therefore, $y = e^{-x}\left[\ln\left(1+e^x\right) + e - \ln 2\right]$ is the solution which satisfies the side condition.

26. $y' + y = \dfrac{1}{1+2e^x}; \qquad H(x) = \int dx, \quad$ integrating factor: e^x

$$\frac{d}{dx}(e^x y) = \frac{e^x}{1+2e^x}$$

$$e^x y = \frac{1}{2}\ln(1+2e^x) + C$$

$$y = e^{-x}\left[\frac{1}{2}\ln(1+2e^x) + C\right]$$

$e = y(0) = \frac{1}{2}\ln 3 + C \implies C = e - \dfrac{1}{2}\ln 3$ and $y = e^{-x}\left[\frac{1}{2}\ln(1+2e^x) + e - \frac{1}{2}\ln 3\right]$ is the solution which satisfies the side condition.

27. $y' - \dfrac{2}{x}y = x^2e^x; \qquad H(x) = \int\left(-\dfrac{2}{x}\right)dx = -2\ln x = \ln x^{-2},$

integrating factor: $e^{\ln x^{-2}} = x^{-2}$

$$x^{-2}\,y' - 2x^{-3}\,y = e^x$$

$$\frac{d}{dx}\left[x^{-2}\,y\right] = e^x$$

$$x^{-2}\,y = e^x + C$$

$$y = x^2\left(e^x + C\right)$$

$y(1) = e + C = 0 \quad \implies \quad C = -\,e.$ Therefore, $y = x^2\left(e^x - e\right)$ is the solution which satisfies the side condition.

28. $y' + \frac{2}{x}y = e^{-x}$; $H(x) = \int \frac{2}{x}\,dx$, integrating factor: x^2

$$\frac{d}{dx}(x^2y) = x^2e^{-x}$$

$$x^2y = -e^{-x}(x^2+2x+2) + C$$

$$y = -\frac{e^{-x}}{x^2}(x^2+2x+2) + \frac{C}{x^2}$$

$-1 = y(1) = -5e^{-1} + C \implies C = 5e^{-1} - 1$ and $y = -\frac{e^{-x}}{x^2}(x^2+2x+2) + \frac{5e^{-1}-1}{x^2}$ is the solution which satisfies the side condition.

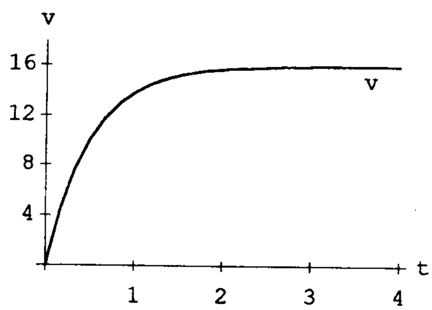
29. (a) You can determine that

$$v(t) = \frac{32}{K}\left(1 - e^{-Kt}\right).$$

(b) At each time t, $1 - e^{-Kt} < 1$. With $K > 0$,

$$v(t) = \frac{32}{K}\left(1 - e^{-Kt}\right) < \frac{32}{K} \quad \text{and} \quad \lim_{t\to\infty} v(t) = \frac{32}{K}$$

(c)



30. (a) $\frac{dP}{dt} + (b-a)P = 0$; $H(t) = \int (b-a)\,dt = (b-a)t$, integrating factor : $e^{(b-a)t}$

$$e^{(b-a)t}\frac{dP}{dt} + (b-a)e^{(b-a)t}P = 0$$

$$\frac{d}{dt}\left[e^{(b-a)t}P\right] = 0$$

$$e^{(b-a)t}P = C$$

$$P = Ce^{(a-b)t}$$

$P(0) = P_0 \implies P(t) = P_0e^{(a-b)t}$.

(b) (i) $a > b \implies P_0e^{(a-b)t}$ is increasing.

$P(t) \to \infty$ as $t \to \infty$.

(ii) $a = b \implies P(t) = P_0$ is a constant.

$P(t) = P_0$ as $t \to \infty$.

(iii) $a < b \implies P_0e^{(a-b)t}$ is decreasing.

$P(t) \to 0$ as $t \to \infty$.

31. (a)

$$\frac{di}{dt}+\frac{R}{L}i=\frac{E}{L}; \quad H(t)=\int\frac{R}{L}\,dt=\frac{R}{L}, \quad \text{integrating factor :} \quad e^{\frac{R}{L}t}$$

$$\begin{aligned} e^{\frac{R}{L}t}\frac{di}{dt}+\frac{R}{L}e^{\frac{R}{L}t}i &= \frac{E}{L}e^{\frac{R}{L}t}\\ \frac{d}{dt}\left[e^{\frac{R}{L}t}i\right] &= \frac{E}{L}e^{\frac{R}{L}t}\\ e^{\frac{R}{L}t}i &= \frac{E}{R}e^{\frac{R}{L}t}+C\\ i(t) &= \frac{E}{R}+Ce^{-\frac{R}{L}t}\end{aligned}$$

$i(0)=0 \implies C=-\frac{E}{R}$, so $i(t)=\frac{E}{R}\left[1-e^{-(R/L)t}\right]$.

(b) $\lim\limits_{t\to\infty} i(t)=\lim\limits_{t\to\infty}\frac{E}{R}\left(1-e^{-(R/L)t}\right)=\frac{E}{R}$ amps

(c) $i(t)=0.9\frac{E}{R} \implies e^{-(R/L)t}=\frac{1}{10} \implies -\frac{R}{L}t=-\ln 10 \implies t=\frac{L}{R}\ln 10$ seconds.

32. (a)

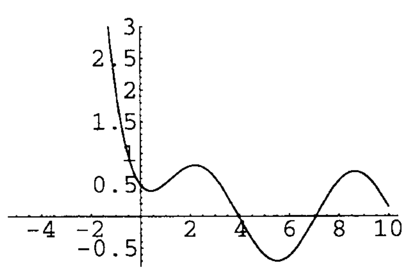
$$\frac{di}{dt}+\frac{R}{L}i=\frac{E}{L}\sin\omega t; \quad H(t)=\int\frac{R}{L}\,dt=\frac{R}{L}, \quad \text{integrating factor :} \quad e^{\frac{R}{L}t}$$

$$\begin{aligned} e^{\frac{R}{L}t}\frac{di}{dt}+\frac{R}{L}e^{\frac{R}{L}t}i &= \frac{E}{L}e^{\frac{R}{L}t}\sin\omega t\\ \frac{d}{dt}\left[e^{\frac{R}{L}t}i\right] &= \frac{E}{L}e^{\frac{R}{L}t}\sin\omega t\\ e^{\frac{R}{L}t}i &= \frac{E}{L}e^{\frac{R}{L}t}\frac{L^2}{R^2+\omega^2L^2}\left[\frac{R}{L}\sin\omega t-\omega\cos\omega t\right]+C\\ i(t) &= \frac{EL}{R^2+\omega^2L^2}\left[\frac{R}{L}\sin\omega t-\omega\cos\omega t\right]+Ce^{-\frac{R}{L}t}\end{aligned}$$

$$i(0)=0 \implies \text{i(t)}=\frac{EL}{R^2+\omega^2L^2}\left[\frac{R}{L}\sin\omega t-\omega\cos\omega t\right]+\left[i_0+\omega\frac{EL}{R^2+\omega^2L^2}\right]e^{-\frac{R}{L}t}.$$

(b) $\lim\limits_{t\to\infty}$ does not exist because the trigonometric functions continue to oscillate.

(c)



33. (a)

$$\begin{aligned} V'(t) &= ktV(t) \\ V'(t) - ktV(t) &= 0 \\ e^{-kt^2/2}V'(t) - kte^{-kt^2/2}\,V(t) &= 0 \\ \frac{d}{dt}\left[e^{-kt^2/2}\,V(t)\right] &= 0 \\ e^{-kt^2/2}\,V(t) &= C \\ V(t) &= Ce^{kt^2/2}. \end{aligned}$$

Since $V(0) = C = 200$,

$$V(t) = 200e^{kt^2/2}.$$

Since $V(5) = 160$,

$$200e^{k(25/2)} = 160, \quad e^{k(25/2)} = \tfrac{4}{5}, \quad e^k = \left(\tfrac{4}{5}\right)^{2/25}$$

and therefore

$$V(t) = 200\left(\tfrac{4}{5}\right)^{t^2/25} \text{ liters.}$$

(b) $V'(t) = kV(t) \implies V(t) = V_0e^{kt}$

Loses 20% in 5 minutes, so $V(5) = V_0e^{5k} = 0.8V_0 \implies k = \frac{1}{5}\ln 0.8$

$\implies V(t) = V_0e^{\frac{1}{5}(\ln 0.8)t} = V_0\left(e^{\ln 0.8}\right)^{t/5} = V_0(0.8)^{t/5} = V_0\left(\frac{4}{5}\right)^{t/5}.$

Since $V_0 = 200$ liters, we get $V(t) = 200\left(\frac{4}{5}\right)^{t/5}$

34. Let $s(t)$ be the number of pounds of salt present after t minutes. Since

$$s'(t) = \text{ rate in } - \text{ rate out } = 3\,(0.2) - 3\left(\frac{s(t)}{100}\right),$$

we have

$$s'(t) + 0.03s(t) = 0.6.$$

Using the approach in the proof of Theorem 7.6.1, multiply by $e^{\int 0.03dt} = e^{0.03t}$ to obtain

$$\begin{aligned} e^{0.03t}s'(t) + 0.03e^{0.03t}s(t) &= 0.6e^{0.03t} \\ \frac{d}{dt}\left[e^{0.03t}s(t)\right] &= 0.6e^{0.03t} \\ e^{0.03t}s(t) &= 20e^{0.03t} + C \\ s(t) &= 20 + Ce^{-0.03t}. \end{aligned}$$

We use the initial condition $s(0) = 100(0.25) = 25$ to determine C:

$$25 = 20 + Ce^0 \quad \text{so} \quad C = 5.$$

Thus, $s(t) = 20 + 5e^{-0.03t}$ lb.

35. (a) $\dfrac{dP}{dt} = k(M - P)$

(b) $\dfrac{dP}{dt} + kP = kM; \qquad H(t) = \int k\,dt = kt, \quad \text{integrating factor: } \quad e^{kt}$

$$e^{kt}\frac{dP}{dt} + ke^{kt}P = kM\,ekt$$

$$\frac{d}{dt}\left[e^{kt}P\right] = kM\,e^{kt}$$

$$e^{kt}P = M\,e^{kt} + C$$

$$P = M + Ce^{-kt}$$

$$P(0) = M + C = 0 \quad \Longrightarrow \quad C = -M \quad \text{and} \quad P(t) = M\left(1 - e^{-kt}\right)$$

$$P(10) = M\left(1 - e^{-10k}\right) = 0.3M \quad \Longrightarrow \quad k \cong 0.0357 \quad \text{and} \quad P(t) = M\left(1 - e^{-0.0357t}\right)$$

(c) $P(t) = M\left(1 - e^{-0.0357t}\right) = 0.9M \quad \Longrightarrow \quad e^{-0.0357t} = 0.1 \quad \Longrightarrow \quad t \cong 65$

Therefore, it will take approximately 65 days for 90 % of the population to be aware of the product.

36. (a) $\dfrac{dQ}{dt} = \text{rate in} - \text{rate out} = r - kQ, \quad k > 0$

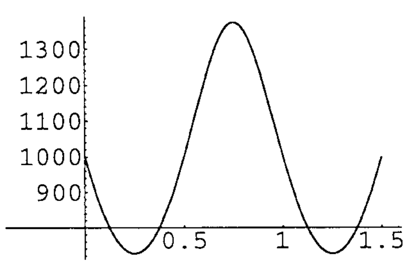
(b) $\dfrac{dQ}{dt} + kQ = r, \quad Q(0) = 0 \quad \Longrightarrow Q(t) = \dfrac{r}{k}\left(1 - e^{-kt}\right)$

(c) $\displaystyle\lim_{t\to\infty} Q(t) = \frac{r}{k}.$

37. (a)

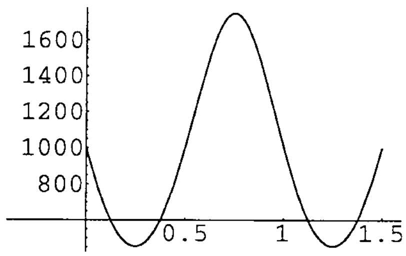
$$\frac{dP}{dt} - 2\cos 2\pi t P = 0 \quad \Longrightarrow P = Ce^{-\frac{1}{\pi}\sin 2\pi t}.$$

$$P(0) = C = 1000 \quad \Longrightarrow P = 1000e^{-\frac{1}{\pi}\sin 2\pi t}.$$



(b) $\frac{dP}{dt} - 2\cos 2\pi t P = 2000\cos 2\pi t \implies P = Ce^{-\frac{1}{\pi}\sin 2\pi t} - 1000.$

$P(0) = 1000 \implies C = 2000 \implies P = 2000e^{-\frac{1}{\pi}\sin 2\pi t} - 1000.$



38. (a) Let $Q = \ln P$. Then $\frac{dQ}{dt} = \frac{1}{P}\frac{dP}{dt} = a - bQ$.

Solving the differential equation $\frac{dQ}{dt} + bQ = a \implies Q = \frac{a}{b} + Ce^{-bt}$, so $P = e^{\frac{a}{b} + Ce^{-bt}}$.

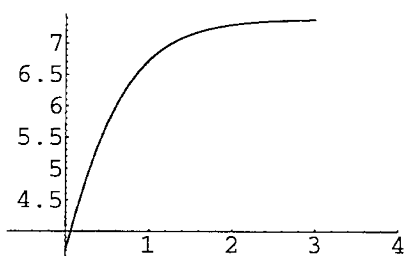
$P(0) = P_0 \implies e^c = P_0 e^{-\frac{a}{b}}$. Thus $P = e^{\frac{a}{b}}\left[P_0 e^{-\frac{a}{b}}\right]^{e^{-bt}}$.

(b) $e^{-bt} \to 0$ as $t \to \infty$, so $P \to e^{\frac{a}{b}}$.

(c) $P' = P(a - b\ln P) \implies P'' = P\left(\frac{-b}{P}\right)P' + P'(a - b\ln P) = P(a - b\ln P)(a - b - b\ln P)$.

Concavity depends on the constants.

(d)



SECTION 8.9

1.

$$y' = y\,\sin(2x+3)$$

$$\frac{1}{y}\,dy = \sin(2x+3)\,dx$$

$$\int \frac{1}{y}\,dy = \int \sin(2x+3)\,dx$$

$$\ln|y| = -\frac{1}{2}\cos(2x+3) + C$$

This solution can also be written: $y = Ce^{-(1/2)\cos(2x+3)}$.

2.

$$\begin{aligned} y' &= (x^2+1)(y^2+y) \\ \int \frac{dy}{y^2+y} &= \int (x^2+1)\,dx \\ \ln\left|\frac{y}{y+1}\right| &= \frac{x^3}{3}+x+C \end{aligned}$$

$y=0$ and $y=-1$ are singular solutions.

3.

$$\begin{aligned} y' &= (xy)^3 \\ \frac{1}{y^3}\,dy &= x^3\,dx, \qquad y \neq 0 \\ \int \frac{1}{y^3}\,dy &= \int x^3\,dx \\ -\frac{1}{2}y^{-2} &= \frac{1}{4}x^4+C \end{aligned}$$

This solution can also be written: $x^4+\dfrac{2}{y^2}=C,$ or $y^2=\dfrac{2}{C-x^4}$;

$y=0$ is a singular solution.

4.

$$\begin{aligned} y' &= 3x^2(1+y^2) \\ \int \frac{dy}{1+y^2} &= \int 3x^2\,dx \\ \tan^{-1} y &= x^3+C \\ y &= \tan(x^3+C) \end{aligned}$$

5.

$$\begin{aligned} y' &= -\frac{\sin(1/x)}{x^2y\cos y} \\ y\cos y\,dy &= -\frac{1}{x^2}\sin(1/x)\,dx \\ \int y\cos y\,dy &= \int -\frac{1}{x^2}\sin(1/x)\,dx \\ y\sin y+\cos y &= -\cos(1/x)+C \end{aligned}$$

6.

$$\begin{aligned} y' &= \frac{y^2+1}{y+yx} \\ \int \frac{y}{1+y^2}\,dy &= \int \frac{1}{1+x}\,dx \\ \ln\sqrt{1+y^2} &= \ln|1+x|+C \end{aligned}$$

7.

$$\begin{aligned} y' &= x\,e^{y-x} \\ e^{-y}\,dy &= xe^{-x}\,dx \\ \int e^{-y}\,dy &= \int xe^{-x}\,dx \\ e^{-y} &= xe^{-x}+e^{-x}+C \end{aligned}$$

8.
$$y' = xy^2 - x - y^2 + 1 = (x-1)(y^2-1)$$
$$\int \frac{dy}{y^2-1} = \int \frac{dx}{x-1}$$
$$\frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| = \ln|x-1| + C$$

$y = 1,\ y = -1$ are singular solutions.

9.
$$(y \ln x)y' = \frac{(y+1)^2}{x}$$
$$\frac{y}{(y+1)^2}\,dy = \frac{1}{x \ln x}\,dx$$
$$\int \frac{y}{(y+1)^2}\,dy = \int \frac{1}{x\ln x}\,dx$$
$$\ln|y+1| + \frac{1}{y+1} = \ln|\ln x| + C$$

10.
$$e^y \sin 2x\,dx + \cos x(e^{2y} - y)\,dy = 0$$
$$\int \frac{\sin 2x}{\cos x}\,dx + \int (e^y - ye^{-y})\,dy = C$$
$$-2\cos x + e^y + e^{-y}(1+y) = C$$

11.
$$y' = x\sqrt{\frac{1-y^2}{1-x^2}}, \qquad y(0) = 0$$
$$\frac{1}{\sqrt{1-y^2}}\,dy = \frac{x}{\sqrt{1-x^2}}\,dx$$
$$\int \frac{1}{\sqrt{1-y^2}}\,dy = \int \frac{x}{\sqrt{1-x^2}}\,dx$$
$$\sin^{-1} y = -\sqrt{1-x^2} + C$$

$$y(0) = 0 \implies \sin^{-1} 0 = -1 + C \implies C = 1$$

Thus, $\sin^{-1} y = 1 - \sqrt{1-x^2}$.

12.
$$y' = \frac{e^{x-y}}{1+e^x}$$
$$\int e^y\,dy = \int \frac{e^x}{1+e^x}$$
$$e^y = \ln(1+e^x) + C$$

$y(1) = 0 \implies 1 = \ln(1+e) + C \implies C = 1 - \ln(1+e)$ and $e^y = \ln(1+e^x) + 1 - \ln(1+e)$

13.
$$y' = \frac{x^2y - y}{y+1}, \qquad y(3) = 1$$
$$\frac{y+1}{y}\,dy = (x^2-1)\,dx, \qquad y \neq 0$$
$$\int \frac{y+1}{y}\,dy = \int (x^2-1)\,dx$$
$$y + \ln|y| = \frac{1}{3}x^3 - x + C$$

$$y(3)=1 \implies 1+\ln 1=\frac{1}{3}(3)^3-3+C \implies C=-5.$$

Thus, $\quad y+\ln|y|=\frac{1}{3}x^3-x-5.$

14.

$$x^2y'=y-xy$$
$$\int\frac{1}{y}\,dy=\int(1-x)x^{-2}\,dx$$
$$\ln|y|=-\frac{1}{x}-\ln|x|+C$$

$-1=y(-1) \implies C=-1 \quad \text{and} \quad \ln|xy|+\frac{1}{x}=-1$

15. $\quad (xy^2+y^2+x+1)\,dx+(y-1)\,dy=0, \qquad y(2)=0$

$$(x+1)(y^2+1)\,dx+(y-1)\,dy=0$$
$$(x+1)\,dx+\frac{y-1}{y^2+1}\,dy=0$$
$$\int(x+1)\,dx+\int\frac{y-1}{y^2+1}\,dy=C$$
$$\frac{x^2}{2}+x+\frac{1}{2}\ln(y^2+1)-\tan^{-1}y=C$$

$y(2)=0 \implies C=4.$ Thus, $\frac{1}{2}x^2+x+\frac{1}{2}\ln(y^2+1)-\tan^{-1}y=4$

16.

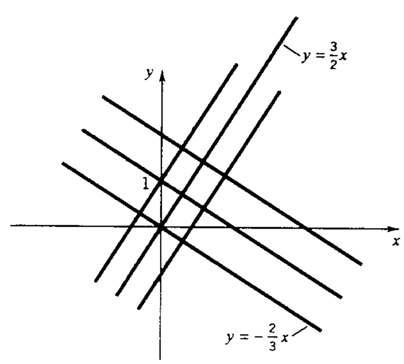
$$\cos y\,dx+(1-e^{-x})\sin y\,dy=0$$
$$\int\frac{dx}{1+e^{-x}}+\int\frac{\sin y}{\cos y}\,dy=C$$
$$\ln(e^x+1)+\ln|\sec y|=C;$$
$$\frac{\pi}{4}=y(0) \implies \ln 2+\ln\sqrt{2}=C$$
$$\ln(e^x+1)+\ln|\sec y|=\frac{3}{2}\ln 2$$

17. $\quad 2x+3y=C \implies 2+3y'=0 \implies y'=-\frac{2}{3}$

The orthogonal trajectories are the solutions of:

$$y'=\frac{3}{2}.$$

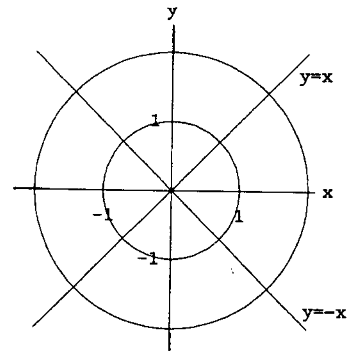
$y'=\frac{3}{2} \implies y=\frac{3}{2}x+C$



18. Curves: $y = Cx, \quad y' = C = \dfrac{y}{x}$

orthogonal trajectories: $y' = -\dfrac{x}{y}$

$$\int y\,dy + \int x\,dx = K_1; \quad x^2 + y^2 = K \quad (= 2K_1)$$



19. $xy = C \implies y + xy' = 0 \implies y' = -\dfrac{y}{x}$

The orthogonal trajectories are the solutions of:

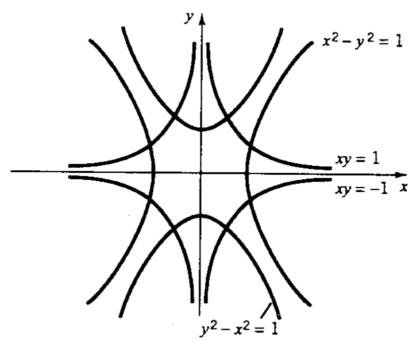
$y' = \dfrac{x}{y}.$

$$y' = \frac{x}{y}$$

$$\int x\,dx = \int y\,dy$$

$$\tfrac{1}{2}x^2 = \tfrac{1}{2}y^2 + C$$

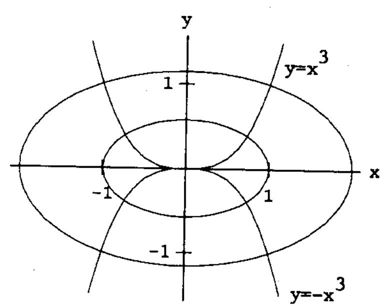
$$\text{or} \quad x^2 - y^2 = C$$



20. $y = Cx^3, \quad y' = 3Cx^2 = \dfrac{3y}{x}$

orthogonal trajectories: $y' = -\dfrac{x}{3y}$

$$\int 3y\,dy + \int x\,dx = K_1; \quad 3y^2 + x^2 = K \quad (= 2K_1)$$



21. $y = Ce^x \implies y' = Ce^x = y$

The orthogonal trajectories are the solutions of:

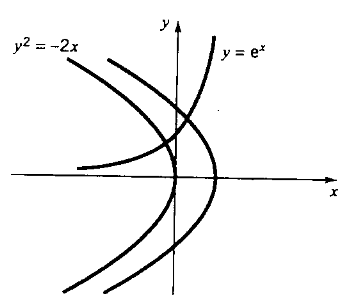
$y' = -\dfrac{1}{y}.$

$$y' = -\frac{1}{y}$$

$$\int y\,dy = -\int dx$$

$$\tfrac{1}{2}y^2 = -x + K$$

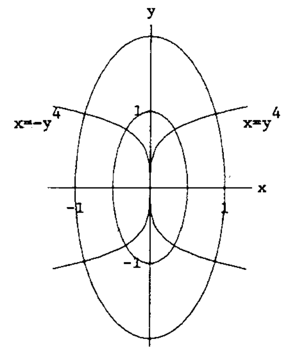
$$\text{or} \quad y^2 = -2x + C$$



22. $x = Cy^4, \quad 1 = 4Cy^3y'; \quad y' = \dfrac{y}{4x}$

orthogonal trajectories: $\dfrac{dy}{dx} = -\dfrac{4x}{y}$

$\displaystyle\int y\,dy + \int 4x\,dx = K_1; \quad y^2 + 4x^2 = K \quad (= 2K_1)$



23. A differential equation for the given family is:

$$y^2 = 2xyy' + y^2(y')^2$$

A differential equation for the family of orthogonal trajectories is found by replacing y' by $-1/y'$. The result is:

$$y^2 = -\frac{2xy}{y'} + \frac{y^2}{(y')^2} \qquad \text{which simplifies to} \qquad y^2 = 2xyy' + y^2(y')^2$$

Thus, the given family is self-orthogonal.

24. $\dfrac{x^2}{C^2} + \dfrac{y^2}{C^2 - 4} = 1 \implies \dfrac{2x}{C^2} + \dfrac{2yy'}{C^2 - 4} = 0 \implies C^2 = \dfrac{4x}{x + yy'}$

A differential equation for the given family is:

$$x^2 + xyy' - \frac{xy}{y'} - y^2 = 4$$

A differential equation for the family of orthogonal trajectories is found by replacing y' by $-1/y'$.

The result is:

$$x^2 - xy\frac{1}{y'} + xyy' - y^2 = 4$$

Thus, the given family is self-orthogonal.

25. We assume that $C = 0$ at time $t = 0$. (a) Let $A_0 = B_0$. Then

$$\frac{dC}{dt} = k(A_0 - C)^2 \quad \text{and} \quad \frac{dC}{(A_0 - C)^2} = k\,dt \qquad \text{see Section 7.6.}$$

Integrating, we get

$$\int \frac{1}{(A_0 - C)^2}\,dC = \int k\,dt$$

$$\frac{1}{A_0 - C} = kt + M \qquad M \text{ an arbitrary constant.}$$

Since $C(0) = 0$, $M = \dfrac{1}{A_0}$ and

$$\frac{1}{A_0 - C} = kt + \frac{1}{A_0}.$$

Solving this equation for C gives

$$C(t) = \frac{kA_0^2 t}{1 + kA_0 t}.$$

(b) Suppose that $A_0 \neq B_0$. Then

$$\frac{dC}{dt} = k(A_0 - C)(B_0 - C) \quad \text{and} \quad \frac{dC}{(A_0 - C)(B_0 - C)} = k\,dt.$$

Integrating, we get

$$\int \frac{1}{(A_0 - C)(B_0 - C)}\,dC = \int k\,dt$$

$$\frac{1}{B_0 - A_0}\int\left(\frac{1}{A_0 - C} - \frac{1}{B_0 - C}\right)dC = \int k\,dt$$

$$\frac{1}{B_0 - A_0}\left[-\ln(A_0 - C) + \ln(B_0 - C)\right] = kt + M$$

$$\frac{1}{B_0 - A_0}\ln\left(\frac{B_0 - C}{A_0 - C}\right) = kt + M \qquad M \text{ an arbitrary constant}$$

Since $C(0) = 0$, $M = \frac{1}{B_0 - A_0}\ln\left(\frac{B_0}{A_0}\right)$ and

$$\frac{1}{B_0 - A_0}\ln\left(\frac{B_0 - C}{A_0 - C}\right) = kt + \frac{1}{B_0 - A_0}.$$

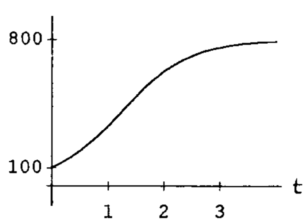
Solving this equation for C, gives

$$C(t) = \frac{A_0 B_0\left(e^{kA_0 t} - e^{kB_0 t}\right)}{A_0 e^{kA_0 t} - B_0 e^{kB_0 t}}.$$

26. (a) From Example 7 with $K = 0.0020$, $M = 800$, $R = P(0) = 100$, we have

$$P(t) = \frac{80,000}{100 + 700e^{-1.6t}}$$

(b)



(c) $\frac{dP}{dt}$ is maximal at $t \cong 1.2162$

Maximum value $= 320$

27. (a)

$$m\frac{dv}{dt} = -\alpha v - \beta v^2$$

$$\frac{dv}{v(\alpha+\beta v)} = -\frac{1}{m}\,dt$$

$$\int \frac{1}{v(\alpha+\beta v)}\,dv = -\int \frac{1}{m}\,dt$$

$$\frac{1}{\alpha}\int \frac{1}{v}\,dv - \frac{\beta}{\alpha}\int \frac{1}{\alpha+\beta v}\,dv = -\int \frac{1}{m}\,dt$$

$$\frac{1}{\alpha}\ln v - \frac{1}{\alpha}\ln(\alpha+\beta v) = -\frac{1}{m}t + M, \quad M \text{ an arbitrary constant}$$

$$\ln\left(\frac{v}{\alpha+\beta v}\right) = -\frac{\alpha}{m}t + M$$

$$\frac{v}{\alpha+\beta v} = Ke^{-\alpha t/m} \quad [K = e^M]$$

Solving this equation for v we get $v(t) = \dfrac{\alpha K}{e^{\alpha t/m} - \beta K} = \dfrac{\alpha}{Ce^{\alpha t/m} - \beta}$ $[C = 1/K]$.

(b) Setting $v(0) = v_0$, we get

$$C = \frac{\alpha + \beta v_0}{v_0} \quad \text{and}$$

$$v(t) = \frac{\alpha v_0}{(\alpha+\beta v_0)e^{\alpha t/m} - \beta v_0}$$

(c) $\lim_{t\to\infty} v(t) = 0$

28. $F = ma = m\dfrac{dv}{dt}$

(a)

$$m\frac{dv}{dt} = mg - \beta v^2$$

$$1 = \frac{m(dv/dt)}{mg - \beta v^2} = \frac{m}{\beta}\left(\frac{dv/dt}{v_c{}^2 - v^2}\right)$$

$$t = \frac{m}{\beta}\int \frac{dv/dt}{v_c{}^2 - v^2}\,dt = \frac{m}{2v_c\beta}\int\left(\frac{dv/dt}{v_c+v} + \frac{dv/dt}{v_c - v}\right)dt$$

$$= \frac{m}{2v_c\beta}\left[\ln\left(\frac{v_c+v}{v_c-v}\right)\right] + C$$

At $t = 0$, $v(0) = v_0$. Therefore

$$C = -\frac{m}{2v_c\beta}\left[\ln\left(\frac{v_c+v_0}{v_c-v_0}\right)\right] = \frac{m}{2v_c\beta}\left[\ln\left(\frac{v_c-v_0}{v_c+v_0}\right)\right].$$

Thus

$$t = \frac{m}{2v_c\beta}\left[\ln\left(\frac{v_c+v}{v_c+v_0}\cdot\frac{v_c-v_0}{v_c-v}\right)\right] = \frac{v_c}{2g}\left[\ln\left(\frac{v_c+v}{v_c+v_0}\cdot\frac{v_c-v_0}{v_c-v}\right)\right].$$

$$v_c = \sqrt{mg/\beta}$$

(b)

$$\frac{v_c+v}{v_c+v_0}\cdot\frac{v_c-v_0}{v_c-v} = e^{2tg/v_c}$$

$$v_c + v = \frac{v_c+v_0}{v_c-v_0}e^{2tg/v_c}(v_c - v)$$

$$v\left[1 + \left(\frac{v_c+v_0}{v_c-v_0}\right)e^{2tg/v_c}\right] = v_c\left[\left(\frac{v_c+v_0}{v_c-v_0}\right)e^{2tg/v_c} - 1\right]$$

$$v = v_c\left[\frac{(v_c+v_0)e^{2tg/v_c}-(v_c-v_0)}{(v_c+v_0)e^{2tg/v_c}+(v_c-v_0)}\right].$$

We can bring the hyperbolic function into play by writing

$$\begin{aligned} v &= v_c\left[\frac{(v_c+v_0)e^{gt/v_c}-(v_c-v_0)e^{-gt/v_c}}{(v_c+v_0)e^{gt/v_c}-(v_c-v_0)e^{-gt/v_c}}\right] \\ &= v_c\left[\frac{v_0\cosh(gt/v_c)+v_c\sinh(gt/v_c)}{v_0\sinh(gt/v_c)+v_c\cosh(gt/v_c)}\right] \end{aligned}$$

(c) $$a = g\left\{\frac{{v_c}^2-{v_0}^2}{[v_0\sinh(gt/v_c)+v_0\cosh(gt/v_c)]^2}\right\}$$

The acceleration can not change sign since the denominator is always positive and the numerator is constant. As $t\to\infty$, the denominator $\to\infty$, and the fraction $\to 0$.

(d) We can write

$$v = v_c\left[\frac{(v_c+v_0)-(v_c-v_0)e^{-2tg/v_c}}{(v_c+v_0)+(v_c-v_0)e^{-2tg/v_c}}\right].$$

As $t\to\infty$, $-2gt/v_c\to-\infty$ and $e^{-2tg/v_c}\to 0$. Thus $v\to v_c$

29. (a) Let $P=P(t)$ denote the number of people who have the disease at time t. Then

$$\begin{aligned} \frac{dP}{dt} &= kP(25{,}000-P) \quad k>0 \text{ constant} \\ \frac{dP}{P(25{,}000-P)} &= k\,dt \\ \int\frac{1}{P(25{,}000-P)}\,dP &= \int k\,dt \\ \frac{1}{25{,}000}\ln\left|\frac{P}{25{,}000-P}\right| &= kt+M \end{aligned}$$

Solving for P, we get

$$P(t) = \frac{25{,}000}{1+Ce^{25{,}000\,kt}}$$

Now, $P(0)=\dfrac{25{,}000}{1+C}=100 \implies C=249.$

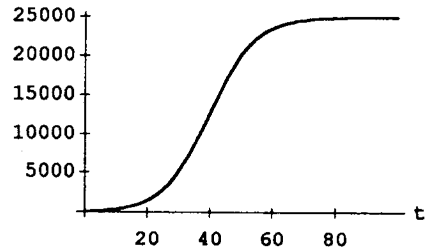
Also, $P(10)=\dfrac{25{,}000}{1+249e^{25{,}000\,(10k)}}=400 \implies 25{,}000k\cong-0.1382.$

Therefore, $P(t)=\dfrac{25{,}000}{1+249e^{-0.1382\,t}}.$

$P(20)=\dfrac{25{,}000}{1+249e^{-0.1382\,(20)}}\cong 1498;$ 1498 people will have the disease after 20 days.

(b) $\dfrac{25,000}{1+249e^{-0.1382\,t}} = 12,500 \implies t \cong 40$; (c)

It will take 40 days for half the population to have the disease.



30. $\dfrac{dy}{dt} = ky(M-y) = kMy - ky^2 \implies \dfrac{d^2y}{dt^2} = (kM - 2ky)\dfrac{dy}{dt} = k^2(M-2y)(M-y)y$

$\dfrac{d^2y}{dt^2} > 0$ for $0 < y < \dfrac{M}{2}$, so $\dfrac{dy}{dt}$ is increasing

$\dfrac{d^2y}{dt^2} < 0$ for $\dfrac{M}{2} < y < M$, so $\dfrac{dy}{dt}$ is decreasing

Therefore $\dfrac{dy}{dt}$ is maximal at $y = \dfrac{M}{2}$. The disease is spreading fastest when half the population is infected.

31. Assume that the package is dropped from rest.

(a) Let $v = v(t)$ be the velocity at time t. Then

$$100\frac{dv}{dt} = 100g - 2v \qquad \text{or} \qquad \frac{dv}{dt} + \frac{1}{50}v = g \quad (g = 9.8\text{m/sec}^2)$$

This is a linear differential equation; $e^{t/50}$ is an integrating factor.

$$e^{t/50}\frac{dv}{dt} + \frac{1}{50}e^{t/50}v = g\,e^{t/50}g$$

$$\frac{d}{dt}\left[e^{t/50}v\right] = g\,e^{t/50}$$

$$e^{t/50}v = 50g\,e^{t/50} + C$$

$$v = 50g + Ce^{-t/50}$$

Now, $v(0) = 0 \implies C = -50g$ and $v(t) = 50g\left(1 - e^{-t/50}\right)$.

At the instant the parachute opens, $v(10) = 50g\left(1 - e^{-1/5}\right) \cong 50g(0.1813) \cong 88.82$ m/sec.

(b) Now let $v = v(t)$ denote the velocity of the package t seconds after the parachute opens. Then

$$100\frac{dv}{dt} = 100g - 4v^2 \qquad \text{or} \qquad \frac{dv}{dt} = g - \frac{1}{25}v^2$$

This is a separable differential equation:

$$\begin{aligned}
\frac{dv}{dt} &= g - \frac{1}{25}v^2 \quad \text{set } u = v/5,\ du = (1/5)dv \\
\frac{du}{g-u^2} &= \frac{1}{5}\,dt \\
\frac{1}{2\sqrt{g}}\ln\left|\frac{u+\sqrt{g}}{u-\sqrt{g}}\right| &= \frac{t}{5} + K \\
\ln\left|\frac{u+\sqrt{g}}{u-\sqrt{g}}\right| &= \frac{2\sqrt{g}}{5}t + M \\
\frac{u+\sqrt{g}}{u-\sqrt{g}} &= Ce^{2\sqrt{g}t/5} \cong Ce^{1.25\,t} \\
u &= \sqrt{g}\,\frac{Ce^{1.25\,t}+1}{Ce^{1.25t}-1} \\
v &= 5\sqrt{g}\,\frac{Ce^{1.25\,t}+1}{Ce^{1.25t}-1}
\end{aligned}$$

Now, $v(0) = 88.82 \implies 5\sqrt{g}\,\dfrac{C+1}{C-1} = 88.82 \implies C \cong 1.43.$

Therefore, $v(t) = 5\sqrt{g}\,\dfrac{1.43e^{1.25\,t}+1}{1.43e^{1.25t}-1} = \dfrac{15.65\left(1+0.70e^{-1.25\,t}\right)}{1-0.70e^{-1.25\,t}}$

(c) From part (b), $\lim\limits_{t\to\inf} v(t) = 15.65\text{m/sec}.$

32. (a) By the hint

$$\begin{aligned}
\int \frac{dC}{\left(A_0 - \frac{1}{2}C\right)^2} &= \int k\,dt \\
\frac{2}{A_0 - \frac{1}{2}C} &= kt + K.
\end{aligned}$$

First, $C(0) = 0 \implies K = 2/A_0$. Then, $C(1) = A_0 \implies k = 2/A_0$. Thus,

$$\frac{2}{A_0 - \frac{1}{2}C} = \frac{2}{A_0}(t+1), \quad \text{which gives} \quad C(t) = 2A_0\left(\frac{t}{t+1}\right).$$

(b) By the hint

$$\begin{aligned}
\int \frac{dC}{\left(A_0 - \frac{1}{2}C\right)\left(2A_0 - \frac{1}{2}C\right)} &= \int k\,dt \\
\frac{1}{A_0}\int\left[\frac{1}{A_0 - \frac{1}{2}C} - \frac{1}{2A_0 - \frac{1}{2}C}\right]dC &= \int k\,dt \\
\frac{1}{A_0}\left[-2\ln|A_0 - \tfrac{1}{2}C| + 2\ln|2A_0 - \tfrac{1}{2}C|\right] &= kt + K \\
\frac{2}{A_0}\ln\left|\frac{2A_0 - \frac{1}{2}C}{A_0 - \frac{1}{2}C}\right| &= kt + K.
\end{aligned}$$

First, $C(0) = 0 \implies K = \dfrac{2}{A_0}\ln 2.$ Then,

$$C(1) = A_0 \implies \frac{2}{A_0}\ln 3 = k + \frac{2}{A_0}\ln 2 \implies k = \frac{2}{A_0}\ln\frac{3}{2}.$$

Thus,

$$\frac{2}{A_0}\ln\left|\frac{2A_0-\frac{1}{2}C}{A_0-\frac{1}{2}C}\right| = \frac{2}{A_0}t\ln\frac{3}{2}+\frac{2}{A_0}\ln 2 = \frac{2}{A_0}\ln\left[2\left(\frac{3}{2}\right)^t\right]$$

so that

$$\frac{2A_0-\frac{1}{2}C}{A_0-\frac{1}{2}C} = 2\left(\frac{3}{2}\right)^t \quad \text{and therefore} \quad C(t) = 4A_0\frac{3^t-2^t}{2(3^t)-2^t}.$$

(c) By the hint

$$\int \frac{dC}{\left(A_0-\frac{m}{m+n}C\right)\left(A_0-\frac{n}{m+n}C\right)} = \int k\,dt$$

$$\int \frac{1}{A_0(m-n)}\left[\frac{m}{A_0-\frac{m}{m+n}C}-\frac{n}{A_0-\frac{n}{m+n}C}\right]dC = \int k\,dt$$

$$\frac{1}{A_0(m-n)}\left[-(m+n)\ln\left|A_0-\frac{m}{m+n}C\right|+(m+n)\ln\left|A_0-\frac{n}{m+n}C\right|\right] = kt+K$$

$$\frac{m+n}{A_0(m-n)}\ln\left|\frac{A_0-\frac{n}{m+n}C}{A_0-\frac{m}{m+n}C}\right| = kt+K.$$

First, $C(0)=0 \implies K = \frac{m+n}{A_0(m-n)}\ln\left|\frac{A_0}{A_0}\right| = 0.$ Then,

$$C(1)=A_0 \implies k = \frac{m+n}{A_0(m-n)}\ln\left|\frac{A_0-\frac{n}{m+n}A_0}{A_0-\frac{m}{m+n}A_0}\right| = \frac{m+n}{A_0(m-n)}\ln\frac{m}{n}.$$

Thus,

$$\frac{m+n}{A_0(m-n)}\ln\left|\frac{A_0-\frac{n}{m+n}C}{A_0-\frac{m}{m+n}C}\right| = \frac{m+n}{A_0(m-n)}\ln\left(\frac{m}{n}\right)(t)+0$$

so that

$$\frac{A_0-\frac{n}{m+n}C}{A_0-\frac{m}{m+n}C} = \left(\frac{m}{n}\right)^t \quad \text{and therefore} \quad C(t) = A_0(m+n)\left[\frac{m^t-n^t}{m^{t+1}-n^{t+1}}\right].$$

CHAPTER 9

SECTION 9.1

1. $8x^6 + 64 = 8(x^2+2)(x^4 - 2x^2 + 4)$

2. $4(x^2 + \frac{1}{2})^2$

3. $4x^2 + 12x + 9 = (2x+3)^2$

4. $-3, 3$

5. $x^2 - x - 2 = (x-2)(x+1) = 0; \quad x = 2, -1$

6. $-\frac{1}{2}, 3$

7. Adjust the sign of A and B so that the equation reads $Ax + By = |C|$. Then we have

$$x\frac{A}{\sqrt{A^2+B^2}} + y\frac{B}{\sqrt{A^2+B^2}} = \frac{|C|}{\sqrt{A^2+B^2}}.$$

Now set $\dfrac{A}{\sqrt{A^2+B^2}} = \cos\alpha, \quad \dfrac{B}{\sqrt{A^2+B^2}} = \sin\alpha, \quad \dfrac{|C|}{\sqrt{A^2+B^2}} = p$

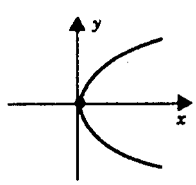
p is the length of $\overline{OQ}$, the distance between the line and the origin; α is the angle from the positive x-axis to the line segment $\overline{OQ}$.

8. Using Exercise 7, $p = \dfrac{|C|}{\sqrt{A^2+B^2}}$ is the distance between the line and the origin.

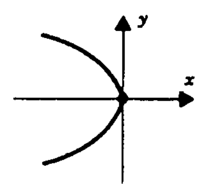
Thus the distance between the two lines is:

$$d = |p' - p| = \left|\frac{|C'|}{\sqrt{A^2+B^2}} - \frac{|C|}{\sqrt{A^2+B^2}}\right| = \frac{||C'| - |C||}{\sqrt{A^2+B^2}}.$$

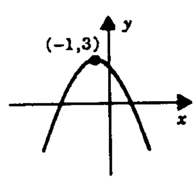
9. $y^2 = 8x$



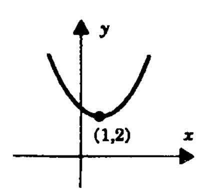
10. $y^2 = -8x$



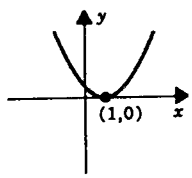
11. $(x+1)^2 = -12(y-3)$



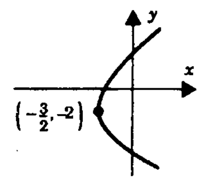
12. $(x-1)^2 = 4(y-2)$



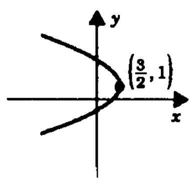
13. $(x-1)^2 = 4y$



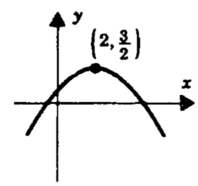
14. $(y+2)^2 = 14(x+\dfrac{3}{2})^2$



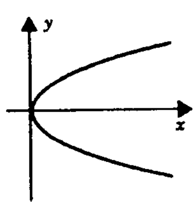
15. $(y-1)^2 = -2(x-\frac{3}{2})$



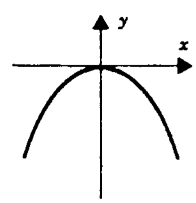
16. $(x-2)^2 = -6(y-\dfrac{3}{2})$



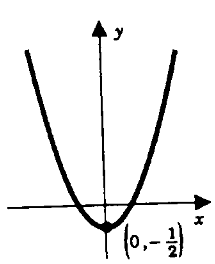
17. $y^2 = 2x$
vertex $(0,0)$
focus $(\frac{1}{2},0)$
axis $y=0$
directrix $x=-\frac{1}{2}$



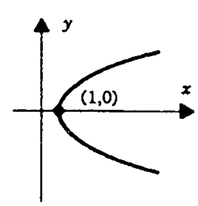
18. $x^2 = -5y$
vertex $(0,0)$
focus $(0,-\frac{5}{4})$
axis $x=0$
directrix $y=\frac{5}{4}$



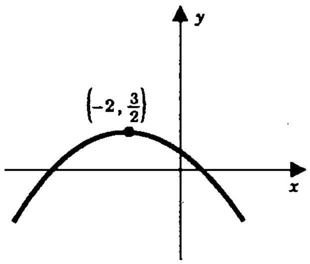
19. $x^2 = \frac{1}{2}(y+\frac{1}{2})$
vertex $(0,-\frac{1}{2})$
focus $(0,-\frac{3}{8})$
axis $x=0$
directrix $y=-\frac{5}{8}$



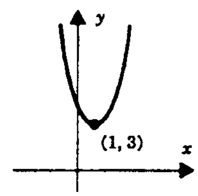
20. $y^2 = 2(x-1)$
vertex $(1,0)$
focus $(\frac{3}{2},0)$
axis $y=0$
directrix $x=\frac{1}{2}$



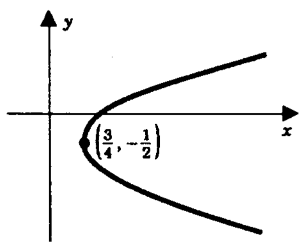
21. $(x+2)^2 = -8(y-\frac{3}{2})$
vertex $(-2, \frac{3}{2})$
focus $(-2, -\frac{1}{2})$
axis $x = -2$
directrix $y = \frac{7}{2}$



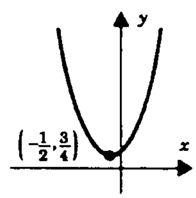
22. $y - 3 = 2(x-1)^2$
vertex $(1, 3)$
focus $(1, \frac{25}{8})$
axis $x = 1$
directrix $y = \frac{23}{8}$



23. $(y+\frac{1}{2})^2 = x - \frac{3}{4}$
vertex $(\frac{3}{4}, -\frac{1}{2})$
focus $(1, -\frac{1}{2})$
axis $y = -\frac{1}{2}$
directrix $x = \frac{1}{2}$



24. $y = x^2 + x + 1$
vertex $(-\frac{1}{2}, \frac{3}{4})$
focus $(-\frac{1}{2}, 1)$
axis $x = -\frac{1}{2}$
directrix $y = \frac{1}{2}$



25. $\sqrt{(x-1)^2 + (y-2)^2} = \dfrac{|x+y+1|}{\sqrt{2}}$ simplifies to $(x-y)^2 = 6x + 10y - 9$

26. $\sqrt{\left(x - \dfrac{18}{5}\right)^2 + \left(y + \dfrac{4}{5}\right)^2} = \dfrac{|2x-y|}{\sqrt{5}}$ simplifies to $(x+2y)^2 = 36x - 8y - 68$

27. Directrix has equation $x - y - 6 = 0$ since it has slope 1 and passes through the point $(4, -2)$.

$\sqrt{x^2 + (y-2)^2} = \dfrac{|x-y-6|}{\sqrt{2}}$ simplifies to $(x+y)^2 = -12x + 20y + 28$.

28. Directrix through $(6, -1)$ with slope 3, so $3x - y - 19 = 0$

$\sqrt{x^2 + (y-1)^2} = \dfrac{|3x - y + 19|}{\sqrt{10}}$ simplifies to $(x+3y)^2 = 58y - 114x + 351$

29. $P(x, y)$ is on the parabola with directrix $l\colon Ax + By + C = 0$ and focus $F(a, b)$ iff $d(P, l) = d(P, F)$

which happens iff $\dfrac{|Ax + By + C|}{\sqrt{A^2 + B^2}} = \sqrt{(x-a)^2 + (y-b)^2}$.

Squaring both sides of this equation and simplifying, we obtain

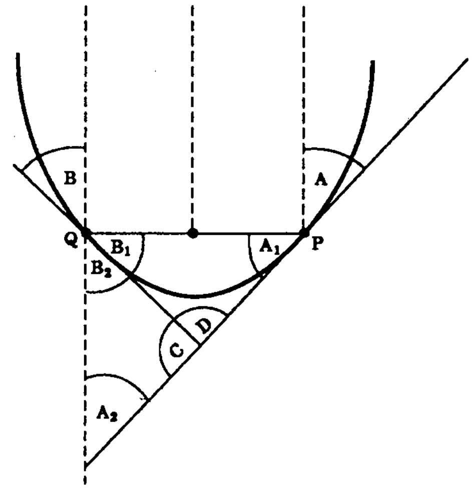
$$(Ay - Bx)^2 = (2aS + 2AC)x + (2bS + 2BC)y + c^2 - (a^2 + b^2)S$$

with $S = A^2 + B^2 \neq 0$.

30. $A = A_1$, $B = B_1$ by the reflection property of parabola;

$A = A_2$, $B = B_2$ by simple geometry. Therefore

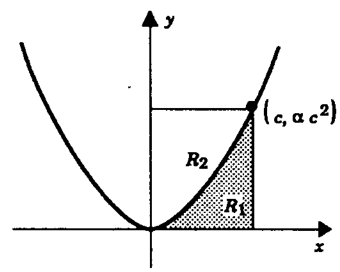
$A_1 = A_2$, $B_1 = B_2$ and $C = D = \frac{1}{2}\pi$



31. We can choose the coordinate system so that the parabola has an equation of the form $y = \alpha x^2, \alpha > 0$. One of the points of intersection is then the origin and the other is of the form $(c, \alpha c^2)$. We will assume that $c > 0$.

$$\text{area of } R_1 = \int_0^c \alpha x^2 dx = \frac{1}{3}\alpha c^3 = \frac{1}{3}A,$$

area of $R_2 = A - \frac{1}{3}A = \frac{2}{3}A$.



32. (a) The equation of every such parabola takes the form

$$(x - x_0)^2 = 4c(y - y_0)$$

This equation can be written

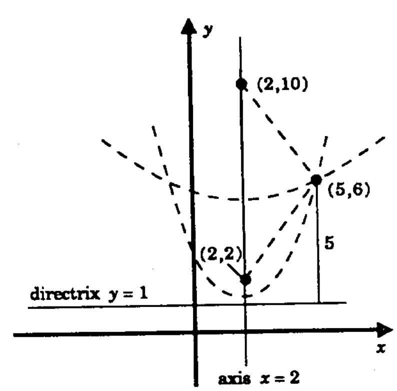
$$y = \left(\frac{1}{4c}\right)x^2 - \left(\frac{x_0}{2c}\right)x + \left(y_0 + \frac{x_0^2}{4c}\right)$$

(b) vertex $\left(-\frac{B}{2A}, \frac{4AC - B^2}{4A}\right)$, focus $\left(-\frac{B}{2A}, \frac{4AC - B^2 + 1}{4A}\right)$, directrix $y = \frac{4AC - B^2 - 1}{4A}$

33. There are two possible positions for the focus:

$$(2, 2) \text{ and } (2, 10).$$

[The point $(5, 6)$ is equidistant from the focus and the directrix. This distance is 5. The points on the line $x = 2$ which are 5 units from $(5, 6)$ are $(2, 2)$ and $(2, 10)$.] These in turn give rise to two parabolas.



Focus $(2,2)$, vertex $(2,3/2)$:

$$(x-2)^2 = 4(\tfrac{1}{2})(y-\tfrac{3}{2}), \quad \text{which simplifies to} \quad x^2-4x+7=2y.$$

Focus $(2,10)$, vertex $(2,11/2)$:

$$(x-2)^2 = 4(\tfrac{9}{2})(y-\tfrac{11}{2}), \quad \text{which simplifies to} \quad x^2-4x+103=18y.$$

34. Horizontal axis, vertex $(-1,1) \implies 4c(x+1)=(y-1)^2$

Goes through $(-6,13) \implies 4c(-5)=(12)^2 \implies 144(x+1)=-5(y-1)^2$

$\implies 144x = -5y^2+10y-149$

35. In this case the length of the latus rectum is the width of the parabola at height $y=c$. With $y=c$, $4c^2=x^2$, and $x=\pm 2c$. The length of the latus rectum is thus $4c$.

36. $y=\dfrac{1}{4c}x^2 \implies \dfrac{dy}{dx}=\dfrac{2x}{4c}=\dfrac{x}{2c}$

At $x=\pm 2c$, we thus get $\dfrac{dy}{dx}=\pm 1$

37.

$$A=\int_{-2c}^{2c}\left(c-\frac{x^2}{4c}\right)dx = 2\int_0^{2c}\left(c-\frac{x^2}{4c}\right)dx = 2\left[cx-\frac{x^3}{12c}\right]_0^{2c}=\frac{8}{3}c^2$$

$\bar{x}=0$ by symmetry

$$\bar{y}A=\int_{-2c}^{2c}\frac{1}{2}\left(c^2-\frac{x^4}{16c^2}\right)dx=\int_0^{2c}\left(c^2-\frac{x^4}{16c^2}\right)dx=\left[c^2x-\frac{x^5}{80c^2}\right]_0^{2c}=\frac{8}{5}c^3$$

$$\bar{y}=(\frac{8}{5}c^3)/(\frac{8}{3}c^2)=\frac{3}{5}c$$

38. $V=\displaystyle\int_0^{2c} 2\pi x\left(c-\frac{x^2}{4c}\right)dx = 2\pi\left[\frac{cx^2}{2}-\frac{x^4}{16c}\right]_0^{2c}=2\pi c^3$

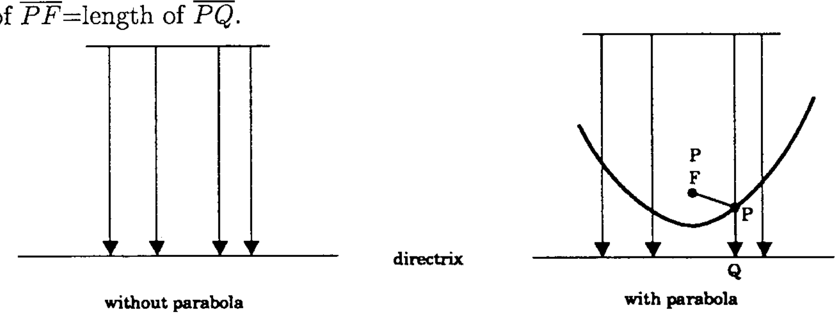
$\bar{y}V=\displaystyle\int_0^{2c}\pi x\left(c^2-\frac{x^4}{16c^2}\right)dx=\pi\left[\frac{c^2x^2}{2}-\frac{x^6}{6\cdot 16c^2}\right]_0^{2c}=\frac{4}{3}\pi c^4$

$\implies \bar{y}=\dfrac{2}{3}c,\quad \bar{x}=0$ by symmetry

39. $\dfrac{kx}{p(0)}=\tan\theta=\dfrac{dy}{dx},\quad y=\dfrac{k}{2\,p(0)}x^2+C$

In our figure $C=y(0)=0$. Thus the equation of the cable is $y=kx^2/2p(0)$, the equation of a parabola.

40. We'll work in two dimensions. The directrix of the parabola is perpendicular to the undisturbed light rays. If the parabola were not there to intercept the rays, the rays would reach the directrix in paths of the same length (this is an assumption we are making). Also true upon reflection by the parabola since length of $\overline{PF}$=length of $\overline{PQ}$.



41. Start with any two parabolas γ_1, γ_2. By moving them we can see to it that they have equations of the following form:

$$\gamma_1 : x^2 = 4c_1 y, \quad c_1 > 0; \qquad \gamma_2 : x^2 = 4c_2 y, \quad c_2 > 0.$$

Now we change the scale for γ_2 so that the equation for γ_2 will look exactly like the equation for γ_1. Set $X = (c_1/c_2)\, x, \quad Y = (c_1/c_2)\, y$. Then

$$x^2 = 4c_2 y \quad \Longrightarrow \quad (c_2/c_1)^2 X^2 = 4c_2\,(c_2/c_1)\, Y \quad \Longrightarrow \quad X^2 = 4c_1 Y.$$

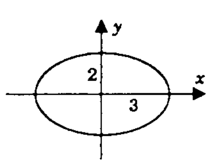
Now γ_2 has exactly the same equation as γ_1; only the scale, the units by which we measure distance, has changed.

42. Placing the parabola with its vertex at the origin, the equation becomes $x^2 = 4cy$.
When $y = 2, \quad x = 2.5,$ so $c = \dfrac{25}{32}$.
Thus the distance from the focus to the center of the mirror is $\dfrac{25}{32}$ ft.

SECTION 9.2

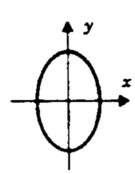
1. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

center $(0, 0)$
foci $(\pm\sqrt{5},\ 0)$
length of major axis 6
length of minor axis 4



2. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

center $(0, 0)$
foci $(0,\ \pm\sqrt{5})$
length of major axis 6
length of minor axis 4



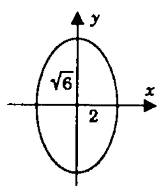
3. $\dfrac{x^2}{4} + \dfrac{y^2}{6} = 1$

center $(0, 0)$

foci $(0, \pm\sqrt{2})$

length of major axis $2\sqrt{6}$

length of minor axis 4



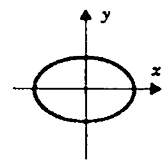
4. $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$

center $(0, 0)$

foci $(\pm 1, 0)$

length of major axis 4

length of minor axis $2\sqrt{3}$



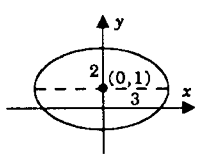
5. $\dfrac{x^2}{9} + \dfrac{(y-1)^2}{4} = 1$

center $(0, 1)$

foci $(\pm\sqrt{5}, 1)$

length of major axis 6

length of minor axis 4



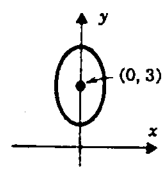
6. $x^2 + \dfrac{(y-3)^2}{4} = 1$

center $(0, 3)$

foci $(0, 3 \pm \sqrt{3})$

length of major axis 4

length of minor axis 2



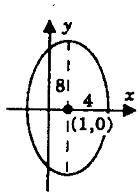
7. $\dfrac{(x-1)^2}{16} + \dfrac{y^2}{64} = 1$

center $(1, 0)$

foci $(1, \pm 4\sqrt{3})$

length of major axis 16

length of minor axis 8



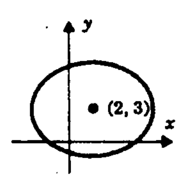
8. $\dfrac{(x-2)^2}{25} + \dfrac{(y-3)^2}{16} = 1$

center $(2, 3)$

foci $(5, 3)$ $(-1, 3)$

length of major axis 10

length of minor axis 8



9. Foci $(-1, 0), (1, 0)$ $\implies$ center $(0, 0)$, $c = 1$, and major axis parallel to x-axis.

Major axis 6 $\implies$ $a = 3$. Thus, $b = \sqrt{8}$.

Equation: $\dfrac{x^2}{9} + \dfrac{y^2}{8} = 1.$

10. Foci $(0,-1),(0,1) \implies$ center $(0,0)$, $c = 1$, major axis vertical.

Major axis 6 $\implies$ $b = 3$. Hence $a = \sqrt{8}$.

Equation: $\dfrac{x^2}{8} + \dfrac{y^2}{9} = 1$

11. Foci at $(1,3)$ and $(1,9)$ $\implies$ center $(1,6)$, $c = 3$, and major axis parallel to y-axis.

Minor axis 8 $\implies$ $b = 4$. Thus, $a = 5$.

Equation: $\dfrac{(x-1)^2}{16} + \dfrac{(y-6)^2}{25} = 1.$

12. Foci $(3,1),(9,1) \implies$ center $(6,1)$, $c = 3$, major axis horizontal.

Major axis 10 $\implies$ $b = 5$. Hence $a = \sqrt{34}$.

Equation: $\dfrac{(x-6)^2}{34} + \dfrac{(y-1)^2}{25} = 1$

13. Focus $(1,1)$ and center $(1,3)$ $\implies$ $c = 2$ and major axis parallel to y-axis.

Major axis 10 $\implies$ $a = 5$. Thus, $b = \sqrt{21}$.

Equation: $\dfrac{(x-1)^2}{21} + \dfrac{(y-3)^2}{25} = 1.$

14. Center $(2,1)$, vertices at $(2,6),(1,1)$ $\implies$ $a = 1$, $b = 5$.

Equation: $(x-2)^2 + \dfrac{(y-1)^2}{25} = 1$

15. Major axis 10 $\implies$ $a = 5$. Vertices at $(3,2)$ and $(3,-4)$ are then on minor axis parallel to y-axis. Then, $b = 3$ and center is $(3,-1)$.

Equation: $\dfrac{(x-3)^2}{25} + \dfrac{(y+1)^2}{9} = 1.$

16. Vertices $(1,7),(1,-3)$ $\implies$ center $(1,2)$, $b = 5$

Focus $(6,2)$ $\implies$ $c = 5$, major axis horizontal, so $a = \sqrt{50}$.

Equation: $\dfrac{(x-1)^2}{50} + \dfrac{(y-2)^2}{25} = 1$

17. Foci $(-5,0)$ and $(5,0)$ $\implies$ $c = 5$ and center $(0,0)$.

Transverse axis 6 $\implies$ $a = 3$. Thus, $b = 4$.

Equation: $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1.$

18. Focus $(-13,0),(13,0)$ $\implies$ center $(0,0)$, $c = 13$

Transverse axis $10 \implies a = 5$, hence $b = 12$.

Equation: $\dfrac{x^2}{25} - \dfrac{y^2}{144} = 1$

19. Foci $(0, -13)$ and $(0, 13) \implies c = 13$ and center $(0, 0)$.

Transverse axis $10 \implies a = 5$. Thus, $b = 12$.

Equation: $\dfrac{y^2}{25} - \dfrac{x^2}{144} = 1.$

20. Foci $(0, -13), (0, 13) \implies$ center $(0, 0)$, $c = 13$, transverse axis vertical

Transverse axis $24 \implies b = 12$, hence $a = 5$. Equation: $\dfrac{y^2}{144} - \dfrac{x^2}{25} = 1$

21. Foci $(-5, 1)$ and $(5, 1) \implies c = 5$ and center $(0, 1)$.

Transverse axis $6 \implies a = 3$. Thus, $b = 4$.

Equation: $\dfrac{x^2}{9} - \dfrac{(y-1)^2}{16} = 1.$

22. Foci $(-3, 1), (7, 1) \implies$ center $(2, 1)$, $c = 5$, transverse axis horizontal

Transverse axis $6 \implies a = 3$, hence $b = 4$. Equation: $\dfrac{(x-2)^2}{9} - \dfrac{(y-1)^2}{16} = 1$

23. Foci $(-1, -1)$ and $(-1, 1) \implies c = 1$ and center $(-1, 0)$.

Transverse axis $\frac{1}{2} \implies a = \frac{1}{4}$. Thus, $b = \frac{1}{4}\sqrt{15}$.

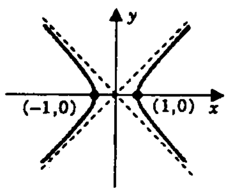
Equation: $\dfrac{y^2}{1/16} - \dfrac{(x+1)^2}{15/16} = 1.$

24. Foci $(2, 1), (2, 5) \implies$ center $(2, 3)$, $c = 2$, transverse axis vertical

Transverse axis $3 \implies b = \dfrac{3}{2}$, hence $a = \dfrac{\sqrt{7}}{2}$. Equation: $\dfrac{(y-3)^2}{9/4} - \dfrac{(x-2)^2}{7/4} = 1$

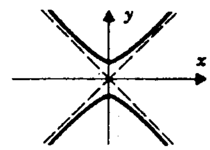
25. $x^2 - y^2 = 1$

center $(0,0)$
transverse axis 2
vertices $(\pm 1,\ 0)$
foci $(\pm\sqrt{2},\ 0)$
asymptotes $y = \pm x$



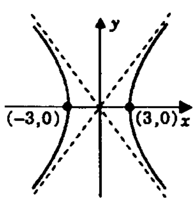
26. $y^2 - x^2 = 1$

center $(0,0)$
transverse axis 2
vertices $(0,\ \pm 1)$
foci $(0,\ \pm\sqrt{2})$
asymptotes $y = \pm x$



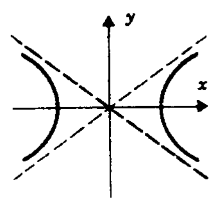
27. $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

center $(0,0)$
transverse axis 6
vertices $(\pm 3,\ 0)$
foci $(\pm 5,\ 0)$
asymptotes $y = \pm\frac{4}{3}x$



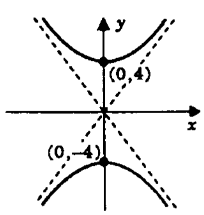
28. $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$

center $(0,0)$
transverse axis 8
vertices $(\pm 4,\ 0)$
foci $(\pm 5,\ 0)$
asymptotes $y = \pm\frac{3}{4}x$



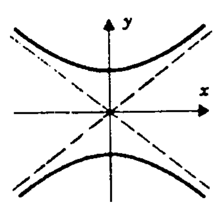
29. $\dfrac{y^2}{16} - \dfrac{x^2}{9} = 1$

center $(0,0)$
transverse axis 8
vertices $(0,\ \pm 4)$
foci $(0,\ \pm 5)$
asymptotes $y = \pm\frac{4}{3}x$



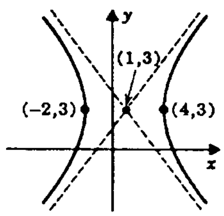
30. $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$

center $(0,0)$
transverse axis 6
vertices $(0,\ \pm 3)$
foci $(0,\ \pm 5)$
asymptotes $y = \pm\frac{3}{4}x$



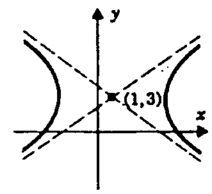
31. $\dfrac{(x-1)^2}{9} - \dfrac{(y-3)^2}{16} = 1$

center $(1,3)$
transverse axis 6
vertices (4, 3) and (−2, 3)
foci (6, 3) and (−4, 3)
asymptotes $y - 3 = \pm\frac{4}{3}(x-1)$

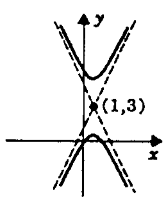


32. $\dfrac{(x-1)^2}{16} - \dfrac{(y-3)^2}{9} = 1$

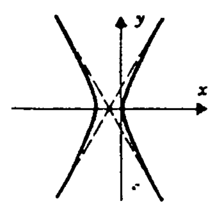
center $(1,3)$
transverse axis 8
vertices (5, 3) and (−3, 3)
foci (6, 3) and (−4, 3)
asymptotes $y = \pm\frac{3}{4}(x-1)+3$



33. $\dfrac{(y-3)^2}{4} - \dfrac{(x-1)^2}{1} = 1$
center (1, 3)
transverse axis 4
vertices (1, 5) and (1, 1)
foci $(1,\ 3 \pm \sqrt{5})$
asymptotes $y - 3 = \pm 2(x-1)$



34. $(x+1)^2 + \dfrac{y^2}{3} = 1$
center $(-1,0)$
transverse axis 2
vertices $(0,0)$ and $(-2,0)$
foci $(1,0)$ and $(-3,0)$
asymptotes $y = \pm\sqrt{3}(x+1)$



35. The length of the string is $d(F_1,F_2) + k = 2(c+a)$.

36. The points lie on the ellipse: $b^2x^2 + a^2y^2 = a^2b^2$

$$b^2(a^2\cos^2 t) + a^2(b^2\sin^2 t) = a^2b^2(\cos^2 t + \sin^2 t) = a^2b^2$$

37. $2\sqrt{\pi^2a^4 - A^2}\,/\pi a$

38. We refer to Figure 9.2.13. The equation of the tangent can be written

$$(b^2x_0)x + (a^2y_0)y - a^2b^2 = 0$$

Thus

$$d(T_1,F_1)d(T_2,F_2) = \frac{|(b^2x_0)(-c) - a^2b^2|}{\sqrt{b^4{x_0}^2 + a^4{y_0}^2}} \cdot \frac{|(b^2x_0)(c) - a^2b^2|}{\sqrt{b^4{x_0}^2 + a^4{y_0}^2}} = b^2\,\frac{|b^2{x_0}^2c^2 - a^4b^2|}{b^4{x_0}^2 + a^4{y_0}^2}$$

The fraction on the right is identically 1:

$$|b^2{x_0}^2c^2 - a^4b^2| = |b^2{x_0}^2c^2 - a^2(b^2{x_0}^2 + a^2{y_0}^2)|$$
$$= |(c^2-a^2)b^2{x_0}^2 - a^4{y_0}^2| = |-b^4{x_0}^2 - a^4{y_0}^2| = b^4{x_0}^2 + a^4{y_0}^2$$

Thus $d(T_1,F_1)d(T_2,F_2) = b^2$.

39. The equation of the ellipse is of the form

$$\frac{(x-5)^2}{25} + \frac{y^2}{25-c^2} = 1.$$

Substitute $x = 3$ and $y = 4$ in that equation and you find that $c = \pm\frac{5}{21}\sqrt{5}$.

The foci are at $\left(5 \pm \frac{5}{21}\sqrt{5}, 0\right)$.

40. Use $y = \dfrac{b}{a}\sqrt{a^2 - x^2}$.

$$A = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx$$

$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{01}\frac{x}{a}\right]_0^a$$

$$= \frac{ba\pi}{4}.$$

Also,

$$\overline{x}A = \int_0^a \frac{b}{a}x\sqrt{a^2 - x^2}\, dx = -\frac{b}{3a}\left[(a^2 - x^2)^{\frac{3}{2}}\right]_0^a$$

$$= \frac{ba^2}{3},$$

and

$$\overline{y}A = \int_0^a \frac{b^2}{2a^2}(a^2 - x^2)\, dx = \frac{b^2}{2a^2}\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= \frac{b^2a}{3},$$

Thus, $\quad \overline{x} = \dfrac{4a}{3\pi}$, and $\quad \overline{y} = \dfrac{4b}{3\pi}$.

41. By the hint, $xy = X^2 - Y^2 = 1$. In the XY-system $a = 1$, $\quad b = 1$, $\quad c = \sqrt{2}$. We have center $(0, 0)$, vertices $(\pm 1, 0)$, foci $(\pm\sqrt{2}, 0)$ and asymptotes $Y = \pm X$. Using

$$x = X + Y \quad \text{and} \quad y = X - Y$$

to convert to the xy-system, we find center $(0, 0)$, vertices $(1, 1)$ and $(-1, -1)$, foci $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$, asymptotes $y = 0$ and $x = 0$, transverse axis $2\sqrt{2}$.

42. (a) Take $x(t) = a\cosh t$, $\quad y(t) = b\sinh t \quad$ Then $\quad \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = \cosh^2 t - \sinh^2 t = 1$

range$(x) = [a, \infty)$, $\quad$ range$(y) = (-\infty, \infty)$

(b) Take $x(t) = -a\cosh t$, $\quad y(t) = b\sinh t \quad$ Then $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$,

range$(x) = (-\infty, -a]$, $\quad$ range $(y) = (-\infty, \infty)$

43.

$$A = \frac{2b}{a}\int_a^{2a}\sqrt{x^2 - a^2}\, dx = \frac{2b}{a}\left[\frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left(x + \sqrt{x^2 - a^2}\right)\right]_a^{2a}$$

$$= [2\sqrt{3} - \ln(2 + \sqrt{3})]ab$$

44. Let $P(x_0, y_0)$ be the point of tangency and let l be the tangent line.

$$\text{slope of } l = \frac{b^2x_0}{a^2y_0}, \quad \text{slope of } \overline{F_1P} = \frac{y_0}{x_0+c}, \quad \text{slope of } \overline{F_2P} = \frac{y_0}{x_0-c}$$

With θ_1 as the angle between l and $\overline{F_1P}$ and with θ_2 as the angle between $\overline{F_2P}$ and l, we have

$$\tan\theta_1 = \frac{\dfrac{b^2x_0}{a^2y_0} - \dfrac{y_0}{x_0+c}}{1 + \dfrac{b^2x_0}{a^2y_0}\left(\dfrac{y_0}{x_0+c}\right)} \quad \text{and} \quad \tan\theta_2 = \frac{\dfrac{y_0}{x_0-c} - \dfrac{b^2x_0}{a^2y_0}}{1 + \left(\dfrac{y_0}{x_0-c}\right)\left(\dfrac{b^2x_0}{a^2y_0}\right)}$$

From the fact that

$$b^2x_0^2 - a^2y_0^2 = a^2b^2, \quad (P \text{ is on the hyperbola})$$

it follows readily that $\tan\theta_1 = \tan\theta_2$.

45. $e = \dfrac{\sqrt{25-16}}{\sqrt{25}} = \dfrac{3}{5}$

46. $e = \dfrac{\sqrt{25-16}}{\sqrt{25}} = \dfrac{3}{5}$

47. $e = \dfrac{\sqrt{25-9}}{\sqrt{25}} = \dfrac{4}{5}$

48. $e = \dfrac{\sqrt{169-144}}{\sqrt{169}} = \dfrac{5}{13}$

49. E_1 is fatter than E_2, more like a circle.

50. The ellipse tends to a circle of radius a (Since b approaches a).

51. The ellipse tends to a line segment of length $2a$.

52. $a = 3, \quad c = ea = \dfrac{1}{3} \cdot 3 = 1, \quad b = \sqrt{a^2 - c^2} = \sqrt{8}$

Center $(0,0)$, so $\dfrac{x^2}{9} + \dfrac{y^2}{8} = 1$

53. $x^2/9 + y^2 = 1$

54. The basic equation reads $\sqrt{x^2+y^2} = e|x-d|$. Square and arrange to

$$\left(x + \frac{e^2d}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{e^2d^2}{(1-e^2)^2}$$

Now set $a = ed/(1-e^2)$ and $c = ea$. This reduces the equation to

$$\frac{(x+c)^2}{a^2} + \frac{y^2}{a^2-c^2} = 1$$

This equation represents an ellipse of eccentricity $e = c/a$.

55. $e = \frac{5}{3}$

56. $e = \frac{5}{4}$

57. $e = \sqrt{2}$

58. $e = \frac{13}{5}$

59. The branches of H_1 open up less quickly than the branches of H_2.

60. The hyperbola tends to the union of two oppositely-directed half lines that begin at the ends of the transverse axis.

61. The hyperbola tends to a pair of parallel lines separated by the transverse axis.

62. The basic equation reads $\sqrt{x^2+y^2} = e|x-d|$. Square and rearrange to

$$\left(x - \frac{e^2 d}{e^2 - a}\right)^2 - \frac{y^2}{e^2-1} = \frac{e^2 d^2}{(e^2-1)^2}$$

Now set $a = ed/(e^2-1)$ and $c = ea$. This reduces the equation to

$$\frac{(x-c)^2}{a^2} - \frac{y^2}{c^2-a^2} = 1$$

This equation represents a hyperbola of eccentricity $e = c/a$

63. Measure distances in miles and time in seconds. Place the origin at A and let $P\,(x,y)$ be the site of the crash. Then

$$d(P,B) - d(P,A) = (4)(0.20) = 0.80.$$

This places P on the right branch of the hyperbola

$$\frac{(x+1)^2}{(0.4)^2} - \frac{y^2}{1-(0.4)^2} = 1.$$

Also

$$d(P,C) - d(P,A) = 6(0.20) = 1.20.$$

This places P on the left branch of the hyperbola

$$\frac{(x-1)^2}{(0.6)^2} - \frac{y^2}{1-(0.6)^2} = 1.$$

Solve the two equations simultaneously keeping in mind the conditions of the problem and you will find that $x \cong -0.248$ and $y \cong 1.459$. The impact takes place about a quarter of a mile west of A and one and a half miles north.

64. Measure distance in miles and time in microseconds. Let $P(x,y)$ be the source of radio signal. Then

$$d(P,P_1) - d(P,P_2) = (0.186)(600) = 2(55.8)$$

This places P on the right branch of the hyperbola

$$(1) \qquad \frac{x^2}{(55.8)^2} - \frac{y^2}{(100)^2-(55.8)^2} = 1$$

Also

$$d(P,P_4) - d(P,P_3) = (0.186)(800) = 2(74.4)$$

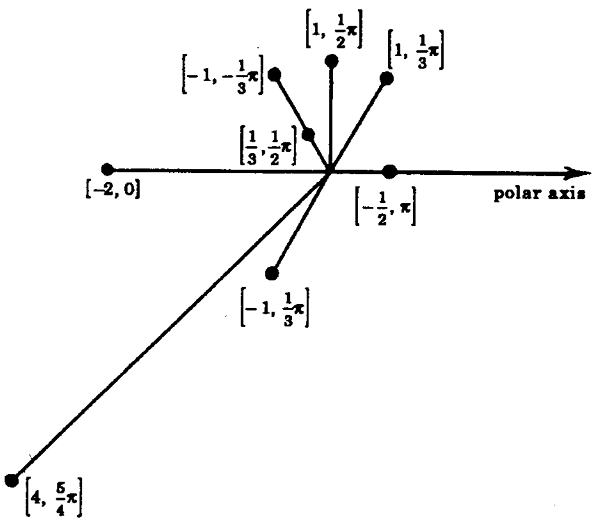
This places P on the upper branch of the hyperbola

$$(2) \qquad \frac{y^2}{(74.4)^2} - \frac{x^2}{(100)^2-(74.4)^2} = 1$$

Solve equations (1) and (2) simultaneously and you will find that $x \cong 113$ and $y \cong 146$

SECTION 9.3

1–8.



9. $x = 3\cos\frac{1}{2}\pi = 0$

$y = 3\sin\frac{1}{2}\pi = 3$

10. $x = 4\cos\dfrac{\pi}{6} = 2\sqrt{3}$

$y = 4\sin\dfrac{\pi}{6} = 2$

11. $x = -\cos(-\pi) = 1$

$y = -\sin(-\pi) = 0$

12. $x = -1\,\cos\dfrac{\pi}{4} = -\sqrt{2}/2$

$y = -1\,\sin\dfrac{\pi}{4} = -\sqrt{2}/2$

13. $x = -3\cos\left(-\frac{1}{3}\pi\right) = -\frac{3}{2}$

$y = -3\sin\left(-\frac{1}{3}\pi\right) = \frac{3}{2}\sqrt{3}$

14. $x = 2\cos 0 = 2$

$y = 2\sin 0 = 0$

15. $x = 3\cos\left(-\frac{1}{2}\pi\right) = 0$

$y = 3\sin\left(-\frac{1}{2}\pi\right) = -3$

16. $x = 2\cos 3\pi = -2$

$y = 2\sin 3\pi = 0$

17. $\left.\begin{array}{l} r^2 = 0^2 + 1^2, \quad r = \pm 1 \\ r = 1: \quad \cos\theta = 0 \text{ and } \sin\theta = 1 \\ \theta = \frac{1}{2}\pi \end{array}\right\} \quad \left[1, \frac{1}{2}\pi + 2n\pi\right], \quad \left[-1, \frac{3}{2}\pi + 2n\pi\right]$

18. $\left.\begin{array}{l} r^2 = 1^2 + 0^2, \quad r = \pm 1 \\ r = 1: \quad \cos\theta = 1 \text{ and } \sin\theta = 0 \\ \theta = 2\pi \end{array}\right\} \quad [1, 2n\pi], \quad [-1, \pi + 2n\pi]$

19. $\left.\begin{array}{l} r^2 = (-3)^2 + 0^2 = 9, \quad r = \pm 3 \\ r = 3: \quad \cos\theta = -1 \text{ and } \sin\theta = 0 \\ \theta = \pi \end{array}\right\} \quad [3, \pi + 2n\pi], \quad [3, 2n\pi]$

20. $\left.\begin{array}{l} r^2 = 4^2 + 4^2, \quad r = \pm 4\sqrt{2} \\ r = 4\sqrt{2}: \quad \cos\theta = \frac{1}{\sqrt{2}} \text{ and } \sin\theta = \frac{1}{\sqrt{2}} \\ \theta = \frac{1}{4}\pi \end{array}\right\} \quad \left[4\sqrt{2}, \frac{1}{4}\pi + 2n\pi\right], \quad \left[-4\sqrt{2}, \frac{5}{4}\pi + 2n\pi\right]$

21. $\left.\begin{array}{l} r^2 = 2^2 + (-2)^2 = 8, \quad r = \pm 2\sqrt{2} \\ r + 2\sqrt{2}: \quad \cos\theta = \frac{1}{2}\sqrt{2}, \quad \sin\theta = -\frac{1}{2}\sqrt{2} \\ \theta = \frac{7}{4}\pi \end{array}\right\}$ $\left[2\sqrt{2}, \frac{7}{4}\pi + 2n\pi\right], \quad \left[-2\sqrt{2}, \frac{3}{4}\pi + 2n\pi\right]$

22. $\left.\begin{array}{l} r^2 = 3^2 + (3\sqrt{3})^2, \quad r = \pm 6 \\ r = 6: \quad \cos\theta = \frac{1}{2} \text{ and } \sin\theta = -\frac{\sqrt{3}}{2} \\ \theta = \frac{5}{3}\pi \end{array}\right\}$ $\left[6, \frac{5}{3}\pi + 2n\pi\right], \quad \left[-6, \frac{2}{3}\pi + 2n\pi\right]$

23. $\left.\begin{array}{l} r^2 = \left(4\sqrt{3}\right)^2 + 4^2 = 64, \quad r \pm 8 \\ r = 8: \quad \cos\theta = \frac{1}{2}\sqrt{3}, \quad \sin\theta = \frac{1}{2} \\ r = \frac{1}{6}\pi \end{array}\right\}$ $\left[8, \frac{1}{6}\pi + 2n\pi\right], \quad \left[-8, \frac{7}{6}\pi + 2n\pi\right]$

24. $\left.\begin{array}{l} r^2 = (\sqrt{3})^2 + 1^2, \quad r = \pm 2 \\ r = 2: \quad \cos\theta = \frac{\sqrt{3}}{2} \text{ and } \sin\theta = -\frac{1}{2} \\ \theta = \frac{11}{6}\pi \end{array}\right\}$ $\left[2, \frac{11}{6}\pi + 2n\pi\right], \quad \left[-2, \frac{5}{6}\pi + 2n\pi\right]$

25.
$$\begin{aligned} d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 &= (r_1\cos\theta_1 - r_2\cos\theta_2)^2 + (r_1\sin\theta_1 - r_2\sin\theta_2)^2 \\ &= {r_1}^2\cos^2\theta_1 - 2r_1r_2\cos\theta_1\cos\theta_2 + {r_2}^2\cos^2\theta_2 \\ &\quad + {r_1}^2\sin^2\theta_1 - 2r_1r_2\sin\theta_1\sin\theta_2 + {r_2}^2\sin^2\theta_2 \\ &= {r_1}^2 + {r_2}^2 - 2r_1r_2\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right) \\ &= {r_1}^2 + {r_2}^2 - 2r_1r_2\cos\left(\theta_1 - \theta_2\right) \end{aligned}$$

$$d = \sqrt{{r_1}^2 + {r_2}^2 - 2r_1r_2\cos\left(\theta_1 - \theta_2\right)}$$

26. Draw a figure; the result is clear.

27. (a) $\left[\frac{1}{2}, \frac{11}{6}\pi\right]$ (b) $\left[\frac{1}{2}, \frac{5}{6}\pi\right]$ (c) $\left[\frac{1}{2}, \frac{7}{6}\pi\right]$

28. (a) $\left[3, \frac{5}{4}\pi\right]$ (b) $\left[3, \frac{1}{4}\pi\right]$ (c) $\left[3, \frac{7}{4}\pi\right]$

29. (a) $\left[2, \frac{2}{3}\pi\right]$ (b) $\left[2, \frac{5}{3}\pi\right]$ (c) $\left[2, \frac{1}{3}\pi\right]$

30. (a) $\left[3, \frac{3}{4}\pi\right]$ (b) $\left[3, \frac{7}{4}\pi\right]$ (c) $\left[3, \frac{1}{4}\pi\right]$

31. about the x-axis?: $r = 2 + \cos(-\theta) \implies r = 2 + \cos\theta$, yes.

about the y-axis?: $r = 2 + \cos(\pi - \theta) \implies r = 2 - \cos\theta$, no.

about the origin?: $r = 2 + \cos(\pi + \theta) \implies r = 2 - \cos\theta$, no.

32. about the x-axis?: $r = \cos[2(-\theta)] \implies r = \cos 2\theta$, yes.

about the y-axis?: $r = \cos[2(\pi - \theta)] \implies r = \cos 2\theta$, yes.

about the origin?: $r = \cos[2(\pi + \theta)] \implies r = \cos 2\theta$, yes.

33. about the x-axis?: $r\,(\sin(-\theta) + \cos(-\theta)) = 1 \implies r\,(-\sin\theta + \cos\theta) = 1$, no.

about the y-axis?: $r\,(\sin(\pi - \theta) + \cos(\pi - \theta)) = 1 \implies r\,(\sin\theta - \cos\theta) = 1$, no.

about the origin?: $r\,(\sin(\pi + \theta) + \cos(\pi + \theta)) = 1 \implies r\,(-\sin\theta - \cos\theta) = 1$, no.

34. about the x-axis?: $r\sin(-\theta) = 1 \implies -r\sin\theta = 1$, no.

about the y-axis?: $r\sin(\pi - \theta) = 1 \implies r\sin\theta = 1$ yes.

about the origin?: $r\sin(\pi + \theta) = 1 \implies -r\sin\theta = 1$, no.

35. about the x-axis?: $r^2\sin(-2\theta) = 1 \implies -r^2\sin 2\theta = 1$, no.

about the y-axis?: $r^2\sin(2(\pi - \theta)) = 1 \implies -r^2\sin 2\theta = 1$, no.

about the origin?: $r^2\sin(2(\pi + \theta)) = 1 \implies r^2\sin 2\theta = 1$, yes.

36. about the x-axis?: $r^2\cos[2(-\theta)] = 1 \implies r^2\cos 2\theta = 1$, yes.

about the y-axis?: $r^2\cos[2(\pi - \theta)] = 1 \implies r^2\cos 2\theta = 1$, yes.

about the origin?: $r^2\cos[2(\pi + \theta)] = 1 \implies r^2\cos 2\theta = 1$, yes.

37.
$$x = 2$$
$$r\cos\theta = 2$$

38.
$$r\sin\theta = 3$$

39.
$$2xy = 1$$
$$2\,(r\cos\theta)(r\sin\theta) = 1$$
$$r^2\sin 2\theta = 1$$

40.
$$r^2 = 9$$
$$r = 3$$

41.
$$x^2 + (y-2)^2 = 4$$
$$x^2 + y^2 - 4y = 0$$
$$r^2 - 4r\sin\theta = 0$$
$$r = 4\sin\theta$$

42.
$$(x-a)^2 + y^2 = a^2$$
$$x^2 - 2ax + y^2 = 0$$
$$r^2 = 2ar\cos\theta$$
$$r = 2a\cos\theta$$

43.
$$y = x$$
$$r\sin\theta = r\cos\theta$$
$$\tan\theta = 1$$
$$\theta = \pi/4$$

[note: division by r okay since $[0, 0,]$ is on the curve]

44.
$$x^2 - y^2 = 4$$
$$r^2(\cos^2\theta - \sin^2\theta) = 4$$
$$r^2\cos 2\theta = 4$$

45.
$$x^2+y^2+x=\sqrt{x^2+y^2}$$
$$r^2+r\cos\theta=r$$
$$r=1-\cos\theta$$

46.
$$y=mx$$
$$r\sin\theta=mr\cos\theta$$
$$\tan\theta=m$$

47.
$$(x^2+y^2)^2=2xy$$
$$r^4=2(r\cos\theta)(r\sin\theta)$$
$$r^2=\sin 2\theta$$

48.
$$(x^2+y^2)^2=x^2-y^2$$
$$r^4=r^2(\cos^2\theta-\sin^2\theta)$$
$$r^2=\cos 2\theta$$

49. The horizontal line $y=4$

50. The vertical line $x=4$

51. the line $y=\sqrt{3}x$

52. $\theta=\pm\frac{\pi}{3}$: the lines $y=\pm\sqrt{3}x$

53.
$$r=2\,(1-\cos\theta)^{-1}$$
$$r-r\cos\theta=2$$
$$\sqrt{x^2+y^2}-x=2$$
$$x^2+y^2=(x+2)^2$$
$$y^2=4(x+1)$$
a parabola

54.
$$r=4\sin(\theta+\pi)=-4\sin\theta$$
$$r^2=-4r\sin\theta$$
$$x^2+y^2=-4y$$
$$x^2+(y+2)^2=4$$
a circle

55.
$$r=3\cos\theta$$
$$r^2=3r\cos\theta$$
$$x^2+y^2=3x$$

56. The y − axis

57. the line $y=2x$

58. circle $x^2+y^2=2y$

59.
$$r=\frac{4}{2-\cos\theta}$$
$$2r-r\cos\theta=4$$
$$2\sqrt{x^2+y^2}-x=4$$
$$4(x^2+y^2)=(x+4)^2$$
$$3x^2+4y^2-8x=16$$
an ellipse

60.
$$\frac{2}{1+2\sin\theta}=r$$
$$2=r+2r\sin\theta$$
$$\sqrt{x^2+y^2}=2-2y$$
$$x^2+y^2=4-8y+4y^2$$
$$x^2-3y^2+8y=4$$
a hyperbola

61.
$$r=\frac{4}{1-\cos\theta}$$
$$r-r\cos\theta=4$$
$$\sqrt{x^2+y^2}-x=4$$
$$x^2+y^2=(x+4)^2$$
$$y^2=8x+16$$
a parabola

62.
$$r=\frac{2}{3+2\sin\theta}$$
$$3r+2r\sin\theta=2$$
$$3\sqrt{x^2+y^2}=2-2y$$
$$9x^2+5y^2=4-8y$$
an ellipse

63.

$$r = a\sin\theta + b\cos\theta$$
$$r^2 = ar\sin\theta + br\cos\theta$$
$$x^2 + y^2 = ay + bz$$
$$\left(x - \frac{b}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2 + b^2}{4}$$

center: $(b/2, a/2)$; radius: $\frac{1}{2}\sqrt{a^2 + b^2}$

64. $r = d + r\cos\theta$

65. $\frac{1}{2}(r\cos\theta + d) = r$

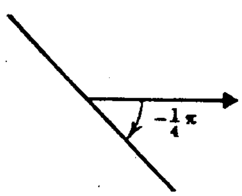
$$r = \frac{d}{2 - \cos\theta}$$

66. $r = 2(d + r\cos\theta)$

$$= 2d + 2r\cos\theta$$

SECTION 9.4

1.



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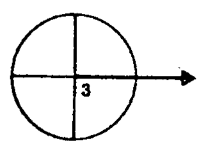

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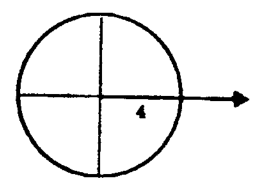

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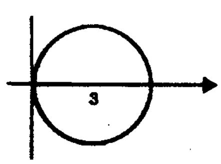

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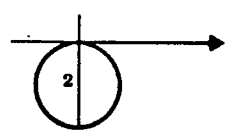

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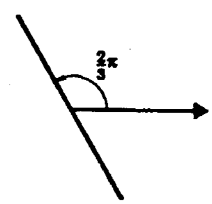

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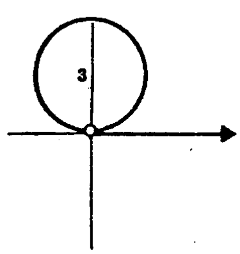

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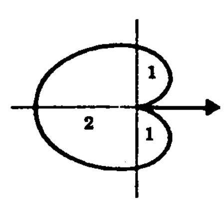

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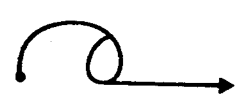

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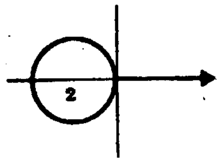

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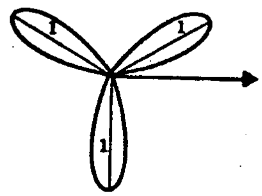

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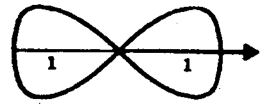

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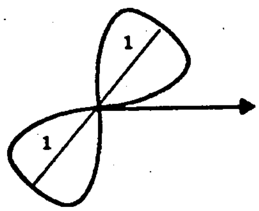

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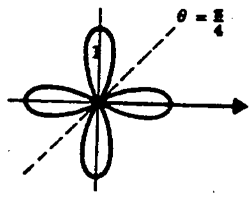

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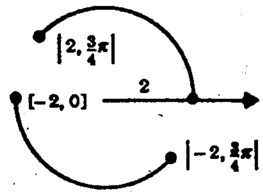

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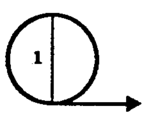

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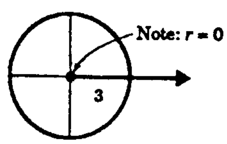

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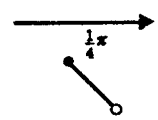

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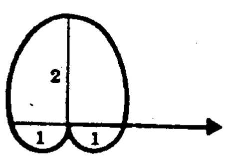

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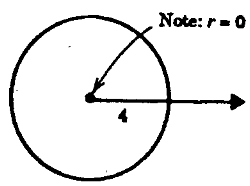

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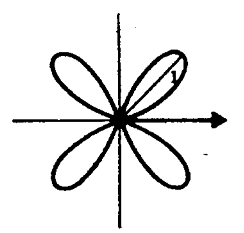

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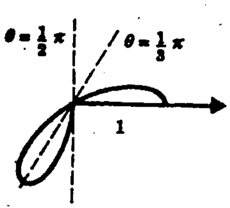

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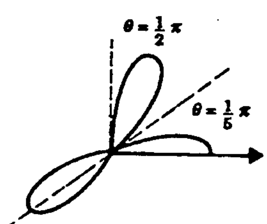

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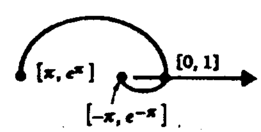

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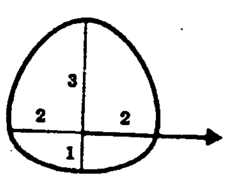

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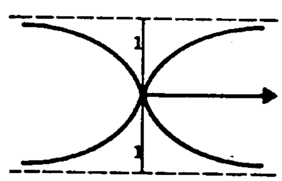
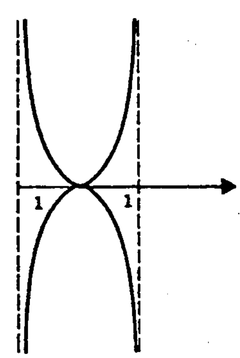
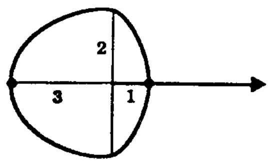


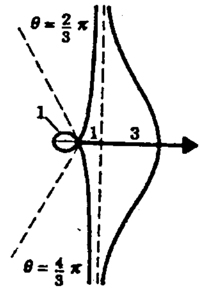
27.



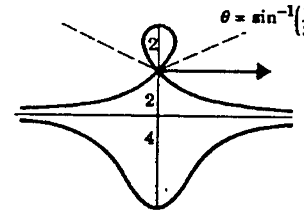
28.



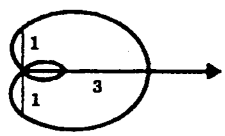
29.



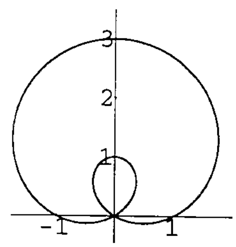
30.



31.



32.



33. yes; $[1, \pi] = [-1, 0]$ and the pair $r = -1,\ \theta = 0$ satisfies the equation

34. No.

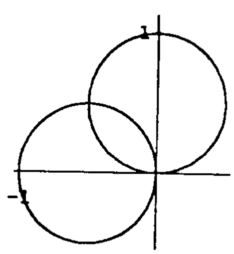
35. yes; the pair $r = \frac{1}{2},\ \theta = \frac{1}{2}\pi$ satisfies the equation

36. yes; $[1, -\frac{5}{6}\pi] = [-1, \frac{1}{6}\pi]$, which lies on the curve.

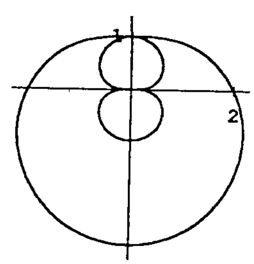
37. $[2, \pi] = [-2, 0]$. The coordinates $[-2, 0]$ satisfy the equation $r^2 = 4\cos\theta$, and the coordinates $[2, \pi]$ satisfy the equation $r = 3 + \cos\theta$.

38. $[2, \pi/2] = [-2, -\pi/2]$ The coordinates $[2, \pi/2]$ satisfy $r^2 \sin\theta = 4$, and the coordinates $[-2, -\pi/2]$ satisfy $r = 2\cos 2\theta$.

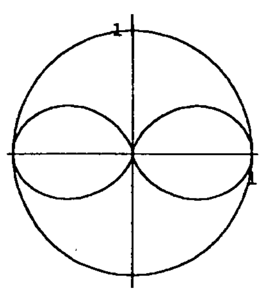
39. $(0, 0),\ (-\frac{1}{2}, \frac{1}{2})$



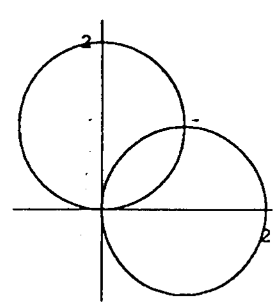
40. $(0, 1)$



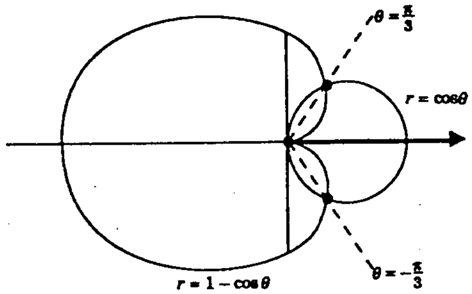
41. $(1, 0),\ (-1, 0)$



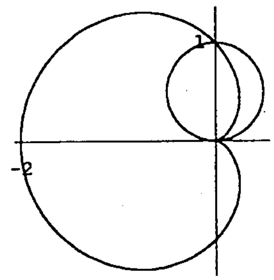
42. $(0, 0),\ (1, 1)$



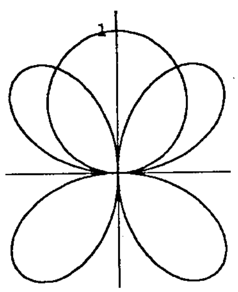
43. $(0,0)$, $\left(\frac{1}{4}, \frac{1}{4}\sqrt{3}\right)$, $\left(\frac{1}{4}, -\frac{1}{4}\sqrt{3}\right)$



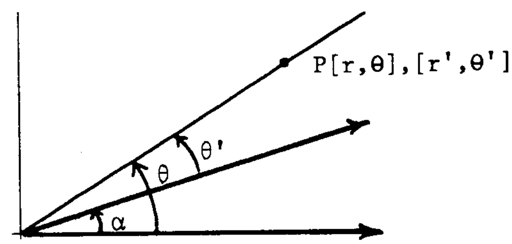
44. $(0,0)$, $(0,1)$



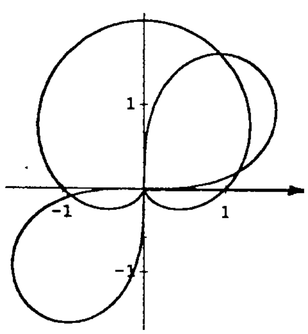
45. $(0,0)$, $\left(\pm\frac{\sqrt{3}}{4}, \frac{3}{4}\right)$



46.



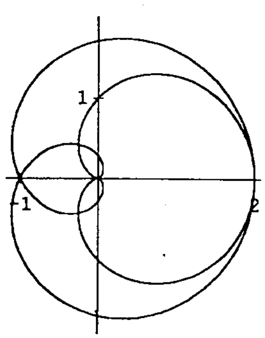
47. (a)



(b) The curves intersect at the pole and at:

$[1.172, 0.173]$, $[1.86, 1.036]$, $[0.90, 3.245]$

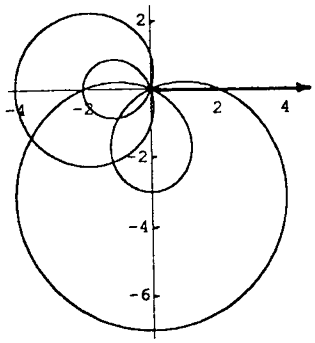
48. (a)



(b) The curves intersect at:

$(0,0)$, $(2,0)$, $(-1,0)$, $(-0.2500, \pm 0.4330)$

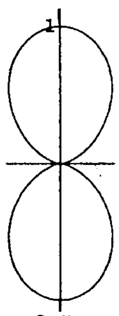
49. (a)



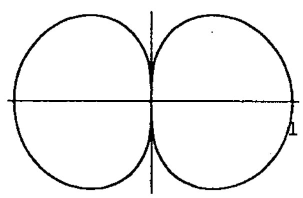
(b) The curves intersect at the pole and at:

$r = 1 - 3\sin\theta$	$r = 2 - 5\sin\theta$
$[-2, 0]$	$[2, \pi]$
$[3.800, 3.510]$	$[3.800, 3.510]$
$[2.412, 4.223]$	$[-2.412, 1.081]$
$[-1.267, 0.713]$	$[-1.267, 0.713]$

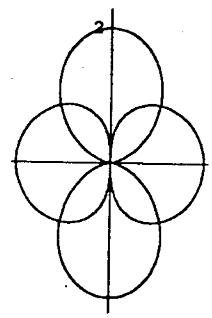
50. (a)



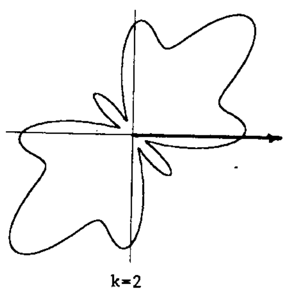
(b)



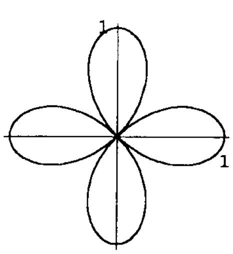
(c) $(\pm 0.7484, \pm 0.8651)$



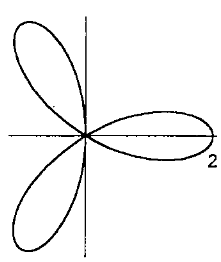
51. "Butterfly" curves. The graph for the case $k = 2$ is:



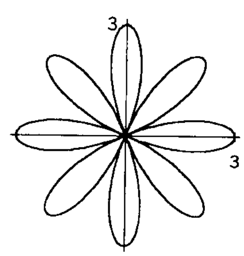
52. (a) $r = \cos 2\theta$



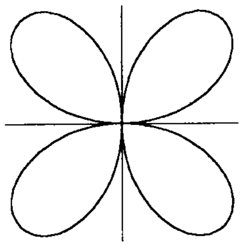
$r = 2\cos 3\theta$

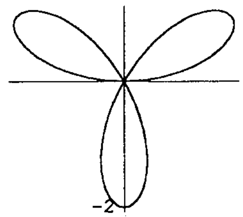


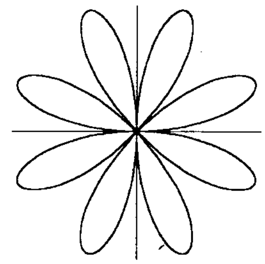
$r = 3\cos 4\theta$



$r = 3\sin 2\theta$ | $r = 2\sin 3\theta$ | $r = \sin 4\theta$

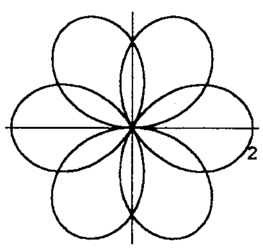




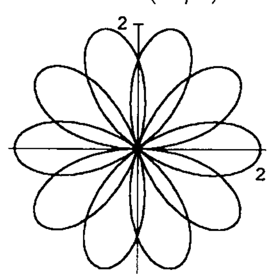


(b) A changes the length of the petals. The sine curves are rotated $\frac{\pi}{4}$ from the corresponding cosine curves.

(c) $k = \frac{3}{2}$:
same graph for $r = 2\cos(3\theta/2)$
and $r = 2\sin(3\theta/2)$



$k = \frac{5}{2}$:
same graph for $r = 2\cos(5\theta/2)$
and $r = 2\sin(5\theta/2)$



For $k = m/2$, m odd, you get a $2m$-petal flower.

PROJECT 9.4

1. $e = \dfrac{r}{d - e\cos\theta} \implies r = ed - er\cos\theta \implies r(1 + e\cos\theta) = ed \implies r = \dfrac{ed}{1 + e\cos\theta}$

2. (a) Suppose $r = \dfrac{ed}{1 + e\cos\theta}$. Then

$$\begin{aligned}
r + er\cos\theta &= ed \\
r &= ed - er\cos\theta \\
r^2 &= e^2d^2 - 2e^2dr\cos\theta + e^2r^2\cos^2\theta \\
x^2 + y^2 &= e^2d^2 - 2e^2dx + e^2x^2 \\
(1 - e^2)x^2 + 2e^2dx + y^2 &= e^2d^2 \\
\left(x + \frac{e^2d}{1 - e^2}\right)^2 + \frac{y^2}{1 - e^2} &= \frac{e^2d^2}{(1 - e^2)^2} \\
(x + c)^2 + \frac{y^2}{1 - e^2} &= a^2 \quad \left(a = \frac{ed}{1 - e^2}, \quad c = ea\right) \\
\frac{(x + c)^2}{a^2} + \frac{y^2}{(1 - e^2)a^2} &= 1 \\
\frac{(x + c)^2}{a^2} + \frac{y^2}{a^2 - c^2} &= 1
\end{aligned}$$

(b) From part (a),

$$x^2 + y^2 = e^2d^2 - 2e^2dx + e^2x^2$$
$$\text{becomes} \quad y^2 = d^2 - 2dx$$
$$\text{so} \quad y^2 = -4\frac{d}{2}\left(x - \frac{d}{2}\right)$$

(c) From part (a),

$$\left(x + \frac{e^2d}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{e^2d^2}{(1-e^2)^2}$$
$$(x-c)^2 - \frac{y^2}{e^2-1} = a^2 \qquad \left(a = \frac{ed}{e^2-1}, \quad c = ea\right)$$
$$\frac{(x-c)^2}{a^2} - \frac{y^2}{(e^2-1)a^2} = 1$$
$$\frac{(x-c)^2}{a^2} - \frac{y^2}{c^2-a^2} = 1$$

3. (a) ellipse: $r = \dfrac{8}{4 + 3\cos\theta} = \dfrac{2}{1 + \frac{3}{4}\cos\theta}$.

Thus $e = \dfrac{3}{4}$ and $\dfrac{3}{4}d = 2 \implies d = \dfrac{8}{3}$.

Rectangular equation:

$a = \dfrac{32}{7}, \quad c = \dfrac{24}{7}, \quad$ so $\dfrac{\left(x + \frac{24}{7}\right)^2}{\left(\frac{32}{7}\right)^2} + \dfrac{y^2}{\left(\frac{24}{7}\right)^2} = 1$

(b) hyperbola: $r = \dfrac{6}{1 + 2\cos\theta}$

Thus $e = 2$ and $2d = 6 \implies d = 3$.

Rectangular equation:

$a = 2, \quad c = 4, \quad$ so $\dfrac{(x-4)^2}{4} - \dfrac{y^2}{12} = 1$

(c) parabola: $r = \dfrac{6}{2 + 2\cos\theta} = \dfrac{3}{1 + \cos\theta}$

Thus $e = 1$ and $d = 3$.

Rectangular equation:

$y^2 = -4\left(\dfrac{3}{2}\right)\left(x - \dfrac{3}{2}\right) + -6\left(x - \dfrac{3}{2}\right)$

4. $r = \dfrac{\alpha}{1 - \beta\cos\theta}$ is the conic section $\quad r = \dfrac{\alpha}{1 + \beta\cos\theta}$ rotated π radians in the counter clockwise direction:

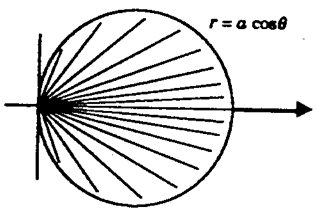
$$\frac{\alpha}{1 - \beta\cos\theta} = \frac{\alpha}{1 + \beta\cos(\theta - \pi)}$$

$r = \dfrac{\alpha}{1 - \beta\sin\theta}$ is the conic section $\quad r = \dfrac{\alpha}{1 + \beta\cos\theta}$ rotated $\dfrac{\pi}{2}$ radians in the clockwise direction:

$$\frac{\alpha}{1-\beta\sin\theta}=\frac{\alpha}{1+\beta\cos(\theta+\frac{\pi}{2})}$$

SECTION 9.5

1.

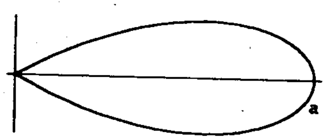


$$A=\int_{-\pi/2}^{\pi/2}\frac{1}{2}\left[a\cos\theta\right]^2\,d\theta$$

$$=a^2\int_0^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta$$

$$=a^2\left[\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]_0^{\pi/2}=\frac{1}{4}\pi a^2$$

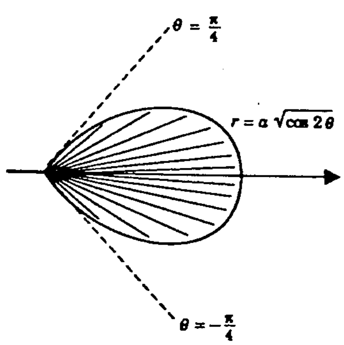
2.



$$A=\int_{-\pi/6}^{\pi/6}\frac{1}{2}a^2\cos^2 3\theta\,d\theta$$

$$=\frac{a^2}{2}\left[\frac{\theta}{2}+\frac{\sin 6\theta}{12}\right]_{-\pi/6}^{\pi/6}=\frac{1}{12}\pi a^2$$

3.

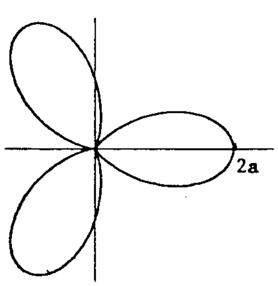


$$A=\int_{-\pi/4}^{\pi/4}\frac{1}{2}\left[a\sqrt{\cos 2\theta}\right]^2\,d\theta$$

$$=a^2\int_0^{\pi/4}\cos 2\theta\,d\theta$$

$$=a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4}=\frac{1}{2}a^2$$

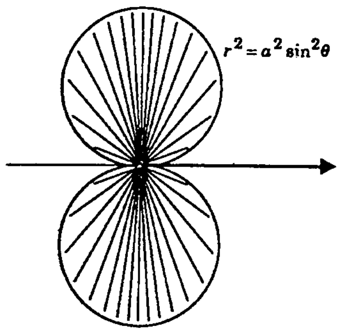
4.



$$A=\int_{-\pi/3}^{3}\frac{1}{2}a^2(1+2\cos 3\theta+\cos^2 3\theta)\,d\theta$$

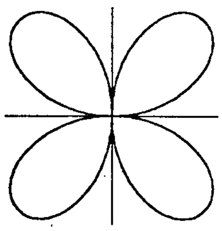
$$=\frac{a^2}{2}\left[\theta+\frac{2}{3}\sin 3\theta+\frac{\theta}{2}+\frac{\sin 6\theta}{12}\right]_{-\pi/3}^{\pi/3}=\frac{1}{2}\pi a^2$$

5.



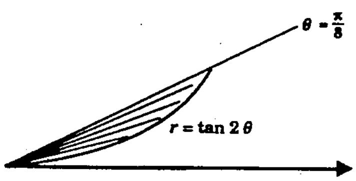
$$A = 2\int_0^{\pi} \frac{1}{2}\left(a^2 \sin^2\theta\right) d\theta$$
$$= a^2 \int_0^{\pi} \frac{1-\cos 2\theta}{2}\, d\theta$$
$$= a^2 \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi} = \frac{1}{2}\pi a^2$$

6.



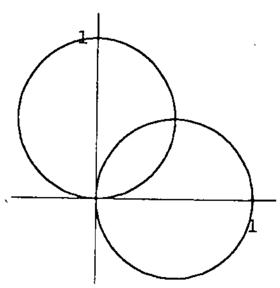
$$A = 4\int_0^{\pi/2} \frac{1}{2}a^2 \sin^2 2\theta\, d\theta$$
$$= 2a^2 \left[\frac{\theta}{2} - \frac{\sin 4\theta}{8}\right]_0^{\pi/2} = \frac{1}{2}\pi a^2$$

7.



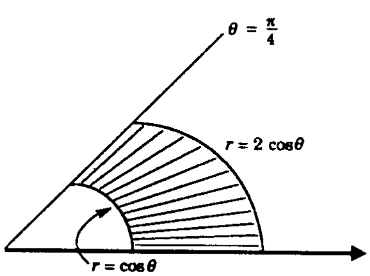
$$A = \int_0^{\pi/8} \frac{1}{2}\left[\tan 2\theta\right]^2 d\theta$$
$$= \frac{1}{2}\int_0^{\pi/8} \left(\sec^2 2\theta - 1\right) d\theta$$
$$= \frac{1}{2}\left[\frac{1}{2}\tan 2\theta - \theta\right]_0^{\pi/8} = \frac{1}{4} - \frac{\pi}{16}$$

8.



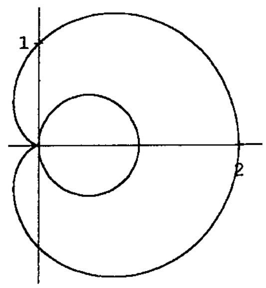
$$A = \int_0^{\pi/4} \frac{1}{2}(\cos^2\theta - \sin^2\theta)\, d\theta$$
$$= \frac{1}{2}\int_0^{\pi/4} \cos 2\theta\, d\theta$$
$$= \left[\frac{\sin 2\theta}{4}\right]_0^{\pi/4} = \frac{1}{4}$$

9.



$$A = \int_0^{\pi/4} \frac{1}{2}\left(\left[2\cos\theta\right]^2 - \left[\cos\theta\right]^2\right) d\theta$$
$$= \frac{3}{2}\int_0^{\pi/4} \frac{1+\cos 2\theta}{2}\, d\theta$$
$$= \frac{3}{2}\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/4} = \frac{3}{16}\pi + \frac{3}{8}$$

10.

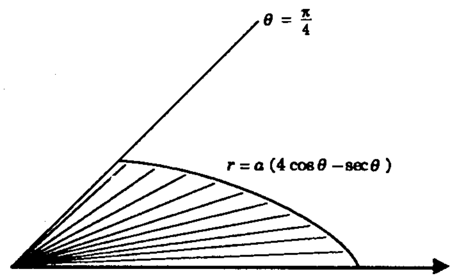


$$A = \int_0^{\pi/2} \frac{1}{2}(1 + 2\cos\theta + \cos^2\theta - \cos^2\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} (1 + 2\cos\theta)\, d\theta$$

$$= \frac{1}{2}\left[\theta + 2\sin\theta\right]_0^{\pi/2} = 1 + \frac{\pi}{4}$$

11.



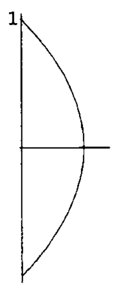
$$A = \int_0^{\pi/4} \frac{1}{2}\left[a\,(4\cos\theta - \sec\theta)\right]^2 d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/4} \left[16\cos^2\theta - 8 + \sec^2\theta\right] d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/4} \left[8\,(1 + \cos 2\theta) - 8 + \sec^2\theta\right] d\theta$$

$$= \frac{a^2}{2}\left[4\sin 2\theta + \tan\theta\right]_0^{\pi/4} = \frac{5}{2}a^2$$

12.

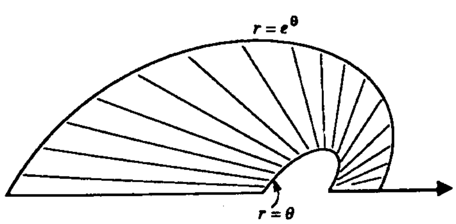


$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\cdot\frac{1}{4}\cdot\sec^4\frac{\theta}{2}\, d\theta$$

$$= \frac{1}{8}\int_{-\pi/2}^{\pi/2} (1 + \tan^2\frac{\theta}{2})\sec^2\frac{\theta}{2}\, d\theta$$

$$= \frac{1}{8}\left[2\tan\frac{\theta}{2} + \frac{2}{3}\tan^3\frac{\theta}{2}\right]_{-\pi/2}^{\pi/2} = \frac{2}{3}$$

13.

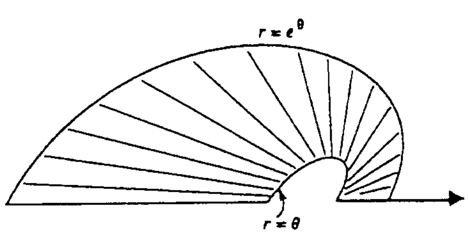


$$A = \int_0^{\pi} \frac{1}{2}\left(\left[e^{\theta}\right]^2 - [\theta]^2\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \left(e^{2\theta} - \theta^2\right) d\theta$$

$$= \tfrac{1}{2}\left[\tfrac{1}{2}e^{2\theta} - \tfrac{1}{3}\theta^3\right]_0^{\pi} = \tfrac{1}{12}\left(3e^{2\pi} - 3 - 2\pi^3\right)$$

14.

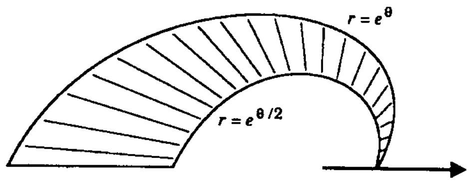


$$A = \int_0^{\pi} \frac{1}{2}\left[\left(e^{2\pi+\theta}\right)^2 - \theta^2\right] d\theta$$

$$= \frac{1}{2}\left[\frac{e^{4\pi+2\theta}}{2} - \frac{\theta^3}{3}\right]_0^{\pi}$$

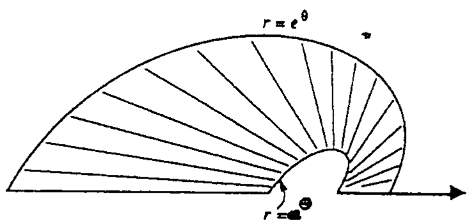
$$= \frac{1}{4}e^{6\pi} - \frac{1}{4}e^{4\pi} - \frac{1}{6}\pi^3$$

15.



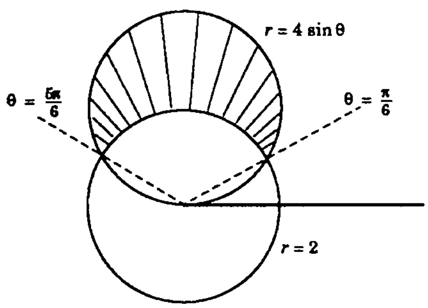
$$A = \int_0^{\pi} \frac{1}{2}\left(\left[e^{\theta}\right]^2 - \left[e^{\theta/2}\right]^2\right) d\theta$$
$$= \frac{1}{2}\int_0^{\pi} \left(e^{2\theta} - e^{\theta}\right) d\theta$$
$$= \frac{1}{2}\left[\frac{1}{2}e^{2\theta} - e^{\theta}\right]_0^{\pi} = \frac{1}{4}\left(e^{2\pi} + 1 - 2e^{\pi}\right)$$

16.



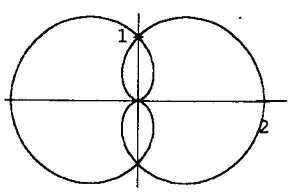
$$A = \int_0^{\pi} \frac{1}{2}\left[\left(e^{2\pi+\theta}\right)^2 - \left(e^{\theta}\right)^2\right] d\theta$$
$$= \frac{1}{2}\left[\frac{e^{4\pi+2\theta}}{2} - \frac{e^{2\theta}}{2}\right]_0^{\pi}$$
$$= \frac{1}{4}(e^{6\pi} - e^{4\pi} - e^{2\pi} + 1)$$

17.



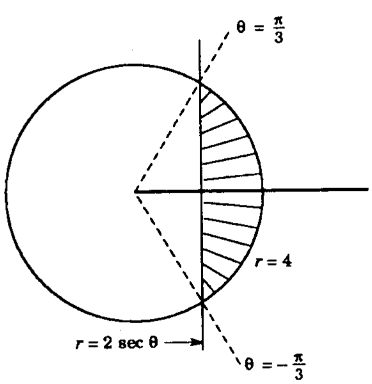
$$A = \int_{\pi/6}^{5\pi/6} \frac{1}{2}\left([\,4\sin\theta\,]^2 - [2]^2\right) d\theta$$

18.



$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left[(1+\cos\theta)^2 - (1-\cos\theta)^2\right] d\theta$$

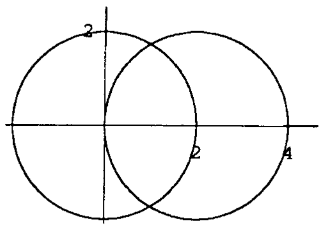
19.



$$A = \int_{-\pi/3}^{\pi/3} \frac{1}{2}\left([\,4\,]^2 - [\,2\sec\theta]^2\right) d\theta$$

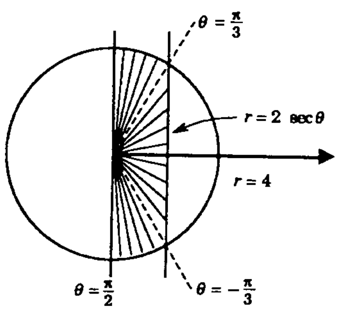
20.

$$A = 2\int_{\pi/3}^{\pi/2} \frac{1}{2}\left[2^2 - (4\cos\theta)^2\right] d\theta + 2\int_{\pi/2}^{\pi} \frac{1}{2}\cdot 2^2\, d\theta$$



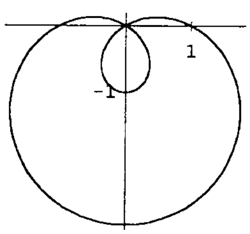
21.

$$A = 2\left\{\int_0^{\pi/3} \frac{1}{2}(2\sec\theta)^2\, d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2}(4)^2\, d\theta\right\}$$



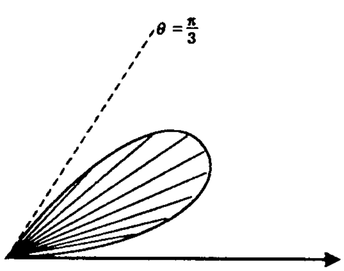
22.

$$A = \int_{\pi/6}^{5\pi/6} \frac{1}{2}(1 - 2\sin\theta)^2\, d\theta$$



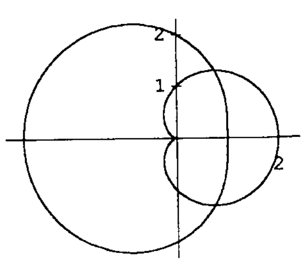
23.

$$A = \int_0^{\pi/3} \frac{1}{2}(2\sin 3\theta)^2\, d\theta$$



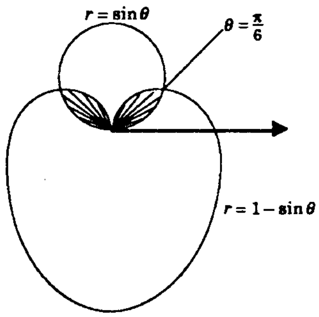
24.

$$A = \int_{\pi/3}^{5\pi/3} \frac{1}{2}\left[(2-\cos\theta)^2 - (1+\cos\theta)^2\right] d\theta$$



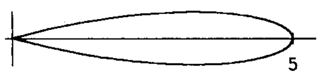
25.

$$A = 2\left\{\int_0^{\pi/6} \frac{1}{2}(\sin\theta)^2\, d\theta + \int_{\pi/6}^{\pi/2} \frac{1}{2}(1-\sin\theta)^2\, d\theta\right\}$$



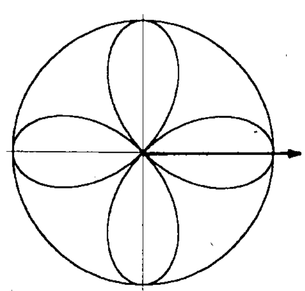
26.

$$A = 2\int_0^{\pi/12} \frac{1}{2}(5\cos 6\theta)^2\, d\theta$$



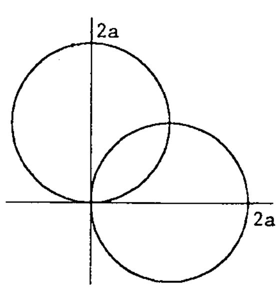
27.

$$A = \pi - 8\int_0^{\pi/4} \frac{1}{2}(\cos 2\theta)^2\, d\theta$$



28.

$$A = \int_0^{\pi/4} \frac{1}{2}(2a\cos\theta)^2\, d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}(2a\sin\theta)^2\, d\theta$$



29. The area of one petal of the curve $r = a\cos 2n\theta$ is given by:

$$2\int_0^{\pi/4n} \tfrac{1}{2}(a\cos 2n\theta)^2\,d\theta = a^2\int_0^{\pi/4n} \cos^2 2n\theta\,d\theta$$

$$= a^2\int_0^{\pi/4n}\left(\frac{1}{2}+\frac{\cos 4n\theta}{2}\right)d\theta$$

$$= a^2\left[\frac{1}{2}\theta+\frac{\sin 4n\theta}{8n}\right]_0^{\pi/4n} = \frac{\pi\, a^2}{8n}$$

The total area enclosed by $r = a\cos 2n\theta$ is $\dfrac{\pi\, a^2}{2}$.

The area of one petal of the curve $r = a\,\sin 2n\theta$ is given by:

$$A = 2\int_0^{\pi/4n} \tfrac{1}{2}(a\sin 2n\theta)^2\,d\theta = a^2\int_0^{\pi/4n}\left(\frac{1}{2}+\frac{\cos 4n\theta}{2}\right)d\theta = \frac{\pi\, a^2}{8n}$$

and the total area enclosed by the curve is $\dfrac{\pi\, a^2}{2}$.

30. Since there are $2n+1$ petals, total area $= (2n+1)\cdot$ (area of one petal). So

$$A = (2n+1)\cdot 2\int_0^{\pi/[2(2n+1)]} \frac{a^2}{2}\cos^2[(2n+1)\theta]\,d\theta = (2n+1)a^2\left[\frac{\theta}{2}+\frac{\sin[2(2n+1)\theta]}{4(2n+1)}\right]_0^{\pi/[2(2n+1)]}$$

$$= (2n+1)\cdot\frac{\pi a^2}{4(2n+1)} = \frac{\pi}{4}a^2, \quad \text{independent of } n.$$

31. Let $P = \{\alpha = \theta_0, \theta_1, \theta_2, \ldots,\ \theta_n = \beta\}$ be a partition of the interval $[\alpha, \beta]$. Let θ_i^* be the midpoint of $[\theta_{i-1}, \theta_i]$ and let $r_i^* = f(\theta_i^*)$. The area of the $i\,th$ "triangular" region is $\frac{1}{2}\,(r_i^*)\Delta\theta_i$, where $\Delta\theta_i = \theta_i - \theta_{i-1}$, and the rectangular coordinates of its centroid are(approximately) $\left(\frac{2}{3}\,r_i^*\cos\theta_i^*, \frac{2}{3}\,r_i^*\sin\theta_i^*\right)$.

The centroid $(\overline{x}_p, \overline{y}_p)$ of the union of the triangular regions satisfies the following equations

$$\overline{x}_p\,A_p = \frac{1}{3}\,(r_1^*)^3\cos\theta_1\Delta\theta_1 + \frac{1}{3}\,(r_2^*)^3\cos\theta_2\Delta\theta_2 + \cdots + \frac{1}{3}\,(r_n^*)^3\cos\theta_n\Delta\theta_n$$

$$\overline{y}_p\,A_p = \frac{1}{3}\,(r_1^*)^3\sin\theta_1\Delta\theta_1 + \frac{1}{3}\,(r_2^*)^3\sin\theta_2\Delta\theta_2 + \cdots + \frac{1}{3}\,(r_n^*)^3\sin\theta_n\Delta\theta_n$$

As $||P|| \to 0$, the union of the triangular regions tends to the region Ω and the equations above tend to

$$\overline{x}\,A = \int_\alpha^\beta \frac{1}{3}r^3\cos\theta\,d\theta$$

$$\overline{x}\,A = \int_\alpha^\beta \frac{1}{3}r^3\cos\theta\,d\theta$$

The result follows from the fact that $A = \displaystyle\int_\alpha^\beta \frac{1}{2}r^2\cos\theta\,d\theta$.

32. $\int_{-\alpha}^{\alpha} r^2\,d\theta = 2\alpha r^2 = 32\alpha$

$\overline{x} = \frac{1}{32\alpha}\cdot\frac{2}{3}\int_{-\alpha}^{\alpha} r^3\cos\theta\,d\theta = \frac{1}{48\alpha}\int_{-\alpha}^{\alpha} 64\cos\theta\,d\theta = \frac{4}{3\alpha}[\sin\theta]_{-\alpha}^{\alpha} = \frac{8\sin\alpha}{3\alpha}.$

$\overline{y} = \frac{1}{32\alpha}\cdot\frac{2}{3}\int_{-\alpha}^{\alpha} r^3\sin\theta\,d\theta = 0 \quad (\sin\theta \text{ an odd function.})$

33. Since the region enclosed by the cardioid $r = 1+\cos\theta$ is symmetric with respect to the x-axis, $\overline{y} = 0$.

To find $\overline{x}$:

$$\begin{aligned} A &= \int_0^{2\pi} r^2\,d\theta = \int_0^{2\pi} (1+\cos\theta)^2\,d\theta \\ &= \int_0^{2\pi} (1+2\cos\theta+\cos^2\theta)\,d\theta \\ &= \int_0^{2\pi} \left(\frac{3}{2}+2\cos\theta\frac{1}{2}\cos 2\theta\right)\,d\theta \\ &= \left[\frac{3}{2}+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{2\pi} = 3\pi \end{aligned}$$

and

$$\begin{aligned} \frac{2}{3}\int_0^{2\pi} r^3\cos\theta\,d\theta &= \frac{2}{3}\int_0^{2\pi} (1+\cos\theta)^3\cos\theta\,d\theta \\ &= \frac{2}{3}\int_0^{2\pi} \left(\cos\theta+3\cos^2\theta+3\cos^3\theta+\cos^4\theta\right)\,d\theta \\ &= \frac{2}{3}\int_0^{2\pi} \left(\frac{15}{8}+4\cos\theta+2\cos 2\theta+\frac{1}{8}\cos 4\theta-3\sin^2\theta\,\cos\theta\right)\,d\theta \\ &= \frac{2}{3}\left[\frac{15}{8}\theta+4\sin\theta+\sin 2\theta+\frac{1}{32}\sin 4\theta-\sin^3\theta\right]_0^{2\pi} = \frac{5}{2}\pi \end{aligned}$$

Thus $\quad \overline{x} = \frac{5\pi/2}{3\pi} = \frac{5}{6}.$

34. $\int_0^{2\pi} r^2\,d\theta = \int_0^{2\pi} (4+4\sin\theta+\sin^2\theta)\,d\theta = \left[4\theta-4\cos\theta+\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]_0^{2\pi} = 9\pi$

$\overline{x} = \frac{1}{9\pi}\cdot\frac{2}{3}\int_0^{2\pi} r^3\cos\theta\,d\theta = \frac{2}{27\pi}\int_0^{2\pi} (2+\sin\theta)^3\cos\theta\,d\theta = 0$

$\overline{y} = \frac{2}{27\pi}\int_0^{2\pi} (2+\sin\theta)^3\sin\theta\,d\theta = \frac{2}{27\pi}\cdot\frac{51}{8}\cdot 2\pi = \frac{17}{18}$

35. (a) $y^2 = x^2\left(\dfrac{a-x}{a+x}\right)$

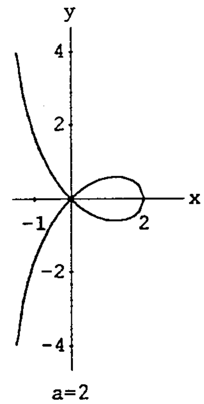
$r^2\sin^2\theta = r^2\cos^2\theta\left(\dfrac{a-r\cos\theta}{a+r\cos\theta}\right)$

$\sin^2\theta(a+r\cos\theta) = \cos^2\theta(a-r\cos\theta)$

$r\cos\theta = a\cos 2\theta$

$r = a\cos 2\theta\sec\theta$

(b) Let $a=2$



(c) $A = \displaystyle\int_{3\pi/4}^{5\pi/4}\frac{1}{2}a^2\cos^2 2\theta\sec^2\theta\,d\theta$

$= 2\displaystyle\int_{3\pi/4}^{5\pi/4}\cos^2 2\theta\sec^2\theta\,d\theta \qquad (a=2)$

$= 2\displaystyle\int_{3\pi/4}^{5\pi/4}\frac{(2\cos^2\theta-1)^2}{\cos^2\theta}\,d\theta$

$= 2\displaystyle\int_{3\pi/4}^{5\pi/4}\left(4\cos^2\theta-4+\sec^2\theta\right)\,d\theta$

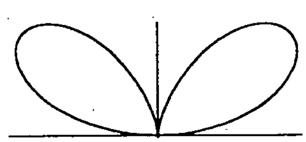
$= 2\displaystyle\int_{3\pi/4}^{5\pi/4}\left(-2+2\cos 2\theta+\sec^2\theta\right)\,d\theta$

$= 2\left[-2\theta+\sin 2\theta+\tan\theta\right]_{3\pi/4}^{5\pi/4} = 8-2\pi$

36. (a) $(x^2+y^2)^2 = ax^2y \implies r^4 = ar^2\cos^2\theta r\sin\theta \implies r = a\sin\theta\cos^2\theta$

(b) same for all values of a,

with different scale



(c) $A = \displaystyle\int_0^{\pi/2}\frac{1}{2}a^2\sin^2\theta\cos^4\theta\,d\theta$

$= \dfrac{a^2}{2}\displaystyle\int_0^{\pi/2}(\cos^4\theta-\cos^6\theta)\,d\theta$

$= \dfrac{a^2}{2}\cdot\dfrac{\pi}{32} = \dfrac{\pi a^2}{64}$

SECTION 9.6

1. $4x = (y-1)^2$

2. $2x+3y = 13$

3. $y = 4x^2+1, \quad x \ge 0$

4. $y = (x+1)^3-5$

5. $9x^2+4y^2 = 36$

6. $x = (y-2)^2+1$

7. $1 + x^2 = y^2$

8. $(x-2)^2 + y^2 = 1$

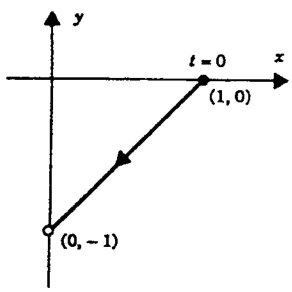
9. $y = 2 - x^2, \quad -1 \le x \le 1$

10. $y = 4 - x^2, \quad x > 0$

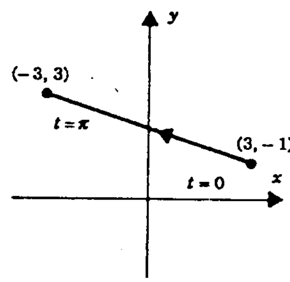
11. $2y - 6 = x, \quad -4 \le x \le 4$

12. $1 + y^2 = x^2$

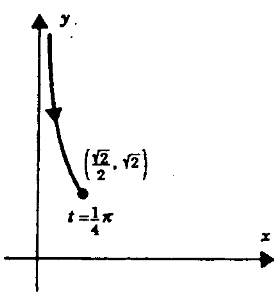
13. $y = x - 1$



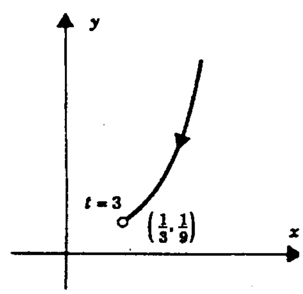
14. $x = 6 - 3y$



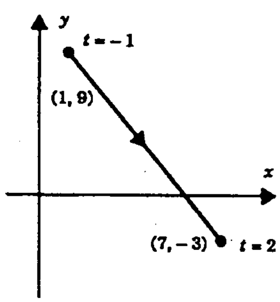
15. $xy = 1$



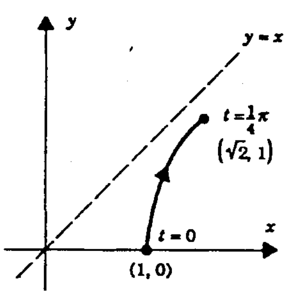
16. $y = x^2$



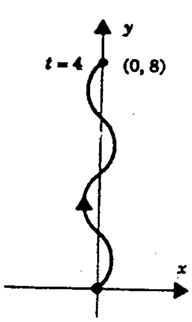
17. $2x + y = 11$



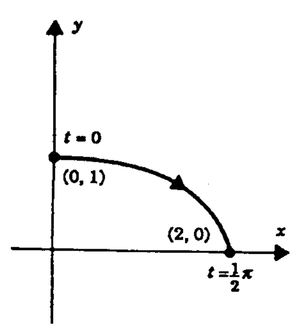
18. $1 + y^2 = x^2$



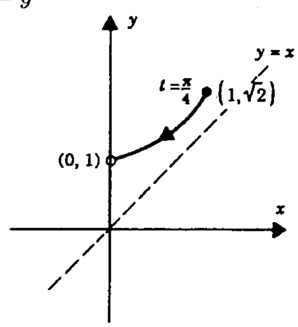
19. $x = \sin \frac{1}{2}\pi y$



20. $x^2 + 4y^2 = 4$



21. $1 + x^2 = y^2$



22. (a) $x(t) = t, \quad y(t) = f(t), \quad t \in [a, b]$

(b) $x(t) = f(t)\cos t, \quad y(t) = f(t)\sin t, \quad t \in [\alpha, \beta]$

23. (a) $x\,(t) = -\sin 2\pi t, \quad y\,(t) = \cos 2\pi t$ (b) $x\,(t) = \sin 4\pi t, \quad y\,(t) = \cos 4\pi t$

(c) $x\,(t) = \cos \frac{1}{2}\pi t, \quad y\,(t) = \sin \frac{1}{2}\pi t$ (d) $x\,(t) = \cos \frac{3}{2}\pi t, \quad y\,(t) = -\sin \frac{3}{2}\pi t$

24. (a) $x(t) = 3\cos 2\pi t, \quad y(t) = -4\sin 2\pi t$ (b) $x(t) = 3\sin 2\pi t, \quad y(t) = 4\cos 2\pi t$

(c) $x(t) = -3\cos 4\pi t, \quad y(t) = -4\sin \pi t$ (d) $x(t) = 3\cos \frac{1}{2}\pi t, \quad y(t) = 4\sin \frac{1}{2}\pi t$

25. $x\,(t) = \tan \frac{1}{2}\pi t, \quad y\,(t) = 2$

26. Any continuous function unbounded on $(0, 1)$ will do. For example, $f(t) = \dfrac{1}{t}$

27. $x\,(t) = 3 + 5t, \quad y\,(t) = 7 - 2t$

28. $x(t) = 2 + 4t, \quad y(t) = 6 - 3t$

29. $x\,(t) = \sin^2 \pi t, \quad y\,(t) = -\cos \pi t$

30. $x(t) = 4(1 - t)^2, \quad y(t) = 2(1 - t)$

31. $x\,(t) = (2 - t)^2, \quad y\,(t) = (2 - t)^3$

32. $x(t) = (t + 1)^3, \quad y(t) = (t + 1)^2$

33. $x\,(t) = t(b - a) + a, \quad y\,(t) = f(t\,(b - a) + a)$

34. (a) $$\int_c^d y(t)x'(t)\,dt = \int_c^d f(x(t))x'(t)\,dt = \int_a^b f(x)\,dx = \text{area below } C$$

(b) $$\int_c^d x(t)y(t)x'(t)\,dt = \int_c^d x(t)f(x(t))x'(t)\,dt = \int_a^b xf(x)\,dx = \overline{x}A$$

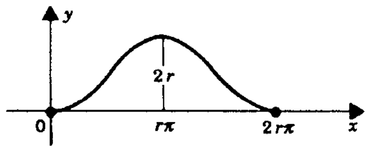
$$\int_c^d \frac{1}{2}\,[y(t)]^2\,x'(t)\,dt = \int_c^d \frac{1}{2}\,[f(x(t))]^2\,x'(t)\,dt = \int_c^d \frac{1}{2}\,[f(x)]^2\,dx = \overline{y}A$$

(c) $$\int_c^d \pi\,[y(t)]^2\,x'(t)\,dt = \int_c^d \pi\,[f(x(t))]^2\,x'(t)\,dt = \int_a^b \pi\,[f(x)]^2\,dx = V_x$$

$$\int_c^d 2\pi x(t)y(t)x'(t)\,dt = \int_c^d 2\pi x(t)f(x(t))x'(t)\,dt = \int_a^b 2\pi xf(x)\,dx = V_y$$

$$\int_c^d \pi x(t)\,[y(t)]^2\,x'(t)\,dt = \int_c^d \pi x(t)\,[f(x(t))]^2\,x'(t)\,dt = \int_a^b \pi x\,[f(x)]^2\,dx$$

35.



$$A = \int_0^{2\pi} x(t)\, y'(t)\, dt = r^2 \int_0^{2\pi} (1-\cos t)\, dt = r^2 \left[t - \sin t\right]_0^{2\pi} = 2\pi r^2$$

36. $\overline{x}A = \int_0^{2\pi} r\,t\,r(1-\cos t)\,r\,dt = r^3 \int_0^{2\pi} (t - t\cos t)\,dt = r^3\left[\frac{t^3}{2} - t\sin t - \cos t\right]_0^{2\pi} = 2r^3\pi^2$

$\implies \overline{x} = \dfrac{2r^3\pi^2}{2\pi r^2} = \pi r$

$\overline{y}A = \int_0^{2\pi} \frac{1}{2}\left[r^2(1-\cos t)^2\right] r\,dt = \frac{r^3}{2}\int_0^{2\pi}\left(1 - 2\cos t + \cos^2 t\right)\,dt = \frac{r^3}{2}\left[\frac{3t}{2} - 2\sin t + \frac{\sin 2t}{2}\right]_0^{2\pi}$

$= \dfrac{3\pi r^3}{2} \implies \overline{y} = \dfrac{3\pi r^3/2}{2\pi r^2} = \dfrac{3}{4}r$

37. (a) $V_x = 2\pi\overline{y}A = 2\pi\left(\frac{3}{4}r\right)\left(2\pi r^2\right) = 3\pi^2 r^3$

(b) $V_y = 2\pi\overline{x}A = 2\pi(\pi r)\,2\pi r^2 = 4\pi^3 r^3$

38. (a) $\overline{x}V_x = \int_0^{2\pi} \pi r\,t\,r^2(1-\cos t)^2 r\,dt = \pi r^4 \int_0^{2\pi} (t - 2t\cos t + t\cos^2 t)\,dt = 3\pi^3 r^4$

$\implies \overline{x} = \pi r, \quad \overline{y} = 0$

$\implies \overline{y}V_y = \int_0^{2\pi} \pi r\,t\,r^2(1-\cos t)^2 r\,dt = 3\pi^3 r^4 \implies \overline{y} = \frac{3}{4}r, \quad \overline{x} = 0$

39. $x(t) = -a\cos t, \quad y(t) = b\sin t \qquad t\in[0,\pi]$

40. (a) $A = 2\int_0^{\pi} y(t)x'(t)\,dt = 2\int_0^{\pi}(b\sin t)(a\sin t)\,dt = 2ab\left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{\pi} = \pi ab$

(b) $\overline{x}A = \int_0^{\pi}(-\cos t)(b\sin t)(a\sin t)\,dt = -a^2 b\int_0^{\pi}\sin^2 t\cos t\,dt = 0 \implies \overline{x} = 0$

$\overline{y}A = \int_0^{\pi}\frac{1}{2}b^2\sin^2 t(a\sin t)\,dt = \frac{a^2 b}{2}\int_0^{\pi}(1-\cos^2 t)\sin t\,dt = \frac{4}{3}ab^2$

$\implies \overline{y} = \dfrac{2}{3}\dfrac{ab^2}{\pi ab/2} = \dfrac{4b}{3\pi}$

41. (a) Equation for the ray: $y + 2x = 17, \quad x \geq 6.$

Equation for the circle: $(x-3)^2 + (y-1)^2 = 25.$

Simultaneous solution of these equations gives the points of intersection: $(6,5)$ and $(8,1)$.

(b) The particle on the ray is at $(6,5)$ when $t = 0$. However, when $t = 0$ the particle on the circle is at the point $(-2,1)$. Thus, the intersection point $(6,5)$ is not a collision point.

The particle on the ray is at $(8,1)$ when $t=1$. Since the particle on the circle is also at $(8,1)$ when $t=1$, the intersection point $(8,1)$ is a collision point.

42. (a) Equation of ellipse: $\dfrac{(x-2)^2}{9}+\dfrac{(y-3)^2}{49}=1$

Equation of parabola: $y=-\dfrac{7}{15}(x-1)^2+\dfrac{157}{15}$

Solving simultaneously gives the points of intersection $(2,10)$ and $(5,3)$

(b) Particle 2 is at $(2,10)$ at $t=0$, but particle 1 is at $(-1,3)$ at $t=0$, so no collision at $(2,10)$.

Both particles are at $(5,3)$ when $t=1$, so $(5,3)$ is a collision point.

43. If $x(r)=x(s)$ and $r\neq s$, then

$$r^2-2r=s^2-2s$$

$$r^2-s^2=2r-2s$$

$$(1) \qquad r+s=2.$$

If $y(r)=y(s)$ and $r\neq s$, then

$$r^3-3r^2+2r=s^3-3s^2+2s$$

$$\left(r^3-s^3\right)-3\left(r^2-s^2\right)+2\left(r-s\right)=0$$

$$(2) \qquad \left(r^2+rs+s^2\right)-3\left(r+s\right)+2=0.$$

Simultaneous solution of (1) and (2) gives $r=0$ and $r=2$. Since $(x(0),y(0))=(0,0)=(x(2),y(2))$, the curve intersects itself at the origin.

44. $x(r)=x(s)$ and $r\neq s \implies \cos r(1-2\sin r)=\cos s(1-2\sin s)$

$y(r)=y(s)$ and $r\neq s \implies \sin r(1-2\sin r)=\sin s(1-2\sin s)$

Both equations can be satisfied simultaneously with $r\neq s$ in $[0,\pi]$ only for $r=\dfrac{\pi}{6},\ s=\dfrac{5\pi}{6}$, or $r=\dfrac{5\pi}{6},\ r=\dfrac{\pi}{6}$

So the curve intersects itself at $(0,0)$.

45. Suppose that $r,s\in[0,4]$ and $r\neq s$.

$$x(r)=x(s) \implies \sin 2\pi r=\sin 2\pi s.$$

$$y(r)=y(s) \implies 2r-r^2=2s-s^2 \implies 2(r-s)=r^2-s^2 \implies 2=r+s.$$

Now we solve the equations simultaneously:

$$\sin 2\pi r = \sin[2\pi(2-r)] = -\sin 2\pi r$$

$$2\sin 2\pi r = 0$$

$$\sin 2\pi r = 0.$$

Since $r \in [0,4]$, $r = 0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ 2,\ \frac{5}{2},\ 3,\ \frac{7}{2},\ 4.$

Since $s \in [0,4]$ and $r \neq s$ and $r+s=2$, we are left with $r = 0,\ \frac{1}{2},\ \frac{3}{2},\ 2.$ Note that

$$(x(0),\,y(0)) = (0,0) = (x(2),\,y(2)) \quad \text{and} \quad \left(x\left(\tfrac{1}{2}\right),\,y\left(\tfrac{1}{2}\right)\right) = \left(0,\,\tfrac{3}{4}\right) = \left(x\left(\tfrac{3}{2}\right),\,y\left(\tfrac{3}{2}\right)\right).$$

The curve intersects itself at $(0,0)$ and $\left(0,\frac{3}{4}\right)$.

46.

$$x(r) = x(s) \text{ and } r \neq s \implies \begin{cases} r^3 - 4r = s^3 - 4s \\ r^3 - s^3 = 4(r-s) \\ r^2 + rs + s^2 = 4 \end{cases}$$

$$y(r) = y(s) \text{ and } r \neq s \implies \begin{cases} r^3 - 3r^2 + 2r = s^3 - 3s^2 + 2s \\ r^3 - s^3 - 3(r^2 - s^2) + 2(r-s) = 0 \\ (r^2 + rs + s^2) - 3(r+s) + 2 = 0 \end{cases}$$

Solving simultaneously gives $r = 0$, $s = 2$, so curve intersect itself at $(0,0)$

47. (a) The coefficient a affects the amplitude and the period.

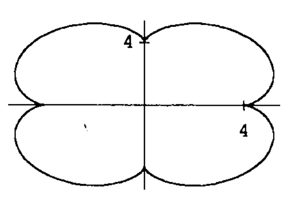
(b) $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}$$

You can verify that $\dfrac{dy}{dx} \to -\infty$ as $\theta \to 2\pi^-$; $\dfrac{dy}{dx} \to \infty$ as $\theta \to 2\pi^+$.

(c) The curve has a vertical cusp at $\theta = 2\pi$.

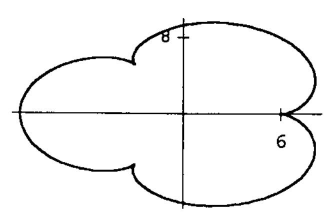
48. (a) $x(\theta) = 5\cos\theta - \cos 5\theta$

$y(\theta) = 5\sin\theta - \sin 5\theta$



(b) $x(\theta) = 8\cos\theta - 2\cos 4\theta$

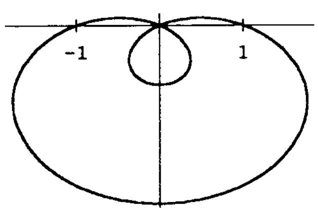
$y(\theta) = 8\sin\theta - 2\sin 4\theta$



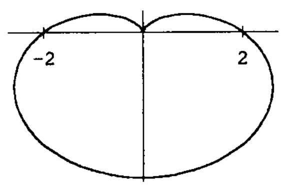
49. See the answer section in the text.

50. $x(\theta) = \cos\theta(a - b\sin\theta), \quad y(\theta) = \sin\theta(a - b\sin\theta)$

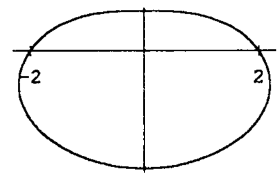
(a) $a = 1, \ b = 2$



(b) $a = 2, \ b = 2$



(c) $a = 2, \ b = 1$



(d) The curves are limacons; the curve has an inner loop if $a < b$ and no loop if $a > b$.

PROJECT 9.6

1. Since $x"(t) = 0$, we have $x'(t) = C$.

Since $x'(0) = v_0 \cos\theta$, we have $x'(t) = v_0 \cos\theta$.

Integrating again, $x(t) = (v_0 \cos\theta)t + x_0$ (since $x(0) = x_0$)

Similarly, since $y"(t) = -g$, we have $y'(t) = -gt + C$.

Since $y'(0) = v_0 \sin\theta$, we have $y'(t) = -gt + v_0 \sin\theta$.

Integrating again, $y(t) = -\frac{1}{2}gt^2 + (v_0 \sin\theta)t + y_0$ (since $y(0) = y_0$)

2. From the first equation, we get $t = \dfrac{1}{v_0 \cos\theta}[x(t) - x_0]$.

Substituting this into the second equation gives the desired result.

3. (a) Using $x_0 = 0, \ y_0 = 0, \ g = 32:$ $\quad y = -\dfrac{16}{{v_0}^2}(\sec^2\theta)x^2 + (\tan\theta)x$.

(b) At maximum height, $y'(t) = 0$. $\ y(t) = -16t^2 + (v_0 \sin\theta)\,t$; $\ y'(t) = -32t + v_0 \sin\theta$.

$$y'(t) = 0 \implies t = \frac{v_0 \sin\theta}{32}; \quad \text{max height: } \ y\left(\frac{v_0 \sin\theta}{32}\right) = \frac{{v_0}^2}{64}\sin^2\theta.$$

(c) $y = 0$ (and $x \neq 0$) $\quad \Longrightarrow \quad \dfrac{v_0^2}{16}\cos\theta\sin\theta$

(d) $y(t) = 0$ (and $t \neq 0$) when $t = \dfrac{v_0}{16}\sin\theta$

(e) The range $\frac{1}{16}{v_0}^2 \sin\theta\cos\theta = \frac{1}{32}{v_0}^2 \sin 2\theta$ is clearly maximal when $\theta = \frac{1}{4}\pi$ for then $\sin 2\theta = 1$.

(f) We want $\frac{1}{32}{v_0}^2 \sin 2\theta = b, \quad \theta = \frac{1}{2}\sin^{-1}\left(\frac{32b}{{v_0}^2}\right).$

SECTION 9.7

1. $x'(1) = 1, \quad y'(1) = 3, \quad$ slope 3, $\quad$ point $(1, 0)$; $\quad$ tangent: $y = 3(x-1)$

2. $x'(2) = 4, \quad y'(2) = 1, \quad$ slope $= \frac{1}{4}$, $\quad$ point $(4, 7)$; $\quad$ tangent: $y - 7 = \frac{1}{4}(x-4)$

3. $x'(0) = 2, \quad y'(0) = 0, \quad$ slope 0, $\quad$ point $(0, 1)$; $\quad$ tangent: $y = 1$

4. $x'(1) = 2, \quad y'(1) = 4, \quad$ slope $= 2$, $\quad$ point $(1, 1)$; $\quad$ tangent: $y - 1 = 2(x-1)$

5. $x'(1/2) = 1, \quad y'(1/2) = -3, \quad$ slope -3, $\quad$ point $\left(\frac{1}{4}, \frac{9}{4}\right)$; $\quad$ tangent: $y - \frac{9}{4} = -3\left(x - \frac{1}{4}\right)$

6. $x'(1) = -1, \quad y'(1) = 2, \quad$ slope $= -2$, $\quad$ point $(1, 2)$; $\quad$ tangent: $y - 2 = -2(x-1)$

7. $x'\left(\frac{\pi}{4}\right) = -\frac{3}{4}\sqrt{2}, \quad y'\left(\frac{\pi}{4}\right) = \frac{3}{4}\sqrt{2}, \quad$ slope -1, $\quad$ point $\left(\frac{1}{4}\sqrt{2}, \frac{1}{4}\sqrt{2}\right)$;

tangent: $y - \frac{1}{4}\sqrt{2} = -\left(x - \frac{1}{4}\sqrt{2}\right)$

8. $x'(0) = 1, \quad y'(0) = -3, \quad$ slope $= -3$, $\quad$ point $(1, 3)$; $\quad$ tangent: $y - 3 = -3(x-1)$

9. $x(\theta) = \cos\theta\,(4 - 2\sin\theta), \quad y(\theta) = \sin\theta\,(4 - 2\sin\theta), \quad$ point $(4, 0)$

$x'(\theta) = -4\sin\theta - 2\left(\cos^2\theta - \sin^2\theta\right), \quad y'(\theta) = 4\cos\theta - 4\sin\theta\cos\theta$

$x'(0) = -2, \quad y'(0) = 4, \quad$ slope -2, $\quad$ tangent: $y = -2\,(x-4)$

10. $x(\theta) = 4\cos 2\theta\cos\theta, \quad y(\theta) = 4\cos 2\theta\sin\theta, \quad$ point $(0, -4)$

$x'(\theta) = -8\sin 2\theta\cos\theta - 4\cos 2\theta\sin\theta, \quad y'(\theta) = -8\sin 2\theta\sin\theta + 4\cos 2\theta\cos\theta$

$x'(\frac{\pi}{2}) = 4, \quad y'(\frac{\pi}{2}) = 0, \quad m = 0$; $\quad$ tangent: $y = -4$.

11. $x(\theta) = \frac{4\cos\theta}{5 - \cos\theta}, \quad y(\theta) = \frac{4\sin\theta}{5 - \cos\theta}, \quad$ point $\left(0, \frac{4}{5}\right)$

$x'(\theta) = \frac{-20\sin\theta}{(5 - \cos\theta)^2}, \quad y'(\theta) = \frac{4\,(5\cos\theta - 1)}{(5 - \cos\theta)^2}$

$x'\left(\frac{\pi}{2}\right) = -\frac{4}{5}, \quad y'\left(\frac{\pi}{2}\right) = -\frac{4}{25}, \quad$ slope $\frac{1}{5}$, $\quad$ tangent: $y - \frac{4}{5} = \frac{1}{5}x$

12. $x(\theta) = \frac{5\cos\theta}{4 - \cos\theta}, \quad y(\theta) = \frac{5\sin\theta}{4 - \cos\theta}, \quad$ point $\left(\frac{5\sqrt{3}}{8 - \sqrt{3}}, \frac{5}{8 - \sqrt{3}}\right)$

$x'(\theta) = \frac{-20\sin\theta}{(4 - \cos\theta)^2}, \quad y'(\theta) = \frac{5(4\cos\theta - 1)}{(4 - \cos\theta)^2}, \quad$ slope $\frac{1 - 2\sqrt{3}}{2}$

tangent: $y - \frac{5}{8 - \sqrt{3}} = \left(\frac{1 - 2\sqrt{3}}{2}\right)\left(x - \frac{5\sqrt{3}}{8 - \sqrt{3}}\right)$

13. $x(\theta) = \dfrac{\cos\theta\,(\sin\theta - \cos\theta)}{\sin\theta + \cos\theta}, \quad y(\theta) = \dfrac{\sin\theta\,(\sin\theta - \cos\theta)}{\sin\theta + \cos\theta}, \quad$ point $(-1, 0)$

$$x'(\theta) = \frac{\sin\theta\,\cos 2\theta + 2\cos\theta}{(\sin\theta + \cos\theta)^2}, \quad y'(\theta) = \frac{2\sin\theta - \cos\theta\,\cos 2\theta}{(\sin\theta + \cos\theta)^2}$$

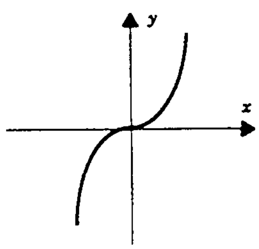
$x'(0) = 2, \quad y'(0) = -1, \quad$ slope $-\frac{1}{2}, \quad$ tangent $y = -\frac{1}{2}(x+1)$

14. $x(\theta) = \dfrac{\sin\theta + \cos\theta}{\sin\theta - \cos\theta}\cos\theta, \quad y(\theta) = \dfrac{\sin\theta + \cos\theta}{\sin\theta - \cos\theta}\sin\theta, \quad$ point $(0, 1)$

$$x'(\theta) = \frac{-2\cos\theta}{(\sin\theta - \cos\theta)^2} - \frac{\sin\theta + \cos\theta}{\sin\theta - \cos\theta}\sin\theta, \quad y'(\theta) = \frac{-2\sin\theta}{(\sin\theta - \cos\theta)^2} + \frac{\sin\theta + \cos\theta}{\sin\theta - \cos\theta}\cos\theta$$

$x'(\pi/2) = -1, \quad y'(\pi/2) = -2, \quad$ slope 2; $\quad$ tangent: $y - 1 = 2x$

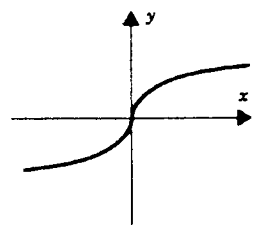
15. $x(t) = t, \quad y(t) = t^3$



$x'(0) = 1, \quad y'(0) = 0, \quad$ slope 0

tangent $y = 0$

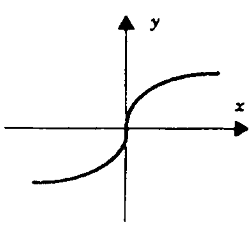
16. $x(t) = t^3, \quad y(t) = t$



$x'(0) = 0, \quad y'(0) = 1$

tangent $x = 0$

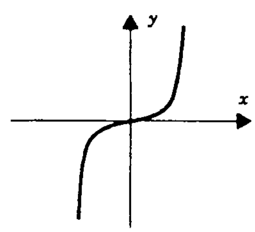
17. $x(t) = t^{5/3}, \quad y(t) = t$



$x'(0) = 0, \quad y'(0) = 1, \quad$ slope undefined

tangent $x = 0$

18. $x(t) = t, \quad y(t) = t^{5/3}$



$x'(0) = 1, \quad y'(0) = 0$

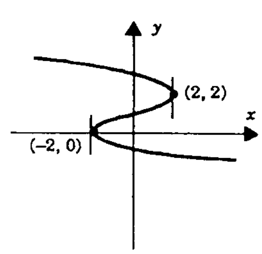
tangent $y = 0$

19. $x'(t) = 3 - 3t^2, \quad y'(t) = 1$

$x'(t) = 0 \quad \Longrightarrow \quad t = \pm 1; \quad y'(t) \neq 0$

(a) none

(b) at (2, 2) and (−2, 0)



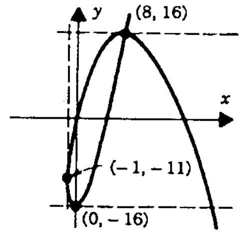
20. $x'(t) = 2t - 2, \quad y'(t) = 3t^2 - 12$

$x'(t) = 0 \implies t = 1$

$y'(t) = 0 \implies t = \pm 2$

(a) at $(0, -16)$ and $(8, 16)$

(b) at $(-1, -11)$



21. curve traced once completely with $t \in [0, 2\pi)$

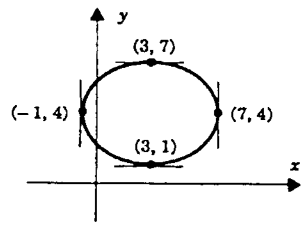
$x'(t) = -4\cos t, \quad y'(t) = -3\sin t$

$x'(t) = 0 \implies t = \frac{\pi}{2}, \frac{3\pi}{2};$

$y'(t) = 0 \implies t = 0, \pi$

(a) at $(3, 7)$ and $(3, 1)$

(b) at $(-1, 4)$ and $(7, 4)$



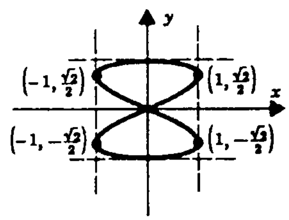
22. $x'(t) = 2\cos 2t, \quad y'(t) = \cos t$

$x'(t) = 0 \implies t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$y'(t) = 0 \implies t = \frac{\pi}{2}, \frac{3\pi}{2}$

(a) at $(0, \pm 1)$

(b) at $(\pm 1, \pm\frac{\sqrt{2}}{2})$



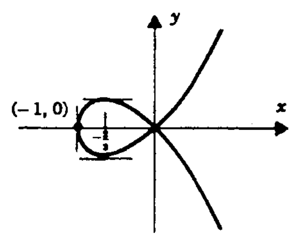
23. $x'(t) = 2t - 2, \quad y'(t) = 3t^2 - 6t + 2$

$x'(t) = 0 \implies t = 1$

$y'(t) = 0 \implies t = 1 \pm \frac{1}{3}\sqrt{3}$

(a) at $\left(-\frac{2}{3}, \pm\frac{2}{9}\sqrt{3}\right)$

(b) at $(-1, 0)$



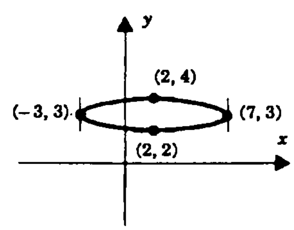
24. $x'(t) = 5\sin t, \quad y'(t) = \cos t$

$x'(t) = 0 \implies t = 0, \pi$

$y'(t) = 0 \implies t = \frac{\pi}{2}, \frac{3\pi}{2}$

(a) at $(2, 2)$ and $(2, 4)$

(b) at $(-3, 3)$ and $(7, 3)$

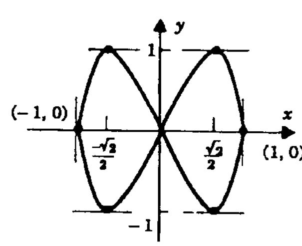


25. curve traced completely with $t \in [0, 2\pi)$

$x'(t) = -\sin t, \quad y'(t) = 2\cos 2t$

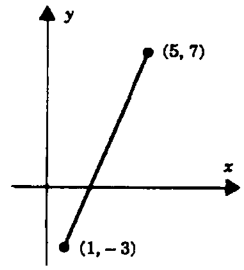
$x'(t) = 0 \implies t = 0, \pi$

$y'(t) = 0 \implies t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$



(a) at $\left(\pm\frac{1}{2}\sqrt{2}, \pm 1\right)$

(b) at $(\pm 1, 0)$



26. $x'(t) = 2\cos t, \quad y'(t) = 5\cos t$

$x'(t)$ and $y'(t)$ are near zero separately.

(a) none, (b) none.

27. First, we find the values of t when the curve passes through $(2, 0)$.

$$y(t) = 0 \implies t^4 - 4t^2 = 0 \implies t = 0, \pm 2.$$

$$x(-2) = 2, \quad x(0) = 2, \quad x(2) = -2.$$

The curve passes through $(2, 0)$ at $t = -2$ and $t = 0$.

$$x'(t) = -1 - \frac{\pi}{2}\sin\frac{\pi t}{4}, \quad y'(t) = 4t^3 - 8t.$$

At $t = -2$, $x'(-2) = \frac{\pi}{2} - 1$, $y'(t) = -16$, tangent: $y = \frac{32}{2-\pi}(x-2)$.

At $t = 0$, $x'(0) = -1$, $y'(0) = 0$, tangent: $y = 0$.

28. Passes through $(0, 1)$ at $t = 1$ and $t = -1$

$x'(t) = 3t^2 - 1, \quad y'(t) = \sin\frac{\pi}{2}t + \frac{\pi}{2}t\cos\frac{\pi}{2}t;$

at $t = 1:$ $x'(1) = 2$, $y'(1) = 1$, tangent: $y - 1 = \frac{x}{2}$

at $t = -1:$ $x'(-1) = 2$, $y'(-1) = -1$, tangent: $y - 1 = -\frac{x}{2}$

29. The slope of $\overline{OP}$ is $\tan\theta_1$. The curve $r = f(\theta)$ can be parameterized by setting

$$x(\theta) = f(\theta)\cos\theta, \qquad y(\theta) = f(\theta)\sin\theta.$$

Differentiation gives

$$x'(\theta) = -f(\theta)\sin\theta + f'(\theta)\cos\theta, \qquad y'(\theta) = f(\theta)\cos\theta + f'(\theta)\sin\theta.$$

If $f'(\theta_1) = 0$, then

$$x'(\theta_1) = -f(\theta_1)\sin\theta_1, \qquad y'(\theta_1) = f(\theta_1)\cos\theta_1.$$

Since $f(\theta_1) \neq 0$, we have

$$m = \frac{y'(\theta_1)}{x'(\theta_1)} = -\cot\theta_1 = -\frac{1}{\text{slope of } \overline{OP}}.$$

30. $x(\theta) = (a - \cos\theta)\cos\theta$, $y'(\theta) = (a - \cos\theta)\sin\theta$ goes through $(0, 0)$ when $\theta = \pm\cos^{-1} a$

$x'(\theta) = -a\sin\theta + 2\cos\theta\sin\theta, \quad y'(\theta) = a\cos\theta + \sin^2\theta - \cos^2\theta$

At $\theta = \cos^{-1} a$, $\sin\theta = \sqrt{1-a^2} \implies x'(\cos^{-1} a) = a\sqrt{1-a^2}$, $y'(\cos^{-1} a) = 1 - a^2$

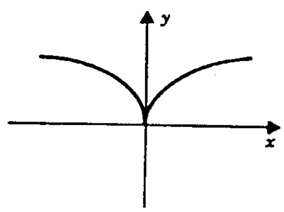
$$\implies m_1 = \frac{\sqrt{1-a^2}}{a}$$

At $\theta = -\cos^{-1} a$, $\sin\theta = -\sqrt{1-a^2} \implies x'(-\cos^{-1} a) = -a\sqrt{1-a^2}$, $y'(-\cos^{-1} a) = 1 - a^2$

$$\implies m_2 = -\frac{\sqrt{1-a^2}}{a}$$

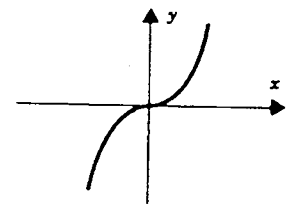
We want $m_1 = -\dfrac{1}{m_2}$: $\dfrac{\sqrt{1-a^2}}{a} = -\dfrac{a}{\sqrt{1-a^2}} \implies a = \dfrac{\sqrt{2}}{2}$

31. $x'(t) = 3t^2, \quad y'(t) = 2t$



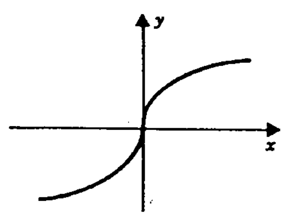
$x^2 = y^3$

32. $x'(t) = 3t^2, \quad y'(t) = 5t^4$



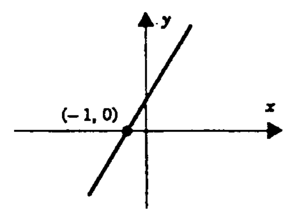
$x^5 = y^3$

33. $x'(t) = 5t^4, \quad y'(t) = 3t^2$



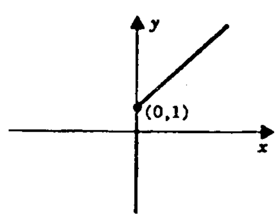
$x^3 = y^5$

34. $x'(t) = 3t^2, \quad y'(t) = 6t^2$



$y = 2x + 2$

35. $x'(t) = 2t, \quad y'(t) = 2t$



ray: $y = x + 1, \quad x \geq 0$

36. $\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) \cdot \dfrac{dt}{dx} = \dfrac{d}{dt}\left[\dfrac{y'(t)}{x'(t)}\right] \cdot \dfrac{1}{x'(t)} = \dfrac{x'(t)y''(t) - y'(t)x''(t)}{x'(t)^3}$

37. By (9.7.5), $\dfrac{d^2y}{dx^2} = \dfrac{(-\sin t)(-\sin t) - (\cos t)(-\cos t)}{(-\sin t)^3} = \dfrac{-1}{\sin^3 t}$. At $t = \dfrac{\pi}{6}$, $\dfrac{d^2y}{dx^2} = -8$.

38. $\dfrac{d^2y}{dx^2} = \dfrac{3t^2 \cdot 0 - 1 \cdot 6t}{(3t^2)^3} = -\dfrac{2}{9t^3}$. At $t = 1$, $\dfrac{d^2y}{dx^2} = -\dfrac{2}{9}$

39. By (9.7.5), $\dfrac{d^2y}{dx^2} = \dfrac{(e^t)(e^{-t}) - (-e^{-t})(e^t)}{(e^t)^3} = 2e^{-3t}$. At $t = 0$, $\dfrac{d^2y}{dx^2} = 2$.

40. $\dfrac{d^2y}{dx^2} = \dfrac{-2\sin t\cos\cos t + \sin t(2\cos 2t)}{(2\sin t\cos t)^3}$ At $t = \dfrac{\pi}{4}$, $\dfrac{d^2y}{dx^2} = -\dfrac{\sqrt{2}}{2}$

SECTION 9.8

1. $L = \displaystyle\int_0^1 \sqrt{1+2^2}\,dx = \sqrt{5}$

2. $L = \displaystyle\int_0^1 \sqrt{1+3^2}\,dx = \sqrt{10}$

3. $L = \displaystyle\int_1^4 \sqrt{1+\left[\frac{3}{2}\left(x-\frac{4}{9}\right)^{1/2}\right]^2}\,dx = \int_1^4 \frac{3}{2}\sqrt{x}\,dx = \left[x^{3/2}\right]_1^4 = 7$

4. $L = \displaystyle\int_0^{44} \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}\,dx = \int_0^{44}\sqrt{1+\frac{9}{4}x}\,dx = \frac{8}{27}\left[(1+\frac{9}{4}x)^{3/2}\right]_0^{44} = 296$

5. $L = \displaystyle\int_0^3 \sqrt{1+\left(\frac{1}{2}\sqrt{x}-\frac{1}{2\sqrt{x}}\right)^2}\,dx = \int_0^3 \left(\frac{1}{2}\sqrt{x}+\frac{1}{2\sqrt{x}}\right)\,dx = \left[\frac{1}{3}x^{3/2}+x^{1/2}\right]_0^3 = 2\sqrt{3}$

6. $L = \displaystyle\int_1^2 \sqrt{1+(x-1)}\,dx = \frac{2}{3}\left[x^{3/2}\right]_1^2 = \frac{2}{3}(2\sqrt{2}-1) \cong 1.22$

7. $L = \displaystyle\int_0^1 \sqrt{1+\left[x\left(x^2+2\right)^{1/2}\right]^2}\,dx = \int_0^1 \left(x^2+1\right)\,dx = \left[\frac{1}{3}x^3+x\right]_0^1 = \frac{4}{3}$

8. $L = \displaystyle\int_2^4 \sqrt{1+x^2(x^2-2)}\,dx = \int_2^4 (x^2-1)\,dx = \left[\frac{x^3}{3}-x\right]_2^4 = \frac{50}{3}$

9. $L = \displaystyle\int_1^5 \sqrt{1+\left[\frac{1}{2}\left(x-\frac{1}{x}\right)\right]^2}\,dx = \int_1^5 \frac{1}{2}\left(x+\frac{1}{x}\right)\,dx = \left[\frac{1}{2}\left(\frac{1}{2}x^2+\ln x\right)\right]_1^5 = 6+\frac{1}{2}\ln 5 \cong 6.80$

10. $L = \displaystyle\int_1^4 \sqrt{1+\left(\frac{x}{4}-\frac{1}{x}\right)^2}\,dx = \int_1^4 \left(\frac{x}{4}+\frac{1}{x}\right)\,dx = \left[\frac{x^2}{8}+\ln x\right]_1^4 = \frac{15}{8}+\ln 4 \cong 3.26$

11. $L = \displaystyle\int_1^8 \sqrt{1+\left[\frac{1}{2}\left(x^{1/3}-x^{-1/3}\right)\right]^2}\,dx = \int_1^8 \frac{1}{2}\left(x^{1/3}+x^{-1/3}\right)\,dx = \frac{1}{2}\left[\frac{3}{4}x^{4/3}+\frac{3}{2}x^{2/3}\right]_1^8 = \frac{63}{8}$

12. $L = \displaystyle\int_1^2 \sqrt{1+\left(\frac{x^4}{2}-\frac{1}{2}x^{-4}\right)^2}\,dx = \int_1^2 \left(\frac{x^4}{2}+\frac{x^{-4}}{2}\right)\,dx = \frac{1}{2}\left[\frac{x^5}{5}-\frac{1}{3x^3}\right]_1^2 = \frac{779}{240}$

13. $L = \displaystyle\int_0^{\pi/4} \sqrt{1+\tan^2 x}\,dx = \int_0^{\pi/4} \sec x\,dx = \left[\ln|\sec x+\tan x|\right]_0^{\pi/4} = \ln\left(1+\sqrt{2}\right) \cong 0.88$

14. $L = \int_0^1 \sqrt{1+x^2}\,dx = \int_0^{\pi/4} \sec^3 u\,du = \frac{1}{2}\left[\sec u \tan u + \ln|\sec u + \tan u|\right]_0^{\pi/4}$

$= \frac{1}{2}\left[\sqrt{2} + \ln(1+\sqrt{2})\right] \cong 1.15$

15. $L = \int_1^2 \sqrt{1+\left(\sqrt{x^2-1}\right)^2}\,dx = \int_1^2 x\,dx = \left[\frac{1}{2}x^2\right]_1^2 = \frac{3}{2}$

16. $L = \int_0^{\ln 2} \sqrt{1+\sinh^2 x}\,dx = \int_0^{\ln 2} \cosh x\,dx = [\sinh x]_0^{\ln 2} = \frac{3}{4}$

17. $L = \int_0^1 \sqrt{1+\left[\sqrt{3-x^2}\right]^2}\,dx = \int_0^1 \sqrt{4-x^2}\,dx = \int_0^{\pi/6} \sqrt{4\cos^2 u}\,du$

$(x = 2\sin u)$

$= 2\int_0^{\pi/6} \cos u\,du = 2\left[\sin u\right]_0^{\pi/6} = 1$

18. $L = \int_{\pi/6}^{\pi/2} \sqrt{1+\cot^2 x}\,dx = \int_{\pi/6}^{\pi/2} \csc x\,dx = [\ln|\csc x - \cot x|]_{\pi/6}^{\pi/2} = -\ln(2-\sqrt{3}) \cong 1.32$

19. $v(t) = \sqrt{(2t)^2 + 2^2} = 2\sqrt{t^2+1}$

initial speed $= v(0) = 2,$ terminal speed $= v\left(\sqrt{3}\right) = 4$

$$s = \int_0^{\sqrt{3}} 2\sqrt{t^2+1}\,dt = 2\int_0^{\pi/3} \sec^3 u\,du = 2\left[\frac{1}{2}\sec u \tan u + \frac{1}{2}\ln\left|\sec u + \tan u\right|\right]_0^{\pi/3}$$

$(t = \tan u)$ (by parts)

$$= 2\sqrt{3} + \ln\left(2+\sqrt{3}\right) \cong 4.78$$

20. $v(t) = \sqrt{1+t^2}$ initial speed $= v(0) = 1,$ terminal speed $= v(1) = \sqrt{2}$

$s = \int_0^1 \sqrt{1+t^2}\,dt = \frac{1}{2}\left[\sqrt{2} + \ln(1+\sqrt{2})\right] \cong 1.15$

21. $v(t) = \sqrt{(2t)^2 + (3t^2)^2}\,dt = t\left(4+9t^2\right)^{1/2}$

initial speed $= v(0) = 0,$ terminal speed $= v(1) = \sqrt{13}$

$s = \int_0^1 t\left(4+9t^2\right)^{1/2}\,dt = \left[\frac{1}{27}\left(4+9t^2\right)^{3/2}\right]_0^1 = \frac{1}{27}\left(13\sqrt{13} - 8\right)$

22. $v(t) = \sqrt{9a^2\cos^4 t \sin^2 t + 9a^2 \sin^4 t \cos^2 t} = 3a\cos t \sin t$

initial speed $= v(0) = 0$, terminal speed $= v(\pi/2) = 0$

$$s = \int_0^{\pi/2} 3a\cos t\sin t\,dt = \left[\frac{3a\sin^2 t}{2}\right]_0^{\pi/2} = \frac{3}{2}a$$

23. $v(t) = \sqrt{[e^t\cos t + e^t\sin t]^2 + [e^t\cos t - e^t\sin t]^2} = \sqrt{2}\,e^t$

initial speed $= v(0) = \sqrt{2}$, terminal speed $= \sqrt{2}\,e^{\pi}$

$$s = \int_0^{\pi} \sqrt{2}\,e^t\,dt = \left[\sqrt{2}\,e^t\right]_0^{\pi} = \sqrt{2}\,(e^{\pi} - 1)$$

24. $v(t) = \sqrt{(t\cos t)^2 + (t\sin t)^2} = t$ initial speed $= v(0) = 0$, terminal speed $= v(\pi) = \pi$

$$s = \int_0^{\pi} t\,dt = \left[\frac{t^2}{2}\right]_0^{\pi^2} = \frac{\pi^2}{2}$$

25. $$L = \int_0^{2\pi} \sqrt{[x'(\theta)]^2 + [y'(\theta)]^2}\,d\theta = \int_0^{2\pi} \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}\,d\theta$$

$$= a\int_0^{2\pi} \sqrt{2(1-\cos\theta}\,d\theta = 2a\int_0^{2\pi} \sin\frac{\theta}{2}\,d\theta = -4a\left[\cos\frac{\theta}{2}\right]_0^{2\pi} = 8a$$

26. $\sqrt{[x'(\theta)]^2 + [y'(\theta)]^2} = \sqrt{(-2a\sin\theta + 2a\sin 2\theta)^2 + (2a\cos\theta - 2a\cos 2\theta)^2} = 2a\sqrt{2 - 2\cos\theta}$

$$L = \int_0^{2\pi} 2a\sqrt{2-2\cos\theta}\,d\theta = 2a\int_0^{2\pi} 2\sin\frac{\theta}{2}\,d\theta = 4a\left[-2\cos\frac{\theta}{2}\right]_0^{2\pi} = 16a$$

27. (a) $$L = \int_0^{2\pi} \sqrt{(-3a\sin\theta - 3a\sin 3\theta)^2 + (3a\cos\theta - 3a\cos 3\theta)^2}\,d\theta$$

$$= 3a\int_0^{2\pi} \sqrt{\sin^2\theta + 2\sin\theta\sin 3\theta + \sin^2 3\theta + \cos^2\theta - 2\cos\theta\cos 3\theta + \cos^2 3\theta}\,d\theta$$

$$= 3a\int_0^{2\pi} \sqrt{2(1-\cos 4\theta)}\,d\theta = 6a\int_0^{2\pi} |sin\,2\theta|\,d\theta$$

$$= 24a\int_0^{\pi/2} \sin 2\theta\,d\theta = -12a\,[\cos 2\theta]_0^{\pi/2} = 24a$$

(b) The result follows from the identities: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$; $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

28. $\sqrt{[x'(\theta)]^2 + [y'(\theta)]^2} = \sqrt{(\cos\theta - \theta\sin\theta)^2 + (\sin\theta + \theta\cos\theta)^2} = \sqrt{1+\theta^2}$

$$L = \int_0^{2\pi} \sqrt{1+\theta^2}\,d\theta = \left[\frac{1}{2}\theta\sqrt{1+\theta^2} + \frac{1}{2}\ln|\theta + \sqrt{1+\theta^2}|\right]_0^{2\pi} = \pi\sqrt{1+4\pi^2} + \frac{1}{2}\ln(2\pi + \sqrt{1+4\pi^2})$$

29. $L =$ circumference of circle of radius $1 = 2\pi$

30. $L =$ half circumference of circle of radius $3 = 3\pi$

31. $L = \int_0^{4\pi} \sqrt{[e^\theta]^2 + [e^\theta]^2}\, d\theta = \int_0^{4\pi} \sqrt{2}\, e^\theta d\theta = \left[\sqrt{2}\, e^\theta\right]_0^{4\pi} = \sqrt{2}\,(e^{4\pi} - 1)$

32. $L = \int_{-2\pi}^{2\pi} \sqrt{(ae^\theta)^2 + (ae^\theta)^2}\, d\theta = \int_{-2\pi}^{2\pi} \sqrt{2}ae^\theta\, d\theta = a\sqrt{2}(e^{2\pi} - e^{-2\pi}) \quad (a > 0)$

33. $L = \int_0^{2\pi} \sqrt{[e^{2\theta}]^2 + [2e^{2\theta}]^2}\, d\theta = \int_0^{2\pi} \sqrt{5}\, e^{2\theta} d\theta = \left[\frac{1}{2}\sqrt{5}\, e^{2\theta}\right]_0^{2\pi} = \frac{1}{2}\sqrt{5}\,(e^{4\pi} - 1)$

34. $L = \int_0^{2\pi} \sqrt{(1 + \cos\theta)^2 + (-\sin\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta = \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} = 8$

35. $L = \int_0^{\pi/2} \sqrt{(1 - \cos\theta)^2 + \sin^2\theta}\, d\theta = \int_0^{\pi/2} \sqrt{2 - 2\cos\theta}\, d\theta$

$= \int_0^{\pi/2} \left(2\sin\frac{1}{2}\theta\right) d\theta = \left[-4\cos\frac{1}{2}\theta\right]_0^{\pi/2} = 4 - 2\sqrt{2}$

36. $L = \int_0^{\pi/4} \sqrt{(2a\sec\theta)^2 + (2a\sec\theta\tan\theta)^2}\, d\theta = \int_0^{\pi/4} 2a\sec^2\theta\, d\theta = [2a\tan\theta]_0^{\pi/4} = 2a$

37. $s = \int_0^1 \sqrt{\left[\frac{1}{1+t^2}\right]^2 + \left[\frac{-t}{1+t^2}\right]^2}\, dt = \int_0^1 \frac{dt}{\sqrt{1+t^2}}$

$= \int_0^{\pi/4} \sec u\, du = [\ln|\sec u + \tan u|]_0^{\pi/4} = \ln\left(1 + \sqrt{2}\right)$

$(t = \tan u)$

initial speed $= v(0) = 1$, terminal speed $= v(1) = \frac{1}{2}\sqrt{2}$

38. $s = \int_0^{2\pi} \sqrt{(\sin t)^2 + (1 - \cos t)^2}\, dt = \int_0^{2\pi} \sqrt{2 - 2\cos t}\, d\theta = \left[-4\cos\frac{t}{2}\right]_0^{2\pi} = 8$

initial speed $= v(0) = 0$, terminal speed $= v(2\pi) = 0$

39. $c = 1$; the curve $y = e^x$ is the curve $y = \ln x$ reflected in the line $y = x$

40. By the hint:

$$L = \int_1^4 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\, dx = \int_1^4 \sqrt{1 + \frac{9}{4}x}\, dx = \frac{8}{27}\left[\left(1 + \frac{9}{4}x\right)^{3/2}\right]_1^4$$

$$= \frac{1}{27}\left[80\sqrt{10} - 13\sqrt{13}\right] \cong 7.63$$

41. $L = \int_a^b \sqrt{1+\sinh^2 x}\, dx = \int_a^b \sqrt{\cosh^2 x}\, dx = \int_a^b \cosh x\, dx = A$

42. The line segment that joins P_{i-1} to P_i has length

$$\sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2}$$

This can be written

$$\sqrt{1 + \left[\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right]^2}\,(x_i - x_{i-1}) = \sqrt{1 + \left[\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right]^2}\,\Delta x_i$$

The mean value theorem gives the existence of x_i^* in $[x_{i-1}, x_i]$ such that

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = f'(x_i^*).$$

The length of $\overline{P_{i-1}P_i}$ can then be written

$$\sqrt{1 + [f'(x_i^*)]^2}\,\Delta x_i$$

The length of the polygonal approximation is the sum of such expressions from $i = 1$ to $i = n$.

43. $\sqrt{1 + [f(x)]^2} = \sqrt{1 + \tan^2[\alpha(x)]} = |\sec[\alpha(x)]|$

44. Balancing the vertical forces we have

$$k\int_0^x \sqrt{1 + [f'(t)]^2}\, dt = p(x)\sin\theta. \quad \text{(weight=vertical pull at } x\text{)}$$

Balancing the horizontal forces we have

$$p(0) = p(x)\cos\theta. \quad \text{(pull at } 0 = \text{horizontal pull at } x\text{)}$$

Together these equations give

$$k\int_0^x \sqrt{1 + [f'(x)]^2}\, dt = p(0)\tan\theta = p(0)f'(\theta).$$

Differentiating with respect to x, we have

$$k\sqrt{1 + [f'(x)]^2} = p(0)f''(x)$$

$$\frac{f''(x)}{\sqrt{1 + [f'(x)]^2}} = b. \quad \text{(set } \; p(0)/h = b\text{)}$$

Integration gives

$$\ln\left(f'(x) + \sqrt{1 + [f'(x)]^2}\right) = bx + C.$$

In the coordinate system of the figure, we have $f'(0) = 0$. Therefore $C = 0$ and

$$\ln\left(f'(x)+\sqrt{1+[f'(x)]^2}\right)=bx$$

Exponentiating both sides we have

$$\begin{aligned}f'(x)+\sqrt{1+[f'(x)]^2}&=e^{bx}\\ \sqrt{1+[f'(x)]^2}&=e^{bx}-f'(x)\\ 1+[f'(x)]^2&=e^{bx}-2e^{bx}f'(x)+[f'(x)]^2\\ 2e^{bx}f'(x)&=e^{2bx}-1\\ f'(x)&=\frac{1}{2}(e^{bx}-e^{-bx})=\sinh bx\end{aligned}$$

A final integration gives

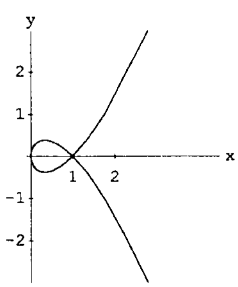
$$f(x)=\frac{1}{b}\cosh bx+K.$$

Adjusting the coordinate system vertically so that $f(0)=1/b$ we have

$$f(x)=\frac{1}{b}\cosh bx$$

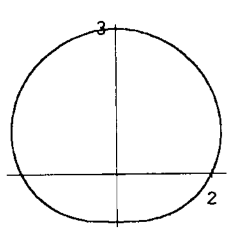
Now set $a=1/b$

45. (a)



(b) $L=\displaystyle\int_{-1}^{1}\sqrt{9t^4-2t^2+1}\,dt\cong 2.7156$

46. (a)



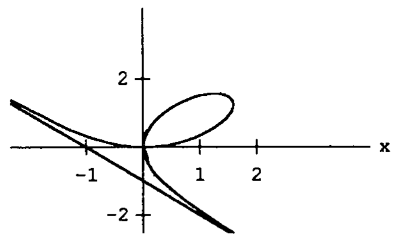
(b) $$\begin{aligned}L&=\int_0^{2\pi}\sqrt{(2+\sin\theta)^2+(\cos\theta)^2}\,d\theta\\&=\int_0^{2\pi}\sqrt{5+4\sin\theta}\,d\theta\cong 13.38\end{aligned}$$

47. (a) $L=\displaystyle\int_0^{2\pi}\sqrt{a^2\sin^2 t+b^2\cos^2 t}\,dt=4\int_0^{\pi/2}\sqrt{a^2(1-\cos^2 t)+b^2\cos^2 t}\,dt$

$$= 4a\int_0^{\pi/2} \sqrt{1 - e^2\cos^2 t}\, dt, \qquad \text{where} \quad e = \frac{\sqrt{a^2 - b^2}}{a}$$

(b) $L = 4\int_0^{\pi/2} \sqrt{25 - 9\cos^2 t}\, dt \cong 28.3617$

48. (a)



(b) $L = 2\int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$

$$= 2\int_0^1 \sqrt{\left(\frac{1-6t^3}{(t^3+1)^2}\right)^2 + \left(\frac{6t-3t^4}{(t^3+1)^2}\right)^2}\, dt$$

$$= 3.5485$$

SECTION 9.9

1. $L =$ length of the line segment $= 1$

$(\overline{x}, \overline{y}) = \left(\frac{1}{2}, 4\right)$ (the midpoint of the line segment)

$A_x =$ lateral surface area of cylinder of radius 4 and side $1 = 16\pi$.

2.
$$L = \int_0^1 \sqrt{1 + 2^2}\, dx = \sqrt{5}$$
$$\overline{x}L = \int_0^1 x\sqrt{1+2^2}\, dx = \frac{\sqrt{5}}{2}, \quad \overline{x} = \frac{1}{2}$$
$$\overline{y}L = \int_0^1 2x\sqrt{1+2^2}\, dx = \sqrt{5}, \quad \overline{y} = 1$$
$$A_x = 2\pi\overline{y}L = 2\pi\sqrt{5}$$

3.
$$L = \int_0^3 \sqrt{1 + \left(\frac{4}{3}\right)^2}\, dx = \left(\frac{5}{3}\right)3 = 5$$
$$\overline{x}L = \int_0^3 x\sqrt{1 + \left(\frac{4}{3}\right)^2}\, dx = \frac{5}{3}\left[\frac{1}{2}x^2\right]_0^3 = \frac{15}{2}, \quad \overline{x} = \frac{3}{2}$$
$$\overline{y}L = \int_0^3 \frac{4}{3}x\sqrt{1 + \left(\frac{4}{3}\right)^2}\, dx = \left(\frac{4}{3}\right)\left(\frac{15}{2}\right) = 10, \quad \overline{y} = 2$$
$$A_x = 2\pi\overline{y}L = 2\pi(2)(5) = 20\pi$$

4. $$L = \int_0^5 \sqrt{1 + \left(-\frac{12}{5}\right)^2}\, dx = 5\sqrt{1 + \frac{144}{25}} = 13$$

$$\overline{x}L = \int_0^5 x\sqrt{1 + \frac{144}{25}}\, dx = \frac{25}{2}\sqrt{1 + \frac{144}{25}}, \quad \overline{x} = \frac{5}{2}$$

$$\overline{y}L = \int_0^5 \left(12 - \frac{12}{5}x\right)\sqrt{1 + \frac{144}{25}}\, dx = \frac{12}{5}\cdot\frac{25}{2}\sqrt{1 + \frac{144}{25}}, \quad \overline{y} = 6$$

$$A_x = 2\pi\overline{y}L = 156\pi$$

5. $$L = \int_0^2 \sqrt{(3)^2 + (4)^2}\, dt = (2)(5) = 10$$

$$\overline{x}L = \int_0^2 3t\sqrt{(3)^2 + (4)^2}\, dt = 15\left[\frac{1}{2}t^2\right]_0^2 = 30, \quad \overline{x} = 3$$

$$\overline{y}L = \int_0^2 4t\sqrt{(3)^2 + (4)^2}\, dt = 20\left[\frac{1}{2}t^2\right]_0^2 = 40, \quad \overline{y} = 4$$

$$A_x = 2\pi\overline{y}L = 2\pi(4)(10) = 80\pi$$

6. $$L = \frac{1}{8}\ \text{circumference of circle of radius } 5 = \frac{5}{4}\pi$$

$$\overline{x}L = \int_0^{\pi/4} x(\theta)\sqrt{x'(\theta)^2 + y'(\theta)^2}\, d\theta = \int_0^{\pi/4} 5\cos\theta\cdot 5\, d\theta = \frac{25\sqrt{2}}{2}, \quad \overline{x} = \frac{10\sqrt{2}}{\pi}$$

$$\overline{y}L = \int_0^{\pi/4} y(\theta)\sqrt{x'(\theta)^2 + y'(\theta)^2}\, d\theta = \int_0^{\pi/4} 5\sin\theta\cdot 5\, d\theta = 25\left(1 - \frac{\sqrt{2}}{2}\right), \quad \overline{y} = \frac{10}{\pi}(2 - \sqrt{2})$$

$$A_x = 2\pi\overline{y}L = 25\pi(2 - \sqrt{2})$$

7. $$L = \int_0^{\pi/6} \sqrt{4\sin^2 t + 4\cos^2 t}\, dt = 2\left(\frac{\pi}{6}\right) = \frac{1}{3}\pi$$

$$\overline{x}L = \int_0^{\pi/6} 2\cos t\sqrt{4\sin^2 t + 4\cos^2 t}\, dt = 4\left[\sin t\right]_0^{\pi/6} = 2, \quad \overline{x} = \frac{6}{\pi}$$

$$\overline{y}L = \int_0^{\pi/6} 2\sin t\sqrt{4\sin^2 t + 4\cos^2 t}\, dt = 4\left[-\cos t\right]_0^{\pi/6} = 4 - 2\sqrt{3}, \quad \overline{y} = 6\left(2 - \sqrt{3}\right)/\pi$$

$$A_x = 2\pi\overline{y}L = 2\pi(6(2 - \sqrt{3})/\pi)\tfrac{1}{3}\pi = 4\pi(2 - \sqrt{3})$$

8. $$L = \int_0^{\pi/2} \sqrt{(-3\cos^2 t\sin t)^2 + (3\sin^2 t\cos t)^2}\, dt = \int_0^{\pi/2} 3\cos t\sin t\, dt = \left[\frac{3\sin^2 t}{2}\right]_0^{\pi/2} = \frac{3}{2}$$

$$\overline{x}L = \int_0^{\pi/2} \cos^3 t\cdot 3\cos t\sin t\, dt = \left[-\frac{3}{5}\cos^5 t\right]_0^{\pi/2} = \frac{3}{5}, \quad \overline{x} = \frac{2}{5}$$

$$\overline{y}L = \int_0^{\pi/2} \sin^3 t\cdot 3\cos t\sin t\, dt = \left[\frac{3}{5}\sin^5 t\right]_0^{\pi/2} = \frac{3}{5}, \quad \overline{y} = \frac{2}{5}$$

$$A_x = 2\pi\overline{y}L = \frac{6}{5}\pi$$

9. $x(t) = a\cos t, \quad y = a\sin t; \quad t \in [\frac{1}{3}\pi, \frac{2}{3}\pi]$

$$L = \int_{\pi/3}^{2\pi/3} \sqrt{a^2\sin^2 t + a^2\cos^2 t}\, dt = \frac{1}{3}\pi a$$

by symmetry $\bar{x} = 0$

$$\bar{y}L = \int_{\pi/3}^{2\pi/3} a\sin t\sqrt{a^2\sin^2 t + a^2\cos^2 t}\, dt = a^2\int_{\pi/3}^{2\pi/3} \sin t\, dt$$

$$= a^2\left[-\cos t\right]_{\pi/3}^{2\pi/3} = a^2, \quad \bar{y} = 3a/\pi$$

$$A_x = 2\pi\bar{y}L = 2\pi a^2$$

10. $$L = \int_0^\pi \sqrt{(1+\cos\theta)^2 + \sin^2\theta}\, d\theta = \int_0^\pi \sqrt{2 + 2\cos\theta}\, d\theta$$

$$= \int_0^\pi 2\sqrt{\frac{1+\cos\theta}{2}}\, d\theta = \int_0^\pi 2\cos\frac{1}{2}\theta\, d\theta = 4$$

$$\bar{x}L = \int_0^\pi \cos\theta(1+\cos\theta)\left(2\cos\frac{1}{2}\theta\right) d\theta = \int_0^\pi \left(\cos^2\frac{1}{2}\theta - \sin^2\frac{1}{2}\theta\right)\left(2\cos^2\frac{1}{2}\theta\right)\left(2\cos\frac{1}{2}\theta\right) d\theta$$

$$= 4\int_0^\pi \left(1 - 2\sin^2\frac{1}{2}\theta\right)\left(1 - \sin^2\frac{1}{2}\theta\right)\left(\cos\frac{1}{2}\theta\right) d\theta$$

$$= 4\int_0^\pi \left(\cos\frac{1}{2}\theta - 3\sin^2\frac{1}{2}\theta\cos\frac{1}{2}\theta + 2\sin^4\frac{1}{2}\theta\cos\frac{1}{2}\theta\right) d\theta = \frac{16}{5}; \quad \bar{x} = \frac{4}{5}$$

$$\bar{y}L = \int_0^\pi \sin\theta(1+\cos\theta)\left(2\cos\frac{1}{2}\theta\right) d\theta$$

$$= \int_0^\pi \left(2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta\right)\left(2\cos^2\frac{1}{2}\theta\right)\left(2\cos\frac{1}{2}\theta\right) d\theta$$

$$= 8\int_0^\pi \left(\cos^4\frac{1}{2}\theta\sin\frac{1}{2}\theta\right) d\theta = \frac{16}{5}; \quad \bar{y} = \frac{4}{5}$$

$$A_x = 2\pi\bar{y}L = \frac{32}{5}\pi$$

11. $$A_x = \int_0^2 \frac{2}{3}\pi x^3\sqrt{1+x^4}\, dx = \frac{1}{9}\pi\left[(1+x^4)^{3/2}\right]_0^2 = \frac{1}{9}(17\sqrt{17} - 1) \cong 7.68$$

12. $$A_x = \int_1^2 2\pi\sqrt{x}\sqrt{1 + \frac{1}{4x}}\, dx = \int_1^2 \pi\sqrt{4x+1}\, dx = \pi\left[\frac{1}{6}(4x+1)^{3/2}\right]_1^2 = \frac{1}{6}\pi(27 - 5\sqrt{5})$$

13. $$A_x = \int_0^1 \frac{1}{2}\pi x^3\sqrt{1 + \frac{9}{16}x^4}\, dx = \frac{4}{27}\pi\left[\left(1 + \frac{9}{16}x^4\right)\right]_0^1 = \frac{61}{432}\pi$$

14. $$A_x = \int_0^4 2\pi\cdot 3\sqrt{x}\sqrt{1 + \frac{9}{4x}}\, dx = \int_0^4 3\pi\sqrt{4x+9}\, dx = \frac{\pi}{2}\left[(4x+9)^{3/2}\right]_0^4 = 49\pi$$

15. $A_x = \int_0^{\pi/2} 2\pi \cos x \sqrt{1+\sin^2 x}\, dx = \int_0^1 2\pi \sqrt{1+u^2}\, du$

$u = \sin x$

$= 2\pi \left[\tfrac{1}{2} u\sqrt{1+u^2} + \tfrac{1}{2}\ln\left(u + \sqrt{1+u^2}\right)\right]_0^1 = \pi\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]$

(8.5.1)

16. $A_x = \int_{-1}^{0} 4\pi\sqrt{1-x}\sqrt{1+\dfrac{1}{1-x}}\, dx = 4\pi \int_{-1}^{0} \sqrt{2-x}\, dx = -\dfrac{8}{3}\left[(2-x)^{3/2}\right]_{-1}^{0} = \dfrac{8}{3}\pi(3\sqrt{3} - 2\sqrt{2})$

17.

$$\begin{aligned} A_x &= \int_0^{\pi/2} 2\pi(e^\theta \sin\theta)\sqrt{[e^\theta\cos\theta - e^\theta\sin\theta]^2 + [e^\theta\sin\theta + e^\theta\cos\theta]^2}\, d\theta \\ &= 2\pi\sqrt{2}\int_0^{\pi/2} e^{2\theta}\sin\theta\, d\theta \\ &= 2\pi\sqrt{2}\left[\tfrac{1}{5}\left(2e^{2\theta}\sin\theta - e^{2\theta}\cos\theta\right)\right]_0^{\pi/2} = \tfrac{2}{5}\sqrt{2}\,\pi\,(2e^\pi + 1) \end{aligned}$$

(by parts twice)

18.

$$\begin{aligned} A_x &= \int_0^{\ln 2} 2\pi \cosh x \sqrt{1+\sinh^2 x}\, dx \\ &= 2\pi \int_0^{\ln 2} \cosh^2 x\, dx \\ &= \frac{1}{2}\pi \int_0^{\ln 2} (e^{2x} + 2 + e^{-2x})\, dx = \frac{1}{2}\pi\left[\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x}\right]_0^{\ln 2} = \frac{1}{16}\pi(15 + 16\ln 2) \end{aligned}$$

19. (a) $A = \int_0^{2\pi} y(\theta)x'(\theta)\, d\theta$ [see (9.6.4)]

$$\begin{aligned} &= \int_0^{2\pi} a^2(1-\cos\theta)^2\, d\theta \\ &= a^2\int_0^{2\pi}(1 - 2\cos\theta + \cos^2\theta)\, d\theta \\ &= a^2\int_0^{2\pi}\left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta \\ &= a^2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} \\ &= 3\pi a^2 \end{aligned}$$

(b) $A = \int_0^{2\pi} 2\pi\, y(\theta)\sqrt{[x'(\theta)]^2 + [y'(\theta)]^2}\, d\theta$ (9.9.2)

$$= \int_0^{2\pi} 2\pi\, a(1-\cos\theta)\sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}\; d\theta$$

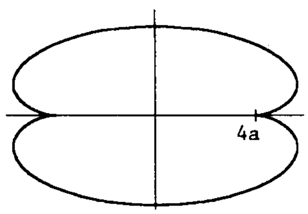
$$= 2\pi\, a^2 \int_0^{2\pi} (1-\cos\theta)\sqrt{2-2\,cos\theta}\; d\theta$$

$$= 4\pi\, a^2 \int_0^{2\pi} (1-\cos\theta)\sin\frac{\theta}{2}\, d\theta$$

$$= 4\pi\, a^2 \int_0^{2\pi} \left(2\,\sin\frac{\theta}{2} - 2\,\cos^2\frac{\theta}{2}\,\sin\frac{\theta}{2}\right) d\theta$$

$$= 4\pi\, a^2 \left[-4\,\cos\frac{\theta}{2}\right]_0^{2\pi} + \frac{16\pi\, a^2}{3}\left[\cos^3(\theta/2)\right]_0^{2\pi} = \frac{64\pi\, a^2}{3}$$

20. (a)



(b) $$A = 2\int_\pi^0 y(\theta)x'(\theta)\,d\theta$$

$$= 2\int_\pi^0 (3a\sin\theta - a\sin 3\theta)(-3a\sin\theta - 3a\sin 3\theta)\,d\theta$$

$$= -6a^2\int_\pi^0 (3\sin^2\theta + 2\sin\theta\sin 3\theta - \sin^2 3\theta)\,d\theta = 6a^2\pi$$

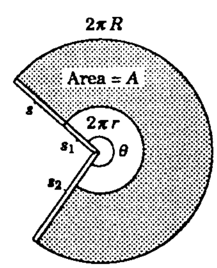
(c) $$A = \int_0^\pi 2\pi y(\theta)\sqrt{x'(\theta)^2 + y'(\theta)^2}\,d\theta = 2\pi\int_0^\pi (3a\sin\theta - a\sin 3\theta)3a\sqrt{2-2\cos 4\theta}\,d\theta$$

21.

$$A = \tfrac{1}{2}\theta {s_2}^2 - \tfrac{1}{2}\theta {s_1}^2$$

$$= \tfrac{1}{2}(\theta s_2 + \theta s_1)(s_2 - s_1)$$

$$= \tfrac{1}{2}(2\pi R + 2\pi r)s = \pi(R+r)s$$



22. (a) $$\overline{x} = \frac{1}{\frac{\pi}{4}r^2 - \frac{\pi}{4}a^2}\cdot\left(-\frac{4a}{3\pi}\cdot\frac{\pi}{4}a^2 + \frac{4r}{3\pi}\cdot\frac{\pi}{4}r^2\right) = \frac{4}{3\pi}\left(\frac{r^3-a^3}{r-a}\right) = \frac{4}{3\pi}\left(\frac{r^2+ar+a^2}{r+a}\right)$$

$\overline{y} = \overline{x}$

(b) $$\lim_{a\to r}\frac{4}{3\pi}\left(\frac{r^2+ar+a^2}{r+a}\right) = \frac{2r}{\pi}; \quad = \overline{x} = \overline{y} = \frac{2r}{\pi}.$$

23. (a) The centroids of the 3, 4, 5 sides are the midpoints $\left(\frac{3}{2},0\right)$, $(3,2)$, $\left(\frac{3}{2},2\right)$.

(b) $\overline{x}(3+4+5) = \frac{3}{2}(3) + 3(4) + \frac{3}{2}(5), \quad 12\overline{x} = 24, \quad \overline{x} = 2$

$\overline{y}(3+4+5) = 0(3) + 2(4) + 2(5), \quad 12\overline{y} = 18, \quad \overline{y} = \dfrac{3}{2}$

(c) $A = \frac{1}{3}(3)(4) = 6$

$$\overline{x}A = \int_0^3 x\left(\frac{4}{3}x\right)\,dx = \int_0^3 \frac{4}{3}x^2dx = \frac{4}{9}\left[x^3\right]_0^3 = 12, \quad \overline{x} = 2$$

$$\overline{y}A = \int_0^3 \frac{1}{2}\left(\frac{4}{3}x\right)^2\,dx = \int_0^3 \frac{8}{9}x^2dx = \frac{8}{27}\left[x^3\right]_0^3 = 8, \quad \overline{x} = \frac{4}{3}$$

(d) $\overline{x}\,(4+5) = 3(4) + \frac{3}{2}(5), \quad 9\overline{x} = \frac{39}{2}, \quad \overline{x} = \frac{13}{6}$

$\overline{y}\,(4+5) = 2(4) + 2(5), \quad 9\overline{y} = 18, \quad \overline{y} = 2$

(e) $A_x = 2\pi(2)(5) = 20\pi$ (f) $A_x = 2\pi(2)(4+5) = 36\pi$

24. (a) Set $y(t) = t, \quad x(t) = \frac{1}{2}\left[t\sqrt{t^2-1} + \ln|t - \sqrt{t^2-1}|\right], \quad t \in [2,5]$

$x'(t) = \sqrt{t^2-1} \quad y'(t) = 1$

$$A = \int_2^5 2\pi t\sqrt{(\sqrt{t^2-1})^2 + 1^2}\,dt = \int_2^5 2\pi t^2\,dt = \left[2\pi\frac{t^3}{3}\right]_2^5 = 78\pi$$

(b) Set $y(t) = t, \quad x(t) = \frac{1}{6a^2}\left(t^3 + \frac{3a^4}{t}\right) \quad t \in [a, 3a]$

$$x'(t) = \frac{1}{6a^2}\left(3t^2 - \frac{3a^4}{t^2}\right), \quad y'(t) = 1$$

$$A = \int_a^{3a} 2\pi t\sqrt{\frac{1}{36a^4}\left(3t^2 - \frac{3a^4}{t^2}\right)^2 + 1}\,dt = 2\pi\int_a^{3a}\frac{t}{6a^2}\left(3t^2 + \frac{3a^4}{t^2}\right)\,dt$$

$$= \frac{\pi}{a^2}\int_a^{3a}\left(t^3 + \frac{a^4}{t}\right)\,dt = \frac{\pi}{a^2}\left[\frac{t^4}{4} + a^4\ln 4\right]_a^{3a} = \pi a^2(20 + \ln 3)$$

25. $A_x = 2\pi\overline{y}L = 2\pi(b)(2\pi a) = 4\pi^2ab$

26. (a) No: $f'(x) = -x/\sqrt{r^2-x^2}$ is not defined at $x = \pm r$. We can however begin with a smaller interval $[-r+\epsilon, r-\epsilon]$, integrate, and take the limit as $\epsilon \to 0$.

(b) $A = \int_0^{2\pi} 2\pi r\sqrt{(\sin t)^2}\,dt = 2\pi r\int_0^{2\pi}|\sin t|\,dt = 8\pi r$

C is not simple. The curve [in this case the line segment that joins $(1,r)$ to $(-1,r)$] is traced out twice.

27. The band can be obtained by revolving about the x-axis the graph of the function

$$f(x) = \sqrt{r^2-x^2}, \qquad x \in [a,b].$$

A straightforward calculation shows that the surface area of the band is $2\pi r(b-a)$.

28. (We use the solution to Exercise 27.) C revolved about the y-axis generates a surface of area

$$A_y = 2\pi\overline{x}L = 2\pi\overline{x}r(\theta_2 - \theta_1).$$

C revolved about the x-axis generates a surface of area

$$A_x = 2\pi x\overline{y}L = 2\pi\overline{y}r(\cos\theta_1 - \cos\theta_2).$$

By our solution to Exercise 27.

$$A_y = 2\pi r^2(\sin\theta_2 - \sin\theta_1)$$

and

$$A_x = 2\pi r^2(\cos\theta_1 - \cos\theta_2).$$

Therefore

$$\overline{x} = r\left(\frac{\sin\theta_2 - \sin\theta_1}{\theta_2 - \theta_1}\right), \quad \overline{y} = r\left(\frac{\cos\theta_1 - \cos\theta_2}{\theta_2 - \theta_1}\right).$$

29. (a) Parameterize the upper half of the ellipse by

$$x(t) = a\cos t, \quad y(t) = b\sin t; \qquad t \in [0, \pi].$$

Here

$$\sqrt{[x'(t)]^2 + [y'(t)]^2} = \sqrt{a^2\sin^2 t + b^2\cos^2 t} = \sqrt{a^2 - (a^2 - b^2)\cos^2 t},$$

which, with $c = \sqrt{a^2 - b^2}$, can be written $\sqrt{a^2 - c^2\cos^2 t}$. Therefore,

$$A = \int_0^{\pi} 2\pi b\sin t\sqrt{a^2 - c^2\cos^2 t}\,dt = 4\pi b\int_0^{\pi/2} \sin t\sqrt{a^2 - c^2\cos^2 t}\,dt.$$

Setting $u = c\cos t$, we have $du = -c\sin t$ and

$$A = -\frac{4\pi b}{c}\int_c^0 \sqrt{a^2 - u^2}\,du = \frac{4\pi b}{c}\left[\frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right)\right]_0^c$$
$$= 2\pi b^2 + \frac{2\pi a^2 b}{c}\sin^{-1}\left(\frac{c}{a}\right) = 2\pi b^2 + \frac{2\pi ab}{e}\sin^{-1} e$$

where e is the eccentricity of ellipse: $e = c/a$.

(b) Parameterize the right half of the ellipse by

$$x(t) = a\cos t, \quad y(t) = b\sin t; \qquad t \in [-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi].$$

Again $\sqrt{[x'(t)]^2 + [y'(t)]^2} = \sqrt{a^2 - c^2\cos^2 t}$ where $c = \sqrt{a^2 - b^2}$.

Therefore

$$A = \int_{-\pi/2}^{\pi/2} 2\pi a\cos t\sqrt{a^2 - c^2\cos^2 t}\,dt.$$

Set $u = c\sin t$. Then $du = c\cos t\,dt$ and

$$A = \frac{2\pi a}{c}\int_{-c}^{c}\sqrt{b^2+u^2}\,du = \frac{2\pi a}{c}\left[\frac{u}{2}\sqrt{b^2+u^2}+\frac{b^2}{2}\ln\left|u+\sqrt{b^2+u^2}\right|\right]_{-c}^{c}$$

Routine calculation gives

$$A = 2\pi a^2 + \frac{\pi b^2}{e}\ln\left|\frac{1+e}{1-e}\right|.$$

30. Set $s'(t) = \sqrt{[x'(t)]^2+[y'(t)]^2}$. Let $P = \{t_0, t_1, \cdots, t_n\}$ be a partition of [c,d]. The partition breaks up the surface into n surfaces of revolution with areas

$$A_i = \int_{t_{i-1}}^{t_i} 2\pi y(t)s'(t)\,dt$$

and centroids $\bar{x}_i = x(t_i^*)$ with $t_i^* \in [t_{i-1}, t_i]$.

$$\bar{x}A = \bar{x}_1A_1 + \cdots + \bar{x}_nA_n$$

$$= x(t_1^*)\int_{t_0}^{t_1} 2\pi y(t)s'(t)\,dt + \cdots + x(t_n^*)\int_{t_{n-1}}^{t_n} 2\pi y(t)s'(t)\,dt$$

$$\cong x(t_1^*)[2\pi y(t_1^*)s'(t_1^*)\Delta t_1] + \cdots + x(t_n^*)[2\pi y(t_n^*)s'(t_n^*)\Delta t_n]$$

$$= 2\pi x(t_1^*)y(t_1^*)s'(t_1^*)\Delta t_1 + \cdots + 2\pi x(t_n^*)y(t_n^*)s'(t_n^*)\Delta t_n$$

As $\| P \| \to 0$, the expression on the right tends to

$$\int_c^d 2\pi x(t)y(t)s'(t)\,dt = \int_c^d 2\pi x(t)y(t)\sqrt{[x'(t)]^2+[y'(t)]^2}\,dt.$$

31. Such a hemisphere can be obtained by revolving about the x-axis the curve

$$x(t) = r\cos t,\quad y(t) = r\sin t;\quad t\in[0,\tfrac{1}{2}\pi].$$

Therefore,
$$\bar{x}A = \int_0^{\pi/2} 2\pi(r\cos t)(r\sin t)\sqrt{r^2\sin^2 t + r^2\cos^2 t}\,dt$$

$$= \int_0^{\pi/2} 2\pi r^3 \sin t\cos t\,dt = \pi r^3\left[\sin^2 t\right]_0^{\pi/2} = \pi r^3.$$

$A = 2\pi r^2$; $\bar{x} = \bar{x}A/A = \frac{1}{2}r$.

The centroid lies on the midpoint of the axis of the hemisphere.

32. The cone can be generated by revolving about the x-axis the graph of the function

$$f(x) = (r/h)x,\qquad x\in[0,h].$$

Formula 9.9.8 gives

$$\bar{x}A = \int_0^h 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx = \frac{2\pi r}{h^2}\sqrt{h^2+r^2}\int_0^h x^2\,dx = \frac{2}{3}\pi rh\sqrt{h^2+r^2}.$$

$A = \pi r\sqrt{h^2 + r^2}; \quad \overline{x} = \overline{x}A/A = \frac{2}{3}h$

The centroid of the surface lies on the axis of the cone at a distance of $\frac{2}{3}h$ from the vertex of the cone.

33. Such a surface can be obtained by revolving about the x-axis the graph of the function

$$f(x) = \left(\frac{R-r}{h}\right)x + r, \quad x \in [0, h].$$

Formula (9.9.8) gives

$$\begin{aligned}\overline{x}A &= \int_0^h 2\pi x f(x)\sqrt{1 + [f'(x)]^2}\, dx \\ &= \frac{2\pi}{h}\sqrt{h^2 + (R-h)^2}\int_0^h \left[\left(\frac{R-r}{h}\right)x^2 + rx\right] dx \\ &= \frac{\pi}{3}\sqrt{h^2 + (R-r)^2}\,(2R + r)h\end{aligned}$$

$A = \pi(R+r)s = \pi(R+r)\sqrt{h^2 + (R-r)^2}$ and $\overline{x} = \dfrac{\overline{x}A}{A} = \left(\dfrac{2R+r}{R+r}\right)\dfrac{h}{3}.$

The centroid of the surface lies on the axis of the cone $\left(\dfrac{2R+r}{R+r}\right)\dfrac{h}{3}$ units from the base of radius r.

34. (a) Clear since this is just the circle $x^2 + y^2 = b^2$ translated horizontally by a.

(The circle can be parameterized by $x(t) = b\cos t, \quad y(t) = b\sin t$)

(b) Area of disc is πb^2, centroid is $(2, 0)$. So $V_y = 2\pi\overline{x}A = 2\pi \cdot 2 \cdot \pi b^2 = 4\pi^2 b^2$

(c) Length of circle is $2\pi b$, centroid is $(2, 0)$. So $A_y = 2\pi\overline{x}L = 2\pi \cdot 2 \cdot 2\pi b = 8\pi^2 b^2$

PROJECT 9.9

1. Referring to the figure we have

$$x(\theta) = \overline{OB} - \overline{AB} = R\theta - R\sin\theta = R(\theta - \sin\theta)$$
$$y(\theta) = \overline{BQ} - \overline{QC} = R - R\cos\theta = R(1 - \cos\theta).$$

2. (a) $x'(\theta) = R(1 - \cos\theta), \quad y'(\theta) = R\sin\theta.$

The arches end at $\theta = 2n\pi$, and $x'(2n\pi) = y'(2n\pi) = 0$

(b) $A = \int_0^{2\pi} y(\theta)x'(\theta)\, d\theta = R^2\int_0^{2\pi}(1 - \cos\theta)^2\, d\theta = 3\pi R^2$

(c)

$$L = \int_0^{2\pi} \sqrt{[R\,(1-\cos\theta)]^2 + [R\sin\theta]^2}\,d\theta$$
$$= R\int_0^{2\pi} \sqrt{2-2\cos\theta}\,d\theta$$
$$= R\int_0^{2\pi} \sqrt{4\sin^2\left(\frac{\theta}{2}\right)}\,d\theta = 2R\int_0^{2\pi} \sin\frac{\theta}{2}\,d\theta = 4R\left[-\cos\frac{\theta}{2}\right]_0^{2\pi} = 8R$$

3. (a)

$\overline{x} = \pi R$ by symmetry

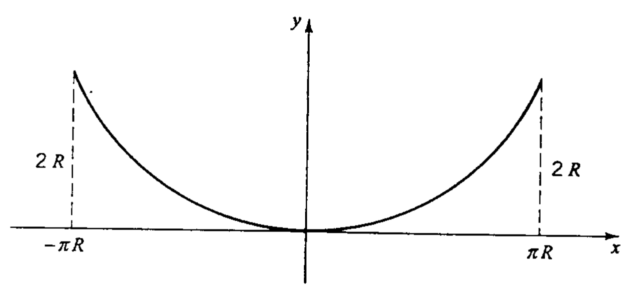
$$\overline{y}A = \int_0^{2\pi} \frac{1}{2}[y(\theta)]^2 x'(\theta)\,d\theta$$
$$= \int_0^{2\pi} \frac{1}{2}R^2(1-\cos\theta)^2[R(1-\cos\theta)]\,d\theta$$
$$= \frac{1}{2}R^3\int_0^{2\pi} \left(1-3\cos\theta+3\cos^2\theta-\cos^3\theta\right)\,d\theta = \frac{5}{2}\pi R^3$$

$A = 3\pi R^2$ (by Exercise 2b) $\overline{y} = \left(\frac{5}{2}\pi R^3\right) / \left(3\pi R^2\right) = \frac{5}{6}R$

(b) $V_x = 2\pi\overline{y}A = 2\pi\left(\frac{5}{6}R\right)\left(3\pi R^2\right) = 5\pi^2R^3$

(c) $V_y = 2\pi\overline{x}A = 2\pi(\pi R)(3\pi R^2) = 6\pi^3R^3$

4.



5. (a) $\dfrac{dy}{dx} = \dfrac{y'(\phi)}{x'(\phi)} = \dfrac{R\sin\phi}{R\,(1+\cos\phi)} = \dfrac{2\sin\frac{1}{2}\phi\cos\frac{1}{2}\phi}{2\cos^2\frac{1}{2}\phi} = \tan\dfrac{1}{2}\phi; \quad \alpha = \dfrac{1}{2}\phi$

(b)

$$s = \int_0^{\phi} \sqrt{[x'(t)]^2+[y'(t)]^2}\,dt = \int_0^{\phi} \sqrt{[R(1+\cos t)]^2+[R\sin t]^2}\,dt$$
$$= R\int_0^{\phi} \sqrt{2+2\cos t}\,dt = R\int_0^{\phi} 2\cos\frac{1}{2}t\,dt = 4R\left[\sin\frac{1}{2}t\right]_0^{\phi}$$
$$= 4R\sin\frac{1}{2}\phi = 4R\sin\alpha$$

Now note that the tangent at $(x(\phi), y(\phi))$ has slope

$$m = \frac{y'(\phi)}{x'(\phi)} = \frac{\sin\phi}{1+\cos\phi} = \frac{\sin\phi(1-\cos\phi)}{1-\cos^2\phi} = \frac{1-\cos\phi}{\sin\phi} = \frac{2\sin^2\frac{\phi}{2}}{2\sin\frac{\phi}{2}\cos\frac{\phi}{2}} = \tan\frac{\phi}{2}$$

So the inclination of the tangent is $\dfrac{\phi}{2} = \alpha$

6. (a) Already shown more generally in Example 6 of Section 9.9.

(b) Combining $d^2s/dt^2 = -g\sin\alpha$ with $s = 4R\sin\alpha$, we have

$$\frac{d^2s}{dt^2} = -\frac{g}{4R}s.$$

This is simple harmonic motion with angular frequency $\omega = \frac{1}{2}\sqrt{g/R}$ (see Section 18.7)

and period $T = 2\pi/\omega = 4\pi\sqrt{R/g}$.

7. $\dfrac{d^2s}{dt^2} = -g\sin\alpha = -g\cdot\dfrac{2R}{\sqrt{\pi^2R^2+4R^2}} = -\dfrac{2g}{\sqrt{\pi^2+4}}$

Since $\dfrac{ds}{dt} = 0$ and $s = R\sqrt{\pi^2+4}$ when $t = 0$, integrating twice gives

$$s = -\frac{g}{\sqrt{\pi^2+4}}t^2 + R\sqrt{\pi^2+4}$$

$s = 0$ at $t^2 = \dfrac{1}{g}R(\pi^2+4)$, so $t = \sqrt{R(\pi^2+4)/g}$

CHAPTER 10

SECTION 10.1

1. lub = 2; glb = 0

2. lub = 2; glb = 0

3. no lub; glb = 0

4. lub = 1, no glb

5. lub = 2; glb = −2

6. lub = 3; glb = −1

7. no lub; glb = 2

8. lub = 2; glb = −2

9. lub = $2\frac{1}{2}$; glb = 2

10. lub = 0; glb = −1

11. lub = 1; glb = 0.9

12. lub = $2\frac{1}{9}$, glb = $-2\frac{1}{9}$

13. lub = e; glb = 0

14. no lub, glb = 1

15. lub = $\frac{1}{2}(-1+\sqrt{5})$; glb = $\frac{1}{2}(-1-\sqrt{5})$

16. no lub, no glb

17. no lub; no glb

18. no lub; no glb

19. no lub; no glb

20. lub = 0; no glb

21. glb $S = 0$, $0 \le \left(\frac{1}{11}\right)^3 < 0 + 0.001$

22. glb=1; $1 \le 1 < 1 + 0.0001$

23. glb $S = 0$, $0 \le \left(\dfrac{1}{10^{2n-1}}\right) < 0 + \left(\dfrac{1}{10^k}\right) \quad \left(n > \dfrac{k+1}{2}\right)$

24. glb=0; $0 \le \left(\dfrac{1}{2}\right)^n < 0 + \left(\dfrac{1}{4}\right)^k$ for $n > 2k$

25. Let $\epsilon > 0$. The condition $m \le s$ is satisfied by all numbers s in S. All we have to show therefore is that there is some number s in S such that

$$s < m + \epsilon.$$

Suppose on the contrary that there is no such number in S. We then have

$$m + \epsilon \le x \quad \text{for all} \quad x \in S.$$

This makes $m + \epsilon$ a lower bound for S. But this cannot be, for then $m + \epsilon$ is a lower bound for S that is *greater* than m, and by assumption, m is the *greatest* lower bound.

26. (a) Let $M = |a_1| + \cdots + |a_n|$. Then for any i, $|a_i| < M$, so S is bounded

(b) lub $S = \max\{a_1, a_2, \ldots, a_n\} \in S$

glb $S = \min\{a_1, a_2, \ldots, a_n\} \in S$

27. Let $c =$ lub S. Since $b \in S$, $b \le c$. Since b is an upper bound for S, $c \le b$. Thus, $b = c$.

28. S consists of a single element, equal to lub S.

29. (a) Suppose that K is an upper bound for S and k is a lower bound. Let t be any element of T. Then $t \in S$ which implies that $k \leq t \leq K$. Thus K is an upper bound for T and k is a lower bound, and T is bounded.

(b) Let $a = \text{glb } S$. Then $a \leq t$ for all $t \in T$. Therefore, $a \leq \text{glb } T$. Similarly, if $b = \text{lub } S$, then $t \leq b$ for all $t \in T$, so lub $T \leq b$. It now follows that glb $S \leq$ glb $T \leq$ lub $T \leq$ lub S.

30. (a) Any $y \in T$ is an upper bound for S, so by definition of least upper bound lub S $\leq y$.

(b) By (a), lub S is a lower bound for T, so by definition of greatest lower bound lub $S \leq$ glb T

31. Let c be a positive number and let $S = \{c, 2c, 3c, \ldots\}$. Choose any positive number M and consider the positive number M/c. Since the set of positive integers is not bounded above, there exists a positive integer k such that $k \geq M/c$. This implies that $kc \geq M$. Since $kc \in S$, it follows that S is not bounded above.

32. Since a and $b - a$ are positive, we can choose n large enough that $na > 1$ and $n(b-a) > 1$.

Let m be the least integer greater that na; then $m > na$ and $m - 1 \leq na$. Then $\frac{m}{n} > a$, and

$1 < nb - na \leq nb - (m-1) \implies m < nb$, so $\frac{m}{n} < b$. Therefore $r = \frac{m}{n}$ satisfies $a < r < b$

33. (a)

a_1	a_2	a_3	a_4	a_5
1.4142	1.6818	1.8340	1.9152	1.9571

a_6	a_7	a_8	a_9	a_{10}
1.9785	1.9892	1.9946	1.9973	1.9986

(b) Let S be the set of positive integers for which $a_n < 2$. Then $1 \in S$ since

$$a_1 = \sqrt{2} \cong 1.4142 < 2.$$

Assume that $k \in S$. Since $a_{k+1}^2 = 2a_k < 4$, it follows that $a_{k+1} < 2$. Thus $k+1 \in S$ and S is the set of positive integers.

(c) Yes, $a_n \to 2$ as $n \to \infty$.

(d) Let c be a positive number. Then c is the least upper bound of the set

$$S = \left\{\sqrt{c},\ \sqrt{c\sqrt{c}},\ \sqrt{c\sqrt{c\sqrt{c}}}, \ldots\right\}.$$

34. (a) $a_1 \cong 1.4142136$ $a_2 \cong 1.8477591$ $a_3 \cong 1.9615706$ $a_4 \cong 1.9903695$ $a_5 \cong 1.9975909$

$a_6 \cong 1.9993976$ $a_7 \cong 1.9998494$ $a_8 \cong 1.9999624$ $a_9 \cong 1.9999906$ $a_{10} \cong 1.9999976$

(b) $a_1 = \sqrt{2} < 2$. Assume true for a_n. Then $a_{n+1} = \sqrt{2 + a_n} < \sqrt{2+2} = 2$.

(c) Yes, lub $S = 2$. For any positive number c, lub S is the positive number satisfying

$$x = \sqrt{c + x}, \quad \text{that is,} \quad x = (1 + \sqrt{1 + 4c})/2$$

SECTION 10.2

1. $a_n = 2 + 3(n-1), \quad n = 1, 2, 3, \dots$

2. $a_n = 1 - (-1)^n$

3. $a_n = \dfrac{(-1)^{n-1}}{2n-1}, \quad n = 1, 2, 3, \dots$

4. $a_n = \dfrac{2^n - 1}{2^n}$

5. $a_n = \dfrac{n^2+1}{n}, \quad n = 1, 2, 3, \dots$

6. $a_n = (-1)^n \dfrac{n}{(n+1)^2}$

7. $a_n = \begin{cases} n, & \text{if } n = 2k-1 \\ 1/n, & \text{if } n = 2k, \end{cases} \quad \text{where } k = 1, 2, 3, \dots$

8. $a_n = \begin{cases} n, & \text{if } n = 2k \\ 1/n^2, & \text{if } n = 2k-1, \end{cases} \quad \text{where } k = 1, 2, 3, \dots$

9. decreasing; bounded below by 0 and above by 2

10. not monotonic; bounded below by -1 and above by $\frac{1}{2}$.

11. $\dfrac{n + (-1)^n}{n} = 1 + (-1)^n \dfrac{1}{n}$: not monotonic; bounded below by 0 and above by $\dfrac{3}{2}$

12. increasing; bounded below by 1.001 but not bounded above.

13. decreasing; bounded below by 0 and above by 0.9

14. increasing; bounded below by 0 and above by 1

15. $\dfrac{n^2}{n+1} = n - 1 + \dfrac{1}{n+1}$: increasing; bounded below by $\dfrac{1}{2}$ but not bounded above

16. increasing; bounded below by $\sqrt{2}$, but not bounded above: $\sqrt{n^2+1} > n$

17. $\dfrac{4n}{\sqrt{4n^2+1}} = \dfrac{2}{\sqrt{1 + 1/4n^2}}$ and $\dfrac{1}{4n^2}$ decreases to 0: increasing;

bounded below by $\frac{4}{5}\sqrt{5}$ and above by 2

18. $\dfrac{2^n}{4^n+1} = \dfrac{2^n}{(2^n)^2+1} = \dfrac{1}{2^n + \frac{1}{2^n}}$. decreasing; bounded below by 0 and above by $\frac{2}{5}$.

19. increasing; bounded below by $\frac{2}{51}$ but not bounded above

20. $\dfrac{n^2}{\sqrt{n^3+1}} = \dfrac{1}{\sqrt{\frac{1}{n} + \frac{1}{n^4}}}$ and $\dfrac{1}{n} + \dfrac{1}{n^4}$ decreases to 0 $\Longrightarrow$ increasing; bounded below by $\dfrac{\sqrt{2}}{2}$,

but not bounded above.

21. $\dfrac{2n}{n+1} = 2 - \dfrac{2}{n+1}$ increases toward 2: increasing; bounded below by 0 and above by $ln2$.

22. increasing ($n \geq 2$); bounded below by $\dfrac{2\sqrt{2}}{3^{10}}$, but not bounded above.

23. decreasing; bounded below by 1 and above by 4

24. not monotonic; not bounded below and not bounded above

25. increasing; bounded below by $\sqrt{3}$ and above by 2

26. decreasing (since $\dfrac{n+1}{n}$ decreases to 1); bounded below by 0 and above by $\ln 2$.

27. $(-1)^{2n+1}\sqrt{n} = -\sqrt{n}$: decreasing; bounded above by -1 but not bounded below

28. $\dfrac{\sqrt{n+1}}{\sqrt{n}} = \sqrt{1+\dfrac{1}{n}}$ decreasing; bounded below by 1 and above by $\sqrt{2}$

29. $\dfrac{2^n - 1}{2^n} = 1 - \dfrac{1}{2^n}$: increasing; bounded below by $\dfrac{1}{2}$ and above by 1

30. $\dfrac{1}{2n} - \dfrac{1}{2n+3} = \dfrac{3}{2n(2n+3)}$ decreasing; bounded below by 0 and above by $\frac{3}{10}$.

31. consider $\sin x$ as $x \to 0^+$: decreasing; bounded below by 0 and above by 1

32. not monotonic; bounded below by $-\frac{1}{2}$ and above by $\frac{1}{4}$

33. decreasing; bounded below by 0 and above by $\frac{5}{6}$

34. increasing (because $\dfrac{x}{\ln x}$ is an increasing function on $[4, \infty)$.); bounded below by $\dfrac{4}{\ln 4}$ but not bounded above.

35. $\dfrac{1}{n} - \dfrac{1}{n+1} = \dfrac{1}{n\,(n+1)}$: decreasing; bounded below by 0 and above by $\dfrac{1}{2}$

36. not monotonic; bounded below by -1 and above by 1

37. Set $f(x) = \dfrac{\ln x}{x}$. Then, $f'(x) = \dfrac{1 - \ln x}{x^2} < 0$ for $x > e$: decreasing; bounded below by 0 and above by $\frac{1}{3}\ln 3$.

38. not monotonic; not bounded below nor above (because exponentials grow faster than polynomials).

39. Set $a_n = \dfrac{3^n}{(n+1)^2}$. Then, $\dfrac{a_{n+1}}{a_n} = 3\left(\dfrac{n+1}{n+2}\right)^2 > 1$: increasing; bounded below by $\frac{3}{4}$ but not bounded above.

40. $\dfrac{1 - (\frac{1}{2})^n}{(\frac{1}{2})^n} = 2^n - 1$ increasing; bounded below by 1 but not bounded above.

41. For $n \geq 5$

$$\frac{a_{n+1}}{a_n} = \frac{5^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n} = \frac{5}{n+1} < 1 \quad \text{and thus} \quad a_{n+1} < a_n.$$

Sequence is not nonincreasing: $a_1 = 5 < \frac{25}{2} = a_2$.

42. For $n \geq M$, $\dfrac{a_{n+1}}{a_n} = \dfrac{M^{n+1}}{(n+1)!} \cdot \dfrac{n!}{M^n} = \dfrac{M}{n+1} < 1$, so the sequence decreases for $n \geq M$.

43. boundedness: $0 < (c^n + d^n)^{1/n} < (2d^n)^{1/n} = 2^{1/n}d \leq 2d$

monotonicity :
$$\begin{aligned} a_{n+1}^{n+1} = c^{n+1} + d^{n+1} &= cc^n + dd^n \\ &< (c^n + d^n)^{1/n}c^n + (c^n + d^n)^{1/n}d^n \\ &= (c^n + d^n)^{1+1/n} \\ &= (c^n + d^n)^{(n+1)/n} \\ &= a_n^{n+1} \end{aligned}$$

Taking the $(n+1)$st root of each side we have $a_{n+1} < a_n$. The sequence is monotonic decreasing.

44. If for all n

$$|a_n| \leq M \quad \text{and} \quad |b_n| \leq N,$$

then for all n

$$|\alpha a_n + \beta b_n| \leq |\alpha||a_n| + |\beta||b_n| \leq |\alpha|M + |\beta|N$$

and

$$|a_n b_n| = |a_n||b_n| \leq MN$$

45. $a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{6}, \quad a_4 = \frac{1}{24}, \quad a_5 = \frac{1}{120}, \quad a_6 = \frac{1}{720}; \quad a_n = \frac{1}{n!}$

46. $a_1 = 1, \quad a_2 = 8, \quad a_3 = 27, \quad a_4 = 64, \quad a_5 = 125, \quad a_6 = 216; \quad a_n = n^3$

47. $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 1; \quad a_n = 1$

48. $a_1 = 1, \quad a_2 = \frac{3}{2}, \quad a_3 = \frac{7}{4}, \quad a_4 = \frac{15}{8}, a_5 = \frac{31}{16}, \quad a_6 = \frac{63}{32}; \quad a_n = (2^n - 1)/2^{n-1}$

49. $a_1 = 1, \quad a_2 = 3, \quad a_3 = 5, \quad a_4 = 7, \quad a_5 = 9, \quad a_6 = 11; \quad a_n = 2n - 1$

50. $a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{3}, \quad a_4 = \frac{1}{4}, \quad a_5 = \frac{1}{5}, \quad a_6 = \frac{1}{6}; \quad a_n = 1/n$

51. $a_1 = 1, \quad a_2 = 4, \quad a_3 = 9, \quad a_4 = 16, \quad a_5 = 25, \quad a_6 = 36; \quad a_n = n^2$

52. $a_1 = 1, \quad a_2 = 3, \quad a_3 = 7, \quad a_4 = 15, \quad a_5 = 31, \quad a_6 = 63; \quad a_n = 2^n - 1$

53. $a_1 = 1, \quad a_2 = 3, \quad a_3 = 4, \quad a_4 = 8, \quad a_5 = 16, \quad a_6 = 32; \quad a_n = 2^{n-1} \quad (n \geq 3)$

54. $a_1 = 3, \quad a_2 = 1, \quad a_3 = 3, \quad a_4 = 1, \quad a_5 = 3, \quad a_6 = 1; \quad a_n = 2 - (-1)^n$

55. $a_1 = 1, \quad a_2 = 3, \quad a_3 = 5, \quad a_4 = 7, \quad a_5 = 9, \quad a_6 = 11; \quad a_n = 2n - 1$

56. $a_1 = 1, \quad a_2 = 3, \quad a_3 = 4, \quad a_4 = 5, \quad a_5 = 6, \quad a_6 = 7; \quad a_n = n + 1 \quad (n \geq 2)$

57. First $a_1 = 2^1 - 1 = 1$. Next suppose $a_k = 2^k - 1$ for some $k \geq 1$. Then

$$a_{k+1} = 2a_k + 1 = 2\left(2^k - 1\right) + 1 = 2^{k+1} - 1.$$

58. True for $n = 1$, Assume true for n. Then $a_{n+1} = a_n + 5 = 5n - 2 + 5 = 5(n+1) - 2$.

59. First $a_1 = \dfrac{1}{2^0} = 1$. Next suppose $a_k = \dfrac{k}{2^{k-1}}$ for some $k \geq 1$. Then

$$a_{k+1} = \frac{k+1}{2k} a_k = \frac{k+1}{2k}\frac{k}{2^{k-1}} = \frac{k+1}{2^k}.$$

60. True for $n = 1$. Assume true for n. Then $a_{n+1} = a_n - \dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n(n+1)} = \dfrac{n+1-1}{n(n+1)} = \dfrac{1}{n+1}$

61. (a) If $r = 1$ then $S_n = n$ for $n = 1, 2, 3, \ldots$

(b)

$$\begin{aligned} S_n &= 1 + r + r^2 + \cdots + r^{n-1} \\ rS_n &= r + r^2 + \cdots + r^n \\ S_n - rS_n &= 1 - r^n \\ S_n &= \frac{1 - r^n}{1 - r}, \qquad r \neq 1. \end{aligned}$$

62. $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$, so

$$\begin{aligned} S_n &= a_1 + a_2 + \cdots + a_{n-1} + a_n \\ &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1)n} + \frac{1}{n(n+1)} \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} = \frac{n}{n+1} \quad \text{since all middle terms cancel out.} \end{aligned}$$

63. (a) Let S_n denote the distance traveled between the nth and $(n+1)st$ bounce. Then

$$S_1 = 75 + 75 = 150, \qquad S_2 = \tfrac{3}{4}(75) + \tfrac{3}{4}(75) = 150\left(\frac{3}{4}\right), \ldots, \qquad S_n = 150\left(\frac{3}{4}\right)^{n-1}.$$

(b) An object dropped from rest from a height h above the ground will hit the ground in $\frac{1}{4}\sqrt{h}$ seconds. Therefore it follows that the ball will be in the air

$$T_n = 2\left(\tfrac{1}{4}\right)\sqrt{\frac{S_n}{2}} = \frac{5\sqrt{3}}{2}\left(\frac{3}{4}\right)^{(n-1)/2} \quad \text{seconds.}$$

64. $P_0 = 500.$ $P_{12} = 500 \cdot 2,$ $P_{24} = 500 \cdot 2^2;$ so $P_n = 500\, 2^{n/12}$

65. (a) Let S be the set of positive integers for which $a_{n+1} > a_n$. Since $a_2 = 1 + \sqrt{a_1} = 2 > 1,$ $1 \in S$. Assume that $a_k = 1 + \sqrt{a_{k-1}} > a_{k-1}$. Then

$$a_{k+1} = 1 + \sqrt{a_k} > 1 + \sqrt{a_{k-1}} = a_k.$$

Thus, $k \in S$ implies $k+1 \in S$. It now follows that $\{a_n\}$ is an increasing sequence.

(b) Since $\{a_n\}$ is an increasing sequence,

$$a_n = 1 + \sqrt{a_{n-1}} < 1 + \sqrt{a_n}, \quad \text{or} \quad a_n - \sqrt{a_n} - 1 < 0.$$

Rewriting the second inequality as

$$\left(\sqrt{a_n}\right)^2 - \sqrt{a_n} - 1 < 0$$

and solving for $\sqrt{a_n}$ it follows that $\sqrt{a_n} < \frac{1}{2}(1 + \sqrt{5})$. Hence, $a_n < \frac{1}{2}(3 + \sqrt{5})$ for all n.

(c) $a_2 = 2,$ $a_3 \cong 2.4142,$ $a_4 \cong 2.5538,$ $a_5 \cong 2.6118, \ldots,$ $a_9 \cong 2.6179,$ $\ldots,$ $a_{15} \cong 2.6180;$ $\text{lub}\,\{a_n\} = \frac{1}{2}(3 + \sqrt{5}) \cong 2.6180$

66. (a) We show that $a_n < a_{n+1}$ for all n. True for $n = 1$ since $a_1 = 1 < \sqrt{3} = a_2$.

Assume true for n, that is, $a_n < a_{n+1}$; we need to show that $a_{n+1} < a_{n+2}$.

But $a_{n+1} = \sqrt{3a_n} < \sqrt{3a_{n+1}} = a_{n+2},$ as required.

(b) True since $a_1 < 3,$ and $a_n < 3 \implies a_{n+1} < \sqrt{3 \cdot 3} = 3$

(c) $a_1 = 1,$ $a_2 = \sqrt{3},$ $a_3 \cong 2.2795, \ldots, a_{14} = 2.9996,$ $a_{15} = 2.9998;$ lub $= 3$

SECTION 10.3

1. diverges

2. converges to 0

3. converges to 0

4. diverges

5. converges to 1: $\dfrac{n-1}{n} = 1 - \dfrac{1}{n} \to 1$

6. converges to 1: $\dfrac{n + (-1)^n}{n} = 1 + \dfrac{(-1)^n}{n} \to 1$

7. converges to 0: $\dfrac{n+1}{n^2} = \dfrac{1}{n} + \dfrac{1}{n^2} \to 0$

8. converges to 0: $\dfrac{\pi}{2n} \to 0,$ so $\sin\left(\dfrac{\pi}{2n}\right) \to \sin 0 = 0$

9. converges to 0: $0 < \dfrac{2^n}{4^n+1} < \dfrac{2^n}{4^n} = \dfrac{1}{2^n} \to 0$

10. diverges: $\dfrac{n^2}{n+1} \geq \dfrac{n^2}{2n} = \dfrac{n}{2}$

11. diverges

12. converges to 4: $\dfrac{4n}{\sqrt{n^2+1}} = \dfrac{4}{\sqrt{1+(1/n^2)}} \to 4$

13. converges to 0

14. diverges: for n large, $10^6 < 2^n$, so $\dfrac{4^n}{2^n+10^6} > \dfrac{4^n}{2 \cdot 2^n} = 2^{n-1}$

15. converges to 1: $\dfrac{n\pi}{4n+1} \to \dfrac{\pi}{4}$ so $\tan \dfrac{n\pi}{4n+1} \to \tan \dfrac{\pi}{4} = 1$

16. converges to 0: $\dfrac{10^{10}\sqrt{n}}{n+1} = \dfrac{10^{10}}{\sqrt{n}+1/\sqrt{n}} \to 0$

17. converges to $\dfrac{4}{9}$: $\dfrac{(2n+1)^2}{(3n-1)^2} = \dfrac{4+4/n+1/n^2}{9-6/n+1/n^2} \to \dfrac{4}{9}$

18. converges to $\ln 2$: $\dfrac{2n}{n+1} \to 2$, so $\ln\left(\dfrac{2n}{n+1}\right) \to \ln 2$

19. converges to $\dfrac{1}{2}\sqrt{2}$: $\dfrac{n^2}{\sqrt{2n^4+1}} = \dfrac{1}{\sqrt{2+1/n^4}} \to \dfrac{1}{\sqrt{2}}$

20. converges to 1: $\dfrac{n^4-1}{n^4+n-6} = \dfrac{1-\frac{1}{n^4}}{1+\frac{1}{n^3}-\frac{6}{n^4}} \to 1$

21. diverges: $\cos n\pi = (-1)^n$

22. diverges: $\dfrac{n^5}{17n^4+12} = n\left(\dfrac{1}{17+\frac{12}{n^4}}\right)$

23. converges to 1: $\dfrac{1}{\sqrt{n}} \to 0$ so $e^{1/\sqrt{n}} \to e^0 = 1$

24. converges to $\sqrt{4} = 2$

25. diverges

26. converges to 1: $\dfrac{2^n-1}{2^n} = 1 - \dfrac{1}{2^n} \to 1$

27. converges to 0 : $\ln n - \ln(n+1) = \ln\left(\dfrac{n}{n+1}\right) \to \ln 1 = 0$

28. converges to 0: $\dfrac{1}{n} - \dfrac{1}{n+1} = \dfrac{1}{n(n+1)} \to 0$

29. converges to $\dfrac{1}{2}$: $\dfrac{\sqrt{n+1}}{2\sqrt{n}} = \dfrac{1}{2}\sqrt{1+\dfrac{1}{n}} \to \dfrac{1}{2}$

30. converges to 0: $(0.9)^n = \left(\dfrac{9}{10}\right)^n \to 0$

31. converges to e^2: $\left(1+\dfrac{1}{n}\right)^{2n} = \left[\left(1+\dfrac{1}{n}\right)^n\right]^2 \to e^2$

32. converges to $\sqrt{e}$: $\quad \left(1+\frac{1}{n}\right)^{n/2} = \sqrt{\left(1+\frac{1}{n}\right)^{n}} \to \sqrt{e}$

33. diverges; since $2^n > n^3$ for $n \geq 10$, $\quad \frac{2^n}{n^2} > \frac{n^3}{n^2} = n$

34. converges to 0: $\quad -\frac{1+\frac{1}{n}}{1+\sqrt{n}} \leq \frac{(n+1)\cos\sqrt{n}}{n(1+\sqrt{n})} \leq \frac{1+\frac{1}{n}}{1+\sqrt{n}} \to 0$

35. converges to 0: $\quad \left|\frac{\sqrt{n}\sin(e^n\pi)}{n+1}\right| = \frac{|\sin(e^n\pi)|}{\sqrt{n}+\frac{1}{\sqrt{n}}} \leq \frac{1}{\sqrt{n}+\frac{1}{\sqrt{n}}} \to 0$

36. converges to $\ln 9$: $\quad 2\ln 3n - \ln(n^2+1) = \ln\left(\frac{9n^2}{n^2+1}\right) \to \ln 9$

37. Set $\epsilon > 0$. Since $a_n \to L$, there exists N_1 such that

$$\text{if} \quad n \leq N_1, \quad \text{then} \quad |a_n - L| < \epsilon/2.$$

Since $b_n \to M$, there exists N_2 such that

$$\text{if} \quad n \geq N_2, \quad \text{then} \quad |b_n - M| < \epsilon/2.$$

Now set $N = \max\{N_1, N_2\}$. Then, for $n \geq N$,

$$|(a_n + b_n) - (L+M)| \leq |a_n - L| + |b_n - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

38. Let $\epsilon > 0$, choose k such that $n \geq k \Longrightarrow |a_n - L| < \frac{\epsilon}{|\alpha|+1}$.

Then for $n \geq k$, $\quad |\alpha a_n - \alpha L| = |\alpha||a_n - L| \leq (|\alpha|+1)|a_n - L| < \epsilon$

Therefore $\quad \alpha\, a_n \to \alpha\, L$

39. Since $\left(1+\frac{1}{n}\right) \to 1$ and $\left(1+\frac{1}{n}\right)^n \to e$,

$$\left(1+\frac{1}{n}\right)^{n+1} = \left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right) \to (e)(1) = e.$$

40. (a) If $k = j$, $\quad a_n = \frac{\alpha_k + \alpha_{k-1}\cdot\frac{1}{n} + \cdots + \alpha_0\cdot\frac{1}{n^k}}{\beta_k + \beta_{k-1}\cdot\frac{1}{n} + \cdots + \beta_0\cdot\frac{1}{n^k}} \to \frac{\alpha_k}{\beta_k}$

(b) If $k < j$, $\quad a_n = \frac{\alpha_k + \alpha_{k-1}\cdot\frac{1}{n} + \cdots + \alpha_0\cdot\frac{1}{n^k}}{\beta_j\cdot n^{j-k} + \beta_{j-1}\cdot n^{j-1-k} + \cdots + \beta_0\cdot\frac{1}{n^k}} \quad \to \quad 0$

(c) If $k > j$, $\quad a_n = \frac{\alpha_k\cdot n^{k-j} + \alpha_{k-1}\cdot n^{k-1-j} + \cdots + \alpha_0\cdot\frac{1}{n^j}}{\beta_j + \beta_{j-1}\cdot\frac{1}{n} + \cdots + \beta_0\cdot\frac{1}{n^j}}$ diverges

41. Suppose that $\{a_n\}$ is bounded and non-increasing. If L is the greatest lower bound of the range of this sequence, then $a_n \geq L$ for all n. Set $\epsilon > 0$. By Theorem 10.1.5 there exists a_k such that $a_k < L + \epsilon$. Since the sequence is non-increasing, $a_n \leq a_k$ for all $n \geq k$. Thus,

$$L \leq a_n < L + \epsilon \quad \text{or} \quad |a_n - L| < \epsilon \quad \text{for all} \quad n \geq k$$

and $\quad a_n \to L$.

42. Let $\epsilon > 0$. If $a_n \to L$, then there exists a positive integer k such that

$$|a_n - L| < \epsilon \quad \text{for all} \quad n \geq k$$

If $n \geq k$, then $2n \geq k$ and $2n - 1 \geq k$, and thus

$$|e_n - L| = |a_{2n} - L| < \epsilon \quad \text{and} \quad |o_n - L| = |a_{2n-1} - L| < \epsilon$$

It follows that $e_n \to L$ and $o_n \to L$. If $e_n \to L$ and $o_n \to L$, then there exist k_1 and k_2 such that

$$\text{if} \quad m \geq k_1, \quad \text{then} \quad |e_m - L| = |a_{2m} - L| < \epsilon$$

and

$$\text{if} \quad m \geq k_2, \quad \text{then} \quad |o_m - L| = |a_{2m-1} - L| < \epsilon$$

Let $k = max\{2k_1,\ 2k_2 - 1\}$. If $n \geq k$ then

$$\text{either} \quad a_n = a_{2m} \quad \text{with} \quad m > k_1 \quad \text{or} \quad a_n = a_{2m-1} \quad \text{with} \quad m \geq k_2$$

In either case, $|a_n - L| < \epsilon$. This shows that $a_n \to L$.

43. Let $\epsilon > 0$. Choose k so that, for $n \geq k$,

$$L - \epsilon < a_n < L + \epsilon, \quad L - \epsilon < c_n < L + \epsilon \quad \text{and} \quad a_n \leq b_n \leq c_n.$$

For such n,

$$L - \epsilon < b_n < L + \epsilon.$$

44. Let M be a bound for $\{b_n\}$. Then $|a_n b_n| \leq |a_n| M$.

Given $\epsilon > 0$, choose k such that $|a_n| < \epsilon/M$ for $n \geq k$. Then $|a_n b| < \epsilon$ for $n \geq k$.

45. Let $\epsilon > 0$. Since $a_n \to L$, there exists a positive integer N such that $L - \epsilon < a_n < L + \epsilon$ for all $n \geq N$. Now $a_n \leq M$ for all n, so $L - \epsilon < M$, or $L < M + \epsilon$. Since ϵ is arbitrary, $L \leq M$.

46. The converse is false. For example, let $a_n = (-1)^n$. Then $|a_n| \to 1$, but $\{a_n\}$ diverges.

47. By the continuity of f, $f(L) = f\left(\lim_{n\to\infty} a_n\right) = \lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} a_{n+1} = L$.

48. $\dfrac{2^n}{n!} = \dfrac{2}{1} \cdot \dfrac{2}{2} \cdot \dfrac{2}{3} \cdots \dfrac{2}{n} = 2 \cdot \dfrac{2}{n} \cdot (\text{terms that are } \leq 1) \leq \dfrac{4}{n}.$

Since $\dfrac{4}{n} \to 0$ and $0 < \dfrac{2^n}{n!} \leq \dfrac{4}{n}$, $\dfrac{2^n}{n!} \to 0$ as well

49. Set $f(x) = x^{1/p}$. Since $\frac{1}{n} \to 0$ and f is continuous at 0, it follows by Theorem 10.3.12 that $\left(\frac{1}{n}\right)^{1/p} \to 0$.

50. Since $|a_n - L| = |(a_n - L) - 0| = ||a_n - L| - 0|$,

$|a_n - L| < \epsilon$ iff $|(a_n - L) - 0| < \epsilon$ iff $||a_n - L| - 0| < \epsilon$,

So $a_n \to L$ iff $a_n - L \to 0$ iff $|a_n - L| \to 0$.

51. $a_n = e^{1-n} \to 0$

52. diverges

53. $a_n = \dfrac{1}{n!} \to 0$

54. $a_n = 1 \cdot \dfrac{1}{2} \cdot \dfrac{2}{3} \cdots \dfrac{n-1}{n} = \dfrac{1}{n}$ converges to 0

55. $a_n = \frac{1}{2}[1 - (-1)^n]$ diverges

56. $a_n = (-1)^{n-1}$ diverges.

57. $a_n = \dfrac{2^n - 1}{2^{n-1}} \to 2$

58. $a_n = 1 + \dfrac{1}{3} + \dfrac{1}{3^2} + \cdots + \dfrac{1}{3^n}$, geometric series, converges to $\frac{3}{2}$.

59. $L = 0, \quad n = 32$

60. $\dfrac{1}{\sqrt{n}} \to 0. \quad \dfrac{1}{\sqrt{n}} < 0.001$ for $n \geq 1000^2 + 1$

61. $L = 0, \quad n = 4$

62. $\dfrac{n^{10}}{10^n} \to 0. \quad \dfrac{n^{10}}{10^n} < 0.001$ for $n \geq 15$

63. $L = 0, \quad n = 7$

64. $\dfrac{2^n}{n!} \to 0. \quad \dfrac{2^n}{n!} < 0.001$ for $n \geq 10$

65. $L = 0, \quad n = 65$

66. $\dfrac{\ln n}{n} \to 0. \quad \dfrac{\ln n}{n} < 0.001$ for $n \geq 9119$

67. (a) $a_n = 1 + \sqrt{a_{n-1}}$ Suppose that $a_n \to L$ as $n \to \infty$. Then $a_{n-1} \to L$ as $n \to \infty$. Therefore $L = 1 + \sqrt{L}$ which implies that $L = \frac{1}{2}(3 + \sqrt{5})$.

(b) $a_n = \sqrt{3a_{n-1}}$ Suppose that $a_n \to L$ as $n \to \infty$. Then $a_{n-1} \to L$ as $n \to \infty$. Therefore $L = \sqrt{3L}$ which implies that $L = 3$.

68. (a) $a_2 \cong 2.645751, \quad a_3 \cong 2.940366, \quad a_4 \cong 2.990044, \quad a_5 \cong 2.998340, \quad a_6 \cong 2.999723$

(b) True for $n = 1$. Assume true for n. Then $a_{n+1} = \sqrt{6 + a_{n-1}} \leq \sqrt{6+3} = 3$

(c) ${a_{n+1}}^2 - {a_n}^2 = 6 + a_n - {a_n}^2 = (3 - a_n)(2 + a_n) \geq 0$ since $a_n \leq 3$.

Since $a_k \geq 0$ for all k, this implies $a_{n+1} \geq a_n$

(d) $a_n \to 3$

69. (a)

a_2	a_3	a_4	a_5	a_6
0.540302	0.857553	0.654290	0.793480	0.701369

a_7	a_8	a_9	a_{10}
0.763960	0.722102	0.750418	0.73140

(b) L is a fixed point of $f(x) = \cos x$, that is, $\cos L = L$; $L \cong 0.739085$.

70. (a) $a_2 \cong 1.540302$, $a_3 \cong 1.570792$, $a_4 \cong a_5 \cong \cdots \cong a_{10} \cong 1.570796$.

(b) $L \cong 1.570796$ Let $f(x) = x + \cos x$. L must satisfy $L = f(L)$, so $L = L + \cos L$, and $\cos L = 0$. Indeed, the L we found is just $\frac{\pi}{2} \cong 1.570796327$

71. (a)

a_2	a_3	a_4	a_5	a_6	a_7	a_8
2.000000	1.750000	1.732143	1.732051	1.732051	1.732051	1.732051

(b) $L = \frac{1}{2}\left(L + \frac{3}{L}\right)$ which implies $L^2 = 3$ or $L = \sqrt{3}$.

(c) Newton's method applied to the function $f(x) = x^2 - R$ gives

$$\begin{aligned} a_n &= a_{n-1} - \frac{f(a_{n-1})}{f'(a_{n-1})} = a_{n-1} - \frac{a_{n-1}^2 - R}{2a_{n-1}} \\ &= \frac{1}{2}a_{n-1} + \frac{1}{2}\frac{R}{a_{n-1}} = \frac{1}{2}\left(a_{n-1} + \frac{R}{a_{n-1}}\right), \quad n = 2, 3, \ldots . \end{aligned}$$

72. (a) $f(x) = x^3 - 8$, so $x_n \to 2$

(b) $f(x) = \sin x - \frac{1}{2}$, so $x_n \to \frac{\pi}{6}$

(c) $f(x) = \ln x - 1$, so $x_n \to e$

PROJECT 10.3

1. (a)

a_2	a_3	a_4	a_5	a_6	a_7	a_8
2.000000	1.750000	1.732143	1.732051	1.732051	1.732051	1.732051

(b) $L = \frac{1}{2}\left(L + \frac{3}{L}\right)$ which implies $L^2 = 3$ or $L = \sqrt{3}$.

2. Newton's method applied to the function $f(x) = x^2 - R$ gives

$$\begin{aligned} a_n &= a_{n-1} - \frac{f(a_{n-1})}{f'(a_{n-1})} = a_{n-1} - \frac{a_{n-1}^2 - R}{2a_{n-1}} \\ &= \frac{1}{2}a_{n-1} + \frac{1}{2}\frac{R}{a_{n-1}} = \frac{1}{2}\left(a_{n-1} + \frac{R}{a_{n-1}}\right), \quad n = 2, 3, \ldots . \end{aligned}$$

3. (a) $f(x) = x^3 - 8$, so $x_n \to 2$

(b) $f(x) = \sin x - \frac{1}{2}$, so $x_n \to \frac{\pi}{6}$

(c) $f(x) = \ln x - 1$, so $x_n \to e$

SECTION 10.4

1. converges to 1: $2^{2/n} = (2^{1/n})^2 \to 1^2 = 1$

2. converges to 1: $e^{-\alpha/n} \to e^0 = 1$

3. converges to 0: for $n > 3$, $0 < \left(\frac{2}{n}\right)^n < \left(\frac{2}{3}\right)^n \to 0$

4. converges to 0: $\frac{\log_{10} n}{n} = \frac{1}{\ln 10} \cdot \frac{\ln n}{n} \to 0$

5. converges to 0: $\frac{\ln(n+1)}{n} = \left[\frac{\ln(n+1)}{n+1}\right]\left(\frac{n+1}{n}\right) \to (0)(1) = 0$

6. converges to 0: $\frac{3^n}{4^n} = \left(\frac{3}{4}\right)^n \to 0$

7. converges to 0: $\frac{x^{100n}}{n!} = \frac{(x^{100})^n}{n!} \to 0$

8. converges to 1: $n^{1/n+2} = \left(n^{1/n}\right)^{n/n+2} \to 1$

9. converges to 1: $n^{\alpha/n} = (n^{1/n})^\alpha \to 1^\alpha = 1$

10. $\ln\left(\frac{n+1}{n}\right)$ converges to $\ln 1 = 0$

11. converges to 0: $\frac{3^{n+1}}{4^{n-1}} = 12\left(\frac{3^n}{4^n}\right) = 12\left(\frac{3}{4}\right)^n \to 12(0) = 0$

12. converges to $\frac{1}{2}$: $\int_{-n}^{0} e^{2x}\,dx = \frac{1}{2} - \frac{e^{-2n}}{2} \to \frac{1}{2}$

13. converges to 1: $(n+2)^{1/n} = e^{\frac{1}{n}\ln(n+2)}$ and, since

$$\frac{1}{n}\ln(n+2) = \left[\frac{\ln(n+2)}{n+2}\right]\left(\frac{n+2}{n}\right) \to (0)(1) = 0,$$

it follows that

$$(n+2)^{1/n} \to e^0 = 1.$$

14. converges to e^{-1}: $\left(1 - \frac{1}{n}\right)^n = \left(1 + \frac{(-1)}{n}\right)^n \to e^{-1}$ (by (10.4.7)

15. converges to 1: $\int_0^n e^{-x}\,dx = 1 - \frac{1}{e^n} \to 1$

16. diverges.

17. converges to π: integral $= 2\int_0^n \frac{dx}{1+x^2} = 2\tan^{-1} n \to 2\left(\frac{\pi}{2}\right) = \pi$

18. converges to 0: $\int_0^n e^{-nx}\,dx = -\frac{e^{-n^2}}{n} + \frac{1}{n} \to 0$

19. converges to 1: recall (10.4.6)

20. converges to 0: $n^2 \sin n\pi = 0$ for all n

21. converges to 0: $\dfrac{\ln(n^2)}{n} = 2\dfrac{\ln\, n}{n} \to 2(0) = 0$

22. converges to π : $\displaystyle\int_{-1+1/n}^{1-1/n} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}\left(1-\frac{1}{n}\right) - \sin^{-1}\left(-1+\frac{1}{n}\right) \to \sin^{-1}(1) - \sin^{-1}(-1) = \pi$

23. diverges: since $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1,$

$$\frac{n}{\pi}\sin\frac{\pi}{n} = \frac{\sin(\pi/n)}{\pi/n} \to 1$$

and, for n sufficiently large,

$$n^2 \sin\frac{\pi}{n} = n\pi\left(\frac{n}{\pi}\sin\frac{\pi}{n}\right) > n\pi\left(\frac{1}{2}\right) = \frac{n\pi}{2}$$

24. diverges

25. converges to 0: $\dfrac{5^{n+1}}{4^{2n-1}} = 20\left(\dfrac{5}{16}\right)^n \to 0$

26. converges to e^{3x} : $\left(1+\dfrac{x}{n}\right)^{3n} = \left[\left(1+\dfrac{x}{n}\right)^n\right]^3 \to (e^x)^3 = e^{3x}$

27. converges to e^{-1}: $\left(\dfrac{n+1}{n+2}\right)^n = \left(1-\dfrac{1}{n+2}\right)^n = \dfrac{\left(1+\dfrac{(-1)}{n+2}\right)^{n+2}}{\left(1+\dfrac{(-1)}{n+2}\right)^2} \to \dfrac{e^{-1}}{1} = e^{-1}$

28. converges to 2: $\displaystyle\int_{1/n}^{1} \frac{dx}{\sqrt{x}} = 2 - \frac{2}{\sqrt{n}} \to 2.$

29. converges to 0: $\displaystyle 0 < \int_n^{n+1} e^{-x^2}\,dx \le e^{-n^2}[(n+1)-n] = e^{-n^2} \to 0$

30. converges to 1: $\left(1+\dfrac{1}{n^2}\right)^n = \left[\left(1+\dfrac{1}{n^2}\right)^{n^2}\right]^{1/n} \to (e^1)^0 = 1$

31. converges to 0: $\dfrac{n^n}{2^{n^2}} = \left(\dfrac{n}{2^n}\right)^n \to 0$ since $\dfrac{n}{2^n} \to 0$

32. converges to 0: $\displaystyle\int_0^{1/n} \cos e^x\,dx \quad\to\quad \int_0^0 \cos e^x\,dx = 0$

33. converges to e^x: use (10.4.7)

34. diverges: $\left(1+\dfrac{1}{n}\right)^{n^2} = \left[\left(1+\dfrac{1}{n}\right)^n\right]^n \approx e^n$

35. converges to 0: $\displaystyle\left|\int_{-1/n}^{1/n} \sin x^2 dx\right| \le \int_{-1/n}^{1/n} |\sin x^2|\,dx \le \int_{-1/n}^{1/n} 1\,dx = \frac{2}{n} \to 0$

36. $\left(t+\dfrac{x}{n}\right) = t^n\left(1+\dfrac{x/t}{n}\right)^n$; converges to 0 if $t<1$, converges to e^x if $t=1$, diverges if $t>1$.

37. $\sqrt{n+1}-\sqrt{n}=\dfrac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\left(\sqrt{n+1}+\sqrt{n}\right)=\dfrac{1}{\sqrt{n+1}+\sqrt{n}}\to 0$

38. $\sqrt{n^2+n}-n=\dfrac{\sqrt{n^2+n}-n}{\sqrt{n^2+n}+n}(\sqrt{n^2+n}+n)=\dfrac{n}{\sqrt{n^2+n}+n}=\dfrac{1}{\sqrt{1+1/n}+1}\to\dfrac{1}{2}$

39. (a) The length of each side of the polygon is $2r\sin(\pi/n)$. Therefore the perimeter, p_n, of the polygon is given by: $p_n=2rn\sin(\pi/n)$.

(b) $2rn\sin(\pi/n)\to 2\pi r$ as $n\to\infty$: The number $2rn\sin(\pi/n)$ is the perimeter of a regular polygon of n sides inscribed in a circle of radius r. As n tends to ∞, the perimeter of the polygon tends to the circumference of the circle.

40. Since $0<c<d$, $\quad d<(c^n+d^n)^{1/n}<(2d^n)^{1/n}=2^{1/n}d\to d,\quad$ so by the pinching theorem

$$(c^n+d^n)^{1/n}\to d.$$

41. By the hint, $\displaystyle\lim_{n\to\infty}\frac{1+2+\cdots+n}{n^2}=\lim_{n\to\infty}\frac{n(n+1)}{2n^2}=\lim_{n\to\infty}\frac{1+1/n}{2}=\frac{1}{2}.$

42. diverges: $\dfrac{1^2+2^2+\cdots+n^2}{(1+n)(2+n)}=\dfrac{n(n+1)(2n+1)}{6(1+n)(2+n)}=\dfrac{2n^3+3n^2+n}{6n^2+18n+12}\to\infty$

43. By the hint, $\displaystyle\lim_{n\to\infty}\frac{1^3+2^3+\cdots+n^3}{2n^4+n-1}=\lim_{n\to\infty}\frac{n^2(n+1)^2}{4(2n^4+n-1)}=\lim_{n\to\infty}\frac{1+2/n+1/n^2}{8+4/n^3-4/n^4}=\frac{1}{8}.$

44. Here we show that every convergent sequence is a Cauchy sequence. Let $\epsilon>0$. If $a_n\to L$, then there exists a positive integer k such that

$$|a_p-L|<\frac{\epsilon}{2}\quad\text{for all}\quad p\ge k$$

With $m,n\ge k$ we have

$$|a_m-a_n|\le|a_m-L|+|L-a_n|=|a_m-L|+|a_n-L|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

45. (a)

$$\begin{aligned}m_{n+1}-m_n&=\frac{1}{n+1}(a_1+\cdots+a_n+a_{n+1})-\frac{1}{n}(a_1+\cdots+a_n)\\&=\frac{1}{n(n+1)}\left[na_{n+1}-(\overbrace{a_1+\cdots+a_n}^{n})\right]\\&>0\quad\text{since }\{a_n\}\text{ is increasing.}\end{aligned}$$

(b) We begin with the hint

$$m_n<\frac{|a_1+\cdots+a_j|}{n}+\frac{\epsilon}{2}\left(\frac{n-j}{n}\right).$$

Since j is fixed,

$$\frac{|a_1+\cdots+a_j|}{n}\to 0$$

and therefore for n sufficiently large

$$\frac{|a_1 + \cdots + a_j|}{n} < \frac{\epsilon}{2}.$$

Since

$$\frac{\epsilon}{2}\left(\frac{n-j}{n}\right) < \frac{\epsilon}{2},$$

we see that, for n sufficiently large, $|m_n| < \epsilon$. This shows that $m_n \to 0$.

46. (a) Since $\{a_n\}$ converges, it is a Cauchy sequence (see Exercise 44), so given $\epsilon > 0$ we can find k such that $|a_n - a_m| < \epsilon$ for $n, m \geq k$.

In particular, $|a_{n+1} - a_n| < \epsilon$, so $\lim_{n\to\infty}(a_n - a_{n-1}) = 0$.

(b) $\{a_n\}$ does not necessarily converge. For example let $a_n = \ln n$. This diverges, but

$$a_n - a_{n-1} = \ln n - \ln(n-1) = \ln\left(\frac{n}{n-1}\right) \to \ln 1 = 0$$

47. (a) Let S be the set of positive integers n $(n \geq 2)$ for which the inequalities hold. Since

$$\left(\sqrt{b}\right)^2 - 2\sqrt{ab} + \left(\sqrt{a}\right)^2 > 0 = \left(\sqrt{b} - \sqrt{a}\right)^2 > 0,$$

it follows that $\dfrac{a+b}{2} > \sqrt{ab}$ and so $a_1 > b_1$. Now,

$$a_2 = \frac{a_1 + b_1}{2} < a_1 \quad \text{and} \quad b_2 = \sqrt{a_1 b_1} > b_1.$$

Also, by the argument above,

$$a_2 = \frac{a_1 + b_1}{2} > \sqrt{a_1 b_1} = b_2,$$

and so $a_1 > a_2 > b_2 > b_1$. Thus $2 \in S$. Assume that $k \in S$. Then

$$a_{k+1} = \frac{a_k + b_k}{2} < \frac{a_k + a_k}{2} = a_k, \quad b_{k+1} = \sqrt{a_k b_k} > \sqrt{b_k^2} = b_k,$$

and

$$a_{k+1} = \frac{a_k + b_k}{2} > \sqrt{a_k b_k} = b_{k+1}.$$

Thus $k + 1 \in S$. Therefore, the inequalities hold for all $n \geq 2$.

(b) $\{a_n\}$ is a decreasing sequence which is bounded below.

$\{b_n\}$ is an increasing sequence which is bounded above.

Let $L_a = \lim_{n\to\infty} a_n$, $L_b = \lim_{n\to\infty} b_n$. Then

$$a_n = \frac{a_{n-1} + b_{n-1}}{2} \quad \text{implies} \quad L_a = \frac{L_a + L_b}{2} \quad \text{and} \quad L_a = L_b.$$

48. $\dfrac{e-\left(1+\frac{1}{1000}\right)^{1000}}{e} \cong 0.0004995$: within 0.05%

$\dfrac{e^5-\left(1+\frac{5}{1000}\right)^{1000}}{e^5} \cong 0.01238$: within 1.3%

49. The numerical work suggests $L \cong 1$. Justification: Set $f(x) = \sin x - x^2$. Note that $f(0) = 0$ and $f'(x) = \cos x - 2x > 0$ for x close to 0. Therefore $\sin x - x^2 > 0$ for x close to 0 and $\sin 1/n - 1/n^2 > 0$ for n large. Thus, for n large,

$$\frac{1}{n^2} < \sin\frac{1}{n} < \frac{1}{n}$$

$$|\sin x| \le |x| \quad \text{for all } x$$

$$\left(\frac{1}{n^2}\right)^{1/n} < \left(\sin\frac{1}{n}\right)^{1/n} < \left(\frac{1}{n}\right)^{1/n}$$

$$\left(\frac{1}{n^{1/n}}\right)^2 < \left(\sin\frac{1}{n}\right)^{1/n} < \frac{1}{n^{1/n}}.$$

As $n \to \infty$ both bounds tend to 1 and therefore the middle term also tends to 1.

50. Numerical work suggests $L \cong 1/3$. conjecture: $L = 1/k$.

Proof: $(n^k + n^{k-1})^{1/k} - n = n(1+1/n)^{1/k} - n = \dfrac{(1+1/n)^{1/k}-1}{1/n}$;

$$\lim_{n\to\infty} \frac{(1+1/n)^{1/k}-1}{1/n} = \lim_{h\to 0} \frac{(1+h)^{1/k}-1}{h} = f'(1),$$

where $f(x) = x^{1/k}$. Since $f'(x) = (1/k)x^{(1/k)-1}$, $f'(1) = 1/k$.

51. (a)

a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
2	3	5	8	13	21	34	55

(b)

r_1	r_2	r_3	r_4	r_5	r_6
1	2	1.2	1.667	1.600	1.625

(c) Following the hint,

$$1 + \frac{1}{r_{n-1}} = 1 + \frac{1}{\dfrac{a_n}{a_{n-1}}} = 1 + \frac{an-1}{a_n} = \frac{a_n + a_{n-1}}{a_n} = \frac{a_{n+1}}{a_n} = r_n.$$

Now, if $r_n \to L$, then $r_{n-1} \to L$ and

$$1 + \frac{1}{L} = L \quad \text{which implies} \quad L = \frac{1+\sqrt{5}}{2} \cong 1.618034.$$

52. With the partition $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\}$ and $f(x) = x$, we have

$$a_n = \frac{1}{n}\left(\frac{1}{n} + \frac{2}{n} + \cdots + \frac{n}{n}\right) = \frac{1}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n}{n}\right)\right] = \sum_{i=1}^{n} f(x_i)\Delta x_i,$$

so it is a Riemann sum for $\displaystyle\int_0^1 x\,dx$, and therefore $\displaystyle\lim_{n\to\infty} a_n = \int_0^1 x\,dx = \frac{1}{2}$

SECTION 10.5

(We'll use $\star$ to indicate differentiation of numerator and denominator.)

1. $\lim_{x\to 0^+} \frac{\sin x}{\sqrt{x}} \overset{\star}{=} \lim_{x\to 0^+} 2\sqrt{x}\cos x = 0$

2. $\lim_{x\to 1} \frac{\ln x}{1-x} \overset{\star}{=} \lim_{x\to 1} \frac{1/x}{-1} = -1$

3. $\lim_{x\to 0} \frac{e^x - 1}{\ln(1+x)} \overset{\star}{=} \lim_{x\to 0}(1+x)e^x = 1$

4. $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} \overset{\star}{=} \lim_{x\to 4} \frac{\frac{1}{2\sqrt{x}}}{1} = \frac{1}{4}$

5. $\lim_{x\to \pi/2} \frac{\cos x}{\sin 2x} \overset{\star}{=} \lim_{x\to \pi/2} \frac{-\sin x}{2\cos 2x} = \frac{1}{2}$

6. $\lim_{x\to a} \frac{x-a}{x^n - a^n} \overset{\star}{=} \lim_{x\to a} \frac{1}{nx^{n-1}} = \frac{1}{na^{n-1}}$

7. $\lim_{x\to 0} \frac{2^x - 1}{x} \overset{\star}{=} \lim_{x\to 0} 2^x \ln 2 = \ln 2$

8. $\lim_{x\to 0} \frac{\tan^{-1} x}{x} \overset{\star}{=} \lim_{x\to 0} \frac{\frac{1}{1+x^2}}{1} = 1$

9. $\lim_{x\to 1} \frac{x^{1/2} - x^{1/4}}{x-1} \overset{\star}{=} \lim_{x\to 1} \left(\frac{1}{2}x^{-1/2} - \frac{1}{4}x^{-3/4}\right) = \frac{1}{4}$

10. $\lim_{x\to 0} \frac{e^x - 1}{x(1+x)} \overset{\star}{=} \lim_{x\to 0} \frac{e^x}{1+2x} = 1$

11. $\lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x} \overset{\star}{=} \lim_{x\to 0} \frac{e^x + e^{-x}}{\cos x} = 2$

12. $\lim_{x\to 0} \frac{1-\cos x}{3x} \overset{\star}{=} \lim_{x\to 0} \frac{\sin x}{3} = 0$

13. $\lim_{x\to 0} \frac{x + \sin \pi x}{x - \sin \pi x} \overset{\star}{=} \lim_{x\to 0} \frac{1 + \pi\cos \pi x}{1 - \pi \cos \pi x} = \frac{1+\pi}{1-\pi}$

14. $\lim_{x\to 0} \frac{a^x - (a+1)^x}{x} \overset{\star}{=} \lim_{x\to 0} \frac{a^x \ln a - (a+1)^x \ln(a+1)}{1} = \ln\left(\frac{a}{a+1}\right)$

15. $\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos 2x} \overset{\star}{=} \lim_{x\to 0} \frac{e^x - e^{-x}}{2\sin 2x} \overset{\star}{=} \lim_{x\to 0} \frac{e^x + e^{-x}}{4\cos 2x} = \frac{1}{2}$

16. $\lim_{x\to 0} \frac{x - \ln(x+1)}{1 - \cos 2x} \overset{\star}{=} \lim_{x\to 0} \frac{1 - \frac{1}{x+1}}{2\sin 2x} \overset{\star}{=} \lim_{x\to 0} \frac{\frac{1}{(x+1)^2}}{4\cos 2x} = \frac{1}{4}$

17. $\lim_{x\to 0} \frac{\tan \pi x}{e^x - 1} \overset{\star}{=} \lim_{x\to 0} \frac{\pi \sec^2 \pi x}{e^x} = \pi$

18. $\lim_{x\to 0} \frac{\cos x - 1 + x^2/2}{x^4} \overset{\star}{=} \lim_{x\to 0} \frac{-\sin x + x}{4x^3} \overset{\star}{=} \lim_{x\to 0} \frac{-\cos x + 1}{12x^2} \overset{\star}{=} \lim_{x\to 0} \frac{\sin x}{24x} = \frac{1}{24}$

19. $\lim_{x\to 0} \frac{1 + x - e^x}{x(e^x - 1)} \overset{\star}{=} \lim_{x\to 0} \frac{1 - e^x}{xe^x + e^x - 1} \overset{\star}{=} \lim_{x\to 0} \frac{-e^x}{xe^x + 2e^x} = -\frac{1}{2}$

20. $\lim_{x\to 0} \frac{\ln(\sec x)}{x^2} \overset{\star}{=} \lim_{x\to 0} \frac{\tan x}{2x} \overset{\star}{=} \lim_{x\to 0} \frac{\sec^2 x}{2} = \frac{1}{2}$

21. $\lim_{x\to 0} \dfrac{x-\tan x}{x-\sin x} \overset{\star}{=} \lim_{x\to 0} \dfrac{1-\sec^2 x}{1-\cos x} \overset{\star}{=} \lim_{x\to 0} \dfrac{-2\sec^2 x \tan x}{\sin x} = \lim_{x\to 0} \dfrac{-2\sec^2 x}{\cos x} = -2$

22. $\lim_{x\to 0} \dfrac{xe^{nx}-x}{1-\cos nx} \overset{\star}{=} \lim_{x\to 0} \dfrac{e^{nx}+nxe^{nx}-1}{n\sin nx} \overset{\star}{=} \lim_{x\to 0} \dfrac{e^{nx}(2n+n^2x)}{n^2\cos nx} = \dfrac{2}{n}$

23. $\lim_{x\to 1^-} \dfrac{\sqrt{1-x^2}}{\sqrt{1-x^3}} = \lim_{x\to 1^-} \sqrt{\dfrac{1-x^2}{1-x^3}} = \sqrt{\dfrac{2}{3}} = \dfrac{1}{3}\sqrt{6}$ since $\lim_{x\to 1^-} \dfrac{1-x^2}{1-x^3} \overset{\star}{=} \lim_{x\to 1^-} \dfrac{2x}{3x^2} = \dfrac{2}{3}$

24. $\lim_{x\to 0} \dfrac{2x-\sin \pi x}{4x^2-1} = 0$

25. $\lim_{x\to \pi/2} \dfrac{\ln(\sin x)}{(\pi-2x)^2} \overset{\star}{=} \lim_{x\to \pi/2} \dfrac{-\cot x}{4(\pi-2x)} \overset{\star}{=} \lim_{x\to \pi/2} \dfrac{\csc^2 x}{-8} = -\dfrac{1}{8}$

26. $\lim_{x\to 0^+} \dfrac{\sqrt{x}}{\sqrt{x}+\sin\sqrt{x}} \overset{\star}{=} \lim_{x\to 0^+} \dfrac{\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{x}}+\frac{\cos\sqrt{x}}{2\sqrt{x}}} \overset{\star}{=} \lim_{x\to 0^+} \dfrac{1}{1+\cos\sqrt{x}} = \dfrac{1}{2}$

27. $\lim_{x\to 0} \dfrac{\cos x-\cos 3x}{\sin(x^2)} \overset{\star}{=} \lim_{x\to 0} \dfrac{-\sin x+3\sin 3x}{2x\cos(x^2)} \overset{\star}{=} \lim_{x\to 0} \dfrac{-\cos x+9\cos 3x}{2\cos(x^2)-4x^2\sin(x^2)} = 4$

28. $\lim_{x\to 0} \dfrac{\sqrt{a+x}-\sqrt{a-x}}{x} \overset{\star}{=} \lim_{x\to 0} \dfrac{\frac{1}{2\sqrt{a+x}}+\frac{1}{2\sqrt{a-x}}}{1} = \dfrac{1}{\sqrt{a}} = \dfrac{\sqrt{a}}{a}$

29. $\lim_{x\to \pi/4} \dfrac{\sec^2 x-2\tan x}{1+\cos 4x} \overset{\star}{=} \lim_{x\to \pi/4} \dfrac{2\sec^2 x\ \tan x-2\sec^2 x}{-4\sin 4x}$

$\overset{\star}{=} \lim_{x\to \pi/4} \dfrac{2\sec^4 x+4\sec^2 x\tan^2 x-4\sec^2 x\tan x}{-16\cos 4x} = \dfrac{1}{2}$

30. $\lim_{x\to 0} \dfrac{x-\sin^{-1} x}{\sin^3 x} \overset{\star}{=} \lim_{x\to 0} \dfrac{1-\frac{1}{\sqrt{1-x^2}}}{3\sin^2 x\cos x} \overset{\star}{=} \lim_{x\to 0} \dfrac{\frac{-x}{(1-x^2)^{3/2}}}{6\sin x\cos^2 x-3\sin^3 x}$

$\overset{\star}{=} \lim_{x\to 0} \dfrac{\frac{-1-2x^2}{(1-x^2)^{5/2}}}{6\cos^3 x-12\sin^2 x\cos x-9\sin^2 x\cos x} = -\dfrac{1}{6}$

31. $\lim_{x\to 0} \dfrac{\tan^{-1} x}{\tan^{-1} 2x} \overset{\star}{=} \lim_{x\to 0} \dfrac{1}{1+x^2}\,\dfrac{1+4x^2}{2} = \dfrac{1}{2}$

32. $\lim_{x\to 0} \dfrac{\sin^{-1} x}{x} \overset{\star}{=} \lim_{x\to 0} \dfrac{\frac{1}{\sqrt{1-x^2}}}{1} = 1$

33. $1:$ $\lim_{x\to \infty} \dfrac{\pi/2-\tan^{-1} x}{1/x} \overset{\star}{=} \lim_{x\to \infty} \dfrac{x^2}{1+x^2} = 1$

34. $-1:$ $\lim_{n\to \infty} \dfrac{\ln(1-\frac{1}{n})}{\sin(\frac{1}{n})} = \lim_{x\to \infty} \dfrac{\ln(1-\frac{1}{x})}{\sin(\frac{1}{x})} \overset{\star}{=} \lim_{x\to \infty} \dfrac{\frac{1}{1-1/x}(\frac{1}{x^2})}{\cos(\frac{1}{x})(-\frac{1}{x^2})} = -1$

35. 1 : $\lim_{x\to\infty} \frac{1}{x[\ln(x+1)-\ln x]} = \lim_{x\to\infty} \frac{1/x}{\ln(1+1/x)} = \lim_{t\to 0^+} \frac{t}{\ln(1+t)} \overset{\star}{=} \lim_{t\to 0^+}(1+t) = 1$

36. $\frac{1}{3}$: $\lim_{n\to\infty} \frac{\sinh(\pi/n)-\sin(\pi/n)}{\sin^3(\pi/n)} = \lim_{x\to\infty} \frac{\sinh(\pi/x)-\sin(\pi/x)}{\sin^3(\pi/x)} \overset{\star}{=} \lim_{u\to 0^+} \frac{\sinh u - \sin u}{\sin^3 u}$

$\overset{\star}{=} \lim_{u\to 0^+} \frac{\cosh u - \cos u}{3\sin^2 u \cos u} \overset{\star}{=} \lim_{u\to 0^+} \frac{\sinh u + \sin u}{6\sin u\cos^2 u - 3\sin^3 u} \overset{\star}{=} \lim_{u\to 0^+} \frac{\cosh u + \cos u}{6\cos^3 u - 21\sin^2 u\cos u} = \frac{2}{6} = \frac{1}{3}$

37. $\lim_{x\to 0}(2+x+\sin x) \neq 0, \quad \lim_{x\to 0}(x^3+x-\cos x) \neq 0$

38. $\lim_{n\to\infty} n\left(a^{1/n}-1\right) = \lim_{x\to\infty} \frac{a^{1/x}-1}{1/x} \overset{\star}{=} \lim_{x\to\infty} \frac{a^{1/x}\ln a\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} = \ln a$

39. The limit does not exist if $b \neq 1$. Therefore, $b = 1$.

$\lim_{x\to 0} \frac{\cos ax - 1}{2x^2} \overset{\star}{=} \lim_{x\to 0} \frac{-a\sin ax}{4x} \overset{\star}{=} \lim_{x\to 0} \frac{-a^2\cos ax}{4} = -\frac{a^2}{4}$

Now, $-\frac{a^2}{4} = -4$ implies $a = \pm 4$.

40. $\lim_{x\to 0} \frac{\sin 2x + ax + bx^3}{x^3} \overset{\star}{=} \lim_{x\to 0} \frac{2\cos 2x + a + 3bx^2}{3x^2}$ need $a = -2$ to keep numerator 0

$\overset{\star}{=} \lim_{x\to 0} \frac{-4\sin 2x + 6bx}{6x} \overset{\star}{=} \lim_{x\to 0} \frac{-8\cos 2x + 6b}{6} = 0$ if $6b = 8$

$\Longrightarrow a = -2, \quad b = \frac{4}{3}$

41. $\lim_{x\to 0} \frac{1}{x}\int_0^x f(t)\,dt \overset{\star}{=} \lim_{x\to 0} \frac{f(x)}{1} = f(0)$

42. (a) $\lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h} \overset{\star}{=} \lim_{h\to 0} \frac{f'(x+h)-f'(x-h)(-1)}{2} = f'(x)$

(note that here we differentiated with respect to h, not x.)

(b) $\lim_{h\to 0} \frac{f(x+h)-2f(x)+f(x-h)}{h^2} \overset{\star}{=} \lim_{h\to 0} \frac{f'(x+h)-f'(x-h)}{2h}$

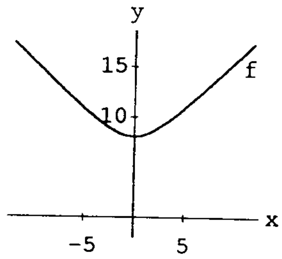
$\overset{\star}{=} \lim_{h\to 0} \frac{f''(x+h)+f''(x-h)}{2} = f''(x)$

43. $A(b) = 2\int_0^{\sqrt{b}} (b-x^2)\,dx = 2\left[bx - x^2\right]_0^{\sqrt{b}} = \frac{4}{3}b\sqrt{b}$ and $T(b) = \frac{1}{2}\left(2\sqrt{b}\right)b = b\sqrt{b}$.

Thus, $\lim_{b\to 0} \frac{T(b)}{A(b)} = \frac{b\sqrt{b}}{\frac{4}{3}b\sqrt{b}} = \frac{3}{4}$.

44. $T(\theta) = \frac{1}{2}\cdot 1\cdot\sin\theta = \frac{\sin\theta}{2}; \quad S(\theta) = \frac{\theta}{2}\cdot 1^2 = \frac{\theta}{2}$: $\quad \lim_{\theta\to 0^+} \frac{T(\theta)}{S(\theta)} = \lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1$

45. (a)

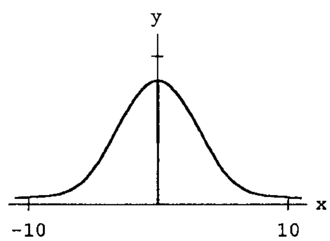


$f(x) \to \infty$ as $x \to \pm\infty$

(b) $f(x) \to 10$ as $x \to 4$

Confirmation: $\lim_{x\to 4} \dfrac{x^2-16}{\sqrt{x^2+9}-5} \overset{\star}{=} \lim_{x\to 4} \dfrac{2x}{x\,(x^2+9)^{-1/2}} = \lim_{x\to 4} 2\sqrt{x^2+9} = 10$

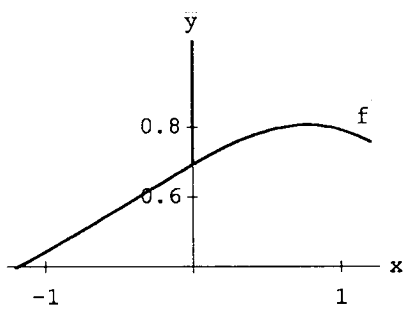
46. (a)



$\lim_{x\to\pm\infty} f(x) = \infty$

(b) $f(x) \to \dfrac{1}{6}$ as $x \to 0$; $\lim_{x\to 0} \dfrac{x-\sin x}{x^3} \overset{\star}{=} \lim_{x\to 0} \dfrac{1-\cos x}{3x^2} \overset{\star}{=} \lim_{x\to 0} \dfrac{\sin x}{6x} = \dfrac{1}{6}$

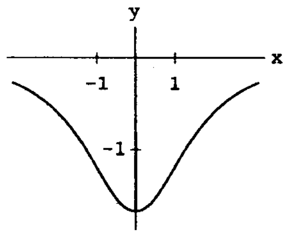
47. (a)



$f(x) \to 0.7$ as $x \to 0$

(b) Confirmation: $\lim_{x\to 0} \dfrac{2^{\sin x}-1}{x} \overset{\star}{=} \lim_{x\to 0} \dfrac{\ln(2)\, 2^{\sin x}\cos x}{1} = \ln 2 \cong 0.6931$

48. (a)



$g(x) \to -1.6$ as $x \to 0$

(b) Confirmation: $\displaystyle\lim_{x\to 0}\frac{3^{\cos x}-3}{x^2} \overset{\star}{=} \lim_{x\to 0}\frac{3^{\cos x}(-\sin x)\ln(3)}{2x} = -\lim_{x\to 0} 3^{\cos x}\ln 3\,\frac{\sin x}{2x}$

$= -\frac{3\ln 3}{2} \cong -1.6479$

SECTION 10.6

(We'll use $\star$ to indicate differentiation of numerator and denominator.)

1. $\displaystyle\lim_{x\to-\infty}\frac{x^2+1}{1-x} \overset{\star}{=} \lim_{x\to-\infty}\frac{2x}{-1} = \infty$

2. $\displaystyle\lim_{x\to\infty}\frac{20x}{x^2+1} = 0$

3. $\displaystyle\lim_{x\to\infty}\frac{x^3}{1-x^3} = \lim_{x\to\infty}\frac{1}{1/x^3-1} = -1$

4. $\displaystyle\lim_{x\to\infty}\frac{x^3-1}{2-x} = -\infty$

5. $\displaystyle\lim_{x\to\infty} x^2\sin\frac{1}{x} = \lim_{h\to 0^+}\left[\left(\frac{1}{h}\right)\left(\frac{\sin h}{h}\right)\right] = \infty$

6. $\displaystyle\lim_{x\to\infty}\frac{\ln(x^k)}{x} \overset{\star}{=} \lim_{x\to\infty}\frac{k/x}{1} = 0$

7. $\displaystyle\lim_{x\to\frac{\pi}{2}^-}\frac{\tan 5x}{\tan x} = \lim_{x\to\frac{\pi}{2}^-}\left[\left(\frac{\sin 5x}{\sin x}\right)\left(\frac{\cos x}{\cos 5x}\right)\right] = \frac{1}{5}$ since

$$\lim_{x\to\frac{\pi}{2}^-}\frac{\sin 5x}{\sin x} = 1 \quad\text{and}\quad \lim_{x\to\frac{\pi}{2}^-}\frac{\cos x}{\cos 5x} \overset{\star}{=} \lim_{x\to\frac{\pi}{2}^-}\frac{\sin x}{5\sin 5x} = \frac{1}{5}$$

8. $\displaystyle\lim_{x\to 0}(x\ln|\sin x|) = \lim_{t\to 0}\frac{\ln|\sin x|}{1/x} \overset{\star}{=} \lim_{x\to 0}\frac{\frac{\cos x}{\sin x}}{-\frac{1}{x^2}} = \lim_{x\to 0}\left(-\frac{x}{\sin x}\right)(x\cos x) = 0$

9. $\displaystyle\lim_{x\to 0^+} x^{2x} = \lim_{x\to 0^+}(x^x)^2 = 1^2 = 1$ [see (10.6.4)]

10. $\displaystyle\lim_{x\to\infty} x\sin\frac{\pi}{x} = \lim_{t\to 0^+}\frac{\sin\pi t}{t} \overset{\star}{=} \lim_{t\to 0^+}\frac{\pi\cos\pi t}{1} = \pi$

11. $\lim_{x\to 0} x(\ln|x|)^2 = \lim_{x\to 0} \frac{(\ln|x|)^2}{1/x} \overset{\star}{=} \lim_{x\to 0} \frac{2\ln|x|}{-1/x} \overset{\star}{=} \lim_{x\to 0} \frac{2}{1/x} = 0$

12. $\lim_{x\to 0+} \frac{\ln x}{\cot x} \overset{\star}{=} \lim_{x\to 0+} \frac{1/x}{-\csc^2 x} = \lim_{x\to 0+} -\frac{\sin^2 x}{x} = \lim_{x\to 0+} -\frac{\sin x}{x} \cdot \sin x = 0$

13. $\lim_{x\to\infty} \frac{1}{x}\int_0^x e^{t^2}\,dt \overset{\star}{=} \lim_{x\to\infty} \frac{e^{x^2}}{1} = \infty$

14. $\lim_{x\to\infty} \frac{\sqrt{1+x^2}}{x} = \lim_{x\to\infty} \sqrt{\frac{1}{x^2}+1} = 1$

15.
$$\lim_{x\to 0}\left[\frac{1}{\sin^2 x} - \frac{1}{x^2}\right] = \lim_{x\to 0} \frac{x^2 - \sin^2 x}{x^2\sin^2 x} \overset{\star}{=} \lim_{x\to 0} \frac{2x - 2\sin x\cos x}{2x^2\sin x\cos x + 2x\sin^2 x}$$
$$= \lim_{x\to 0} \frac{2x - \sin 2x}{x^2\sin 2x + 2x\sin^2 x}$$
$$\overset{\star}{=} \lim_{x\to 0} \frac{2 - 2\cos 2x}{2x^2\cos 2x + 4x\sin 2x + 2\sin^2 x}$$
$$\overset{\star}{=} \lim_{x\to 0} \frac{4\sin 2x}{-4x^2\sin 2x + 12x\cos 2x + 6\sin 2x}$$
$$\overset{\star}{=} \lim_{x\to 0} \frac{8\cos 2x}{-8x^2\cos 2x - 32x\sin 2x + 24\cos 2x} = \frac{1}{3}$$

16. Since $\lim_{x\to 0} \ln(|\sin x|^x) = \lim_{x\to 0}(x\ln|\sin x|) = 0$ by Exercise 8, $\lim_{x\to 0} |\sin x|^x = e^0 = 1$

17. $\lim_{x\to 1} x^{1/(x-1)} = e$ since $\lim_{x\to 1} \ln\left[x^{1/(x-1)}\right] = \lim_{x\to 1} \frac{\ln x}{x-1} \overset{\star}{=} \lim_{x\to 1} \frac{1}{x} = 1$

18. Take log:
$$\lim_{x\to 0+} \ln\left(x^{\sin x}\right) = \lim_{x\to 0+} (\sin x\,\ln x) = \lim_{x\to 0+} \left(\frac{\ln x}{\sec x}\right)$$
$$\overset{\star}{=} \lim_{x\to 0+} \frac{1/x}{-\csc x\cot x} = \lim_{x\to 0+} \frac{-\sin^2 x}{x\cos x} = 0, \quad \text{so} \quad \lim_{x\to 0+} x^{\sin x} = e^0 = 1$$

19.
$$\lim_{x\to\infty}\left(\cos\frac{1}{x}\right)^x = 1 \quad \text{since} \quad \lim_{x\to\infty} \ln\left[\left(\cos\frac{1}{x}\right)^x\right] = \lim_{x\to\infty} \frac{\ln\left(\cos\frac{1}{x}\right)}{(1/x)}$$
$$\overset{\star}{=} \lim_{x\to\infty}\left(-\frac{\sin(1/x)}{\cos(1/x)}\right) = 0$$

20. Take log:
$$\lim_{x\to\pi/2} \ln\left(|\sec x|^{\cos x}\right) = \lim_{x\to\pi/2} \cos x\ln|\sec x| = \lim_{x\to\pi/2} \frac{\ln|\sec x|}{\sec x}$$
$$\overset{\star}{=} \lim_{x\to\pi/2} \frac{\tan x}{\sec x\tan x} = \lim_{x\to\pi/2} \cos x = 0, \quad \text{so} \quad \lim_{x\to\pi/2} |\sec x|^{\cos x} = e^0 = 1$$

21.
$$\lim_{x\to 0}\left[\frac{1}{\ln(1+x)}-\frac{1}{x}\right]=\lim_{x\to 0}\frac{x-\ln(1+x)}{x\ln(1+x)}\overset{\star}{=}\lim_{x\to 0}\frac{x}{x+(1+x)\ln(1+x)}$$
$$\overset{\star}{=}\lim_{x\to 0}\frac{1}{1+1+\ln(1+x)}=\frac{1}{2}$$

22. Take log: $\lim\limits_{x\to\infty}\ln(x^2+a^2)^{(1/x)^2}=\lim\limits_{x\to\infty}\dfrac{\ln(x^2+a^2)}{x^2}\overset{\star}{=}\lim\limits_{x\to\infty}\dfrac{\frac{2x}{x^2+a^2}}{2x}=0,$

so $\lim\limits_{x\to\infty}(x^2+a^2)^{(1/x)^2}=e^0=1$

23.
$$\lim_{x\to 0}\left[\frac{1}{x}-\cot x\right]=\lim_{x\to 0}\frac{\sin x-x\cos x}{x\sin x}\overset{\star}{=}\lim_{x\to 0}\frac{x\sin x}{\sin x+x\cos x}$$
$$\overset{\star}{=}\lim_{x\to 0}\frac{\sin x+x\cos x}{2\cos x-x\sin x}=0$$

24. $\lim\limits_{x\to\infty}\ln\left(\dfrac{x^2-1}{x^2+1}\right)^3=3\lim\limits_{x\to\infty}\ln\left(\dfrac{x^2-1}{x^2+1}\right)=0$

25.
$$\lim_{x\to\infty}\left(\sqrt{x^2+2x}-x\right)=\lim_{x\to\infty}\left[\left(\sqrt{x^2+2x}-x\right)\left(\frac{\sqrt{x^2+2x}+x}{\sqrt{x^2+2x}+x}\right)\right]$$
$$=\lim_{x\to\infty}\frac{2x}{\sqrt{x^2+2x}+x}=\lim_{x\to\infty}\frac{2}{\sqrt{1+2/x}+1}=1$$

26. $\lim\limits_{x\to\infty}\dfrac{1}{x}\displaystyle\int_0^x\sin\left(\frac{1}{t+1}\right)\overset{\star}{=}\lim\limits_{x\to\infty}\sin\left(\dfrac{1}{x+1}\right)=0$

27. $\lim\limits_{x\to\infty}(x^3+1)^{1/\ln x}=e^3$ since

$$\lim_{x\to\infty}\ln\left[(x^3+1)^{1/\ln x}\right]=\lim_{x\to\infty}\frac{\ln(x^3+1)}{\ln x}\overset{\star}{=}\lim_{x\to\infty}\frac{\left(\frac{3x^2}{x^3+1}\right)}{1/x}=\lim_{x\to\infty}\frac{3}{1+1/x^3}=3.$$

28. Take log: $\lim\limits_{x\to\infty}\dfrac{\ln(e^x+1)}{x}\overset{\star}{=}\lim\limits_{x\to\infty}\dfrac{\frac{e^x}{e^x+1}}{1}\overset{\star}{=}\lim\limits_{x\to\infty}\dfrac{e^x}{e^x}=1,$

so $\lim\limits_{x\to\infty}(e^x+1)^{1/x}=e$

29. $\lim\limits_{x\to\infty}(\cosh x)^{1/x}=e$ since

$$\lim_{x\to\infty}\ln\left[(\cosh x)^{1/x}\right]=\lim_{x\to\infty}\frac{\ln(\cosh x)}{x}\overset{\star}{=}\lim_{x\to\infty}\frac{\sinh x}{\cosh x}=1.$$

30. Take log: $\lim\limits_{x\to\infty}3x\ln\left(1+\dfrac{1}{x}\right)=3\lim\limits_{x\to\infty}\dfrac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}\overset{\star}{=}3\lim\limits_{x\to\infty}\dfrac{\frac{-1/x^2}{1+1/x}}{-\frac{1}{x^2}}=3,$

so $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x} = e^3$

31. $$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) = \lim_{x\to 0}\frac{x-\sin x}{x\,\sin x} \stackrel{\star}{=} \lim_{x\to 0}\frac{1-\cos x}{\sin x + x\,\cos x}$$
$$\stackrel{\star}{=} \lim_{x\to 0}\frac{\sin x}{2\cos x - x\,\sin x} = 0$$

32. Take log: $\lim_{x\to 0}\frac{\ln(e^x+3x)}{x} \stackrel{\star}{=} \lim_{x\to 0}\frac{\frac{e^x+3}{e^x+3x}}{1} = 4,$

so $\lim_{x\to 0}(e^x+3x)^{1/x} = e^4$

33. $$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{x}{x-1}\right) = \lim_{x\to 1}\frac{x-1\ln x}{(x-1)\,\ln x} \stackrel{\star}{=} \lim_{x\to 1}\frac{-\ln x}{(x-1)(1/x)+\ln x}$$
$$= \lim_{x\to 1}\frac{-x\,\ln x}{x-1+x\,\ln x} \stackrel{\star}{=} \lim_{x\to 1}\frac{-\ln x - 1}{2+\ln x} = -\frac{1}{2}$$

•

34. Take log: $\lim_{x\to 0}\frac{\ln(1+2^x)-\ln 2}{x} \stackrel{\star}{=} \lim_{x\to 0}\frac{\frac{2^x\ln 2}{1+2^x}}{1} = \frac{\ln 2}{2}$

35. $0:\quad \frac{1}{n}\ln\frac{1}{n} = -\frac{\ln n}{n} \to 0$

36. $0:\quad \lim_{n\to\infty}\frac{n^k}{2^n}\to 0$

37. $1:\quad \ln\left[(\ln n)^{1/n}\right] = \frac{1}{n}\ln(\ln n)\to 0$

38. $0:\quad \lim_{n\to\infty}\frac{\ln n}{n^p} \stackrel{\star}{=} \lim_{n\to\infty}\frac{\cdot 1/n}{p\,n^{p-1}} = \lim_{n\to\infty}\frac{1}{p\,n^p} = 0$

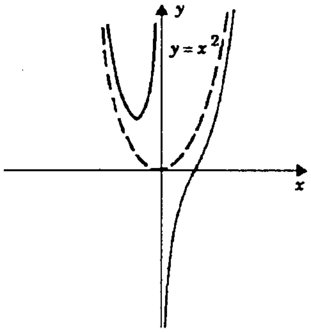
39. $1:\quad \ln\left[(n^2+n)^{1/n}\right] = \frac{1}{n}\ln[n(n+1)] = \frac{\ln n}{n}+\frac{\ln(n+1)}{n}\to 0$

40. $1:\quad \lim_{n\to\infty}\ln\left(n^{\sin(\pi/n)}\right) = \lim_{n\to\infty}[\sin(\pi/n)\ln n] = \lim_{n\to\infty}\left(\frac{\sin(\pi/n)}{1/n}\right)\left(\frac{\ln n}{n}\right) = 0,$ so $n^{\sin(\pi/n)}\to 1$

41. $0:\quad 0\le\frac{n^2\ln n}{e^n}<\frac{n^3}{e^n},\quad \lim_{x\to\infty}\frac{x^3}{e^x}=0$

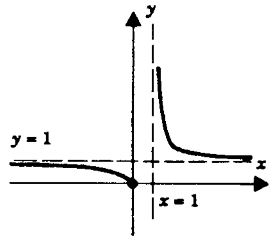
42. $1:\quad \ln(\sqrt{n}-1)^{1/\sqrt{n}} = \frac{\ln(\sqrt{n}-1)}{\sqrt{n}}\to 0,$ so $(\sqrt{n}-1)^{1/\sqrt{n}}\to 1$

43.



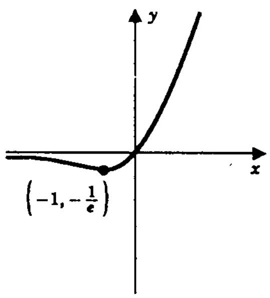
vertical asymptote y-axis

44.



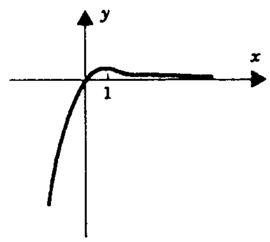
vertical asymptote $x = 1$
horizontal asymptote $y = 1$

45.



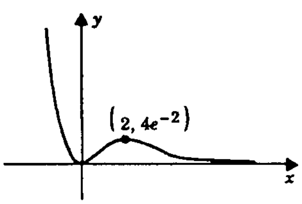
horizontal asymptote x-axis

46.



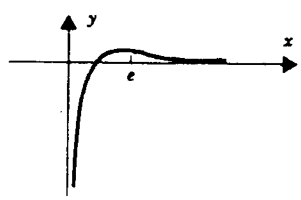
horizontal asymptote x-axis

47.



horizontal asymptote x-axis

48.



vertical asymptote y-axis
horizontal asymptote x-axis

49. $\frac{b}{a}\sqrt{x^2 - a^2} - \frac{b}{a}x = \frac{\sqrt{x^2 - a^2} + x}{\sqrt{x^2 - a^2} + x}\left(\frac{b}{a}\right)\left(\sqrt{x^2 - a^2} - x\right) = \frac{-ab}{\sqrt{x^2 - a^2} + x} \to 0 \quad \text{as} \quad x \to \infty$

50. $\cosh x - \sinh x = \frac{1}{2}(e^x + e^{-x}) - \frac{1}{2}(e^x - e^{-x}) = e^{-x} \to 0, \quad \text{as } x \to \infty$

51. for instance, $f(x) = x^2 + \frac{(x-1)(x-2)}{x^3}$

52. for instance, $F(x) = x + \frac{\sin x}{x}$

53. $\lim_{x\to 0^-} -\dfrac{2x}{\cos x} \neq \lim_{x\to 0^-} \dfrac{2}{-\sin x}$. L'Hospital's rule does not apply here since $\lim_{x\to 0^-} \cos x = 1$.

54. Let k be the nonnegative integer such that $k < \alpha \le k+1$. Then

$$\lim_{x\to\infty} \frac{x^\alpha}{e^x} \overset{\star}{=} \lim_{x\to\infty} \frac{\alpha\, x^{\alpha-1}}{e^x} \overset{\star}{=} \cdots \overset{\star}{=} \lim_{x\to\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k)\, x^{\alpha-(k+1)}}{e^x} = 0$$

55. (a) Let S be the set of positive integers for which the statement is true. Since $\lim_{x\to\infty} \dfrac{\ln x}{x} = 0$, $1 \in S$. Assume that $k \in S$. By L'Hospital's rule,

$$\lim_{x\to\infty} \frac{(\ln x)^{k+1}}{x} \overset{\star}{=} \lim_{x\to\infty} \frac{(k+1)(\ln x)^k}{x} = 0 \quad (\text{since} \quad k \in S).$$

Thus $k+1 \in S$, and S is the set of positive integers.

(b) Choose any positive number α. Let $k-1$ and k be positive integers such that $k-1 \le \alpha \le k$. Then, for $x > e$,

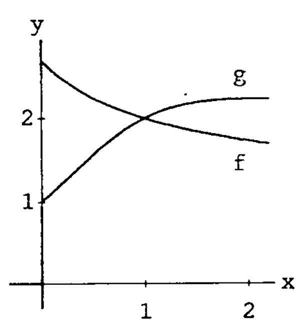
$$\frac{(\ln x)^{k-1}}{x} \le \frac{(\ln x)^\alpha}{x} \le \frac{(\ln x)^k}{x}$$

and the result follows by the pinching theorem.

56. (a) $\lim_{k\to 0^+} v(t) = \lim_{k\to 0^+} \dfrac{mg\left(1 - e^{-(k/m)t}\right)}{k} \overset{\star}{=} \lim_{k\to 0^+} \dfrac{gte^{-(k/m)t}}{1} = gt$

(b) $\dfrac{dv}{dt} = gt \implies v(t) = gt + C; \quad v(0) = 0 \implies C = 0 \quad \text{and} \quad v(t) = gt.$

57. (a)

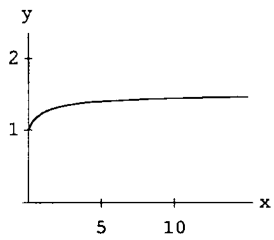


$\lim_{x\to 0^+} \left(1+x^2\right)^{1/x} = 1.$

(b) $\lim_{x\to 0^+} \left(1+x^2\right)^{1/x} = 1.$ since

$$\lim_{x\to 0^+} \ln\left[\left(1+x^2\right)^{1/x}\right] = \lim_{x\to 0^+} \frac{\ln\left(1+x^2\right)}{x} \overset{\star}{=} \lim_{x\to 0^+} \frac{2x}{(1+x^2)} = 0$$

58. (a)



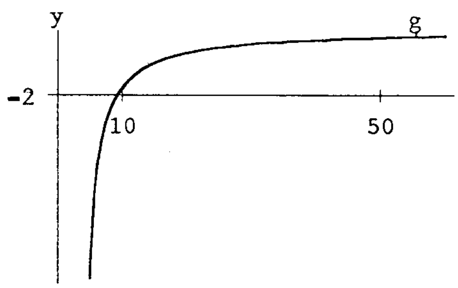
$\lim_{x\to\infty} f(x) \cong 1.5$

(b)

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left[\sqrt{x^2+3x+1} - x\right]$$

$$= \lim_{x\to\infty} \frac{3x+1}{\sqrt{x^2+3x+1}+x} = \frac{3x+1}{x\sqrt{1+(3/x)+1/x^2)}+x} = \frac{3}{2}$$

59. (a)



$\lim_{x\to\infty} g(x) \cong -1.7.$

(b)

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \left[\sqrt[3]{x^3-5x^2+2x+1} - x\right]$$

$$= \lim_{x\to\infty} \frac{-5x^2+2x+1}{\left(\sqrt[3]{x^3-5x^2+2x+1}\right)^2 + x\sqrt[3]{x^3-5x^2+2x+1} + x^2}$$

$$= -\frac{5}{3} \cong -1.667$$

62.

$$[P(x)]^{1/n} - x = [P(x)]^{1/n} - x \cdot \frac{[P(x)]^{(n-1)/n} + x[P(x)]^{(n-2)/n} + \cdots + x^{n-2}[P(x)]^{1/n} + x^{n-1}}{[P(x)]^{(n-1)/n} + x[P(x)]^{(n-2)/n} + \cdots + x^{n-2}[P(x)]^{1/n} + x^{n-1}}$$

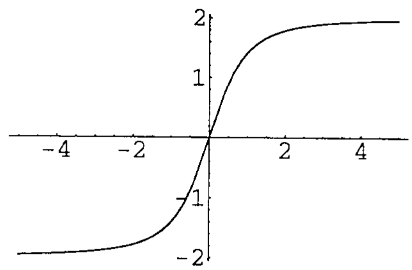
$$= \frac{P(x) - x^n}{[P(x)]^{(n-1)/n} + x[P(x)]^{(n-2)/n} + \cdots + x^{n-2}[P(x)]^{1/n} + x^{n-1}}$$

$$= \frac{b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n}{[P(x)]^{(n-1)/n} + x[P(x)]^{(n-2)/n} + \cdots + x^{n-2}[P(x)]^{1/n} + x^{n-1}} \to \frac{b_1}{n} \quad \text{as } x \to \infty$$

PROJECT 10.6

1. (a) $\displaystyle \lim_{x\to\infty} \frac{2x}{\sqrt{x^2+1}} \overset{*}{=} \lim_{x\to\infty} \frac{2}{\frac{x}{\sqrt{x^2+1}}} = \lim_{x\to\infty} \frac{2\sqrt{x^2+1}}{x} \overset{*}{=} \lim_{x\to\infty} \frac{\frac{2x}{\sqrt{x^2+1}}}{1}$ and so on.

(b)



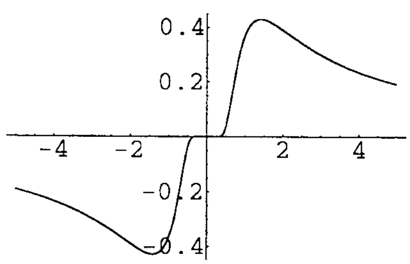
(c) $\lim_{x\to\infty} \frac{2x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{2x}{x\sqrt{1+(1/x^2)}} = \lim_{x\to\infty} \frac{2}{\sqrt{1+(1/x^2)}} = 2$

2. (a) L'Hospital's rule applied to the given limit results in $\lim_{x\to 0} \frac{2\,e^{-1/x^2}}{x^3}$.

(b) Rewrite the quotient as $\frac{1/x}{e^{1/x^2}}$. Then

$$\lim_{x\to 0} \frac{1/x}{e^{1/x^2}} \stackrel{*}{=} \lim_{x\to\infty} \frac{-1/x^2}{(-2/x^3)\,e^{1/x^2}} = \lim_{x\to 0} \frac{x}{2\,e^{1/x^2}} = 0$$

(c)



3. $\lim_{x\to 0} \frac{f(x)-f(0)}{x} = \lim_{x\to 0} \frac{e^{-1/x^2}}{x} = 0.$ Therefore, $f'(0) = 0$.

SECTION 10.7

1. $1: \quad \int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2} = \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty} \left[1 - \frac{1}{b}\right] = 1$

2. $\frac{\pi}{2}: \quad \int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty} \tan^{-1} b = \frac{\pi}{2}$

3. $\frac{\pi}{4}: \quad \int_0^\infty \frac{dx}{4+x^2} = \lim_{b\to\infty} \int_0^b \frac{dx}{4+x^2} = \lim_{b\to\infty} \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^b = \lim_{b\to\infty} \frac{1}{2}\tan^{-1}\left(\frac{b}{2}\right) = \frac{\pi}{4}$

4. $\frac{1}{p}: \quad \int_0^\infty e^{-px}\,dx = \lim_{b\to 0} \left(-\frac{e^{-px}}{p} + \frac{1}{p}\right) = \frac{1}{p}$

5. diverges: $\int_0^\infty e^{px}dx = \lim_{b\to\infty}\int_0^b e^{px}\,dx = \lim_{b\to\infty}\left[\frac{1}{p}\,e^{px}\right]_0^b = \lim_{b\to\infty}\frac{1}{p}\left(e^{pb}-1\right) = \infty$

6. 2 : $\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+}\int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+}(2-2\sqrt{a}) = 2$

7. 6: $\int_0^8 \frac{dx}{x^{2/3}} = \lim_{a\to 0^+}\int_a^8 x^{-2/3}\,dx = \lim_{a\to 0^+}\left[3x^{1/3}\right]_a^8 = \lim_{a\to 0^+}\left[6-3a^{1/3}\right] = 6$

8. diverges: $\int_0^1 \frac{dx}{x^2} = \lim_{a\to 0^+}\int_a^1 \frac{dx}{x^2} = \lim_{a\to 0^+}\left(-1+\frac{1}{a}\right) = \infty$

9. $\frac{\pi}{2}$: $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b\to 1^-}\int_0^b \frac{dx}{\sqrt{1-x^2}} = \lim_{b\to 1^-}\sin^{-1} b = \frac{\pi}{2}$

10. 2: $\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{a\to 0^+}\int_a^1 \frac{dx}{\sqrt{1-x}} = \lim_{a\to 0^+}\left[-2\sqrt{1-x}\right]_a^1 = \lim_{a\to 0^+}(0+2\sqrt{1-a}) = 2$

11. 2:

$$\int_0^2 \frac{x}{\sqrt{4-x^2}}\,dx = \lim_{b\to 2^-}\int_0^b x\left(4-x^2\right)^{-1/2}\,dx = \lim_{b\to 2^-}\left[-\left(4-x^2\right)^{1/2}\right]_0^b$$

$$= \lim_{b\to 2^-}\left(2-\sqrt{4-b^2}\right) = 2$$

12. $\frac{\pi}{2}$: $\int_0^a \frac{dx}{\sqrt{a^2-x^2}} = \lim_{b\to a^-}\int_0^b \frac{dx}{\sqrt{a^2-x^2}} = \lim_{b\to a^-}\left[\sin^{-1}\left(\frac{b}{a}\right)-\sin^{-1}(0)\right] = \frac{\pi}{2}$

13. diverges;

$$\int_e^\infty \frac{\ln x}{x}\,dx = \lim_{b\to\infty}\int_e^b \frac{\ln x}{x}\,dx = \lim_{b\to\infty}\left[\frac{1}{2}\left(\ln x\right)^2\right]_e^b$$

$$= \lim_{b\to\infty}\left[\frac{1}{2}\left(\ln b\right)^2-\frac{1}{2}\right] = \infty$$

14. diverges: $\int_e^\infty \frac{dx}{x\ln x} = \lim_{b\to\infty}\int_e^b \frac{dx}{x\ln x} = \lim_{b\to\infty}\left[\ln(\ln x)\right]_e^b = \infty$

15. $-\frac{1}{4}$:

$$\int_0^1 x\ln x\,dx = \lim_{a\to 0^+}\int_a^1 x\ln x\,dx = \lim_{a\to 0^+}\left[\frac{1}{2}x^2\ln x-\frac{1}{4}x^2\right]_a^1$$

(by parts)

$$= \lim_{a\to 0^+}\left[\frac{1}{4}a^2-\frac{1}{2}a^2\ln a-\frac{1}{4}\right] = -\frac{1}{4}$$

Note: $\lim_{t\to 0^+} t^2\ln t = \lim_{t\to 0^+}\frac{\ln t}{1/t^2} \overset{\star}{=} \lim_{t\to 0^+}\frac{1/t}{-2/t^3} = -\frac{1}{2}\lim_{t\to 0^+} t^2 = 0.$

16. 1: $$\int_e^\infty \frac{dx}{x(\ln x)^2} = \lim_{b\to\infty}\int_e^b \frac{dx}{x(\ln x)^2} = \lim_{b\to\infty}\left[-\frac{1}{\ln x}\right]_e^b = \lim_{b\to\infty}\left[-\frac{1}{\ln b}+1\right] = 1$$

17. π : $$\int_{-\infty}^\infty \frac{dx}{1+x^2} = \lim_{a\to-\infty}\int_a^0 \frac{dx}{1+x^2} + \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2}$$

$$= \lim_{a\to-\infty}\left[\tan^{-1}x\right]_a^0 + \lim_{b\to\infty}\left[\tan^{-1}x\right]_0^b = -\left(-\frac{\pi}{2}\right)+\frac{\pi}{2} = \pi$$

18. $\frac{\ln 3}{2}$: $$\int_2^\infty \frac{dx}{x^2-1} = \lim_{b\to\infty}\int_2^b \frac{dx}{x^2-1} = \lim_{b\to\infty}\left[\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|\right]_2^b$$

$$= \lim_{b\to\infty}\left[\frac{1}{2}\ln\left(\frac{b-1}{b+1}\right)+\frac{1}{2}\ln 3\right] = \frac{1}{2}\ln 3$$

19. diverges: $$\int_{-\infty}^\infty \frac{dx}{x^2} = \lim_{a\to-\infty}\int_a^{-1}\frac{dx}{x^2} + \lim_{b\to 0^-}\int_{-1}^b \frac{dx}{x^2} + \lim_{c\to 0^+}\int_c^1 \frac{dx}{x^2} + \lim_{d\to\infty}\int_1^d \frac{dx}{x^2};$$

and, $$\lim_{c\to 0^+}\int_c^1 \frac{dx}{x^2} = \lim_{c\to 0^+}\left[-\frac{1}{x}\right]_c^1 = \lim_{c\to 0^+}\left[\frac{1}{c}-1\right] = \infty$$

20. 2: $$\int_{1/3}^3 \frac{dx}{\sqrt[3]{3x-1}} = \lim_{a\to\frac{1}{3}^+}\int_a^3 \frac{dx}{(3x-1)^{1/3}} = \lim_{a\to\frac{1}{3}^+}\left[\frac{3(3x-1)^{2/3}}{2\cdot 3}\right]_a^3$$

$$= \lim_{a\to\frac{1}{3}^+}\left[\frac{8^{2/3}}{2}-\frac{(3a-1)^{2/3}}{2}\right] = 2$$

21. $\ln 2$; $$\int_1^\infty \frac{dx}{x(x+1)} = \lim_{b\to\infty}\int_1^b \left[\frac{1}{x}-\frac{1}{x+1}\right]dx$$

$$= \lim_{b\to\infty}\left[\ln\left(\frac{x}{x+1}\right)\right]_1^b = \lim_{b\to\infty}\left[\ln\left(\frac{b}{b+1}\right)-\ln\left(\frac{1}{2}\right)\right]$$

$$= 0-\ln\tfrac{1}{2} = \ln 2$$

22. -1 : $$\int_{-\infty}^0 xe^x\,dx = \lim_{a\to-\infty}\int_a^0 xe^x\,dx = \lim_{a\to-\infty}\left[xe^x-e^x\right]_a^0 = \lim_{a\to-\infty}\left[-1-ae^a-e^a\right] = -1$$

23. 4: $$\int_3^5 \frac{x}{\sqrt{x^2-9}}\,dx = \lim_{a\to 3^-}\int_a^5 x\left(x^2-9\right)^{-1/2}dx$$

$$= \lim_{a\to 3^-}\left[\left(x^2-9\right)^{1/2}\right]_a^5 = \lim_{a\to 3^-}\left[4-\left(a^2-9\right)^{1/2}\right] = 4$$

24.

$$\int_1^4 \frac{dx}{x^2-4} = \lim_{b\to 2^-}\int_1^b \frac{dx}{x^2-4} + \lim_{a\to 2^+}\int_a^4 \frac{dx}{x^2-4} = \lim_{b\to 2^-}\left[\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|\right]_1^b + \lim_{a\to 2^+}\left[\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|\right]_a^4$$

$$= \lim_{b\to 2^-}\left(\frac{1}{4}\ln\left|\frac{b-2}{b+2}\right|-\frac{1}{4}\ln\frac{1}{3}\right) + \lim_{a\to 2^+}\left(\frac{1}{4}\ln\frac{2}{6}-\frac{1}{4}\ln\left|\frac{a-2}{a+2}\right|\right) = \infty \quad \text{diverges}$$

25. $\int_{-3}^{3} \frac{dx}{x(x+1)}$ diverges since $\int_{0}^{3} \frac{dx}{x(x+1)}$ diverges:

$$\int_0^3 \frac{dx}{x(x+1)} = \int_0^3 \left(\frac{1}{x} - \frac{1}{x+1}\right) dx = \lim_{a\to 0^+} \left[\ln|x| - \ln|x+1|\right]_a^3$$
$$= \lim_{a\to 0^+} \left[\ln 3 - \ln 4 - \ln a + \ln(a+1)\right] = \infty.$$

26. $\frac{1}{4}$: $\int_1^\infty \frac{x}{(1+x^2)^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{x}{(1+x^2)^2}\,dx = \lim_{b\to\infty}\left[\frac{-1}{2(1+x^2)}\right]_1^b = \lim_{b\to\infty}\left[\frac{-1}{2(1+b^2)} + \frac{1}{4}\right] = \frac{1}{4}$

27. $\int_{-3}^{1} \frac{dx}{x^2-4}$ diverges since $\int_{-2}^{1} \frac{dx}{x^2-4}$ diverges:

$$\int_{-2}^1 \frac{dx}{x^2-4} = \int_{-2}^1 \frac{1}{4}\left[\frac{1}{x-2} - \frac{1}{x+2}\right] dx$$
$$= \lim_{a\to -2^+} \left[\frac{1}{4}\left(\ln|x-2| - \ln|x+2|\right)\right]_a^1$$
$$= \lim_{a\to -2^+} \frac{1}{4}\left[-\ln 3 - \ln|a-2| + \ln|a+2|\right] = -\infty.$$

28. diverges: $\int_0^\infty \sinh x\,dx = \lim_{b\to\infty} (\cosh b - 1) = \infty$

29. diverges: $\int_0^\infty \cosh x\,dx = \lim_{b\to\infty} \int_0^b \cosh x\,dx = \lim_{b\to\infty} \left[\sinh x\right]_0^b = \infty$

30. Since $\int_1^2 \frac{dx}{x^2-5x+6} = \lim_{b\to 2^-} \int_1^b \frac{dx}{(x-2)(x-3)} = \lim_{b\to 2^-} \left[\ln\left|\frac{x-3}{x-2}\right|\right]_1^b = \lim_{b\to 2^-}\left(\ln\left|\frac{b-3}{b-2}\right| - \ln 2\right)$

diverges, so does $\int_1^4 \frac{dx}{x^2-5x+6}$

31. $\frac{1}{2}$: $\int_0^\infty e^{-x}\sin x\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\sin x\,dx = \lim_{b\to\infty} -\frac{1}{2}\left[e^{-x}\cos x + e^{-x}\sin x\right]_0^b$

(by parts)

$$= \lim_{b\to\infty} \frac{1}{2}\left[1 - e^{-b}\cos b - e^{-b}\sin b\right] = \frac{1}{2}$$

32. diverges: $\int_0^\infty \cos^2 x\,dx = \lim_{b\to\infty}\left(\frac{b}{2} + \frac{\sin 2b}{4} - \frac{1}{2}\right) = \infty$

33. $2e-2$: $\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\left[2\,e^{\sqrt{x}}\right]_a^1 = 2(e-1)$

34. 2: $\int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\,dx = \lim_{a\to 0^+}\int_a^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\,dx = \lim_{a\to 0^+}\left[2\sqrt{\sin x}\right]_a^{\pi/2} = 2$

35. $\int_0^1 \sin^{-1} x\,dx = \left[x\,\sin^{-1} x\right]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2} - \lim_{a\to 1^-}\int_0^a \frac{x}{\sqrt{1-x^2}}\,dx$

(by parts)

Now, $\displaystyle\int_0^a \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int_1^{1-a^2} \frac{1}{\sqrt{u}}\,du = \left[-\sqrt{u}\right]_1^{1-a^2} = 1-\sqrt{1-a^2}$

$u = 1-x^2$

Thus, $\displaystyle\int_0^1 \sin^{-1} x\,dx = \frac{\pi}{2} - \lim_{a\to 1^-}\left(1-\sqrt{1-a^2}\right) = \frac{\pi}{2} - 1.$

36. (a) For any r, we can find k such that $x^r < e^{x/2}$ for $x \geq k$

(Since $e^{x/2}$ grows faster than any power of x).

Then $\displaystyle\int_0^\infty x^r e^{-x}\,dx < \int_0^k x^r e^{-x}\,dx + \int_k^\infty e^{-x/2}\,dx,$ which converges.

Thus $\displaystyle\int_0^\infty x^r e^{-x}\,dx$ converges for all r

(b) For $n = 1$: $\displaystyle\int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\left[-xe^{-x} - e^{-x}\right]_0^b = \lim_{b\to\infty}\left[-be^{-b} - e^{-b} + 1\right] = 1$

Assume true for n. $\displaystyle\int_0^\infty x^{n+1}e^{-x}\,dx = \lim_{b\to\infty}\left(\left[-x^{n+1}e^{-x}\right]_0^b + (n+1)\int_0^b x^n e^{-x}\,dx\right)$

$$= \lim_{b\to\infty}(-b^{n+1}e^{-b}) + (n+1)\int_0^\infty x^n e^{-x}\,dx = 0 + (n+1)n! = (n+1)!$$

37.

$$\int_0^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx = \int_0^1 \frac{1}{\sqrt{x}\,(1+x)}\,dx + \int_1^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx$$
$$= \lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}\,(1+x)}\,dx + \lim_{b\to\infty}\int_1^b \frac{1}{\sqrt{x}\,(1+x)}\,dx$$

Now, $\displaystyle\int \frac{1}{\sqrt{x}\,(1+x)}\,dx = \int \frac{2}{1+u^2}\,du = 2\tan^{-1} u + C = 2\tan^{-1}\sqrt{x} + C.$

$u = \sqrt{x}$

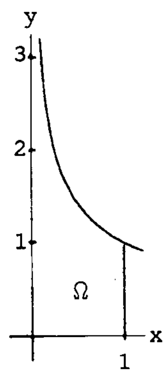
Therefore, $\displaystyle\lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}\,(1+x)}\,dx = \lim_{a\to 0^+}\left[2\tan^{-1}\sqrt{x}\right]_a^1 = \lim_{a\to 0^+} 2\left[\pi/4 - \tan^{-1} a)\right] = \frac{\pi}{2}$

and $\displaystyle\lim_{b\to\infty}\int_1^b \frac{1}{\sqrt{x}\,(1+x)}\,dx = \lim_{b\to\infty}\left[2\tan^{-1}\sqrt{x}\right]_1^b = \lim_{b\to\infty} 2\left[\tan^{-1} b - \pi/4\right] = \frac{\pi}{2}.$

Thus, $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx = \pi.$

38.
$$\int_1^\infty \frac{1}{x\sqrt{x^2-1}}\,dx = \lim_{a\to1^+}\int_a^2 \frac{1}{x\sqrt{x^2-1}}\,dx + \lim_{b\to\infty}\int_2^\infty \frac{1}{x\sqrt{x^2-1}}\,dx$$
$$= \lim_{a\to1^+}\left[\sec^{-1}x\right]_a^2 + \lim_{b\to\infty}\left[\sec^{-1}x\right]_2^\infty$$
$$= \lim_{a\to1^+}(\sec^{-1}2 - \sec^{-1}a) + \lim_{b\to\infty}(\sec^{-1}b - \sec^{-1}2)$$
$$= (\sec^{-1}2 - 0) + (\tfrac{\pi}{2} - \sec^{-1}2) = \frac{\pi}{2}$$

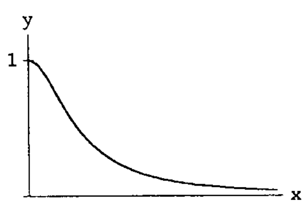
39. (a)



(b)
$$A = \int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to0^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx$$
$$= \lim_{a\to0^+}\left[2\sqrt{x}\right]_a^1 = 2$$

(c) $V = \int_0^1 \pi\left(\frac{1}{\sqrt{x}}\right)^2 dx = \pi\int_0^1 \frac{1}{x}\,dx = \pi\lim_{a\to0^+}\int_a^1 \frac{1}{x}\,dx = \pi\lim_{a\to0^+}\left[\ln x\right]_a^1$ diverges

40. (a)

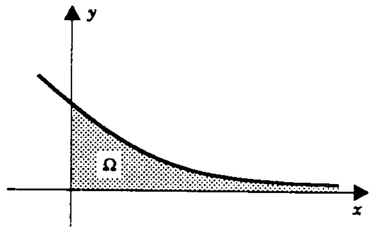


(b) $A = \int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\tan^{-1}b = \frac{\pi}{2}$

(c)
$$V_x = \int_0^\infty \pi\cdot\frac{1}{(1+x^2)^2}\,dx = \lim_{b\to\infty}\frac{\pi}{2}\left[\tan^{-1}x + \frac{x}{1+x^2}\right]_0^b$$
$$= \lim_{b\to\infty}\frac{\pi}{2}\left(\tan^{-1}b + \frac{b}{1+b^2} - 0\right) = \frac{\pi^2}{4}$$

(d)
$$V_y = \int_0^\infty \frac{2\pi x}{1+x^2}\,dx = \lim_{b\to\infty}\pi\left[\ln(1+x^2)\right]_0^b = \infty$$

41. (a)



(b) $A = \int_0^\infty e^{-x}\,dx = 1$

(c) $V_x = \int_0^\infty \pi e^{-2x}\,dx = \pi/2$

(d) $$V_y = \int_0^\infty 2\pi x e^{-x}\,dx = \lim_{b\to\infty} \int_0^b 2\pi x e^{-x}\,dx = \lim_{b\to\infty} \left[2\pi(-x-1)e^{-x}\right]_0^b$$
(by parts)
$$= 2\pi\left(1 - \lim_{b\to\infty} \frac{b+1}{e^b}\right) = 2\pi(1-0) = 2\pi$$

(e) $$A = \int_0^\infty 2\pi e^{-x}\sqrt{1+e^{-2x}}\,dx = \lim_{b\to\infty} \int_0^b 2\pi e^{-x}\sqrt{1+e^{-2x}}\,dx$$
$$\int_0^b 2\pi e^{-x}\sqrt{1+e^{-2x}}\,dx = -2\pi \int_1^{e^{-b}} \sqrt{1+u^2}\,du$$
$u = e^{-x}$
$$= -\pi\left[u\sqrt{1+u^2} + \ln\left(1+\sqrt{1+u^2}\right)\right]_1^{e^{-b}}$$
$$= \pi\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right) - e^{-b}\sqrt{1+e^{-2b}} - \ln\left(1+\sqrt{1+e^{-2b}}\right)\right]$$

Taking the limit of this last expression as $b \to \infty$, we have
$$A = \pi\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right].$$

42. $$\overline{x}A = \int_0^\infty x e^{-x}\,dx = 1, \quad \overline{x} = \frac{\overline{x}A}{A} = 1$$
$$\overline{y}A = \int_0^\infty \frac{1}{2}e^{-2x}\,dx = \frac{1}{4}, \quad \overline{y} = \frac{\overline{y}A}{A} = \frac{1}{4}$$
Yes: $2\pi\overline{x}A = 2\pi = V_y, \quad 2\pi\overline{y}A = \frac{1}{2}\pi = V_x$

43. (a) The interval $[0,1]$ causes no problem. For $x \ge 1$, $e^{-x^2} \le e^{-x}$ and $\int_1^\infty e^{-x}\,dx$ is finite.

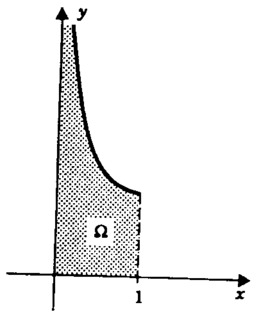
(b) $$V_y = \int_0^\infty 2\pi x e^{-x^2}\,dx = \lim_{b\to\infty} \int_0^b 2\pi x e^{-x^2}\,dx = \lim_{b\to\infty} \pi\left[-e^{-x^2}\right]_0^b = \lim_{b\to\infty} \pi\left(1 - e^{-b^2}\right) = \pi$$

44. (a) $$A = \int_1^\infty \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx = \frac{1}{2}\ln 2$$

(b) $$V_x = \int_1^\infty \left(\frac{1}{x} - \frac{x}{x^2+1}\right)^2 dx < \int_1^\infty \frac{dx}{x^2} \quad \text{finite}$$

(c) $$V_y = \int_1^\infty 2\pi x\left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx = \frac{1}{2}\pi^2$$

45. (a)



(b) $A = \lim_{a\to 0^+} \int_a^1 x^{-1/4}\,dx = \lim_{a\to 0^+} \left[\frac{4}{3}x^{3/4}\right]_a^1 = \frac{4}{3}$

(c) $V_x = \lim_{a\to 0^+} \int_a^1 \pi x^{-1/2}\,dx = \lim_{a\to 0^+} \left[2\pi x^{1/2}\right]_a^1 = 2\pi$

(d) $V_y = \lim_{a\to 0^+} \int_a^1 2\pi x^{3/4}\,dx = \lim_{a\to 0^+} \left[\frac{8\pi}{7}x^{7/4}\right]_a^1 = \frac{8}{7}\pi$

46. (i) Suppose that $\int_a^\infty g(x)\,dx = L$. Since $f(x) \geq 0$ for $x \in [a, \infty)$, $\int_a^x f(t)\,dt$ is increasing. Therefore it is sufficient to show that $\int_a^x f(t)\,dt$ is bounded above. For any number $M \geq a$, we have

$$\int_a^M f(x)\,dx \leq \int_a^M g(x)\,dx \leq \int_a^\infty g(x)\,dx = L$$

Therefore, $\int_a^x f(t)\,dt$ is bounded and $\int_a^\infty f(x)\,dx$ converges.

(ii) If $\int_0^\infty f(x)\,dx$ diverges, then $\int_0^\infty g(x)\,dx$ can not converge, by (i)

47. converges by comparison with $\int_1^\infty \frac{dx}{x^{3/2}}$

48. converges by comparison with $\int_1^\infty e^{-x}\,dx$

49. diverges since for x large the integrand is greater than $\frac{1}{x}$ and $\int_1^\infty \frac{dx}{x}$ diverges

50. Converges by comparison with $\int_\pi^\infty \frac{dx}{x^2}$

51. converges by comparison with $\int_1^\infty \frac{dx}{x^{3/2}}$

52. Diverges by comparison with $\int_e^\infty \frac{dx}{(x+1)\ln(x+1)}$

53. $r(\theta) = ae^{c\theta}, \quad r'(\theta) = ace^{c\theta}$

$$L = \int_{-\infty}^{\theta_1} \sqrt{a^2e^{2c\theta} + a^2c^2e^{2c\theta}}\, d\theta$$

(9.9.3)

$$= \left(a\sqrt{1+c^2}\right)\left(\lim_{b\to-\infty}\int_b^{\theta_1} e^{c\theta}\, d\theta\right)$$

$$= \left(a\sqrt{1+c^2}\right)\left(\lim_{b\to-\infty}\left[\frac{e^{c\theta}}{c}\right]_b^{\theta_1}\right)$$

$$= \left(\frac{a\sqrt{1+c^2}}{c}\right)\left(\lim_{b\to-\infty}\left[e^{c\theta_1} - e^{cb}\right]\right) = \left(\frac{a\sqrt{1+c^2}}{c}\right)e^{c\theta_1}$$

54. For all real t, $-\dfrac{t^2}{2} < t+1$. Therefore $\displaystyle\int_{-\infty}^{x} e^{-t^2/2}\, dt$ converges by comparison with $\displaystyle\int_{-\infty}^{x} e^{t+1}\, dt$.

55. $$F(s) = \int_0^\infty e^{-sx}\cdot 1\, dx = \lim_{b\to\infty}\int_0^b e^{-sx}\, dx = \lim_{b\to\infty}\left[-\frac{1}{s}e^{-sx}\right]_0^b = \frac{1}{s} \qquad \text{provided } s > 0.$$

Thus, $F(s) = \dfrac{1}{s}; \qquad \text{dom}(F) = (0,\infty).$

56. $$F(s) = \int_0^\infty xe^{-sx}\, dx = \lim_{b\to\infty}\left[\frac{xe^{-sx}}{s} - \frac{e^{-sx}}{s^2}\right]_0^b = \lim_{b\to\infty}\left(\frac{be^{-sb}}{s} - \frac{e^{-sb}}{s^2} + \frac{1}{s^2}\right)$$

$$= \frac{1}{s^2} \quad \text{if } s > 0, \quad \text{diverges for } s \le 0, \quad \text{so dom}(F) = (0,\infty)$$

57. $$F(s) = \int_0^\infty e^{-sx}\cos 2x\, dx = \lim_{b\to\infty}\int_0^b e^{-sx}\cos 2x\, dx$$

Using integration by parts $\displaystyle\int e^{-sx}\cos 2x\, dx = \frac{4}{s^2+4}\left[\frac{1}{2}e^{-sx}\sin 2x - \frac{s}{4}e^{-sx}\cos 2x\right] + C.$

Therefore,

$$F(s) = \lim_{b\to\infty}\frac{4}{s^2+4}\left[\frac{1}{2}e^{-sx}\sin 2x - \frac{s}{4}e^{-sx}\cos 2x\right]_0^b$$

$$= \frac{4}{s^2+4}\lim_{b\to\infty}\left[\frac{1}{2}e^{-sb}\sin 2b - \frac{s}{4}e^{-sb}\cos 2b + \frac{s}{4}\right] = \frac{4}{s^2+4}\cdot\frac{s}{4} = \frac{s}{s^2+4} \qquad \text{provided } s > 0.$$

Thus, $F(s) = \dfrac{s}{s^2+4}; \qquad \text{dom}(F) = (0,\infty).$

58. $$F(s)=\int_0^\infty e^{ax}e^{-sx}\,dx=\int_0^\infty e^{(a-s)x}\,dx=\lim_{b\to\infty}\left[\frac{e^{(a-s)b}}{a-s}-\frac{1}{a-s}\right]$$

$$=\frac{1}{s-a}\quad\text{if } s>a,\quad \text{diverges if } s\le a.\quad \text{so dom}F=(a,\infty)$$

59. The function f is nonnegative on $(-\infty,\infty)$ and

$$\int_{-\infty}^{\infty} f(x)\,dx=\int_{-\infty}^{0} 0\,dx+\int_0^\infty \frac{6x}{(1+3x^2)^2}\,dx=\int_0^\infty \frac{6x}{(1+3x^2)^2}\,dx$$

Now, $\displaystyle\int \frac{6x}{(1+3x^2)^2}\,dx=-\frac{1}{1+3x^2}+C.$

Therefore,

$$\int_{-\infty}^{\infty} f(x)\,dx=\lim_{b\to\infty}\left[-\frac{1}{1+3x^2}\right]_0^b=\lim_{b\to\infty}\left(1-\frac{1}{1+3b^2}\right)=1.$$

60. F is nonnegative, and

$$\int_{-\infty}^{\infty} f(x)\,dx=\int_0^\infty ke^{-kx}\,dx=\lim_{b\to\infty}\left[-e^{-kx}\right]_0^b=\lim_{b\to\infty}\left(-e^{-kb}+1\right)=1$$

So, f is a probability density function.

61. $$\mu=\int_{-\infty}^{\infty} xf(x)\,dx=\int_{-\infty}^{0} 0\,dx+\int_0^\infty kxe^{-kx}\,dx=\lim_{b\to\infty}\int_0^b kxe^{-kx}\,dx$$

Using integration by parts, $\displaystyle\int kxe^{-kx}\,dx=-xe^{-kx}-\frac{1}{k}e^{-kx}+C.$

Therefore,

$$\mu=\int_{-\infty}^{\infty} xf(x)\,dx=\lim_{b\to\infty}\left[-xe^{-kx}-\frac{1}{k}e^{-kx}\right]_0^b=\lim_{b\to\infty}\left[-be^{-kb}-\frac{1}{k}e^{-kb}+\frac{1}{k}\right]=\frac{1}{k}$$

62. $$\sigma=\int_{-\infty}^{\infty}(x-u)^2f(x)\,dx=\int_0^\infty\left(x-\frac{1}{k}\right)^2ke^{-kx}\,dx$$

$$=\int_0^\infty kx^2e^{-kx}\,dx-2\int_0^\infty xe^{-kx}\,dx+\frac{1}{k}\int_0^\infty e^{-kx}\,dx=\frac{1}{k^2}$$

63. Observe that $F(t) = \int_1^t f(x)\,dx$ is continuous and increasing, that $a_n = \int_1^n f(x)\,dx$ is increasing, and that $(*)\ \ a_n \le \int_1^t f(x)\,dx \le a_{n+1}$ for $t \in [n, n+1]$.

If $\int_1^\infty f(x)\,dx$ converges, then F, being continuous, is bounded and, by $(*)$, $\{a_n\}$ is bounded and therefore convergent. If $\{a_n\}$ converges, then $\{a_n\}$ is bounded and, by $(*)$, F is bounded. Being increasing, F is also convergent; i.e., $\int_1^\infty f(x)\,dx$ converges.

CHAPTER 11

SECTION 11.1

1. $1+4+7=12$

2. $2+5+8+11=26$

3. $1+2+4+8=15$

4. $2-4+8-16=-10$

5. $-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}=-\frac{2}{15}$

6. $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}=\frac{13}{27}$

7. $\sum_{n=1}^{11}(2n-1)$

8. $\sum_{k=1}^{10}(-1)^{k+1}(2k-1)$

9. $\sum_{k=1}^{35}k(k+1)$

10. $\sum_{k=1}^{n} m_k\,\Delta x_k$

11. $\sum_{k=1}^{n} M_k\Delta x_k$

12. $\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$

13. $\sum_{k=3}^{10}\frac{1}{2^k},\quad \sum_{i=0}^{7}\frac{1}{2^{i+3}}$

14. $\sum_{k=3}^{10}\frac{k^k}{k!},\quad \sum_{i=0}^{7}\frac{(i+3)^{i+3}}{(i+3)!}$

15. $\sum_{k=3}^{10}(-1)^{k+1}\frac{k}{k+1},\quad \sum_{i=0}^{7}(-1)^i\frac{i+3}{i+4}$

16. $\sum_{k=3}^{10}\frac{1}{2k-3},\quad \sum_{i=0}^{7}\frac{1}{2i+3}$

17. Set $k=n+3$. Then $n=-1$ when $k=2$ and $n=7$ when $k=10$.

$$\sum_{k=2}^{10}\frac{k}{k^2+1}=\sum_{n=-1}^{7}\frac{n+3}{(n+3)^2+1}=\sum_{n=-1}^{7}\frac{n+3}{n^2+6n+10}$$

18. $\sum_{n=2}^{12}\frac{(-1)^n}{n-1}=\sum_{k=1}^{11}\frac{(-1)^{k+1}}{k+1-1}=\sum_{k=1}^{11}\frac{(-1)^{k+1}}{k}$

19. Set $k=n-3$. Then $n=7$ when $k=4$ and $n=28$ when $k=25$.

$$\sum_{k=4}^{25}\frac{1}{k^2-9}=\sum_{n=7}^{28}\frac{1}{(n-3)^2-9}=\sum_{n=7}^{28}\frac{1}{n^2-6n}$$

20. $\sum_{k=0}^{15}\frac{3^{2k}}{k!}=\sum_{n=-2}^{13}\frac{3^{2(n+2)}}{(n+2)!}=81\sum_{n=-2}^{13}\frac{3^{2n}}{(n+2)!}$

21. $\sum_{k=1}^{10}(2k+3)=2\sum_{k=1}^{10}k+\sum_{k=1}^{10}3=2\cdot\frac{(10)(11)}{2}+3\cdot 10=140$

22. $\sum_{k=1}^{10}(2k^2+3k)=2\sum_{k=1}^{10}k^2+3\sum_{k=1}^{10}k=2\cdot\frac{1}{6}\cdot 10(10+1)(2\cdot 10+1)+3\cdot\frac{1}{2}\cdot 10(10+1)=935$

23. $\sum_{k=1}^{8}(2k-1)^2=\sum_{k=1}^{8}(4k^2-4k+1)=4\sum_{k=1}^{8}k^2-4\sum_{k=1}^{8}k+\sum_{k=1}^{8}1$

$$=4\cdot\frac{(8)(9)(17)}{6}-4\cdot\frac{(8)(9)}{2}+8=680$$

24. $\sum_{k=1}^{n}k(k^2-5)=\sum_{k=1}^{n}k^3-5\sum_{k=1}^{n}k=\left[\frac{1}{2}n(n+1)\right]^2-\frac{5}{2}n(n+1)$

25. $\frac{1}{2}$; $$s_n = \frac{1}{2}\left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(n)(n+1)}\right]$$
$$= \frac{1}{2}\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right] = \frac{1}{2}\left[1-\frac{1}{n+1}\right] \to \frac{1}{2}$$

26. $\frac{1}{2}$; $\displaystyle\sum_{k=3}^{\infty}\frac{1}{k^2-k} = \sum_{k=3}^{\infty}\left(\frac{1}{k-1}-\frac{1}{k}\right) = \lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{n}\right) = \frac{1}{2}$

27. $\frac{11}{18}$; $$s_n = \frac{1}{1\cdot 4}+\frac{1}{2\cdot 5}+\cdots+\frac{1}{n(n+3)}$$
$$= \frac{1}{3}\left[\left(1-\frac{1}{4}\right)+\left(\frac{1}{2}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+3}\right)\right]$$
$$= \frac{1}{3}\left[1+\frac{1}{2}+\frac{1}{3}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}\right] \to \frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{3}\right) = \frac{11}{18}$$

28. $\frac{3}{4}$; $\displaystyle\sum_{k=0}^{\infty}\frac{1}{(k+1)(k+3)} = \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k+3}\right) = \frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{2}-\frac{1}{n+2}-\frac{1}{n+3}\right) = \frac{3}{4}$

29. $\frac{10}{3}$; $\displaystyle\sum_{k=0}^{\infty}\frac{3}{10^k} = 3\sum_{k=0}^{\infty}\left(\frac{1}{10}\right)^k = 3\left(\frac{1}{1-1/10}\right) = \frac{30}{9} = \frac{10}{3}$

30. $\frac{5}{6}$; $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{5^k} = \sum_{k=0}^{\infty}\left(-\frac{1}{5}\right)^k = \frac{1}{1+\frac{1}{5}} = \frac{5}{6}$

31. $-\frac{3}{2}$; $\displaystyle\sum_{k=0}^{\infty}\frac{1-2^k}{3^k} = \sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^k - \sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k = \frac{1}{1-1/3}-\frac{1}{1-2/3} = \frac{3}{2}-3 = -\frac{3}{2}$

32. $\frac{2150}{99}$; $\displaystyle\sum_{k=0}^{\infty}\left(\frac{25}{10^k}-\frac{6}{100^k}\right) = \frac{25}{1-\frac{1}{10}}-\frac{6}{1-\frac{1}{100}} = \frac{2150}{99}$

33. $\frac{1}{2}$; geometric series with $a=\frac{1}{4}$ and $r=\frac{1}{2}$, sum $=\frac{a}{1-r}=\frac{1}{2}$

34. $\frac{1}{4}$; $\displaystyle\sum_{k=0}^{\infty}\frac{1}{2^{k+3}} = \frac{1}{8}\sum_{k=0}^{\infty}\frac{1}{2^k} = \frac{1}{8}\cdot\frac{1}{1-\frac{1}{2}} = \frac{1}{4}$

35. 24; geometric series with $a=8$ and $r=\frac{2}{3}$, sum $=\frac{a}{1-r}=24$

36. $\frac{3}{15,616}$; $\displaystyle\sum_{k=2}^{\infty}\frac{3^{k-1}}{4^{3k+1}} = \sum_{k=0}^{\infty}\frac{3^{k+1}}{4^{3k+7}} = \frac{3}{4^7}\sum_{k=0}^{\infty}\left(\frac{3}{4^3}\right)^k = \frac{3}{4^7}\cdot\frac{1}{1-\frac{3}{4^3}} = \frac{3}{15,616}$

37. $\displaystyle\sum_{k=1}^{\infty}\frac{7}{10^k} = \frac{7/10}{1-1/10} = \frac{7}{9}$

38. $\displaystyle\sum_{k=1}^{\infty}\frac{9}{10^k} = 1$

39. $\displaystyle\sum_{k=1}^{\infty} \frac{24}{100^k} = \frac{24/100}{1-1/100} = \frac{8}{33}$

40. $\displaystyle\sum_{k=1}^{\infty} \frac{89}{100^k} = \frac{89}{99}$

41. $\displaystyle\frac{62}{100} + \frac{1}{100}\sum_{k=1}^{\infty} \frac{45}{100^k} = \frac{62}{100} + \frac{1}{100}\left(\frac{45/100}{1-1/100}\right) = \frac{687}{1100}$

42. $\displaystyle\frac{112}{1000} + \frac{1}{1000}\sum_{k=1}^{\infty} \frac{19}{1000^k} = \frac{111,907}{999,000}$

43. Let $x = 0.\overbrace{a_1a_2\cdots a_n}\overbrace{a_1a_2\cdots a_n}\cdots$. Then

$$x = \sum_{k=1}^{\infty} \frac{a_1a_2\cdots a_n}{(10^n)^k} = a_1a_2\cdots a_n \sum_{k=1}^{\infty}\left(\frac{1}{10^n}\right)^k$$

$$= a_1a_2\cdots a_n\left[\frac{1}{1-1/10^n} - 1\right] = \frac{a_1a_2\cdots a_n}{10^n-1}.$$

44. Denote the partial sums of the first series by s_n and those of the second series by t_n and observe that $s_n = (a_0 + a_1 + \cdots + a_j) + t_n$. Obviously $s_n \to L$ iff

$$t_n = s_n - (a_0 + a_1 + \cdots + a_j) \to L - (a_0 + a_1 + \cdots + a_j).$$

45. $\displaystyle\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{k=0}^{\infty}(-x)^k = \sum_{k=0}^{\infty}(-1)^k x^k$

46. $\displaystyle\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{k=0}^{\infty}(-x^2)^k = \sum_{k=0}^{\infty}(-1)^k x^{2k}$

47. $\displaystyle\frac{x}{1-x} = x\left(\frac{1}{1-x}\right) = x\sum_{k=0}^{\infty}(x^k) = \sum_{k=0}^{\infty} x^{k+1}$

48. $\displaystyle\frac{x}{1+x} = x\cdot\frac{1}{1-(-x)} = x\sum_{k=0}^{\infty}(-x)^k = \sum_{k=0}^{\infty}(-1)^k x^{k+1}$

49. $\displaystyle\frac{x}{1+x^2} = x\left[\frac{1}{1-(-x^2)}\right] = x\sum_{k=0}^{\infty}(-x^2)^k = \sum_{k=0}^{\infty}(-1)^k x^{2k+1}$

50. $\displaystyle\frac{1}{1+4x^2} = \frac{1}{1-(-4x^2)} = \sum_{k=0}^{\infty}(-4x^2)^k = \sum_{k=0}^{\infty}(-1)^k(2x)^{2k}$

51. $\displaystyle 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \frac{81}{16} + \cdots = \sum_{k=0}^{\infty}\left(\frac{3}{2}\right)^k$

This is a geometric series with $x = \frac{3}{2} > 1$. Therefore the series diverges.

52. $a_k = \dfrac{1}{4}\left(\dfrac{-5}{4}\right)^k$ does not go to zero

53. $\lim\limits_{k\to\infty}\left(\dfrac{k+1}{k}\right)^k = e \neq 0$

54. $a_k = \dfrac{k^{k-2}}{3^k} = \left(\dfrac{k}{3}\right)^k \cdot \dfrac{1}{k^2} > \dfrac{2^k}{k^2}$ for $k > 6$, so $a_k \to \infty$

55. Rebounds to half its previous height:

$$s = 6 + 3 + 3 + \frac{3}{2} + \frac{3}{2} + \frac{3}{4} + \frac{3}{4} + \cdots = 6 + 6\sum_{k=0}^{\infty}\frac{1}{2^k} = 6 + \frac{6}{1-\frac{1}{2}} = 18 \text{ ft.}$$

56. $s = 6 + 2h + 2h\left(\dfrac{h}{6}\right) + 2h\left(\dfrac{h}{6}\right)^2 + \cdots = 6 + 2h\displaystyle\sum_{k=0}^{\infty}\left(\dfrac{h}{6}\right)^k = 6 + \dfrac{12h}{6-h} = 21$

$\Longrightarrow 12h = 15(6-h) \Longrightarrow h = \frac{10}{3}$

57. A principal x deposited now at $r\%$ interest compounded annually will grow in k years to

$$x\left(1 + \frac{r}{100}\right)^k.$$

This means that in order to be able to withdraw n_k dollars after k years one must place

$$n_k\left(1 + \frac{r}{100}\right)^{-k}$$

dollars on deposit today. To extend this process in perpetuity as described in the text, the total deposit must be

$$\sum_{k=1}^{\infty} n_k\left(1 + \frac{r}{100}\right)^{-k}.$$

58. (a) $\displaystyle\sum_{k=1}^{\infty} 5000\left(\frac{1}{2}\right)^{k-1}(1.05)^{-k} = \frac{5000}{1.05}\sum_{k=1}^{\infty}\left[\frac{1}{2(1.05)}\right]^{k-1} = \frac{5000}{1.05}\cdot\frac{1}{1-\frac{1}{2\cdot 1}} \cong \$9090.91$

(b) $\dfrac{1000}{1.06}\cdot\dfrac{1}{1-\frac{0.8}{1.06}} \cong \$3846.15$

(c) $\dfrac{N}{1.05}\cdot\dfrac{1}{1-\frac{1}{1.05}} = 20N$

59. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{9}{10}\right)^n = \frac{\frac{9}{10}}{1-\frac{9}{10}} = 9$ or \$9

60. Total length removed $= \dfrac{1}{3} + \dfrac{2}{9} + \dfrac{4}{27} + \cdots = \dfrac{1}{3}\displaystyle\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k = \frac{1}{3}\cdot\frac{1}{1-\frac{2}{3}} = 1$

Some points: $0, 1, \frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \frac{7}{9}, \frac{8}{9}$.

61. $$A = 4^2 + (2\sqrt{2})^2 + 2^2 + (\sqrt{2})^2 + 1^2 + \cdots + \left[4\left(\frac{1}{\sqrt{2}}\right)^n\right]^2 + \cdots$$
$$= \sum_{n=0}^{\infty}\left[4\left(\frac{1}{\sqrt{2}}\right)^n\right]^2 = 16\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 16\cdot\frac{1}{1-\frac{1}{2}} = 32$$

62. (a) If $\sum(a_k + b_k)$ converges, then $\sum b_k = \sum(a_k+b_k) - \sum a_k$ would also converge.

(b) If $a_k = b_k = 2^k$, $\sum a_k$, $\sum b_k$ and $\sum(a_k+b_k)$ diverge, but $\sum(a_k-b_k) = \sum 0$ converges.

If $a_k = 2^k, b_k = -2^k$, $\sum a_k, \sum b_k$ and $\sum(a_k - b_k)$ diverge, but $\sum(a_k+b_k) = \sum 0$ converges.

63. Let $L = \sum_{k=0}^{\infty} a_k$. Then
$$L = \sum_{k=0}^{\infty} a_k = \sum_{k=0}^{n} a_k + \sum_{k=n+1}^{\infty} a_k = s_n + R_n.$$
Therefore, $R_n = L - s_n$ and since $s_n \to L$ as $n \to \infty$, it follows that $R_n \to 0$ as $n \to \infty$.

64. (a) By convergence, $a_k \to 0$, so $\dfrac{1}{a_k}$ diverges, hence $\sum \dfrac{1}{a_k}$ diverges.

(b) If $a_k = \sqrt{k}$, then $\sum a_k$ diverges and $\sum \dfrac{1}{a_k}$ diverges (Example 6)

If $a_k = 2^k$, then $\sum a_k$ diverges and $\sum \dfrac{1}{a_k} = \sum \dfrac{1}{2^k}$ converges.

65. $s_0 = 1,\ s_1 = 0,\ s_2 = 1,\ \ldots;\quad s_n = \dfrac{1+(-1)^n}{2},\ n = 0, 1, 2, \ldots$.

66. $s_1 = \ln\left(\frac{1}{2}\right)$, $s_2 = \ln\left(\frac{1}{2}\right) + \ln\left(\frac{2}{3}\right) = \ln\left(\frac{1}{3}\right)$, $s_3 = \ln\left(\frac{1}{2}\right) + \ln\left(\frac{2}{3}\right) + \ln\left(\frac{3}{4}\right) = \ln\left(\frac{1}{4}\right)$
$\implies s_n = \ln\left(\dfrac{1}{n+1}\right)$.

67. $s_n = \sum_{k=1}^{n} \ln\left(\dfrac{k+1}{k}\right) = [\ln(n+1) - \ln(n)] + [\ln n - \ln(n-1)] + \cdots + [\ln 2 - \ln 1] = \ln(n+1) \to \infty$

68. $a_k = \left(1 + \dfrac{1}{k}\right)^k \to e \neq 0$

69. (a) $s_n = \sum_{k=1}^{n}(d_k - d_{k+1}) = d_1 - d_{n+1} \to d_1$

(b) We use part (a).

(i) $$\sum_{k=1}^{\infty} \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k(k+1)}} = \sum_{k=1}^{\infty}\left[\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right] = 1$$

(ii) $\displaystyle\sum_{k=1}^{\infty} \frac{2k+1}{2k^2(k+1)^2} = \sum_{k=1}^{\infty} \frac{1}{2}\left[\frac{1}{k^2} - \frac{1}{(k+1)^2}\right] = \frac{1}{2}$

70. Use induction to verify the hint. Then

$$s_n = \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2} \to \frac{1}{(1-x)^2}$$

Since $-(n+1)x^n + nx^{n+1} \to 0$ for $|x| < 1$. This last statement follows from observing that $nx^n \to 0$.

To see this, choose $\epsilon > 0$ so that $(1+\epsilon)|x| < 1$. Since $n^{1/n} \to 1$, there exists k so that

$$n^{1/n} < 1 + \epsilon \quad \text{for} \quad n \geq k.$$

Then for $n \geq k$

$$|nx^n| = |n^{1/n}x|^n \leq ((1+\epsilon)|x|)^n \to 0$$

71. $\displaystyle R_n = \sum_{k=n+1}^{\infty} \frac{1}{4^k} = \frac{\left(\frac{1}{4}\right)^{n+1}}{1 - \frac{1}{4}} = \frac{1}{3 \cdot 4^n};$

$$\frac{1}{3 \cdot 4^n} < 0.0001 \implies 4^n > 3333.33 \implies n > \frac{\ln 3333.33}{\ln 4} \cong 5.85$$

Take $N = 6$.

72. $\displaystyle R_n = \sum_{k=n+1}^{\infty} (0.9)^k = (0.9)^{n+1}\frac{1}{1-0.9} = 9(0.9)^n < 0.0001 \implies n \geq \frac{\ln\left(\frac{0.0001}{9}\right)}{\ln 0.9} \cong 109$

73. $\displaystyle R_n = \sum_{k=n+1}^{\infty} \frac{1}{k(k+2)} = \frac{1}{2}\sum_{k=n+1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+2}\right) = \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n+2}\right);$

$\displaystyle\frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n+2}\right) < 0.0001 \implies n \geq 9999$. Take $N = 9999$.

74. $\displaystyle R_n = \sum_{k=n+1}^{\infty} \left(\frac{2}{3}\right)^k = \left(\frac{2}{3}\right)^{n+1} \frac{1}{1-\frac{2}{3}} = 2\left(\frac{2}{3}\right)^n < 0.0001 \implies n \geq \frac{\ln\left(\frac{0.0001}{2}\right)}{\ln\left(\frac{2}{3}\right)} \cong 25$

75. $\displaystyle |R_n| = \left|\sum_{k=n+1}^{\infty} x^k\right| = \left|\frac{x^{n+1}}{1-x}\right| = \frac{|x|^{n+1}}{1-x};$

$$\frac{|x|^{n+1}}{1-x} < \epsilon$$

$$|x|^{n+1} < \epsilon(1-x)$$

$$(n+1)\ln|x| < \ln\epsilon(1-x)$$

$$n+1 > \frac{\ln\epsilon(1-x)}{\ln|x|} \qquad [\text{recall } \ln|x| < 0]$$

$$n > \frac{\ln\epsilon(1-x)}{\ln|x|} - 1$$

Take N to be smallest integer which is greater than $\dfrac{\ln\epsilon(1-x)}{\ln|x|}$.

76. $s_n = a_n - a_1$. Thus $\{s_n\}$ converges iff $\{a_n\}$ converges.

Hence $\sum\limits_{k=1}^{\infty}(a_{k+1}-a_k)$ converges iff $\{a_n\}$ converges.

SECTION 11.2

1. converges; basic comparison with $\sum \frac{1}{k^2}$

2. diverges; limit comparison with $\sum \frac{1}{k}$

3. converges; basic comparison with $\sum \frac{1}{k^2}$

4. diverges; basic comparison with $\sum \frac{1}{k}$

5. diverges; basic comparison with $\sum \frac{1}{k+1}$

6. converges; basic comparison with $\sum \frac{1}{k^2}$

7. diverges; limit comparison with $\sum \frac{1}{k}$

8. converges; geometric with $x = \frac{2}{5}$

9. converges; integral test, $\displaystyle\int_1^{\infty} \frac{\tan^{-1}x}{1+x^2}\,dx = \lim_{b\to\infty}\left[\frac{1}{2}(\tan^{-1}x)^2\right]_1^b = \frac{3\pi^2}{32}$

10. converges; basic comparison with $\sum \frac{1}{k^2}$

11. diverges; p-series with $p = \frac{2}{3} \le 1$

12. converges; basic comparison with $\sum \frac{1}{k^3}$

13. diverges; divergence test, $(\frac{3}{4})^{-k} \not\to 0$

14. diverges; basic comparison with $\sum \frac{1}{1+2k}$

15. diverges; basic comparison with $\sum \frac{1}{k}$

16. converges; integral test, $\displaystyle\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{b\to\infty}\left[\frac{-1}{\ln x}\right]_2^{\infty} = \frac{1}{\ln 2}$

17. diverges; divergence test, $\dfrac{1}{2+3^{-k}} \to \dfrac{1}{2} \neq 0$

18. converges; limit comparison with $\sum \frac{1}{k^4}$

19. converges; limit comparison with $\sum \frac{1}{k^2}$ **20.** diverges; $a_k \not\to 0$.

21. diverges; integral test, $\int_2^\infty \frac{dx}{x\ln x} = \lim_{b\to\infty} [\ln(\ln x)]_2^b = \infty$

22. converges; limit comparison with $\sum \frac{1}{2^k}$ **23.** converges; limit comparison with $\sum \frac{1}{k^2}$

24. diverges; limit comparison with $\sum \frac{1}{k}$ **25.** diverges; limit comparison with $\sum \frac{1}{k}$

26. diverges; limit comparison with $\sum \frac{1}{\sqrt{k}}$ **27.** converges; limit comparison with $\sum \frac{1}{k^{3/2}}$

28. diverges; limit comparison with $\sum \frac{1}{k}$

29. converges; integral test, $\int_1^\infty xe^{-x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_1^b = \frac{1}{2e}$

30. converges; integral test: $\int_1^\infty x^2 2^{-x^3}\,dx = \lim_{b\to\infty}\left[\frac{2^{-x^3}}{-3\ln 2}\right]_1^\infty = \frac{1}{6\ln 2}$

31. Converges; basic comparison with $\sum \frac{3}{k^2}$, $2 + \sin k \le 3$ for all k.

32. diverges; basic comparison with $\sum \frac{1}{\sqrt{k}}$, $\frac{2+\cos k}{\sqrt{k+1}} > \frac{1}{\sqrt{k}}$

33. Recall that $1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$. Therefore

$\sum \frac{1}{1+2+3+\cdots+k} = \sum \frac{2}{k(k+1)}$. This series converges; direct comparison with $\sum \frac{2}{k^2}$

34. $\sum \frac{n}{1+2^2+3^2+\cdots+n^2} = \sum \frac{n}{\frac{1}{6}n(n+1)(n+2)}$ converges: limit comparison with $\sum \frac{1}{n^2}$

35. Use the integral test:
Let $u = \ln x$, $du = \frac{1}{x}\,dx$: $\int \frac{1}{x(\ln x)^p}\,dx = \int u^{-p}\,du = \frac{u^{1-p}}{1-p} + C.$

$$\int_1^\infty \frac{1}{x(\ln x)^p}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x(\ln x)^p}\,dx = \lim_{b\to\infty}\frac{1}{1-p}(\ln a)^{1-p}$$

The series converges for $p > 1$.

36. If $p \le 1$, $\sum \frac{\ln k}{k^p} > \sum \frac{1}{k^p}$ diverges.

If $p > 1$, then $\frac{p-1}{2} > 0$, so for large k, $\ln k < k^{\frac{p-1}{2}}$

Then $\dfrac{\ln k}{k^p} < \dfrac{k^{\frac{p-1}{2}}}{k^p} = \dfrac{1}{k^{\frac{p+1}{2}}}$. Since $\dfrac{p+1}{2} > 1, \sum \dfrac{1}{k^{\frac{p+1}{2}}}$ converges

Hence so does $\sum \dfrac{\ln k}{k^p}$. so converges iff $p > 1$.

37. (a) Use the integral test: $\displaystyle\int_0^\infty e^{-\alpha x}\,dx = \lim_{b\to\infty}\left[-\frac{1}{\alpha}e^{-\alpha x}\right]_0^b = \frac{1}{\alpha}$ converges.

(b) Use the integral test: $\displaystyle\int_0^\infty xe^{-\alpha x}\,dx = \lim_{b\to\infty}\left[-\frac{1}{\alpha x e^{-\alpha x}} - \frac{1}{\alpha^2}e^{-\alpha x}\right]_0^b = \frac{1}{\alpha^2}$ converges.

(c) The proof follows by induction using parts (a) and (b) and the reduction formula

$$\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx \quad \text{[see Exercise 67, Section 8.2]}$$

38. $\displaystyle\int_{n+1}^\infty \frac{dx}{x^p} < \sum_{k=n+1}^\infty \frac{1}{k^p} < \int_n^\infty \frac{dx}{x^p}$

$\displaystyle\Longrightarrow \frac{1}{(p-1)(n+1)^{p-1}} < \sum_{k=1}^\infty \frac{1}{k^p} - \sum_{k=1}^n \frac{1}{k^p} < \frac{1}{(p-1)n^{p-1}}$

39. (a) $\displaystyle\sum_{k=1}^4 \frac{1}{k^3} \cong 1.1777$ (b) $\dfrac{1}{2\cdot 5^2} < R_4 < \dfrac{1}{2\cdot 4^2}$

$0.02 < R_4 < 0.0313$

(c) $\displaystyle 1.1777 + 0.02 = 1.1977 < \sum_{k=1}^\infty \frac{1}{k^3} < 1.1777 + 0.0313 = 1.2090$

40. (a) $\displaystyle\sum_{k=1}^4 \frac{1}{k^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} \cong 1.0788$

(b) $\dfrac{1}{3.5^3} < R_4 < \dfrac{1}{3.4^3} \Longrightarrow 0.0027 < R_4 < 0.0052$

(c) $\displaystyle 1.0815 < \sum_{k=1}^\infty \frac{1}{k^4} < 1.0840$

41. (a) Put $p = 2$ and $n = 100$ in the estimates in Exercise 38. The result is: $\dfrac{1}{101} < R_{100} < \dfrac{1}{100}$.

(b) $R_n < \dfrac{1}{(2-1)n^{2-1}} < 0.0001 \Longrightarrow n > 10,000$ Take $n = 10,001$.

42. (a) $\dfrac{1}{2(101)^2} < R_{100} < \dfrac{1}{2\cdot 100^2} \Longrightarrow 0.000049 < R_{100} < 0.00005$

(b) $R_n < \dfrac{1}{2n^2} < 0.0001 \Longrightarrow n \geq 71$

43. (a) $R_n < \dfrac{1}{(4-1)n^{4-1}} < 0.0001 \implies n^3 > 3333 \implies n > 14.94$: Take $n = 15$.

(b) $R_n < \dfrac{1}{(4-1)n^{4-1}} < 0.0005 \implies n^3 > 666.67 \implies n > 8.74$: Take $n = 9$.

$$\sum_{k=1}^{\infty} \frac{1}{k^4} \cong \sum_{k=1}^{9} \frac{1}{k^4} \cong 1.082$$

44. (a) $R_n < \dfrac{1}{4n^4} < 0.0001 \implies n \geq 8$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^5} \cong \sum_{k=1}^{8} \frac{1}{k^5} \cong 1.037$

45. (a) If $a_k/b_k \to 0$, then $a_k/b_k < 1$ for all $k \geq K$ for some K. But then $a_k < b_k$ for all $k \geq K$ and, since $\sum b_k$ converges, $\sum a_k$ converges. [The Basic Comparison Theorem 11.2.5.]

(b) Similar to (a) except that this time we appeal to part (ii) of Theorem 11.2.5.

(c) $\sum a_k = \sum \dfrac{1}{k^2}$ converges, $\sum b_k = \sum \dfrac{1}{k^{3/2}}$ converges, $\dfrac{1/k^2}{1/k^{3/2}} = \dfrac{1}{\sqrt{k}} \to 0$

$\sum a_k = \sum \dfrac{1}{k^2}$ converges, $\sum b_k = \sum \dfrac{1}{\sqrt{k}}$ diverges, $\dfrac{1/k^2}{1/\sqrt{k}} = \dfrac{1}{k^{3/2}} \to 0$

(d) $\sum b_k = \sum \dfrac{1}{\sqrt{k}}$ diverges, $\sum a_k = \sum \dfrac{1}{k^2}$ converges, $\dfrac{1/k^2}{1/\sqrt{k}} = 1/k^{3/2} \to 0$

$\sum b_k = \sum \dfrac{1}{\sqrt{k}}$ diverges, $\sum a_k = \sum \dfrac{1}{k}$ diverges, $\dfrac{1/k}{1/\sqrt{k}} = \dfrac{1}{\sqrt{k}} \to 0$

46. (a) Since $a_k/b_k \to \infty$, $b_k/a_k \to 0$, so this follows from Exercise 45(b)

(b) Follows from Exercise 45(a)

(c) $\sum a_k = \sum \dfrac{1}{\sqrt{k}}$ diverges, $\sum b_k = \sum \dfrac{1}{k^2}$ converges, $\dfrac{1/\sqrt{k}}{1/(k^2)} = k^{3/2} \to \infty$

$\sum a_k = \sum \dfrac{1}{\sqrt{k}}$ diverges, $\sum b_k = \sum \dfrac{1}{k}$ diverges, $\dfrac{1/\sqrt{k}}{1/k} = \sqrt{k} \to \infty$

(d) $\sum b_k = \sum \dfrac{1}{k^2}$ converges, $\sum a_k = \sum \dfrac{1}{k^{3/2}}$ converges, $\dfrac{1/k^{3/2}}{1/(k^2)} = \sqrt{k} \to \infty$

$\sum b_k = \sum \dfrac{1}{k^2}$ converges, $\sum a_k = \sum \dfrac{1}{\sqrt{k}}$ diverges, $\dfrac{1/\sqrt{k}}{1/(k^2)} = k^{3/2} \to \infty$

47. (a) Since $\sum a_k$ converges, $a_k \to 0$. Therefore there exists a positive integer N such that $0 < a_k < 1$ for $k \geq N$. Thus, for $k \geq N$, $a_k^2 < a_k$ and so $\sum a_k^2$ converges by the comparison test.

(b) $\sum a_k$ may either converge or diverge: $\sum 1/k^4$ and $\sum 1/k^2$ both converge; $\sum 1/k^2$ converges and $\sum 1/k$ diverges.

48. Since $0 < \left(a_k - \frac{1}{k}\right)^2 < {a_k}^2 + \frac{1}{k^2}$, $\sum \left(a_k - \frac{1}{k}\right)^2$ converges by comparison with $\sum {a_k}^2 + \sum \frac{1}{k^2}$ But $\sum \left(a_k - \frac{1}{k}\right)^2 = \sum a_k^2 - 2\sum \frac{a_k}{k} + \sum \frac{1}{k^2}$,

so $\sum \frac{a_k}{k}$ must converge.

49. $0 < L - \sum_{k=1}^{n} f(k) = L - s_n = \sum_{k=n+1}^{\infty} f(k) < \int_n^\infty f(x)\,dx$ [see the proof of the integral test]

50. $0 < L - S_n < \int_n^\infty \frac{1}{x^2+1}\,dx = \frac{\pi}{2} - \tan^{-1} n < 0.001 \Longrightarrow n > \tan\left(\frac{\pi}{2} - 0.001\right) \cong 1000$

51.

$$\begin{aligned} L - s_n < \int_n^\infty xe^{-x^2}\,dx &= \lim_{b\to\infty} \int_n^b xe^{-x^2}\,dx \\ &= \lim_{b\to\infty} \left[-\frac{1}{2}e^{-x^2}\right]_n^b = \frac{1}{2}e^{-n^2} \end{aligned}$$

$\frac{1}{2}e^{-n^2} < 0.001 \quad \Longrightarrow \quad e^{n^2} > 500 \quad \Longrightarrow \quad n > 2.49$; take n=3.

52. (a) Let $\sum a_k$ be a series with nonpositive terms.

(i) $\sum a_k$ converges if there exists a convergent series $\sum c_k$ with nonpositive terms such that $c_k \le a_k$ for all k sufficiently large.

(ii) $\sum a_k$ diverges if there exists a divergent series $\sum d_k$ with nonpositive terms such that $a_k \le d_k$ for all k sufficiently large.

(b) If f is continuous, increasing, and negative on $[1, \infty)$ then

$$\sum_{k=1}^{\infty} f(k) \quad \text{converges iff} \quad \int_1^\infty f(x)\,dx \quad \text{converges}$$

53. (a) Set $f(x) = x^{1/4} - \ln x$. Then

$$f'(x) = \frac{1}{4}x^{-3/4} - \frac{1}{x} = \frac{1}{4x}(x^{1/4} - 4).$$

Since $f(e^{12}) = e^3 - 12 > 0$ and $f'(x) > 0$ for $x > e^{12}$, we have that

$$n^{1/4} > \ln n \quad \text{and therefore} \quad \frac{1}{n^{5/4}} > \frac{\ln n}{n^{3/2}}$$

for sufficiently large n. Since $\sum \frac{1}{n^{5/4}}$ is a convergent p-series, $\sum \frac{\ln n}{n^{3/2}}$ converges by the basic comparison test.

(b) By L'Hospital's rule

$$\lim_{x\to\infty}\frac{(\ln x)/x^{3/2}}{1/x^{5/4}}=\lim_{x\to\infty}\frac{\ln x}{x^{1/4}}\overset{*}{=}\lim_{x\to\infty}\frac{1/x}{\frac{1}{4}x^{-3/4}}=\lim_{x\to\infty}\frac{4}{x^{1/4}}=0.$$

Thus, the limit comparison test does not apply.

SECTION 11.3

1. converges; ratio test: $\dfrac{a_{k+1}}{a_k}=\dfrac{10}{k+1}\to 0$

2. converges; root test: $\left(\dfrac{1}{k2^k}\right)^{1/k}=\dfrac{1}{2k^{1/k}}\to\dfrac{1}{2}$

3. converges; root test: $(a_k)^{1/k}=\dfrac{1}{k}\to 0$

4. converges; root test: $a_k^{1/k}=\dfrac{k}{2k+1}\to\dfrac{1}{2}$

5. diverges; divergence test: $\dfrac{k!}{100^k}\to\infty$

6. diverges; comparison with $\sum\dfrac{1}{k}$

7. diverges; limit comparison with $\sum\dfrac{1}{k}$

8. converges; root test $(a_k)^{1/k}=\dfrac{1}{\ln k}\to 0$

9. converges; root test: $(a_k)^{1/k}=\dfrac{2}{3}k^{1/k}\to\dfrac{2}{3}$

10. diverges; comparison with $\sum\dfrac{1}{k}$

11. diverges; limit comparison with $\sum\dfrac{1}{\sqrt{k}}$

12. converges; limit comparison with $\sum\dfrac{1}{k^2}$

13. diverges; ratio test: $\dfrac{a_{k+1}}{a_k}=\dfrac{k+1}{10^4}\to\infty$

14. converges; root test: $(a_k)^{1/k}=\dfrac{k^{2/k}}{e}\to\dfrac{1}{e}$

15. converges; basic comparison with $\sum\dfrac{1}{k^{3/2}}$

16. converges; ratio test, $\dfrac{a_{k+1}}{a_k}=\dfrac{2^{k+1}(k+1)!}{(k+1)^{k+1}}\cdot\dfrac{k^k}{2^k k!}=2\left(\dfrac{k}{k+1}\right)^k=\dfrac{2}{\left(\frac{k+1}{k}\right)^k}\to\dfrac{2}{e}$

17. converges; basic comparison with $\sum\dfrac{1}{k^2}$

18. converges; integral test $\displaystyle\int_2^\infty\frac{dx}{x(\ln x)^{3/2}}=\frac{2}{\sqrt{\ln 2}}$

19. diverges; integral test: $\displaystyle\int_2^\infty\frac{1}{x}(\ln x)^{-1/2}dx=\lim_{b\to\infty}\left[2(\ln x)^{1/2}\right]_2^b=\infty$

20. converges; limit comparison with $\sum\dfrac{1}{k^{3/2}}$

21. diverges; divergence test: $\left(\dfrac{k}{k+100}\right)^k=\left(1+\dfrac{100}{k}\right)^{-k}\to e^{-100}\neq 0$

22. converges; ratio test: $\dfrac{[(k+1)!]^2}{(2k+2)!}\cdot\dfrac{(2k)!}{(k!)^2}=\dfrac{(k+1)^2}{(2k+1)(2k+2)}\to\dfrac{1}{4}$

23. diverges; limit comparison with $\sum \frac{1}{k}$ **24.** diverges; $a_k \not\to 0$

25. converges; ratio test: $\frac{a_{k+1}}{a_k} = \frac{\ln(k+1)}{e \ln k} \to \frac{1}{e}$

26. converges; ratio test: $\frac{(k+1)!}{(k+1)^{k+1}} \cdot \frac{k^k}{k!} = \frac{1}{\left(\frac{k+1}{k}\right)^k} \to \frac{1}{e}$

27. converges; basic comparison with $\sum \frac{1}{k^{3/2}}$

28. converges: ratio test $\frac{(k+1)!}{1 \cdot 2 \cdot \ldots \cdot (2k+1)} \cdot \frac{1 \cdot 3 \ldots \cdot (2k-1)}{k!} = \frac{k+1}{2k+1} \to \frac{1}{2}$

29. converges; ratio test: $\frac{a_{k+1}}{a_k} = \frac{2(k+1)}{(2k+1)(2k+2)} \to 0$

30. converges; root test: $(a_k)^{1/k} = \frac{(2k+1)^2}{5k^2+1} \to \frac{4}{5}$

31. converges; ratio test: $\frac{a_{k+1}}{a_k} = \frac{(k+1)(2k+1)(2k+2)}{(3k+1)(3k+2)(3k+3)} \to \frac{4}{27}$

32. converges by Exercise 36, section 11.2

33. converges; ratio test: $\frac{a_{k+1}}{a_k} = \frac{1}{(k+1)^{1/2}}\left(\frac{k+1}{k}\right)^{k/2} \to 0 \cdot \sqrt{e} = 0$

34. diverges: $a_k = \left(\frac{k}{9}\right)^k \not\to 0$

35. converges; root test: $(a_k)^{1/k} = \frac{k}{3^k} \to 0$

36. converges; root test: $(a_k)^{1/k} = \sqrt{k} - \sqrt{k-1} = \frac{1}{\sqrt{k} + \sqrt{k+1}} \to 0$

37. $\frac{1}{2} + \frac{2}{3^2} + \frac{4}{4^3} + \frac{8}{5^4} + \cdots = \sum_{k=0}^{\infty} \frac{2^k}{(k+2)^{k+1}}$

converges; root test: $(a_k)^{1/k} = \frac{2}{(k+2)^{1+1/k}} \to 0$

38. converges: ratio test (see Exercise 28)

39. $\frac{1}{4} + \frac{1 \cdot 3}{4 \cdot 7} + \frac{1 \cot 3 \cdot 5}{4 \cdot 7 \cdot 10} + \cdots = \sum_{k=0}^{\infty} \frac{1 \cdot 3 \cdots (1+2k)}{4 \cdot 7 \cdots (4+3k)}$

converges; ratio test: $\frac{a_{k+1}}{a_k} = \frac{3+2k}{7+3k} \to \frac{2}{3}$

40. converges; ratio test : $\frac{a_{k+1}}{a_k} = \frac{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2(k+1)}{3 \cdot 7 \cdot 11 \cdot \ldots \cdot [4(k+1)-1]} \cdot \frac{3 \cdot 7 \cdot 11 \cdot \ldots \cdot (4k-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2k} = \frac{2k+2}{4k+3} \to \frac{1}{2}$

41. By the hint

$$\sum_{k=1}^{\infty} k\left(\frac{1}{10}\right)^k = \frac{1}{10}\sum_{k=1}^{\infty} k\left(\frac{1}{10}\right)^{k-1} = \frac{1}{10}\left[\frac{1}{1-1/10}\right]^2 = \frac{10}{81}.$$

42. (a) If $\lambda > 1$, then for k sufficiently large

$$\frac{a_{k+1}}{a_k} > 1 \quad \text{and thus} \quad a_{k+1} > a_k$$

This shows that the kth term cannot tend to 0 and thus the series cannot converge.

(b) $\sum \frac{1}{k}$ diverges, $\frac{a_{k+1}}{a_k} = \frac{k}{k+1} \to 1$

$\sum \frac{1}{k^2}$ converges, $\frac{a_{k+1}}{a_k} = \frac{k^2}{(k+1)^2} \to 1$

43. The series $\sum_{k=0}^{\infty} \frac{k!}{k^k}$ converges (see Exercise 26). Therefore, $\lim_{k\to\infty} \frac{k!}{k^k} = 0$ by Theorem 11.1.6.

44. $\frac{r^{n+1}}{(n+1)!} \cdot \frac{n!}{r^n} = \frac{r}{n+1} \to 0$, so by ratio test $\sum \frac{r^n}{n!}$ converges, and therefore $\frac{r^n}{n!} \to 0$

45. Use the ratio test:

$$\frac{a_{k+1}}{a_k} = \frac{\dfrac{[(k+1)!]^2}{[p(k+1)]!}}{\dfrac{(k!)^2}{(pk)!}} = (k+1)^2 \frac{(pk)!}{(pk)!(pk+1)\cdots(pk+p)} = \frac{(k+1)^2}{(pk+1)\cdots(pk+p)}$$

Thus

$$\frac{a_{k+1}}{a_k} \to \begin{cases} \frac{1}{4}, & \text{if } p = 2 \\ 0, & \text{if } p > 2 \end{cases}$$

The series converges for all $p \geq 2$.

46. By root test: $(a_k)^{1/k} = \frac{r}{k^{r/k}} = \frac{r}{\left(k^{1/k}\right)^r} \to r$ converges if $r < 1$, diverges if $r > 1$.

If $r = 1$, we get $\sum \frac{1}{k}$, which diverges.

47. Set $b_k = a_k r^k$. If $(a_k)^{1/k} \to \rho$ and $\rho < \frac{1}{r}$, then

$$(b_k)^{1/k} = (a_k r^k)^{1/k} = (a_k)^{1/k} r \to \rho r < 1$$

and thus, by the root test, $\Sigma b_k = \Sigma a_k r^k$ converges.

48. (a)

$$a_k = \begin{cases} (\frac{1}{2})^k, & k \text{ is odd} \\ (\frac{1}{2})^{k-2}, & k \text{ is even} \end{cases}$$

Clearly, $(a_k)^{1/k} \to \frac{1}{2} < 1$.

(b) $\lim_{k\to\infty} \frac{a_{k+1}}{a_k}$ does not exist since

$$\frac{a_{k+1}}{a_k} = \begin{cases} \frac{1}{8}, & k \text{ is even} \\ 2, & k \text{ is odd} \end{cases}$$

SECTION 11.4

1. diverges; $a_k \not\to 0$

2. (a) $\sum |a_k| = \sum \frac{1}{2k}$ diverges, so not absolutely convergent.

(b) $\frac{1}{2(k+1)} < \frac{1}{2k}$, $a_k \to 0$: converges conditionally; Theorem 11.4.4.

3. diverges; $\frac{k}{k+1} \to 1 \neq 0$

4. (a) $\sum |a_k| = \sum \frac{1}{k \ln k}$, does not converge absolutely.

(b) converges conditionally; Theorem 11.4.4.

5. (a) does not converge absolutely; integral test,

$$\int_1^\infty \frac{\ln x}{x} dx = \lim_{b\to\infty} \left[\frac{1}{2}(\ln x)^2\right]_1^b = \infty$$

(b) converges conditionally; Theorem 11.4.4

6. diverges; $a_k \not\to 0$

7. diverges; limit comparison with $\sum \frac{1}{k}$

another approach: $\sum \left(\frac{1}{k} - \frac{1}{k!}\right) = \sum \frac{1}{k} - \sum \frac{1}{k!}$ diverges since $\sum \frac{1}{k}$ diverges and $\sum \frac{1}{k!}$ converges

8. converges absolutely (terms already positive): ratio test, $\frac{a_{k+1}}{a_k} = \frac{(k+1)^3}{2^{k+1}} \cdot \frac{2^k}{k^3} = \left(\frac{k+1}{k}\right)^3 \cdot \frac{1}{2} \to \frac{1}{2}$

9. (a) does not converge absolutely; limit comparison with $\sum \frac{1}{k}$

(b) converges conditionally; Theorem 11.4.4

10. converges absolutely by ratio test.

11. diverges; $a_k \not\to 0$

12. diverges: $a_k \not\to 0$

13. (a) does not converge absolutely;

$$(\sqrt{k+1}-\sqrt{k})\cdot\frac{(\sqrt{k+1}+\sqrt{k})}{(\sqrt{k+1}+\sqrt{k})}=\frac{1}{\sqrt{k+1}+\sqrt{k}}$$

and

$$\sum\frac{1}{\sqrt{k}+\sqrt{k+1}}>\sum\frac{1}{2\sqrt{k+1}}=2\sum\frac{1}{\sqrt{k+1}}\qquad\text{(a }p\text{-series with }p<1)$$

(b) converges conditionally; Theorem 11.4.4

14. (a) does not converge absolutely: $\dfrac{k}{k^2+1}>\dfrac{k}{2k^2}=\dfrac{1}{2k}$, comparison with $\sum\dfrac{1}{2k}$

(b) $\dfrac{k+1}{(k+1)^2+1}<\dfrac{k}{k^2+1}$; converges conditionally; Theorem 11.4.4.

15. converges absolutely (terms already positive); basic comparison,

$$\sum\sin\left(\frac{\pi}{4k^2}\right)\le\sum\frac{\pi}{4k^2}=\frac{\pi}{4}\sum\frac{1}{k^2}\qquad(|\sin x|\le|x|)$$

16. (a) does not converges absolutely: $\sum\dfrac{1}{\sqrt{k(k+1)}}>\sum\dfrac{1}{k+1}$

(b) converges conditionally by Theorem 11.4.4

17. converges absolutely; ratio test, $\dfrac{a_{k+1}}{a_k}=\dfrac{k+1}{2k}\to\dfrac{1}{2}$

18. terms all positive, converges absolutely: $a_k=\dfrac{1}{\sqrt{k}\sqrt{k+1}(\sqrt{k}+\sqrt{k+1})}$, comparison with $\sum\dfrac{1}{k^{3/2}}$

19. (a) does not converge absolutely; limit comparison with $\sum\dfrac{1}{k}$

(b) converges conditionally; Theorem 11.4.4

20. (a) does not converge absolutely: $\dfrac{k+2}{k^2+k}>\dfrac{k}{2k^2}=\dfrac{1}{2k}$, comparison with $\sum\dfrac{1}{2k}$

(b) converges conditionally; Theorem 11.4.4.

21. diverges; $a_k=\dfrac{4^{k-2}}{e^k}=\dfrac{1}{16}\left(\dfrac{4}{e}\right)^k\not\to 0$

22. converges absolutely by integral test: $\displaystyle\int_1^\infty x^2 2^{-x}\,dx$ converges

23. diverges; $a_k=k\sin(1/k)=\dfrac{\sin(1/k)}{1/k}\to 1\neq 0$

24. diverges: $\left|\dfrac{a_{k+1}}{a_k}\right|=\dfrac{(k+1)^{k+1}}{(k+1)!}\cdot\dfrac{k!}{k^k}=\left(\dfrac{k+1}{k}\right)^k>1$, so $a_k\not\to 0$

25. converges absolutely; ratio test, $\dfrac{a_{k+1}}{a_k}=\dfrac{(k+1)e^{-(k+1)}}{k\,e^{-k}}=\dfrac{k+1}{k}\,\dfrac{1}{e}\to\dfrac{1}{e}$

26. (a) $\sum\dfrac{\cos\pi k}{k}=\sum\dfrac{(-1)^k}{k}$ does not converge absolutely.

(b) converges conditionally; Theorem 11.4.4.

27. diverges; $\sum(-1)^k \dfrac{\cos \pi k}{k} = \sum(-1)^k \dfrac{(-1)^k}{k} = \sum \dfrac{1}{k}$

28. Converges absolutely; $|a_k| = \left|\dfrac{\sin(\pi k/2)}{k\sqrt{k}}\right| < \dfrac{1}{k^{3/2}}$

29. converges absolutely; basic comparison

$$\sum \left|\frac{\sin(\pi k/4)}{k^2}\right| \leq \sum \frac{1}{k^2}$$

30. The series $\sum \left(\dfrac{1}{3k+2} - \dfrac{1}{3k+3}\right) = \sum \dfrac{1}{(3k+2)(3k+3)}$ converges by comparison with $\sum \dfrac{1}{k^2}$.

If $\sum \left(\dfrac{1}{3k+2} - \dfrac{1}{3k+3} - \dfrac{1}{3k+4}\right)$ converged, then

$\sum \dfrac{1}{3k+4} = \sum \left(\dfrac{1}{3k+2} - \dfrac{1}{3k+3}\right) - \sum \left(\dfrac{1}{3k+2} - \dfrac{1}{3k+3} - \dfrac{1}{3k+4}\right)$ would converge, which is not the case.

31. diverges; $a_k \not\to 0$

32. error $< a_{21} = \dfrac{1}{21}$

33. Use (11.4.5); $|s - s_{80}| < a_{81} = \dfrac{1}{\sqrt{82}} \cong 0.1104$

34. error $< a_5 = \dfrac{1}{10^5} = 0.00001$

35. Use (11.4.5); $|s - s_9| < a_{10} = \dfrac{1}{10^3} = 0.001$

36. error $< a_{n+1} = \dfrac{1}{10^{n+1}}$ (a) $\dfrac{1}{10^{n+1}} < 10^{-3} \implies n \geq 3$ (b) $\dfrac{1}{10^{n+1}} < 10^{-4} \implies n \geq 4$

37. $\dfrac{10}{11}$; geometric series with $a = 1$ and $r = -\dfrac{1}{10}$, sum $= \dfrac{a}{1-r} = \dfrac{10}{11}$

38. $\dfrac{(0.9)^{N+1}}{N+1} < 0.001 \quad N \geq 32$

39. Use (11.4.5); $|s - s_n| < a_{n+1} = \dfrac{1}{\sqrt{n+2}} < 0.005 \implies n \geq 39,998$

40. The series diverges because among the partial sums are all sums of the form

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

Thus for instance,

$$s_1 = \frac{1}{2}, \quad s_5 = \frac{1}{2} + \frac{1}{3}, \quad s_{11} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4}, \quad \text{and so on.}$$

This does not violate the theorem on alternating series because, in the notation of the theorem, it is not true that $\{a_k\}$ decreases.

41. Use (11.4.5).

(a) $n = 4$; $\dfrac{1}{(n+1)!} < 0.01 \implies 100 < (n+1)!$

(b) $n = 6$; $\dfrac{1}{(n+1)!} < 0.001 \implies 1000 < (n+1)!$

42. Yes. This can be shown by making slight changes in the proof of Theorem 11.4.4. The even partial sums s_{2m} are now nonnegative. Since $s_{2m+2} \le s_{2m}$, the sequence converges; say, $s_{2m} \to l$. Since $s_{2m+1} = s_{2m} - a_{2m+1}$ and $a_{2m+1} \to 0$, we have $s_{2m+1} \to l$. Thus, $s_n \to l$.

43. No. For instance, set $a_{2k} = 2/k$ and $a_{2k+1} = 1/k$.

44. If $\sum a_k$ is absolutely convergent, then $\sum |a_k|$ converges. Therefore $\sum |b_k|$ by comparison with $\sum |a_k|$. Thus $\sum b_k$ is absolutely convergent.

45. (a) Since $\sum |a_k|$ converges, $\sum |a_k|^2 = \sum a_k^2$ converges (Exercise 47, Section 11.2).

(b) $\sum \dfrac{1}{k^2}$ converges, $\sum (-1)^k \dfrac{1}{k}$ is not absolutely convergent.

46. (a) write down just enough positive terms so that the sum is greater than L:

$$p_1 + p_2 + \cdot + p_{j_1} > L$$

Now add just enough negative terms so that the sum is less than L:

$$p_1 + p_2 + \cdots + p_{j_1} + n_1 + n_2 + \cdots + n_{j_2} < L.$$

Now add just enough positive terms so that the sum is greater than L, and so on.

The resulting rearrangement converges to L.

(b) Write down n_1 plus enough positive terms so that the sum exceeds 1:

$$n_1 + p_1 + p_2 + \cdot + p_{k_1} > 1.$$

Now add n_2 plus enough positive terms so that the sum exceeds 2:

$$n_1 + p_1 + p_2 + \cdots + p_{k_1} + n_2 + p_{k_1+1} + p_{k_1+2} + \cdots + p_{k_2} > 2.$$

Go on in this manner at the q'th stage adding n_q plus enough positive terms so that the sum exceeds q. The resulting series diverges to $+\infty$.

(c) Imitate the procedure used in part (b).

47. See the proof of Theorem 11.7.2.

48.

$$\begin{aligned} s_{2m+1} &= a_0 - a_1 + a_2 - a_3 + a_4 + \cdots - a_{2m+1} \\ &= a_0 + (-a_1 + a_2) + (-a_3 + a_4) + \cdots + (-a_{2m-1} + a_{2m}) - a_{2m+1} \\ &= a_0 + \text{negative terms.} \end{aligned}$$

Then $s_{2m+1} < a_0$: and $\{s_{2m+1}\}$ is bounded above.

$s_{2m+3} = s_{2m+1} + (a_{2m+2} - a_{2m+3}) > s_{2m+1}$, thus $\{s_{2m+1}\}$ is increasing.

49. (a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(a+b)+(a-b)}{2k} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(a+b)}{2k} + \sum_{k=1}^{\infty} \frac{a-b}{2k}$

(b) The series is absolutely convergent if $a = b = 0$; conditionally convergent if $a = b \neq 0$;

divergent if $a \neq b$.

PROJECT 11.4

n	a_n	s_n	err s_n	t_n	err t_n	b_n	u_n
9	0.112329	−0.053					
10	0.100899	0.394036	0.047898	0.346242	0.000103	0.525290	0.343568
19	0.052770	0.320450	−0.025690				
20	0.050119	0.370569	0.024431	0.346152	$1.4e-05$	0.512544	0.345510
29	0.034522	0.329175	−0.016960				
30	0.033369	0.362544	0.016416	0.346142	$4.2e-06$	0.508350	0.345860
39	0.025657	0.333474	−0.012660				
40	0.025015	0.358489	0.012351	0.346140	$1.7e-06$	0.506260	0.345982
49	0.020416	0.336034	−0.0101				
50	0.020008	0.356042	0.009904	0.346139	$8.3e-07$	0.505009	0.346038
59	0.016954	0.337733	−0.00841				
60	0.016671	0.354404	0.008266	0.346139	$4.3e-07$	0.504176	0.346069
69	0.014496	0.338943	−0.0072				
70	0.014289	0.353231	0.007903	0.346138	$2.3e-07$	0.503581	0.346087
79	0.012660	0.338848	−0.00629				
80	0.012502	0.352350	0.006212	0.346138	$1.1e-07$	0.503135	0.346099
89	0.011237	0.340551	−0.00559				
90	0.011112	0.351663	0.005525	0.346138	$4.5e-08$	0.502789	0.346107
99	0.010102	0.341113	−0.00503				
100	0.010001	0.351114	0.004975	0.346138	0	0.502513	0.346113

1. Column s_n gives these estimates.

2. We want n such that $a_n < 5 \times 10^{-4}$; $n = 2000$.

3. $t_n = \dfrac{s_n + s_{n-1}}{2} = \dfrac{1}{2}(s_n + s_{n-1})$

$$\lim_{n\to\infty} t_n = \frac{1}{2}\left(\lim_{n\to\infty} s_n + \lim_{n\to\infty} s_{n-1}\right) = \frac{1}{2}(2s) = s.$$

$$u_n = \frac{a_n s_n + a_{n-1}s_{n-1}}{a_n + a_{n-1}}$$

Let $\epsilon > 0$. Since $\lim\limits_{n\to\infty} = s$, there exists a positive integer N such that

$$|s_n - s| < \epsilon \text{ whenever } n - 1 > N.$$

Choose any integer $k > N + 1$. Then

$$\begin{aligned}|u_k - s| &= \left|\frac{a_k s_k + a_{k-1}s_{k-1}}{a_k + a_{k-1}} - s\right| \\ &= \left|\frac{a_k(s_k - s) + a_{k-1}(s_{k-1} - s)}{a_k + a_{k-1}}\right| \\ &\le \frac{a_k|s_k - s| + a_{k-1}|s_{k-1} - s|}{a_k + a_{k-1}} \\ &< \frac{\epsilon(a_k + a_{k-1})}{a_k + a_{k-1}} = \epsilon\end{aligned}$$

Therefore $\lim\limits_{n\to\infty} u_n = s$.

4. (a) See column t_n

(b) See column u_n.

(c) $\{u_n\}$ converges much faster than $\{s_n\}$ and $\{t_n\}$.

SECTION 11.5

1. $-1 + x + \frac{1}{2}x^2 - \frac{1}{24}x^4$

2. $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4$

3. $-\frac{1}{2}x^2 - \frac{1}{12}x^4$

4. $1 + \frac{1}{2}x^2 + \frac{5}{24}x^4$

5. $1 - x + x^2 - x^3 + x^4 - x^5$

6. $x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5$

7. $x + \frac{1}{3}x^3 + \frac{2}{15}x^5$

8. $x - \frac{1}{2}x^5$

9. $P_0(x) = 1, \quad P_1(x) = 1 - x, \quad P_2(x) = 1 - x + 3x^2, \quad P_3(x) = 1 - x + 3x^2 + 5x^3$

10. $P_0(x) = 1, \quad P_1(x) = 1 + 3x, \quad p_2(x) = 1 + 3x + 3x^2, \quad P_3(x) = 1 + 3x + 3x^2 + x^3$

11. $\sum\limits_{k=0}^{n}(-1)^k\dfrac{x^k}{k!}$

12. $\sum\limits_{k=0}^{m}\dfrac{x^{2k+1}}{(2k+1)!}$ where $m = \dfrac{n-1}{2}$ and n is odd.

13. $\sum\limits_{k=0}^{m}\dfrac{x^{2k}}{(2k)!}$ where $m = \dfrac{n}{2}$ and n is even

14. $-\sum\limits_{k=1}^{n}\dfrac{x^k}{k}$

15. $f^{(k)}(x) = r^k e^{rx}$ and $f^{(k)}(0) = r^k$, $k = 0, 1, 2, \ldots$. Thus, $P_n(x) = \sum\limits_{k=0}^{n}\dfrac{r^k}{k!}x^k$

16. $\sum\limits_{k=0}^{m}\dfrac{(-1)^k}{(2k)!}(bx)^{2k}$ where $m = \dfrac{n}{2}$ and n is even.

17. The Taylor polynomial

$$P_n(0.5) = 1 + (0.5) + \frac{(0.5)^2}{2!} + \cdots + \frac{(0.5)^n}{n!}$$

estimates $e^{0.5}$ within

$$|R_{n+1}(0.5)| \le e^{0.5}\frac{|0.5|^{n+1}}{(n+1)!} < 2\frac{(0.5)^{n+1}}{(n+1)!}.$$

Since

$$2\frac{(0.5)^4}{4!} = \frac{1}{8(24)} < 0.01,$$

we can take $n = 3$ and be sure that

$$P_3(0.5) = 1 + (0.5) + \frac{(0.5)^2}{2} + \frac{(0.5)^3}{6} = \frac{79}{48}$$

differs from $\sqrt{e}$ by less than 0.01. Our calculator gives

$$\tfrac{79}{48} \cong 1.645833 \quad \text{and} \quad \sqrt{e} \cong 1.6487213.$$

18. At $x = 0.3$ the sine series gives

$$\sin 0.3 = 0.3 - \frac{(0.3)^3}{3!} + \frac{(0.3)^5}{5!} - \frac{(0.3)^7}{7!} + \cdots.$$

This is a convergent alternating series with decreasing terms. The first term of magnitude less than 0.01 is $(0.3)^3/3! = 0.0045$. Thus 0.3 differs from $\sin 0.3$ by less than 0.01. Our calculator gives $\sin 0.3 \cong 0.2955202$. The estimate

$$0.3 - \frac{(0.3)^3}{3!} = 0.2955$$

is much more accurate. The series converges very rapidly for small values of x.

19. At $x = 1$, the sine series gives

$$\sin 1 = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots.$$

This is a convergent alternating series with decreasing terms. The first term of magnitude less than 0.01 is $1/5! = 1/120$. Thus

$$1 - \frac{1}{3!} = 1 - \frac{1}{6} = \frac{5}{6}$$

differs from $\sin 1$ by less than 0.01. Our calculator gives

$$\tfrac{5}{6} \cong 0.8333333 \quad \text{and} \quad \sin 1 \cong 0.84114709.$$

The estimate

$$1 - \frac{1}{3!} + \frac{1}{5!} = \frac{101}{120} \cong 0.8416666$$

is much more accurate.

20. At $x = 1.2$ the logarithm series (11.5.8) gives

$$\ln 1.2 = \ln(1 + 0.2) = 0.2 - \frac{1}{2}(0.2)^2 + \frac{1}{3}(0.2)^3 - \frac{1}{4}(0.2)^4 + \cdots.$$

This is a convergent alternating series with decreasing terms. The first term of magnitude less than 0.01 is $(0.2)^3/3 \cong 0.00267$. Thus

$$0.2 - \frac{1}{2}(0.2)^2 = 0.18$$

differs from $\ln 1.2$ by less than 0.01. Our calculator gives $\ln 1.2 \cong 0.1823215$.

21. At $x = 1$, the cosine series gives

$$\cos 1 = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} + \cdots.$$

This is a convergent alternating series with decreasing terms. The first term of magnitude less than 0.01 is $1/6! = 1/720$. Thus

$$1 - \frac{1}{2!} + \frac{1}{4!} = 1 - \frac{1}{2} + \frac{1}{24} = \frac{13}{24}$$

differs from $\cos 1$ by less than 0.01. Our calculator gives

$$\tfrac{13}{24} \cong 0.5416666 \quad \text{and} \quad \cos 1 \cong 0.5403023.$$

22. The Taylor polynomial

$$P_n(0.8) = 1 + (0.8) + \frac{(0.8)^2}{2!} + \cdots + \frac{(0.8)^n}{n!}$$

estimates $e^{0.8}$ within

$$|R_{n+1}(0.8)| \le e^{0.8}\frac{(0.8)^{n+1}}{(n+1)!} < 3\frac{(0.8)^{n+1}}{(n+1)!}.$$

Since $3\dfrac{(0.8)^5}{5!} < 0.009 < 0.01$ we can take $n = 4$ and be sure that

$$P_4(0.8) = 1 + (0.8) + \frac{(0.8)^2}{2!} + \frac{(0.8)^3}{3!} + \frac{(0.8)^4}{4!} = 2.224$$

differs from $e^{0.8}$ by less than 0.01. Our calculator gives $e^{0.8} \cong 2.2255409$.

23. First convert $10°$ to radians: $10° = \dfrac{10}{180}\pi \cong 0.1745$ radians

At $x = 0.1745$, the sine series gives

$$\sin 0.1745 = 0.1745 - \frac{(0.1745)^3}{3!} + \frac{(0.1745)^5}{5!} - \cdots.$$

This is a convergent alternating series with decreasing terms. The first term of magnitude less than 0.01 is $(0.1745)^3/3! \cong 0.00089$. Thus 0.1745 differs from $\sin 10°$ by less than 0.01. Our calculator gives $\sin 10° \cong 0.1736$

24. At $x = 6° = \dfrac{\pi}{30}$, the cosine series gives $\cos\dfrac{\pi}{30} = 1 - \dfrac{1}{2}\left(\dfrac{\pi}{30}\right)^2 + \dfrac{1}{4!}\left(\dfrac{\pi}{30}\right)^4 - \dfrac{1}{6!}\left(\dfrac{\pi}{30}\right)^6 + \cdots$

The first term less than 0.01 is $\frac{1}{2}\left(\frac{\pi}{30}\right)^2 \cong 0.0055$, so 1 differs from $\cos 6°$ by less than 0.01.

Calculator gives $\cos 6° \cong 0.9945219$

25. $f(x) = e^{2x}$; $f^{(5)}(x) = 2^5e^{2x}$; $R_5(x) = \dfrac{2^5e^{2c}}{5!}x^5 = \dfrac{4}{15}e^{2c}x^5$, where c is between 0 and x.

26. $R_{n+1}(x) = R_6(x) = \dfrac{f^6(c)}{(5+1)!}x^6 = \dfrac{-120(1+c)^{-6}}{6!}x^6 = -\dfrac{1}{6}\left(\dfrac{x}{1+c}\right)^6$, where c is between 0 and x.

27. $f(x) = \cos 2x$; $f^{(5)}(x) = -2^5 \sin 2x$

$$R_5(x) = \frac{-2^5 \sin 2c}{5!}\, x^5 = -\frac{4}{15}\sin(2c)\, x^5,$$

where c is between 0 and x.

28. $R_{n+1}(x) = R_4(x) = \dfrac{f^4(c)}{4!}x^4 = \dfrac{-\frac{15}{16}(c+1)^{-7/2}}{4!}x^4 = \dfrac{-5x^4}{128(c+1)^{7/2}}$, where c is between 0 and x.

29. $f(x) = \tan x;\quad f'''(x) = 6\sec^4 x - 4\sec^2 x$

$$R_3(x) = \frac{6\sec^4 c - 4\sec^2 c}{3!}\, x^3 = \frac{3\sec^4 c - 2\sec^2 c}{3}\, x^3,$$

where c is between 0 and x.

30. $R_{n+1}(x) = R_6(x) = \dfrac{f^6(c)}{6!}x^6 = \dfrac{-\sin c}{6!}x^6$, where c is between 0 and x.

31. $f(x) = \tan^{-1} x;\quad f'''(x) = \dfrac{6x^2-2}{(1+x^2)^3}$

$$R_3(x) = \frac{6c^2-2}{3!\,(1+c^2)^3}\, x^3 = \frac{3c^2-1}{3\,(1+c^2)^3}\, x^3,$$

where c is between 0 and x.

32. $R_{n+1}(x) = R_5(x) = \dfrac{f^5(c)}{5!}x^5 = \dfrac{-120(1+c)^{-6}}{5!}x^5 = \dfrac{-x^5}{(1+c)^6}$, where c is between 0 and x.

33. $f(x) = e^{-x};\quad f^{(k)}(x) = (-1)^k e^{-x},\ k = 0, 1, 2, \ldots$

$$R_{n+1}(x) = \frac{(-1)^{n+1}e^{-c}}{(n+1)!}\, x^{n+1},$$

where c is between 0 and x.

34.
$$R_{n+1}(x) = \frac{f^{n+1}(c)}{(n+1)!}x^{n+1} = \begin{cases} \dfrac{(-1)^{\frac{n-1}{2}}2^{n+1}\cos 2c}{(n+1)!}x^{n+1} & n \text{ odd} \\[2ex] \dfrac{(-1)^{\frac{n}{2}}2^{n+1}\sin 2c}{(n+1)!}x^{n+1} & n \text{ even, where } c \text{ is between 0 and } x. \end{cases}$$

35. $f(x) = \dfrac{1}{1-x};\quad f^{(k)}(x) = \dfrac{k!}{(1-x)^{k+1}},\ k = 0, 1, 2, \ldots$

$$R_{n+1}(x) = \frac{(n+1)!}{(1-c)^{n+2}(n+1)!}\, x^{n+1} = \frac{1}{(1-c)^{n+2}}\, x^{n+1}, \text{ where } c \text{ is between 0 and } x.$$

36. $R_{n+1}(x) = \dfrac{f^{n+1}(c)}{(n+1)!}x^{n+1} = \dfrac{(-1)^{n+1}(n!)/(1+c)^{n+1}}{(n+1)!}x^{n+1} = \dfrac{(-1)^{n+1}}{n+1}\left(\dfrac{x}{1+c}\right)^{n+1},$

where c is between 0 and x.

37. By (11.5.8)

$$P_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n}.$$

For $0 \le x \le 1$ we know from (11.4.5) that

$$|P_n(x) - \ln(1+x)| < \frac{x^{n+1}}{n+1}.$$

(a) $n = 4; \dfrac{(0.5)^{n+1}}{n+1} \le 0.01 \implies 100 \le (n+1)2^{n+1} \implies n \ge 4$

(b) $n = 2; \dfrac{(0.3)^{n+1}}{n+1} \le 0.01 \implies 100 \le (n+1)\left(\dfrac{10}{3}\right)^{n+1} \implies n \ge 2$

(c) $n = 999; \dfrac{(1)^{n+1}}{n+1} \le 0.001 \implies 1000 \le n+1 \implies n \ge 999$

38. (a) Since $\dfrac{1^7}{7!} \cong 0.0002$, use P_5.

(b) Since $\dfrac{2^{11}}{11!} \cong 0.00005$ is the first term less than 0.001, use P_9.

(c) Since $\dfrac{3^{13}}{13!} \cong 0.0002$ is the first term less than 0.001, use P_{11}.

39. $f(x) = e^x; \quad f^{(n)}(x) = e^x; \quad R_{n+1}(x) = \dfrac{e^c}{(n+1)!}x^{n+1}, \quad |c| < |x|$

(a) We want $|R_{n+1}(1/2)| < .00005$: for $0 < c < \frac{1}{2}$, we have

$$|R_{n+1}(1/2)| = \frac{e^c}{(n+1)!}\left(\frac{1}{2}\right)^{n+1} < \frac{e^{1/2}}{(n+1)!}\left(\frac{1}{2}\right)^{n+1} < \frac{2}{2^{n+1}(n+1)!} < 0.00005$$

You can verify that this inequality is satisfied if $n \ge 5$.

$$P_5(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

$$P_5(1/2) = 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \frac{1}{320} + \frac{1}{3840} \cong 1.6492$$

(b) We want $|R_{n+1}(-1)| < .0005$: for $-1 < c < 0$, we have

$$|R_{n+1}(-1)| = \frac{e^c}{(n+1)!}\left|(-1)^{n+1}\right| < \frac{1}{(n+1)!} < 0.0005$$

You can verify that this inequality is satisfied if $n \ge 7$.

$$P_7(x) = \sum_{k=0}^{7}\frac{x^k}{k!}; \quad P_7(-1) = \sum_{k=0}^{7}\frac{(-1)^k}{k!} \cong 0.368$$

40. (a) $\dfrac{1}{4!}\left(\dfrac{\pi}{30}\right)^4$ is the first term less than 0.0005, so use $P_2\left(\frac{\pi}{30}\right) = 1 - \frac{1}{2}\left(\frac{\pi}{30}\right)^2 \cong 0.995$

(b) $9° = \frac{\pi}{20}$ $\quad \frac{1}{4!}\left(\frac{\pi}{20}\right)^4$ is the first term less than 0.00005, so use $P_2\left(\frac{\pi}{20}\right) = 1 - \frac{1}{2}\left(\frac{\pi}{20}\right)^2 \cong 0.9877$

41. The result follows from the fact that $P^{(k)}(0) = \left\{ \begin{array}{ll} k!a_k, & 0 \le k \le n \\ 0, & n < k \end{array} \right]$.

42. Straightforward

43.
$$\frac{d^k}{dx^k}(\sinh x) = \begin{cases} \sinh x, & \text{if } k \text{ is odd} \\ \cosh x, & \text{if } k \text{ is even} \end{cases}$$

Thus
$$\frac{d^k}{dx^k}(\sinh x)\Big|_{x=0} = \begin{cases} 0, & \text{if } k \text{ is odd} \\ 1, & \text{if } k \text{ is even} \end{cases}$$

and
$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{(2k+1)}}{(2k+1)!}$$

44.
$$\cosh x = \frac{1}{2}(e^x + e^{-x}) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{n!} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}$$
$$= \sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!} \quad \text{because the odd terms cancel out}$$

45. Set $t = ax$. Then, $e^{ax} = e^t = \sum_{k=0}^{\infty}\frac{t^k}{k!} = \sum_{k=0}^{\infty}\frac{a^k}{k!}x^k, \quad (-\infty, \infty).$

46. $\sin ax = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(ax)^{2k+1} = \sum_{k=0}^{\infty}\frac{(-1)^k a^{2k+1}}{(2k+1)!}x^{2k+1}; \quad (-\infty, \infty)$

47. Set $t = ax$. Then, $\cos ax = \cos t = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}t^{2k} = \sum_{k=0}^{\infty}\frac{(-1)^k a^{2k}}{(2k)!}x^{2k}, \quad (-\infty, \infty).$

48. $\ln(1 - ax) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}(-ax)^k = -\sum_{k=1}^{\infty}\frac{a^k}{k}x^k; \quad \left[-\frac{1}{a}, \frac{1}{a}\right)$

49. By the hint $\ln(a + x) = \ln\left[a\left(1 + \frac{x}{a}\right)\right] = \ln a + \ln\left(1 + \frac{x}{a}\right) = \ln a + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{ka^k}x^k.$

By (11.5.8) the series converges for $-1 < \frac{x}{a} \le 1$; that is, $-a < x \le a$.

50. $f(x) = \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x); \quad f(0) = 0$

$f'(x) = \frac{1}{1+x} + \frac{1}{1-x}, \quad f'(0) = 2$

$$f''(x) = \frac{-1}{(1+x)^2} + \frac{1}{(1-x)^2}, \quad f''(0) = 0$$

$$f'''(x) = \frac{2}{(1+x)^3} + \frac{2}{(1-x)^3}, \quad f'''(0) = 4$$

In general, $f^n(x) = \dfrac{(-1)^{n+1}(n-1)!}{(1+x)^n} + \dfrac{(n-1)!}{(1-x)^n}, \quad f^{(n)}(0) = 2(n-1)!$ for n odd,

0 for n even The result follows.

51. $\ln 2 = \ln\left(\dfrac{1+1/3}{1-1/3}\right) \cong 2\left[\dfrac{1}{3} + \dfrac{1}{3}\left(\dfrac{1}{3}\right)^3 + \dfrac{1}{5}\left(\dfrac{1}{3}\right)^5\right] = \dfrac{842}{1215}.$

Our calculator gives $\dfrac{842}{1215} \cong 0.6930041$ and $\ln 2 \cong 0.6931471.$

52. $\dfrac{1+x}{1-x} = 1.4$ gives $x = \dfrac{1}{6}$; $\ln 1.4 \cong 2\left[\dfrac{1}{6} + \dfrac{1}{3}\left(\dfrac{1}{6}\right)^3\right] = \dfrac{109}{324} \cong 0.336$

53. Set $u = (x-t)^k, \qquad dv = f^{(k+1)}(t)\,dt$

$du = -k(x-t)^{k-1}\,dt, \quad v = f^{(k)}(t).$

Then,

$$\begin{aligned} &-\frac{1}{k!}\int_0^x f^{(k+1)}(t)(x-t)^k\,dt \\ &= -\frac{1}{k!}\left[(x-t)^k f^{(k)}(t)\right]_0^x - \frac{1}{k!}\int_0^x k(x-t)^{k-1} f^{(k)}(t)\,dt \\ &= \frac{f^{(k)}(0)}{k!}x^k - \frac{1}{(k-1)!}\int_0^x f^{(k)}(t)(x-t)^{k-1}\,dt. \end{aligned}$$

The given identity follows.

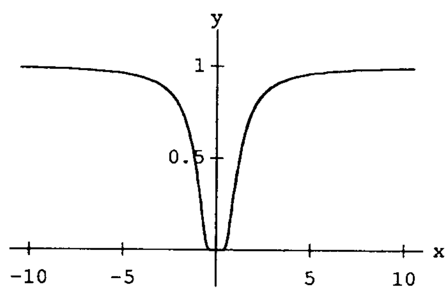
54. If $x > 0$, then

$$\begin{aligned} |R_{n+1}(x)| &= \frac{1}{n!}\left|\int_0^x f^{(n+1)}(t)(x-t)^n\,dt\right| \\ &\le \frac{1}{n!}\int_0^x \left|f^{(n+1)}(t)\right|(x-t)^n\,dt \\ &\le \frac{1}{n!}\int_0^x M(x-t)^n\,dt \quad \text{where} \quad M = \max_{t\in I}|f^{n+1}(t)| \\ &= \frac{M}{n!}\int_0^x (x-t)^n\,dt \\ &= \frac{M}{n!}\left[\frac{-(x-t)^{n+1}}{n+1}\right]_0^x = \frac{M}{n!}\frac{x^{n+1}}{n+1} = M\frac{|x|^{n+1}}{(n+1)!}. \end{aligned}$$

If $x < 0$, then

$$\begin{aligned}|R_{n+1}(x)| &= \frac{1}{n!}\left|\int_0^x f^{(n+1)}(t)(x-t)^n\,dt\right|\\ &\le \frac{1}{n!}\int_x^0 \left|f^{(n+1)}(t)\right|(t-x)^n\,dt\\ &\le \frac{1}{n!}\int_x^0 M(t-x)^n\,dt\\ &= \frac{M}{n!}\int_x^0 (t-x)^n\,dt\\ &= \frac{M}{n!}\left[\frac{(t-x)^{n+1}}{n+1}\right]_x^0 = \frac{M}{n!}\frac{(-x)^{n+1}}{n+1} = M\frac{|x|^{n+1}}{(n+1)!}.\end{aligned}$$

55. (a)



(b) Let $g(x) = \dfrac{x^{-n}}{e^{1/x^2}}$. Then $\lim_{x\to 0} g(x)$ has the form ∞/∞. Successive applications of L'Hospital's rule will finally produce a quotient of the form $\dfrac{cx^k}{e^{1/x^2}}$, where k is a nonnegative integer and c is a constant. It follows that $\lim_{x\to 0} g(x) = 0$.

(c) $f'(0) = \lim_{x\to 0}\dfrac{e^{-1/x^2} - 0}{x} = 0$ by part (b). Assume that $f^{(k)}(0) = 0$. Then

$$f^{(k+1)}(0) = \lim_{x\to 0}\frac{f^{(k)}(x) - 0}{x} = \lim_{x\to 0}\frac{f^{(k)}(x)}{x}.$$

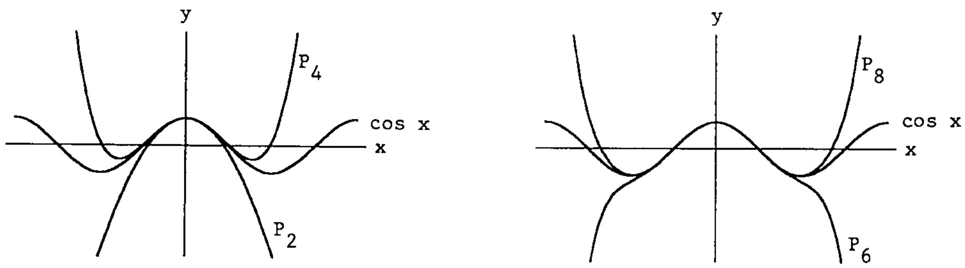
Now, $f^{(k)}(x)/x$ is a sum of terms of the form $ce^{-1/x^2}/x^n$, where n is a positive integer and c is a constant.

Again by part (b), $f^{(k+1)}(0) = 0$. Therefore, $f^{(n)}(0) = 0$ for all n.

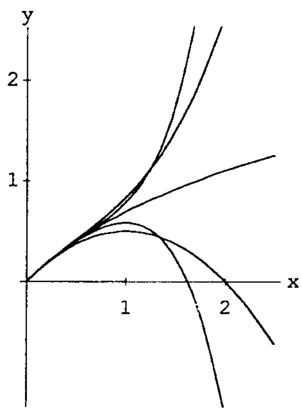
(d) 0

(e) x=0

56.



57.



58. (1)

$$0 < q!(e - s_q) = (q!)\sum_{k=q+1}^{\infty}\frac{1}{k!} = \frac{q!}{(q+1)!} + \frac{q!}{(q+2)!} + \frac{q!}{(q+3)!} + \cdots$$

$$\leq \frac{1}{q+1} + \frac{1}{(q+1)^2} + \frac{1}{(q+3)^3} + \cdots$$

$$\leq \frac{1}{q+1}\left[1 + \frac{1}{q+1} + \frac{1}{(q+1)^2} + \cdots\right]$$

geometric series

$$= \frac{1}{q+1}\left[\frac{1}{1 - 1/(q+1)}\right] = \frac{1}{q}$$

(2) If e equaled p/q, then $q!e$ would be an integer, and since $q!s_q$ is an integer, so would $q!(e - s_q)$. But $0 < q!(e - s_q) < \dfrac{1}{q} < 1$, impossible.

SECTION 11.6

1.

$f(x) = \sqrt{x} = x^{1/2};$ $\quad f(4) = 2$

$f'(x) = \dfrac{1}{2}x^{-1/2};$ $\quad f'(4) = \dfrac{1}{4}$

$f''(x) = -\dfrac{1}{4}x^{-3/2};$ $\quad f''(4) = -\dfrac{1}{32}$

$f'''(x) = \dfrac{3}{8}x^{-5/2};$ $\quad f'''(4) = \dfrac{3}{256}$

$f^{(4)}(x) = -\dfrac{15}{16}x^{-7/2}$

$$P_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1/32}{2!}(x-4)^2 + \frac{3/256}{3!}(x-4)^3$$
$$= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$$

$$R_4(x) = \frac{f^{(4)}(c)}{4!}(x-4)^4 = -\frac{15}{16}\cdot\frac{1}{4!}c^{-7/2}(x-4)^4 = -\frac{5}{128c^{7/2}}(x-4)^4,$$ where c is between 4 and x.

2. $f\left(\frac{\pi}{3}\right) = \frac{1}{2}$, $f'\left(\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$, $f''\left(\frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}$,

$f'''\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $f^{(4)}\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$, $f^{(5)}(c) = -\sin c$

$$P_4(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right) - \frac{1}{4}\left(x-\frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{2\cdot 3!}\left(x-\frac{\pi}{3}\right)^3 + \frac{1}{2\cdot 4!}\left(x-\frac{\pi}{3}\right)^4$$

$$R_5(x) = \frac{-\sin c}{5!}\left(x-\frac{\pi}{3}\right)^5,$$ where c is between $\pi/3$ and x.

3.

$f(x) = \sin x;$	$f(\pi/4) = \frac{\sqrt{2}}{2}$
$f'(x) = \cos x;$	$f'(\pi/4) = \frac{\sqrt{2}}{2}$
$f''(x) = -\sin x;$	$f''(\pi/4) = -\frac{\sqrt{2}}{2}$
$f'''(x) = -\cos x;$	$f'''(\pi/4) = -\frac{\sqrt{2}}{2}$
$f^{(4)}(x) = \sin x;$	$f^{(4)}(\pi/4) = \frac{\sqrt{2}}{2}$
$f^{(5)}(x) = \cos x$	

$$P_4(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) - \frac{\sqrt{2}/2}{2!}\left(x-\frac{\pi}{4}\right)^2 - \frac{\sqrt{2}/2}{3!}\left(x-\frac{\pi}{4}\right)^3 + \frac{\sqrt{2}/2}{4!}\left(x-\frac{\pi}{4}\right)^4$$
$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x-\frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{48}\left(x-\frac{\pi}{4}\right)^4$$

$$R_5(x) = \frac{f^{(5)}(c)}{5!}\left(x-\frac{\pi}{4}\right)^5 = \frac{\cos c}{120}\left(x-\frac{\pi}{4}\right)^5,$$ where c is between $\pi/4$ and x.

4. $f(1) = 0$, $f'(1) = \frac{1}{1} = 1$, $f''(1) = \frac{-1}{1^2} = -1$,

$f'''(1) = \frac{2}{1^3} = 2$, $f^{(4)}(1) = \frac{-6}{1^4} = -6$, $f^{(5)}(1) = \frac{4!}{1^5} = 4!$, $f^{(6)}(c) = \frac{-5!}{c^6}$

$$P_5(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5$$

$$R_6(x) = \frac{-5!}{c^6}\cdot\frac{1}{6!}(x-1)^6 = -\frac{1}{6}\left(\frac{x-1}{c}\right)^6,$$ where c is between 1 and x.

5. $f(x) = \tan^{-1}(x)$ $\qquad f(1) = \frac{\pi}{4}$

$f'(x) = \dfrac{1}{1+x^2}$ $\qquad f'(1) = \frac{1}{2}$

$f''(x) = \dfrac{-2x}{(1+x^2)^2}$ $\qquad f''(1) = -\frac{1}{2}$

$f'''(x) = \dfrac{6x^2-2}{(1+x^2)^2}$ $\qquad f'''(1) = \frac{1}{2}$

$f^{(4)}(x) = \dfrac{24(x-x^3)}{(1+x^2)^3}$

$$P_3(x) = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1/2}{2!}(x-1)^2 + \frac{1/2}{3!}(x-1)^3 = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3$$

$$R_4(x) = \frac{f^{(4)}(c)}{4!}(x-1)^4 = \frac{24(c-c^3)}{(1+c^2)^3}\cdot\frac{1}{4!}(x-1)^4 = \frac{c-c^3}{(1+c^2)^3}(x-1)^4, \quad \text{where } c \text{ is between 1 and } x.$$

6. $f\left(\frac{1}{2}\right) = 0, \quad f'\left(\frac{1}{2}\right) = -\pi\sin\frac{\pi}{2} = -\pi, \quad f''\left(\frac{1}{2}\right) = -\pi^2\cos\frac{\pi}{2} = 0,$

$f'''\left(\frac{1}{2}\right) = \pi^3\sin\frac{\pi}{2} = \pi^3, \quad f^{(4)}\left(\frac{1}{2}\right) = \pi^4\cos\frac{\pi}{2} = 0, \quad f^{(5)}(c) = -\pi^5\sin\pi c$

$P_4(x) = -\pi\left(x-\frac{1}{2}\right) + \frac{\pi^3}{3!}\left(x-\frac{1}{2}\right)^3; \quad R_5(x) = \dfrac{-\pi^5\sin c}{5!}\left(x-\frac{1}{2}\right)^5,$

where c is between $1/2$ and x.

7. $g(x) = 6 + 9(x-1) + 7(x-1)^2 + 3(x-1)^3, \quad (-\infty,\infty)$

8. $11 + 23(x-2) + 19(x-2)^2 + 7(x-2)^3 + (x-2)^4; \quad (-\infty,\infty)$

9. $g(x) = -3 + 5(x+1) - 19(x+1)^2 + 20(x+1)^3 - 10(x+1)^4 + 2(x+1)^5, \quad (-\infty,\infty)$

10. $\dfrac{1}{x} = \dfrac{1}{1+(x-1)} = \displaystyle\sum_{k=0}^{\infty}(-1)^k(x-1)^k; \quad (0,2)$

11. $$g(x) = \frac{1}{1+x} = \frac{1}{2+(x-1)} = \frac{1}{2}\left[\frac{1}{1+\left(\frac{x-1}{2}\right)}\right] = \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k\left(\frac{x-1}{2}\right)^k$$

(geometric series)

$$= \sum_{k=0}^{\infty}(-1)^k\frac{(x-1)^k}{2^{k+1}} \quad \text{for} \quad \left|\frac{x-1}{2}\right| < 1 \quad \text{and thus for} \quad -1 < x < 3$$

12.

$$\frac{1}{b+x} = \frac{1}{b+a+x-a} = \frac{1}{b+a}\cdot\frac{1}{1+\frac{x-a}{b+a}}$$

$$= \frac{1}{b+a}\sum_{k=0}^{\infty}(-1)^k\left(\frac{x-a}{b+a}\right)^k = \sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{a+b}\right)^{k+1}(x-a)^k, \quad (a-|a+b|,\, a+|a+b|)$$

13. $g(x)=\dfrac{1}{1-2x}=\dfrac{1}{5-2(x+2)}=\dfrac{1}{5}\left[\dfrac{1}{1-\frac{2}{5}(x+2)}\right]=\dfrac{1}{5}\sum_{k=0}^{\infty}\left[\dfrac{2}{5}(x+2)\right]^k$

(geometric series)

$$=\sum_{k=0}^{\infty}\frac{2^k}{5^{k+1}}(x+2)^k \quad \text{for} \quad \left|\frac{2}{5}(x+2)\right|<1 \quad \text{and thus for} \quad -\frac{9}{2}<x<\frac{1}{2}$$

14. $e^{-4x}=e^{-4(x+1)}e^4=e^4\sum_{k=0}^{\infty}\dfrac{(-1)^k4^k}{k!}(x+1)^k; \quad (-\infty,\infty)$

15. $g(x)=\sin x=\sin\left[(x-\pi)+\pi\right]=\sin(x-\pi)\cos\pi+\cos(x-\pi)\sin\pi$

$$=-\sin(x-\pi)=-\sum_{k=0}^{\infty}(-1)^k\frac{(x-\pi)^{2k+1}}{(2k+1)!}$$

(11.5.6)

$$=\sum_{k=0}^{\infty}(-1)^{k+1}\frac{(x-\pi)^{2k+1}}{(2k+1)!}, \quad (-\infty,\infty)$$

16. $\sin x=\cos\left(x-\dfrac{\pi}{2}\right)=\sum_{k=0}^{\infty}\dfrac{(-1)^k}{(2k)!}\left(x-\dfrac{\pi}{2}\right)^{2k}; \quad (-\infty,\infty)$

17. $g(x)=\cos x=\cos\left[(x-\pi)+\pi\right]=\cos(x-\pi)\cos\pi-\sin(x-\pi)\sin\pi$

$$=-\cos(x-\pi)=-\sum_{k=0}^{\infty}(-1)^k\frac{(x-\pi)^{2k}}{(2k)!}=\sum_{k=0}^{\infty}(-1)^{k+1}\frac{(x-\pi)^{2k}}{(2k)!}, \quad (-\infty,\infty)$$

(11.5.7)

18. $\cos x=-\sin\left(x-\dfrac{\pi}{2}\right)=\sum_{k=0}^{\infty}\dfrac{(-1)^{k+1}}{(2k+1)!}\left(x-\dfrac{\pi}{2}\right)^{2k+1}; \quad (-\infty,\infty)$

19. $g(x)=\sin\dfrac{1}{2}\pi x=\sin\left[\dfrac{\pi}{2}(x-1)+\dfrac{\pi}{2}\right]$

$$=\sin\left[\frac{\pi}{2}(x-1)\right]\cos\frac{\pi}{2}+\cos\left[\frac{\pi}{2}(x-1)\right]\sin\frac{\pi}{2}$$

$$=\cos\left[\frac{\pi}{2}(x-1)\right]=\sum_{k=0}^{\infty}(-1)^k\left(\frac{\pi}{2}\right)^{2k}\frac{(x-1)^{2k}}{(2k)!}, \ (-\infty,\infty)$$

(11.5.7)

20. $\sin\pi x=-\sin\pi(x-1)=-\sum_{k=0}^{\infty}\dfrac{(-1)^k}{(2k+1)!}\left[\pi(x-1)\right]^{2k+1}=\sum_{k=0}^{\infty}\dfrac{(-1)^{k+1}\pi^{2k+1}}{(2k+1)!}(x-1)^{2k+1}; \quad (-\infty,\infty)$

21. $g(x)=\ln(1+2x)=\ln\left[3+2(x-1)\right]=\ln\left[3\left(1+\frac{2}{3}(x-1)\right)\right]$

$$= \ln 3 + \ln\left[1 + \frac{2}{3}(x-1)\right] = \ln 3 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\left[\frac{2}{3}(x-1)\right]^k$$

(11.5.8)

$$= \ln 3 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\left(\frac{2}{3}\right)^k (x-1)^k.$$

This result holds if $-1 < \frac{2}{3}(x-1) \le 1$, which is to say, if $-\frac{1}{2} < x \le \frac{5}{2}$.

22. $\ln(2+3x) = \ln[14 + 3(x-4)] = \ln 14 + \ln\left[1 + \frac{3}{14}(x-4)\right] = \ln 14 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\left(\frac{3}{14}\right)^k (x-4)^k;$
$\left(-\frac{2}{3}, \frac{26}{3}\right]$

23.

$$\begin{aligned} g(x) &= x \ln x \\ g'(x) &= 1 + \ln x \\ g''(x) &= x^{-1} \\ g'''(x) &= -x^{-2} \\ g^{(\text{iv})}(x) &= 2x^{-3} \\ &\vdots \\ g^{(k)}(x) &= (-1)^k (k-2)! x^{1-k}, \quad k \ge 2. \end{aligned}$$

Then, $g(2) = 2\ln 2$, $g'(2) = 1 + \ln 2$, and $g^{(k)}(2) = \dfrac{(-1)^k (k-2)!}{2^{k-1}}$, $k \ge 2$.

Thus, $g(x) = 2 \ln 2 + (1 + \ln 2)(x-2) + \sum_{k=2}^{\infty} \dfrac{(-1)^k}{k(k-1)2^{k-1}}(x-2)^k.$

24. $g(2) = 4 + e^6$; $g'(x) = 2x + 3e^{3x}$, $g'(2) = 4 + 3e^6$, $g''(x) = 2 + 9e^{3x}$,

$g''(2) = 2 + 9e^6$, $g'''(x) = 27e^{3x}$, $g'''(2) = 27e^6$, $g^n(x) = 3^n e^{3x}$

$\Longrightarrow g(x) = (4 + e^6) + (4 + 3e^6)(x-2) + (1 + \frac{9}{2}e^6)(x-2)^2 + e^6 \sum_{k=3}^{\infty} \frac{3^k}{k!}(x-2)^k$

25. $g(x) = x \sin x = x \sum_{k=0}^{\infty} (-1)^k \dfrac{x^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} (-1)^k \dfrac{x^{2k+2}}{(2k+1)!}$

26. $\ln(x^2) = 2\ln x = 2\ln[1 + (x-1)] = 2\sum_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{k}(x-1)^k$

27.

$$\begin{aligned} g(x) &= (1-2x)^{-3} \\ g'(x) &= -2(-3)(1-2x)^{-4} \\ g''(x) &= (-2)^2(4\cdot 3)(1-2x)^{-5} \\ g'''(x) &= (-2)^3(-5\cdot 4\cdot 3)(1-2x)^{-6} \\ &\vdots \\ g^{(k)}(x) &= (-2)^k\left[(-1)^k\frac{(k+2)!}{2}\right](1-2x)^{-k-3}, \quad k\geq 0. \end{aligned}$$

Thus, $g^{(k)}(-2) = (-2)^k\left[(-1)^k\dfrac{(k+2)!}{2}\right]5^{-k-3} = \dfrac{2^{k-1}}{5^{k+3}}(k+2)!$

and $g(x) = \displaystyle\sum_{k=0}^{\infty}(k+2)(k+1)\frac{2^{k-1}}{5^{k+3}}(x-2)^k.$

28. $\sin^2 x = \dfrac{1}{2} - \dfrac{\cos 2x}{2} = \dfrac{1}{2} + \dfrac{1}{2}\cos 2(x-\dfrac{\pi}{2}) = \dfrac{1}{2} + \dfrac{1}{2}\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}2^{2k}(x-\frac{\pi}{2})^{2k}$

$$= 1 + \sum_{k=1}^{\infty}\frac{(-1)^k 2^{2k-1}}{(2k)!}\left(x-\frac{\pi}{2}\right)^{2k}$$

29. $g(x) = \cos^2 x = \dfrac{1+\cos 2x}{2} = \dfrac{1}{2} + \dfrac{1}{2}\cos[2(x-\pi)+2\pi]$

$$= \frac{1}{2} + \frac{1}{2}\cos[2(x-\pi)] = \frac{1}{2} + \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k\frac{[2(x-\pi)]^{2k}}{(2k)!}$$

$$= 1 + \sum_{k=1}^{\infty}\frac{(-1)^k 2^{2k-1}}{(2k)!}(x-\pi)^{2k}$$

($k=0$ term is $\frac{1}{2}$)

30. $g(x) = (1+2x)^{-4},\ g'(x) = -4(1+2x)^{-5}\cdot 2,\ g''(x) = 20(1+2x)^{-6}\cdot 4,\ g'''(x) = -120(1+2x)^{-7}\cdot 2^3,$

$g^n(x) = (-1)^n\dfrac{(n+3)!}{3!}\cdot 2^n\cdot(1+2x)^{-n-4} \quad g^n(2) = (-1)^n\dfrac{(n+3)!}{3!}\cdot\dfrac{2^n}{5^{n+4}}.$

$g(x) = \displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\cdot\frac{(k+3)!}{3!}\cdot\frac{2^k}{5^{k+4}}\cdot(x-2)^k = \sum_{k=0}^{\infty}\frac{(-1)^k}{3}(k+3)(k+2)(k+1)\frac{2^{k-1}}{5^{k+4}}(x-2)^k$

31.

$$\begin{aligned} g(x) &= x^n \\ g'(x) &= nx^{n-1} \\ g''(x) &= n(n-1)x^{n-2} \\ g'''(x) &= n(n-1)(n-2)x^{n-3} \\ &\vdots \\ g^{(k)}(x) &= n(n-1)\cdots(n-k+1)x^{n-k}, \qquad 0\leq k\leq n \\ g^{(k)}(x) &= 0, \qquad k>n. \end{aligned}$$

Thus,

$$g^{(k)}(1) = \left\{\begin{array}{ll} \dfrac{n!}{(n-k)!}, & 0\leq k\leq n \\ 0, & k>n \end{array}\right] \quad \text{and} \quad g(x) = \sum_{k=0}^{n}\frac{n!}{(n-k)!k!}(x-1)^k.$$

32. $(x-1)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} x^k (-1)^{n-k}$

33. (a) $\frac{e^x}{e^a} = e^{x-a} = \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!}, \quad e^x = e^a \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!}$

(b) $e^{a+(x-a)} = e^x = e^a \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!}, \quad e^{x_1+x_2} = e^{x_1} \sum_{k=0}^{\infty} \frac{x_2^k}{k!} = e^{x_1} e^{x_2}$

(c) $e^{-a} \sum_{k=0}^{\infty} (-1)^k \frac{(x-a)^k}{k!}$

34. (a) $\sin x = \sin a + (x-a)\cos a - \frac{(x-a)^2}{2!} \sin a + \frac{(x-a)^3}{3!} \cos a + \cdots$

$\cos x = \cos a - (x-a)\sin a - \frac{(x-a)^2}{2!} \cos a + \frac{(x-a)^3}{3!} \sin a + \cdots$

(b) in both instances $\sum_{k=0}^{\infty} |a_k| \le \sum_{k=0}^{\infty} \frac{|x-a|^k}{k!}$

(c)

$$\begin{aligned}
&\sin(x_1 + x_2) \\
&= \sin x_1 + x_2 \cos x_1 - \frac{(x_2)^2}{2!} \sin x_1 - \frac{(x_2)^3}{3!} \cos x_1 + \cdots \\
&= \left(\sin x_1 - \frac{{x_2}^2}{2!} \sin x_1 + \frac{{x_2}^4}{4!} \sin x_1 - \cdots\right) + \left(x_2 \cos x_1 - \frac{{x_2}^2}{3!} \cos x_1 + \frac{{x_2}^5}{5!} \cos x_1 - \cdots\right) \\
&= \sin x_1 \left(1 - \frac{{x_2}^2}{2!} + \frac{{x_2}^4}{4!} - \cdots\right) + \cos x_1 \left(x_2 - \frac{x_2}{3!} + \frac{{x_2}^5}{5!} - \cdots\right) \\
&= \sin x_1 \cos x_2 + \cos x_1 \sin x_2
\end{aligned}$$

The other formula can be derived in a similar manner.

35. (a) Let $g(x) = \sin x$ and $a = \pi/6$. Then

$$\begin{aligned}
|R_{n+1}(x)| &= \frac{|g^{(n+1)}(c)|}{(n+1)!} \left| \left(x - \frac{\pi}{6}\right)^{n+1} \right| \\
&\le \frac{\left|\left(x - \frac{\pi}{6}\right)\right|^{n+1}}{(n+1)!} \quad (g^{(n+1)}(c) = \pm \sin c \text{ or } \pm \cos c)
\end{aligned}$$

Now, $35^\circ = \frac{35\pi}{180}$ radians. We want to find the smallest positive integer n such that

$|R_{n+1}(35\pi/180)| < 0.00005$.

$$|R_{n+1}(35\pi/180)| \le \frac{\left(\frac{35\pi}{180} - \frac{\pi}{6}\right)^{n+1}}{(n+1)!} \cong \frac{(0.087266)^{n+1}}{(n+1)!} < 0.00005 \implies n \ge 3$$

$g(x) = \sin x; \qquad g(\pi/6) = \frac{1}{2}$

$g'(x) = \cos x;$ $\qquad g'(\pi/6) = \dfrac{\sqrt{3}}{2}$

$g''(x) = -\sin x;$ $\qquad g(\pi/6) = -\dfrac{1}{2}$

$g'''(x) = -\cos x;$ $\qquad g(\pi/6) = -\dfrac{\sqrt{3}}{2}$

$$\begin{aligned} P_3(x) &= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1/2}{2!}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}/2}{3!}\left(x - \frac{\pi}{6}\right)^3 \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)^3 \end{aligned}$$

(b) $P_3(35\pi/180) \cong 0.5736$

36. (a) $\cos x = \dfrac{1}{2} - \dfrac{\sqrt{3}}{2}\left(x - \dfrac{\pi}{3}\right) - \dfrac{1}{4}\left(x - \dfrac{\pi}{3}\right)^2 + \dfrac{\sqrt{3}}{2\cdot 3!}\left(x - \dfrac{\pi}{3}\right)^3 + \dfrac{1}{2\cdot 4!}\left(x - \dfrac{\pi}{3}\right)^4 - \cdots$

$57^\circ = \dfrac{\pi}{3} - \dfrac{\pi}{60}$ $\qquad \dfrac{\sqrt{3}}{2\cdot 3!}\left(\dfrac{\pi}{60}\right)^3 \cong 0.00002,$ so use $P_2(x)$

(b) $\cos 57^\circ \cong \dfrac{1}{2} - \dfrac{\sqrt{3}}{2}\left(-\dfrac{\pi}{60}\right) - \dfrac{1}{4}\left(\dfrac{\pi}{60}\right)^2 \cong 0.5446$

37. Let $g(x) = \sqrt{x} = x^{1/2}$ and $a = 36$.

(a) $g(x) = x^{1/2}$ $\qquad g(36) = 6$

$g'(x) = \dfrac{1}{2}x^{-1/2}$ $\qquad g'(36) = \dfrac{1}{12}$

$g''(x) = -\dfrac{1}{4}x^{-3/2}$ $\qquad g''(36) = -\dfrac{1}{864}$

$g'''(x) = \dfrac{3}{8}x^{-5/2}$ $\qquad g'''(36) = \dfrac{1}{20,736}$

We want to find the smallest positive integer n such that $|R_{n+1}(38)| < 0.0005$:

$$n = 1: \quad |R_2(38)| = \frac{c^{-3/2}/4}{2!}(38-36)^2 = \frac{4}{8c^{3/2}} = \frac{1}{2c^{3/2}}, \quad \text{where } 36 \le c \le 38,$$

and

$$|R_2(38)| \le \frac{1}{2(36)^{3/2}} = \frac{1}{432} \cong 0.0023.$$

$$n = 2: \quad |R_3(38)| = \frac{3c^{-5/2}/8}{3!}(38-36)^3 = \frac{8}{16c^{5/2}} = \frac{1}{2c^{5/2}}, \quad \text{where } 36 \le c \le 38,$$

and

$$|R_3(38)| \le \frac{1}{2(36)^{5/2}} = \frac{1}{15,552} \cong 0.000064.$$

Thus, we take $n = 2$:

$$P_2(x) = 6 + \frac{1}{12}(x-36) - \frac{1}{1728}(x-36)^2 \quad \text{and} \quad P_2(38) \cong 6.164$$

38. $\sqrt{x} = 8 + \frac{1}{2} \cdot \frac{1}{\sqrt{64}}(x-64) - \frac{1}{2} \cdot \frac{1}{4 \cdot 64^{3/2}}(x-64)^2 + \frac{1}{3!} \cdot \frac{3}{8 \cdot 64^{5/2}}(x-64)^3 - \cdots$

$|R_4(61)| \le \left| \frac{1}{4!} \cdot \frac{15}{16 \cdot 61^{7/2}} \cdot (-3)^4 \right| < 0.0001, \quad$ so take P_3

$\sqrt{61} \cong 8 + \frac{1}{16}(-3) - \frac{1}{8^4}(-3)^2 + \frac{1}{16 \cdot 8^5}(-3)^3 \cong 7.8102$

SECTION 11.7

1. (a) converges

(b) absolutely converges

(c) ?

(d) diverges

2. (a) absolutely converges

(b) absolutely converges

(c) ?

(d) ?

3. $(-1, 1)$; ratio test: $\frac{b_{k+1}}{b_k} = \frac{k+1}{k}|x| \to |x|$, series converges for $|x| < 1$.

At the endpoints $x = 1$ and $x = -1$ the series diverges since at those points $b_k \not\to 0$.

4. $[-1, 1)$; ratio test: $\left| \frac{x^{k+1}}{k+1} \cdot \frac{k}{x^k} \right| = \frac{k}{k+1}|x| \to |x| \implies r = 1$

At $x = 1 : \sum \frac{1}{k}$, diverges; at $x = -1, \sum \frac{(-1)^k}{k}$ converges.

5. $(-\infty, \infty)$; ratio test: $\frac{b_{k+1}}{b_k} = \frac{|x|}{(2k+1)(2k+2)} \to 0$, series converges all x.

6. $\left[-\frac{1}{2},\frac{1}{2}\right]$; root test: $\left|\frac{2^k}{k^2}x^k\right|^{1/k} = \frac{2\cdot|x|}{k^{2/k}} \to 2|x| \implies r = \frac{1}{2}$

At $x = \frac{1}{2}: \sum \frac{1}{k^2}$, converges; at $x = -\frac{1}{2}: \sum \frac{(-1)^k}{k^2}$ converges.

7. Converges only at 0; divergence test: $(-k)^{2k}x^{2k} \to 0$ only if $x = 0$, and series clearly converges at $x = 0$.

8. $(-1,1]$; root test: $\left|\frac{x^k}{\sqrt{k}}\right|^{1/k} = \frac{|x|}{k^{k/2}} \to |x| \implies r = 1$

At $x = 1: \sum \frac{(-1)^k}{\sqrt{k}}$, converges; at $x = -1: \sum \frac{1}{\sqrt{k}}$ diverges.

9. $[-2,2)$; root test: $(b_k)^{1/k} = \frac{|x|}{2k^{1/k}} \to \frac{|x|}{2}$, series converges for $|x| < 2$.

At $x = 2$ series becomes $\sum \frac{1}{k}$, the divergent harmonic series.

At $x = -2$ series becomes $\sum (-1)^k \frac{1}{k}$, a convergent alternating series.

10. $[-2,2]$; root test: $\left|\frac{x^k}{k^2 2^k}\right|^{1/k} = \frac{|x|}{k^{2/k}2} \to \frac{|x|}{2} \implies r = 2$

At $x = 2: \sum \frac{1}{k^2}$, converges; at $x = -2: \sum \frac{(-1)^k}{k^2}$ converges.

11. Converges only at 0; divergence test: $\left(\frac{k}{100}\right)^k x^k \to 0$ only if $x = 0$, and series clearly converges at $x = 0$.

12. $(-1,1)$; ratio test: $\frac{(k+1)^2|x|^{k+1}}{1+(k+1)^2} \cdot \frac{1+k^2}{k^2|x|^k} \to |x| \implies r = 1$

At $x = 1: \sum \frac{k^2}{1+k^2}$, diverges; at $x = -1: \sum \frac{(-1)^k k^2}{1+k^2}$ diverges.

13. $\left[-\frac{1}{2},\frac{1}{2}\right)$; root test: $(b_k)^{1/k} = \frac{2|x|}{\sqrt{k^{1/k}}} \to 2|x|$, series converges for $|x| < \frac{1}{2}$.

At $x = \frac{1}{2}$ series becomes $\sum \frac{1}{\sqrt{k}}$, a divergent p-series.

At $x = -\frac{1}{2}$ series becomes $\sum (-1)^k \frac{1}{\sqrt{k}}$, a convergent alternating series.

14. $[-1,1)$; ratio test: $\frac{|x|^{k+1}}{\ln(k+1)} \cdot \frac{\ln k}{|x|^k} = |x| \frac{\ln k}{\ln(k+1)} \to |x| \implies r = 1$

At $x=1$: $\sum \frac{1}{\ln k}$, diverges; at $x=-1$: $\sum \frac{(-1)^k}{\ln k}$ converges.

15. $(-1,1)$; ratio test: $\frac{b_{k+1}}{b_k} = \frac{k^2}{(k+1)(k-1)}|x| \to |x|$, series converges for $|x|<1$.

At the endpoints $x=1$ and $x=-1$ the series diverges since there $b_k \not\to 0$.

16. $\left(-\frac{1}{|a|}, \frac{1}{|a|}\right)$; root test: $\left|ka^k x^k\right|^{1/k} = k^{1/k}|a|\cdot|x| \to |a|\cdot|x| \implies r = \frac{1}{|a|}$

At $x = \frac{1}{|a|}$: $\sum ka^k \frac{1}{|a|^k}$, diverges, similarly at $x = -\frac{1}{|a|}$.

17. $(-10,10)$; root test: $(b_k)^{1/k} = \frac{k^{1/k}}{10}|x| \to \frac{|x|}{10}$, series converges for $|x|<10$.

At the endpoints $x=10$ and $x=-10$ the series diverges since there $b_k \not\to 0$.

18. $(-e,e)$; ratio test: $\left|\frac{3k^2x^k}{e^k}\right|^{1/k} = \frac{3^{1/k}k^{2/k}}{e}|x| \to \frac{|x|}{e} \implies r=e$

Diverges at $x=\pm e$.

19. $(-\infty,\infty)$; root test: $(b_k)^{1/k} = \frac{|x|}{k} \to 0$, series converges all x.

20. $(-\infty,\infty)$; ratio test: $\frac{7^{k+1}|x|^{k+1}}{(k+1)!}\cdot\frac{k!}{7^k|x|^k} = \frac{7|x|}{k+1} \to 0, \implies r=\infty$

21. $(-\infty,\infty)$; root test: $(b_k)^{1/k} = \frac{|x-2|}{k} \to 0$, series converges all x.

22. converges only at 0, ratio test: $\frac{(k+1)!|x|^{k+1}}{k!|x|^k} = (k+1)|x| \to \infty \implies r=0$

23. $\left(-\frac{3}{2}, \frac{3}{2}\right)$; ratio test: $\frac{b_{k+1}}{b_k} = \frac{\frac{2^{k+1}}{3^{k+2}}|x|}{\frac{2^k}{3^{k+1}}} = \frac{2}{3}|x|$, series converges for $|x| < \frac{3}{2}$.

At the endpoints $x=3/2$ and $x=-3/2$, the series diverges since there $b_k \not\to 0$.

24. $(-\infty,\infty)$; ratio test: $\frac{2^{k+1}|x|^{k+1}}{(2k+2)!}\cdot\frac{(2k)!}{2^k|x|^k} = \frac{2|x|}{(2k+1)(2k+2)} \to 0 \implies r=\infty$

25. Converges only at $x = 1$; ratio test: $\dfrac{b_{k+1}}{b_k} = \dfrac{k^3}{(k+1)^2}|x-1| \to \infty$ if $x \neq 1$

The series clearly converges at $x = 1$; otherwise it diverges.

26. $\left[-\dfrac{1}{e}, \dfrac{1}{e}\right]$ root test: $\left|\dfrac{(-e)^k}{k^2}x^k\right|^{1/k} = \dfrac{e|x|}{k^{2/k}} \to e|x| \implies r = \dfrac{1}{e}$.

Converges at $x = \pm\dfrac{1}{e}$;

27. $(-4, 0)$; ratio test: $\dfrac{b_{k+1}}{b_k} = \dfrac{k^2-1}{2k^2}|x+2| \to \dfrac{|x+2|}{2}$, series converges for $|x+2| < 2$.

At the endpoints $x = 0$ and $x = -4$, the series diverges since there $b_k \not\to 0$.

28. $[-2, 0)$; ratio test: $\dfrac{\ln(k+1)}{k+1}|x+1|^{k+1} \cdot \dfrac{k}{\ln k|x+1|^k} = \dfrac{\ln(k+1)}{\ln k} \cdot \dfrac{k}{k+1}|x+1| \to |x+1| \implies r = 1$

At $x = 0$, $\sum \dfrac{\ln k}{k}$ diverges, at $x = -2$, $\sum \dfrac{\ln k}{k}(-1)^k$ converges.

29. $(-\infty, \infty)$; ratio test: $\dfrac{b_{k+1}}{b_k} = \dfrac{(k+1)^2}{k^2(k+2)}|x+3| \to 0$, series converges for all x.

30. $(a-e, a+e)$; root test: $\left|\dfrac{k^3}{e^k}(x-a)^k\right|^{1/k} = \dfrac{k^{3/k}}{e}|x-a| \to \dfrac{|x-a|}{e} \implies r = e$

At $x = a+e$, $\sum k^3$ diverges, at $x = a-e$, $\sum(-1)^k k^3$ diverges.

31. $(-1, 1)$; root test: $(b_k)^{1/k} = \left(1 + \dfrac{1}{k}\right)|x| \to |x|$, series converges for $|x| < 1$.

At the endpoints $x = 1$ and $x = -1$, the series diverges since there $b_k \not\to 0$

[recall $\left(1 + \dfrac{1}{k}\right)^k \to e$]

32. $\left[a - \dfrac{1}{|a|}, a + \dfrac{1}{|a|}\right]$; root test: $\left|\dfrac{(-1)^k a^k}{k^2}(x-a)^k\right|^{1/k} = \dfrac{|a|}{k^{2/k}}|x-a| \to |a| \cdot |x-a| \implies r = \dfrac{1}{|a|}$

Converges at both $a - \dfrac{1}{|a|}, a + \dfrac{1}{|a|}$ $\left(\text{compare with } \sum \dfrac{1}{k^2}\right)$.

33. $(0, 4)$; ratio test: $\dfrac{b_{k+1}}{b_k} = \dfrac{\ln(k+1)}{\ln k}\dfrac{|x-2|}{2} \to \dfrac{|x-2|}{2}$, series converges for $|x-2| < 2$.

At the endpoints $x = 0$ and $x = 4$ the series diverges since there $b_k \not\to 0$.

34. $(-\infty, \infty)$; root test: $\dfrac{|x-1|}{\ln k} \to 0 \implies r = \infty$

35. $\left(-\frac{5}{2}, \frac{1}{2}\right)$; root test: $(b_k)^{1/k} = \frac{2}{3}|x+1| \to \frac{2}{3}|x+1|$, series converges for $|x+1| < \frac{3}{2}$.

At the endpoints $x = -\frac{5}{2}$ and $x = \frac{1}{2}$ the series diverges since there $b_k \not\to 0$.

36. $\left[2 - \frac{1}{\pi}, 2 + \frac{1}{\pi}\right]$; ratio test: $\dfrac{2^{\frac{1}{k+1}}\pi^{k+1}|x-2|^{k+1}}{(k+1)(k+2)(k+3)} \cdot \dfrac{k(k+1)(k+2)}{2^{\frac{1}{k}}\pi^k|x-2|^k} \to \pi|x-2| \implies r = \dfrac{1}{\pi}$

Converges at $x = 2 \pm \dfrac{1}{\pi}$ $\left(\text{compare with } \sum \dfrac{1}{k^3}\right)$.

37. $1 - \dfrac{x}{2} + \dfrac{2x^2}{4} - \dfrac{3x^3}{8} + \dfrac{4x^4}{16} - \cdots = 1 + \sum_{k=1}^{\infty} (-1)^k \dfrac{kx^k}{2^k}$

$(-2, 2)$; ratio test: $\dfrac{b_{k+1}}{b_k} = \dfrac{k+1}{2k}|x| \to \dfrac{|x|}{2}$, series converges for $|x| < 2$.

At the endpoints $x = 2$ and $x = -2$ the series diverges since there $b_k \not\to 0$.

38. $(-24, 26)$; root test: $\left|\dfrac{k^2}{5^{2k}}(x-1)^k\right|^{1/k} = \dfrac{k^{2/k}}{5^2}|x-1| \to \dfrac{|x-1|}{25} \implies r = 25$

At $x = -24$, $\sum (-1)^k k^2$ diverges, at $x = 26$, $\sum k^2$ diverges.

39. $\dfrac{3x^2}{4} + \dfrac{9x^4}{9} + \dfrac{27x^6}{16} + \dfrac{81x^8}{25} + \cdots = \sum_{k=1}^{\infty} \dfrac{3^k}{(k+1)^2} x^{2k}$

$\left[-\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right]$; ratio test: $\dfrac{b_{k+1}}{b_k} = \dfrac{3(k+1)^2}{(k+2)^2}x^2 \to 3x^2$, series converges for $x^2 < \dfrac{1}{3}$ or $|x| < \dfrac{1}{\sqrt{3}}$.

At $x = \pm\dfrac{1}{\sqrt{3}}$, the series becomes $\sum \dfrac{1}{(k+1)^2} \cong \sum \dfrac{1}{n^2}$, a convergent series p-series.

40. $(-2, 0]$; ratio test: $\dfrac{(k+1)|x+1|^{k+1}}{(k+4)^2} \cdot \dfrac{(k+3)^2}{k|x+1|^k} \to |x+1| \implies r = 1$

At $x = 0$, $\sum \dfrac{k(-1)^{k+1}}{(k+3)^2}$ converges; at $x = -2$, $-\sum \dfrac{k}{(k+3)^2}$ diverges.

41. (a) absolutely converges

(b) absolutely converges

(c) ?

42. It must converge absolutely for $-8 < x < 4$.

43. Examine the convergence of $\sum |a_k x^k|$; for (a) use the root test and for (b) use the ratio rest.

44. $(-1,1)$; $s_k \le k$ and $\sum kx^k$ converges for $|x|<1$; for $|x| \ge 1$, $s_k x^k \not\to 0$.

45. $\sum |a_k r^k| = \sum |a_k(-r)^k|$

46. By ratio test: $\left|\dfrac{a_{k+1}}{a_k}\right| \cdot |x| \to \dfrac{|x|}{r}$, so $\left|\dfrac{a_{k+1}x^{2(k+1)}}{a_k x^{2k}}\right| = \left|\dfrac{a_{k+1}}{a_k}\right| |x|^2 \to \dfrac{|x|^2}{r}$

$\Longrightarrow$ radius of convergence is $\sqrt{r}$.

SECTION 11.8

1. Use the fact that $\dfrac{d}{dx}\left(\dfrac{1}{1-x}\right) = \dfrac{1}{(1-x)^2}$:

$$\frac{1}{(1-x)^2} = \frac{d}{dx}(1+x+x^2+x^3+\cdots+x^n+\cdots) = 1+2x+3x^2+\cdots+nx^{n-1}+\cdots.$$

2

$$\begin{aligned}\frac{1}{(1-x)^3} &= \frac{1}{2}\frac{d^2}{dx^2}\left[\frac{1}{1-x}\right] = \frac{1}{2}\frac{d^2}{dx^2}\left[1+x+x^2+\cdots+x^n+\cdots\right]\\ &= \frac{1}{2}\left[2+6x+12x^2+\cdots+n(n-1)x^{n-2}+\cdots\right]\\ &= 1+3x+6x^2+\cdots+\frac{n(n-1)}{2}x^{n-2}+\cdots\end{aligned}$$

3. Use the fact that $\dfrac{d^{(k-1)}}{dx^{(k-1)}}\left[\dfrac{1}{1-x}\right] = \dfrac{(k-1)!}{(1-x)^k}$:

$$\begin{aligned}\frac{1}{(1-x)^k} &= \frac{1}{(k-1)!}\frac{d^{(k-1)}}{dx^{(k-1)}}\left[1+x+\cdots+x^{k-1}+x^k+x^{k+1}+\cdots+x^{n+k-1}+\cdots\right]\\ &= \frac{1}{(k-1)!}\frac{d^{(k-1)}}{dx^{(k-1)}}\left[x^{k-1}+x^k+x^{k+1}+\cdots+x^{n+k-1}+\cdots\right]\\ &= 1+kx+\frac{(k+1)k}{2}x^2+\cdots+\frac{(n+k-1)(n+k-2)\cdots(n+1)}{(k-1)!}x^n+\cdots\\ &= 1+kx+\frac{(k+1)k}{2!}x^2+\cdots+\frac{(n+k-1)!}{n!(k-1)!}x^n+\cdots.\end{aligned}$$

4. $\ln(1-x) = -\displaystyle\int \frac{dx}{1-x} = -\left[x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{x^{n+1}}{n+1}+\cdots\right]+C$; $\ln 1 = 0 \Longrightarrow C=0$

$\Longrightarrow \ln(1-x) = -x-\dfrac{x^2}{2}-\dfrac{x^3}{3}-\cdots-\dfrac{x^{n+1}}{n+1}-\cdots$

5. Use the fact that $\dfrac{d}{dx}[\ln(1-x^2)] = \dfrac{-2x}{1-x^2}$:

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + \cdots + x^{2n} + \cdots$$

$$\frac{-2x}{1-x^2} = -2x - 2x^3 - 2x^5 - \cdots - 2x^{2n+1} - \cdots.$$

By integration

$$\ln(1-x^2) = \left(-x^2 - \frac{1}{2}x^4 - \frac{1}{3}x^6 - \cdots - \frac{x^{2n+2}}{n+1} - \cdots\right) + C.$$

At $x = 0$, both $\ln(1-x^2)$ and the series are 0. Thus, $C = 0$ and

$$\ln(1-x^2) = -x^2 - \frac{1}{2}x^4 - \frac{1}{3}x^6 - \cdots - \frac{1}{n+1}x^{2n+2} - \cdots.$$

6. $\ln(2-3x) = \ln 2 + \ln\left(1 - \frac{3}{2}x\right) = \ln 2 - \frac{3}{2}x - \frac{1}{2}\left(\frac{3}{2}\right)^2 x^2 - \frac{1}{3}\left(\frac{3}{2}\right)^3 x^3 - \cdots - \frac{1}{n+1}\left(\frac{3}{2}\right)^{n+1} x^{n+1} - \cdots$

7. $\sec^2 x = \dfrac{d}{dx}(\tan x) = \dfrac{d}{dx}\left(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots\right) = 1 + x^2 + \frac{2}{3}x^4 + \frac{17}{45}x^6 + \cdots$

8. $\ln\cos x = -\displaystyle\int \frac{\sin x}{\cos x}\,dx = -\int \tan x\,dx = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17}{2520}x^8 - \cdots + C$

$\ln\cos 0 = 0 \implies C = 0$

9. On its interval of convergence a power series is the Taylor series of its sum. Thus,

$$\begin{aligned} f(x) = x^2\sin^2 x &= x^2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\ &= x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \frac{x^9}{7!} + \cdots = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!} \end{aligned}$$

implies $f^{(9)}(0) = -9!/7! = -72$.

10. $f(x) = x\cos x^2 = x\left(1 - \dfrac{(x^2)^2}{2!} + \dfrac{(x^2)^4}{4!} - \dfrac{(x^2)^6}{6!} + \cdots\right)$

$\implies \dfrac{f^{(9)}(0)}{9!}x^9 = \dfrac{x^9}{4!} \implies f^{(9)}(0) = \dfrac{9!}{4!} = 15120.$

11. $\sin x^2 = \displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{(x^2)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty}(-1)^k\frac{x^{4k+2}}{(2k+1)!}$

12. $$x^2 \tan^{-1} x = x^2 \int \frac{1}{1+x^2}\,dx = x^2 \int \left(\sum_{k=0}^{\infty}(-1)^k x^{2k}\right) dx = x^2 \left[\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} + C\right]$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+3} \quad (\tan^{-1} 0 = 0 \implies C = 0)$$

13. $$e^{3x^3} = \sum_{k=0}^{\infty} \frac{(3x^3)^k}{k!} = \sum_{k=0}^{\infty} \frac{3^k}{k!} x^{3k}$$

14. $$\frac{1-x}{1+x} = \frac{1}{1+x} - \frac{x}{1+x} = \sum_{k=0}^{\infty}(-1)^k x^k - \sum_{k=0}^{\infty}(-1)^k x^{k+1} = 1 + 2\sum_{k=0}^{\infty}(-1)^{k+1} x^{k+1}$$

15. $$\frac{2x}{1-x^2} = 2x\left(\frac{1}{1-x^2}\right) = 2x\sum_{k=0}^{\infty}(x^2)^k = \sum_{k=0}^{\infty} 2x^{2k+1}$$

16. $$x \sinh x^2 = \frac{x}{2}\left(e^{x^2} - e^{-x^2}\right) = \frac{x}{2}\left[\sum_{k=0}^{\infty} \frac{x^{2k}}{k!} - \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!}\right] = \frac{x}{2}\left[2\sum_{k=0}^{\infty} \frac{x^{4k+2}}{(2k+1)!}\right]$$
$$= \sum_{k=0}^{\infty} \frac{x^{4k+3}}{(2k+1)!}$$

17. $$\frac{1}{1-x} + e^x = \sum_{k=0}^{\infty} x^k + \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{(k!+1)}{k!} x^k$$

18. $$\cosh x \sinh x = \frac{1}{2}\sinh 2x = \frac{1}{2}\sum_{k=0}^{\infty} \frac{(2x)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{4^k}{(2k+1)!} x^{2k+1}$$

19. $$x \ln(1+x^3) = x\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(x^3)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^{3k+1}$$
(11.5.8)

20. $$(x^2+x)\ln(1+x) = (x^2+x)\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x^2 + \sum_{k=3}^{\infty} \frac{(-1)^{k+1}}{(k-1)(k-2)} x^k$$

21. $$x^3 e^{-x^3} = x^3 \sum_{k=0}^{\infty} \frac{(-x^3)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{3k+3}$$

22. $$x^5(\sin x + \cos 2x) = x^5\left[\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} 2^{2k} x^{2k}\right]$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left[(2k+1)4^k x^{2k+5} + x^{2k+5}\right]$$

23. (a) $\lim_{x\to 0}\dfrac{1-\cos x}{x^2} \overset{\star}{=} \lim_{x\to 0}\dfrac{\sin x}{2x} = \dfrac{1}{2}$ ($\star$ indicates differentiation of numerator and denominator).

(b) $\lim_{x\to 0}\dfrac{1-\cos x}{x^2} = \lim_{x\to 0}\dfrac{\dfrac{x^2}{2!}-\dfrac{x^4}{4!}+\dfrac{x^6}{6!}-\cdots}{x^2} = \lim_{x\to 0}\left(\dfrac{1}{2}-\dfrac{x^2}{4!}+\dfrac{x^4}{6!}-\cdots\right) = \dfrac{1}{2}$

24. (a) $\lim_{x\to 0}\dfrac{\sin x - x}{x^2} \overset{\star}{=} \lim_{x\to 0}\dfrac{\cos x - 1}{2x} \overset{\star}{=} \lim_{x\to 0} -\dfrac{\sin x}{2} = 0$

(b) $\dfrac{\sin x - x}{x^2} = \dfrac{-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\cdots}{x^2} \to 0 \quad \text{as} \quad x\to 0$

25. (a) $\lim_{x\to 0}\dfrac{\cos x - 1}{x\sin x} \overset{\star}{=} \lim_{x\to 0}\dfrac{-\sin x}{\sin x + x\cos x} \overset{\star}{=} \lim_{x\to 0}\dfrac{-\cos x}{2\cos x - x\sin x} = -\dfrac{1}{2}$

(b)

$$\lim_{x\to 0}\frac{\cos x - 1}{x\sin x} = \frac{-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\cdots}{x^2-\dfrac{x^4}{3!}+\dfrac{x^6}{5!}\cdots}$$
$$= \frac{-\dfrac{1}{2}+\dfrac{x^2}{4!}-\dfrac{x^4}{6!}+\cdots}{1-\dfrac{x^2}{3!}+\dfrac{x^4}{5!}\cdots} = -\frac{1}{2}$$

26. (a) $\lim_{x\to 0}\dfrac{e^x-1-x}{x\tan^{-1}x} \overset{\star}{=} \lim_{x\to 0}\dfrac{e^x-1}{\tan^{-1}x + x/(1+x^2)} \overset{\star}{=} \lim_{x\to 0}\dfrac{e^x}{\dfrac{1}{1+x^2}+\dfrac{1-x^2}{(1+x^2)^2}} = \dfrac{1}{2}$

(b) $\dfrac{e^x-1-x}{x\tan^{-1}x} = \dfrac{\dfrac{x^2}{2}+\dfrac{x^3}{3!}+\dfrac{x^4}{4!}+\cdots}{x^2-\dfrac{x^4}{3}+\dfrac{x^6}{5}-\cdots} \to \dfrac{1}{2}.$

27.

$$\int_0^x \frac{\ln(1+t)}{t}\,dt = \int_0^x \frac{1}{t}\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}t^k\right)dt = \int_0^x\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}t^{k-1}\right)dt$$
$$= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\int_0^x t^{k-1}\,dt = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}x^k,\ -1\le x\le 1$$

28.

$$\int_0^x \frac{1-\cos t}{t^2}\,dt = \int_0^x \frac{1}{t^2}\left[\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k!)}t^{2k}\right]dt$$
$$= \int_0^x\left[\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k)!}t^{2k-2}\right]dt = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k)!}\cdot\frac{x^{2k-1}}{2k-1}$$

29. $$\int_0^x \frac{\tan^{-1} t}{t}\,dt = \int_0^x \frac{1}{t}\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} t^{2k+1}\right) dt = \int_0^x \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} t^{2k}\right) dt$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \int_0^x t^{2k}\,dt$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} x^{2k+1}, \quad -1 \le x \le 1$$

30. $$\int_0^x \frac{\sinh t}{t}\,dt = \int_0^x \frac{1}{t}\left[\sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}\right] dt = \int_0^x \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k+1)!}\,dt = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)^2(2k)!}$$

31. $0.804 \le I \le 0.808$; $$I = \int_0^1 \left(1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + \cdots\right) dx$$
$$= \left[x - \frac{x^4}{4} + \frac{x^7}{14} - \frac{x^{10}}{60} + \frac{x^{13}}{(13)(24)} - \cdots\right]_0^1$$
$$= 1 - \frac{1}{4} + \frac{1}{14} - \frac{1}{60} + \frac{1}{312} - \cdots.$$

Since $\dfrac{1}{312} < 0.01$, we can stop there:

$$1 - \frac{1}{4} + \frac{1}{14} - \frac{1}{60} \le I \le 1 - \frac{1}{4} + \frac{1}{14} - \frac{1}{60} + \frac{1}{312} \quad \text{gives} \quad 0.804 \le I \le 0.808.$$

32. $$I = \int_0^1 \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots\right) dx = \frac{1}{3} - \frac{1}{7(3!)} + \frac{1}{11(5!)} - \frac{1}{15(7!)} + \cdots.$$

Since $\dfrac{1}{11(5!)} = \dfrac{1}{840} < 0.01$, we can stop there:

$$\frac{1}{3} - \frac{1}{7(3!)} \le I \le \frac{1}{3} - \frac{1}{7(3!)} + \frac{1}{11(5!)} \quad \text{gives} \quad 0.309 \le I \le 0.311.$$

33. $0.600 \le I \le 0.603$; $$I = \int_0^1 \left(x^{1/2} - \frac{x^{3/2}}{3!} + \frac{x^{5/2}}{5!} - \cdots\right) dx$$
$$= \left[\frac{2}{3}x^{3/2} - \frac{1}{15}x^{5/2} + \frac{1}{420}x^{7/2} - \cdots\right]_0^1$$
$$= \frac{2}{3} - \frac{1}{15} + \frac{1}{420} - \cdots.$$

Since $\dfrac{1}{420} < 0.01$, we can stop there:

$$\frac{2}{3} - \frac{1}{15} \le I \le \frac{2}{3} - \frac{1}{15} + \frac{1}{420} \quad \text{gives} \quad 0.600 \le I \le 0.603.$$

34. $$I = \int_0^1 \left(x^4 - x^6 + \frac{x^8}{2!} - \frac{x^{10}}{3!} + \frac{x^{12}}{4!} - \cdots\right) dx = \frac{1}{5} - \frac{1}{7} + \frac{1}{18} - \frac{1}{66} + \frac{1}{312} - \cdots.$$

Since $\dfrac{1}{312} < 0.01$, we can stop there:

$$\frac{1}{5} - \frac{1}{7} + \frac{1}{18} - \frac{1}{16} \le I \le \frac{1}{5} - \frac{1}{7} + \frac{1}{18} - \frac{1}{66} + \frac{1}{312} \quad \text{gives} \quad 0.097 \le I \le 0.101.$$

35. $0.294 \le I \le 0.304;$ $\quad I = \int_0^1 \left(x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} + \cdots\right) dx$

(11.8.7)

$$= \left[\frac{1}{3}x^3 - \frac{1}{21}x^7 + \frac{1}{55}x^{11} - \frac{1}{105}x^{15} + \cdots\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{21} + \frac{1}{55} - \frac{1}{105} + \cdots.$$

Since $\frac{1}{105} < 0.01$, we can stop there:

$$\frac{1}{3} - \frac{1}{21} + \frac{1}{55} - \frac{1}{105} \le I \le \frac{1}{3} - \frac{1}{21} + \frac{1}{55} \quad \text{gives} \quad 0.294 \le I \le 0.304.$$

36. $I = \int_1^2 \left(\frac{x^2}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \frac{x^7}{8!} + \cdots\right) dx$

$$= \frac{3}{2(2!)} - \frac{15}{4(4!)} + \frac{63}{6(6!)} - \frac{255}{8(8!)} + \cdots = \frac{3}{4} - \frac{15}{96} + \frac{63}{4320} - \frac{255}{322560} + \cdots.$$

Since $\frac{255}{322560} < 0.01$, we can stop there:

$$\frac{3}{4} - \frac{15}{96} + \frac{63}{4320} - \frac{255}{322560} \le I \le \frac{3}{4} - \frac{15}{96} + \frac{63}{4320} \quad \text{gives} \quad 0.607 \le I \le 0.609.$$

37. $I \cong 0.9461;$ $\quad I = \int_0^1 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right) dx$

$$= \left[x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \cdots\right]_0^1$$

$$= 1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!} - \frac{1}{7\cdot 7!}\cdots.$$

Since $\frac{1}{7\cdot 7!} = \frac{1}{35,280} \cong 0.000028 < 0.0001$, we can stop there:

$$1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!} - \frac{1}{7\cdot 7!} < I < 1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!}; \quad I \cong 0.9461$$

38. $I = \int_0^{0.5} \left(\frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \frac{x^7}{8!} + \cdots\right) dx = \frac{0.5}{2!} - \frac{(0.5)^3}{3\cdot 4!} + \frac{(0.5)^5}{5\cdot 6!} - \frac{(0.5)^7}{7\cdot 8!} + \cdots$

$\frac{(0.5)^5}{5\cdot 6!} < 0.0001;$ $\quad I \cong 0.2483$

39. $I \cong 0.4485;$ $\quad I = \int_0^{0.5} \left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots\right) dx$

$$= \left[x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \cdots\right]_0^{1/2}$$

$$= \frac{1}{2} - \frac{1}{2^2\cdot 2^2} + \frac{1}{3^2\cdot 2^3} - \frac{1}{4^2\cdot 2^4} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2\cdot 2^k}$$

Now, $\frac{1}{8^2\cdot 2^8} = \frac{1}{16,384} \cong 0.000061$ is the first term which is less than 0.0001. Thus

$$\sum_{k=1}^{7}\frac{(-1)^{k-1}}{k^2\cdot 2^k} < I < \sum_{k=1}^{8}\frac{(-1)^{k-1}}{k^2\cdot 2^k};\quad I \cong 0.4485$$

40. $I=\int_0^{0.2}\left(x^2-\frac{x^4}{3!}+\frac{x^6}{5!}-\frac{x^8}{7!}+\cdots\right)dx = \frac{(0.2)^3}{3}-\frac{(0.2)^5}{5\cdot 3!}+\frac{(0.2)^7}{7\cdot 5!}-\cdots$

$\frac{(0.2)^5}{5\cdot 3!} < 0.0001,$ so $I \cong 0.0027$

41. e^{x^3}; by (11.5.5)

42. $\sum_{k=0}^{\infty}\frac{1}{k!}x^{3k+1} = x\sum_{k=0}^{\infty}\frac{1}{k!}(x^3)^k = xe^{x^3}$

43. $3x^2e^{x^3} = \frac{d}{dx}(e^{x^3})$

44. (a) $f(x)=\frac{e^x-1}{x}=\frac{1}{x}\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)=1+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\cdots$

(b) $f'(x)=\frac{xe^x-e^x+1}{x^2}=\frac{1}{2}+\frac{2x}{3!}+\frac{3x^2}{4!}+\cdots+\frac{nx^{n-1}}{(n+1)!}+\cdots$

$$f'(1)=1=\sum_{k=1}^{\infty}\frac{k}{(k+1)!}$$

45. (a) $f(x)=xe^x=x\sum_{k=0}^{\infty}\frac{x^k}{k!}=\sum_{k=0}^{\infty}\frac{x^{k+1}}{k!}$

(b) Using integration by parts: $\int_0^1 xe^x\,dx = [xe^x-e^x]_0^1 = e-e+1=1.$

Using the power series representation:

$$\int_0^1 xe^x\,dx = \int_0^1\left(\sum_{k=0}^{\infty}\frac{x^{k+1}}{k!}\right)dx = \sum_{k=0}^{\infty}\int_0^1\left(\frac{x^{k+1}}{k!}\right)dx$$
$$=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{x^{k+2}}{k+2}\right]_0^1$$
$$=\sum_{k=0}^{\infty}\frac{1}{k!(k+2)}$$
$$=\frac{1}{2}+\sum_{k=1}^{\infty}\frac{1}{k!(k+2)}$$

Thus, $1=\frac{1}{2}+\sum_{k=1}^{\infty}\frac{1}{k!(k+2)}$ and $\sum_{k=1}^{\infty}\frac{1}{k!(k+2)}=\frac{1}{2}.$

46. $\frac{d}{dx}(\sinh x)=\frac{d}{dx}\left[\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}\right]=\sum_{k=0}^{\infty}\frac{(2k+1)x^{2k}}{(2k+1)!}=\sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!}=\cosh x$

$\frac{d}{dx}(\cosh x)=\frac{d}{dx}\left[\sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!}\right]=\sum_{k=1}^{\infty}\frac{2kx^{2k-1}}{(2k)!}=\sum_{k=1}^{\infty}\frac{x^{2k-1}}{(2k-1)!}=\sinh x.$

47. Let $f(x)$ be the sum of these series; a_k and b_k are both $\dfrac{f^{(k)}(0)}{k!}$.

48. As $k \to \infty$,

$$k^{1/k}|x|^{(k-1)/k} \to |x|.$$

Thus, for k sufficiently large,

$$k^{1/k}|x|^{(k-1)/k} < |x| + \epsilon \quad \text{and} \quad |kx^{k-1}| = k|x|^{k-1} < (|x| + \epsilon)^k.$$

49. (a) If f is even, then the odd ordered derivatives $f^{(2k-1)}$, $k = 1, 2, \ldots$ are odd. This implies that $f^{(2k-1)}(0) = 0$ for all k and so $a_{2k-1} = f^{(2k-1)}(0)/(2k-1)! = 0$ for all k.

(b) If f is odd, then all the even ordered derivatives $f^{(2k)}$, $k = 1, 2, \ldots$ are odd. This implies that $f^{(2k)}(0) = 0$ for all k and so $a_{2k} = f^{(2k)}(0)/(2k)! = 0$ for all k.

50. $f(0) = 1, \quad f'(0) = -2f(0) = -2, \quad f''(x) = -2f'(x) = 4f(x), \quad f''(0) = 4$

$$f^n(x) = (-2)^n f(x), \quad f^n(0) = (-2)^n, \quad f(x) = \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} x^k = \sum_{k=0}^{\infty} \frac{(-2x)^k}{k!} = e^{-2x}$$

51. $0.0352 \le I \le 0.0359$;

$$I = \int_0^{1/2} \left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \cdots \right) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{8} + \frac{x^5}{15} - \frac{x^6}{24} + \cdots \right]_0^{1/2}$$

$$= \frac{1}{3(2^3)} - \frac{1}{8(2^4)} + \frac{1}{15(2^5)} - \frac{1}{24(2^6)} + \cdots.$$

Since $\dfrac{1}{24(2^6)} = \dfrac{1}{1536} < 0.001$, we can stop there:

$$\frac{1}{3(2^3)} - \frac{1}{8(2^4)} + \frac{1}{15(2^5)} - \frac{1}{24(2^6)} \le I \le \frac{1}{3(2^3)} - \frac{1}{8(2^4)} + \frac{1}{15(2^5)}$$

gives $0.0352 \le I \le 0.0359$. Direct integration gives

$$I = \int_0^{1/2} x \ln(1+x)\, dx = \left[\frac{1}{2}(x^2 - 1)\ln(1+x) - \frac{1}{4}x^2 + \frac{1}{2}x \right]_0^{1/2} = \frac{3}{16} - \frac{3}{8}\ln 1.5 \cong 0.0354505.$$

52. $$I = \int_0^1 \left(x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \cdots \right) = \frac{1}{3} - \frac{1}{5(3!)} + \frac{1}{7(5!)} - \frac{1}{9(7!)} + \cdots.$$

Since $\dfrac{1}{9(7!)} = \dfrac{1}{5040} < 0.001$, we can stop there:

$$\frac{1}{2} - \frac{1}{5(3!)} + \frac{1}{7(5!)} - \frac{1}{9(7!)} \le I \le \frac{1}{3} - \frac{1}{5(3!)} + \frac{1}{7(5!)} \quad \text{gives} \quad 0.3009 \le I \le 0.3012.$$

Direct integration gives

$$I=\int_0^1 x\sin x\,dx=[-x\cos x+\sin x]_0^1=\sin 1-\cos 1\cong 0.3011686.$$

53. $0.2640\le I\le 0.2643$;

$$I=\int_0^1\left(x-x^2+\frac{x^3}{2!}-\frac{x^4}{3!}+\frac{x^5}{4!}-\frac{x^6}{5!}+\frac{x^7}{6!}-\cdots\right)dx$$
$$=\left[\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^4}{4(2!)}-\frac{x^5}{5(3!)}+\frac{x^6}{6(4!)}-\frac{x^7}{7(5!)}+\frac{x^8}{8(6!)}-\cdots\right]_0^1$$
$$=\frac{1}{2}-\frac{1}{3}+\frac{1}{4(2!)}-\frac{1}{5(3!)}+\frac{1}{6(4!)}-\frac{1}{7(5!)}+\frac{1}{8(6!)}-\cdots.$$

Note that $\frac{1}{8(6!)}=\frac{1}{5760}<0.001.$ The integral lies between

$$\frac{1}{2}-\frac{1}{3}+\frac{1}{4(2!)}-\frac{1}{5(3!)}+\frac{1}{6(4!)}-\frac{1}{7(5!)}$$

and

$$\frac{1}{2}-\frac{1}{3}+\frac{1}{4(2!)}-\frac{1}{5(3!)}+\frac{1}{6(4!)}-\frac{1}{7(5!)}+\frac{1}{8(6!)}.$$

The first sum is greater than 0.2640 and the second sum is less than 0.2643.

Direct integration gives

$$\int_0^1 xe^{-x}\,dx=\left[-xe^{-x}-e^{-x}\right]_0^1=1-2/e\cong 0.2642411.$$

54. For $x\in[0,4]$

$$0\le e^x-\left(1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}\right)\le\frac{e^4 4n+1}{(n+1)!}$$

(11.5.3)

Thus for $x\in[0,2]$

$$0\le e^x-\left(1+x^2+\frac{x^4}{2!}+\cdots+\frac{x^{2n}}{n!}\right)\le\frac{e^4 4n+1}{(n+1)!}$$

It follows that

$$0\le\int_0^2 e^{x^2}dx-\int_0^2\left(1+x^2+\frac{x^4}{2!}+\cdots+\frac{x^{2n}}{n!}\right)dx\le\int_0^2\frac{e^4 4n+1}{(n+1)!}dx$$
$$0\le\int_0^2 e^{x^2}dx-\left[x+\frac{x^3}{3}+\frac{x^5}{5(2!)}+\cdots+\frac{x^{2n+1}}{(2n+1)n!}\right]\le\frac{2e^4 4n+1}{(n+1)!}$$
$$0\le\int_0^2 e^{x^2}dx-\left(2+\frac{2^3}{3}+\frac{2^5}{5(2!)}+\cdots+\frac{2^{2n+1}}{(2n+1)n!}\right)\le\frac{e^4 2^{2n+3}}{(n+1)!}$$

PROJECT 11.8

1. $f(x)=\frac{4}{1+x^2}$

(a) $T_8=\frac{1-0}{2\cdot 8}\left[f(0)+2\sum_{I=1}^{7}f\left(\frac{i}{8}\right)+f(1)\right]\simeq 3.13899$

(b) $S_4 = \dfrac{1-0}{6\cdot 4}\left[f(0) + 2\sum_{I=1}^{3} f\left(\frac{i}{4}\right) + 4\sum_{I=1}^{4} f\left(\frac{2i-1}{8}\right) + f(1)\right] \simeq 3.141592$

2. $K = 8, \quad M = 96$

3. (a) $|E_n^T| \le \dfrac{(1-0)^3}{12n^2}(8) < 0.000005$

$$12n^2 > \frac{8}{0.00005} = 1,600,000 \quad \Longrightarrow \quad n^2 > 133,333.33 \quad \Longrightarrow \quad n > 365.15$$

Thus n must be at least 366.

(b) $|E_n^S| \le \dfrac{(1-0)^5}{2880n^4}(96) < 5\cdot 10^{-11}$

$$n^4 > \frac{96(10)^{11}}{5(2880)} \quad \Longrightarrow \quad n > 160.68.$$

Thus n must be at least 161.

4. $\tan\left[2\tan^{-1}\left(\frac{1}{5}\right)\right] = \dfrac{2\tan\left[\tan^{-1}\left(\frac{1}{5}\right)\right]}{1-\tan^2\left[tan^{-1}(\frac{1}{5})\right]} = \dfrac{\frac{2}{5}}{1-\frac{1}{25}} = \dfrac{5}{12}$

$2\tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}\left(\frac{5}{12}\right)$

$4\tan^{-1}\left(\frac{1}{5}\right) = 2\tan^{-1}\left(\frac{5}{12}\right)$

$\tan\left([\tan^{-1}\left(\frac{1}{5}\right)\right] = \tan\left[2\tan^{-1}\left(\frac{5}{12}\right)\right] = \dfrac{\frac{10}{12}}{1-\frac{25}{144}} = \dfrac{120}{119}$

$\tan\left[4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right)\right] = \dfrac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\cdot\frac{1}{239}} = \dfrac{120(239)-119}{119(239)+120} = 1$

Thus $4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right) = \dfrac{\pi}{4}$.

5. (a) $4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right) < 4\left[\frac{1}{5} - \frac{1}{3}\left(\frac{1}{5}\right)^3 + \frac{1}{5}\left(\frac{1}{5}\right)^5\right] - \left[\frac{1}{239} - \frac{1}{3}\left(\frac{1}{239}\right)^3\right]$

$4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right) > 4\left[\frac{1}{5} - \frac{1}{3}\left(\frac{1}{5}\right)^3\right] - \frac{1}{239}$

These inequalities imply $\quad 3.1406 < \pi < 3.1416.$

(b) $4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right) < 4\sum_{k=1}^{5}\dfrac{(-1)^{k-1}}{2k-1}\left(\frac{1}{5}\right)^{2k-1} - \left[\frac{1}{239} - \frac{1}{3}\left(\frac{1}{239}\right)^3\right]$

$= 0.789582246 - 0.004184076 = 0.78539817.$

$4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} > 4\sum_{k=1}^{5}\dfrac{(-1)^{k-1}}{2k-1}\left(\frac{1}{5}\right)^{2k-1} - \frac{1}{239}$

$= 0.789582238 - 0.0041841 = 0.785398138.$

These inequalities imply $\quad 3.14159255 < \pi < 3.14159268.$

SECTION 11.9

1. Take $\alpha = 1/2$ in (11.9.2) to obtain $1 + \dfrac{1}{2}x - \dfrac{1}{8}x^2 + \dfrac{1}{16}x^3 - \dfrac{5}{128}x^4.$

2. $\sqrt{1-x} = [1+(-x)]^{1/2} = 1 - \frac{x}{2} - \frac{\frac{1}{2}(-\frac{1}{2})}{2!}x^2 - \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!}x^4$

$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4 - \cdots$

3. In (11.9.2), replace x by x^2 and take $\alpha = 1/2$ to obtain $1 + \frac{1}{2}x^2 - \frac{1}{8}x^4$.

4. $\sqrt{1-x^2} = 1 - \frac{x^2}{2} - \frac{1}{8}x^4$

5. Take $\alpha = -1/2$ in (11.9.2) to obtain $1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4$.

6. $\alpha = -\frac{1}{3}$: $\quad \frac{1}{\sqrt[3]{1+x}} = 1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \frac{35}{243}x^4 + \cdots$

7. In (11.9.2), replace x by $-x$ and take $\alpha = 1/4$ to obtain $1 - \frac{1}{4}x - \frac{3}{32}x^2 - \frac{7}{128}x^3 - \frac{77}{2048}x^4$.

8. $\alpha = -\frac{1}{4}$: $\quad \frac{1}{\sqrt[4]{1+x}} = 1 - \frac{1}{4}x + \frac{5}{32}x^2 - \frac{15}{128}x^3 + \frac{195}{2048}x^4 + \cdots$

9. $f(x) = (4+x)^{3/2} = 8\left(1+\frac{x}{4}\right)^{3/2}$

In 11.9.2, replace x by $x/4$ and take $\alpha = 3/2$ to obtain

$$8\left[1 + \frac{3}{8}\left(\frac{x}{4}\right) + \frac{1}{2!}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(\frac{x}{4}\right)^2 + \frac{1}{3!}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{x}{4}\right)^3\right]$$

$$\left[+\frac{1}{4!}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x}{4}\right)^4\right]$$

$$= 8 + 3x + \frac{3}{16}x^2 - \frac{1}{128}x^3 + \frac{3}{4096}x^4$$

10. $\sqrt{1+x^4} = 1 + \frac{x^4}{2} + \cdots$

11. (a) $f(x) = \frac{1}{\sqrt{1-x^2}} = \left(1-x^2\right)^{-1/2}$

In 11.9.2, replace x by x^2 and take $\alpha = -1/2$ to obtain

$$\frac{1}{\sqrt{1-x^2}} = \sum_{k=0}^{\infty}\binom{-1/2}{k}(-1)^k x^{2k}$$

By Problem 2, this series has radius of convergence $r = 1$.

(b)
$$\sin^{-1} x = \int_0^x \frac{1}{\sqrt{1-x^2}}\,dt = \int_0^x \sum_{k=0}^{\infty}\binom{-1/2}{k}(-1)^k t^{2k}\,dt$$

$$= \sum_{k=0}^{\infty}\binom{-1/2}{k}(-1)^k \int_0^x t^{2k}\,dt$$

$$= \sum_{k=0}^{\infty}\binom{-1/2}{k}\frac{(-1)^k}{2k+1}x^{2k+1}$$

By Theorem 11.8.4, the radius of convergence of this series is $r = 1$.

12. (a) $$\alpha = -\frac{1}{2}: \quad \frac{1}{\sqrt{1+x^2}} = 1 - \frac{x^2}{2} + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}x^4 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}x^6 + \cdots$$
$$= 1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{5}{16}x^6 + \cdots$$

(b) $$\sinh^{-1} x = \int_0^x \frac{1}{\sqrt{1+t^2}}\,dt = \int_0^x \left(1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 - \frac{5}{16}t^6 + \cdots\right) dt$$
$$= x - \frac{1}{6}x^3 + \frac{3}{40}x^4 - \frac{5}{112}x^7 + \cdots; \quad r = 1$$

13. $\sqrt{98} = (100-2)^{1/2} = 10\left(1 - \frac{1}{50}\right)^{1/2} \cong 10\left[1 - \frac{1}{100} - \frac{1}{20000}\right] = 9.8995$

14. $\sqrt[5]{36} = (32+4)^{1/5} = 2\left(1+\frac{1}{8}\right)^{1/5} \cong 2\left[1 + \frac{1}{5}\left(\frac{1}{8}\right) - \frac{4}{25}\cdot\frac{1}{2}\cdot\left(\frac{1}{8}\right)^2\right] \cong 2.0475$

15. $\sqrt[3]{9} = (8+1)^{1/3} = 2\left(1+\frac{1}{8}\right)^{1/3} \cong 2\left[1 + \frac{1}{24} - \frac{1}{576}\right] \cong 2.0799$

16. $\sqrt[4]{620} = (625-5)^{1/4} = 5\left(1 - \frac{1}{125}\right)^{1/4} \cong 5\left[1 - \frac{1}{4}\cdot\frac{1}{125} - \frac{3}{16}\cdot\frac{1}{2}\cdot\left(\frac{1}{125}\right)^2\right] \cong 4.9900$

17. $17^{-1/4} = (16+1)^{-1/4} = \frac{1}{2}\left(1+\frac{1}{16}\right)^{-1/4} \cong \frac{1}{2}\left[1 - \frac{1}{64} + \frac{5}{8192}\right] \cong 0.4925$

18. $9^{-1/3} = (8+1)^{-1/3} = \frac{1}{2}\left(1+\frac{1}{8}\right)^{-1/3} \cong \frac{1}{2}\left[1 - \frac{1}{3}\cdot\frac{1}{8} + \frac{4}{9}\cdot\frac{1}{2}\left(\frac{1}{8}\right)^2\right] \cong 0.4809$

19. $$I = \int_0^{1/3} \sqrt{1+x^3}\,dx = \int_0^{1/3} \sum_{k=0}^{\infty} \binom{1/2}{k} x^{3k}\,dx$$
$$= \sum_{k=0}^{\infty} \binom{1/2}{k} \int_0^{1/3} x^{3k}\,dx$$
$$= \sum_{k=0}^{\infty} \binom{1/2}{k} \left[\frac{x^{3k+1}}{3k+1}\right]_0^{1/3}$$
$$= \sum_{k=0}^{\infty} \binom{1/2}{k} \frac{1}{3k+1}\left(\frac{1}{3}\right)^{3k+1}$$
$$= \frac{1}{3} + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{3}\right)^4 + \frac{1}{2!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{7}\right)\left(\frac{1}{3}\right)^7 + \cdots$$

Now, $I - \frac{1}{3} = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{3}\right)^4 + \frac{1}{2!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{7}\right)\left(\frac{1}{3}\right)^7 + \cdots$

is an alternating series and $\frac{1}{2!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{7}\right)\left(\frac{1}{3}\right)^7 \cong 8.2 \times 10^{-6} < 0.001$

Therefore, $\int_0^{1/3} \sqrt{1+x^3}\,dx \cong \frac{1}{3} + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{3}\right)^4 \cong 0.3349$

20.
$$\int_0^{1/5}(1+x^4)^{1/2}\,dx=\int_0^{1/5}\left(1+\frac{1}{2}x^4-\frac{1}{8}x^8+\frac{3}{8}\cdot\frac{1}{3!}x^{12}-\cdots\right)dx$$
$$=(0.2)+\frac{(0.2)^5}{10}-\frac{(0.2)^9}{72}+\frac{(0.2)^{13}}{16\cdot 13}-\cdots\cong 0.200$$

21.
$$\int_0^{1/2}\frac{1}{\sqrt{1+x^2}}\,dx=\int_0^{1/2}\left(1+x^2\right)^{-1/2}dx=\int_0^{1/2}\sum_{k=0}^{\infty}\binom{-1/2}{k}x^{2k}\,dx$$
$$=\sum_{k=0}^{\infty}\binom{-1/2}{k}\int_0^{1/2}x^{2k}\,dx$$
$$=\sum_{k=0}^{\infty}\binom{-1/2}{k}\left[\frac{1}{2k+1}x^{2k+1}\right]_0^{1/2}$$
$$=\sum_{k=0}^{\infty}\binom{-1/2}{k}\frac{1}{2k+1}\left(\frac{1}{2}\right)^{2k+1}$$

Now

$$\sum_{k=0}^{\infty}\binom{-1/2}{k}\frac{1}{2k+1}\left(\frac{1}{2}\right)^{2k+1}=\frac{1}{2}+\left(-\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^3+\frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{1}{5}\right)\left(\frac{1}{2}\right)^5$$
$$+\frac{1}{3!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{1}{7}\right)\left(\frac{1}{2}\right)^7+\cdots$$

is an alternating series and

$$\frac{1}{3!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{1}{7}\right)\left(\frac{1}{2}\right)^7\cong 3.5\times 10^{-4}<0.001$$

Therefore,

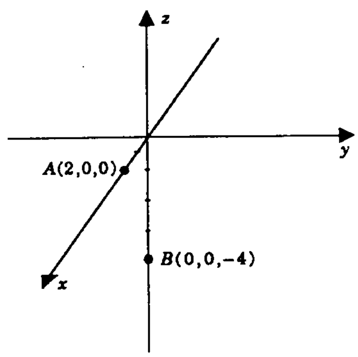
$$\int_0^{1/2}\frac{1}{\sqrt{1+x^2}}\,dx\cong\frac{1}{2}+\left(-\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^3+\frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{1}{5}\right)\left(\frac{1}{2}\right)^5\cong 0.4815$$

22. $$\int_0^{1/2}\frac{1}{\sqrt{1-x^3}}\,dx=\int_0^{1/2}\left(1+\frac{1}{2}x^3+\frac{3}{8}x^6+\cdots\right)dx=0.5+\frac{(0.5)^4}{8}+\frac{3(0.5)^7}{56}+\cdots\cong 0.508$$

CHAPTER 12

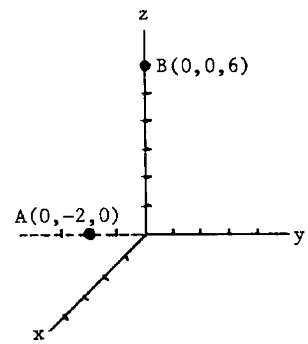
SECTION 12.1

1.



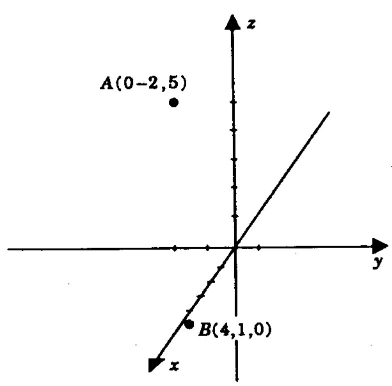
length $\overline{AB}$: $2\sqrt{5}$
midpoint: $(1,0,-2)$

2.



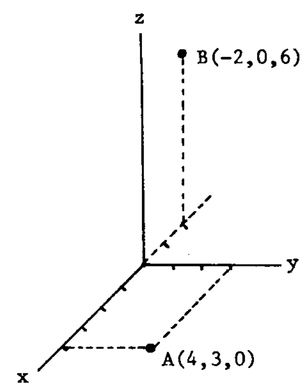
length $\overline{AB}$: $2\sqrt{10}$
midpoint: $(0,-1,3)$

3.



length $\overline{AB}$: $5\sqrt{2}$
midpoint: $\left(2,-\frac{1}{2},\frac{5}{2}\right)$

4.



length $\overline{AB}$: 9
midpoint: $(1,\frac{3}{2},3)$

5. $z=-2$

6. $y=1$

7. $y=1$

8. $z=-2$

9. $x=3$

10. $x=3$

11. $x^2+(y-2)^2+(z+1)^2=9$

12. $(x-1)^2+y^2+(z+2)^2=16$

13. $(x-2)^2+(y-4)^2+(z+4)^2=36$

14. $x^2+y^2+z^2=9$

15. $(x-3)^2+(y-2)^2+(z-2)^2=13$

16. $(x-2)^2+(y-3)^2+(z+4)^2=16$

17. $(x-2)^2+(y-3)^2+(z+4)^2=25$

18. $(x-2)^2+(y-3)^2+(z+4)^2=4$

19.
$$x^2+y^2+z^2+4x-8y-2z+5=0$$
$$x^2+4x+4+y^2-8y+16+z^2-2z+1=-5+4+16+1$$
$$(x+2)^2+(y-4)^2+(z-1)^2=16$$

center: $(-2,4,1)$, radius: 4

20. Rewrite as $x^2-4x+4+y^2+z^2-2z+1=-1+4+1=4$

$\implies (x-2)^2+y^2+(z-1)^2=4$ center $(2,0,1)$; radius 2

21. $(2,3,-5)$ **22.** $(2,-3,5)$ **23.** $(-2,3,5)$

24. $(2,-3,-5)$ **25.** $(-2,3,-5)$ **26.** $(-2,-3,5)$

27. $(-2,-3,-5)$ **28.** $(0,3,5)$ **29.** $(2,-5,5)$

30. $(2,3,3)$ **31.** $(-2,1,-3)$ **32.** $(6,-3,-3)$

33. Each such sphere has an equation of the form
$$(x-a)^2+(y-a)^2+(z-a)^2=a^2.$$
Substituting $x=5,\quad y=1,\quad z=4$ we get
$$(5-a)^2+(1-a)^2+(4-a)^2=a^2.$$
This reduces to $a^2-10a+21=0$ and gives $a=3$ or $a=7$. The equations are:
$$(x-3)^2+(y-3)^2+(z-3)^2=9;\quad (x-7)^2+(y-7)^2+(z-7)^2=49$$

34. Farthest point from $(2,1,-2)$ on $x^2+y^2+z^2=1$ is $(-\frac{2}{3},-\frac{1}{3},\frac{2}{3})$ [on line through $(2,1,-2)$ and origin]. Distance from $(2,1,-2)$ to $(-\frac{2}{3},-\frac{1}{3},\frac{2}{3})$ is 4, so equation of largest sphere is
$$(x-2)^2+(y-1)^2+(z+2)^2=16.$$

35. Not a sphere; this equation is equivalent to:
$$(x-2)^2+(y+2)^2+(z+3)^2=-3$$
which has no (real) solutions.

36. $x^2+y^2+z^2+Ax+By+Cz+D=0$
$$\implies \left(x+\frac{A}{2}\right)^2+\left(y+\frac{B}{2}\right)^2+\left(z+\frac{C}{2}\right)^2=\frac{A^2}{4}+\frac{B^2}{4}+\frac{C^2}{4}-D,$$
so you get a sphere if $\dfrac{A^2}{4}+\dfrac{B^2}{4}+\dfrac{C^2}{4}-D>0.$

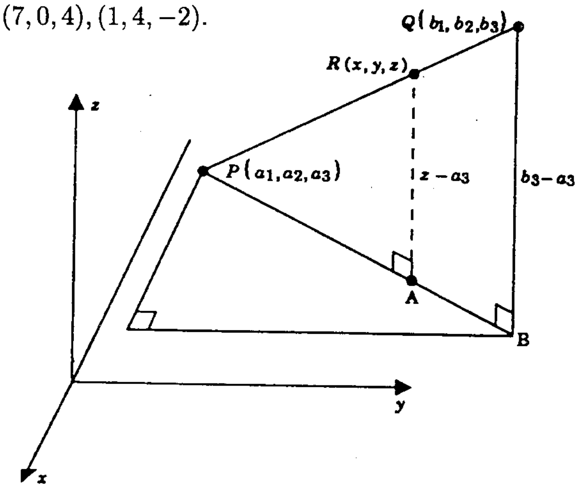
37. $d(PR)=\sqrt{14},\quad d(QR)=\sqrt{45},\quad d(PQ)=\sqrt{59};\qquad [d(PR)]^2+[d(QR)]^2=[d(PQ)]^2$

38. Let the vertices be $(x_i,y_i,z_i),\quad i=1,2,3.$ Then

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right) = (5,-1,3); \quad \left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right) = (4,2,1);$$

$$\left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}, \frac{z_1+z_3}{2}\right) = (2,1,0)$$

Solving simultaneously gives vertices $(3,-2,2), (7,0,4), (1,4,-2)$.



39. (a) Take R as (x,y,z). Since

$$d(P,R) = t\,d(P,Q)$$

we conclude by similar triangles that

$$d(AR) = t\,d(B,Q)$$

and therefore

$$z - a_3 = t(b_3 - a_3).$$

Thus

$$z = a_3 + t(b_3 - a_3).$$

In similar fashion

$$x = a_1 + t(b_1 - a_1) \quad \text{and} \quad y = a_2 + t(b_2 - a_2).$$

(b) The midpoint of PQ, $\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \frac{a_3+b_3}{2}\right)$, occurs at $t = \frac{1}{2}$.

40. (a) $d(P,R) = rd(R,Q) = r\,[d(P,Q) - d(P,Q)] \implies d(P,R) = \frac{r}{r+1}d(P,Q),$

so by Exercise 39,

$$x = a_1 + \frac{r}{r+1}(b_1 - a_1), \quad y = a_2 + \frac{r}{r+1}(b_2 - a_2), \quad z = a_3 + \frac{r}{r+1}(b_3 - a_3).$$

(b) Need $\frac{r}{r+1} = \frac{1}{2}$, so $r = 1$

SECTION 12.3

1. $\overrightarrow{PQ} = (3,4,-2); \quad \|\overrightarrow{PQ}\| = \sqrt{29}$

2. $\overrightarrow{PQ} = (-2,6); \quad \|\overrightarrow{PQ}\| = 2\sqrt{10}$

3. $\overrightarrow{PQ} = (-2,1); \quad \|\overrightarrow{PQ}\| = \sqrt{5}$

4. $\overrightarrow{PQ} = (4,3,-8); \quad \|\overrightarrow{PQ}\| = \sqrt{89}$

5. $2\mathbf{a} - \mathbf{b} = (2\cdot 1 - 3, 2\cdot[-2] - 0, 2\cdot 3 + 1) = (-1,-4,7)$

6. $2\mathbf{b} + 3\mathbf{c} = (6,0,-2) + (-12,6,3) = (-6,6,1)$

7. $-2\mathbf{a} + \mathbf{b} - \mathbf{c} = [-2(\mathbf{a} - \mathbf{b})] - \mathbf{c} = (1+4, 4-2, -7-1) = (5,2,-8)$

8. $\mathbf{a} + 3\mathbf{b} - 2\mathbf{c} = (1,-2,3) + 3(3,0,-1) - 2(-4,2,1) = (18,-6,-2).$

9. $3\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}$

10. $3\mathbf{i} + 5\mathbf{j} + \mathbf{k}$

11. $-3\mathbf{i} - \mathbf{j} + 8\mathbf{k}$

12. $14\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}$

13. 5

14. $\sqrt{2}$

15. 3

16. $\sqrt{41}$

17. $\sqrt{6}$

18. $\sqrt{2}$

19. (a) $\mathbf{a}$, $\mathbf{c}$, and $\mathbf{d}$ since $\mathbf{a} = \frac{1}{3}\mathbf{c} = -\frac{1}{2}\mathbf{d}$

(b) $\mathbf{a}$ and $\mathbf{c}$ since $\mathbf{a} = \frac{1}{3}\mathbf{c}$

(c) $\mathbf{a}$ and $\mathbf{c}$ both have direction opposite to $\mathbf{d}$

20. $\|\mathbf{a}\| - \|\mathbf{b}\| \le \|\mathbf{a} - \mathbf{b}\|$ since

$$\|\mathbf{a}\| = \|(\mathbf{a} - \mathbf{b}) + \mathbf{b}\| \le \|\mathbf{a} - \mathbf{b}\| + \|\mathbf{b}\|.$$

Similarly $\|\mathbf{b}\| - \|\mathbf{a}\| \le \|\mathbf{b} - \mathbf{a}\| = \|\mathbf{a} - \mathbf{b}\|$.

21. $\|\mathbf{a}\| = 5;\quad \dfrac{\mathbf{a}}{\|\mathbf{a}\|} = \left(\dfrac{3}{5}, -\dfrac{4}{5}\right)$

22. $\left(\dfrac{-2}{\sqrt{13}}, \dfrac{3}{\sqrt{13}}\right)$

23. $\|\mathbf{a}\| = 3;\quad \dfrac{\mathbf{a}}{\|\mathbf{a}\|} = \dfrac{1}{3}\mathbf{i} - \dfrac{2}{3}\mathbf{j} + \dfrac{2}{3}\mathbf{k}$

24. $\left(\dfrac{2}{3}, \dfrac{1}{3}, \dfrac{2}{3}\right)$

25. $\|\mathbf{a}\| = \sqrt{14};\quad -\dfrac{\mathbf{a}}{\|\mathbf{a}\|} = \dfrac{1}{\sqrt{14}}\mathbf{i} - \dfrac{3}{\sqrt{14}}\mathbf{j} - \dfrac{2}{\sqrt{14}}\mathbf{k}$

26. $-\dfrac{2}{\sqrt{5}}\mathbf{i} + \dfrac{1}{\sqrt{5}}\mathbf{k}$

27. (i) $\mathbf{a} + \mathbf{b}$ (ii) $-(\mathbf{a} + \mathbf{b})$ (iii) $\mathbf{a} - \mathbf{b}$ (iv) $\mathbf{b} - \mathbf{a}$

28. (a) $6\mathbf{i} + 3\mathbf{j} + 12\mathbf{k}$

(b) $A(1,1,1) + B(-1,3,2) + C(-3,0,1) = (4,-1,1)$.

Solve simultaneously to get $A = \frac{26}{7}$, $B = -\frac{11}{7}$, $C = \frac{3}{7}$

29. (a) $\mathbf{a} - 3\mathbf{b} + 2\mathbf{c} + 4\mathbf{d} = (2\mathbf{i} - \mathbf{k}) - 3(\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) + 2(-\mathbf{i} + \mathbf{j} + \mathbf{k}) + 4(\mathbf{i} + \mathbf{j} + 6\mathbf{k})$

$$= \mathbf{i} - 3\mathbf{j} + 10\mathbf{k}$$

(b) The vector equation

$$(1,1,6) = A(2,0,-1) + B(1,3,5) + C(-1,1,1)$$

implies

$$\begin{aligned} 1 &= 2A + B - C, \\ 1 &= 3B + C, \\ 6 &= -A + 5B + C. \end{aligned}$$

Simultaneous solution gives $A = -2$, $B = \frac{3}{2}$, $C = -\frac{7}{2}$.

30. $\alpha = -12$

31. $\|3\mathbf{i} + \mathbf{j}\| = \|\alpha\mathbf{j} - \mathbf{k}\| \implies 10 = \alpha^2 + 1$ so $\alpha = \pm 3$

32. $\dfrac{1}{3}(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$

33. $$\|\alpha\mathbf{i} + (\alpha - 1)\,\mathbf{j} + (\alpha + 1)\,\mathbf{k}\| = 2 \implies \alpha^2 + (\alpha - 1)^2 + (\alpha + 1)^2 = 4$$
$$\implies 3\alpha^2 = 2 \quad \text{so} \quad \alpha = \pm\tfrac{1}{3}\sqrt{6}$$

34. $\dfrac{2}{\sqrt{6}}(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = \dfrac{\sqrt{6}}{3}\,(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

35. $\pm\frac{2}{13}\sqrt{13}\,(3\mathbf{j} + 2\mathbf{k})$ since $\|\alpha(3\mathbf{j} + 2\mathbf{k})\| = 2 \implies \alpha = \pm\frac{2}{13}\sqrt{13}$

36. (i) $\mathbf{c} = \mathbf{a} + \dfrac{1}{2}(\mathbf{b} - \mathbf{a}) = \dfrac{1}{2}(\mathbf{a} + \mathbf{b})$

(ii) $\mathbf{a} + \mathbf{c} = \dfrac{1}{2}(\mathbf{a} + \mathbf{b}) \implies \mathbf{c} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$

37. $\mathbf{v} = (2\cos 30^\circ)\,\mathbf{i} + (2\sin 30^\circ)\,\mathbf{j} = \sqrt{3}\,\mathbf{i} + \mathbf{j}$

38. $\mathbf{v} = 5\left(-\dfrac{\sqrt{3}}{2}\mathbf{i} - \dfrac{1}{2}\mathbf{j}\right)$

39. Since the $\mathbf{i}$ component is twice the $\mathbf{j}$ component, $\mathbf{v} = 2y\,\mathbf{i} + y\,\mathbf{j}$. Now, $\|\mathbf{v}\| = \sqrt{4y^2 + y^2} = 3$ which implies that $y = \dfrac{3}{\sqrt{5}}$. Thus, $\mathbf{v} = \dfrac{6}{\sqrt{5}}\mathbf{i} + \dfrac{3}{\sqrt{5}}\mathbf{j}$ or $\mathbf{v} = -\dfrac{6}{\sqrt{5}}\mathbf{i} - \dfrac{3}{\sqrt{5}}\mathbf{j}$.

40. $b = \dfrac{3}{4}a,\quad a^2 + b^2 + 25 \implies a = \pm 4 \implies \mathbf{v} = 4\mathbf{i} + 3\mathbf{j}$, or $\mathbf{v} = -4\mathbf{i} - 3\mathbf{j}$

41. If $\mathbf{a}$ and $\mathbf{b}$ are the sides of a triangle, then $\mathbf{b} - \mathbf{a}$ is the third side. Now $\|\mathbf{a}\| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$, $\|\mathbf{b}\| = \sqrt{1^2 + 2^2} = \sqrt{5}$, and $\|\mathbf{b} - \mathbf{a}\| = \sqrt{(1-2)^2 + (2+1)^2} = \sqrt{10}$. The triangle is a right triangle since $\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 = \|\mathbf{b} - \mathbf{a}\|^2$.

42. $\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 = 13 + 56 = 69 = \|\mathbf{b} - \mathbf{a}\|^2$, right triangle, so $\mathbf{a} \perp \mathbf{b}$.

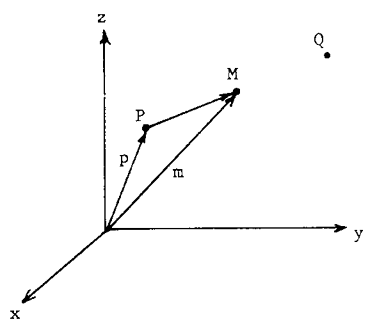
43. (a) Since $\|\mathbf{a} - \mathbf{b}\|$ and $\|\mathbf{a} + \mathbf{b}\|$ are the lengths of the diagonals of the parallelogram, the parallelogram must be a rectangle.

(b) Simplify
$$\sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2} = \sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2 + (a_3 + b_3)^2}.$$

44. (a) If $\mathbf{a}$ and $\mathbf{b}$ have the same direction, then $\mathbf{a} = \alpha\mathbf{b}$ for some scalar $\alpha > 0$. Then,
$$\|\mathbf{a} + \mathbf{b}\| = \|\alpha\mathbf{b} + \mathbf{b}\| = (\alpha + 1)\|\mathbf{b}\| = \alpha\|\mathbf{b}\| + \|\mathbf{b}\| = \|\alpha\mathbf{b}\| + \|\mathbf{b}\| = \|\mathbf{a}\| + \|\mathbf{b}\|.$$

(b) No. For the choice $\mathbf{b} = -\mathbf{a} \neq 0$, we have
$$\|\mathbf{a} + \mathbf{b}\| = 0 \quad \text{but} \quad \|\mathbf{a}\| + \|\mathbf{b}\| = 2\|\mathbf{a}\| > 0$$

45. (a)



(b) Let $P = (x_1, y_1, z_1)$, $Q = (x_2, y_2, z_2)$, and $M = (x_m, y_m, z_m)$. Then

$$(x_m, y_m, z_m) = (x_1, y_1, z_1) + \tfrac{1}{2}(x_2 - x_1, y_2 - y_1, z_2 - z_1)$$
$$= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

46. $\mathbf{r} - \mathbf{p} = 2(\mathbf{q} - \mathbf{r}) \quad 3\mathbf{r} = \mathbf{p} + 2\mathbf{q}$

$\mathbf{r} = \frac{1}{3}\mathbf{p} + \frac{2}{3}\mathbf{q}$

47. $\|\mathbf{F}_1\| \sin 40^\circ + \|\mathbf{F}_2\| \sin 25^\circ = 200$ and $\|\mathbf{F}_1\| \cos 40^\circ = \|\mathbf{F}_2\| \cos 25^\circ$

$\implies \|\mathbf{F}_1\| = 200.02$ and $\|\mathbf{F}_2\| = 169.05$

$\mathbf{F}_1 = -\|\mathbf{F}_1\| \cos 40^\circ\, \mathbf{i} + \|\mathbf{F}_1\| \sin 40^\circ\, \mathbf{j} = -153.21\,\mathbf{i} + 128.56\,\mathbf{j}$

$\mathbf{F}_2 = \|\mathbf{F}_2\| \cos 25^\circ\, \mathbf{i} + \|\mathbf{F}_2\| \sin 25^\circ\, \mathbf{j} = 153.21\,\mathbf{i} + 71.44\,\mathbf{j}$

48. Need forces perpendicular to l to counterbalance, so $F_A \sin 40^\circ = F_B \sin 35^\circ$.

$\implies F_B = 5000\dfrac{\sin 40^\circ}{\sin 35^\circ} \cong 5603$ pounds.

49. $\mathbf{V}_1 = 600 \sin 30^\circ\, \mathbf{i} + 600 \cos 30^\circ\, \mathbf{j} = 300\,\mathbf{i} + 300\sqrt{3}\,\mathbf{j}$ and

$\mathbf{V}_2 = 50 \sin 45^\circ\, \mathbf{i} - 50 \cos 45^\circ\, \mathbf{j} = 25\sqrt{2}\,\mathbf{i} - 25\sqrt{2}\,\mathbf{j}$

$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 = (300 + 25\sqrt{2})\,\mathbf{i} + (300\sqrt{3} - 25\sqrt{2})\,\mathbf{j} \cong 335.36\,\mathbf{i} + 484.26\,\mathbf{j}$

true course: $\theta = \tan^{-1} \dfrac{335.36}{484.26} = 34.70^\circ$; or $N\,34.70^\circ\,E$.

ground speed: $\|\mathbf{V}\| = \sqrt{(335.56)^2 + (484.26)^2} \cong 589.05$ mi/hr

50. $\mathbf{v}_g = (-550 \sin 40^\circ, 550 \cos 40^\circ)$, $\mathbf{v}_w = (-70, 0)$

$\mathbf{v}_a = \mathbf{v}_g - \mathbf{v}_w = (-550 \sin 40^\circ + 70,\ 550 \cos 40^\circ)$

airspeed$=\|\mathbf{v}_a\| \cong 508$ mph.

51. (a) $\|\mathbf{r} - \mathbf{a}\| = 3$ where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

(b) $\|\mathbf{r}\| \le 2$ (c) $\|\mathbf{r} - \mathbf{a}\| \le 1$ where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

(d) $\|\mathbf{r} - \mathbf{a}\| = \|\mathbf{r} - \mathbf{b}\|$ (e) $\|\mathbf{r} - \mathbf{a}\| + \|\mathbf{r} - \mathbf{b}\| = k$

SECTION 12.4

1. $\mathbf{a}\cdot\mathbf{b}=(2)(-2)+(-3)(0)+(1)(3)=-1$

2. $\mathbf{a}\cdot\mathbf{b}=(4)(-2)+(2)(2)+(-1)(1)=-5$

3. $\mathbf{a}\cdot\mathbf{b}=(2)(1)+(-4)(1/2)=0$

4. $\mathbf{a}\cdot\mathbf{b}=(-2)(3)+(5)(1)=-1$

5. $\mathbf{a}\cdot\mathbf{b}=(2)(1)+(1)(1)-(2)(2)=-1$

6. $\mathbf{a}\cdot\mathbf{b}=(2)(1)+(3)(4)+(1)(0)=14$

7. $\mathbf{a}\cdot\mathbf{b}$

8. $\mathbf{a}\cdot(\mathbf{a}-\mathbf{b})+\mathbf{b}\cdot(\mathbf{b}+\mathbf{a})=\mathbf{a}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}=\|\mathbf{a}\|^2+\|\mathbf{b}\|^2$

9. $(\mathbf{a}-\mathbf{b})\cdot\mathbf{c}+\mathbf{b}\cdot(\mathbf{c}+\mathbf{a})=\mathbf{a}\cdot\mathbf{c}-\mathbf{b}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{a}=\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})$

10. $\mathbf{a}\cdot(\mathbf{a}+2\mathbf{c})+(2\mathbf{b}-\mathbf{a})\cdot(\mathbf{a}+2\mathbf{c})-2\mathbf{b}\cdot(\mathbf{a}+2\mathbf{c})=(\mathbf{a}+2\mathbf{b}-\mathbf{a}-2\mathbf{b})\cdot(\mathbf{a}+2\mathbf{c})=0$

11. (a) $\mathbf{a}\cdot\mathbf{b}=(2)(3)+(1)(-1)+(0)(2)=5$

$\mathbf{a}\cdot\mathbf{c}=(2)(4)+(1)(0)+(0)(3)=8$

$\mathbf{b}\cdot\mathbf{c}=(3)(4)+(-1)(0)+(2)(3)=18$

(b) $\|\mathbf{a}\|=\sqrt{5},\quad \|\mathbf{b}\|=\sqrt{14},\quad \|\mathbf{c}\|=5.$ Then,

$$\cos\sphericalangle(\mathbf{a},\mathbf{b})=\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}=\frac{5}{(\sqrt{5})\,(\sqrt{14})}=\frac{1}{14}\sqrt{70},$$
$$\cos\sphericalangle(\mathbf{a},\mathbf{c})=\frac{8}{(\sqrt{5})\,(5)}=\frac{8}{25}\sqrt{5},$$
$$\cos\sphericalangle(\mathbf{b},\mathbf{c})=\frac{18}{(\sqrt{14})\,(5)}=\frac{9}{35}\sqrt{14}.$$

(c) $\mathbf{u_b}=\dfrac{1}{\sqrt{14}}(3\mathbf{i}-\mathbf{j}+2\mathbf{k}),\quad \text{comp}_\mathbf{b}\,\mathbf{a}=\mathbf{a}\cdot\mathbf{u_b}=\dfrac{1}{\sqrt{14}}(6-1)=\dfrac{5}{14}\sqrt{14},$

$\mathbf{u_c}=\dfrac{1}{5}(4\mathbf{i}+3\mathbf{k}),\quad \text{comp}_\mathbf{c}\,\mathbf{a}=\mathbf{a}\cdot\mathbf{u_c}=\dfrac{8}{5}$

(d) $\mathbf{proj_b}\,\mathbf{a}=(\text{comp}_\mathbf{b}\,\mathbf{a})\,\mathbf{u_b}=\dfrac{5}{14}(3\mathbf{i}-\mathbf{j}+2\mathbf{k}),\quad \mathbf{proj_c}\,\mathbf{a}=(\text{comp}_\mathbf{c}\,\mathbf{a})\,\mathbf{u_c}=\dfrac{8}{25}(4\mathbf{i}+3\mathbf{k})$

12. (a) $\mathbf{a}\cdot\mathbf{b}=5,\quad \mathbf{a}\cdot\mathbf{c}=-3,\quad \mathbf{b}\cdot\mathbf{c}=4$

(b) $\cos\sphericalangle(\mathbf{a},\mathbf{b})=\frac{1}{6}\sqrt{10},\quad \cos\sphericalangle(\mathbf{a},\mathbf{c})=-\frac{3}{10},\quad \cos\sphericalangle(\mathbf{b},\mathbf{c})=\frac{2}{15}\sqrt{10}$

(c) $\text{comp}_\mathbf{b}\mathbf{a}=\frac{5}{3},\quad \text{comp}_\mathbf{c}\mathbf{a}=-\frac{3}{10}\sqrt{10}$

(d) $\mathbf{proj_b a}=\frac{5}{9}(2\mathbf{i}-\mathbf{j}+2\mathbf{k}),\quad \mathbf{proj_c a}=-\frac{3}{10}(3\mathbf{i}-\mathbf{k})$

13. $\mathbf{u}=\cos\dfrac{\pi}{3}\,\mathbf{i}+\cos\dfrac{\pi}{4}\,\mathbf{j}+\cos\dfrac{2\pi}{3}\,\mathbf{k}=\dfrac{1}{2}\mathbf{i}+\dfrac{1}{2}\sqrt{2}\,\mathbf{j}-\dfrac{1}{2}\mathbf{k}$

14. $\mathbf{v}=2(\cos\dfrac{\pi}{4}\mathbf{i}+\cos\dfrac{\pi}{4}\mathbf{j}+\cos\dfrac{\pi}{2}\mathbf{k})=\sqrt{2}\mathbf{i}+\sqrt{2}\mathbf{j}.$

15. $\cos\theta=\dfrac{(3\mathbf{i}-\mathbf{j}-2\mathbf{k})\cdot(\mathbf{i}+2\mathbf{j}-3\mathbf{k})}{\|3\mathbf{i}-\mathbf{j}-2\mathbf{k}\|\,\|\mathbf{i}+2\mathbf{j}-3\mathbf{k}\|}=\dfrac{7}{\sqrt{14}\sqrt{14}}=\dfrac{1}{2},\quad \theta=\dfrac{\pi}{3}$

16. $\cos\theta=\dfrac{(2\mathbf{i}-3\mathbf{j}+\mathbf{k})\cdot(-3\mathbf{i}+\mathbf{j}+9\mathbf{k})}{\|2\mathbf{i}-3\mathbf{j}+\mathbf{k}\|\cdot\|-3\mathbf{i}+\mathbf{j}+9\mathbf{k}\|}=0\implies\theta=\dfrac{\pi}{2}$

17. Since $\|\mathbf{i}-\mathbf{j}+\sqrt{2}\,\mathbf{k}\|=2$, we have $\cos\alpha=\frac{1}{2}$, $\cos\beta=-\frac{1}{2}$, $\cos\gamma=\frac{1}{2}\sqrt{2}$.
The direction angles are $\frac{1}{3}\pi$, $\frac{2}{3}\pi$, $\frac{1}{4}\pi$.

18. $\alpha=\cos^{-1}\dfrac{1}{2}=\dfrac{\pi}{3}$, $\beta=\cos^{-1}0=\dfrac{\pi}{2}$, $\gamma=\cos^{-1}\dfrac{-\sqrt{3}}{2}=\dfrac{5}{6}\pi$

19. $\theta=\cos^{-1}\dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}=\cos^{-1}\left(\dfrac{-1}{\sqrt{231}}\right)\cong 2.2$ radians or $126.3°$

20. $\theta=\cos^{-1}\dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}=\cos^{-1}\dfrac{12}{\sqrt{13}\sqrt{52}}\cong 1.09$ radians or $62.5°$.

21. $\theta=\cos^{-1}\dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}=\cos^{-1}\left(\dfrac{-13}{5\sqrt{10}}\right)\cong 2.5$ radians or $145.3°$

22. $\theta=\cos^{-1}\dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}=\cos^{-1}\dfrac{-4}{\sqrt{11}\sqrt{2}}\cong 2.59$ radians or $148.5°$

23. $\|\mathbf{a}\|=\sqrt{1^2+2^2+2^2}=3$; $\cos\alpha=\dfrac{1}{3}$, $\cos\beta=\dfrac{2}{3}$, $\cos\gamma=\dfrac{2}{3}$
$\alpha\cong 70.5°$ $\beta\cong 48.2°$, $\gamma\cong 48.2°$

24. $\|\mathbf{a}\|=\sqrt{41}$; $\cos\alpha=\dfrac{2}{\sqrt{41}}$, $\cos\beta=\dfrac{6}{\sqrt{41}}$, $\cos\gamma=\dfrac{-1}{\sqrt{41}}$
$\alpha\cong 71.8°$ $\beta\cong 20.4°$, $\gamma\cong 99.0°$

25. $\|\mathbf{a}\|=\sqrt{3^2+(12)^2+4^2}=13$; $\cos\alpha=\dfrac{3}{13}$, $\cos\beta=\dfrac{12}{13}$ $\cos\gamma=\dfrac{4}{13}$
$\alpha\cong 76.7°$ $\beta\cong 22.6°$, $\gamma\cong 72.1°$

26. $\|\mathbf{a}\|=\sqrt{50}$; $\cos\alpha=\dfrac{3}{\sqrt{50}}$, $\cos\beta=\dfrac{5}{\sqrt{50}}$, $\cos\gamma=\dfrac{-4}{\sqrt{50}}$
$\alpha\cong 64.9°$ $\beta\cong 45°$, $\gamma\cong 124.4°$

27. (a) $\mathbf{proj_b}\,\alpha\mathbf{a}=(\alpha\mathbf{a}\cdot\mathbf{u_b})\mathbf{u_b}=\alpha(\mathbf{a}\cdot\mathbf{u_b})\mathbf{u_b}=\alpha\,\mathbf{proj_b}\,\mathbf{a}$

(b)
$$\begin{aligned}\mathbf{proj_b}\,(\mathbf{a}+\mathbf{c})&=[(\mathbf{a}+\mathbf{c})\cdot\mathbf{u_b}]\,\mathbf{u_b}\\&=(\mathbf{a}\cdot\mathbf{u_b}+\mathbf{c}\cdot\mathbf{u_b})\mathbf{u_b}\\&=(\mathbf{a}\cdot\mathbf{u_b})\mathbf{u_b}+(\mathbf{c}\cdot\mathbf{u_b})\mathbf{u_b}=\mathbf{proj_b}\,\mathbf{a}+\mathbf{proj_b}\,\mathbf{c}\end{aligned}$$

28. (a) If $\beta>0$, then $\mathbf{u}_{\beta\mathbf{b}}=\mathbf{u_b}$ and
$$\mathbf{proj}_{\beta\mathbf{b}}\mathbf{a}=(\mathbf{a}\cdot\mathbf{u}_{\beta\mathbf{b}})\mathbf{u}_{\beta\mathbf{b}}=(\mathbf{a}\cdot\mathbf{u_b})\mathbf{u_b}=\mathbf{proj_b}\mathbf{a}.$$
If $\beta<0$, then $\mathbf{u}_{\beta\mathbf{b}}=-\mathbf{u_b}$ and
$$\mathbf{proj}_{\beta\mathbf{b}}\mathbf{a}=(\mathbf{a}\cdot\mathbf{u}_{\beta\mathbf{b}})\mathbf{u}_{\beta\mathbf{b}}=[\mathbf{a}\cdot(-\mathbf{u_b})](-\mathbf{u_b})=(\mathbf{a}\cdot\mathbf{u_b})\mathbf{u_b}=\mathbf{proj_b}\mathbf{a}.$$

(b) If $\beta>0$,
$$\text{comp}_{\beta\mathbf{b}}\mathbf{a}=(\mathbf{a}\cdot\mathbf{u}_{\beta\mathbf{b}})=(\mathbf{a}\cdot\mathbf{u_b})=\text{comp}_{\mathbf{b}}\mathbf{a}.$$
If $\beta<0$,

$$\text{comp}_{\beta\mathbf{b}}\mathbf{a} = (\mathbf{a}\cdot\mathbf{u}_{\beta\mathbf{b}}) = [\mathbf{a}\cdot(-\mathbf{u}_{\mathbf{b}})] = -(\mathbf{a}\cdot\mathbf{u}_{\mathbf{b}}) = -\text{comp}_{\mathbf{b}}\mathbf{a}.$$

29. (a) For $\mathbf{a}\neq\mathbf{0}$ the following statements are equivalent:

$$\mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c},\quad \mathbf{b}\cdot\mathbf{a}=\mathbf{c}\cdot\mathbf{a},$$

$$\mathbf{b}\cdot\frac{\mathbf{a}}{\|\mathbf{a}\|}=\mathbf{c}\cdot\frac{\mathbf{a}}{\|\mathbf{a}\|},\quad \mathbf{b}\cdot\mathbf{u_a}=\mathbf{c}\cdot\mathbf{u_a}$$

$$(\mathbf{b}\cdot\mathbf{u_a})\mathbf{u_a}=(\mathbf{c}\cdot\mathbf{u_a})\mathbf{u_a},$$

$$\mathbf{proj_a}\,\mathbf{b}=\mathbf{proj_a}\,\mathbf{c}.$$

$$\mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c}\quad\text{but}\quad\mathbf{b}\neq\mathbf{c}$$

(b) $\underset{(12.4.13)}{\mathbf{b}=(\mathbf{b}\cdot\mathbf{i})\mathbf{i}+(\mathbf{b}\cdot\mathbf{j})\mathbf{j}+(\mathbf{b}\cdot\mathbf{k})\mathbf{k}}=\underset{(12.4.13)}{(\mathbf{c}\cdot\mathbf{i})\mathbf{i}+(\mathbf{c}\cdot\mathbf{j})\mathbf{j}+(\mathbf{c}\cdot\mathbf{k})\mathbf{k}=\mathbf{c}}$

30. (a) $\|\mathbf{a}+\mathbf{b}\|^2=\|\mathbf{a}\|^2+\|\mathbf{b}\|^2+2\mathbf{a}\cdot\mathbf{b}=\|\mathbf{a}\|^2+\|\mathbf{b}\|^2 \implies \mathbf{a}\perp\mathbf{b}.$

(b) $\|\mathbf{a}-\mathbf{b}\|^2=\|\mathbf{a}\|^2+\|\mathbf{b}\|^2-2\mathbf{a}\cdot\mathbf{b}=\|\mathbf{a}\|^2+\|\mathbf{b}\|^2 \implies \mathbf{a}\perp\mathbf{b}.$

31. (a) $\|\mathbf{a}+\mathbf{b}\|^2-\|\mathbf{a}-\mathbf{b}\|^2=(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})-(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})$

$$=[(\mathbf{a}\cdot\mathbf{a})+2(\mathbf{a}\cdot\mathbf{b})+(\mathbf{b}\cdot\mathbf{b})]-[(\mathbf{a}\cdot\mathbf{a})-2(\mathbf{a}\cdot\mathbf{b})+(\mathbf{b}\cdot\mathbf{b})]=4(\mathbf{a}\cdot\mathbf{b})$$

(b) The following statements are equivalent:

$$\mathbf{a}\perp\mathbf{b},\quad \mathbf{a}\cdot\mathbf{b}=0,\quad \|\mathbf{a}+\mathbf{b}\|^2-\|\mathbf{a}-\mathbf{b}\|^2=0,\quad \|\mathbf{a}+\mathbf{b}\|=\|\mathbf{a}-\mathbf{b}\|.$$

(c) By (b), the relation $\|\mathbf{a}+\mathbf{b}\|=\|\mathbf{a}-\mathbf{b}\|$ gives $\mathbf{a}\perp\mathbf{b}$. The relation $\mathbf{a}+\mathbf{b}\perp\mathbf{a}-\mathbf{b}$ gives

$$0=(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=\|\mathbf{a}\|^2-\|\mathbf{b}\|^2\quad\text{and thus}\quad\|\mathbf{a}\|=\|\mathbf{b}\|.$$

The parallelogram is a square since it has two adjacent sides of equal length and these meet at right angles.

32. $|\mathbf{a}\cdot\mathbf{b}|=\|\mathbf{a}\|\|\mathbf{b}\||\cos\theta|=\|\mathbf{a}\|\|\mathbf{b}\|$ iff $\theta=0$ or $\theta=\pi$

33. $\|\mathbf{a}+\mathbf{b}\|^2=(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})=\mathbf{a}\cdot\mathbf{a}+2\,\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b}=\|\mathbf{a}\|^2+2\mathbf{a}\cdot\mathbf{b}+\|\mathbf{b}\|^2$

$\|\mathbf{a}-\mathbf{b}\|^2=(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=\mathbf{a}\cdot\mathbf{a}-2\,\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b}=\|\mathbf{a}\|^2-2\mathbf{a}\cdot\mathbf{b}+\|\mathbf{b}\|^2$

Add the two equations and the result follows.

34. If $\mathbf{v}=a\mathbf{i}+a\mathbf{j}+a\mathbf{k}$, then $\alpha=\beta=\gamma=\cos^{-1}\dfrac{a}{a\sqrt{3}}=\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)\cong 54.7^\circ.$

35. Let $\theta_1,\theta_2,\theta_3$ be the direction angles of $-\mathbf{a}$. Then

$$\theta_1=\cos^{-1}\left[\frac{(-\mathbf{a}\cdot\mathbf{i})}{\|-\mathbf{a}\|}\right]=\cos^{-1}\left[-\frac{(\mathbf{a}\cdot\mathbf{i})}{\|\mathbf{a}\|}\right]=\cos^{-1}(-\cos\alpha)=\cos^{-1}(\pi-\alpha)=\pi-\alpha.$$

Similarly $\theta_2=\pi-\beta$ and $\theta_3=\pi-\gamma$.

36.

$$\cos t=\frac{\mathbf{a}\cdot\beta\mathbf{b}}{\|\mathbf{a}\|\|\beta\mathbf{b}\|}=\left(\frac{\beta}{|\beta|}\right)\left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}\right)=-\cos\theta$$

$$t=\pi-\theta$$

37. If $\mathbf{a} \perp \mathbf{b}$ and $\mathbf{a} \perp \mathbf{c}$, then

$$\mathbf{a}\cdot\mathbf{b}=0,\quad \mathbf{a}\cdot\mathbf{c}=0$$
$$\mathbf{a}\cdot(\alpha\mathbf{b}+\beta\mathbf{c})=\alpha(\mathbf{a}\cdot\mathbf{b})+\beta(\mathbf{a}\cdot\mathbf{c})=0$$
$$\mathbf{a}\perp(\alpha\mathbf{b}+\beta\mathbf{c}).$$

38. Suppose that $\mathbf{a} \parallel \mathbf{b}$ and $\mathbf{a} \parallel \mathbf{c}$. If $\mathbf{a}=\mathbf{0}$, then certainly $\mathbf{a} \parallel \beta\mathbf{b}+\beta\mathbf{c}$. Let's assume therefore that $\mathbf{a}\neq\mathbf{0}$. Then we can write $\mathbf{b}=\gamma_1\mathbf{a}$ and $\mathbf{c}=\gamma_2\mathbf{a}$. Thus

$$\alpha\mathbf{b}+\beta\mathbf{c}=\alpha\gamma_1\mathbf{a}+\beta\gamma_2\mathbf{a}=(\alpha\gamma_1+\beta\gamma_2)\mathbf{a},$$

Which shows that $\mathbf{a} \parallel \alpha\mathbf{b}+\beta\mathbf{c}$.

39. Existence of decomposition:

$$\mathbf{a}=(\mathbf{a}\cdot\mathbf{u_b})\mathbf{u_b}+[\mathbf{a}-(\mathbf{a}\cdot\mathbf{u_b})\mathbf{u_b}].$$

Uniqueness of decomposition: suppose that

$$\mathbf{a}=\mathbf{a}_{\parallel}+\mathbf{a}_{\perp}=\mathbf{A}_{\parallel}+\mathbf{A}_{\perp}.$$

Then the vector $\mathbf{a}_{\parallel}-\mathbf{A}_{\parallel}=\mathbf{A}_{\perp}-\mathbf{a}_{\perp}$ is both parallel to $\mathbf{b}$ and perpendicular to $\mathbf{b}$. (Exercises 37 and 38.) Therefore it is zero. Consequently $\mathbf{A}_{\parallel}=\mathbf{a}_{\parallel}$ and $\mathbf{A}_{\perp}=\mathbf{a}_{\perp}$.

40. $(x\mathbf{i}+11\mathbf{j}-3\mathbf{k})\cdot(2x\mathbf{i}-x\mathbf{j}-5\mathbf{k})=0 \implies 2x^2-11x+15=0$

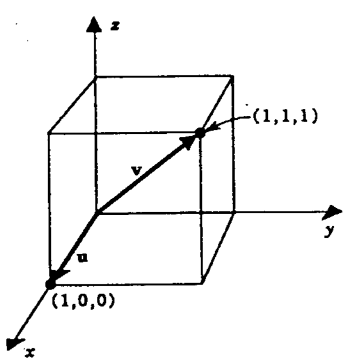
$\implies x=3,\quad x=\dfrac{5}{2}$

41. $\cos\dfrac{\pi}{3}=\dfrac{\mathbf{c}\cdot\mathbf{d}}{\|\mathbf{c}\|\,\|\mathbf{d}\|},\quad \dfrac{1}{2}=\dfrac{2x+1}{x^2+2},\quad x^2=4x;\qquad x=0,\quad x=4$

42. $(\mathbf{i}+x\mathbf{j}+\mathbf{k})\cdot(2\mathbf{i}-\mathbf{j}+y\mathbf{k})=0 \implies 2-x+y=0$

$1+x^2+1=4+1+y^2 \implies x^2-y^2=3 \implies x=\dfrac{7}{4},\quad y=-\dfrac{1}{4}$

43.



We take $\mathbf{u}=\mathbf{i}$ as an edge and $\mathbf{v}=\mathbf{i}+\mathbf{j}+\mathbf{k}$ as a diagonal of a cube. Then,

$$\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}=\frac{1}{3}\sqrt{3},$$

$$\theta=\cos^{-1}\left(\tfrac{1}{3}\sqrt{3}\right)\cong 0.96 \text{ radians}.$$

44. Take $\mathbf{a}=\mathbf{i}+\mathbf{j},\quad \mathbf{b}=\mathbf{i}+\mathbf{j}+\mathbf{k}.$

$$\theta=\cos^{-1}\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}=\cos^{-1}\frac{2}{\sqrt{2}\cdot\sqrt{3}}=\cos^{-1}\left(\frac{\sqrt{6}}{3}\right)\cong 0.62 \text{ radians}.$$

45. (a) The direction angles of a vector always satisfy

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

and, as you can check,

$$\cos^2 \tfrac{1}{4}\pi + \cos^2 \tfrac{1}{6}\pi + \cos^2 \tfrac{2}{3}\pi \neq 1.$$

(b) The relation

$$\cos^2\alpha + \cos^2 \tfrac{1}{4}\pi + \cos^2 \tfrac{1}{4}\pi = 1$$

gives

$$\cos^2\alpha + \tfrac{1}{2} + \tfrac{1}{2} = 1, \quad \cos\alpha = 0, \quad a_1 = \|\mathbf{a}\|\cos\alpha = 0.$$

46. (a) $\|\mathbf{u}_r\|^2 = \cos^2\theta + \sin^2\theta = 1, \quad \|\mathbf{u}_\theta\|^2 = \sin^2\theta + \cos^2\theta = 1.$

$\mathbf{u}_r \cdot \mathbf{u}_\theta = -\cos\theta\sin\theta + \sin\theta\cos\theta = 0, \quad$ so $\quad \mathbf{u}_r \perp \mathbf{u}_\theta.$

(b) $P = (r\cos\theta, r\sin\theta) = r\mathbf{u}_r, \quad$ so $\overrightarrow{OP}$ has same direction as $\mathbf{u}_r$.

To see that $\mathbf{u}_\theta$ is $90°$ counterclockwise from $\mathbf{u}_r$, check the sign of the coefficient in all four quadrants.

47. Set $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. The relations

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 0 \quad \text{and} \quad (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 0$$

give

$$a + 2b + c = 0 \qquad 3a - 4b + 2c = 0$$

so that $b = \frac{1}{8}a$ and $c = -\frac{5}{4}a$.

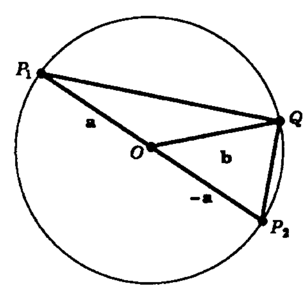
Then, since $\mathbf{u}$ is a unit vector,

$$a^2 + b^2 + c^2 = 1, \quad a^2 + \left(\frac{a}{8}\right)^2 + \left(\frac{-5a}{4}\right)^2 = 1, \quad \frac{165}{64}a^2 = 1.$$

Thus, $\quad a = \pm\dfrac{8}{165}\sqrt{165} \quad$ and $\quad \mathbf{u} = \pm\dfrac{\sqrt{165}}{165}(8\mathbf{i} + \mathbf{j} - 10\mathbf{k}).$

48. $\pm\mathbf{k}, \quad \pm\dfrac{\sqrt{13}}{13}(3\mathbf{i} - 2\mathbf{j})$

49. Place center of sphere at the origin.



$$\overrightarrow{P_1Q} \cdot \overrightarrow{P_2Q} = (-\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$$
$$= -\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2$$
$$= 0.$$

50. Take λ arbitrary, $\mathbf{b} \neq \mathbf{0}$.

$$0 \leq \|\mathbf{a} - \lambda\mathbf{b}\|^2 = (\mathbf{a} - \lambda\mathbf{b}) \cdot (\mathbf{a} - \lambda\mathbf{b}) = \mathbf{a} \cdot \mathbf{a} - \lambda(\mathbf{b} \cdot \mathbf{a}) - \lambda(\mathbf{a} \cdot \mathbf{b}) + \lambda^2(\mathbf{b} \cdot \mathbf{b})$$
$$= \|\mathbf{a}\|^2 - 2\lambda(\mathbf{a} \cdot \mathbf{b}) + \lambda^2\|\mathbf{b}\|^2$$

Setting $\lambda = (\mathbf{a}\cdot\mathbf{b})/\|\mathbf{b}\|^2$ we have

$$0 \le \|\mathbf{a}\|^2 - 2\frac{|(\mathbf{a}\cdot\mathbf{b})|^2}{\|\mathbf{b}\|^2} + \frac{|(\mathbf{a}\cdot\mathbf{b})|^2}{\|\mathbf{b}\|^2}$$
$$0 \le \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - |(\mathbf{a}\cdot\mathbf{b})|^2.$$

Thus $|(\mathbf{a}\cdot\mathbf{b})|^2 \le \|\mathbf{a}\|^2\|\mathbf{b}\|^2$ and $|(\mathbf{a}\cdot\mathbf{b})| \le \|\mathbf{a}\|\cdot\|\mathbf{b}\|$.

PROJECT 12.4

1. (a) $W = \mathbf{F}\cdot\mathbf{r}$ (b) 0 (c) $\|\mathbf{F}\|\,\mathbf{i}\cdot(b-a)\,\mathbf{i} = \|\mathbf{F}\|(b-a)$

2. (a) $W = (15\cos 35^\circ\,\mathbf{i} + 15\sin 35^\circ\,\mathbf{j})\cdot(50\,\mathbf{i}) = 15\cdot\cos 35^\circ\cdot 50 = 614.36$ joules

(b) $W = (15\cos 50^\circ\,\mathbf{i} + 15\sin 50^\circ\,\mathbf{j})\cdot(50\cos 15^\circ\,\mathbf{i} + 50\sin 15^\circ\,\mathbf{j})$

$= 15\cdot 50(\cos 50^\circ\cos 15^\circ + \sin 50^\circ\sin 15^\circ) = 15\cdot 50\cos 35^\circ = 614.36$ joules

3. $F\cos 40^\circ = 50 \implies F \cong 65.3$ pounds.

4. Let $\|\mathbf{F}_1\| = \|\mathbf{F}_2\| = C.$

(a) $W_1 = C\cos\theta_1;\quad W_2 = C\cos\theta_2 = C\cos(-\theta_1) = C\cos\theta_1 = W_1$ Thus, $W_1 = W_2$

(b) $W_1 = C\cdot\cos(\pi/3)\cdot\|\mathbf{r}\| = \frac{1}{2}C\|\mathbf{r}\|$ and $W_2 = C\cdot\cos(\pi/6)\cdot\|\mathbf{r}\| = \frac{\sqrt{3}}{2}C\|\mathbf{r}\|$

Thus, $W_2 = \sqrt{3}\,W_1$

5. Since the object returns to its starting point, the total displacement is zero, so the work done is zero.

SECTION 12.5

1. $(\mathbf{i}+\mathbf{j})\times(\mathbf{i}-\mathbf{j}) = [\mathbf{i}\times(\mathbf{i}-\mathbf{j})] + [\mathbf{j}\times(\mathbf{i}-\mathbf{j})] = (\mathbf{0}-\mathbf{k}) + (-\mathbf{k}-\mathbf{0}) = -2\mathbf{k}$

2. $\mathbf{0}$

3. $(\mathbf{i}-\mathbf{j})\times(\mathbf{j}-\mathbf{k}) = [\mathbf{i}\times(\mathbf{j}-\mathbf{k})] - [\mathbf{j}\times(\mathbf{j}-\mathbf{k})] = (\mathbf{j}+\mathbf{k}) - (\mathbf{0}-\mathbf{i}) = \mathbf{i}+\mathbf{j}+\mathbf{k}$

4. $\mathbf{j}\times(2\mathbf{i}-\mathbf{k}) = \mathbf{j}\times 2\mathbf{i} - \mathbf{j}\times\mathbf{k} = -2\mathbf{k}-\mathbf{i} = -\mathbf{i}-2\mathbf{k}$

5. $(2\mathbf{j}-\mathbf{k})\times(\mathbf{i}-3\mathbf{j}) = [2\mathbf{j}\times(\mathbf{i}-3\mathbf{j})] - [\mathbf{k}\times(\mathbf{i}-3\mathbf{j})] = (-2\mathbf{k}) - (\mathbf{j}+3\mathbf{i}) = -3\mathbf{i}-\mathbf{j}-2\mathbf{k}$

or

$$(2\mathbf{j}-\mathbf{k})\times(\mathbf{i}-3\mathbf{j}) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 2 & -1\\ 1 & -3 & 0\end{vmatrix} = \mathbf{i}\begin{vmatrix}2 & -1\\ -3 & 0\end{vmatrix} - \mathbf{j}\begin{vmatrix}0 & -1\\ 1 & -3\end{vmatrix} + \mathbf{k}\begin{vmatrix}0 & 2\\ 1 & -3\end{vmatrix} = -3\mathbf{i}-\mathbf{j}-2\mathbf{k}$$

6. $\mathbf{i}\cdot(\mathbf{j}\times\mathbf{k}) = \mathbf{i}\cdot\mathbf{i} = 1$ **7.** $\mathbf{j}\cdot(\mathbf{i}\times\mathbf{k}) = \mathbf{j}\cdot(-\mathbf{j}) = -1$ **8.** $(\mathbf{j}\times\mathbf{i})\cdot(\mathbf{i}\times\mathbf{k}) = (-\mathbf{k})\cdot(-\mathbf{j}) = 0$

9. $(\mathbf{i}\times\mathbf{j})\times\mathbf{k}=\mathbf{k}\times\mathbf{k}=\mathbf{0}$ **10.** $\mathbf{k}\cdot(\mathbf{j}\times\mathbf{i})=\mathbf{k}\cdot(-\mathbf{k})=-1$ **11.** $\mathbf{j}\cdot(\mathbf{k}\times\mathbf{i})=\mathbf{j}\cdot(\mathbf{j})=1$

12. $\mathbf{j}\times(\mathbf{k}\times\mathbf{i})=\mathbf{j}\times\mathbf{j}=\mathbf{0}$

13. $$(\mathbf{i}+3\mathbf{j}-\mathbf{k})\times(\mathbf{i}+\mathbf{k})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&3&-1\\2&0&1\end{vmatrix}=[(3)(1)-(-1)(0)]\,\mathbf{i}-[(1)(1)-(-1)(1)]\,\mathbf{j}+[(1)0-(3)(1)]\,\mathbf{k}$$
$$=3\,\mathbf{i}-2\,\mathbf{j}-3\,\mathbf{k}$$

14. $$(3\mathbf{i}-2\mathbf{j}+\mathbf{k})\times(\mathbf{i}-\mathbf{j}+\mathbf{k})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&-2&1\\1&-1&1\end{vmatrix}=-\mathbf{i}-2\mathbf{j}-\mathbf{k}$$

15. $$(\mathbf{i}+\mathbf{j}+\mathbf{k})\times(2\,\mathbf{i}+\mathbf{k})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&1\\2&0&1\end{vmatrix}=[(1)(1)-(1)(0)]\,\mathbf{i}-[(1)(1)-(1)(2)]\,\mathbf{j}+[(1)(0)-(1)(2)]\,\mathbf{k}$$
$$=\mathbf{i}+\mathbf{j}-2\,\mathbf{k}$$

16. $$(2\mathbf{i}-\mathbf{k})\times(\mathbf{i}-2\mathbf{j}+2\mathbf{k})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&0&-1\\1&-2&2\end{vmatrix}=-2\mathbf{i}-5\mathbf{j}-4\mathbf{k}$$

17. $$[2\mathbf{i}+\mathbf{j}]\cdot[(\mathbf{i}-3\mathbf{j}+\mathbf{k})\times(4\mathbf{i}+\mathbf{k})]=\begin{vmatrix}1&-3&1\\4&0&1\\2&1&0\end{vmatrix}=$$
$$[(0)(0)-(1)(1)]-(-3)[(4)(0)-(1)(2)]+[(4)(1)-(0)(2)]=-3$$

18. $$[(-2\mathbf{i}+\mathbf{j}-3\mathbf{k})\times\mathbf{i}]\times[\mathbf{i}+\mathbf{j}]=(-3\mathbf{j}-\mathbf{k})\times(\mathbf{i}+\mathbf{j})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&-3&-1\\1&1&0\end{vmatrix}=\mathbf{i}-\mathbf{j}+3\mathbf{k}$$

19. $$\begin{aligned}[(\mathbf{i}-\mathbf{j})\times(\mathbf{j}-\mathbf{k})]\times[\mathbf{i}+5\mathbf{k}]&=\{[\mathbf{i}\times(\mathbf{j}-\mathbf{k})]-[\mathbf{j}\times(\mathbf{j}-\mathbf{k})]\}\times[\mathbf{i}+5\mathbf{k}]\\&=[(\mathbf{k}+\mathbf{j})-(-\mathbf{i})]\times[\mathbf{i}+5\mathbf{k}]\\&=(\mathbf{i}+\mathbf{j}+\mathbf{k})\times(\mathbf{i}+5\mathbf{k})\\&=[(\mathbf{i}+\mathbf{j}+\mathbf{k})\times\mathbf{i}]\;+\;[(\mathbf{i}+\mathbf{j}+\mathbf{k})\times5\mathbf{k}]\\&=(-\mathbf{k}+\mathbf{j})+(-5\mathbf{j}+5\mathbf{i})\\&=5\mathbf{i}-4\mathbf{j}-\mathbf{k}\end{aligned}$$

20. $$(\mathbf{i}-\mathbf{j})\times[(\mathbf{j}-\mathbf{k})\times(\mathbf{j}+5\mathbf{k})]=(\mathbf{i}-\mathbf{j})\times\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&1&-1\\0&1&5\end{vmatrix}=(\mathbf{i}-\mathbf{j})\times6\mathbf{i}=6\mathbf{k}$$

21. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} 1 & -3 & 1 \\ 4 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} = 3\mathbf{i} - 3\mathbf{j} - 6\mathbf{k}$

$$\frac{\mathbf{a}\times\mathbf{b}}{\|\mathbf{a}\times\mathbf{b}\|} = \frac{1}{\sqrt{6}}\mathbf{i} - \frac{1}{\sqrt{6}}\mathbf{j} - \frac{2}{\sqrt{6}}\mathbf{k}; \quad \frac{\mathbf{b}\times\mathbf{a}}{\|\mathbf{b}\times\mathbf{a}\|} = -\frac{1}{\sqrt{6}}\mathbf{i} + \frac{1}{\sqrt{6}}\mathbf{j} + \frac{2}{\sqrt{6}}\mathbf{k}$$

22. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & 1 & 1 \end{vmatrix} = -\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$, so take $\pm\frac{\sqrt{35}}{35}(-\mathbf{i} + 5\mathbf{j} - 3\mathbf{k})$

23. Set $\mathbf{a} = \overrightarrow{PQ} = -\mathbf{i} + 2\mathbf{k}$ and $\mathbf{b} = \overrightarrow{PR} = 2\mathbf{i} - \mathbf{k}$. Then

$$\mathbf{N} = \overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 2 \\ 2 & 0 & -1 \end{vmatrix} = 3\mathbf{j}$$

and $A = \frac{1}{2}\|\mathbf{a}\times\mathbf{b}\| = \frac{1}{2}\|3\mathbf{j}\| = \frac{3}{2}$.

24. $\mathbf{N} = \overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & -1 \\ 2 & -3 & -1 \end{vmatrix} = -4\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$;

$A = \frac{1}{2}\|\mathbf{N}\| = 2\sqrt{3}$

25. Set $\mathbf{a} = \overrightarrow{PQ} = \mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = \overrightarrow{PR} = -\mathbf{i} + 3\mathbf{j} - \mathbf{k}$. Then

$$\mathbf{N} = \overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix} = 8\mathbf{j} + 4\mathbf{j} + 4\mathbf{k}$$

and $A = \frac{1}{2}\|\mathbf{a}\times\mathbf{b}\| = \frac{1}{2}\|8\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}\| = \frac{1}{2}\sqrt{8^2 + 4^2 + 4^2} = 2\sqrt{6}$.

26. $\mathbf{N} = \overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -4 \\ -5 & 1 & 2 \end{vmatrix} = 8\mathbf{i} + 16\mathbf{j} + 12\mathbf{k}$

Area$=\frac{1}{2}\|\mathbf{N}\| = 2\sqrt{29}$

27. $V = \left|[(\mathbf{i} + \mathbf{j})\times(2\mathbf{i} - \mathbf{k})]\cdot(3\mathbf{j} + \mathbf{k})\right| = \left|(-\mathbf{i} + \mathbf{j} - 2\mathbf{k})\cdot(3\mathbf{j} + \mathbf{k})\right| = 1$

28. $V = (\mathbf{i} - 3\mathbf{j} + \mathbf{k})\times(2\mathbf{j} - \mathbf{k})]\cdot(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = \begin{vmatrix} 1 & -3 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & -2 \end{vmatrix} = -2; \qquad V = |-2| = 2$

29. $V = \overrightarrow{OP}\cdot\left(\overrightarrow{OQ}\times\overrightarrow{OR}\right) = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{vmatrix} = 2$

30. $\overrightarrow{PQ}\cdot\left(\overrightarrow{PR}\times\overrightarrow{PS}\right) = \begin{vmatrix} 1 & 1 & -3 \\ -1 & 3 & -1 \\ 2 & 6 & 3 \end{vmatrix} = 52; \qquad V = 52$

31. $$\begin{aligned}(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b}) &= [\mathbf{a}\times(\mathbf{a}-\mathbf{b})]+[\mathbf{b}\times(\mathbf{a}-\mathbf{b})]\\ &= [\mathbf{a}\times(-\mathbf{b})]+[\mathbf{b}\times\mathbf{a}]\\ &= -(\mathbf{a}\times\mathbf{b})-(\mathbf{a}\times\mathbf{b}) = -2(\mathbf{a}\times\mathbf{b})\end{aligned}$$

32. $(\mathbf{a}+\mathbf{b})\times\mathbf{c} = -[\mathbf{c}\times(\mathbf{a}+\mathbf{b})] = -(\mathbf{c}\times\mathbf{a})-(\mathbf{c}\times\mathbf{b}) = (\mathbf{a}\times\mathbf{c})+(\mathbf{b}\times\mathbf{c})$.

33. $\mathbf{a}\times\mathbf{i}=0, \quad \mathbf{a}\times\mathbf{j}=0 \quad \Longrightarrow \quad \mathbf{a}\,\|\,\mathbf{i} \text{ and } \mathbf{a}\,\|\,\mathbf{j} \quad \Longrightarrow \quad \mathbf{a}=\mathbf{0}$

34. $\mathbf{a}\times\mathbf{b} = (a_1b_2 - b_1a_2)\mathbf{k}$

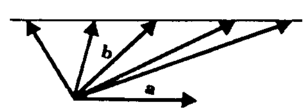
35. $$\begin{aligned}(\alpha\mathbf{a}+\beta\mathbf{b})\times(\gamma\mathbf{a}+\delta\mathbf{b}) &= (\alpha\mathbf{a}\times\delta\mathbf{b})+(\beta\mathbf{b}\times\gamma\mathbf{a})\\ &= \alpha\delta(\mathbf{a}\times\mathbf{b})-\beta\gamma(\mathbf{a}\times\mathbf{b})\\ &= (\alpha\delta-\beta\gamma)(\mathbf{a}\times\mathbf{b}) = \begin{vmatrix}\alpha & \beta\\ \gamma & \delta\end{vmatrix}(\mathbf{a}\times\mathbf{b})\end{aligned}$$

36. (a) The following statements are equivalent:

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= \mathbf{a}\times\mathbf{c}\\ (\mathbf{a}\times\mathbf{b})-(\mathbf{a}\times\mathbf{c}) &= \mathbf{0}\\ (\mathbf{a}\times\mathbf{b})+[\mathbf{a}\times(-\mathbf{c})] &= \mathbf{0}\\ \mathbf{a}\times(\mathbf{b}-\mathbf{c}) &= \mathbf{0}\end{aligned}$$

The last equation holds iff $\mathbf{a}$ and $\mathbf{b}-\mathbf{c}$ are parallel.

(b)



The tip of $\mathbf{c}$ must lie on the line which passes through the tip of $\mathbf{b}$ and is parallel to $\mathbf{a}$.

37. $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = (\mathbf{c}\times\mathbf{a})\cdot\mathbf{b} = (\mathbf{b}\times\mathbf{c})\cdot\mathbf{a} = (\mathbf{a}\times -\mathbf{c})\cdot\mathbf{b}$

$\mathbf{a}\cdot(\mathbf{c}\times\mathbf{b}) = \mathbf{c}\cdot(\mathbf{b}\times\mathbf{a}) = (-\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$

38. Since $\mathbf{a}\times\mathbf{b}\perp\mathbf{b}$, $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=0$.

39. $\mathbf{a}\times\mathbf{b}$ is perpendicular to the plane determined by $\mathbf{a}$ and $\mathbf{b}$;

$\mathbf{c}$ is in this plane iff $\mathbf{a}\times\mathbf{b}\cdot\mathbf{c}=0$.

40. $\mathbf{b}\times\mathbf{c}\perp\mathbf{b}$ and $\mathbf{b}\times\mathbf{c}\perp\mathbf{c}$, so $\mathbf{b}\times\mathbf{c}$ must be parallel to $\mathbf{a}$. Hence $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{0}$.

41. $\mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c} \implies \mathbf{a}\cdot(\mathbf{b}-\mathbf{c})=0$; $\mathbf{a}$ is perpendicular to $\mathbf{b}-\mathbf{c}$.

$\mathbf{a}\times\mathbf{b}=\mathbf{a}\times\mathbf{c} \implies \mathbf{a}\times(\mathbf{b}-\mathbf{c})=\mathbf{0}$; $\mathbf{a}$ is parallel to $\mathbf{b}-\mathbf{c}$.

Since $\mathbf{a}\neq\mathbf{0}$ it follows that $\mathbf{b}-\mathbf{c}=\mathbf{0}$ or $\mathbf{b}=\mathbf{c}$.

42. (a) $\mathbf{i}$ – component of $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=a_2(\mathbf{b}\times\mathbf{c})_3-a_3(\mathbf{b}\times\mathbf{c})_2$

$$=a_2(b_1c_2-b_2c_1)-a_3(b_3c_1-b_1c_3)=(a_2c_2+a_3c_3)b_1-(a_2b_2+a_3b_3)c_1$$

$$=(a_1c_1+a_2c_2+a_3c_3)b_1-(a_1b_1+a_2b_2+a_3b_3)c_1$$

$$=(\mathbf{a}\cdot\mathbf{c})b_1-(\mathbf{a}\cdot\mathbf{c})c_1=\mathbf{i}\text{-component of }(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{c})\mathbf{c}$$

(b) $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=-\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=-[(\mathbf{c}\cdot\mathbf{b})\mathbf{a}-(\mathbf{c}\cdot\mathbf{a})\mathbf{b}]=(\mathbf{c}\cdot\mathbf{a})\mathbf{b}-(\mathbf{c}\cdot\mathbf{b})\mathbf{a}$

(c) with $\mathbf{r}=\mathbf{c}\times\mathbf{d}$

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{r}&=(\mathbf{r}\times\mathbf{a})\cdot\mathbf{b}\\&=[(\mathbf{c}\times\mathbf{d})\times\mathbf{a}]\cdot\mathbf{b}\\&=[(\mathbf{a}\cdot\mathbf{c})\mathbf{d}-(\mathbf{a}\cdot\mathbf{d})\mathbf{c}]\cdot\mathbf{b}\\&=(\mathbf{a}\cdot\mathbf{c})(\mathbf{d}\cdot\mathbf{b})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{c}\cdot\mathbf{b})\\&=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})\end{aligned}$$

43. $\mathbf{c}\times\mathbf{a}=(\mathbf{a}\times\mathbf{b})\times\mathbf{a}=(\mathbf{a}\cdot\mathbf{a})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{a}=(\mathbf{a}\cdot\mathbf{a})\mathbf{b}=\|\mathbf{a}\|^2\mathbf{b}$

Exercise 42(a) $\qquad$ $\mathbf{a}\cdot\mathbf{b}=0$

44.

$$(\mathbf{a}\cdot\mathbf{u})\mathbf{u}+(\mathbf{u}\times\mathbf{a})\times\mathbf{u}=(\mathbf{a}\cdot\mathbf{u})\mathbf{u}+[(\mathbf{u}\cdot\mathbf{u})\mathbf{a}-(\mathbf{u}\cdot\mathbf{a})\mathbf{u}]$$

Exercise 42(b)

$$=(\mathbf{a}\cdot\mathbf{u})\mathbf{u}+\mathbf{a}-(\mathbf{a}\cdot\mathbf{u})\mathbf{u}=\mathbf{a}$$

45. Expanding the determinant by the bottom row gives

$$\begin{vmatrix}a_1&a_2&a_3\\b_1&b_2&b_3\\c_1&c_2&c_3\end{vmatrix}=c_1\begin{vmatrix}a_2&a_3\\b_2&b_3\end{vmatrix}-c_2\begin{vmatrix}a_1&a_3\\b_1&b_3\end{vmatrix}+c_3\begin{vmatrix}a_1&a_2\\b_1&b_2\end{vmatrix}$$

PROJECT 12.5

1. $\mathbf{r} = \overrightarrow{OP} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$||\tau|| = ||\mathbf{r} \times \mathbf{F}|| = ||(\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (\mathbf{i} + 2\mathbf{j} + \mathbf{k})|| = || - \mathbf{i} + \mathbf{k}|| = \sqrt{2}$

2. $||\tau|| = ||\mathbf{r}|| \cdot ||\mathbf{F}|| \sin\theta = (10)(20)\sin 50° = 153.21$ inch-lb $= 12.77$ ft-lb;

the bolt moves into the plane of the paper.

3. $||\tau|| = ||\mathbf{r}||||\mathbf{F}|| \sin\theta = (10)(20)\sin 130° \cong 152.2$ inch-lb $= 12.68$ ft-lb;

the bolt will move into the plane of the paper.

4. Using the figure, the origin is at the center of mass. We measure distance in meters and force in Newtons.

(a) $\tau = \mathbf{r} \times \mathbf{F}$ where $\mathbf{r} = 0\mathbf{i} + 0.7\mathbf{j} - 0.9\mathbf{k}$ and $\mathbf{F} = 0\mathbf{i} - 600\mathbf{j} + 0\mathbf{k}$. Thus

$$\tau = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0.7 & -0.9 \\ 0 & -600 & 0 \end{vmatrix} = -540\mathbf{i} \text{ Newton-meters.}$$

(b) Into the plane of the page

(c) This torque produces a clockwise rptation, i.e. the braking force will flip the bicycle over the front wheel.

SECTION 12.6

1. P (when $t = 0$) and Q (when $t = -1$)

2. l_1, l_3 and l_4 are parallel.

3. Take $\mathbf{r}_0 = \overrightarrow{OP} = 3\mathbf{i} + \mathbf{j}$ and $\mathbf{d} = \mathbf{k}$. Then, $\mathbf{r}(t) = (3\mathbf{i} + \mathbf{j}) + t\mathbf{k}$.

4. $\mathbf{r}(t) = \mathbf{i} - \mathbf{j} + 2\mathbf{k} + t(3\mathbf{i} - \mathbf{j} + \mathbf{k})$

5. Take $\mathbf{r}_0 = \mathbf{0}$ and $\mathbf{d} = \overrightarrow{OQ}$. Then, $\mathbf{r}(t) = t(x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k})$.

6. $\mathbf{r}(t) = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k} + t\,[(x_1 - x_0)\mathbf{i} + (y_1 - y_0)\mathbf{j} + (z_1 - z_0)\mathbf{k}]$

7. $\overrightarrow{PQ} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ so direction numbers are $1, -1, 1$. Using P as a point on the line, we have

$$x(t) = 1 + t, \quad y(t) = -t, \quad z(t) = 3 + t.$$

8. $x(t) = x_0 + t(x_1 - x_0), \quad y(t) = y_0 + t(y_1 - y_0), \quad z(t) = z_0 + t(z_1 - z_0).$

9. The line is parallel to the y-axis so we can take $0, 1, 0$ as direction numbers. Therefore

$$x(t) = 2, \quad y(t) = -2 + t, \quad z(t) = 3.$$

10. $x(t) = t, \quad y(t) = 4, \quad z(t) = -3$

11. Since the line $2(x+1) = 4(y-3) = z$ can be written

$$\frac{x+1}{2} = \frac{y-3}{1} = \frac{z}{4},$$

it has direction numbers $2, 1, 4$. The line through $P(-1, 2, -3)$ with direction vector $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ can be parameterized

$$\mathbf{r}(t) = (-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + t(2\mathbf{i} + \mathbf{j} + 4\mathbf{k}).$$

12. $\dfrac{x}{x_0} = \dfrac{y}{y_0} = \dfrac{z}{z_0}$, so $\dfrac{x - x_0}{x_0} = \dfrac{y - y_0}{y_0} = \dfrac{z - z_0}{z_0}$ provided $x_0 y_0 z_0 \neq 0$

13. We set $\mathbf{r}_1(t) = \mathbf{r}_2(u)$ and solve for t and u:

$$\mathbf{i} + t\mathbf{j} = \mathbf{j} + u(\mathbf{i} + \mathbf{j}),$$

$$(1-u)\,\mathbf{i} + (-1 - u + t)\,\mathbf{j} = \mathbf{0}.$$

Thus,

$$1 - u = 0 \quad \text{and} \quad -1 - u + t = 0.$$

These equations give $u = 1, \ t = 2$. The point of intersection is $P(1, 2, 0)$.

As direction vectors for the lines we can take $\mathbf{u} = \mathbf{j}$ and $\mathbf{v} = \mathbf{i} + \mathbf{j}$. Thus

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{1}{(1)(\sqrt{2})} = \frac{1}{2}\sqrt{2}.$$

The angle of intersection is $\frac{1}{4}\pi$ radians.

14. Set $\mathbf{r}_1(t) = \mathbf{r}_2(u)$:

$$\mathbf{i} - 4\sqrt{3}\mathbf{j} + t(\mathbf{i} + \sqrt{3}\mathbf{j}) = 4\mathbf{i} + 3\sqrt{3}\mathbf{j} + u(\mathbf{i} - \sqrt{3}\mathbf{j})$$

$$(-3 + t - u)\mathbf{i} + (-7\sqrt{3} + t\sqrt{3} + u\sqrt{3})\mathbf{j} = \mathbf{0}$$

$\implies t - u = 3, \quad t + u = 7, \quad \implies \quad t = 5, \quad u = 2$

$\mathbf{r}_1(5) = \mathbf{r}_2(2) = 6\mathbf{i} + \sqrt{3}\mathbf{j}$: the point of intersection is: $P(6, \sqrt{3}, 0)$

$\cos\theta = \dfrac{|(\mathbf{i} + \sqrt{3}\mathbf{j}) \cdot (\mathbf{i} - \sqrt{3}\mathbf{j})|}{\|\mathbf{i} + \sqrt{3}\mathbf{j}\|\|\mathbf{i} - \sqrt{3}\mathbf{j}\|} = \dfrac{1}{2} \quad \implies \theta = \dfrac{\pi}{3}$ radians.

15. We solve the system

$$3 + t = 1, \quad 1 - t = 4 + u, \quad 5 + 2t = 2 + u$$

for t and u to find that $t = -2, \ u = -1$. The point of intersection is $(1, 3, 1)$.

Since $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ is a direction vector for l_1 and $\mathbf{j} + \mathbf{k}$ is a direction vector for l_2,

$$\cos\theta = \frac{(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \cdot (\mathbf{j} + \mathbf{k})}{\sqrt{6}\sqrt{2}} = \frac{1}{2\sqrt{3}} = \frac{1}{6}\sqrt{3} \quad \text{and} \quad \theta \cong 1.28 \text{ radians}.$$

16. $x_1(t) = x_2(u) \implies 1 + t = 1 - u \implies t = -u$

$y_1(t) = y_2(t) \implies -1 - t = 1 + 3u \implies t = -2 - 3u$

$\Longrightarrow u = -1,\ t = 1.\quad z_1(1) = z_2(-1),$ so they intersect at $P(2,-2,-2)$.

$$\cos\theta = \frac{|(1,-1,2)\cdot(-1,3,2)|}{\|(1,-1,2)\|\|(-1,3,2)\|} = 0 \Longrightarrow \theta = \frac{\pi}{2} \text{ radians.}$$

17. $\left(x_0 - \frac{d_1}{d_3}z_0,\ y_0 - \frac{d_2}{d_3}z_0,\ 0\right)$

18. The lines meet at (x_0, y_0, z_0), and since $\mathbf{d}\cdot\mathbf{D} = \mathbf{0}$, they are perpendicular.

19. The lines are parallel.

20. Note that $\mathbf{r}(0) = \mathbf{r}_0$ and $\mathbf{r}(1) = \mathbf{r}_1$, so we need $0 \le t \le 1$

21. $\mathbf{r}(t) = (2\mathbf{i} + 7\mathbf{j} - \mathbf{k}) + t(2\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}),\quad 0 \le t \le 1$

22. $-1 \le t \le 2$.

23. Set $$\mathbf{u} = \frac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} = \frac{-4\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}}{\|-4\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}\|} = -\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}.$$

Then $\mathbf{r}(t) = (6\mathbf{i} - 5\mathbf{j} + \mathbf{k}) + t\mathbf{u}$ is $\overrightarrow{OP}$ at $t = 9$ and it is $\overrightarrow{OQ}$ at $t = 15$. (Check this.)

Answer: $\mathbf{u} = -\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k},\quad 9 \le t \le 15.$

24. Since the two lines intersect, there exist numbers t_0 and u_0 such that
$$\mathbf{r}(t_0) = \mathbf{R}(u_0)$$
Suppose first that
$$\mathbf{r}(t_0) = \mathbf{R}(u_0) = 0.$$
Then
$$\mathbf{r}(t) = \mathbf{r}(t) - \mathbf{r}(t_0) = (\mathbf{r}_0 + t\mathbf{d}) - (\mathbf{r}_0 + t_0\mathbf{d}) = (t - t_0)\mathbf{d}.$$
Similarly $R(u) = (u - u_0)\mathbf{D}$. Since $l_1 \perp l_2$, we have $\mathbf{d}\cdot\mathbf{D} = \mathbf{0}$ and thus
$$\mathbf{r}(t)\cdot\mathbf{R}(u) = (t - t_0)(u - u_0)(\mathbf{d}\cdot\mathbf{D}) = 0 \quad \text{for all } t, u$$
Suppose now that
$$\mathbf{r}(t_0) = \mathbf{R}(u_0) \ne \mathbf{0}$$
Then
$$\mathbf{r}(t_0)\cdot\mathbf{R}(u_0) = \mathbf{r}(t_0)\cdot\mathbf{r}(t_0) = \|\mathbf{r}(t_0)\|^2 \ne 0$$
and it is therefore not true that
$$\mathbf{R}(t)\cdot\mathbf{R}(u) = 0, \quad \text{for all } t, u.$$

25. The given line, call it l, has direction vector $2\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}$.

If $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a direction vector for a line perpendicular to l, then
$$(2\mathbf{i} - 4\mathbf{j} + 6\mathbf{k})\cdot(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = 2a - 4b + 6c = 0.$$
The lines through $P(3,-1,8)$ perpendicular to l can be parameterized
$$X(u) = 3 + au, \quad Y(u) = -1 + bu, \quad Z(u) = 8 + cu$$

with $2a - 4b + 6c = 0$.

26. (a) Since $\mathbf{R}(0) = \mathbf{R}_0$ is on the line, there exists a number t_0 such that $\mathbf{r}(t_0) = \mathbf{r}_0 + t_0\mathbf{d} = \mathbf{R}_0$.

(b) Since $\mathbf{d}$ and $\mathbf{D}$ are both direction vectors for the same line, they are parallel. Since $\mathbf{d} \neq \mathbf{0}$, there exists a scalar α such that $\mathbf{D} = \alpha\mathbf{d}$. It follows that, for all real u,

$$\mathbf{R}(u) = \mathbf{R}_0 + u\mathbf{D} = (\mathbf{r}_0 + t_0\mathbf{d}) + u(\alpha\mathbf{d}) = r_0 + (t_0 + \alpha u)\mathbf{d}.$$

27. $d(P,l) = \dfrac{\|(\mathbf{i}+2\mathbf{k}) \times (2\mathbf{i}-\mathbf{j}+2\mathbf{k})\|}{\|2\mathbf{i}-\mathbf{j}+2\mathbf{k}\|} = 1$

28. $d(P,l) = \dfrac{\|(\mathbf{j}+\mathbf{k}) \times (\mathbf{i}-2\mathbf{j}-2\mathbf{k})\|}{\|\mathbf{i}-2\mathbf{j}-2\mathbf{k}\|} = \dfrac{1}{3}\sqrt{2} \cong 0.47$

29. The line contains the point $P_0(1,0,2)$. Therefore

$$d(P,l) = \frac{\|(2\mathbf{j}+\mathbf{k}) \times (\mathbf{i}-2\mathbf{j}+3\mathbf{k})\|}{\|\mathbf{i}-2\mathbf{j}+3\mathbf{k}\|} = \sqrt{\frac{69}{14}} \cong 2.22$$

30. $P_0 = (1,0,0), \quad \mathbf{d} = \mathbf{j}, \quad d(P,l) = \dfrac{\|-\mathbf{i}\times\mathbf{j}\|}{\|\mathbf{j}\|} = 1.$

31. The line contains the point $P_0(2,-1,0)$. Therefore

$$d(P,l) = \frac{\|(\mathbf{i}-\mathbf{j}-\mathbf{k}) \times (\mathbf{i}+\mathbf{j})\|}{\|\mathbf{i}+\mathbf{j}\|} = \sqrt{3} \cong 1.73.$$

32. The line can be parameterized $\mathbf{r}(t) = b\mathbf{j} + t(\mathbf{i} + m\mathbf{j})$. Since the line contains the point $(0,b,0)$

$$\begin{aligned} d(P,l) &= \frac{\|[x_0\mathbf{i} + (y_0 - b)\mathbf{j} + z_0\mathbf{k}] \times (\mathbf{i} + m\mathbf{j})\|}{\|\mathbf{i} + m\mathbf{j}\|} \\ &= \sqrt{\frac{(1+m^2){z_0}^2 + [y_0 - (mx_0 + b)]^2}{1+m^2}} \end{aligned}$$

If $P(x_0, y_0, z_0)$ lies directly above or below the line, then $y_0 = mx_0 + b$ and $d(P,l)$ reduces to $\sqrt{{z_0}^2} = |z_0|$. This is evident geometrically.

33. (a) The line passes through $P(1,1,1)$ with direction vector $\mathbf{i}+\mathbf{j}$. Therefore

$$d(0,l) = \frac{\|(\mathbf{i}+\mathbf{j}+\mathbf{k}) \times (\mathbf{i}+\mathbf{j})\|}{\|\mathbf{i}+\mathbf{j}\|} = 1.$$

(b) The distance from the origin to the line segment is $\sqrt{3}$.

Solution. The line segment can be parameterized

$$\mathbf{r}(t) = \mathbf{i}+\mathbf{j}+\mathbf{k}+t(\mathbf{i}+\mathbf{j}), \quad t \in [0,1].$$

This is the set of all points $P(1+t, 1+t, 1)$ with $t \in [0,1]$.

The distance from the origin to such a point is

$$f(t) = \sqrt{2(1+t^2)+1}.$$

The minimum value of this function is $f(0) = \sqrt{3}$.

Explanation. The point on the line through P and Q closest to the origin is not on the line segment $\overline{PQ}$.

34. (a) We want $(\mathbf{r}_0 + t_0\mathbf{d}) \cdot \mathbf{d} = 0$, so $\mathbf{r}_0 \cdot \mathbf{d} + t_0\|\mathbf{d}\|^2 = 0 \implies t_0 = -\dfrac{\mathbf{r}_0 \cdot \mathbf{d}}{\|\mathbf{d}\|^2}$

(b) $\mathbf{R}(t) = \mathbf{r}(t_0) \pm t\dfrac{\mathbf{d}}{\|\mathbf{d}\|^2}$, where t_0 is as in part (a).

35. We begin with $\mathbf{r}(t) = \mathbf{j} - 2\mathbf{k} + t(\mathbf{i} - \mathbf{j} + 3\mathbf{k})$. The scalar t_0 for which $\mathbf{r}(t_0) \perp l$ can be found by solving the equation

$$[\mathbf{j} - 2\mathbf{k} + t_0(\mathbf{i} - \mathbf{j} + 3\mathbf{k})] \cdot [\mathbf{i} - \mathbf{j} + 3\mathbf{k}] = 0.$$

This equation gives $-7 + 11t_0 = 0$ and thus $t_0 = 7/11$. Therefore

$$\mathbf{r}(t_0) = \mathbf{j} - 2\mathbf{k} + \tfrac{7}{11}(\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = \tfrac{7}{11}\mathbf{i} + \tfrac{4}{11}\mathbf{j} - \tfrac{1}{11}\mathbf{k}.$$

The vectors of norm 1 parallel to $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ are

$$\pm\frac{1}{\sqrt{11}}(\mathbf{i} - \mathbf{j} + 3\mathbf{k}).$$

The standard parameterizations are

$$\mathbf{R}(t) = \frac{7}{11}\mathbf{i} + \frac{4}{11}\mathbf{j} - \frac{1}{11}\mathbf{k} \pm \frac{t}{\sqrt{11}}(\mathbf{i} - \mathbf{j} + 3\mathbf{k})$$

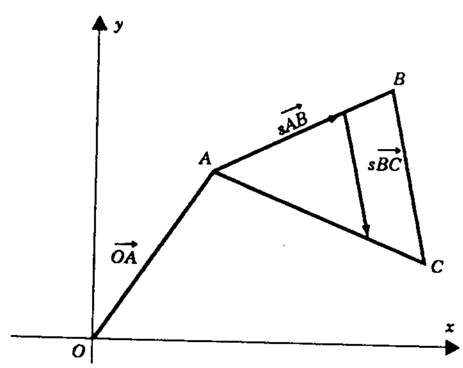
$$= \frac{1}{11}(7\mathbf{i} + 4\mathbf{j} - \mathbf{k}) \pm t\left[\frac{\sqrt{11}}{11}(\mathbf{i} - \mathbf{j} + 3\mathbf{k})\right].$$

36. Start with $\mathbf{r}(t) = \sqrt{3}\mathbf{i} + t(\mathbf{i} + \mathbf{j} + \mathbf{k})$ to get $t_0 = -\dfrac{\sqrt{3}}{3}$, so

$$\mathbf{R}(t) = \frac{\sqrt{3}}{3}(2\mathbf{i} - \mathbf{j} - \mathbf{k}) \pm t\,\frac{\sqrt{3}}{3}(\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

37. $0 < t < s$

By similar triangles, if $0 < s < 1$, the tip of $\overrightarrow{OA} + s\overrightarrow{AB} + s\overrightarrow{BC}$ falls on $\overline{AC}$. If $0 < t < s$, then the tip of $\overrightarrow{OA} + s\overrightarrow{AB} + t\overrightarrow{BC}$ falls short of $\overline{AC}$ and stays within the triangle. Clearly all points in the interior of the triangle can be reached in this manner.



SECTION 12.7

1. Q

2. An equation for the plane is: $(x-4)-3(y-1)+(z+1)=0$; R and S lie on the plane.

3. Since $\mathbf{i}-4\mathbf{j}+3\mathbf{k}$ is normal to the plane, we have

$$(x-2)-4(y-3)+3(z-4)=0 \quad \text{and thus} \quad x-4y+3z-2=0.$$

4. $\mathbf{N}=\mathbf{j}+2\mathbf{k}$, $P(1,-2,3) \implies (y+2)+2(z-3)=0$ or $y+2z-4=0$.

5. The vector $3\mathbf{i}-2\mathbf{j}+5\mathbf{k}$ is normal to the given plane and thus to every parallel plane: the equation we want can be written

$$3(x-2)-2(y-1)+5(z-1)=0, \quad 3x-2y+5z-9=0.$$

6. $\mathbf{N}=4\mathbf{i}+2\mathbf{j}-7\mathbf{k}$, $P(3,-1,5) \implies 4(x-3)+2(y+1)-7(z-5)=0$

7. The point $Q\,(0,0,-2)$ lies on the line l; and $\mathbf{d}=\mathbf{i}+\mathbf{j}+\mathbf{k}$ is a direction vector for l.

We want an equation for the plane which has the vector

$$\mathbf{N}=\overrightarrow{PQ}\times\mathbf{d}=(\mathbf{i}+3\mathbf{j}+3\mathbf{k})\times(\mathbf{i}+\mathbf{j}+\mathbf{k})$$

as a normal vector:

$$\mathbf{N}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 3 & 3\\ 1 & 1 & 1\end{vmatrix}=2\mathbf{j}-2\mathbf{k}$$

An equation for the plane is: $2(y-3)-2(z-1)=0$ or $y-z-2=0$

8. Another point is $Q(1,1,2)$, and the plane is parallel to the vectors $\mathbf{d}=-2\mathbf{i}+4\mathbf{j}+\mathbf{k}$ and

$\overrightarrow{PQ}=-\mathbf{i}+\mathbf{j}+\mathbf{k}$. Thus, $\mathbf{N}=\mathbf{d}\times\overrightarrow{PQ}=3\mathbf{i}+\mathbf{j}+2\mathbf{k}$ is a normal to the plane.

An equation for the plane is: $3(x-2)+y+2(z-1)=0$ or $3x+y+2z-8=0$

9. $\overrightarrow{OP_0}=x_0\mathbf{i}+y_0\mathbf{j}+z_0\mathbf{k}$ An equation for the plane is:

$$x_0\,(x-x_0)+y_0\,(y-y_0)+z_0\,(z-z_0)$$

10. $\mathbf{N}=2\mathbf{i}-3\mathbf{j}+7\mathbf{k}$, unit normals: $\mathbf{u}_N=\pm\dfrac{1}{\sqrt{62}}(2\mathbf{i}-3\mathbf{j}+7\mathbf{k})$

11. The vector $\mathbf{N}=2\mathbf{i}-\mathbf{j}+5\mathbf{k}$ is normal to the plane $2x-y+5z-10=0$. The unit normals are:

$$\frac{\mathbf{N}}{\|\mathbf{N}\|}=\frac{1}{\sqrt{30}}(2\mathbf{i}-\mathbf{j}+5\mathbf{k}) \quad \text{and} \quad -\frac{\mathbf{N}}{\|\mathbf{N}\|}=-\frac{1}{\sqrt{30}}(2\mathbf{i}-\mathbf{j}+5\mathbf{k})$$

12. $(a,0,0)$, $(0,b,0)$, and $(0,0,c)$ satisfy the equation.

13. Intercept form: $\frac{x}{15}+\frac{y}{12}-\frac{z}{10}=1$

x-intercept: $(15,0,0)$

y-intercept: $(0,12,0)$

z-intercept: $(0,0,10)$

14. $\frac{x}{-2/3}+\frac{y}{2}+\frac{z}{-1/2}=1;\quad (-\frac{2}{3},0,0),\ (0,2,0),\ (0,0,-\frac{1}{2}).$

15. $\mathbf{u_{N_1}}=\frac{\sqrt{38}}{38}(5\mathbf{i}-3\mathbf{j}+2\mathbf{k}),\quad \mathbf{u_{N_2}}=\frac{\sqrt{14}}{14}(\mathbf{i}+3\mathbf{j}+2\mathbf{k}),\quad \cos\theta=|\mathbf{u_{N_1}}\cdot\mathbf{u_{N_2}}|=0.$

Therefore $\theta=\pi/2$ radians.

16. $\cos\theta=\frac{|(2\mathbf{i}-\mathbf{j}+3\mathbf{k})\cdot(5\mathbf{i}+5\mathbf{j}-\mathbf{k})|}{\|2\mathbf{i}-\mathbf{j}+3\mathbf{k}\|\|5\mathbf{i}+5\mathbf{j}-\mathbf{k}\|}=\frac{2}{\sqrt{14}\sqrt{51}};\quad \theta\cong 1.50$ radians.

17. $\mathbf{u_{N_1}}=\frac{\sqrt{3}}{3}(\mathbf{i}-\mathbf{j}+\mathbf{k}),\quad \mathbf{u_{N_2}}=\frac{\sqrt{14}}{14}(2\mathbf{i}+\mathbf{j}+3\mathbf{k}),\quad \cos\theta=|\mathbf{u_{N_1}}\cdot\mathbf{u_{N_2}}|=\frac{2}{21}\sqrt{42}\cong 0.617.$

Therefore $\theta\cong 0.91$ radians.

18. $\cos\theta=\frac{|(4\mathbf{i}+4\mathbf{j}-2\mathbf{k})\cdot(2\mathbf{i}+\mathbf{j}+\mathbf{k})|}{\|4\mathbf{i}+4\mathbf{j}-2\mathbf{k}\|\|2\mathbf{i}+\mathbf{j}+\mathbf{k}\|}=\frac{10}{6\sqrt{6}};\quad \theta\cong 0.82$ radian.

19. coplanar since $0(4\mathbf{j}-\mathbf{k})+0(3\mathbf{i}+\mathbf{j}+2\mathbf{k})+1(\mathbf{0})=0$

20. $s\mathbf{i}+t(\mathbf{i}-2\mathbf{j})+u(3\mathbf{j}+\mathbf{k})=(s+t)\mathbf{i}+(-2t+3u)\mathbf{j}+u\mathbf{k}=\mathbf{0}$ only if $s=t=u=0$, so not coplanar.

21. We need to determine whether there exist scalars s,t,u not all zero such that

$$s(\mathbf{i}+\mathbf{j}+\mathbf{k})+t(2\mathbf{i}-\mathbf{j})+u(3\mathbf{i}-\mathbf{j}-\mathbf{k})=\mathbf{0}$$

$$(s+2t+3u)\,\mathbf{i}+(s-t-u)\,\mathbf{j}+(s-u)\,\mathbf{k}=\mathbf{0}.$$

The only solution of the system

$$s+2t+3u=0,\quad s-t-u=0,\quad s-u=0$$

is $s=t=u=0$. Thus, the vectors are not coplanar.

22. coplanar since $1(\mathbf{j}-\mathbf{k})-1(3\mathbf{i}-\mathbf{j}+2\mathbf{k})+1(3\mathbf{i}-2\mathbf{j}+3\mathbf{k})=\mathbf{0}$

23. By (12.7.7), $d(P,p)=\frac{|2(2)+4(-1)-(3)+1|}{\sqrt{4+16+1}}=\frac{2}{\sqrt{21}}=\frac{2}{21}\sqrt{21}.$

24. $d=\frac{|8(3)-2(-5)+2-5|}{\sqrt{8^2+(-2)^2+1^2}}=\frac{31}{\sqrt{69}}$

25. By (12.7.7), $d(P,p)=\frac{|(-3)(1)+0(-3)+4(5)+5|}{\sqrt{9+16}}=\frac{22}{5}.$

26. $d = \dfrac{|1+3-2\cdot 4|}{\sqrt{1^2+1^2+(-2)^2}} = \dfrac{2}{3}\sqrt{6}$

27. $\overrightarrow{P_1P} = (x-1)\,\mathbf{i} + y\mathbf{j} + (z-1)\,\mathbf{k}, \quad \overrightarrow{P_1P_2} = \mathbf{i}+\mathbf{j}-\mathbf{k}, \quad \overrightarrow{P_1P_3} = \mathbf{j}.$

Therefore

$$(\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}) = (\mathbf{i}+\mathbf{j}-\mathbf{k}) \times \mathbf{j} = \mathbf{i}+\mathbf{k}$$

and

$$\overrightarrow{P_1P} \cdot (\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}) = [(x-1)\,\mathbf{i} + y\mathbf{j} + (z-1)\,\mathbf{k}] \cdot [\mathbf{i}+\mathbf{k}] = x-1+z-1.$$

An equation for the plane can be written $x+z=2$.

28. $\overrightarrow{P_1P_2} = (1,-3,-2), \quad \overrightarrow{P_1P_3} = (-1,1,0), \quad \mathbf{N} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = (2,2,-2)$

$\implies 2(x-1)+2(y-1)-2(z-1)=0$ or $x+y-z-1=0$

29. $\overrightarrow{P_1P} = (x-3)\,\mathbf{i} + (y+4)\,\mathbf{j} + (z-1)\,\mathbf{k}, \quad \overrightarrow{P_1P_2} = 6\mathbf{j}, \quad \overrightarrow{P_1P_3} = -4\mathbf{i}+5\mathbf{j}-3\mathbf{k}.$

Therefore

$$(\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}) = 6\mathbf{j} \times (-4\mathbf{i}+5\mathbf{j}-3\mathbf{k}) = -18\mathbf{i}+24\mathbf{k}$$

and

$$\begin{aligned}\overrightarrow{P_1P} \cdot (\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}) &= [(x-3)\,\mathbf{i} + (y+4)\,\mathbf{j} + (z-1)\,\mathbf{k}] \cdot [-18\mathbf{i}+24\mathbf{k}] \\ &= -18(x-3)+24(z-1)\end{aligned}$$

An equation for the plane can be written $-18(x-3)+24(z-1)=0$ or $3x-4z-5=0$.

30. $\overrightarrow{P_1P_2} = (0,-4,5), \quad \overrightarrow{P_1P_3} = (-2,-3,4) \quad \mathbf{N} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = (-1,-10,-8)$

$\implies (x-3)+10(y-2)+8(z+1)=0$

31. The line passes through the point $P_0\,(x_0,y_0,z_0)$ with direction numbers: A, B, C. Equations for the line written in symmetric form are:

$$\frac{x-x_0}{A} = \frac{y-y_0}{B} = \frac{z-z_0}{C}, \quad \text{provided} \quad A \neq 0,\ B \neq 0,\ C \neq 0.$$

32. Take a point $P_1(x_1,y_1,z_1)$ on plane 1 (so $Ax_1+By_1+Cz_1+D_1=0$) and find the distance to plane 2 :

$$d = \frac{|Ax_1+By_1+Cz_1+D_2|}{\sqrt{A^2+B^2+C^2}} = \frac{|D_2-D_1|}{\sqrt{A^2+B^2+C^2}}$$

33. $\dfrac{x-x_0}{d_1} = \dfrac{y-y_0}{d_2}, \qquad \dfrac{y-y_0}{d_2} = \dfrac{z-z_0}{d_3}$

34. No, The procedure may fail if the line of intersection is parallel to yz-plane. In this case begin by selecting a value for y rather than x and then solve for x and z.

35. Following the hint we take $x = 0$ and find that $P_0(0,0,0)$ lies on the line of intersection. As normals to the plane we use

$$\mathbf{N}_1 = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \mathbf{N}_2 = -3\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$

Note that

$$\mathbf{N}_1 \times \mathbf{N}_2 = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \times (-3\mathbf{i} + 4\mathbf{j} + \mathbf{k}) = -10\mathbf{i} - 10\mathbf{j} + 10\mathbf{k}.$$

We take $-\frac{1}{10}(\mathbf{N}_1 \times \mathbf{N}_2) = \mathbf{i} + \mathbf{j} - \mathbf{k}$ as a direction vector for the line through $P_0(0,0,0)$. Then

$$x(t) = t, \quad y(t) = t, \quad z(t) = -t.$$

36. Using the hint, we find $P(0, \frac{1}{2}, -\frac{3}{2})$ on the line of intersection.

For the direction vector, consider $N_1 \times N_2 = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (\mathbf{i} - \mathbf{j} + \mathbf{k}) = 2\mathbf{i} - 2\mathbf{k}$, so we can use $\mathbf{d} = \mathbf{i} - \mathbf{k}$. Thus

$$x(t) = t, \quad y(t) = \frac{1}{2}, \quad z(t) = -\frac{3}{2} - t.$$

37. Straightforward computations give us

$$l\colon x(t) = 1 - 3t, \quad y(t) = -1 + 4t, \quad z(t) = 2 - t$$

and

$$p\colon x + 4y - z = 6.$$

Substitution of the scalar parametric equations for l in the equation for p gives

$$(1 - 3t) + 4(-1 + 4t) - (2 - t) = 6 \quad \text{and thus} \quad t = 11/14.$$

Using $t = 11/14$, we get $\quad x = -19/14, \quad y = 15/7, \quad z = 17/14.$

38. $l : x(t) = 4 - 2t, \quad y(t) = -3 + t, \quad z(t) = 1 + 2t \quad P: \; x + 4y - z = 6$

Note that $\mathbf{d} \cdot \mathbf{N} = (-2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \cdot (\mathbf{i} + 4\mathbf{j} - \mathbf{k}) = 0$, so the line is parallel to the plane, and since P_1 does not lie in the plane, l and P do not intersect.

39. Let $\mathbf{N} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ be normal to the plane. Then

$$\mathbf{N} \cdot \mathbf{d} = (\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}) = 1 + 2B + 4C = 0$$

and

$$\mathbf{N} \cdot \mathbf{d} = (\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot (-\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = -1 - B + 3C = 0.$$

This gives $B = -7/10$ and $C = 1/10$. The equation for the plane can be written

$$1(x - 0) - \tfrac{7}{10}(y - 0) + \tfrac{1}{10}(z - 0) = 0, \quad \text{which simplifies to} \quad 10x - 7y + z = 0.$$

40. If the two lines intersect, then there exist number t_0 and u_0 such that

$$\mathbf{r}_0 + t_0\mathbf{d} = \mathbf{R}_0 + u_0\mathbf{D}.$$

It follows then that

$$1(\mathbf{r}_0 - \mathbf{R}_0) + t_0\mathbf{d} - u_0\mathbf{D} = 0$$

and therefore the vectors $\mathbf{r}_0 - \mathbf{R}_0, \mathbf{d},$ and $\mathbf{D}$ are coplanar. Then there exist number α, β, γ not all zero such that

$$(*) \qquad \alpha(\mathbf{r}_0 - \mathbf{R}_0) + \beta\mathbf{d} + \gamma\mathbf{D} = \mathbf{0}.$$

We assert now that $\alpha \neq 0$. [If α were 0, then we would have $\beta\mathbf{d} + \gamma\mathbf{D} = \mathbf{0}$ so that $\mathbf{d}$ and $\mathbf{D}$ would have to be parallel, which (by assumption) they are not.] With $\alpha \neq 0$ we can divide equation (*) by α and obtain

$$\mathbf{r}_0 + \mathbf{R}_0 + \frac{\beta}{\alpha}\mathbf{d} + \frac{\gamma}{\alpha}\mathbf{D} = \mathbf{0}.$$

This gives

$$\mathbf{r}_0 + \frac{\beta}{\alpha}\mathbf{d} = \mathbf{R}_0 - \frac{\gamma}{\alpha}\mathbf{D},$$

which means that

$$\mathbf{r}\left(\frac{\beta}{\alpha}\right) = \mathbf{R}\left(-\frac{\gamma}{\alpha}\right)$$

and the two lines intersect.

41. $\mathbf{N} + \overrightarrow{PQ}$ and $\mathbf{N} - \overrightarrow{PQ}$ are the diagonals of a rectangle with sides $\mathbf{N}$ and $\overrightarrow{PQ}$. Since the diagonals are perpendicular, the rectangle is a square; that is $||\mathbf{N}|| = ||\overrightarrow{PQ}||$. Thus, the points Q form a circle centered at P with radius $||\mathbf{N}||$.

42. No, each vector is normal to the plane of the other two. Another way to look at it: if they were coplanar, then together with a normal to that plane we would have four mutually perpendicular nonzero vectors. Three-space would then have four dimensions. Here is an algebraic argument:

We are given that $||\mathbf{a}||, ||\mathbf{b}||, ||\mathbf{c}||$ are nonzero and $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{c} = \mathbf{b}\cdot\mathbf{c} = 0$. The vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar iff there exist scalars s, t, u not all zero such that $s\mathbf{a} + t\mathbf{b} + u\mathbf{c} = 0$. Now assume that

$$s\mathbf{a} + t\mathbf{b} + u\mathbf{c} = 0.$$

The relation

$$0 = \mathbf{a}\cdot(s\mathbf{a} + t\mathbf{b} + u\mathbf{c}) = s||\mathbf{a}||^2 + t(\mathbf{a}\cdot\mathbf{b}) + u(\mathbf{a}\cdot\mathbf{c}) = s||\mathbf{a}||^2$$

gives $s = 0$. Similarly, we can show that $t = 0$ and $u = 0$. The vectors $\mathbf{a}, \mathbf{b}.\mathbf{c}$ can not be coplanar.

43. If $\alpha > 0$, then P_1 lies on the same side of the plane as the tip of $\mathbf{N}$; if $\alpha < 0$, then P_1 and the tip of $\mathbf{N}$ lie on opposite sides of the plane.

To see this, suppose that the tip of $\mathbf{N}$ is at $P_0(x_0, y_0, z_0)$. Then

$$\mathbf{N}\cdot\overrightarrow{P_0P_1} = A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0) = Ax_1 + By_1 + Cz_1 + D = \alpha.$$

If $\alpha > 0$, $0 \leq \measuredangle\left(\mathbf{N}, \overrightarrow{P_0P_1}\right) < \pi/2$; if $\alpha < 0$, then $\pi/2 < \measuredangle\left(\mathbf{N}, \overrightarrow{P_0P_1}\right) < \pi$. Since $\mathbf{N}$ is perpendicular to the plane, the result follows.

44. Denote the tips of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ by A, B, C. The determinant equation gives the set of all points $P(x, y, z)$ that lie on the plane determined by A, B, C.

Proof: The equation of that plane,

$$\overrightarrow{CP} \cdot (\overrightarrow{CA} \times \overrightarrow{CB}) = 0$$

can be written

$$(1) \qquad (\overrightarrow{CA} \times \overrightarrow{CB}) \cdot \overrightarrow{CP} = 0.$$

The determinant equation can be written

$$(2) \qquad (\overrightarrow{AP} \times \overrightarrow{BP}) \cdot \overrightarrow{CP} = 0$$

We will show that equations (1) and (2) are equivalent.

From (1) we subtract $(\overrightarrow{CA} \times \overrightarrow{CP}) \cdot \overrightarrow{CP} = 0$ and obtain

$$(\overrightarrow{CA} \times \overrightarrow{BP}) \cdot \overrightarrow{CP} = 0. \qquad (\overrightarrow{CB} - \overrightarrow{CP} = \overrightarrow{BP})$$

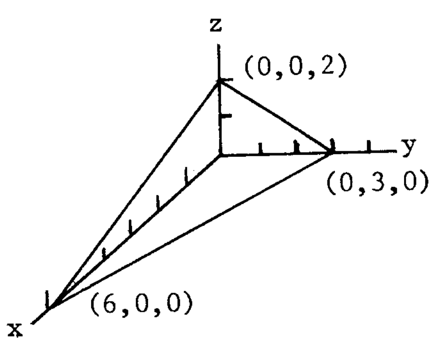
From this we subtract $(\overrightarrow{CP} \times \overrightarrow{BP}) \cdot \overrightarrow{CP} = 0$ and obtain

$$(\overrightarrow{AP} \times \overrightarrow{BP}) \cdot \overrightarrow{CP} = 0. \qquad (\overrightarrow{CA} - \overrightarrow{CP} = \overrightarrow{AP})$$

which is equation (2). We can obtain (1) from (2) by reversing the process.

45. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$

46.



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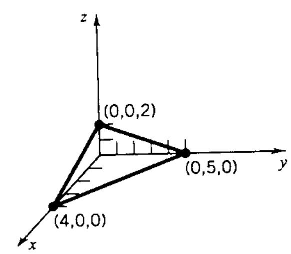

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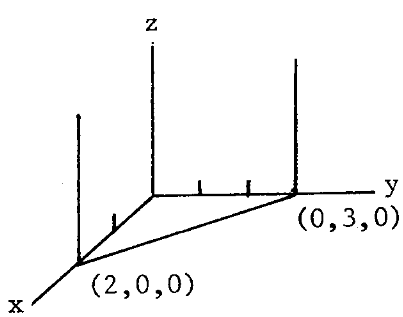

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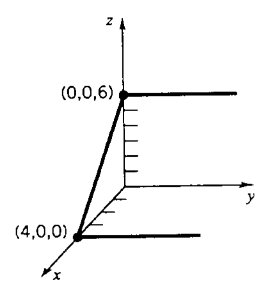


50. $4x + 3y + 6z - 12 = 0$

51. $\dfrac{x}{2} + \dfrac{y}{5} + \dfrac{z}{4} = 1$

$10x + 4y + 5z = 20$

52. $3y + 4z - 12 = 0$

53. $\dfrac{x}{3} + \dfrac{y}{5} = 1$

$5x + 3y = 15$

CHAPTER 13

SECTION 13.1

1. $\mathbf{f}'(t) = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$

2. $\mathbf{f}'(t) = \sin t\mathbf{k}$

3. $\mathbf{f}'(t) = -\dfrac{1}{2\sqrt{1-t}}\,\mathbf{i} + \dfrac{1}{2\sqrt{1+t}}\,\mathbf{j} + \dfrac{1}{(1-t)^2}\,\mathbf{k}$

4. $\mathbf{f}'(t) = e^t\mathbf{i} + \dfrac{1}{t}\mathbf{j} + \dfrac{1}{1+t^2}\mathbf{k}$

5. $\mathbf{f}'(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{j} + \sec^2 t\,\mathbf{k}$

6. $\mathbf{f}'(t) = e^t\left[\mathbf{i} + (1+t)\mathbf{j} + (2t+t^2)\mathbf{k}\right]$

7. $\mathbf{f}'(t) = \dfrac{-1}{1-t}\,\mathbf{i} - \sin t\,\mathbf{j} + 2t\,\mathbf{k}$

8. $\mathbf{f}'(t) = -\dfrac{2}{(t-1)^2}\mathbf{i} + e^{2t}(1+2t)\mathbf{j} + \sec t\tan t\mathbf{k}$

9. $\mathbf{f}'(t) = 4\,\mathbf{i} + 6t^2\mathbf{j} + (2t+2)\mathbf{k};\qquad \mathbf{f}''(t) = 12t\,\mathbf{j} + 2\,\mathbf{k}$

10. $\mathbf{f}'(t) = (\sin t + t\cos t)\mathbf{i} + (\cos t - t\sin t)\mathbf{k}$

$\mathbf{f}''(t) = (2\cos t - t\sin t)\mathbf{i} + (-2\sin t - t\cos t)\mathbf{k}$

11. $\mathbf{f}'(t) = -2\sin 2t\,\mathbf{i} + 2\cos 2t\,\mathbf{j} + 4t\,\mathbf{k};\quad \mathbf{f}''(t) = -4\cos 2t\,\mathbf{i} - 4\sin 2t\,\mathbf{j}$

12. $\mathbf{f}'(t) = \dfrac{1}{2}t^{-1/2}\mathbf{i} + \dfrac{3}{2}t^{1/2}\mathbf{j} + \dfrac{1}{t}\mathbf{k}$

$\mathbf{f}''(t) = -\dfrac{1}{4}t^{-3/2}\mathbf{i} + \dfrac{3}{4}t^{-1/2}\mathbf{j} - \dfrac{1}{t^2}\mathbf{k}$

13. $\displaystyle\int_1^2 (\mathbf{i} + 2t\,\mathbf{j})\,dt = \left[t\,\mathbf{i} + t^2\,\mathbf{j}\right]_1^2 = \mathbf{i} + 3\,\mathbf{j}$

14. $\displaystyle\int_0^{\pi} \mathbf{r}(t)\,dt = \left[-\cos t\mathbf{i} + \sin t\mathbf{j} + \frac{t^2}{2}\mathbf{k}\right]_0^{\pi} = 2\mathbf{i} + \frac{1}{2}\pi^2\mathbf{k}$

15. $\displaystyle\int_0^1 \left(e^t\mathbf{i} + e^{-t}\mathbf{k}\right)dt = \left[e^t\mathbf{i} - e^{-t}\mathbf{k}\right]_0^1 = (e-1)\,\mathbf{i} + \left(1 - \frac{1}{e}\right)\mathbf{k}$

16. $\displaystyle\int_0^1 \mathbf{h}(t)\,dt = \left[-e^{-t}(t^2+2t+2)\mathbf{i} - e^{-t}\sqrt{2}(t+1)\mathbf{j} - e^{-t}\mathbf{k}\right]_0^1 = \frac{1}{e}\left[(2e-5)\mathbf{i} + \sqrt{2}(e-2)\mathbf{j} + (e-1)\mathbf{k}\right]$

17. $\displaystyle\int_0^1 \left(\frac{1}{1+t^2}\,\mathbf{i} + \sec^2 t\,\mathbf{j}\right)dt = \left[\tan^{-1} t\,\mathbf{i} + \tan t\,\mathbf{j}\right]_0^1 = \frac{\pi}{4}\,\mathbf{i} + \tan(1)\,\mathbf{j}$

18. $\displaystyle\int_1^3 \mathbf{F}(t)\,dt = \left[\ln t\mathbf{i} + \frac{1}{2}(\ln t)^2\mathbf{j} - \frac{1}{2}e^{-2t}\mathbf{k}\right]_1^3 = \ln 3\mathbf{i} + \frac{1}{2}(\ln 3)^2\mathbf{j} + \frac{1}{2}(e^{-2} - e^{-6})\mathbf{k}$

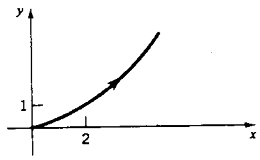
19. $\displaystyle\lim_{t\to 0}\mathbf{f}(t) = \left(\lim_{t\to 0}\frac{\sin t}{2t}\right)\mathbf{i} + \left(\lim_{t\to 0} e^{2t}\right)\mathbf{j} + \left(\lim_{t\to 0}\frac{t^2}{e^t}\right)\mathbf{k} = \frac{1}{2}\,\mathbf{i} + \mathbf{j}$

20. Does not exist (because of $\dfrac{t}{|t|}\mathbf{k}$)

21. $\displaystyle\lim_{t\to 0}\mathbf{f}(t) = \left(\lim_{t\to 0} t^2\right)\mathbf{i} + \left(\lim_{t\to 0}\frac{1-\cos t}{3t}\right)\mathbf{j} + \left(\lim_{t\to 0}\frac{t}{t+1}\right)\mathbf{k} = 0\,\mathbf{i} + \frac{1}{3}\left(\lim_{t\to 0}\frac{1-\cos t}{t}\right)\mathbf{j} + 0\,\mathbf{k} = \mathbf{0}$

22. $\displaystyle\lim_{t\to 0}\mathbf{f}(t) = \mathbf{j} + \mathbf{k}$

23.



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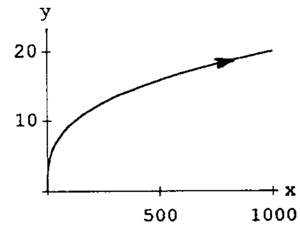

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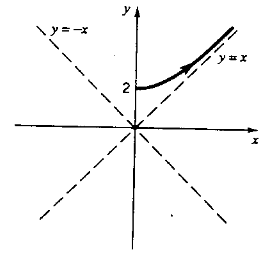

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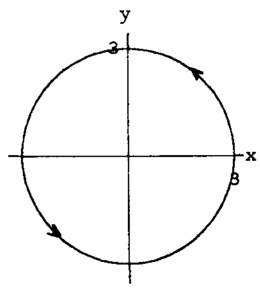

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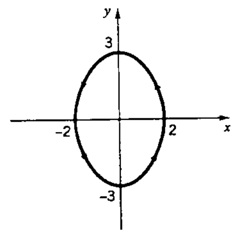

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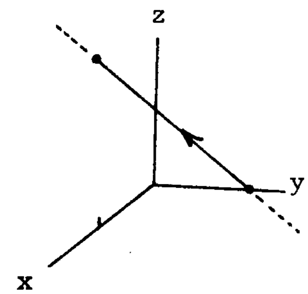

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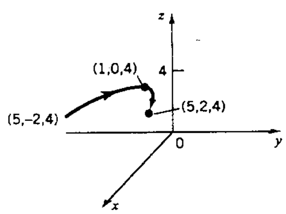

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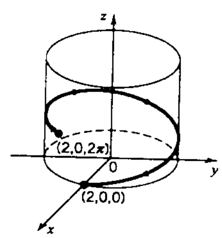

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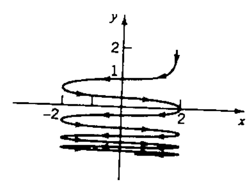

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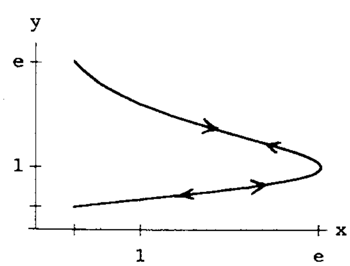

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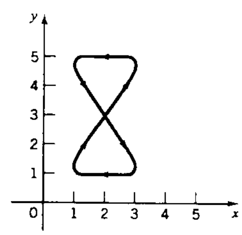
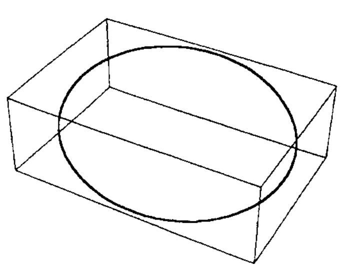
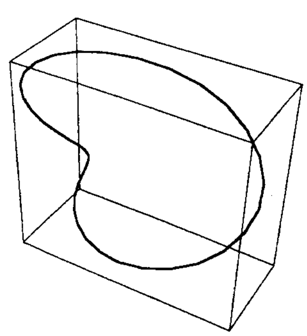


34. (a)



(b)



35. (a) $\mathbf{f}(t) = 3\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$ (b) $\mathbf{f}(t) = 3\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j}$

36. (a) $\mathbf{f}(t) = (1+\cos t)\mathbf{i} + \sin t\mathbf{j}$ (b) $\mathbf{f}(t) = (1+\cos t)\mathbf{i} - \sin t\mathbf{j}$

37. (a) $\mathbf{f}(t) = t\mathbf{i} + t^2\mathbf{j}$ (b) $\mathbf{f}(t) = -t\mathbf{i} + t^2\mathbf{j}$

38. (a) $\mathbf{f}(t) = t\mathbf{i} + t^3\mathbf{j}$ (b) $\mathbf{f}(t) = -t\mathbf{i} - t^3\mathbf{j}$

39. $\mathbf{f}(t) = (1+2t)\,\mathbf{i} + (4+5t)\,\mathbf{j} + (-2+8t)\,\mathbf{k}, \quad 0 \le t \le 1$

40. $\mathbf{f}(t) = (3+4t)\mathbf{i} + 2\mathbf{j} + (-5+14t)\mathbf{k}, \quad 0 \le t \le 1$

41. $\mathbf{f}'(t_0) = \mathbf{i} + m\mathbf{j}$,

$$\int_a^b \mathbf{f}(t)\,dt = \left[\frac{1}{2}t^2\mathbf{i}\right]_a^b + \left[\int_a^b f(t)\,dt\right]\mathbf{j} = \frac{1}{2}\left(b^2 - a^2\right)\mathbf{i} + A\mathbf{j},$$

$$\int_a^b \mathbf{f}'(t)\,dt = [t\mathbf{i} + f(t)\,\mathbf{j}]_a^b = (b-a)\,\mathbf{i} + (d-c)\,\mathbf{j}$$

42. $\mathbf{f}(t) = (\frac{1}{2}t^2 + 1)\mathbf{i} + (\sqrt{1+t^2} + 1)\mathbf{j} + (te^t - e^t + 4)\mathbf{k}$

47.

$$\mathbf{f}'(t) = \mathbf{i} + t^2\mathbf{j}$$

$$\mathbf{f}(t) = (t + C_1)\,\mathbf{i} + \left(\tfrac{1}{3}t^3 + C_2\right)\mathbf{j} + C_3\mathbf{k}$$

$$\mathbf{f}(0) = \mathbf{j} - \mathbf{k} \quad\Longrightarrow\quad C_1 = 0, \quad C_2 = 1, \quad C_3 = -1$$

$$\mathbf{f}(t) = t\,\mathbf{i} + \left(\tfrac{1}{3}\,t^3 + 1\right)\mathbf{j} - \mathbf{k}$$

44. $\mathbf{f}(t) = e^{2t}\mathbf{i} - e^{2t}\mathbf{k} = e^{2t}(\mathbf{i} - \mathbf{k})$

45. $\mathbf{f}'(t) = \alpha\,\mathbf{f}(t) \quad\Longrightarrow\quad \mathbf{f}(t) = e^{\alpha t}\,\mathbf{f}(0) = e^{\alpha t}\mathbf{c}$

46. For each $\epsilon > 0$ there exists $\delta > 0$ such that

if $0 < |t - t_0| < \delta$, then $||\mathbf{f}(t) - \mathbf{L}|| < \epsilon$.

47. (a) If $\mathbf{f}'(t) = \mathbf{0}$ on an interval, then the derivative of each component is 0 on that interval, each component is constant on that interval, and therefore $\mathbf{f}$ itself is constant on that interval.

(b) Set $\mathbf{h}(t) = \mathbf{f}(t) - \mathbf{g}(t)$ and apply part (a).

48.

$$\begin{aligned} ||[\mathbf{f}(t)\cdot\mathbf{g}(t)] - [\mathbf{L}\cdot\mathbf{M}]|| &= ||[\mathbf{f}(t)\cdot\mathbf{g}(t)] - [\mathbf{L}\cdot\mathbf{g}(t)] + [\mathbf{L}\cdot\mathbf{g}(t)] - [\mathbf{L}\cdot\mathbf{M}]|| \\ &= ||[(\mathbf{f}(t) - \mathbf{L})\cdot\mathbf{g}(t)] + [\mathbf{L}\cdot(\mathbf{g}(t) - \mathbf{M})]|| \\ &\le ||(\mathbf{f}(t) - \mathbf{L})\cdot\mathbf{g}(t)|| + ||\mathbf{L}\cdot(\mathbf{g}(t) - \mathbf{M})|| \\ &\le ||\mathbf{f}(t) - \mathbf{L}||\,||\mathbf{g}(t)|| + ||\mathbf{L}||\,||\mathbf{g}(t) - \mathbf{M}|| \end{aligned}$$

by Schwarz's inequality

As $t \to t_0$, the right side tends to $(0)||\mathbf{M}|| + ||\mathbf{L}||(0) = 0$.

49. If $\mathbf{f}$ is differentiable at t, then each component is differentiable at t, each component is continuous at t, and therefore $\mathbf{f}$ is continuous at t.

50. Set $\mathbf{f}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$ and apply the fundamental Theorem of Calculus to f_1, f_2, f_3.

51. no; as a counter-example, set $\mathbf{f}(t) = \mathbf{i} = \mathbf{g}(t)$.

52.
$$\begin{aligned}\int_a^b [c \cdot \mathbf{f}(t)]\,dt &= \int_a^b [c_1 f_1(t) + c_2 f_2(t) + c_3 f_3(t)]\,dt \\ &= c_1 \int_a^b f_1(t)\,dt + c_2 \int_a^b f_2(t)\,dt + c_3 \int_a^b f_3(t)\,dt \\ &= \mathbf{c} \cdot \int_a^b \mathbf{f}(t)\,dt\end{aligned}$$

$\int_a^b [\mathbf{c} \times \mathbf{f}(t)]\,dt$ can be written

$$\int_a^b \{[c_2 f_3(t) - c_3 f_2(t)]\mathbf{i} - [c_1 f_3(t) - c_3 f_1(t)]\mathbf{j} + [c_1 f_2(t) - c_2 f_1(t)]\mathbf{k}\}\,dt.$$

This gives

$$\left[c_2 \int_a^b f_3(t)\,dt - c_3 \int_a^b f_2(t)\,dt\right]\mathbf{i} - \left[c_1 \int_a^b f_3(t)\,dt - c_3 \int_a^b f_1(t)\,dt\right]\mathbf{j} + \left[c_1 \int_a^b f_2(t)\,dt - c_2 \int_a^b f_1(t)\,dt\right]\mathbf{k}$$

which is $\mathbf{c} \times \int_a^b \mathbf{f}(t)\,dt$

53. Suppose $\mathbf{f}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$. Then $\|\mathbf{f}(t)\| = \sqrt{f_1^2(t) + f_2^2(t) + f_3^2(t)}$ and

$$\frac{d}{dt}(\|\mathbf{f}\|) = \frac{1}{2}\left[f_1^2 + f_2^2 + f_3^2\right]^{-1/2}(2f_1 \cdot f_1' + 2f_2 \cdot f_2' + 2f_3 \cdot f_3') = \frac{\mathbf{f}(t) \cdot \mathbf{f}'(t)}{\|\mathbf{f}(t)\|}$$

54.
$$\begin{aligned}\frac{d}{dt}\left(\frac{\mathbf{f}(t)}{\|\mathbf{f}(t)\|}\right) &= \frac{d}{dt}\left(\frac{1}{\|\mathbf{f}(t)\|}\mathbf{f}(t)\right) = \frac{1}{\|\mathbf{f}(t)\|}\mathbf{f}'(t) + \mathbf{f}(t) \cdot \frac{-1}{\|\mathbf{f}(t)\|^2} \cdot \frac{d}{dt}[\|\mathbf{f}(t)\|] \\ &= \frac{\mathbf{f}'(t)}{\|\mathbf{f}(t)\|} - \frac{\mathbf{f}(t) \cdot \mathbf{f}'(t)}{\|\mathbf{f}(t)\|^3}\mathbf{f}(t), \quad \text{using the result of Exercise 53.}\end{aligned}$$

SECTION 13.2

1. $\mathbf{f}'(t) = \mathbf{b}, \quad \mathbf{f}''(t) = \mathbf{0}$

2. $\mathbf{f}'(t) = \mathbf{b} + 2t\,\mathbf{c}, \quad \mathbf{f}''(t) = 2\,\mathbf{c}$

3. $\mathbf{f}'(t) = 2e^{2t}\,\mathbf{i} - \cos t\,\mathbf{j}, \quad \mathbf{f}''(t) = 4e^{2t}\,\mathbf{i} + \sin t\,\mathbf{j}$

4. $\mathbf{f}(t) = 2t^2\mathbf{i} \implies \mathbf{f}'(t) = 4t\mathbf{i}, \quad \mathbf{f}''(t) = 4\mathbf{i}$

5. $\mathbf{f}'(t) = [(t^2\mathbf{i} - 2t\,\mathbf{j}) \cdot (\mathbf{i} + 3t^2\mathbf{j}) + (2t\,\mathbf{i} - 2\,\mathbf{j}) \cdot (t\,\mathbf{i} + t^3\mathbf{j}]\,\mathbf{j} = [3t^2 - 8t^3]\,\mathbf{j}$

$\mathbf{f}''(t) = (6t - 24t^2)\,\mathbf{j}$

6. $\mathbf{f}(t) = (t - t^5)\mathbf{k} \implies \mathbf{f}'(t) = (1 - 5t^4)\mathbf{k}, \quad \mathbf{f}''(t) = -20t^3\mathbf{k}$

7.
$$\begin{aligned}\mathbf{f}'(t) &= \left[\left(e^t\mathbf{i} + t\mathbf{k}\right) \times \frac{d}{dt}\left(t\mathbf{j} + e^{-t}\mathbf{k}\right)\right] + \left[\frac{d}{dt}\left(e^t\mathbf{i} + t\mathbf{k}\right) \times \left(t\mathbf{j} + e^{-t}\mathbf{k}\right)\right] \\ &= \left[\left(e^t\mathbf{i} + t\mathbf{k}\right) \times \left(\mathbf{j} - e^{-t}\mathbf{k}\right)\right] + \left[\left(e^t\mathbf{i} + \mathbf{k}\right) \times \left(t\mathbf{j} + e^{-t}\mathbf{k}\right)\right] \\ &= \left(-t\mathbf{i} + \mathbf{j} + e^t\mathbf{k}\right) + \left(-t\mathbf{i} - \mathbf{j} + te^t\mathbf{k}\right) \\ &= -2t\mathbf{i} + e^t(t+1)\mathbf{k} \\ \mathbf{f}''(t) &= -2\mathbf{i} + e^t(t+2)\mathbf{k}\end{aligned}$$

8. $\mathbf{f}'(t) = (\mathbf{i} - 2t\mathbf{j}) \times (\mathbf{i} + t^3\mathbf{j} + 5t\mathbf{k}) + (t\mathbf{i} - t^2\mathbf{j} + \mathbf{k}) \times (3t^2\mathbf{j} + 5\mathbf{k})$

$\mathbf{f}''(t) = -2\mathbf{j} \times (\mathbf{i} + t^3\mathbf{j} + 5\mathbf{k}) + 2(\mathbf{i} - 2t\mathbf{j}) \times (3t^2\mathbf{j} + 5\mathbf{k}) + (t\mathbf{i} - t^2\mathbf{j} + \mathbf{k}) \times 6t\mathbf{j}$

9. $\mathbf{f}'(t) = (\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \mathbf{k}) \times (2\sin 2t\,\mathbf{i} - 2\cos 2t\,\mathbf{j} + \mathbf{k}) + (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j}) \times (\sin 2t\,\mathbf{i} + \cos 2t\,\mathbf{j} + t\,\mathbf{k})$

$= (\sin t + t\cos t + 2\sin 2t)\mathbf{i} + (2\cos 2t - \cos t + t\sin t)\mathbf{j} - 3\sin 3t\,\mathbf{k}$

$\mathbf{f}''(t) = (2\cos t - t\sin t + 4\cos t)\mathbf{i} + (-4\sin 2t + 2\sin t + t\cos t)\mathbf{j} - 9\cos 3t\,\mathbf{k}$

10. $\mathbf{f}'(t) = \mathbf{g}(t^2) + t\mathbf{g}'(t^2)2t = \mathbf{g}(t^2) + 2t^2\mathbf{g}'(t^2)$

$\mathbf{f}''(t) = \mathbf{g}'(t^2)2t + 4t\mathbf{g}'(t^2) + 2t^2\mathbf{g}''(t^2)2t = 6t\mathbf{g}'(t^2) + 4t^3\mathbf{g}''(t^2)$

11. $\mathbf{f}'(t) = \frac{1}{2}\sqrt{t}\,\mathbf{g}'\left(\sqrt{t}\right) + \mathbf{g}\left(\sqrt{t}\right), \quad \mathbf{f}''(t) = \frac{1}{4}\mathbf{g}''\left(\sqrt{t}\right) + \frac{3}{4\sqrt{t}}\mathbf{g}'\left(\sqrt{t}\right)$

12. $\mathbf{f}(t) = 2e^{-2t}\mathbf{i} - 2\mathbf{k} \implies \mathbf{f}'(t) = -4e^{-2t}\mathbf{i}, \quad \mathbf{f}''(t) = 8e^{-2t}\mathbf{i}.$

13. $-\sin t\, e^{\cos t}\,\mathbf{i} + \cos t\, e^{\sin t}\,\mathbf{j}$

14.
$$\begin{aligned}\frac{d^2}{dt^2}[e^t\cos t\mathbf{i} + e^t\sin t\mathbf{j}] &= \frac{d}{dt}[e^t(\cos t - \sin t)\mathbf{j} + e^t(\sin t + \cos t)\mathbf{j}] \\ &= -2e^t\sin t\mathbf{i} + 2e^t\cos t\mathbf{j}\end{aligned}$$

15. $(e^t\mathbf{i} + e^{-t}\mathbf{j}) \cdot (e^t\mathbf{i} - e^{-t}\mathbf{j}) = e^{2t} - e^{-2t};$ therefore

$$\frac{d^2}{dt^2}\left[(e^t\mathbf{i} + e^{-t}\mathbf{j}) \cdot (e^t\mathbf{i} - e^{-t}\mathbf{j})\right] = \frac{d^2}{dt^2}\left[e^{2t} - e^{-2t}\right] = \frac{d}{dt}\left[2e^{2t} + 2e^{-2t}\right] = 4e^{2t} - 4e^{-2t}$$

16. $\left(\frac{1}{t}\mathbf{i} + \mathbf{j} - 2t\mathbf{k}\right) \times \left(\frac{1}{t}\mathbf{i} + t^2\mathbf{j} - t\mathbf{k}\right) + [\ln t\mathbf{i} + t\mathbf{j} - (t^2+1)\mathbf{k}] \times \left(-\frac{1}{t^2}\mathbf{i} + 2t\mathbf{j} - \mathbf{k}\right)$

17. $\frac{d}{dt}[(a+t\mathbf{b})\times(c+t\mathbf{d})]=[(\mathbf{a}+t\mathbf{b})\times\mathbf{d}]+[\mathbf{b}\times(\mathbf{c}+t\mathbf{d})]=(\mathbf{a}\times\mathbf{d})+(\mathbf{b}\times\mathbf{c})+2t(\mathbf{b}\times\mathbf{d})$

18. $\mathbf{b}\times(\mathbf{a}+t\mathbf{b}+t^2\mathbf{c})+(\mathbf{a}+t\mathbf{b})\times(\mathbf{b}+2t\mathbf{c})=2t(\mathbf{a}\times\mathbf{c})+3t^2(\mathbf{b}\times\mathbf{c})$.

19. $\frac{d}{dt}[(\mathbf{a}+t\mathbf{b})\cdot(\mathbf{c}+t\mathbf{d})]=[(\mathbf{a}+t\mathbf{b})\cdot\mathbf{d}]+[\mathbf{b}\cdot(\mathbf{c}+t\mathbf{d})]=(\mathbf{a}\cdot\mathbf{d})+(\mathbf{b}\cdot\mathbf{c})+2t(\mathbf{b}\cdot\mathbf{d})$

20. $\mathbf{b}\cdot(\mathbf{a}+t\mathbf{b}+t^2\mathbf{c})+(\mathbf{a}+t\mathbf{b})\cdot(\mathbf{b}+2t\mathbf{c})=2(\mathbf{a}\cdot\mathbf{b})+2t(\mathbf{a}\cdot\mathbf{c})+2t\,\|\mathbf{b}\|^2+3t^2(\mathbf{b}\cdot\mathbf{c})$

21. $\mathbf{r}(t)=\mathbf{a}+t\mathbf{b}$

22. $\mathbf{r}(t)=\mathbf{a}+t\mathbf{b}+\frac{1}{2}t^2\mathbf{c}$

23. $\mathbf{r}(t)=\frac{1}{2}t^2\mathbf{a}+\frac{1}{6}t^3\mathbf{b}+t\mathbf{c}+\mathbf{d}$

24. $\mathbf{r}(t)=(1+2t-\frac{1}{4}\cos 2t)\mathbf{i}+(1-\frac{1}{4}\sin 2t)\mathbf{j}$

25. $\mathbf{r}(t)=\sin t\,\mathbf{i}+\cos t\,\mathbf{j}$, $\quad\mathbf{r}'(t)=\cos t\,\mathbf{i}-\sin t\,\mathbf{j}$, $\quad\mathbf{r}''(t)=-\sin t\,\mathbf{i}-\cos t\,\mathbf{j}=-\mathbf{r}(t)$.
Thus $\mathbf{r}(t)$ and $\mathbf{r}''(t)$ are parallel, and they always point in opposite directions.

26. $\mathbf{r}''(t)=k^2e^{kt}\mathbf{i}+k^2e^{-kt}\mathbf{j}=k^2\mathbf{r}(t)$, so $\mathbf{r}''(t)$ and $\mathbf{r}(t)$ are parallel.

27.
$$\begin{aligned}\mathbf{r}(t)\cdot\mathbf{r}'(t)&=(\cos t\,\mathbf{i}+\sin t\,\mathbf{j})\cdot(-\sin t\,\mathbf{i}+\cos t\,\mathbf{j})=0\\ \mathbf{r}(t)\times\mathbf{r}'(t)&=(\cos t\,\mathbf{i}+\sin t\,\mathbf{j})\times(-\sin t\,\mathbf{i}+\cos t\,\mathbf{j})\\ &=\cos^2 t\,\mathbf{k}+\sin^2 t\,\mathbf{k}=(\cos^2 t+\sin^2 t)\,\mathbf{k}=\mathbf{k}\end{aligned}$$

28.
$$\begin{aligned}(\mathbf{g}\times\mathbf{f})'(t)&=[\mathbf{g}(t)\times\mathbf{f}'(t)]+[\mathbf{g}'(t)\times\mathbf{f}(t)]\\ &=-[\mathbf{f}'(t)\times\mathbf{g}(t)]-[\mathbf{f}(t)\times\mathbf{g}'(t)]\\ &=-\{[\mathbf{f}(t)\times\mathbf{g}'(t)]+[\mathbf{f}'(t)\times\mathbf{g}(t)]\}=-(\mathbf{f}\times\mathbf{g})'(t)\end{aligned}$$

29. $\frac{d}{dt}[\mathbf{f}(t)\times\mathbf{f}'(t)]=[\mathbf{f}(t)\times\mathbf{f}''(t)]+\underbrace{[\mathbf{f}'(t)\times\mathbf{f}'(t)]}_{\mathbf{0}}=\mathbf{f}(t)\times\mathbf{f}''(t)$.

30.
$$\begin{aligned}\frac{d}{dt}[u_1(t)\mathbf{r}_1(t)\times u_2(t)\mathbf{r}_2(t)]&=\frac{d}{dt}[(u_1(t)u_2(t))(\mathbf{r}_1(t)\times\mathbf{r}_2(t))]\\ &=u_1(t)u_2(t)\frac{d}{dt}[\mathbf{r}_1(t)\times\mathbf{r}_2(t)]+[\mathbf{r}_1(t)\times\mathbf{r}_2(t)]\frac{d}{dt}[u_1(t)u_2(t)]\end{aligned}$$

31. $[\mathbf{f}\cdot\mathbf{g}\times\mathbf{h}]'=\mathbf{f}'\cdot(\mathbf{g}\times\mathbf{h})+\mathbf{f}\cdot(\mathbf{g}\times\mathbf{h})'=\mathbf{f}'\cdot(\mathbf{g}\times\mathbf{h})+\mathbf{f}\cdot[\mathbf{g}'\times\mathbf{h}+\mathbf{g}\times\mathbf{h}']$
and the result follows.

32. $\frac{d}{dt}(\mathbf{f}\times\mathbf{f}')=\mathbf{f}(t)\times\mathbf{f}''(t)$ by Exercise 29. If $\mathbf{f}(t)$ and $\mathbf{f}''(t)$ are parallel, their cross product is zero, so $\frac{d}{dt}(\mathbf{f}\times\mathbf{f}')=\mathbf{0}$, hence $\mathbf{f}\times\mathbf{f}'$ is constant.

33.
$$\begin{aligned}\|\mathbf{r}(t)\|\text{ is constant}\quad&\iff\quad\|\mathbf{r}(t)\|^2=\mathbf{r}(t)\cdot\mathbf{r}(t)\text{ is constant}\\ &\iff\quad\frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{r}(t)]=2[\mathbf{r}(t)\cdot\mathbf{r}'(t)]=0\text{ identically}\\ &\iff\quad\mathbf{r}(t)\cdot\mathbf{r}'(t)=0\text{ identically}\end{aligned}$$

34. (a) Routine

(b) Write
$$\frac{[\mathbf{f}(t+h)\cdot\mathbf{g}(t+h)]-[\mathbf{f}(t)\cdot\mathbf{g}(t)]}{h}$$
so
$$\left(\mathbf{f}(t+h)\cdot\left[\frac{\mathbf{g}(t+h)-\mathbf{g}(t)}{h}\right]\right)+\left(\left[\frac{\mathbf{f}(t+h)-\mathbf{f}(t)}{h}\right]\cdot\mathbf{g}(t)\right)$$
and take the limit as $h\to 0$. (Appeal to Theorem 13.1.3)

35. Write
$$\frac{[\mathbf{f}(t+h)\times\mathbf{g}(t+h)]-[\mathbf{f}(t)\times\mathbf{g}(t)]}{h}$$
as
$$\left(\mathbf{f}(t+h)\times\left[\frac{\mathbf{g}(t+h)-\mathbf{g}(t)}{h}\right]\right)+\left(\left[\frac{\mathbf{f}(t+h)-\mathbf{f}(t)}{h}\right]\times\mathbf{g}(t)\right)$$
and take the limit as $h\to 0$. (Appeal to Theorem 13.1.3.)

36. (a) and (b) can derived routinely by using components. An ϵ,δ derivation of (a) is also simple:

Let $\epsilon>0$. Since $\mathbf{f}$ is continuous at $u(t_0)$, there exists $\delta_1>0$ such that
$$\text{if}\quad |x-u(t_0)|<\delta_1,\quad\text{then}\quad \|\,\mathbf{f}(x)-\mathbf{f}(u(x_0))\,\|<\epsilon.$$
Since u is continuous at t_0, there exists $\delta>0$ such that
$$\text{if}\quad |t-t_0|<\delta,\quad\text{then}\quad |u(t)-u(t_0)|<\epsilon.$$
Thus
$$|t-t_0|<\epsilon \implies |u(t)-u(t_0)|<\delta_1 \implies \|\,\mathbf{f}(u(t))-\mathbf{f}(u(t_0))\,\|<\epsilon.$$

SECTION 13.3

1. $\mathbf{r}'(t)=-\pi\sin\pi t\,\mathbf{i}+\pi\cos\pi t\,\mathbf{j}+\mathbf{k},\quad \mathbf{r}'(2)=\pi\,\mathbf{j}+\mathbf{k}$

$\mathbf{R}(u)=(\mathbf{i}+2\mathbf{k})+u(\pi\,\mathbf{j}+\mathbf{k})$

2. $\mathbf{r}'(t)=e^t\mathbf{i}-e^{-t}\mathbf{j}-\frac{1}{t}\mathbf{k},\quad \mathbf{r}'(1)=e\mathbf{i}-e^{-1}\mathbf{j}-\mathbf{k};\quad \mathbf{R}(u)=(e\mathbf{i}+e^{-1}\mathbf{j})+u(e\mathbf{i}-e^{-1}\mathbf{j}-\mathbf{k})$

3. $\mathbf{r}'(t)=\mathbf{b}+2t\mathbf{c},\quad \mathbf{r}'(-1)=\mathbf{b}-2\mathbf{c},\quad \mathbf{R}(u)=(\mathbf{a}-\mathbf{b}+\mathbf{c})+u(\mathbf{b}-2\mathbf{c})$

4. $\mathbf{r}'(0)=\mathbf{i};\quad \mathbf{R}(u)=(\mathbf{i}+\mathbf{j}+\mathbf{k})+u\mathbf{i}.$

5. $\mathbf{r}'(t)=4t\mathbf{i}-\mathbf{j}+4t\mathbf{k},\quad P$ is tip of $\mathbf{r}(1),\quad \mathbf{r}'(1)=4\mathbf{i}-\mathbf{j}+4\mathbf{k}$

$\mathbf{R}(u)=(2\mathbf{i}+5\mathbf{k})+u\,(4\mathbf{i}-\mathbf{j}+4\mathbf{k})$

6. $\mathbf{r}'(2)=3\mathbf{a}-4\mathbf{c};\quad \mathbf{R}(u)=(6\mathbf{a}+\mathbf{b}-4\mathbf{c})+u(3\mathbf{a}-4\mathbf{c})$

7. $\mathbf{r}'(t)=-2\sin t\,\mathbf{i}+3\cos t\,\mathbf{j}+\mathbf{k},\quad \mathbf{r}'(\pi/4)=-\sqrt{2}\,\mathbf{i}+\frac{3}{2}\sqrt{2}\,\mathbf{j}+\mathbf{k}$

$\mathbf{R}(u)=\left(\sqrt{2}\,\mathbf{i}+\frac{3}{2}\sqrt{2}\,\mathbf{j}+\frac{\pi}{4}\mathbf{k}\right)+u\left(-\sqrt{2}\,\mathbf{i}+\frac{3}{2}\sqrt{2}\,\mathbf{j}+\mathbf{k}\right)$

8. $\mathbf{r}'(t) = (\sin t + t\cos t)\mathbf{i} + (\cos t - t\sin t)\mathbf{j} + 2\mathbf{k}$

$\mathbf{r}'\left(\frac{\pi}{2}\right) = \mathbf{i} - \frac{\pi}{2}\mathbf{j} + 2\mathbf{k};\quad \mathbf{R}(u) = \left(\frac{\pi}{2}\mathbf{i} + \pi\mathbf{k}\right) + u\left(\mathbf{i} - \frac{\pi}{2}\mathbf{j} + 2\mathbf{k}\right)$

9. The scalar components $x(t) = at$ and $y(t) = bt^2$ satisfy the equation

$$a^2y(t) = a^2(bt^2) = b\,(a^2t^2) = b\,[x(t)]^2$$

and generate the parabola $a^2y = bx^2$.

10. $x(t)^2 - y(t)^2 = \dfrac{a^2}{4}\left[(e^{wt} + e^{-wt})^2 - (e^{wt} - e^{-wt})^2\right] = a^2$, with $x(t) > 0$; the right branch of the hyperbola $x^2 - y^2 = a^2$.

11. $$\mathbf{r}\,(t) = t\mathbf{i} + (1 + t^2)\,\mathbf{j}, \qquad \mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$$

(a) $$\mathbf{r}\,(t) \perp \mathbf{r}'(t) \implies \mathbf{r}\,(t)\cdot\mathbf{r}'(t) = \left[t\mathbf{i} + (1 + t^2)\mathbf{j}\right]\cdot(\mathbf{i} + 2t\mathbf{j})$$

$$= t(2t^2 + 3) = 0 \implies t = 0$$

$\mathbf{r}\,(t)$ and $\mathbf{r}'(t)$ are perpendicular at $(0, 1)$.

(b) and (c) $\mathbf{r}\,(t) = \alpha\,\mathbf{r}'(t)$ with $\alpha \neq 0 \implies t = \alpha$ and $1 + t^2 = 2t\alpha \implies t = \pm 1$.

If $\alpha > 0$, then $t = 1$. $\mathbf{r}\,(t)$ and $\mathbf{r}'(t)$ have the same direction at $(1, 2)$.

If $\alpha < 0$, then $t = -1$. $\mathbf{r}\,(t)$ and $\mathbf{r}'(t)$ have opposite directions at $(-1, 2)$.

12. $\mathbf{r}(t) = e^{\alpha t}(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

13. The tangent line at $t = t_0$ has the form $\mathbf{R}\,(u) = \mathbf{r}\,(t_0) + u\,\mathbf{r}'(t_0)$. If $\mathbf{r}'(t_0) = \alpha\,\mathbf{r}\,(t_0)$, then

$$\mathbf{R}\,(u) = \mathbf{r}\,(t_0) + u\,\alpha\mathbf{r}\,(t_0) = (1 + u\alpha)\,\mathbf{r}\,(t_0).$$

The tangent line passes through the origin at $u = -1/\alpha$.

14. $\mathbf{r}_1'(0) = \mathbf{i}, \quad \mathbf{r}_2'(0) = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$

$\theta = \cos^{-1}\dfrac{\mathbf{r}_1'(0)\cdot\mathbf{r}_2'(0)}{\|\,\mathbf{r}_1(0)\,\|\,\|\,\mathbf{r}_2(0)\,\|} = \cos^{-1}\dfrac{2}{\sqrt{6}} \cong 0.62$ radian, or $35.3°$

15. $\mathbf{r}_1(t)$ passes through $P\,(0,0,0)$ at $t = 0$; $\quad\mathbf{r}_2(u)$ passes through $P\,(0,0,0)$ at $u = -1$.

$\mathbf{r}_1'(t) = e^t\,\mathbf{i} + 2\,\cos t\,\mathbf{j} + \dfrac{1}{t+1}\mathbf{k};\quad \mathbf{r}_1'(0) = \mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$

$\mathbf{r}_2'(u) = \mathbf{i} + 2u\,\mathbf{j} + 3u^2\,\mathbf{k};\quad \mathbf{r}_2'(-1) = \mathbf{i} - 2\,\mathbf{j} + 3\,\mathbf{k}$

$\cos\theta = \dfrac{\mathbf{r}_1'(0)\cdot\mathbf{r}_2'(1)}{\|\mathbf{r}_1'(0)\|\,\|\mathbf{r}_2'(1)\|} = 0;\quad \theta = \dfrac{\pi}{2} \cong 1.57,$ or $90°$.

16. $\mathbf{r}_1'(0) = -1, \quad \mathbf{r}_2'(-1) = \mathbf{i} - 4\mathbf{j} - 8\mathbf{k}$

$\theta = \cos^{-1}\dfrac{\mathbf{r}_1'(0)\cdot\mathbf{r}_2'(-1)}{\|\,\mathbf{r}_1'(0)\,\|\,\|\,\mathbf{r}_2'(-1)\,\|} = \cos^{-1}\left(\dfrac{-1}{9}\right) \cong 1.68$ radians, or $96.4°$

17. $\mathbf{r}_1(t) = \mathbf{r}_2(u)$ implies

$$\left\{\begin{array}{r} e^t = u \\ 2\sin\left(t + \frac{1}{2}\pi\right) = 2 \\ t^2 - 2 = u^2 - 3 \end{array}\right\} \quad \text{so that} \quad t = 0, \quad u = 1.$$

The point of intersection is $(1, 2, -2)$.

$$\mathbf{r}_1'(t) = e^t\mathbf{i} + 2\cos\left(t + \frac{\pi}{2}\right)\mathbf{j} + 2t\mathbf{k}, \quad \mathbf{r}_1'(0) = \mathbf{i}$$

$$\mathbf{r}_2'(u) = \mathbf{i} + 2u\mathbf{k}, \quad \mathbf{r}_2'(1) = \mathbf{i} + 2\mathbf{k}$$

$$\cos\theta = \frac{\mathbf{r}_1'(0) \cdot \mathbf{r}_2'(1)}{\|\mathbf{r}_1'(0)\|\,\|\mathbf{r}_2'(1)\|} = \frac{1}{5}\sqrt{5} \cong 0.447, \quad \theta \cong 1.11 \text{ radians}$$

18. (a) $\mathbf{R}(u) = \mathbf{r}(t_0) + u\mathbf{r}'(t_0) = (t_0\mathbf{i} + f(t_0)\mathbf{j}) + u(\mathbf{i} + f'(t_0)\mathbf{j})$

$\Longrightarrow x(u) = t_0 + u, \quad y(u) = f(t_0) + uf'(t_0)$

(b) From (a), we get $u = x(u) - t_0$ and $y(u) - f(t_0) = f'(t_0)u.$ so

$y - f(t_0) = f'(t_0)(x - t_0),$ as expected.

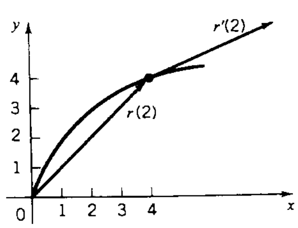
19. (a) $\mathbf{r}(t) = a\cos t\,\mathbf{i} + b\sin t\,\mathbf{j}$ (b) $\mathbf{r}(t) = a\cos t\,\mathbf{i} - b\sin t\,\mathbf{j}$

(c) $\mathbf{r}(t) = a\cos 2t\,\mathbf{i} + b\sin 2t\,\mathbf{j}$ (d) $\mathbf{r}(t) = a\cos 3t\,\mathbf{i} - b\sin 3t\,\mathbf{j}$

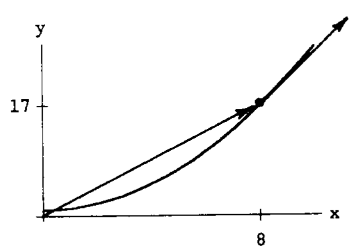
20. (a) $\mathbf{r}(t) = -a\sin t\mathbf{i} + b\cos t\mathbf{j}$ (b) $\mathbf{r}(t) = a\sin t\mathbf{i} + b\cos t\mathbf{j}$

(c) $\mathbf{r}(t) = -a\sin 2t\mathbf{i} + b\cos 2t\mathbf{j}$ (d) $\mathbf{r}(t) = a\sin 3t\mathbf{i} + b\cos 3t\mathbf{j}$

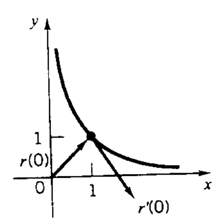
21. $\mathbf{r}'(t) = t^3\,\mathbf{i} + 2t\,\mathbf{j}$



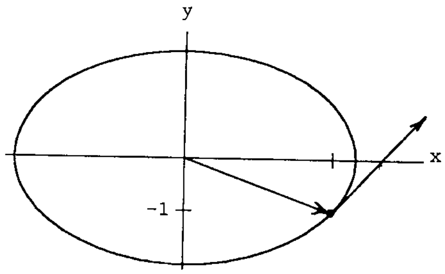
22. $\mathbf{r}'(t) = 2\mathbf{i} + 2t\mathbf{j}$

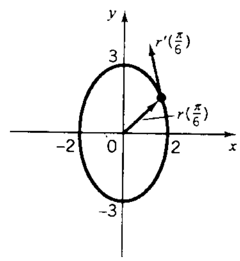


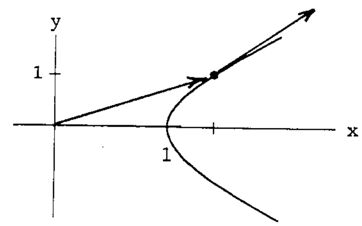
23. $\mathbf{r}'(t) = 2e^{2t}\,\mathbf{i} - 4e^{-4t}\,\mathbf{j}$



24. $\mathbf{r}'(t) = \cos t\mathbf{i} + 2\sin t\mathbf{j}$ **25.** $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j}$ **26.** $\mathbf{r}'(t) = \sec t\tan t\mathbf{i} + \sec^2 t\mathbf{j}$







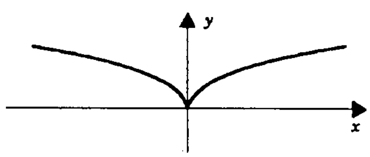
27. $\mathbf{r}(t) = (t^2+1)\,\mathbf{i} + t\mathbf{j},\quad t \geq 1;$ or, $\mathbf{r}(t) = \sec^2 t\,\mathbf{i} + \tan t\,\mathbf{j},\quad t \in \left[\frac{1}{4}\pi,\, \frac{1}{2}\pi\right)$

28. $\mathbf{r}(t) = \cos t(1-\cos t)\mathbf{i} + \sin t(1-\cos t)\mathbf{j},\quad t \in [0, 2\pi]$

29. $\mathbf{r}(t) = \cos t\,\sin 3t\,\mathbf{i} = \sin t\,\sin 3t\,\mathbf{j},\quad t \in [0, \pi]$

30. $\mathbf{r}(t) = t^4\mathbf{i} + t^3\mathbf{j},\quad t \leq 0$

31. $y^3 = x^2$



There is no tangent vector at the origin.

32. (a) $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j} = \mathbf{r}(1)$

(b) $\mathbf{T}(0) = \dfrac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \dfrac{1}{\sqrt{\pi^2+2}}(-\mathbf{i} - \mathbf{j} + \pi\mathbf{k}),\quad \mathbf{T}(1) = \dfrac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \dfrac{1}{\sqrt{\pi^2+5}}(\mathbf{i} + 2\mathbf{j} - \pi\mathbf{k})$

33. We substitute $x = t,\ y = t^2,\ z = t^3$ in the plane equation to obtain

$$4t + 2t^2 + t^3 = 24,\quad (t-2)\left(t^2+4t+12\right) = 0,\quad t = 2.$$

The twisted cubic intersects the plane at the tip of $\mathbf{r}(2)$, the point $(2,4,8)$.

The angle between the curve and the normal line at the point of intersection is the angle between the tangent vector $\mathbf{r}'(2) = \mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$ and the normal $\mathbf{N} = 4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$:

$$\cos\theta = \frac{(\mathbf{i}+4\mathbf{j}+12\mathbf{k})\cdot(4\mathbf{i}+2\mathbf{j}+\mathbf{k})}{\|\mathbf{i}+4\mathbf{j}+12\mathbf{k}\|\,\|4\mathbf{i}+2\mathbf{j}+\mathbf{k}\|} = \frac{24}{\sqrt{161}\sqrt{21}} \cong 0.412,\quad \theta \cong 1.15 \text{ radians.}$$

34. (a) $T(t) = \dfrac{1}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}(-a\sin t\mathbf{i} + b\cos t\mathbf{j})$

$N(t) = \dfrac{1}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}(-b\cos t\mathbf{i} + a\sin t\mathbf{j})$

(b) $\mathbf{r}(u) = \frac{\sqrt{2}}{2}(a\mathbf{i} + b\mathbf{j}) + u(-a\mathbf{i} + b\mathbf{j}), \quad \mathbf{R}(u) = \frac{\sqrt{2}}{2}(a\mathbf{i} + b\mathbf{j}) + u(-b\mathbf{i} - a\mathbf{j})$

35. $\mathbf{r}'(t) = 2\,\mathbf{j} + 2t\,\mathbf{k}, \quad \|\mathbf{r}'(t)\| = 2\sqrt{1+t^2}$

$$\mathbf{T}\,(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{1+t^2}}\,(\mathbf{j} + t\,\mathbf{k}),$$

$$\mathbf{T}'(t) = \frac{1}{(1+t^2)^{3/2}}\,[-t\,\mathbf{j} + \mathbf{k}]$$

at $t = 1$: tip of $\mathbf{r} = (1, 2, 1)$, $\mathbf{T} = \mathbf{T}(1) = \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$;

$\mathbf{T}'(1) = -\frac{1}{2\sqrt{2}}\mathbf{j} + \frac{1}{2\sqrt{2}}\mathbf{k}; \quad \|\mathbf{T}'(1)\| = \frac{1}{2}; \quad \mathbf{N} = \mathbf{N}(1) = \frac{\mathbf{T}'(1)}{\|\mathbf{T}'(1)\|} = -\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{2\sqrt{2}}\mathbf{k}$

normal for osculating plane:

$$\mathbf{T} \times \mathbf{N} = \left(\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}\right) \times \left(-\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{2\sqrt{2}}\mathbf{k}\right) = \frac{1}{2}\mathbf{i}$$

equation for osculating plane:

$$\frac{1}{2}(x-1) + 0(y-2) + 0(z-1) = 0, \quad \text{which gives} \quad x - 1 = 0$$

36. $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\,\mathbf{r}'(t)\,\|} = \frac{\mathbf{i} + 2t\mathbf{j} + 4t\mathbf{k}}{\sqrt{20t^2+1}} \quad \mathbf{T}(1) = \frac{1}{\sqrt{21}}(\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})$

$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\,\mathbf{T}'(t)\,\|} = \frac{-20t\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}}{\sqrt{400t^2+20}} \quad \mathbf{N}(1) = \frac{1}{\sqrt{420}}(-20\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}) = \frac{1}{\sqrt{105}}(-10\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

$\mathbf{T}(1) \times \mathbf{N}(1) = \frac{1}{\sqrt{5}}(-2\mathbf{j} + \mathbf{k})$.

Osculating plane at $(1, 1, 2)$: $\quad -2(y-1) + (z-2) = 0 \implies -2y + 2 = 0$

37. $\mathbf{r}'(t) = -2\sin 2t\,\mathbf{i} + 2\cos 2t\,\mathbf{j} + \mathbf{k}, \quad \|\mathbf{r}'(t)\| = \sqrt{5}$

$$\mathbf{T}\,(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{5}\sqrt{5}\,(-2\sin 2t\,\mathbf{i} + 2\cos 2t\,\mathbf{j} + \mathbf{k})$$

$$\mathbf{T}'(t) = -\tfrac{4}{5}\sqrt{5}\,(\cos 2t\,\mathbf{i} + \sin 2t\,\mathbf{j}), \quad \|\mathbf{T}'(t)\| = \tfrac{4}{5}\sqrt{5}$$

$$\mathbf{N}\,(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = -(\cos 2t\,\mathbf{i} + \sin 2t\,\mathbf{j})$$

at $t = \pi/4$: tip of $\mathbf{r} = (0, 1, \pi/4)$, $\mathbf{T} = \frac{1}{5}\sqrt{5}\,(-2\mathbf{i} + \mathbf{k})$, $\mathbf{N} = -\mathbf{j}$

normal for osculating plane:

$$\mathbf{T} \times \mathbf{N} = \tfrac{1}{5}\sqrt{5}\,(-2\mathbf{i} + \mathbf{k}) \times (-\mathbf{j}) = \tfrac{1}{5}\sqrt{5}\,\mathbf{i} + \tfrac{2}{5}\sqrt{5}\,\mathbf{k}$$

equation for osculating plane:

$$\frac{1}{5}\sqrt{5}\,(x-0) + \frac{2}{5}\sqrt{5}\left(z - \frac{\pi}{4}\right) = 0, \quad \text{which gives} \quad x + 2z = \frac{\pi}{2}$$

38. $\mathbf{T}(t)=\dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}=\dfrac{\mathbf{i}+2\mathbf{j}+2t\mathbf{k}}{\sqrt{4t^2+5}}\qquad \mathbf{T}(2)=\dfrac{1}{\sqrt{21}}(\mathbf{i}+2\mathbf{j}+4\mathbf{k})$

$\mathbf{N}(t)=\dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}=\dfrac{-2t\mathbf{i}-4t\mathbf{j}+5\mathbf{k}}{\sqrt{20t^2+25}}\quad \mathbf{N}(2)=\dfrac{1}{\sqrt{105}}(-4\mathbf{i}-8\mathbf{j}+5\mathbf{k})$

$\mathbf{T}(2)\times\mathbf{N}(2)=\dfrac{1}{\sqrt{5}}(2\mathbf{i}-\mathbf{j})$

Osculating plane at $(2,4,2)$: $\quad 2(x-2)-(y-4)=0 \implies 2x-y=0$

39.
$$\mathbf{r}'(t)=\mathbf{i}+2t\mathbf{j}+3t^2\mathbf{k},\quad \|\mathbf{r}'(t)\|=\sqrt{1+4t^2+9t^4}$$

$$\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}=\frac{1}{\sqrt{1+4t^2+9t^4}}\left(\mathbf{i}+2t\mathbf{j}+3t^2\mathbf{k}\right),$$

$$\mathbf{T}'(t)=\frac{1}{(1+4t^2+9t^4)^{3/2}}\left[\left(-4t-18t^3\right)\mathbf{i}+\left(2-18t^4\right)\mathbf{j}+\left(6t+12t^3\right)\mathbf{k}\right]$$

at $t=1$: tip of $\mathbf{r}=(1,1,1)$, $\quad \mathbf{T}=\dfrac{1}{\sqrt{14}}(\mathbf{i}+2\mathbf{j}+3\mathbf{k})$,

$\mathbf{T}'=\dfrac{1}{7\sqrt{14}}(-11\mathbf{i}-8\mathbf{j}+9\mathbf{k}),\quad \|\mathbf{T}'\|=\dfrac{\sqrt{266}}{7\sqrt{14}},\quad \mathbf{N}=\dfrac{1}{\sqrt{266}}(-11\mathbf{i}-8\mathbf{j}+9\mathbf{k})$

normal for osculating plane:

$$\mathbf{T}\times\mathbf{N}=\frac{1}{\sqrt{14}}(\mathbf{i}+2\mathbf{j}+3\mathbf{k})\times\frac{1}{\sqrt{266}}(-11\mathbf{i}-8\mathbf{j}+9\mathbf{k})=\frac{\sqrt{19}}{19}(3\mathbf{i}-3\mathbf{j}+\mathbf{k})$$

equation for osculating plane:

$$3(x-1)-3(y-1)+(z-1)=0,\quad \text{which gives}\quad 3x-3y+z=1$$

40. $\mathbf{T}(t)=\dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}=\dfrac{-3\sin 3t\mathbf{i}+\mathbf{j}-3\cos 3t\mathbf{k}}{\sqrt{10}}\qquad \mathbf{T}(\frac{\pi}{3})=\mathbf{j}+3\mathbf{k}$

$\mathbf{N}(t)=\dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}=\dfrac{-9\cos 3t\mathbf{i}+9\sin 3t\mathbf{k}}{9}\qquad \mathbf{N}(\frac{\pi}{3})=\mathbf{i}$

$\mathbf{T}(\frac{\pi}{3})\times\mathbf{N}(\frac{\pi}{3})=3\mathbf{j}-\mathbf{k}$

Osculating plane at $(-1,\frac{\pi}{3},0)$: $\quad 3(y-\frac{\pi}{3})-2=0 \implies 3y-z-\pi=0$

41.
$$\mathbf{r}'(t)=e^t\left[(\sin t+\cos t)\,\mathbf{i}+(\cos t-\sin t)\,\mathbf{j}+\mathbf{k}\right],\quad \|\mathbf{r}'(t)\|=e^t\sqrt{3}$$

$$\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}=\frac{1}{\sqrt{3}}\left[(\sin t+\cos t)\,\mathbf{i}+(\cos t-\sin t)\,\mathbf{j}+\mathbf{k}\right],$$

$$\mathbf{T}'(t)=\frac{1}{\sqrt{3}}\left[(\cos t-\sin t)\,\mathbf{i}-(\sin t+\cos t)\,\mathbf{j}\right]$$

at $t=0$: tip of $\mathbf{r}=(0,1,1)$, $\quad \mathbf{T}=\mathbf{T}(0)=\dfrac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$;

$$\mathbf{T}'(0) = \frac{1}{\sqrt{3}}(\mathbf{i}-\mathbf{j}); \quad \|\mathbf{T}'(0)\| = \frac{\sqrt{2}}{\sqrt{3}}; \quad \mathbf{N} = \mathbf{N}(0) = \frac{\mathbf{T}'(0)}{\|\mathbf{T}'(0)\|} = \frac{1}{\sqrt{2}}(\mathbf{i}-\mathbf{j})$$

normal for osculating plane:

$$\mathbf{T}\times\mathbf{N} = \frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k}) \times \frac{1}{\sqrt{2}}(\mathbf{i}-\mathbf{j}) = \frac{1}{\sqrt{6}}(\mathbf{i}+\mathbf{j}-2\mathbf{k})$$

equation for osculating plane:

$$\frac{1}{\sqrt{6}}(x-0) + \frac{1}{\sqrt{6}}(y-1) - \frac{2}{\sqrt{6}}(z-1) = 0, \quad \text{which gives} \quad x+y-2z+1=0$$

42. $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \dfrac{t\cos t\mathbf{i} + t\sin t\mathbf{j}}{t} = \cos t\mathbf{i} + \sin t\mathbf{j} \qquad \mathbf{T}(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = -\sin t\mathbf{i} + \cos t\mathbf{j} \qquad \mathbf{N}(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$

$\mathbf{T}(\frac{\pi}{4}) \times \mathbf{N}(\frac{\pi}{4}) = \mathbf{k}$

Osculating plane at $(\frac{\sqrt{2}}{2}[1+\frac{\pi}{4}],\ \frac{\sqrt{2}}{2}[1-\frac{\pi}{4}],\ 2):\quad z-2=0 \implies z=2$

43. $\mathbf{T}_1 = \dfrac{\mathbf{R}'(u)}{\|\mathbf{R}'(u)\|} = -\dfrac{\mathbf{r}'(a+b-u)}{\|\mathbf{r}'(a+b-u)\|} = -\mathbf{T}.$

Therefore $\mathbf{T}_1'(u) = \mathbf{T}'(a+b-u)$ and $\mathbf{N}_1 = \mathbf{N}$.

44. The condition on ϕ ensure that ϕ increases on J and ϕ^{-1} increases on I.
If

$$\mathbf{r}(t_1) = \mathbf{a} \quad \text{and} \quad \mathbf{r}(t_2) = \mathbf{b} \quad \text{with} \quad t_1 < t_2,$$

then

$$\mathbf{R}(\phi^{-1}(t_1)) = \mathbf{a} \quad \text{and} \quad \mathbf{R}(\phi^{-1}(t_2)) = \mathbf{b} \quad \text{with} \quad \phi^{-1}(t_1) < \phi^{-1}(t_2).$$

Conversely, if

$$\mathbf{R}(u_1) = \mathbf{A} \quad \text{and} \quad \mathbf{R}(u_2) = \mathbf{B} \quad \text{with} \quad u_1 < u_2,$$

then

$$\mathbf{r}(\phi(u_1)) = \mathbf{A} \quad \text{and} \quad \mathbf{r}(\phi(u_2)) = \mathbf{B} \quad \text{with} \quad \phi(u_1) < \phi(u_2).$$

Thus $\mathbf{R}$ and $\mathbf{r}$ take on exactly the same values in exactly the same order.

45. Let $\mathbf{T}$ be the unit tangent at the tip of $\mathbf{R}(u) = \mathbf{r}(\phi(u))$ as calculated from the parametrization $\mathbf{r}$ and let $\mathbf{T}_1$ be the unit tangent at the same point as calculated from the parametrization $\mathbf{R}$. Then

$$\mathbf{T}_1 = \frac{\mathbf{R}'(u)}{\|\mathbf{R}'(u)\|} = \frac{\mathbf{r}'(\phi(u))\,\phi'(u)}{\|\mathbf{r}'(\phi(u))\,\phi'(u)\|} = \frac{\mathbf{r}'(\phi(u))}{\|\mathbf{r}'(\phi(u))\|} = \mathbf{T}.$$

$$\phi'(u) > 0$$

This shows the invariance of the unit tangent.

The invariance of the principal normal and the osculating plane follows directly from the invariance of the unit tangent.

46. (a) $\phi'(u) = 2u > 0$, on $(0, \sqrt{2\pi}]$, so ϕ is increasing on $[0, \sqrt{2\pi}]$.

(b) $\mathbf{T}_1(t) = \dfrac{-\sin t\mathbf{i} + \cos t\mathbf{j} + 2\mathbf{k}}{\sqrt{5}}$ $\qquad \mathbf{T}_2(u) = \dfrac{-\sin u^2\mathbf{i} + \cos u^2\mathbf{j} + 2\mathbf{k}}{\sqrt{5}}$

$$\mathbf{T}_1\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{5}}\left(-\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} + 2\mathbf{k}\right) = \mathbf{T}_2(\tfrac{1}{2}\sqrt{\pi})$$

$\mathbf{N}_1(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}$ $\qquad \mathbf{N}_2(u) = -\cos u^2\mathbf{i} - \sin u^2\mathbf{j}$

$$\mathbf{N}_1\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j} = \mathbf{N}_2(\tfrac{1}{2}\sqrt{\pi}).$$

47. (a) Let $t = \Psi(v) = 2\pi - v^2$. When t increases from 0 to 2π, v decreases from $\sqrt{2\pi}$ to 0.

(b) $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 4t\,\mathbf{k}$, $\quad \mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + 4\,\mathbf{k}$, $\quad \|\mathbf{r}'(t)\| = 2\sqrt{5}$

$$\mathbf{T_r}(t) = -\frac{1}{\sqrt{5}}\sin t\,\mathbf{i} + \frac{1}{\sqrt{5}}\cos t\,\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k},\quad \mathbf{T}_\mathbf{r}'(t) = -\frac{1}{\sqrt{5}}\cos t\,\mathbf{i} - \frac{1}{\sqrt{5}}\sin t\,\mathbf{j},$$

$\|\mathbf{T}_\mathbf{r}'(t)\| = 1/\sqrt{5}$

$\mathbf{N_r}(t) = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j},$

$\mathbf{R}(v) = 2\cos(2\pi - v^2)\,\mathbf{i} + 2\sin(2\pi - v^2)\,\mathbf{j} + 4(2\pi - v^2)\,\mathbf{k}$

$\mathbf{R}'(t) = 4v\sin(2\pi - v^2)\,\mathbf{i} - 4v\cos(2\pi - v^2)\,\mathbf{j} + -8v\,\mathbf{k}$, $\quad \|\mathbf{R}'(t)\| = 4v\sqrt{5}$

$$\mathbf{T_R}(t) = \frac{1}{\sqrt{5}}\sin(2\pi - v^2)\,\mathbf{i} - \frac{1}{\sqrt{5}}\cos(2\pi - v^2)\,\mathbf{j} - \frac{2}{\sqrt{5}}\mathbf{k}$$

$$\mathbf{T}_\mathbf{R}'(t) = -\frac{2v}{\sqrt{5}}\cos(2\pi - v^2)\,\mathbf{i} - \frac{2v}{\sqrt{5}}\sin(2\pi - v^2)\,\mathbf{j},\quad \|\mathbf{T}_\mathbf{R}'(t)\| = 2v/\sqrt{5}$$

$\mathbf{N_R}(t) = -\cos(2\pi - v^2)\,\mathbf{i} - \sin(2\pi - v^2)\,\mathbf{j}$

$$\mathbf{T_r}(\pi/4) = -\frac{1}{\sqrt{5}}\sin(\pi/4)\,\mathbf{i} + \frac{1}{\sqrt{5}}\cos(\pi/4)\,\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k} = -\frac{1}{\sqrt{10}}\mathbf{i} + \frac{1}{\sqrt{10}}\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k}$$

$$\mathbf{T_R}\left(\sqrt{7\pi}/2\right) = \frac{1}{\sqrt{5}}\sin(2\pi - 7\pi/4)\,\mathbf{i} - \frac{1}{\sqrt{5}}\cos(2\pi - 7\pi/4)\,\mathbf{j} - \frac{2}{\sqrt{5}}\mathbf{k} = \frac{1}{\sqrt{10}}\mathbf{i} - \frac{1}{\sqrt{10}}\mathbf{j} - \frac{2}{\sqrt{5}}\mathbf{k} = -\mathbf{T_r}$$

Similarly, $\quad \mathbf{N_r}(\pi/4) = -\dfrac{1}{\sqrt{2}}\mathbf{i} - \dfrac{1}{\sqrt{2}}\mathbf{j} = \mathbf{N_R}\left(\sqrt{7\pi}/2\right)$

48. Since $\mathbf{r}'(t) = \lim\limits_{h\to 0} \dfrac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$, for small positive h we have

$\mathbf{r}'(t) \cong \dfrac{1}{h}[\mathbf{r}(t+h) - \mathbf{r}(t)]$, so $\mathbf{r}'(t)$ has the same direction as $\mathbf{r}(t+h) - \mathbf{r}(t)$, so $\mathbf{r}'(t)$ points in the direction of increasing t.

SECTION 13.4

1. $\mathbf{r}'(t) = \mathbf{i} + t^{1/2}\,\mathbf{j}, \quad \|\mathbf{r}'(t)\| = \sqrt{1+t}$

$$L = \int_0^8 \sqrt{1+t}\,dt = \left[\frac{2}{3}(1+t)^{3/2}\right]_0^8 = \frac{52}{3}$$

2. $\mathbf{r}'(t) = (t^2-1)\mathbf{i} + 2t\mathbf{j}; \quad \|\,\mathbf{r}'(t)\,\| = t^2+1$

$$L = \int_0^2 (t^2+1)\,dt = \frac{14}{3}$$

3. $\mathbf{r}'(t) = -a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j} + b\mathbf{k}, \quad \|\mathbf{r}'(t)\| = \sqrt{a^2+b^2}$

$$L = \int_0^{2\pi} \sqrt{a^2+b^2}\,dt = 2\pi\sqrt{a^2+b^2}$$

4. $\mathbf{r}'(t) = \mathbf{i} + \sqrt{2}t^{1/2}\mathbf{j} + t\mathbf{k}; \quad \|\,\mathbf{r}'(t)\,\| = t+1$

$$L = \int_0^2 (t+1)\,dt = 4$$

5. $\mathbf{r}'(t) = \mathbf{i} + \tan t\,\mathbf{j}, \quad \|\mathbf{r}'(t)\| = \sqrt{1+\tan^2 t} = |\sec t|$

$$L = \int_0^{\pi/4} |\sec t|\,dt = \int_0^{\pi/4} \sec t\,dt = [\ln|\sec t + \tan t|]_0^{\pi/4} = \ln(1+\sqrt{2})$$

6. $\mathbf{r}'(t) = \dfrac{1}{1+t^2}\mathbf{i} + \dfrac{t}{1+t^2}\mathbf{j}; \quad \|\,\mathbf{r}'(t)\,\| = \dfrac{1}{\sqrt{1+t^2}}$

$$L = \int_0^1 \frac{dt}{\sqrt{1+t^2}} = \left[\ln|t+\sqrt{t^2+1}|\right]_0^1 = \ln(1+\sqrt{2})$$

7. $\mathbf{r}'(t) = 3t^2\mathbf{i} + 2t\mathbf{j}, \quad \|\mathbf{r}'(t)\| = \sqrt{9t^4+4t^2} = |t|\sqrt{4+9t^2}$

$$L = \int_0^1 \left|t\sqrt{4+9t2}\right|\,dt = \int_0^1 t\sqrt{4+9t^2}\,dt = \left[\frac{1}{27}\left(4+9t^2\right)^{3/2}\right]_0^1 = \frac{1}{27}\left(13\sqrt{13}-8\right)$$

8. $\mathbf{r}'(t) = \mathbf{i} + \left(\frac{1}{2}t^2 - \frac{1}{2}t^{-2}\right)\mathbf{k}; \quad \|\,\mathbf{r}'(t)\,\| = \frac{1}{2}t^2 + \frac{1}{2}t^{-2}$

$$L = \int_1^3 \left(\frac{1}{2}t^2 + \frac{1}{2}t^{-2}\right)\,dt = \frac{14}{3}$$

9. $\mathbf{r}'(t) = (\cos t - \sin t)e^t\mathbf{i} + (\sin t + \cos t)e^t\mathbf{j}, \quad \|\mathbf{r}'(t)\| = \sqrt{2}\,e^t$

$$L = \int_0^{\pi} \sqrt{2}\,e^t\,dt = \sqrt{2}\,(e^{\pi}-1)$$

10. $\mathbf{r}'(t) = (3\cos t - 3t\sin t)\mathbf{i} + (3\sin t + 3t\cos t)\mathbf{j} + 4\mathbf{k}; \quad \|\,\mathbf{r}'(t)\,\| = \sqrt{9t^2+25}$

$$L = \int_0^4 3\sqrt{t^2 + \frac{25}{9}}\,dt = \left[\frac{3}{2}t\sqrt{t^2+\frac{25}{9}} + \frac{3}{2}\cdot\frac{25}{9}\ln\left|t + \sqrt{t^2+\frac{25}{9}}\right|\right]_0^4 = 26 + \frac{25}{6}\ln 5.$$

11. $\mathbf{r}'(t) = 2\,\mathbf{i} + 2t\mathbf{j} - 2t\mathbf{k}, \quad \|\mathbf{r}'(t)\| = 2\sqrt{1+2t^2}$

$$L = \int_0^2 2\sqrt{1+2t^2}\,dt = \sqrt{2}\int_0^{\tan^{-1}(2\sqrt{2})} \sec^3 u\,du$$

$$(t\sqrt{2} = \tan u)$$

$$= \tfrac{1}{2}\sqrt{2}\,[\sec u\tan u + \ln|\sec u + \tan u|]_0^{\tan^{-1}(2\sqrt{2})} = 6 + \tfrac{1}{2}\sqrt{2}\,\ln(3+2\sqrt{2})$$

12. $\mathbf{r}'(t) = 2t\mathbf{i} + 2t\mathbf{j} - 2t\mathbf{k}; \quad \|\,\mathbf{r}'(t)\,\| = 2t\sqrt{3} \implies L = \int_0^2 2t\sqrt{3}\,dt = 4\sqrt{3}$

13. $\mathbf{r}'(t) = \frac{1}{t}\,\mathbf{i} + 2\,\mathbf{j} + 2t\,\mathbf{k}, \quad \|\mathbf{r}'(t)\| = \sqrt{\frac{1}{t^2} + 4 + 4t^2}$

$$L = \int_1^e \sqrt{\frac{1}{t^2} + 4 + 4t^2}\,dt = \int_1^e \left(\frac{1}{t} + 2t\right)\,dt = \left[\ln|t| + t^2\right]_1^e = e^2$$

14. $\mathbf{r}'(t) = t\cos t\mathbf{i} - t\sin t\mathbf{j}; \quad \|\,\mathbf{r}'(t)\,\| = t \implies L = \int_0^2 t\,dt = 2$

15.

$$s = s(t) = \int_a^t \|\mathbf{r}'(u)\|\,du$$

$$s'(t) = \|\mathbf{r}'(t)\| = \|x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} + z'(t)\,\mathbf{k}\|$$

$$= \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}.$$

In the Leibniz notation this translates to

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}.$$

16. Parameterize the graph of f by setting $\mathbf{r}(x) = x\mathbf{i} + f(x)\mathbf{j}, \quad x \in [a,b]$.

17.

$$s = s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\,dt$$

$$s'(x) = \sqrt{1 + [f'(x)]^2}.$$

In the Leibniz notation this translates to

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

18. $L_1 = \int_1^e \| \mathbf{r}'(t) \| \, dt = \int_1^e \sqrt{2 + \frac{2}{t^2}}\, dt; \quad L_2 = \int_0^1 \sqrt{1 + e^{2x}}\, dx$

Setting $t = e^x$, we get $L_1 = \sqrt{2} L_2$

19. Let L be the length as computed from $\mathbf{r}$ and L^* the length as computed from $\mathbf{R}$. Then

$$L^* = \int_c^d \| \mathbf{R}'(u) \| \, du = \int_c^d \| \mathbf{r}'(\phi(u)) \| \, \phi'(u)\, du = \int_a^b \| \mathbf{r}'(t) \| \, dt = L.$$

$$t = \phi(u)$$

20. (a) $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \| \mathbf{r}'(t) \| > 0,$

so $s(t)$ is increasing. Hence s has an inverse.

(b) $\mathbf{R}'(s) = \mathbf{r}'(\phi(s))\phi'(s) = \mathbf{r}'(\phi(s))\frac{1}{s'(\phi(s))} = \mathbf{r}'(\phi(s)) \cdot \frac{1}{|\mathbf{r}'(\phi(s))|}$, so $\| \mathbf{R}'(s) \| = 1$

21. (a) $s = \int_0^t \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4^2}\, dt = \int_0^t 5\, dt = 5t$

(b) $t = \frac{s}{5}$; $\quad R(s) = 3\cos\left(\frac{s}{5}\right)\mathbf{i} + 3\sin\left(\frac{s}{5}\right)\mathbf{j} + \frac{4s}{5}\mathbf{k}$

(c) $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} + 4t\,\mathbf{k} = 3\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k} \implies t = 0$

From part (a), the arc length $s = 5t$ and $5t = 5\pi \implies t = \pi$; $\mathbf{r}(\pi) = -3\,\mathbf{i} + 0\,wj + 4\pi\,\mathbf{k}$

$\implies Q(-3, 0, 4\pi)$.

(d) $\mathbf{R}'(s) = -\frac{3}{5}\sin\left(\frac{s}{5}\right)\mathbf{i} + \frac{3}{5}\cos\left(\frac{s}{5}\right)\mathbf{j} + \frac{4}{5}\mathbf{k}$;

$$\|\mathbf{R}'(s)\| = \sqrt{\left[-\frac{3}{5}\sin\left(\frac{s}{5}\right)\right]^2 + \left[\frac{3}{5}\cos\left(\frac{s}{5}\right)\right]^2 + \left[\frac{4}{5}\right]^2} = 1$$

22. (a) $s(t) = \int_0^t \sqrt{t^2\sin^2 t + t^2\cos^2 t + t^2}\, dt = \int_0^t t\sqrt{2}\, dt = \frac{\sqrt{2}}{2}t^2$. Invertible on $[0, \infty)$.

(b) $t = \sqrt[4]{2}\sqrt{s}$;

$$\mathbf{R}(s) = \left[\sin(\sqrt[4]{2}\sqrt{s}) - \sqrt[4]{2}\sqrt{s}\cos(\sqrt[4]{2}\sqrt{s})\right]\mathbf{i} + \left[\cos(\sqrt[4]{2}\sqrt{s}) + \sqrt[4]{2}\sqrt{s}\sin(\sqrt[4]{2}\sqrt{s})\right]\mathbf{j} + \frac{\sqrt{2}}{2}s\mathbf{k}$$

$$\mathbf{R}'(s) = \left[\sqrt[4]{2}\sqrt{s}\sin(\sqrt[4]{2}\sqrt{s})\mathbf{i} + \sqrt[4]{2}\sqrt{s}\cos(\sqrt[4]{2}\sqrt{s})\mathbf{j} + \sqrt[4]{2}\sqrt{s}\mathbf{k}\right]\frac{\sqrt[4]{2}}{2\sqrt{s}} \implies \| \mathbf{R}'(s) \| = 1$$

23. $\mathbf{r}'(t) = t^{3/2}\mathbf{j} + \mathbf{k}, \quad \|\mathbf{r}'(t)\| = \sqrt{\left(t^{3/2}\right)^2 + 1} = \sqrt{t^3+1}$

$$s = \int_0^{1/2} \sqrt{t^3+1}\,dt \cong 0.5077$$

24. $\mathbf{r}'(t) = \mathbf{i} + t^2\mathbf{j}; \quad \| \mathbf{r}'(t) \| = \sqrt{1+t^4} \quad L = \int_0^2 \sqrt{1+t^4} \cong 3.6535$

25. $\mathbf{r}'(t) = -3\sin t\,\mathbf{i} + 4\cos t\,\mathbf{j}, \quad \|\mathbf{r}'(t)\| = \sqrt{9\sin^2 t + 16\cos^2 t}\,dt$

$$s = \int_0^{2\pi} \sqrt{9\sin^2 t + 16\cos^2 t}\,dt \cong 22.0939$$

26. $\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + \frac{1}{t}\mathbf{k}; \quad \| \mathbf{r}'(t) \| = \sqrt{1 + 4t^2 + \frac{1}{t^2}} \quad L = \int_1^4 \sqrt{1+4t^2+\frac{1}{t^2}}\,dt \cong 15.4480$

SECTION 13.5

1.

$$\mathbf{r}(t) = r[\cos\theta(t)\,\mathbf{i} + \sin\theta(t)\,\mathbf{j}]$$

$$\mathbf{r}'(t) = r[-\sin\theta(t)\,\mathbf{i} + \cos\theta(t)\,\mathbf{j})]\theta'(t)$$

$$\|\mathbf{r}'(t)\| = v \implies r|\theta'(t)| = v \implies |\theta'(t)| = v/r$$

$$\mathbf{r}''(t) = r[-\cos\theta(t)\,\mathbf{i} - \sin\theta(t)\,\mathbf{j}]\,[\theta'(t)]^2$$

$$\|\mathbf{r}''(t)\| = r[\theta'(t)]^2 = v^2/r$$

2.

$$\mathbf{r}'(t) = (-\pi a\sin\pi t + 2bt)\mathbf{i} + (\pi a\cos\pi t - 2bt)\mathbf{j}$$

$$\mathbf{r}''(t) = (-\pi^2 a\cos\pi t + 2b)\mathbf{i} + (-\pi^2 a\sin\pi t - 2b)\mathbf{j}$$

At $t=1$, $\quad \mathbf{v} = 2b\mathbf{i} - (a\pi + 2b)\mathbf{j}, \quad v = \| \mathbf{v} \| = \sqrt{4b^2 + (a\pi^2 + 2b)^2}$

$$\mathbf{a} = (a\pi^2 + 2b)\mathbf{i} - 2b\mathbf{j}, \quad \| \mathbf{a} \| = \sqrt{4b^2 + (a\pi^2+2b)^2}$$

3.

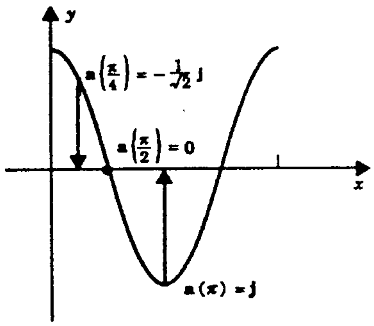
$$\mathbf{r}(t) = at\mathbf{i} + b\sin at\,\mathbf{j}, \quad \mathbf{r}'(t) = a\mathbf{i} + ab\cos at\,\mathbf{j}$$

$$\mathbf{r}''(t) = -a^2 b\sin at\,\mathbf{j}, \quad \|\mathbf{r}''(t)\| = a^2|b\sin at| = a^2|y(t)|$$

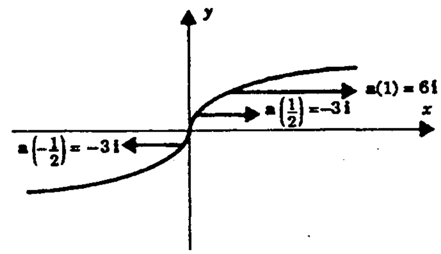
4. $\mathbf{r}'(t) = 2t\mathbf{j} + 2(t-1)\mathbf{k}$; speed is minimum when $\| \mathbf{r}'(t) \|^2$ is minimum

$4t^2 + 4(t-1)^2 = 8t^2 - 8t + 4$ is minimum at $t = \dfrac{1}{2}$

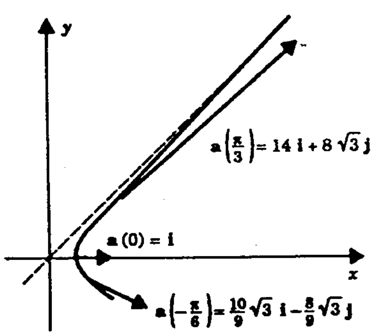
5. $y = \cos \pi x, \quad 0 \le x \le 2$



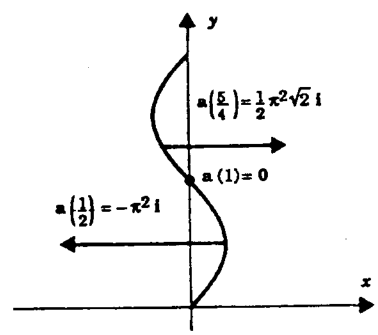
6. $x = y^3$, all real x



7. $x = \sqrt{1+y^2}, \quad y \ge -1$



8. $x = \sin \pi y, \quad 0 \le y \le 2$



9. (a) initial position is tip of $\mathbf{r}(0) = x_0\,\mathbf{i} + y_0\,\mathbf{j} + z_0\,\mathbf{k}$

(b) $\mathbf{r}'(t) = (\alpha \cos\theta)\,\mathbf{j} + (\alpha \sin\theta - 32t)\,\mathbf{k}, \quad \mathbf{r}'(0) = (\alpha\cos\theta)\,\mathbf{j} + (\alpha\sin\theta)\,\mathbf{k}$

(c) $|\mathbf{r}'(0)| = |\alpha|$

(d) $\mathbf{r}''(t) = -32\mathbf{k}$

(e) a parabolic arc from the parabola

$$z = z_0 + (\tan\theta)(y - y_0) - 16\,\frac{(y-y_0)^2}{\alpha^2\cos^2\theta}$$

in the plane $x = x_0$

10. (a) $x(t) = 2cos2t = 4\cos^2 t - 2$ and $y(t) = 3\cos t \implies x = \dfrac{4}{9}y^2 - 2.$ Since

$$-2 \le x(t) \le 2, \quad -3 \le y(t) \le 3,$$

the path consists only of the bounded arc

$$x = \frac{4}{9}y^2 - 2, \quad -3 \le y \le 3.$$

The motion traces out this arc twice on every t-interval of length 2π.

(b) included in part (d)

(c) $\mathbf{r}(t) = 2\cos 2t\mathbf{i} + 3\cos t\mathbf{j}$

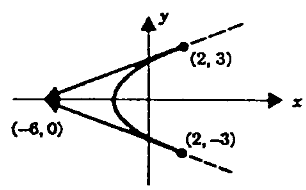
$\mathbf{r}'(t) = -4\sin 2t\mathbf{i} - 3\sin t\mathbf{j}$

$\mathbf{r}''(t) = -8\cos 2t\mathbf{i} - 3\cos t\mathbf{j}$

The velocity $\mathbf{r}'(t)$ is $\mathbf{0}$ when $t = n\pi$. At such points

$$\text{the acceleration } = \begin{cases} -8\mathbf{i} - 3\mathbf{j}, & \text{if } n \text{ is even} \\ -8\mathbf{i} + 3\mathbf{j}, & \text{if } n \text{ is odd} \end{cases}$$

(b) and (d)



11. (a) $\mathbf{r}'(t) = \frac{a\omega}{2}\left(e^{\omega t} - e^{-\omega t}\right)\mathbf{i} + \frac{b\omega}{2}\left(e^{\omega t} + e^{-\omega t}\right)\mathbf{j}, \quad \mathbf{r}'(0) = b\omega\mathbf{j}$

(b) $\mathbf{r}''(t) = \frac{a\omega^2}{2}\left(e^{\omega t} + e^{-\omega t}\right)\mathbf{i} + \frac{b\omega^2}{2}\left(e^{\omega t} - e^{-\omega t}\right)\mathbf{j} = \omega^2\mathbf{r}(t)$

(c) The torque τ is $\mathbf{0}$: $\tau(t) = \mathbf{r}(t) \times m\mathbf{a}(t) = \mathbf{r}(t) \times m\omega^2\mathbf{r}(t) = \mathbf{0}.$

The angular momentum $\mathbf{L}(t)$ is constant since $\mathbf{L}'(t) = \tau(t) = \mathbf{0}.$

12. (a) $\mathbf{F}(t) = m\mathbf{r}''(t) = mb^2\mathbf{r}(t), \quad mb^2 > 0$

(b) $\mathbf{F}(t) = -m\mathbf{r}(t), \quad -m < 0$

(c) $\mathbf{L}(t) = \mathbf{r}(t) \times m\mathbf{v}(t) = m(\mathbf{i} + \mathbf{j} + \mathbf{k})$

13. We begin with the force equation $\mathbf{F}(t) = \alpha\mathbf{k}$. In general, $\mathbf{F}(t) = m\,\mathbf{a}(t)$, so that here

$$\mathbf{a}(t) = \frac{\alpha}{m}\mathbf{k}.$$

Integration gives

$$\mathbf{v}(t) = C_1\mathbf{i} + C_2\mathbf{j} + \left(\frac{\alpha}{m}t + C_3\right)\mathbf{k}.$$

Since $\mathbf{v}(0) = 2\mathbf{j}$, we can conclude that $C_1 = 0,\ C_2 = 2,\ C_3 = 0$. Thus

$$\mathbf{v}(t) = 2\mathbf{j} + \frac{\alpha}{m}t\mathbf{k}.$$

Another integration gives

$$\mathbf{r}(t) = D_1\mathbf{i} + (2t + D_2)\mathbf{j} + \left(\frac{\alpha}{2m}t^2 + D_3\right)\mathbf{k}.$$

Since $\mathbf{r}(0) = y_0\mathbf{j} + z_0\mathbf{k}$, we have $D_1 = 0$, $D_2 = y_0$, $D_3 = z_0$, and therefore

$$\mathbf{r}(t) = (2t + y_0)\,\mathbf{j} + \left(\frac{\alpha}{2m}\,t^2 + z_0\right)\mathbf{k}.$$

The conditions of the problem require that t be restricted to nonnegative values.
To obtain an equation for the path in Cartesian coordinates, we write out the components

$$x(t) = 0, \quad y(t) = 2t + y_0, \quad z(t) = \frac{\alpha}{2m}\,t^2 + z_0. \qquad (t \geq 0)$$

From the second equation we have

$$t = \tfrac{1}{2}\,[y(t) - y_0]. \qquad (y(t) \geq y_0)$$

Substituting this into the third equation, we get

$$z(t) = \frac{\alpha}{8m}\,[y(t) - y_0]^2 + z_0. \qquad (y(t) \geq y_0)$$

Eliminating t altogether, we have

$$z = \frac{\alpha}{8m}\,(y - y_0)^2 + z_0. \qquad (y \geq y_0)$$

Since $x = 0$, the path of the object is a parabolic arc in the yz-plane.

Answers to (a) through (d):

(a) velocity: $\mathbf{v}(t) = 2\mathbf{j} + \dfrac{\alpha}{m}\,t\mathbf{k}$. (b) speed: $v(t) = \dfrac{1}{m}\sqrt{4m^2 + \alpha^2 t^2}$.

(c) momentum: $\mathbf{p}(t) = 2m\mathbf{j} + \alpha\, t\mathbf{k}$.

(d) path in vector form: $\mathbf{r}(t) = (2t + y_0)\,\mathbf{j} + \left(\dfrac{\alpha}{2m}\,t^2 + z_0\right)\mathbf{k}, \quad t \geq 0.$

path in Cartesian coordinates: $z = \dfrac{\alpha}{8m}\,(y - y_0)^2 + z_0, \quad y \geq y_0, \quad x = 0.$

14. $\mathbf{F}(t) \cdot \mathbf{v}(t) = 0$ for all t

$\Longrightarrow \mathbf{a}(t) \cdot \mathbf{v}(t) = \mathbf{v}'(t) \cdot \mathbf{v}(t) = \dfrac{1}{2}\dfrac{d}{dt}[\mathbf{v}(t) \cdot \mathbf{v}(t)] = 0$ for all t

$\Longrightarrow \mathbf{v}(t) \cdot \mathbf{v}(t) = [v(t)]^2$ is constant $\Longrightarrow v(t)$ is constant

15. $\mathbf{F}(t) = m\,\mathbf{a}(t) = m\,\mathbf{r}''(t) = 2m\mathbf{k}$

16. If $\mathbf{v}'(t) = \mathbf{0}$, then $\mathbf{L}'(t) = \mathbf{r}'(t) \times m\mathbf{v}(t) + \mathbf{r}(t) \times m\mathbf{v}'(t)$

$$= \mathbf{v}(t) \times m\mathbf{v}(t) + \mathbf{r}(t) \times \mathbf{0} = \mathbf{0}$$

17. From $\mathbf{F}(t) = m\,\mathbf{a}(t)$ we obtain

$$\mathbf{a}(t) = \pi^2[a\cos\pi t\,\mathbf{i} + b\sin\pi t\,\mathbf{j}].$$

By direct calculation using $\mathbf{v}(0) = -\pi b\mathbf{j} + \mathbf{k}$ and $\mathbf{r}(0) = b\mathbf{j}$ we obtain

$$\mathbf{v}(t) = a\pi\sin\pi t\,\mathbf{i} - b\pi\cos\pi t\,\mathbf{j} + \mathbf{k}$$

$$\mathbf{r}(t) = a(1 - \cos\pi t)\,\mathbf{i} + b(1 - \sin\pi t)\,\mathbf{j} + t\mathbf{k}.$$

(a) $\mathbf{v}(1) = b\pi\mathbf{j} + \mathbf{k}$ (b) $\|\mathbf{v}(1)\| = \sqrt{\pi^2 b^2 + 1}$

(c) $\mathbf{a}(1) = -\pi^2 a\mathbf{i}$ (d) $m\,\mathbf{v}(1) = m\,(\pi b\mathbf{j} + \mathbf{k})$

(e) $\mathbf{L}(1) = \mathbf{r}(1) \times m\,\mathbf{v}(1) = [2a\mathbf{i} + b\mathbf{j} + \mathbf{k}] \times [m\,(b\pi\mathbf{j} + \mathbf{k})]$

$$= m\,[b(1-\pi)\mathbf{i} - 2a\mathbf{j} + 2ab\pi\mathbf{k}]$$

(f) $\tau(1) = \mathbf{r}(1) \times \mathbf{F}(1) = [2a\mathbf{i} + b\mathbf{j} + \mathbf{k}] \times \left[-m\pi^2 a\mathbf{i}\right] = -m\pi^2 a\,[\mathbf{j} - b\mathbf{k}]$

18.
$$\frac{d}{dt}\left(\frac{1}{2}m[v(t)]^2\right) = \frac{1}{2}m\frac{d}{dt}[\mathbf{v}(t)\cdot\mathbf{v}(t)] = \frac{1}{2}m[2\mathbf{v}'(t)\cdot\mathbf{v}(t)]$$
$$= m\mathbf{a}(t)\cdot\mathbf{v}(t) = \mathbf{F}(t)\cdot\mathbf{v}(t)$$

19. We have $\quad m\mathbf{v} = m\mathbf{v}_1 + m\mathbf{v}_2 \quad$ and $\quad \frac{1}{2}mv^2 = \frac{1}{2}mv_1{}^2 + \frac{1}{2}mv_2{}^2.$

Therefore $\quad \mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 \quad$ and $\quad v^2 = v_1{}^2 + v_2{}^2.$

Since $\quad v^2 = \mathbf{v}\cdot\mathbf{v} = (\mathbf{v}_1 + \mathbf{v}_2)\cdot(\mathbf{v}_1 + \mathbf{v}_2) = v_1{}^2 + v_2{}^2 + 2(\mathbf{v}_1\cdot\mathbf{v}_2),$

we have $\quad \mathbf{v}_1\cdot\mathbf{v}_2 = 0 \quad$ and $\quad \mathbf{v}_1 \perp \mathbf{v}_2.$

20. That the path is of the form $\mathbf{r}(t) = \cos\omega t\mathbf{A} + \sin\omega t\mathbf{B}$ can be seen by applying the hint to each component of the equation $\mathbf{F}(t) = -m\omega^2\mathbf{r}(t)$.

$$\mathbf{A} = \mathbf{r}(0) = \text{ the initial position}$$

$$\mathbf{B} = \mathbf{r}'(0)/\omega = \text{ the initial velocity divided by } \omega$$

The path is circular if $\|\mathbf{A}\| = \|\mathbf{B}\|$ and $\mathbf{A} \perp \mathbf{B}$.

21. $\mathbf{r}''(t) = \mathbf{a}, \quad \mathbf{r}'(t) = \mathbf{v}(0) + t\mathbf{a}, \quad \mathbf{r}(t) = \mathbf{r}(0) + t\,\mathbf{v}(0) + \frac{1}{2}t^2\,\mathbf{a}.$

If neither $\mathbf{v}(0)$ nor $\mathbf{a}$ is zero, the displacement $\mathbf{r}(t) - \mathbf{r}(0)$ is a linear combination of $\mathbf{v}(0)$ and $\mathbf{a}$ and thus remains on the plane determined by these vectors. The equation of this plane can be written

$$[\mathbf{a} \times \mathbf{v}(0)]\cdot[\mathbf{r} - \mathbf{r}(0)] = 0.$$

(If either $\mathbf{v}(0)$ or $\mathbf{a}$ is zero, the motion is restricted to a straight line; if both of these vectors are zero, the particle remains at its initial position $\mathbf{r}(0)$.)

22. Clearly we can take $\phi_1 \in [0, 2\pi)$. With $\omega = -1$, the path takes the form

$$\mathbf{r}(t) = [A_1\cos(-t + \phi_1) + D_1]\mathbf{i} - [A_1\sin(-t + \phi_1) + D_2]\mathbf{j} + [Ct + D_3]\mathbf{k}.$$

Differentiation gives

$$\mathbf{v}(t) = A_1\sin(-t + \phi_1)\mathbf{i} + A_1\cos(-t + \phi_1)\mathbf{j} + C\mathbf{k}.$$

$\mathbf{r}(0) = a\mathbf{i} \implies A_1\cos\phi_1 + D_1 = a, \quad A\sin\phi_1 + D_2 = 0, \quad D_3 = 0$

$\mathbf{v}(0) = a\mathbf{j} + b\mathbf{k} \implies A_1\sin\phi_1 = 0, \quad A_1\cos\phi_1 = a, \quad C = b$

From these equations we have

$$D_1 = D_2 = D_3 = 0, \quad c = b, \quad A_1\sin\phi_1 = 0, \quad A_1\cos\phi_1 = a.$$

The last two equations give

$$\phi_1 = 0 \quad \text{and } A_1 = a \quad \text{or} \quad \phi_1 = \pi \text{ and } A_1 = -a.$$

The first possibility gives

$$\begin{aligned}\mathbf{r}(t) &= a\cos(-t)\mathbf{i} - a\sin(-t)\mathbf{j} + bt\mathbf{k}\\ &= \mathbf{a}\cos t\mathbf{i} + a\sin t\mathbf{j} + bt\mathbf{k}.\end{aligned}$$

The second possibility gives the same path:

$$\begin{aligned}\mathbf{r}(t) &= -a\cos(-t+\pi)\mathbf{i} - a\sin(-t+\pi)\mathbf{j} + bt\mathbf{k}\\ &= a\cos t\mathbf{i} + a\sin t\mathbf{j} + bt\mathbf{k}\end{aligned}$$

23. $\mathbf{r}(t) = \mathbf{i} + t\mathbf{j} + \left(\dfrac{qE_0}{2m}\right)t^2\mathbf{k}$

24. Since $\mathbf{v}$ has magnitude $w \,\|\, \mathbf{r} \,\|$, lies in the plane of the wheel, and makes an angle of $90°$ counterclockwise with $\mathbf{r}$, we have $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$

25. $\mathbf{r}(t) = \left(1 + \dfrac{t^3}{6m}\right)\mathbf{i} + \dfrac{t^4}{12m}\mathbf{j} + t\mathbf{k}$

26. $\mathbf{r}(t) = \sin(\omega t + \frac{1}{2}\pi)\mathbf{k}$

27. (13.2.3)

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}mv^2\right) &= mv\frac{dv}{dt} = m\left(\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\right) = m\frac{d\mathbf{v}}{dt}\cdot\mathbf{v} = \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\\ &= 4r^2\left(\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\right) = 4r^2\left(r\frac{dr}{dt}\right) = 4r^3\frac{dr}{dt} = \frac{d}{dt}\left(r^4\right).\end{aligned}$$

Therefore $d/dt\,\left(\frac{1}{2}mv^2 - r^4\right) = 0$ and $\frac{1}{2}mv^2 - r^4$ is a constant E. Evaluating E from $t = 0$, we find that $E = 2m$.

Thus $\frac{1}{2}mv^2 - r^4 = 2m$ and $v = \sqrt{4 + (2/m)\,r^4}$.

28. (a) Since $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\,\mathbf{r}'(t)\,\|}$, $\mathbf{v}(t) = \|\,\mathbf{v}(t)\,\|\,\mathbf{T}(t) = v(t)\mathbf{T}(t)$.

(b) $\mathbf{a}(t) = \dfrac{d}{dt}\left(v(t)\mathbf{T}(t)\right) = v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t) = v'(t)\mathbf{T}(t) + v(t)\,\|\,\mathbf{T}(t)\,\|\,\mathbf{N}(t)$.

SECTION 13.6

1. On Earth: year of length T, average distance from sun d.

On Venus: year of length αT, average distance from sun $0.72d$.

Therefore

$$\frac{(\alpha T)^2}{T^2} = \frac{(0.72d)^3}{d^3}.$$

This gives $\alpha^2 = (0.72)^3 \cong 0.372$ and $\alpha \cong 0.615$. Answer: about 61.5% of an Earth year.

2.
$$\frac{dE}{dt} = \frac{1}{2}m\frac{d}{dt}(v^2) - m\rho\frac{d}{dt}(r^{-1})$$

$$\frac{d}{dt}(v^2) = \frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) = 2(\mathbf{a} \cdot \mathbf{v})$$

$$\frac{d}{dt}(r^{-1}) = -\frac{1}{r^2}\frac{dr}{dt} = -\frac{1}{r^3}\left(r\frac{dr}{dt}\right) = -\frac{1}{r^3}(\mathbf{r} \cdot \mathbf{v}) \quad \text{(using 13.2.3)}$$

$$\frac{dE}{dt} = m(\mathbf{a} \cdot \mathbf{v}) + \frac{m\rho}{r^3}(\mathbf{r} \cdot \mathbf{v}) = (\mathbf{a} \cdot m\mathbf{v}) + \left(\frac{\rho\mathbf{r}}{r^3} \cdot m\mathbf{v}\right) = \left(a + \frac{\rho\mathbf{r}}{r^3}\right) \cdot m\mathbf{v} = 0$$

3.
$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left[\frac{d}{dt}(r\cos\theta)\right]^2 + \left[\frac{d}{dt}(r\sin\theta)\right]^2$$

$$= \left[r(-\sin\theta)\frac{d\theta}{dt} + \frac{dr}{dt}\cos\theta\right]^2 + \left[r\cos\theta\frac{d\theta}{dt} + \frac{dr}{dt}\sin\theta\right]^2$$

$$= r^2\sin^2\theta\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2\cos^2\theta + r^2\cos^2\theta\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2\sin^2\theta$$

$$= \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2$$

4. Using

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta} \quad \text{and} \quad \dot{\theta} = \frac{L}{mr^2}$$

we have

$$E + \frac{m\rho}{r} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{1}{2}\left[\left(\frac{dr}{d\theta}\right)^2\dot{\theta}^2 + r^2\dot{\theta}^2\right]$$

$$= \frac{1}{2}m\left[\left(\frac{dr}{d\theta}\right)^2\frac{L^2}{m^2r^4} + \frac{L^2}{m^2r^2}\right] = \frac{L^2}{2m}\left[\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r^2}\right]$$

and therefore

$$E = \frac{L^2}{2m}\left[\frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2\right] - \frac{m\rho}{r}$$

5. Substitute

$$r = \frac{a}{1 + e\cos\theta}, \quad \left(\frac{dr}{d\theta}\right)^2 = \left[\frac{-a}{(1 + e\cos\theta)^2} \cdot (-e\sin\theta)\right]^2 = \frac{(ae\sin\theta)^2}{(1 + e\cos\theta)^4}$$

into the right side of the equation and you will see that, with a and e^2 as given, the expression reduces to E.

SECTION 13.7

1. $k = \dfrac{e^{-x}}{(1 + e^{-2x})^{3/2}}$

2. $y' = 3x^2, \quad y'' = 6x; \quad k = \dfrac{6|x|}{(1 + 9x^4)^{3/2}}$

3. $y' = \dfrac{1}{2x^{1/2}}; \quad y'' = \dfrac{-1}{4x^{3/2}} \qquad k = \dfrac{|-1/4x^{3/2}|}{\left[1 + (1/2x^{1/2})^2\right]^{3/2}} = \dfrac{2}{(1 + 4x)^{3/2}}$

4. $y' = 1 - 2x, \quad y'' = -2; \quad k = \dfrac{2}{[1 + (1 - 2x)^2]^{3/2}} = \dfrac{\sqrt{2}}{2(1 - 2x + 2x^2)^{3/2}}$

5. $k = \dfrac{\sec^2 x}{(1 + \tan^2 x)^{3/2}} = |\cos x|$

6. $y' = \sec^2 x, \quad y'' = 2\sec^2 x\tan x; \quad k = \dfrac{|2\sec^2 x\tan x|}{(1 + \sec^4 x)^{3/2}}$

7. $k = \dfrac{|\sin x|}{(1 + \cos^2 x)^{3/2}}$

8. $2x - 2yy' = 0 \Longrightarrow y' = \dfrac{x}{y}, \quad y'' = \dfrac{y - x(\frac{x}{y})}{y^2} = -\dfrac{a^2}{y^3}; \quad k = \left|\dfrac{a^2/y^3}{[1 + (\frac{x}{y})^2]^{3/2}}\right| = \dfrac{a^2}{(x^2 + y^2)^{3/2}}$

9. $k = \dfrac{|x|}{(1 + x^4/4)^{3/2}}; \quad \text{at } \left(2, \dfrac{4}{3}\right), \quad \rho = \dfrac{5}{2}\sqrt{5}$

10. $y' = x, \quad y'' = 1, \quad k = \dfrac{1}{(1 + x^2)^{3/2}} \quad \text{At } (0,0), \quad \rho = \dfrac{1}{k} = 1$

11. $k = \dfrac{|-1/y^3|}{(1 + 1/y^2)^{3/2}} = \dfrac{1}{(1 + y^2)^{3/2}}; \quad \text{at } (2,2), \quad \rho = 5\sqrt{5}$

12. $y' = 4\cos 2x, \quad y'' = -8\sin 2x; \quad \text{at } x = \dfrac{\pi}{4}, \quad y' = 0, \quad y'' = -8 \Longrightarrow k = \dfrac{8}{1} \Longrightarrow \rho = \dfrac{1}{8}$

13. $y'(x) = \dfrac{1}{x + 1}, \quad y'(2) = \dfrac{1}{3}; \qquad y''(x) = \dfrac{-1}{(x + 1)^2}, \quad y''(2) = -\dfrac{1}{9}.$

At $x = 2, \quad k = \dfrac{\left|-\dfrac{1}{9}\right|}{\left[1 + \left(\dfrac{1}{3}\right)^2\right]^{3/2}} = \dfrac{3}{10\sqrt{10}}; \quad \rho = \dfrac{10\sqrt{10}}{3}$

14. At $x = \frac{\pi}{4}$, $y' = \sec x \tan x = \sqrt{2}$, $y'' = \sec x \tan^2 x + \sec^3 x = \sqrt{2} + 2^{3/2} = 3\sqrt{2}$

$$\Longrightarrow \; k = \frac{3\sqrt{2}}{(1+2)^{3/2}} = \sqrt{\frac{2}{3}} \;\Longrightarrow\; \rho = \sqrt{\frac{3}{2}}$$

15. $k(x) = \dfrac{|-1/x^2|}{(1+1/x^2)^{3/2}} = \dfrac{x}{(x^2+1)^{3/2}}, \quad x > 0$

$k'(x) = \dfrac{(1-2x^2)}{(x^2+1)^{5/2}}, \qquad k'(x) = 0 \;\Longrightarrow\; x = \frac{1}{2}\sqrt{2}$

Since k increases on $\left(0, \frac{1}{2}\sqrt{2}\right]$ and decreases on $\left[\frac{1}{2}\sqrt{2}, \infty\right)$, k is maximal at $\left(\frac{1}{2}\sqrt{2}, \frac{1}{2}\ln\frac{1}{2}\right)$.

16. $y' = 3 - 3x^2$, $y'' = -6x$. local max at $x = 1:$ $y' = 0$, $y'' = -6$, $k = \dfrac{6}{(1+0)^{3/2}} = 6$

17. $x(t) = t$, $x'(t) = 1$, $x''(t) = 0$; $y(t) = \frac{1}{2}t^2$, $y'(t) = t$, $y''(t) = 1$

$$k = \frac{1}{(1+t^2)^{3/2}}$$

18. $x' = e^t$, $x'' = e^t$, $y' = -e^{-t}$, $y'' = e^{-t}$

$$k = \frac{|x'y'' - x''y'|}{[(x')^2 + (y')^2]^{3/2}} = \frac{2}{(e^{2t} + e^{-2t})^{3/2}}$$

19. $x(t) = 2t$, $x'(t) = 2$, $x''(t) = 0$; $y(t) = t^3$, $y'(t) = 3t^2$, $y''(t) = 6t$;

$$k = \frac{12|t|}{(4+9t^4)^{3/2}}$$

20. $x' = 2t$, $x'' = 2$, $y' = 3t^2$, $y'' = 6t$; $k = \dfrac{|12t^2 - 6t^2|}{(4t^2+9t^4)^{3/2}} = \dfrac{6}{|t|(4+9t^2)^{3/2}}$

21. $x(t) = e^t \cos t$, $x'(t) = e^t(\cos t - \sin t)$, $x''(t) = -2\,e^t \sin t$

$y(t) = e^t \sin t$, $y'(t) = e^t(\sin t + \cos t)$, $y''(t) = 2\,e^t \cos t$

$$k = \frac{\left|2e^{2t}\cos t\,(\cos t - \sin t) + 2e^{2t}\sin t\,(\cos t + \sin t)\right|}{\left[\,e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2\,\right]^{3/2}} = \frac{2e^{2t}}{(2e^{2t})^{3/2}} = \frac{1}{2}\sqrt{2}\,e^{-t}$$

22. $x' = -2\sin t$, $x'' = -2\cos t$, $y' = 3\cos t$, $y'' = -3\sin t$; $k = \dfrac{6}{(4\sin^2 t + 9\cos^2 t)^{3/2}}$

23. $x(t) = t\,\cos t$, $x'(t) = \cos t - t\,\sin t$, $x''(t) = -2\,\sin t - t\,\cos t$

$y(t) = t\,\sin t$, $y'(t) = \sin t + t\,\cos t$, $y''(t) = 2\,\cos t - t\,\sin t$

$$k = \frac{\left|(\cos t - t\,\sin t)(2\,\cos t - t\,\sin t) - (\sin t + t\,\cos t)(-2\,\sin t - t\,\cos t)\right|}{\left[\,(\cos t - t\,\sin t)^2 + (\sin t + t\,\cos t)^2\,\right]^{3/2}} = \frac{2+t^2}{[1+t^2]^{3/2}}$$

24. $x' = t\cos t, \quad x'' = \cos t - t\sin t, \quad y' = t\sin t, \quad y'' = \sin t + t\cos t$

$$k = \left|\frac{t\cos t(\sin t + t\cos t) - t\sin t(\cos t - t\sin t)}{(t^2\cos^2 t + t^2\sin^2 t)^{3/2}}\right| = \frac{t^2}{t^3} = \frac{1}{t}. \quad (t > 0)$$

25. $k = \dfrac{|2/x^3|}{[1+1/x^4]^{3/2}} = \dfrac{2|x^3|}{(x^4+1)^{3/2}}; \quad \text{at } x = \pm 1, \quad \rho = \dfrac{2^{3/2}}{2} = \sqrt{2}$

26. From Exercise 8, $k = \dfrac{1}{(x^2+y^2)^{3/2}}$. At $(\pm 1, 0)$, $k = 1 \implies \rho = 1$

27. We use (13.7.3) and the hint to obtain

$$k = \frac{|ab\sinh^2 t - ab\cosh^2 t|}{\left[a^2\sinh^2 t + b^2\cosh^2 t\right]^{3/2}} = \frac{\left|\frac{a}{b}y^2 - \frac{b}{a}x^2\right|}{\left[\left(\frac{ay}{b}\right)^2 + \left(\frac{bx}{a}\right)^2\right]^{3/2}}$$

$$= \frac{a^3b^3\left|\frac{a}{b}y^2 - \frac{b}{a}x^2\right|}{\left[a^4y^2 + b^4x^2\right]^{3/2}} = \frac{a^4b^4}{\left[a^4y^2 + b^4x^2\right]^{3/2}}.$$

28. $x' = r(1 - \cos t), \quad x'' = r\sin t, \quad y' = r\sin t, \quad y'' = r\cos t$

Highest point when $t = \pi \implies x' = 2r, \quad x'' = 0, \quad y' = 0, \quad y'' = -r$

$$k = \frac{2r^2}{(4r^2)^{3/2}} = \frac{1}{4r}$$

29. By the hint and the fact that $\|\mathbf{T} \times \mathbf{N}\| = 1$,

$$\frac{\|\mathbf{v} \times \mathbf{a}\|}{(ds/dt)^3} = \frac{\left\|\left(\frac{ds}{dt}\mathbf{T}\right) \times \left(\frac{d^2s}{dt^2}\mathbf{T} + k\left(\frac{ds}{dt}\right)^2\mathbf{N}\right)\right\|}{(ds/dt)^3}$$

$$= \frac{\|k\,(ds/dt)^3\,(\mathbf{T} \times \mathbf{N})\|}{(ds/dt)^3} = k.$$

$\mathbf{T} \times \mathbf{T} = \mathbf{0}$

30. $\mathbf{r}'(t) = -2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}, \quad \mathbf{T} = \dfrac{\mathbf{r}'}{\|\mathbf{r}'\|} = \dfrac{-2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}}{\sqrt{29}}$

$\mathbf{T}' = \mathbf{0} \implies k = 0. \quad a_{\mathbf{T}} = \dfrac{d^2s}{dt^2} = \dfrac{d}{dt}\|\mathbf{r}'\| = 0, \quad a_{\mathbf{N}} = k\left(\dfrac{ds}{dt}\right)^2 = 0$

31. $\mathbf{r}'(t) = e^t(\cos t - \sin t)\,\mathbf{i} + e^t(\sin t + \cos t)\,\mathbf{j} + e^t\mathbf{k}$

$\dfrac{ds}{dt} = \|\mathbf{r}'(t)\| = \sqrt{3}\,e^t, \quad \dfrac{d^2s}{dt^2} = \sqrt{3}\,e^t$

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \dfrac{1}{\sqrt{3}}\left[(\cos t - \sin t)\,\mathbf{i} + (\sin t + \cos t)\,\mathbf{j} + \mathbf{k}\right]$

$\mathbf{T}'(t) = \dfrac{1}{\sqrt{3}}\left[(-\sin t - \cos t)\,\mathbf{i} + (\cos t - \sin t)\,\mathbf{j}\right]$

Then, $$k = \frac{\|\mathbf{T}'(t)\|}{ds/dt} = \frac{\sqrt{2/3}}{\sqrt{3}\,e^t} = \frac{1}{3}\sqrt{2}\,e^{-t},$$
$$\mathbf{a_T} = \frac{d^2s}{dt^2} = \sqrt{3}\,e^t, \quad \mathbf{a_N} = k\left(\frac{ds}{dt}\right)^2 = \sqrt{2}\,e^t.$$

32. $\mathbf{r}'(t) = \mathbf{v}(t) = -2\sin t\,\mathbf{i} + \mathbf{j} + \cos t\,\mathbf{k}, \quad \mathbf{a}(t) = -2\cos t\,\mathbf{i} - \sin t\,\mathbf{k}$

$$\frac{ds}{dt} = \|\,\mathbf{v}\,\| = \sqrt{4\sin^2 t + 1 + \cos^2 t} = \sqrt{2 + 3\sin^2 t}; \quad \mathbf{v}\times\mathbf{a} = -\sin t\,\mathbf{i} - 2\mathbf{j} + 2\cos t\,\mathbf{k}$$
$$k = \frac{|\,\mathbf{v}\times\mathbf{a}|}{(ds/dt)^3} = \frac{\sqrt{\sin^2 t + 4 + 4\cos^2 t}}{(2+3\sin^2 t)^{3/2}} = \frac{(5+3\cos^2 t)^{1/2}}{(2+3\sin^2 t)^{3/2}}$$
$$a_{\mathrm{T}} = \frac{d^2s}{dt^2} = \frac{3\sin t\,\cos t}{(2+3\sin^2 t)^{1/2}}; \quad a_{\mathrm{N}} = k\left(\frac{ds}{dt}\right)^2 = \sqrt{\frac{5+3\cos^2 t}{2+3\sin^2 t}}$$

33. $\mathbf{r}'(t) = -2\,\sin 2t\,\mathbf{i} + 2\,\cos 2t\,\mathbf{j}; \quad \frac{ds}{dt} = \|\mathbf{r}'(t)\| = 2, \quad \frac{d^2s}{dt^2} = 0$

$$\mathbf{T}\,(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\,\sin 2t\,\mathbf{i} + \cos 2t\,\mathbf{j}$$
$$\mathbf{T}'(t) = -2\,(\cos 2t\,\mathbf{i} + \sin 2t\,\mathbf{j})$$

Then, $$k = \frac{\|\mathbf{T}'(t)\|}{ds/dt} = \frac{2}{2} = 1,$$
$$\mathbf{a_T} = \frac{d^2s}{dt^2} = 0, \quad \mathbf{a_N} = k\left(\frac{ds}{dt}\right)^2 = 1\cdot 4 = 4.$$

34. $\mathbf{r}'(t) = \mathbf{v}(t) = \mathbf{i} + 2t\,\mathbf{j} + \frac{1}{t}\,\mathbf{k}, \quad \mathbf{a}(t) = 2\,\mathbf{j} - \frac{1}{t^2}\,\mathbf{k}$

$$\frac{ds}{dt} = \|\,\mathbf{v}\,\| = \sqrt{1 + 4t^2 + 1/t^2} = \frac{\sqrt{4t^4+t^2+1}}{t}; \quad \mathbf{v}\times\mathbf{a} = -\frac{4}{t}\,\mathbf{i} + \frac{1}{t^2}\,\mathbf{j} + 2\,\mathbf{k}$$
$$k = \frac{|\,\mathbf{v}\times\mathbf{a}|}{(ds/dt)^3} = \frac{t\sqrt{4t^4+16t^2+1}}{(4t^4+t^2+1)^{3/2}}$$
$$a_{\mathrm{T}} = \frac{d^2s}{dt^2} = \frac{4t^4-1}{t^2\sqrt{4t^4+t^2+1}}; \quad a_{\mathrm{N}} = k\left(\frac{ds}{dt}\right)^2 = \frac{1}{t}\sqrt{\frac{4t^4+16t^2+1}{4t^4+t^2+1}}$$

35. $\mathbf{r}'(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}, \quad \frac{ds}{dt} = \|\mathbf{r}'(t)\| = \sqrt{t^4+t^2+1}, \quad \frac{d^2s}{dt^2} = \frac{2t^3+t}{\sqrt{t^4+t^2+1}}$

$$\mathbf{T}\,(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{t^4+t^2+1}}\left(\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}\right),$$
$$\mathbf{T}'(t) = \frac{1}{\left(t^4+t^2+1\right)^{3/2}}\left[-t\left(2t^2+1\right)\mathbf{i} + \left(1-t^4\right)\mathbf{j} + t\left(t^2+2\right)\mathbf{k}\right].$$

Then, $$k = \frac{\|\mathbf{T}'(t)\|}{ds/dt} = \frac{\sqrt{t^2\left(2t^2+1\right)^2 + \left(1+t^4\right)^2 + t^2\left(t^2+2\right)^2}}{\left(t^4+t^2+1\right)^2}$$
$$= \frac{\sqrt{\left(t^4+4t^2+1\right)\left(t^4+t^2+1\right)}}{\left(t^4+t^2+1\right)^2} = \frac{\sqrt{t^4+4t^2+1}}{\left(t^4+t^2+1\right)^{3/2}},$$
$$\mathbf{a_T} = \frac{d^2s}{dt^2} = \frac{2t^3+t}{\sqrt{t^4+t^2+1}}, \quad \mathbf{a_N} = k\left(\frac{ds}{dt}\right)^2 = \frac{\sqrt{t^4+4t^2+1}}{\sqrt{t^4+t^2+1}}.$$

36. Set $\mathbf{r}(\theta) = \cos\theta f(\theta)\mathbf{i} + \sin\theta f(\theta)\mathbf{j}$

37. By Exercise 36
$$k = \frac{\left|\left(e^{a\theta}\right)^2 + 2\left(ae^{a\theta}\right)^2 - \left(e^{a\theta}\right)\left(a^2e^{a\theta}\right)\right|}{\left[\left(e^{a\theta}\right)^2 + \left(ae^{a\theta}\right)^2\right]^{3/2}} = \frac{e^{-a\theta}}{\sqrt{1+a^2}}.$$

38. $f(\theta) = a\theta, \quad f'(\theta) = a, \quad f''(\theta) = 0 \implies k = \dfrac{a^2\theta^2 + 2a^2}{(a^2\theta^2 + a^2)^{3/2}} = \dfrac{\theta^2+2}{|a|(\theta^2+1)^{3/2}}$

39. By Exercise 36
$$k = \frac{\left|a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a^2(1-\cos\theta)(\cos\theta)\right|}{\left[a^2(1-\cos\theta)^2 + a^2\sin^2\theta\right]^{3/2}}$$
$$= \frac{3a^2(1-\cos\theta)}{\left[2a^2(1-\cos\theta)\right]^{3/2}} = \frac{3ar}{[2ar]^{3/2}} = \frac{3}{2\sqrt{2ar}}.$$

40. By Exercise 36
$$k = \frac{\left|(a\sin 2\theta)^2 + 2(2a\cos 2\theta)^2 - (a\sin 2\theta)(2a\cos 2\theta)\right|}{[(a\sin 2\theta)^2 + (2a\cos 2\theta)^2]^{\frac{3}{2}}}$$
$$= \frac{\left|a^2\sin^2 2\theta + 8a^2\cos^2 2\theta - 3a^2\sin 2\theta\cos 2\theta\right|}{(a^2\sin^2 2\theta + 4a^2\cos^2 2\theta)^{\frac{3}{2}}}$$
$$= \frac{\left|r^2 + 8(a^2-r^2) + 3r\sqrt{a^2-r^2}\right|}{[r^2 + 4(a^2-r^2)]^{\frac{3}{2}}}$$
$$= \frac{\left|8a^2 - 7r^2 + 3r\sqrt{a^2-r^2}\right|}{(4a^2-3r^2)^{\frac{3}{2}}}.$$

41. (a) For $0 \le \theta \le \pi$,
$$s(\theta) = \int_\theta^\pi \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt = \int_\theta^\pi \sqrt{R^2(1-\cos t)^2 + R^2\sin^2 t}\,dt$$
$$= \int_\theta^\pi R\sqrt{2(1-\cos t)}\,dt = \int_\theta^\pi 2R\,\sin\frac{1}{2}t\,dt = 4R\cos\frac{1}{2}\theta = 4R\left|\cos\frac{1}{2}\theta\right|.$$

For $\pi \le \theta \le 2\pi$,
$$s(\theta) = \int_\pi^\theta 2R\,\sin\frac{1}{2}t\,dt = -4R\cos\frac{1}{2}\theta = 4R\left|\cos\frac{1}{2}\theta\right|.$$

(b) $k(\theta) = \dfrac{|x'(\theta)y''(\theta) - y'(\theta)x''(\theta)|}{\{[x'(\theta)]^2 + [y'(\theta)]\}^{3/2}} = \dfrac{|R(1-\cos\theta)R\cos\theta - R\sin\theta\,(R\sin\theta)|}{8R^3\sin^3\frac{1}{2}\theta}.$

This reduces to $k(\theta) = 1/(4R\sin\frac{1}{2}\theta)$ and gives $\rho(\theta) = 4R\,\sin\frac{1}{2}\theta.$

(c) $\rho^2 + s^2 = 16R^2$

42. $\mathbf{r}(\theta) = ae^{c\theta}\cos\theta\mathbf{i} + ae^{c\theta}\sin\theta\mathbf{j}$

$$s(\theta) = \int_0^\theta \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt = \int_0^\theta ae^{ct}\sqrt{1+c^2}\,dt = \frac{a}{c}\sqrt{1+c^2}e^{c\theta}$$

$f(\theta) = ae^{c\theta}, \quad f'(\theta) = ace^{c\theta}, \quad f''(\theta) = ac^2e^{c\theta}.$ by Exercise 36,

$$\rho(\theta) = \frac{1}{k(\theta)} = \frac{\left([f(\theta)]^2 + [f'(\theta)]^2\right)^{3/2}}{|\,[f(\theta)]^2 + 2[f'(\theta)]^2 - f(\theta)f''(\theta)|} = a\sqrt{1+c^2}e^{c\theta}$$

43. Straightforward calculation gives

$$s(\theta) = 4a\left|\cos\tfrac{1}{2}\theta\right| \quad \text{and} \quad \rho(\theta) = \tfrac{4}{3}a\sin\tfrac{1}{2}\theta.$$

Therefore

$$9\rho^2 + s^2 = 16a^2.$$

44. (a) $\dfrac{dB}{ds} = \dfrac{d}{ds}(\mathbf{T}\times\mathbf{N}) = \left(\mathbf{T}\times\dfrac{d\mathbf{N}}{ds}\right) + \left(\dfrac{d\mathbf{T}}{ds}\times\mathbf{N}\right) = \mathbf{T}\times\dfrac{d\mathbf{N}}{ds}.$ Therefore $\dfrac{d\mathbf{B}}{ds}\perp\mathbf{T}.$

Since $\mathbf{B}$ has constant length, $\dfrac{d\mathbf{B}}{ds}\perp\mathbf{B}.$ Being perpendicular to both $\mathbf{T}$ and $\mathbf{B}$, $\dfrac{d\mathbf{B}}{ds}$ is parallel to $\mathbf{N}$ and is therefore a scalar multiple of $\mathbf{N}$.

(b) $$\frac{d\mathbf{N}}{ds} = \frac{d}{ds}(\mathbf{B}\times\mathbf{T}) = \left(\mathbf{B}\times\frac{d\mathbf{T}}{ds}\right) + \left(\frac{d\mathbf{B}}{ds}\times\mathbf{T}\right) = (\mathbf{B}\times k\mathbf{N}) + \tau(\mathbf{N}\times\mathbf{T})$$
$$= -k(\mathbf{N}\times\mathbf{B}) - \tau(\mathbf{T}\times\mathbf{N}) = -k\mathbf{T} - \tau\mathbf{B}$$

(c) We know that $d\mathbf{B}/ds = \tau\mathbf{N}$. It follows that $\|\,d\mathbf{B}/ds\,\| = |\tau|\ \|\,\mathbf{N}\,\| = |\tau|$. Thus $|\tau|$ is the magnitude of the change in direction of $\mathbf{B}$ per unit arc length, or equivalently, the rate per unit arc length at which the curve tends away from the osculating plane.

PROJECT 13.7

1. The system of equations generated by the specified conditions is:

$a+b+c+d=3$

$6a+2b=0$

$729\alpha+81\beta+9\gamma+\delta=-2$

$27a+6b+c=27\alpha+6\beta+\gamma$

$27a+9b+3c+d=7$

$27\alpha+9\beta+3\gamma+\delta=7$

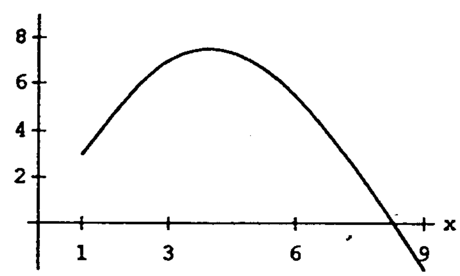
$54\alpha+2\beta=0$

$18a+2b=18\alpha+2\beta$

$a \cong -0.1094$ $\quad b \cong 0.3281$ $\quad c \cong 2.1094$ $\quad d \cong 0.6719$

$\alpha \cong 0.0365$ $\quad \beta \cong -0.9844$ $\quad \gamma \cong 6.0469$ $\quad \delta \cong -3.2656$

2. Clearly p and q are continuous on their respective intervals. The conditions $p(3)=q(3)$, $p'(3)=q'(3)$ and $p''(3)=q''(3)$ imply that F, F', and F'' are continuous on $[1,9]$.



3. $k=\dfrac{|y''|}{[1+(y')^2]^{3/2}}=\dfrac{Cn(n-1)x^{n-2}}{[1+(cnx^{n-1})^2]^{3/2}}$

At $x=0$, $k=0$ as desired. At $x=1$, want $k=\dfrac{Cn(n-1)}{(1+C^2n^2)^{3/2}}=\dfrac{96}{125}$

$y=\dfrac{1}{4}x^3$ works.

CHAPTER 14

SECTION 14.1

1. dom (f) = the first and third quadrants, including the axes; ran $(f) = [0, \infty)$

2. dom (f) =the set of all points (x, y) with $xy \leq 1$; the two branches of the hyperbola $xy = 1$ and all points in between; ran $(f) = [0, \infty)$

3. dom (f) = the set of all points (x, y) except those on the line $y = -x$; ran $(f) = (-\infty, 0) \cup (0, \infty)$

4. dom (f) =the set of all points (x, y) other than the origin; ran $(f) = (0, \infty)$

5. dom (f) = the entire plane; ran $(f) = (-1, 1)$ since

$$\frac{e^x - e^y}{e^x + e^y} = \frac{e^x + e^y - 2e^y}{e^x + e^y} = 1 - \frac{2}{e^{x-y} + 1}$$

and the last quotient takes on all values between 0 and 2.

6. dom (f) = the set of all points (x, y) other than the origin; ran $(f) = [0, 1]$

7. dom (f) = the first and third quadrants, excluding the axes; ran $(f) = (-\infty, \infty)$

8. dom (f) =the set of all points (x, y) between the branches of the hyperbola $xy = 1$; ran $(f) = (-\infty, \infty)$

9. dom (f) = the set of all points (x, y) with $x^2 < y$ —in other words, the set of all points of the plane above the parabola $y = x^2$; ran $(f) = (0, \infty)$

10. dom (f) = the set of all points (x, y) with $-3 \leq x \leq 3$, $-1 \leq y \leq 1$ (a rectangle); ran $(f) = [0, 3]$

11. dom (f) = the set of all points (x, y) with $-3 \leq x \leq 3$, $-2 \leq y \leq 2$ (a rectangle); ran $(f) = [-2, 3]$

12. dom (f) = all of space; ran $(f) = [-3, 3]$

13. dom (f) = the set of all points (x, y, z) not on the plane $x + y + z = 0$; ran $(f) = \{-1, 1\}$

14. dom (f) = the set of all points (x, y, z) with $x^2 \neq y^2$ —that is, all points of space except for those which lie on the plane $x - y = 0$ or on the plane $x + y = 0$; ran $(f) = (-\infty, \infty)$

15. dom (f) = the set of all points (x, y, z) with $|y| < |x|$; ran $(f) = (-\infty, 0]$

16. dom (f) =the set of all points (x, y, z) not on the plane $x - y = 0$; ran $(f) = (-\infty, \infty)$

17. dom (f) = the set of all points (x, y) with $x^2 + y^2 < 9$ —in other words, the set of all points of the plane inside the circle $x^2 + y^2 = 9$; ran $(f) = [2/3, \infty)$

18. dom (f) = all of space; ran $(f) = [0, \infty)$

19. dom (f) = the set of all points (x, y, z) with $x + 2y + 3z > 0$ — in other words, the set of all points in space that lie on the same side of the plane $x + 2y + 3z = 0$ as the point $(1, 1, 1)$; ran $(f) = (-\infty, \infty)$

20. dom (f) = the set of all points (x, y, z) with $x^2 + y^2 + z^2 \leq 4$— in other words, the set of all points inside and on the sphere $x^2 + y^2 + z^2 = 4$; ran $(f) = [1, e^2]$

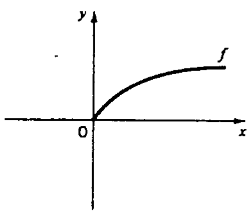
21. dom (f) = all of space; ran $(f) = (0, \infty)$

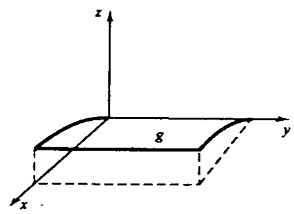
22. dom (f) = the set of all points (x, y, z) with $-1 \leq x \leq 1,\ -2 \leq y \leq 2,\ -3 \leq z \leq 3$ (a rectangular solid); ran $(f) = [0, 3]$

23. dom $(f) = \{x : x \geq 0\}$; range $(f) = [0, \infty)$

dom $(g) = \{(x, y) : x \geq 0,\ y \text{ real}\}$; range $(g) = [0, \infty)$

dom $(h) = \{(x, y, z) : x \geq 0,\ y, z \text{ real}\}$; range $(h) = [0, \infty)$





24. dom (f) = the entire plane, ran $(f) = [-1, 1]$

dom (g) = all of space, ran $(g) = [-1, 1]$

25. $$\lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h} = \lim_{h\to 0} \frac{2(x+h)^2 - y - (2x^2 - y)}{h} = \lim_{h\to 0} \frac{4xh + 2h^2}{h} = 4x$$

$$\lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h} = \lim_{h\to 0} \frac{2x^2 - (y+h) - (2x^2 - y)}{h} = -1$$

26. $$\lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h} = \lim_{h\to 0} \frac{xy + hy + 2y - (xy + 2y)}{h} = \lim_{h\to 0} y = y.$$

$$\lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h} = \lim_{h\to 0} \frac{xy + xh + 2y + 2h - (xy + 2y)}{h} = \lim_{h\to 0} (x + 2) = x + 2$$

27. $$\lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h} = \lim_{h\to 0} \frac{3(x+h) - (x+h)y + 2y^2 - (3x - xy + 2y^2)}{h} = \lim_{h\to 0} \frac{3h - hy}{h} = 3 - y$$

$$\begin{aligned} \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h} &= \lim_{h\to 0} \frac{3x - x(y+h) + 2(y+h)^2 - (3x - xy + 2y^2)}{h} \\ &= \lim_{h\to 0} \frac{-xh + 4yh + 2h^2}{h} = -x + 4y \end{aligned}$$

28. $$\lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h} = \lim_{h\to 0} \frac{x \sin y + h \sin y - x \sin y}{h} = \lim_{h\to 0} \sin y = \sin y.$$

$$\lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h} = \lim_{h\to 0} \frac{x \sin(y+h) - x \sin y}{h} = x \lim_{h\to 0} \frac{\sin(y+h) - \sin y}{h} = x \cos y$$

29. $$\begin{aligned} \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h} &= \lim_{h\to 0} \frac{\cos[(x+h)y] - \cos[xy]}{h} \\ &= \lim_{h\to 0} \frac{\cos[xy]\cos[hy] - \sin[xy]\sin[hy] - \cos[xy]}{h} \\ &= \cos[xy] \left(\lim_{h\to 0} \frac{\cos[hy] - 1}{h} \right) - \sin[xy] \lim_{h\to 0} \frac{\sin hy}{h} \\ &= y \cos[xy] \left(\lim_{h\to 0} \frac{\cos[hy] - 1}{hy} \right) - y \sin[xy] \lim_{h\to 0} \frac{\sin hy}{hy} \\ &= -y \sin[xy] \end{aligned}$$

and

$$\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}=\lim_{h\to 0}\frac{\cos[x(y+h)]-\cos[xy]}{h}$$
$$=\lim_{h\to 0}\frac{\cos[xy]\cos[hx]-\sin[xy]\sin[hx]-\cos[xy]}{h}$$
$$=\cos[xy]\left(\lim_{h\to 0}\frac{\cos[hx]-1}{h}\right)-\sin[xy]\lim_{h\to 0}\frac{\sin hx}{h}$$
$$=x\,\cos[xy]\left(\lim_{h\to 0}\frac{\cos[hx]-1}{hx}\right)-x\,\sin[xy]\lim_{h\to 0}\frac{\sin hx}{hx}$$
$$=-x\,\sin[xy]$$

30. $\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}=\lim_{h\to 0}\frac{(x^2+2xh+h^2)e^y-x^2e^y}{h}=\lim_{h\to 0}(2x+h)e^y=2xe^y.$

$\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}=\lim_{h\to 0}\frac{x^2e^{y+h}-x^2e^y}{h}=x^2\lim_{h\to 0}\frac{e^{y+h}-e^y}{h}=x^2e^y.$

31. (a) $f(x,y)=x^2y$ (b) $f(x,y)=\pi x^2y$ (c) $f(x,y)=|\,2\,\mathbf{i}\times(x\,\mathbf{i}+y\,\mathbf{j})\,|=2|\,y\,|$

32. (a) $f(x,y,z)=xy+2xz+2yz$

(b) $f(x,y,z)=\cos^{-1}\frac{(\mathbf{i}+\mathbf{j})\cdot(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})}{\|\mathbf{i}+\mathbf{j}\|\|x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\|}=\cos^{-1}\frac{x+y}{\sqrt{2}\sqrt{x^2+y^2+z^2}}$

(c) $f(x,y,z)=[\mathbf{i}\times(\mathbf{i}+\mathbf{j})]\cdot(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=z$

33. Surface area: $S=2lw+2lh+2hw=20\implies w=\frac{20-2lh}{2l+2h}=\frac{10-lh}{l+h}$

Volume: $V=lwh=\frac{lh(10-lh)}{l+h}$

34. $wlh=12\implies h=\frac{12}{wl};\quad C=4wl+2(2wh+2lh)=4wl+\frac{48}{l}+\frac{48}{w}$

35. $V=\pi r^2h+\frac{4}{3}\pi r^3$

36. $A=\frac{1}{2}[2(12-2x)+2x\cos\theta]\cdot x\sin\theta=(12-2x+x\cos\theta)\cdot x\sin\theta$

SECTION 14.2

1. a quadric cone

2. an ellipsoid

3. a parabolic cylinder

4. a hyperbolic paraboloid

5. a hyperboloid of one sheet

6. an elliptic cylinder

7. a sphere

8. a hyperboloid of two sheets

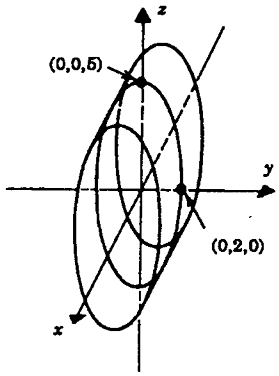
9. an elliptic paraboloid

10. a hyperbolic cylinder

11. a hyperbolic paraboloid

12. an elliptic paraboloid

13.



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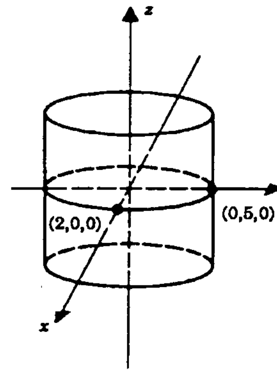

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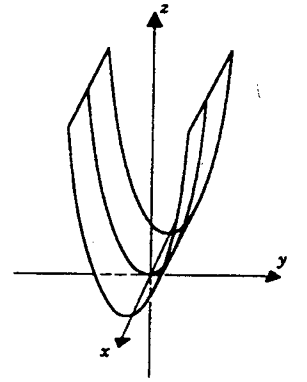

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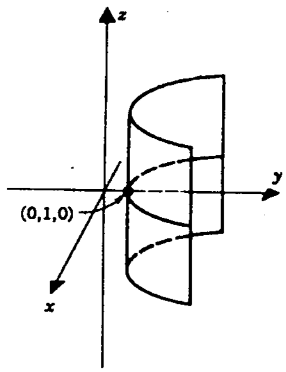

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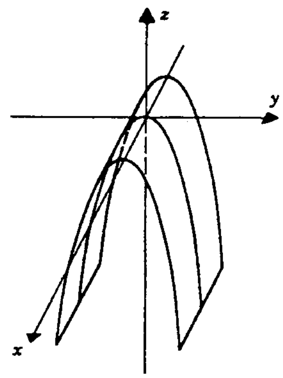

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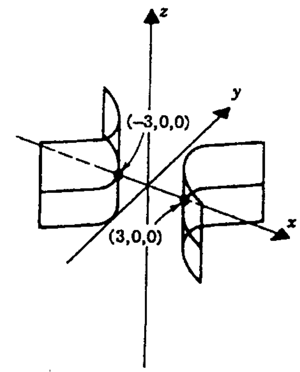

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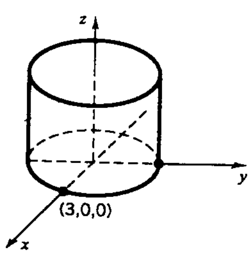

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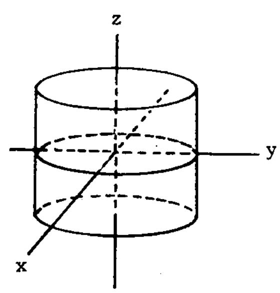

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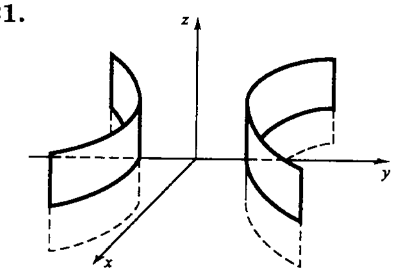

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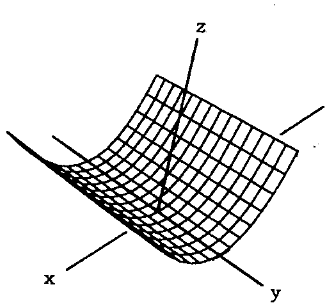

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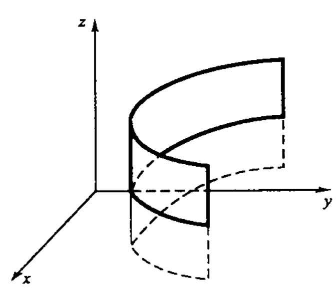

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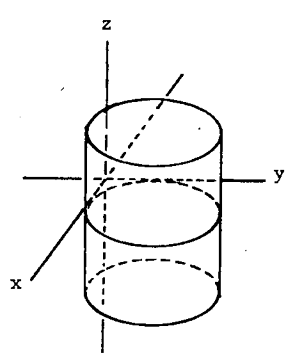


25. elliptic paraboloid
xy-trace: the origin
xz-trace: the parabola $x^2 = 4z$
yz-trace: the parabola $y^2 = 9z$
surface has the form of Figure 14.2.5

26. ellipsoid
xy-trace: the ellipse $9x^2 + 4y^2 = 36$
xz-trace: the ellipse $x^2 + 4z^2 = 4$
yz-trace: the ellipse $y^2 + 9z^2 = 9$
surface has the form of Figure 14.2.1

27. quadric cone
xy-trace: the origin
xz-trace: the lines $x = \pm 2z$
yz-trace: the lines $y = \pm 3z$
surface has the form of Figure 14.2.4

28. hyperboloid of one sheet
xy-trace: the ellipse $9x^2 + 4y^2 = 36$
xz-trace: the hyperbola $x^2 - 4z^2 = 4$
yz-trace: the hyperbola $y^2 - 9z^2 = 9$
surface has the form of Figure 14.2.2

29. hyperboloid of two sheets
xy-trace: none
xz-trace: the hyperbola $4z^2 - x^2 = 4$
yz-trace: the hyperbola $9z^2 - y^2 = 9$
surface has the form of Figure 14.2.3

30. hyperbolic paraboloid
xy-trace: the lines $y = \pm \frac{3}{2}x$
xz-trace: the parabola $x^2 = 4z$
yz-trace: the parabola $y^2 = -9z$
surface has the form Figure 14.2.6

31. hyperboloid of two sheets
xy-trace: the hyperbola $4x^2 - 9y^2 = 36$
xz-trace: the hyperbola $x^2 - 4z^2 = 4$
yz-trace: none
see Figure 14.2.3

32. hyperboloid of one sheet
xy-trace: the hyperbola $x^2 - 4y^2 = 4$
xz-trace: the ellipse $9x^2 + 4z^2 = 36$
yz-trace: the hyperbola $z^2 - 9y^2 = 9$
surface has the form of Figure 14.2.2, rotated 90° about the x-axis.

33. elliptic paraboloid
xy-trace: the origin
xz-trace: the parabola $x^2 = 4z$
yz-trace: the parabola $y^2 = 9z$
surface has the form of Figure 14.2.5

34. quadric cone
xy-trace: the lines $x = \pm 2y$
xz-trace: the origin
yz-trace: the lines $z = \pm 3y$
surface has the form of Figure 14.2.4, rotated 90° about the x-axis

35. hyperboloid of two sheets
xy-trace: the hyperbola $9y^2 - 4x^2 = 36$
xz-trace: none
yz-trace: the hyperbola $y^2 - 4z^2 = 4y$
see Figure 14.2.3

36. elliptic paraboloid
xy-trace: the parabola $y^2 = 4x$
xz-trace: the parabola $z^2 = 9x$
yz-trace: the origin
surface has the form of Figure 14.2.5, but opening along the positive x-axis.

37. paraboloid of revolution
xy-trace: the origin
xz-trace: the parabola $x^2 = 4z$
yz-trace: the parabola $y^2 = 4z$
surface has the form of Figure 14.2.5

38. ellipsoid
xy-trace: the ellipse $4x^2 + y^2 = 4$
xz-trace: the ellipse $9x^2 + z^2 = 9$
yz-trace: the ellipse $9y^2 + 4z^2 = 36$
the surface has the form of Figure 14.2.1, rotated 90° about the x-axis.

39. (a) an elliptic paraboloid (vertex down if A and B are both positive, vertex up if A and B are both negative)

(b) a hyperbolic paraboloid

(c) the xy-plane if A and B are both zero; otherwise a parabolic cylinder

40. The xz-plane and all planes parallel to the xy-plane.

41. $x^2 + y^2 - 4z = 0$ (paraboloid of revolution)

42. $c^2x^2 + c^2y^2 - b^2z^2 = b^2c^2$ (hyperboloid of revolution, one sheet)

43. (a) a circle

(b) (i) $\sqrt{x^2 + y^2} = -3z$ (ii) $\sqrt{x^2 + z^2} = \frac{1}{3}y$

44. (a) the ellipse $b^2x^2 + y^2 = b^2$

(b) ellipse approaches parallel lines $x = \pm 1$ in the plane $z = 1$

(c) paraboloid approaches parabolic cylinder $z = x^2$

45. $x + 2y + 3\left(\dfrac{x + y - 6}{2}\right) = 6$ or $5x + 7y = 30$, a line

46. $3x + y - 2(4 - x + 2y) = 1$, or $5x - 3y = 9$, a line

47. $\left.\begin{array}{r} x^2 + y^2 + (z-1)^2 = \frac{3}{2} \\ x^2 + y^2 - z^2 = 1 \end{array}\right\} (z^2 + 1) + (z - 1)^2 = \dfrac{3}{2}$; $(2z - 1)^2 = 0$, $z = \dfrac{1}{2}$ so that $x^2 + y^2 = \dfrac{5}{4}$

48. $z^2 + (z - 2)^2 = 2 \implies 2(z - 1)^2 = 0 \implies z = 1 \implies x^2 + y^2 = 1$, a circle.

49. $x^2 + y^2 + (x^2 + 3y^2) = 4$ or $x^2 + 2y^2 = 2$, an ellipse

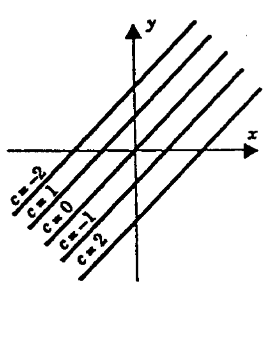
50. $y^2 + (x^2 + 3y^2) = 4 \implies x^2 + 4y^2 = 4$, an ellipse.

51. $x^2 + y^2 = (2 - y)^2$ or $x^2 = -4(y - 1)$, a parabola

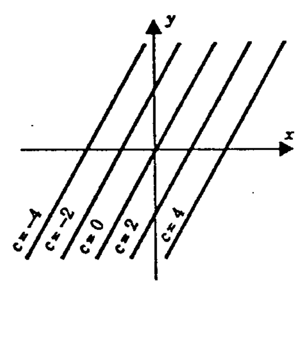
52. $x^2 + y^2 = \left(\dfrac{2 - y}{2}\right)^2 \implies 4x^2 + 3y^2 + 4y = 4$, an ellipse.

SECTION 14.3

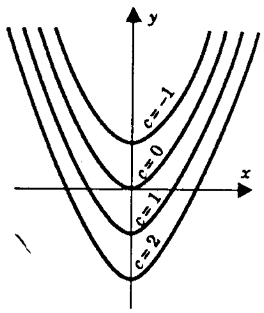
1. lines of slope 1: $y = x - c$



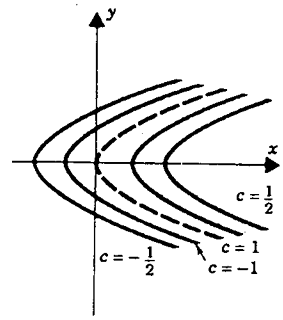
2. lines of slope 2 : $y = 2x - c$



3. parabolas: $y = x^2 - c$

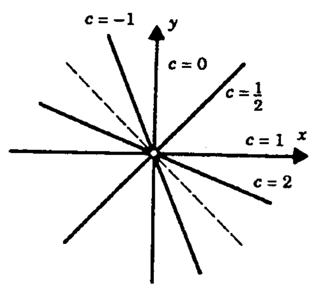


4. parabolas: $x - y^2 = \dfrac{1}{c}$



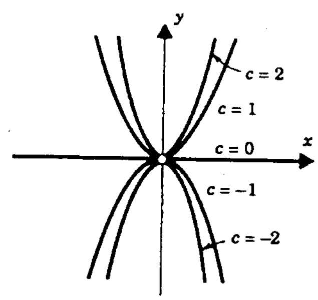
5. the y-axis and the lines $y = \left(\dfrac{1-c}{c}\right) x$

with the origin omitted throughout

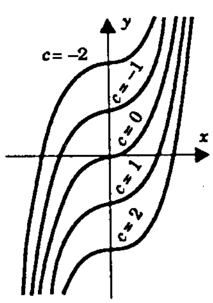


6. the x-axis and the parabolas $y = cx^2$

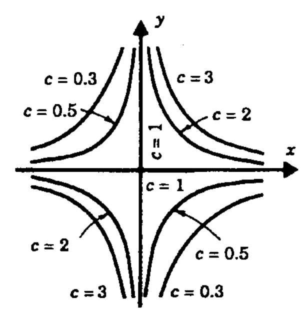
with the origin omitted throughout



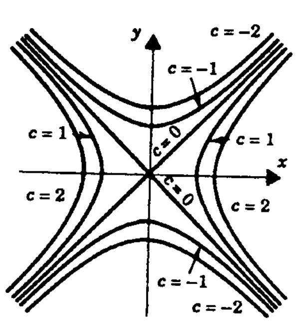
7. the cubics $y = x^3 - c$



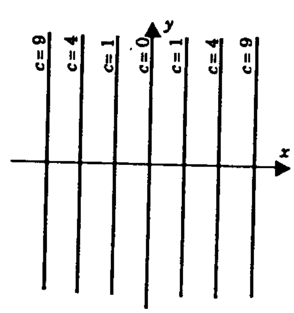
8. the coordinate axes and the hyperbolas $xy = \ln c$



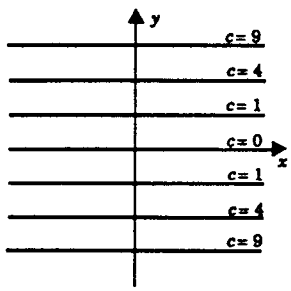
9. the lines $y = \pm x$ and the hyperbolas $x^2 - y^2 = c$



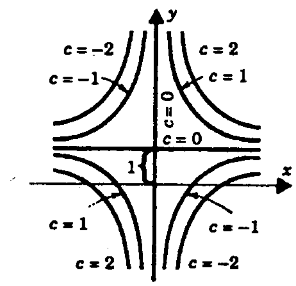
10. pairs of vertical lines $x = \pm\sqrt{c}$ and the y-axis



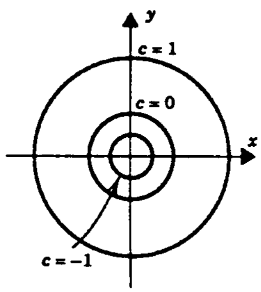
11. pairs of horizontal lines $y = \pm\sqrt{c}$ and the x-axis



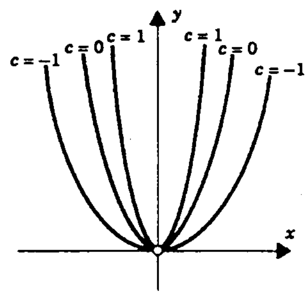
12. the lines $x = 0$, $y = 1$ and the hyperbolas $y = \dfrac{c}{x} + 1$



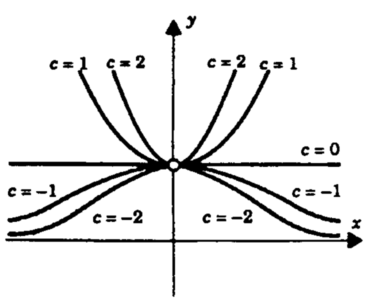
13. the circles $x^2 + y^2 = e^c$, c real



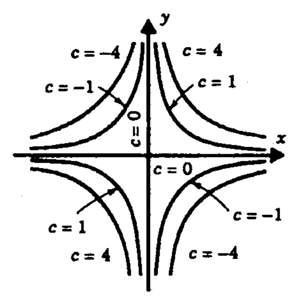
14. the parabolas $y = e^c x^2$ with the origin omitted throughout



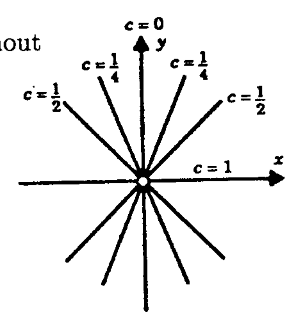
15. the curves $y = e^{cx^2}$ with the point $(0, 1)$ omitted



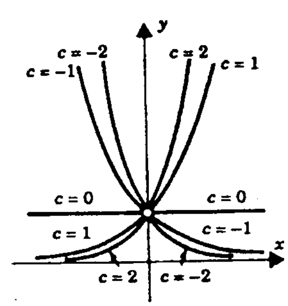
16. the coordinate axes and pairs of hyperbolas $xy = \pm\sqrt{c}$



17. the coordinate axes and pairs of lines $y = \pm\dfrac{\sqrt{1-c}}{\sqrt{c}}\,x$, with the origin omitted throughout



18. the curves $y = e^{cx}$ with the point $(0, 1)$ omitted



19. $x + 2y + 3z = 0$, plane through the origin

20. circular cylinder $x^2 + y^2 = 4$ (Figure 14.2.8)

21. $z = \sqrt{x^2 + y^2}$, the upper nappe of the circular cone $z^2 = x^2 + y^2$ (Figure 14.2.4)

22. ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{6} + \dfrac{z^2}{9} = 1$ (Figure 14.2.1)

23. the elliptic paraboloid $\dfrac{x^2}{(1/2)^2} + \dfrac{y^2}{(1/3)^2} = 72z$ (Figure 14.2.5)

24. hyperboloid of two sheets $\dfrac{x^2}{(1/6)^2} + \dfrac{y^2}{(1/3)^2} - z^2 = -1$ (Figure 14.2.3)

25. (i) hyperboloid of two sheets (Figure 14.2.3)
(ii) circular cone (Figure 14.2.4)
(iii) hyperboloid of one sheet (Figure 14.2.2)

26. (i) hyperboloid of two sheets
(ii) quadric cone
(iii) hyperboloid of one sheet

27. The level curves of f are: $1 - 4x^2 - y^2 = c$. Substituting $P(0,1)$ into this equation, we have

$$1 - 4(0)^2 - (1)^2 = c \quad \Longrightarrow \quad c = 0$$

The level curve that contains P is: $1 - 4x^2 - y^2 = 0$, or $4x^2 + y^2 = 1$.

28. $(x^2 + y^2)e^{xy} = 1$

29. The level curves of f are: $y^2 \tan^{-1} x = c$. Substituting $P(1,2)$ into this equation, we have

$$4^2 \tan^{-1} 1 = c \quad \Longrightarrow \quad c = \pi$$

The level curve that contains P is: $y^2 \tan^{-1} x = \pi$.

30. $(x^2 + y)\ln(2 - x + e^y) = 5$

31. The level surfaces of f are: $x^2 + 2y^2 - 2xyz = c$. Substituting $P(-1,2,1)$ into this equation, we have

$$(-1)^2 + 2(2)^2 - 2(-1)(2)(1) = c \quad \Longrightarrow \quad c = 13$$

The level surface that contains P is: $x^2 + 2y^2 - 2xyz = 13$.

32. $\sqrt{x^2 + y^2} - \ln z = 4$

33. $\dfrac{GmM}{x^2 + y^2 + z^2} = c \quad \Longrightarrow \quad x^2 + y^2 + z^2 = \dfrac{GmM}{c}$; the surfaces of constant gravitational force are concentric spheres.

34. Circular cylinders around the positive y-axis: $x^2 + z^2 = \dfrac{k^2}{c^2}$

35. (a) $T(x,y) = \dfrac{k}{x^2 + y^2}$, where k is a constant.

(b) $\dfrac{k}{x^2+y^2} = c \implies x^2+y^2 = \dfrac{k}{c}$; the level curves are concentric circles.

(c) $T(1,2) = \dfrac{k}{1^2+2^2} = 50 \implies k = 250 \implies T(x,y) = \dfrac{250}{x^2+y^2}$

Now, $T(3,4) = \dfrac{250}{3^2+4^2} = 10°$

36. $x^2+y^2 = r^2 - \dfrac{k^2}{c^2}$; circles about the origin, for $|c| > \dfrac{k}{r}$.

37. $f(x,y) = y^2 - y^3$; F

38. D.

39. $f(x,y) = \cos\sqrt{x^2+y^2}$; A

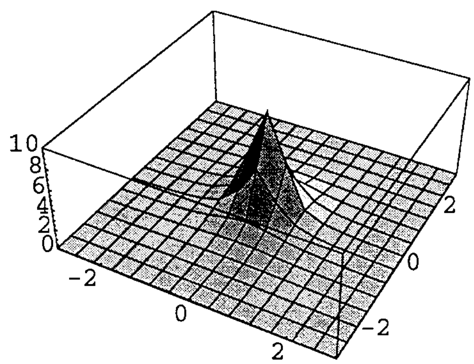
40. B.

41. $f(x,y) = xye^{-(x^2+y^2)/2}$; E

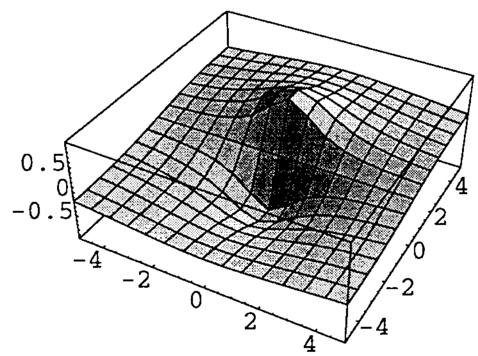
42. C.

PROJECT 14.3

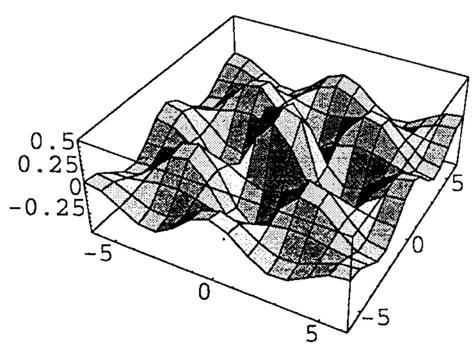
1.



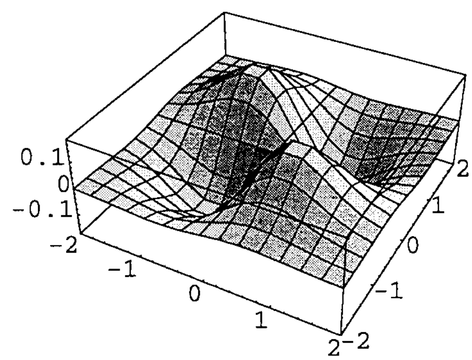
2.



3.



4.



SECTION 14.4

1. $\dfrac{\partial f}{\partial x} = 6x - y, \quad \dfrac{\partial f}{\partial y} = 1 - x$

2. $\dfrac{\partial g}{\partial x} = 2xe^{-y}, \quad \dfrac{\partial g}{\partial y} = -x^2e^{-y}$

3. $\dfrac{\partial \rho}{\partial \phi} = \cos\phi\cos\theta, \quad \dfrac{\partial \rho}{\partial \theta} = -\sin\phi\sin\theta$

4. $\dfrac{\partial \rho}{\partial \theta} = 2\sin(\theta-\phi)\cos(\theta-\phi), \quad \dfrac{\partial \rho}{\partial \phi} = -2\sin(\theta-\phi)\cos(\theta-\phi)$

5. $\dfrac{\partial f}{\partial x} = e^{x-y} + e^{y-x}, \quad \dfrac{\partial f}{\partial y} = -e^{x-y} - e^{y-x}$

6. $\dfrac{\partial z}{\partial x} = \dfrac{x}{\sqrt{x^2-3y}}, \quad \dfrac{\partial z}{\partial y} = \dfrac{-3}{2\sqrt{x^2-3y}}$

7. $\dfrac{\partial g}{\partial x} = \dfrac{(AD-BC)y}{(Cx+Dy)^2}, \quad \dfrac{\partial g}{\partial y} = \dfrac{(BC-AD)x}{(Cx+Dy)^2}$

8. $\dfrac{\partial u}{\partial x} = \dfrac{-e^z}{x^2y^2}, \quad \dfrac{\partial u}{\partial y} = \dfrac{-2e^z}{xy^3}, \quad \dfrac{\partial u}{\partial z} = \dfrac{e^z}{xy^2}$

9. $\dfrac{\partial u}{\partial x} = y+z, \quad \dfrac{\partial u}{\partial y} = x+z, \quad \dfrac{\partial u}{\partial z} = x+y$

10. $\dfrac{\partial z}{\partial x} = 2Ax + By, \quad \dfrac{\partial z}{\partial y} = Bx + 2Cy$

11. $\dfrac{\partial f}{\partial x} = z\cos(x-y), \quad \dfrac{\partial f}{\partial y} = -z\cos(x-y), \quad \dfrac{\partial f}{\partial z} = \sin(x-y)$

12. $\dfrac{\partial g}{\partial u} = \dfrac{2u}{u^2+vw-w^2}, \quad \dfrac{\partial g}{\partial v} = \dfrac{w}{u^2+vw-w^2}, \quad \dfrac{\partial g}{\partial w} = \dfrac{v-2w}{u^2+vw-w^2}$

13. $\dfrac{\partial \rho}{\partial \theta} = e^{\theta+\phi}[\cos(\theta-\phi) - \sin(\theta-\phi)], \quad \dfrac{\partial \rho}{\partial \phi} = e^{\theta+\phi}[\cos(\theta-\phi) + \sin(\theta-\phi)]$

14. $\dfrac{\partial f}{\partial x} = (x+y)\cos(x-y) + \sin(x-y), \quad \dfrac{\partial f}{\partial y} = -(x+y)\cos(x-y) + \sin(x-y)$

15. $\dfrac{\partial f}{\partial x} = 2xy\sec xy + x^2y(\sec xy)(\tan xy)y = 2xy\sec xy + x^2y^2\sec xy\,\tan xy$

$\dfrac{\partial f}{\partial y} = x^2\sec xy + x^2y(\sec xy)(\tan xy)x = x^2\sec xy + x^3y\sec xy\,\tan xy$

16. $\dfrac{\partial g}{\partial x} = \dfrac{2}{1+(2x+y)^2}, \quad \dfrac{\partial g}{\partial y} = \dfrac{1}{1+(2x+y)^2}$

17. $\dfrac{\partial h}{\partial x} = \dfrac{x^2+y^2-x(2x)}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$

$\dfrac{\partial h}{\partial y} = \dfrac{-2xy}{(x^2+y^2)^2}$

18. $\dfrac{\partial z}{\partial x} = \dfrac{x}{x^2+y^2}, \quad \dfrac{\partial z}{\partial y} = \dfrac{y}{x^2+y^2}$

19. $\dfrac{\partial f}{\partial x} = \dfrac{(y\,\cos x)\sin y - (x\,\sin y)(-y\,\sin x)}{(y\,\cos x)^2} = \dfrac{\sin y(\cos x + x\,\sin x)}{y\,\cos^2 x}$

$\dfrac{\partial f}{\partial y} = \dfrac{(y\,\cos x)(x\,\cos y) - (x\,\sin y)\cos x}{(y\,\cos x)^2} = \dfrac{x(y\,\cos y - \sin y)}{y^2\cos x}$

20. $\dfrac{\partial f}{\partial x} = e^{xy}(y\sin xz + z\cos xz), \quad \dfrac{\partial f}{\partial y} = xe^{xy}\sin xz, \quad \dfrac{\partial f}{\partial z} = xe^{xy}\cos xz$

21. $\dfrac{\partial h}{\partial x} = 2f(x)f'(x)g(y), \quad \dfrac{\partial h}{\partial y} = [f(x)]^2g'(y)$

22. $\dfrac{\partial h}{\partial x} = f'(x)g(y)e^{f(x)g(x)}, \quad \dfrac{\partial h}{\partial y} = f(x)g'(y)e^{f(x)g(y)}$

23. $\dfrac{\partial f}{\partial x} = (y^2 \ln z)z^{xy^2}, \quad \dfrac{\partial f}{\partial y} = (2xy \ln z)z^{xy^2}, \quad \dfrac{\partial f}{\partial z} = xy^2 z^{xy^2-1}$

24. $\dfrac{\partial h}{\partial x} = 2[f(x,y)]^3 g(x,z)\dfrac{\partial g}{\partial x} + 3[f(x,y)]^2[g(x,z)]^2\dfrac{\partial f}{\partial x}$

$\dfrac{\partial h}{\partial y} = 3[f(x,y)]^2[g(x,z)]^2\dfrac{\partial f}{\partial y}, \quad \dfrac{\partial h}{\partial z} = 2[f(x,y)]^3 g(x,y)\dfrac{\partial g}{\partial z}$

25. $\dfrac{\partial h}{\partial r} = 2re^{2t}\cos(\theta - t) \qquad \dfrac{\partial h}{\partial \theta} = -r^2e^{2t}\sin(\theta - t)$

$\dfrac{\partial h}{\partial t} = 2r^2e^{2t}\cos(\theta - t) + r^2e^{2t}\sin(\theta - t) = r^2e^{2t}[2\cos(\theta - t) + \sin(\theta - t)]$

26. $\dfrac{\partial u}{\partial x} = \dfrac{1}{x} - yze^{xz}, \quad \dfrac{\partial u}{\partial y} = -\dfrac{1}{y} - e^{xz}, \quad \dfrac{\partial u}{\partial z} = -xye^{xz}$

27. $\dfrac{\partial f}{\partial x} = z\dfrac{1}{1+(y/x)^2}\left(\dfrac{-y}{x^2}\right) = -\dfrac{yz}{x^2+y^2}$

$\dfrac{\partial f}{\partial y} = z\dfrac{1}{1+(y/x)^2}\left(\dfrac{1}{x}\right) = \dfrac{xz}{x^2+y^2}$

$\dfrac{\partial f}{\partial x} = \tan^{-1}(y/x)$

28. $\dfrac{\partial w}{\partial x} = y\sin z - yz\cos x, \quad \dfrac{\partial w}{\partial y} = x\sin z - z\sin x, \quad \dfrac{\partial w}{\partial z} = xy\cos z - y\sin x$

29. $f_x(x,y) = e^x \ln y, \quad f_x(0,e) = 1; \quad f_y(x,y) = \dfrac{1}{y}e^x, \quad f_y(0,e) = e^{-1}$

30. $g_x = e^{-x}[-\sin(x+2y) + \cos(x+2y)], \quad g_x(0, \frac{1}{4}\pi) = -1$

$g_y = 2e^{-x}\cos(x+2y), \quad g_y(0, \frac{1}{4}\pi) = 0$

31. $f_x(x,y) = \dfrac{y}{(x+y)^2}, \quad f_x(1,2) = \dfrac{2}{9}; \quad f_y(x,y) = \dfrac{-x}{(x+y)^2}, \quad f_y(1,2) = -\dfrac{1}{9}$

32. $g_x = \dfrac{-y}{(x+y^2)^2}; \quad g_x(1,2) = -\dfrac{2}{25}$

$g_y = \dfrac{x-y^2}{(x+y^2)^2}; \quad g_y(1,2) = -\dfrac{3}{25}$

33. $f_x(x,y) = \lim_{h\to 0}\dfrac{(x+h)^2y - x^2y}{h} = \lim_{h\to 0} y\left(\dfrac{2xh+h^2}{h}\right) = y\lim_{h\to 0}(2x+h) = 2xy$

$f_x(x,y) = \lim_{h\to 0}\dfrac{x^2(y+h) - x^2y}{h} = \lim_{h\to 0}\dfrac{x^2h}{h} = \lim_{h\to 0} x^2 = x^2$

34. $f_x(x,y) = 0, \quad f_y(x,y) = 2y$

35. $$f_x(x,y)=\lim_{h\to 0}\frac{\ln\left(y\,(x+h)^2\right)-\ln x^2y}{h}=\lim_{h\to 0}\frac{\ln y+2\ln(x+h)-2\ln x-\ln y}{h}$$

$$=2\lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h}=2\frac{d}{dx}(\ln x)=\frac{2}{x}$$

$$f_y(x,y)=\lim_{h\to 0}\frac{\ln\left(x^2\,(y+h)\right)-\ln x^2y}{h}=\lim_{h\to 0}\frac{\ln x^2+\ln(y+h)-\ln x^2-\ln y}{h}$$

$$=\lim_{h\to 0}\frac{\ln(y+h)-\ln y}{h}=\frac{d}{dy}(\ln y)=\frac{1}{y}$$

36. $f_x(x,y)=-(x+4y)^{-2},\quad f_y(x,y)=-4(x+4y)^{-2}$

37. $$f_x(x,y)=\lim_{h\to 0}\frac{1}{h}\left\{\frac{1}{(x+h)-y}-\frac{1}{x-y}\right\}=\lim_{h\to 0}\frac{1}{h}\left\{\frac{-h}{(x+h-y)\,(x-y)}\right\}$$

$$=\lim_{h\to 0}\frac{-1}{(x+h-y)\,(x-y)}=\frac{-1}{(x-y)^2}$$

$$f_x(x,y)=\lim_{h\to 0}\frac{1}{h}\left\{\frac{1}{x-(y+h)}-\frac{1}{x-y}\right\}=\lim_{h\to 0}\frac{1}{h}\left\{\frac{h}{(x-y-h)\,(x-y)}\right\}$$

$$=\lim_{h\to 0}\frac{1}{(x-y-h)\,(x-y)}=\frac{1}{(x-y)^2}$$

38. $f_x(x,y)=2e^{2x+3y},\quad f_y(x,y)=3e^{2x+3y}$

39. $$f_x(x,y,z)=\lim_{h\to 0}\frac{(x+h)y^2z-xy^2z}{h}=\lim_{h\to 0}y^2z=y^2z$$

$$f_y(x,y,z)=\lim_{h\to 0}\frac{x(y+h)^2z-xy^2z}{h}=\lim_{h\to 0}\frac{xz(2yh+h^2)}{h}$$

$$=\lim_{h\to 0}xz(2y+h)=2xyz$$

$$f_z(x,y,z)=\lim_{h\to 0}\frac{xy^2\,(x+h)-xy^2z}{h}=\lim_{h\to 0}xy^2=xy^2$$

40. $f_x(x,y,z)=\dfrac{2xy}{z},\quad f_y(x,y,z)=\dfrac{x^2}{z},\quad f_z(x,y,z)=-\dfrac{x^2y}{z}$

41. (b) The slope of the tangent line to C at the point $P\,(x_0,y_0,f(x_0,y_0))$ is $f_y(x_0,y_0)$

Thus, equations for the tangent line are:

$$y=y_0,\quad z-z_0=f_y(x_0,y_0)(y-y_0)$$

42. $f_x(x,y)=2x,\quad f_x(1,3)=2,\quad$ equations for the tangent line are: $y=3,\quad z-10=2(x-1).$

43. Let $z = f(x,y) = x^2 + y^2$. Then $f(2,1) = 5$, $f_y(x,y) = 2y$ and $f_y(2,1) = 2$;

equations for the tangent line are: $x = 2$, $z - 5 = 2(y - 1)$

44. $f(x,y) = \dfrac{x^2}{y^2 - 3}$, $f_y(x,y) = \dfrac{-x^2 2y}{(y^2-3)^2}$

Tangent line: $x = x_0,\ z - z_0 = f_y(x_0, y_0)(y - y_0) \implies x = 3,\ z - 9 = -36(y - 2)$

45. Let $z = f(x,y) = \dfrac{x^2}{y^2-3}$. Then $f(3,2) = 9$, $f_x(x,y) = \dfrac{2x}{y^2-3}$ and $f_x(3,2) = 6$;
equations for the tangent line are: $y = 2$, $z - 9 = 6(x - 3)$

46. $f(x,y) = (4 - x^2 - y^2)^{1/2}$, $f_x(x,y) = -x(4 - x^2 - y^2)^{-1/2}$, $f_y(x,y) = -y(4 - x^2 - y^2)^{-1/2}$

(a) $f_y(1,1) = -\dfrac{\sqrt{2}}{2} \implies x = 1,\ z - \sqrt{2} = -\dfrac{\sqrt{2}}{2}(y - 1)$

(b) $f_x(1,1) = -\dfrac{\sqrt{2}}{2} \implies y = 1,\quad z - \sqrt{2} = -\dfrac{\sqrt{2}}{2}(x - 1)$

(c) l_1 and l_2 have direction vectors $\mathbf{j} - \frac{\sqrt{2}}{2}\mathbf{k}$, $\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{k}$ respectively. The normal to the plane is

$(\mathbf{j} - \dfrac{\sqrt{2}}{2}\mathbf{k}) \times (\mathbf{i} - \dfrac{\sqrt{2}}{2}\mathbf{k}) = -\dfrac{\sqrt{2}}{2}\mathbf{i} - \dfrac{\sqrt{2}}{2}\mathbf{j} - \mathbf{k}$, so the tangent plane is

$-\dfrac{\sqrt{2}}{2}(x - 1) - \dfrac{\sqrt{2}}{2}(y - 1) - (z - \sqrt{2}) = 0$, or $(x - 1) + (y - 1) + \sqrt{2}(z - \sqrt{2}) = 0$

47. $u_x(x,y) = 2x = v_y(x,y)$; $u_y(x,y) = -2y = -v_x(x,y)$

48. $u_x = e^x \cos y$, $u_y = -e^x \sin y$, $v_x = e^x \sin y = -u_y$, $v_y = e^x \cos y = u_x$

49. $u_x(x,y) = \dfrac{1}{2}\dfrac{1}{x^2+y^2}2x = \dfrac{x}{x^2+y^2}$; $v_y(x,y) = \dfrac{1}{1+(y/x)^2}\left(\dfrac{1}{x}\right) = \dfrac{x}{x^2+y^2}$

Thus, $u_x(x,y) = v_y(x,y)$.

$u_y(x,y) = \dfrac{1}{2}\dfrac{1}{x^2+y^2}2y = \dfrac{y}{x^2+y^2}$; $v_x(x,y) = \dfrac{1}{1+(y/x)^2}\left(\dfrac{-y}{x^2}\right) = \dfrac{-y}{x^2+y^2}$

Thus, $u_y(x,y) = -v_x(x,y)$.

50. $u_x = \dfrac{y^2 - x^2}{(x^2+y^2)^2}$, $u_y = \dfrac{-x^2 y}{(x^2+y^2)^2}$, $v_x = \dfrac{2xy}{(x^2+y^2)^2} = -u_y$, $v_y = \dfrac{y^2 - x^2}{(x^2+y^2)^2} = u_x$

51. (a) f depends only on y. (b) f depends only on x.

52. (a) $a_0 = (b_0^2 + c_0^2 - 2b_0c_0\cos\theta_0)^{1/2} = 5\sqrt{7}$

(b) $\dfrac{\partial a}{\partial b} = (2b - 2c\cos\theta)(\dfrac{1}{2})(b^2 + c^2 - 2bc\cos\theta)^{-1/2} = \dfrac{\sqrt{7}}{14}$

(c) $a \cong a_0 + \dfrac{\sqrt{7}}{14}(b - b_0) = 5\sqrt{7} + \dfrac{\sqrt{7}}{14}\cdot(-1)$ decreases by about $\dfrac{\sqrt{7}}{14}$ inches.

(d) $\dfrac{\partial a}{\partial \theta} = 2bc\sin\theta(\dfrac{1}{2})(b^2 + c^2 - 2bc\cos\theta)^{-1/2} = \dfrac{15}{7}\sqrt{21}$

(e) Differentiate implicitly: $0 = 2c\dfrac{\partial c}{\partial \theta} - 2b\,\dfrac{\partial c}{\partial \theta}\cos\theta + 2bc\sin\theta$

$\dfrac{\partial c}{\partial \theta} = \dfrac{bc\sin\theta}{b\cos\theta - c} = -\dfrac{15}{2}\sqrt{3}$

53. (a) $50\sqrt{3}$ in.^{2}

(b) $\dfrac{\partial A}{\partial b} = \dfrac{1}{2}\,c\,\sin\theta;$ at time t_0, $\dfrac{\partial A}{\partial b} = 5\sqrt{3}$

(c) $\dfrac{\partial A}{\partial \theta} = \dfrac{1}{2}\,bc\,\cos\theta;$ at time t_0, $\dfrac{\partial A}{\partial \theta} = 50$

(d) with $h = \dfrac{\pi}{180}$, $A\,(b,\,c,\,\theta + h) - A\,(b,\,c,\,\theta) \cong h\,\dfrac{\partial A}{\partial \theta} = \dfrac{\pi}{180}\,(50) = \dfrac{5\pi}{18}$ in.^{2}

(e) $0 = \dfrac{1}{2}\,\sin\theta\,\left(b\,\dfrac{\partial c}{\partial b} + c\right);$ at time t_0, $\dfrac{\partial c}{\partial b} = \dfrac{-c}{b} = -2$

54. By theorem 7.6.1, $f(x,y) = Ce^{kx}$ where C is independent of x. Since C may depend on y, we write $C = g(y)$.

55. (a) y_0-section: $\mathbf{r}(x) = x\mathbf{i} + y_0\mathbf{j} + f(x,y_0)\mathbf{k}$

tangent line: $\mathbf{R}(t) = [x_0\mathbf{i} + y_0\mathbf{j} + f(x_0,y_0)\mathbf{k}\,] + t\left[\mathbf{i} + \dfrac{\partial f}{\partial x}(x_0,y_0)\mathbf{k}\right]$

(b) x_0-section: $\mathbf{r}(y) = x_0\mathbf{i} + y\mathbf{j} + f(x_0,y)\mathbf{k}$

tangent line: $\mathbf{R}(t) = [x_0\mathbf{i} + y_0\mathbf{j} + f(x_0,y_0)\,\mathbf{k}] + t\left[\mathbf{j} + \dfrac{\partial f}{\partial y}(x_0,y_0)\mathbf{k}\right]$

(c) For (x,y,z) in the plane

$$[(x-x_0)\mathbf{i} + (y-y_0)\mathbf{j} + (z - f(x_0,y_0))\mathbf{k}\,]\cdot\left[\left(\mathbf{i} + \frac{\partial f}{\partial x}(x_0,y_0)\mathbf{k}\right)\times\left(\mathbf{j} + \frac{\partial f}{\partial y}(x_0,y_0)\mathbf{k}\right)\right] = 0.$$

From this it follows that

$$z - f(x_0,y_0) = (x-x_0)\frac{\partial f}{\partial x}(x_0,y_0) + (y-y_0)\frac{\partial f}{\partial y}(x_0,y_0).$$

56. Fix y and set $F(x) = f(x,y)$. Then, for that value of y, $h(x,y) = g(F(x))$ and thus

$$h_x(x,y) = \frac{d}{dx}[g(F(x))] = g'(F(x))F'(x) = g'(f(x,y))f_x(x,y).$$

The second formula can be derived in the same manner.

57. (a) Set $u = ax + by$. Then

$$b\,\frac{\partial w}{\partial x} - a\,\frac{\partial w}{\partial y} = b(a\,g'(u)) - a(b\,g'(u)) = 0.$$

(b) Set $u = x^m y^n$. Then

$$nx\,\frac{\partial w}{\partial x} - my\,\frac{\partial w}{\partial y} = nx\left[\,mx^{m-1}y^n g'(u)\,\right] - my\left[\,nx^m y^{n-1}g'(u)\,\right] = 0.$$

58. $\dfrac{\partial u}{\partial x} = \dfrac{x^2y^2 + 2xy^3}{(x+y)^2}, \quad \dfrac{\partial u}{\partial y} = \dfrac{x^2y^2 + 2x^3y}{(x+y)^2}$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3x^2y^2(x+y)}{(x+y)^2} = 3u$$

59. $x\,\dfrac{\partial u}{\partial x} + y\,\dfrac{\partial u}{\partial y} = x\left(4Ax^3 + 4Bxy^2\right) + y\left(4Bx^2y + 4Cy^3\right) = 4\left(Ax^4 + 2Bx^2\,y^2 + Cy^4\right) = 4u$

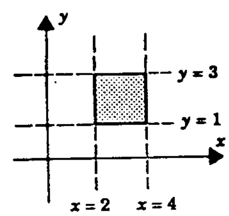
60. $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = (2xy + z^2) + (x^2 + 2yz) + (y^2 + 2zx) = (x+y+z)^2$

61. $\dfrac{\partial x}{\partial r}\dfrac{\partial y}{\partial \theta} - \dfrac{\partial x}{\partial \theta}\dfrac{\partial y}{\partial r} = (\cos\theta)\,(r\,\cos\theta) - (-r\,\sin\theta)\,(\sin\theta) = r$

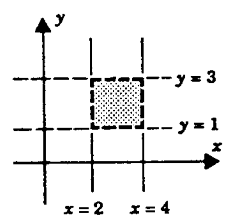
62. $V\dfrac{\partial P}{\partial V} = V\left(-\dfrac{kT}{V^2}\right) = -k\dfrac{T}{V} = -P; \qquad V\dfrac{\partial P}{\partial V} + T\dfrac{\partial P}{\partial T} = -k\dfrac{T}{V} + T\left(\dfrac{k}{V}\right) = 0$

SECTION 14.5

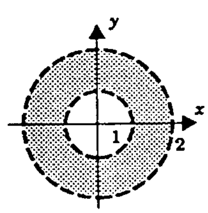
1. interior $= \{(x,y) : 2 < x < 4, \quad 1 < y < 3\}$ (the inside of the rectangle), boundary = the union of the four boundary line segments; set is closed.



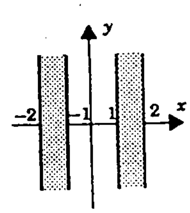
2. same interior and same boundary as in Exercise 1; set is open



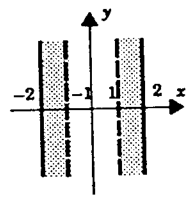
3. interior = the entire set (region between the two concentric circles), boundary = the two circles, one of radius 1, the other of radius 2; set is open.



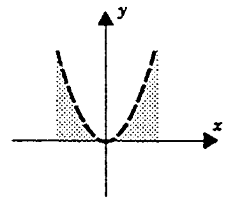
4. interior$=\{(x,y) : 1 < x^2 < 4\} = \{(x,y) : -2 < x < -1\} \cup \{(x,y) : 1 < x < 2\}$ (two vertical stripes without the boundary lines), boundary$=\{(x,y) : x = -2,\ x = -1,\ x = 1, \text{or } x = 2\}$ (four vertical lines); set is closed.



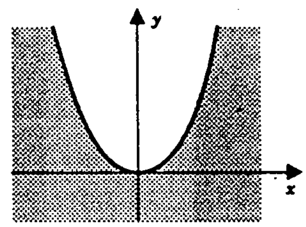
5. interior $= \{(x,y) : 1 < x^2 < 4\} =$ $\{(x,y) : -2 < x < -1\} \cup \{(x,y) : 1 < x < 2\}$ (two vertical strips without the boundary lines), boundary $= \{(x,y) : x = -2, x = -1, x = 1,$ or $x = 2\}$ (four vertical lines); set is neither open nor closed.



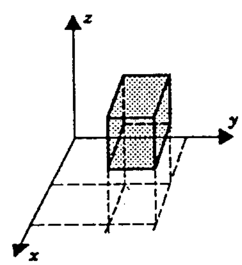
6. interior = the entire set (region below the parabola $y = x^2$), boundary = the parabola $y = x^2$; the set is open.



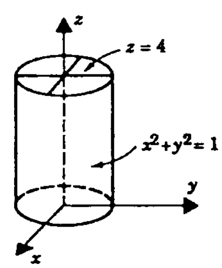
7. interior = region below the parabola $y = x^2$, boundary = the parabola $y = x^2$; the set is closed.



8. interior = the inside of the cube; boundary = the faces of the cube; set is neither open nor closed (upper face of cube is omitted)



9. interior $= \{(x,y,z) : x^2 + y^2 < 1,\ 0 < z \leq 4\}$ (the inside of the cylinder), boundary = the total surface of the cylinder (the curved part, the top, and the bottom); the set is closed.



10. interior = the entire set (the inside of the ball of radius $\frac{1}{2}$, centered at (1,1,1)), boundary = the spherical surface; set is open.

11. (a) ϕ (b) S (c) closed

12. interior = the entire set, boundary $= \{1,3\}$; set is open.

13. interior $= \{x : 1 < x < 3\}$, boundary $= \{1,3\}$; set is closed.

14. interior $= \{x : 1 < x < 3\}$, boundary $= \{1, 3\}$; set is neither open nor closed.

15. interior = the entire set, boundary $= \{1\}$; set is open.

16. interior $=\{x : x < -1\}$, boundary $= \{-1\}$; set is closed.

17. interior $= \{x : |x| > 1\}$, boundary $= \{1, -1\}$; set is neither open nor closed.

18. interior $= \phi$, boundary = the entire set; set is closed.

19. interior $= \phi$, boundary = {the entire set} $\cup \{0\}$; the set is neither open nor closed.

20. (a) ϕ is open because it contains no boundary points,
ϕ is closed because it contains its boundary (the boundary is empty).

(b) X is open because it contains a neighborhood of each of its points,
X is closed because it contains its boundary (the boundary is empty).

(c) Suppose that U is open. Let x be a boundary point of $X - U$. Then every neighborhood of x contains points from $X - U$. The point x can not be in U because U contains a neighborhood of each of its points. Thus $x \in X - U$. This shows that $X - U$ contains its boundary and is therefore closed.

Suppose now that $X - U$ is closed. Let x be a point of U. If no neighborhood of x lies entirely in U, then every neighborhood of x contains points from $X - U$. This makes x a boundary point of $X - U$ and, since $X - U$ is closed, places x in $X - U$. This contradiction shows that some neighborhood of x lies entirely in U. Thus U contains a neighborhood of each of its points and is therefore open.

(e) Set $U = X - F$ and note that $F = X - U$. By (c)

$$F = X - U \text{ is closed} \quad \text{iff} \quad X - F = U \text{ is open.}$$

SECTION 14.6

1. $\dfrac{\partial^2 f}{\partial x^2} = 2A, \quad \dfrac{\partial^2 f}{\partial y^2} = 2C, \quad \dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y} = 2B$

2. $\dfrac{\partial^2 f}{\partial x^2} = 6Ax + 2By, \quad \dfrac{\partial^2 f}{\partial y^2} = 2Cx, \quad \dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y} = 2Bx + 2Cy$

3. $\dfrac{\partial^2 f}{\partial x^2} = Cy^2 e^{xy}, \quad \dfrac{\partial^2 f}{\partial y^2} = Cx^2 e^{xy}, \quad \dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y} = Ce^{xy}(xy + 1)$

4. $\dfrac{\partial^2 f}{\partial x^2} = 2\cos y - y^2 \sin x, \quad \dfrac{\partial^2 f}{\partial y^2} = 2\sin x - x^2 \cos y, \quad \dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y} = 2(y \cos x - x \cos y)$

5. $\dfrac{\partial^2 f}{\partial x^2} = 2, \quad \dfrac{\partial^2 f}{\partial y^2} = 4(x + 3y^2 + z^3), \quad \dfrac{\partial^2 f}{\partial z^2} = 6z(2x + 2y^2 + 5z^3)$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 4y, \quad \frac{\partial^2 f}{\partial z \partial x} = \frac{\partial^2 f}{\partial x \partial z} = 6z^2, \quad \frac{\partial^2 f}{\partial z \partial y} = \frac{\partial^2 f}{\partial y \partial z} = 12yz^2$$

6. $$\frac{\partial^2 f}{\partial x^2} = -\frac{1}{4(x+y^2)^{3/2}}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{x}{(x+y^2)^{3/2}}, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -\frac{y}{2(x+y^2)^{3/2}}$$

7. $$\frac{\partial^2 f}{\partial x^2} = \frac{1}{(x+y)^2} - \frac{1}{x^2}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{1}{(x+y)^2}, \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{1}{(x+y)^2}$$

8. $$\frac{\partial^2 f}{\partial x^2} = -\frac{2C(AD-BC)y}{(Cx+Dy)^3}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{2D(AD-BC)x}{(Cx+Dy)^3}, \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{(AD-BC)(Cx-Dy)}{(Cx+Dy)^3}$$

9. $$\frac{\partial^2 f}{\partial x^2} = 2(y+z), \quad \frac{\partial^2 f}{\partial y^2} = 2(x+z), \quad \frac{\partial^2 f}{\partial z^2} = 2(x+y)$$

all the second mixed partials are $2(x+y+z)$

10. $$\frac{\partial^2 f}{\partial x^2} = \frac{2xy^3z^3}{(1+x^2y^2z^2)^2}, \quad \frac{\partial^2 f}{\partial y^2} = -\frac{2yx^3z^3}{(1+x^2y^2z^2)^2}$$

$$\frac{\partial^2 f}{\partial z^2} = -\frac{2zx^3y^3}{(1+x^2y^2z^2)^2}, \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{z(1-x^2y^2z^2)}{(1+x^2y^2z^2)^2}$$

$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial^2 f}{\partial x \partial z} = \frac{y(1-x^2y^2z^2)}{(1+x^2y^2z^2)^2}, \quad \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y} = \frac{x(1-x^2y^2z^2)}{(1+x^2y^2z^2)^2}$$

11. $$\frac{\partial^2 f}{\partial x^2} = y(y-1)x^{y-2}, \quad \frac{\partial^2 f}{\partial y^2} = (\ln x)^2 x^y, \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = x^{y-1}(1 + y \ln x)$$

12. $$\frac{\partial^2 f}{\partial x^2} = -\sin(x+z^y), \quad \frac{\partial^2 f}{\partial y^2} = z^y(\ln z)^2[\cos(x+z^y) - z^y \sin(x+z^y)]$$

$$\frac{\partial^2 f}{\partial z^2} = y(y-1)z^{y-2}\cos(x+z^y) - y^2 z^{2y-2}\sin(x+z^y)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = -z^y \ln z \sin(x+z^y), \quad \frac{\partial^2 f}{\partial z \partial x} = \frac{\partial^2 f}{\partial x \partial z} = -yz^{y-1}\sin(x+z^y)$$

$$\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial^2 f}{\partial y \partial z} = z^{y-1}(1 + y \ln z)\cos(x+z^y) - yz^{2y-1}(\ln z)\sin(x+z^y)$$

13. $$\frac{\partial^2 f}{\partial x^2} = ye^x, \quad \frac{\partial^2 f}{\partial y^2} = xe^y, \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = e^y + e^x$$

14. $$\frac{\partial^2 f}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2}, \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

15. $$\frac{\partial^2 f}{\partial x^2} = \frac{y^2-x^2}{(x^2+y^2)^2}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}, \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = -\frac{2xy}{(x^2+y^2)^2}$$

16. $\dfrac{\partial^2 f}{\partial x^2} = 6xy^2\cos(x^3y^2) - 9x^4y^4\sin(x^3y^2), \quad \dfrac{\partial^2 f}{\partial y^2} = 2x^3\cos(x^3y^2) - 4x^6y^2\sin(x^3y^2)$

$\dfrac{\partial^2 f}{\partial y\partial x} = \dfrac{\partial^2 f}{\partial x\partial y} = 6x^2y\cos(x^3y^2) - 6x^5y^3\sin(x^3y^2)$

17. $\dfrac{\partial^2 f}{\partial x^2} = -2\,y^2\cos 2xy, \quad \dfrac{\partial^2 f}{\partial y^2} = -2\,x^2\cos 2xy, \quad \dfrac{\partial^2 f}{\partial y\partial x} = \dfrac{\partial^2 f}{\partial x\partial y} = -[\sin 2xy + 2xy\,\cos 2xy]$

18. $\dfrac{\partial^2 f}{\partial x^2} = y^4e^{xy^2}, \quad \dfrac{\partial^2 f}{\partial y^2} = e^{xy^2}(2x + 4x^2y^2), \quad \dfrac{\partial^2 f}{\partial y\partial x} = \dfrac{\partial^2 f}{\partial x\partial y} = e^{xy^2}(2y + 2xy^3)$

19. $\dfrac{\partial^2 f}{\partial x^2} = 0, \quad \dfrac{\partial^2 f}{\partial y^2} = xz\,\sin y, \quad \dfrac{\partial^2 f}{\partial z^2} = -\,xy\,\sin z,$

$\dfrac{\partial^2 f}{\partial y\partial x} = \dfrac{\partial^2 f}{\partial x\partial y} = \sin z - z\,\cos y, \quad \dfrac{\partial^2 f}{\partial x\partial z} = \dfrac{\partial^2 f}{\partial z\partial x} = y\,\cos z - \sin y, \quad \dfrac{\partial^2 f}{\partial y\partial z} = \dfrac{\partial^2 u}{\partial z\partial y} = x\,\cos z - x\,\cos y$

20. $\dfrac{\partial^2 f}{\partial x^2} = ze^x, \quad \dfrac{\partial^2 f}{\partial y^2} = xe^y, \quad \dfrac{\partial^2 f}{\partial z^2} = ye^z, \quad \dfrac{\partial^2 f}{\partial y\partial x} = \dfrac{\partial^2 f}{\partial x\partial y} = e^y,$

$\dfrac{\partial^2 f}{\partial z\partial x} = \dfrac{\partial^2 f}{\partial x\partial z} = e^x, \quad \dfrac{\partial^2 f}{\partial z\partial y} = \dfrac{\partial^2 f}{\partial y\partial z} = e^z$

21. $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = x^2\left(\dfrac{-2y^2}{(x+y)^3}\right) + 2xy\left(\dfrac{2xy}{(x+y)^3}\right) + y^2\left(\dfrac{-2x^2}{(x+y)^3}\right) = 0$

22. (a) mixed partials are 0

(b) mixed partials are $g'(x)\,h'(y)$

(c) by the hint mixed partials for each term x^my^n are $mnx^{m-1}y^{n-1}$

23. (a) no, since $\dfrac{\partial^2 f}{\partial y\partial x} \neq \dfrac{\partial^2 f}{\partial x\partial y}$ (b) no, since $\dfrac{\partial^2 f}{\partial y\partial x} \neq \dfrac{\partial^2 f}{\partial x\partial y}$ for $x \neq y$

24. $\dfrac{\partial h}{\partial x} = g'(x+y) + g'(x-y), \quad \dfrac{\partial h}{\partial y} = g'(x+y) - g'(x-y)$

$\dfrac{\partial^2 h}{\partial x^2} = g''(x+y) + g''(x-y), \quad \dfrac{\partial^2 h}{\partial y^2} = g''(x+y) + g''(x-y) = \dfrac{\partial^2 h}{\partial x^2}$

25.

$$\frac{\partial^3 f}{\partial x^2\partial y} = \underset{\text{by definition}}{\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x\partial y}\right)} = \underset{(14.6.5)}{\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y\partial x}\right)} = \underset{\text{by definition}}{\frac{\partial^2 f}{\partial x\partial y}\left(\frac{\partial f}{\partial x}\right)} = \underset{(14.6.5)}{\frac{\partial^2}{\partial y\partial x}\left(\frac{\partial f}{\partial x}\right)} = \underset{\text{by def.}}{\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right)} = \underset{\text{by def.}}{\frac{\partial^3 f}{\partial y\partial x^2}}$$

26. (a) as (x,y) tends to $(0,0)$ along the x-axis, $f(x,y) = f(x,0) = 1$ tends to 1;

as (x,y) tends to $(0,0)$ along the line $y = x$, $f(x,y) = f(x,x) = 0$ tends to 0;

(b) as (x,y) tends to $(0,0)$ along the x-axis, $f(x,y) = f(x,0) = 0$ tends to 0;

as (x,y) tends to $(0,0)$ along the line $y = x$, $f(x,y) = f(x,x) = \frac{1}{2}$ tends to $\frac{1}{2}$;

27. (a) $\lim_{x\to 0}\dfrac{(x)(0)}{x^2+0}=\lim_{x\to 0}0=0$ (b) $\lim_{y\to 0}\dfrac{(0)(y)}{0+y^2}=\lim_{y\to 0}0=0$

(c) $\lim_{x\to 0}\dfrac{(x)(mx)}{x^2+(mx)^2}=\lim_{x\to 0}\dfrac{m}{1+m^2}=\dfrac{m}{1+m^2}$

(d) $\lim_{\theta\to 0+}\dfrac{(\theta\cos\theta)(\theta\sin\theta)}{(\theta\cos\theta)^2+(\theta\sin\theta)^2}=\lim_{\theta\to 0+}\cos\theta\,\sin\theta=0$

(e) By L'Hospital's rule $\lim_{x\to 0}\dfrac{f(x)}{x}=\lim_{x\to 0}f'(x)=f'(0)$. Thus

$$\lim_{x\to 0}\frac{xf(x)}{x^2+[f(x)]^2}=\lim_{x\to 0}\frac{f(x)/x}{1+[f(x)/x]^2}=\frac{f'(0)}{1+[f'(0)]^2}.$$

(f) $\lim_{\theta\to(\pi/3)^-}=\dfrac{(\cos\theta\,\sin 3\theta)(\sin\theta\,\sin 3\theta)}{(\cos\theta\,\sin 3\theta)^2+(\sin\theta\,\sin 3\theta)^2}=\lim_{\theta\to(\pi/3)^-}\cos\theta\,\sin\theta=\dfrac{1}{4}\sqrt{3}$

(g) $\lim_{t\to\infty}\dfrac{(1/t)(\sin t)/t}{1/t^2+\left(\sin^2 t\right)/t^2}=\lim_{t\to\infty}\dfrac{\sin t}{1+\sin^2 t}$; does not exist

28. (a) $\lim_{x\to 0}\dfrac{x(0)^2}{(x^2+0^2)^{3/2}}=\lim_{x\to 0}0=0$ (b) $\lim_{y\to 0}\dfrac{0\cdot y^2}{(0+y^2)^{3/2}}=\lim_{y\to 0}0=0$

(c) $\lim_{x\to 0}\dfrac{xm^2x^2}{(x^2+m^2x^2)^{3/2}}=\lim_{x\to 0}\dfrac{m^2x^3}{|x|^3(1+m^2)^{3/2}}=\lim_{x\to 0}\dfrac{m^2x}{|x|\,(1+m^2)^{3/2}}$; does not exist

(d) $\lim_{\theta\to 0+}\dfrac{(\theta\cos\theta)(\theta\sin\theta)^2}{[(\theta\cos\theta)^2+(\theta\sin\theta)^2]^{3/2}}=\lim_{\theta\to 0+}\dfrac{\cos^2\theta\sin^2\theta}{\theta}=0$

(e) $\lim_{x\to 0}\dfrac{x[f(x)]^2}{(x^2+[f(x)]^2)^{3/2}}=\lim_{x\to 0}\dfrac{[f(x)/x]^2}{(1+[f(x)/x]^2)^{3/2}}=\lim_{x\to 0}\dfrac{x^3[f'(0)]^2}{|x|^3(1+[f'(0)]^2)^{3/2}}$; does not exist

(f) $\lim_{\theta\to\frac{\pi}{3}-}\dfrac{(\cos\theta\sin 3\theta)(\sin\theta\sin 3\theta)^2}{[(\cos\theta\sin 3\theta)^2+(\sin\theta\sin 3\theta)^2]^{3/2}}=\lim_{\theta\to\frac{\pi}{3}-}\cos\theta\sin^2\theta=\dfrac{3}{8}$

(g) $\lim_{t\to\infty}\dfrac{(1/t)\,(\sin t/t)^2}{\left[(1/t^2)+(\sin^2 t/t^2)\right]^{3/2}}=\lim_{t\to\infty}\dfrac{\sin^2 t}{\left(1+\sin^2 t\right)^{3/2}}$; does not exist

29. (a) $\dfrac{\partial g}{\partial x}(0,0)=\lim_{h\to 0}\dfrac{g(h,0)-g(0,0)}{h}=\lim_{h\to 0}0=0,$

$\dfrac{\partial g}{\partial y}(0,0)=\lim_{h\to 0}\dfrac{g(0,h)-g(0,0)}{h}=\lim_{h\to 0}0=0$

(b) as (x,y) tends to $(0,0)$ along the x-axis, $g(x,y)=g(x,0)=0$ tends to 0;

as (x,y) tends to $(0,0)$ along the line $y=x$, $g(x,y)=g(x,x)=\frac{1}{2}$ tends to $\frac{1}{2}$

30. No; as (x,y) tends to $(1,1)$ along the line $x=1$, $f(x,y)=1$ tends to 1; as (x,y) tends to $(1,1)$ along the line $y=1$,

$$f(x,y)=\frac{x-1}{x^3-1}=\frac{1}{x^2+x+1}\quad\text{tends to}\quad\frac{1}{3}$$

31. For $y \neq 0$, $\quad \dfrac{\partial f}{\partial x}(0,y) = \lim_{h\to 0} \dfrac{f(h,y)-f(0,y)}{h} = \lim_{h\to 0} \dfrac{y(y^2-h^2)}{h^2+y^2} = y.$

Since $\quad \dfrac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} 0 = 0,$

we have $\quad \dfrac{\partial f}{\partial x}(0,y) = y$ for all y.

For $x \neq 0$, $\quad \dfrac{\partial f}{\partial y}(x,0) = \lim_{h\to 0} \dfrac{f(x,h)-f(x,0)}{h} = \lim_{h\to 0} \dfrac{x(h^2-x^2)}{x^2+h^2} = -x.$

Since $\quad \dfrac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \dfrac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} 0 = 0,$

we have $\quad \dfrac{\partial f}{\partial y}(x,0) = -x \quad$ for all x.

Therefore $\quad \dfrac{\partial^2 f}{\partial y \partial x}(0,y) = 1$ for all y and $\dfrac{\partial^2 f}{\partial x \partial y}(x,0) = -1$ for all x.

In particular $\quad \dfrac{\partial^2 f}{\partial y \partial x}(0,0) = 1$ while $\dfrac{\partial^2 f}{\partial x \partial y}(0,0) = -1.$

32.

$$\begin{aligned}\lim_{h\to 0}[f(x_0+h,y_0)-f(x_0,y_0)] &= \lim_{h\to 0}\left[(h)\left(\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}\right)\right]\\ &= \left(\lim_{h\to 0} h\right)\left(\lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}\right)\\ &= 0\cdot\frac{\partial f}{\partial x}(x_0,y_0) = 0\end{aligned}$$

33. Since $f_{xy}(x,y)=0$, $f_x(x,y)$ must be a function of x alone, and $f_y(x,y)$ must be a function of y alone. Then f must be of the form

$$f(x,y) = g(x) + h(y).$$

PROJECT 14.6

1. (a) $\dfrac{\partial u}{\partial x} = \dfrac{x^2y^2+2xy^3}{(x+y)^2}, \quad \dfrac{\partial u}{\partial y} = \dfrac{x^2y^2+2x^3y}{(x+y)^2}$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3x^2y^2(x+y)}{(x+y)^2} = 3u$$

(b) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = x\left(4Ax^3+4Bxy^2\right) + y\left(4Bx^2y+4Cy^3\right)$

$= 4\left(Ax^4+2Bx^2y^2+Cy^4\right) = 4u$

2. (a) (i) $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 6x - 6x = 0$

(ii) $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = (-\cos x\sinh y - \sin x\cosh y) + (\cos x\sinh y + \sin x\cosh y) = 0$

(iii) $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{y^2 - x^2}{(x^2+y^2)^2}, \quad \dfrac{\partial^2 f}{\partial y^2} = \dfrac{x^2 - y^2}{(x^2+y^2)^2}$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{y^2 - x^2}{(x^2+y^2)^2} + \frac{x^2 - y^2}{(x^2+y^2)^2} = 0$$

(b) (i) $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} = \dfrac{2x^2 - y^2 - z^2}{(x^2+y^2+z^2)^{5/2}} + \dfrac{2y^2 - x^2 - z^2}{(x^2+y^2+z^2)^{5/2}} + \dfrac{2z^2 - x^2 - y^2}{(x^2+y^2+z^2)^{5/2}} = 0$

(ii) $\dfrac{\partial^2 f}{\partial x^2} = e^{x+y}\cos\left(\sqrt{2}\,z\right), \quad \dfrac{\partial^2 f}{\partial y^2} = e^{x+y}\cos\left(\sqrt{2}\,z\right), \quad \dfrac{\partial^2 f}{\partial z^2} = -\,2e^{x+y}\cos\left(\sqrt{2}\,z\right)$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = e^{x+y}\cos\left(\sqrt{2}\,z\right) + e^{x+y}\cos\left(\sqrt{2}\,z\right) + \left[-\,2e^{x+y}\cos\left(\sqrt{2}\,z\right)\right] = 0$$

3. (i) $\dfrac{\partial^2 f}{\partial t^2} = \dfrac{\partial^2 f}{\partial x^2} = 0 \implies \dfrac{\partial^2 f}{\partial t^2} - c^2\dfrac{\partial^2 f}{\partial x^2} = 0$

(ii) $\dfrac{\partial^2 f}{\partial t^2} = -\,5c^2\sin(x+ct)\cos(2x+2ct) - 4c^2\cos(x+ct)\sin(2x+2ct)$

$$\frac{\partial^2 f}{\partial x^2} = -\,5\sin(x+ct)\cos(2x+2ct) - 4\cos(x+ct)\sin(2x+2ct)$$

It now follows that $\dfrac{\partial^2 f}{\partial t^2} - c^2\dfrac{\partial^2 f}{\partial x^2} = 0$

(iii) $\dfrac{\partial^2 f}{\partial t^2} = -\dfrac{c^2}{(x+ct)^2}, \quad \dfrac{\partial^2 f}{\partial x^2} = -\dfrac{1}{(x+ct)^2} \implies \dfrac{\partial^2 f}{\partial t^2} - c^2\dfrac{\partial^2 f}{\partial x^2} = 0$

(iv) $\dfrac{\partial^2 f}{\partial t^2} = c^2k^2\left(Ae^{kx} + Be^{-kx}\right)\left(Ce^{ckt} + De^{-ckt}\right), \quad \dfrac{\partial^2 f}{\partial x^2} = k^2\left(Ae^{kx} + Be^{-kx}\right)\left(Ce^{ckt} + De^{-ckt}\right)$

It now follows that $\dfrac{\partial^2 f}{\partial t^2} - c^2\dfrac{\partial^2 f}{\partial x^2} = 0$

4. $\dfrac{\partial^2 f}{\partial t^2} - c^2\dfrac{\partial^2 f}{\partial x^2} = \left[c^2g''(x+ct) + c^2h''(x-ct)\right] - c^2[g''(x+ct) + h''(x-ct)] = 0$

CHAPTER 15

SECTION 15.1

1. $\nabla f = e^{xy}[\,(xy+1)\,\mathbf{i} + x^2\mathbf{j}]$

2. $\nabla f = 3x^2\mathbf{i} + 2y\mathbf{j}$

3. $\nabla f = (6x - y)\,\mathbf{i} + (1 - x)\,\mathbf{j}$

4. $\nabla f = (2Ax + By)\mathbf{i} + (Bx + 2Cy)\mathbf{j}$

5. $\nabla f = 2xy^{-2}\mathbf{i} - 2x^2y^{-3}\mathbf{j}$

6. $\nabla f = (e^{x-y} + e^{y-x})\mathbf{i} + (-e^{x-y} - e^{y-x})\mathbf{j} = (e^{x-y} + e^{y-x})(\mathbf{i} - \mathbf{j})$

7. $\nabla f = z\cos(x-y)\,\mathbf{i} - z\cos(x-y)\,\mathbf{j} + \sin(x-y)\,\mathbf{k}$

8. $\nabla f = e^{-z}(y^2\mathbf{i} + 2xy\mathbf{j} - xy^2\mathbf{k})$

9. $\nabla f = (y+z)\,\mathbf{i} + (x+z)\,\mathbf{j} + (x+y)\mathbf{k}$

10. $\nabla f = \dfrac{1}{2\sqrt{x+y+z}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$

11. $\nabla f = e^{x-y}\,[\,(1+x+y)\,\mathbf{i} + (1-x-y)\,\mathbf{j}\,]$

12. $\nabla f = [\sin(x-y) + (x+y)\cos(x-y)]\mathbf{i} + [\sin(x-y) - (x+y)\cos(x-y)]\mathbf{j}$

13. $\nabla f = e^x\,[\ln y\,\mathbf{i} + y^{-1}\mathbf{j}\,]$

14. $\nabla f = (z^2 + 2xy)\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (y^2 + 2zx)\mathbf{k}$

15. $\nabla f = \dfrac{AD - BC}{(Cx + Dy)^2}\,[\,y\mathbf{i} - x\mathbf{j}\,]$

16. $\nabla f = \dfrac{1}{(x^2+y^2)^2}[(y^2 - x^2 + 2xy)\mathbf{i} + (y^2 - x^2 - 2xy)\mathbf{j}]$

17. $\nabla f = (ye^x + xye^x - ze^y\cos xz)\;\mathbf{i} + (xe^x + e^z - e^y\sin xz)\;\mathbf{j} + (ye^z - xe^y\cos xz)\;\mathbf{k}$

18. $\nabla f = e^{yz^2/x^3}\left(-\dfrac{3yz^2}{x^4}\mathbf{i} + \dfrac{z^2}{x^3}\mathbf{j} + \dfrac{2yz}{x^3}\mathbf{k}\right)$

19. $\nabla f = e^{x+2y}\cos\left(z^2+1\right)\;\mathbf{i} + 2e^{x+2y}\cos\left(z^2+1\right)\;\mathbf{j} - 2ze^{x+2y}\sin\left(z^2+1\right)\;\mathbf{k}$

20. $\nabla f = \left[y\cos(xy) + yze^{xyz} + \dfrac{2}{x}\right]\mathbf{i} + [x\cos(xy) + xze^{xyz}]\mathbf{j} + \left[xye^{xyz} + \dfrac{1}{z}\right]\mathbf{k}$

21. $\nabla f = (4x - 3y)\,\mathbf{i} + (8y - 3x)\,\mathbf{j};$ at $(2,3)$, $\nabla f = -\mathbf{i} + 18\mathbf{j}$

22. $\nabla f = \dfrac{1}{(x-y)^2}(-2y\mathbf{i} + 2x\mathbf{j}),\quad \nabla f(3,1) = -\dfrac{1}{2}\mathbf{i} + \dfrac{3}{2}\mathbf{j}$

23. $\nabla f = \dfrac{2x}{x^2+y^2}\,\mathbf{i} + \dfrac{2y}{x^2+y^2}\,\mathbf{j};$ at $(2,1)$, $\nabla f = \dfrac{4}{5}\,\mathbf{i} + \dfrac{2}{5}\,\mathbf{j}$

24. $\nabla f = \left(\tan^{-1}(y/x) - \dfrac{xy}{x^2+y^2}\right)\mathbf{i} + \left(\dfrac{x^2}{x^2+y^2}\right)\mathbf{j},\quad \nabla f(1,1) = \left(\dfrac{\pi}{4} - \dfrac{1}{2}\right)\mathbf{i} + \dfrac{1}{2}\mathbf{j}$

25. $\nabla f = (\sin xy + xy\,\cos xy)\,\mathbf{i} + x^2\,\cos xy\,\mathbf{j};$ at $(1, \pi/2)$, $\nabla f = \mathbf{i}$

26. $\nabla f = e^{-(x^2+y^2)}[(y-2x^2y)\mathbf{i}+(x-2xy^2)\mathbf{j}], \quad \nabla f(1,-1)=e^{-2}(\mathbf{i}-\mathbf{j})$

27. $\nabla f = -e^{-x}\sin(z+2y)\,\mathbf{i}+2e^{-x}\cos(z+2y)\,\mathbf{j}+e^{-x}\cos(z+2y)\,\mathbf{k};$

at $(0,\pi/4,\pi/4)$, $\quad \nabla f = -\frac{1}{2}\sqrt{2}\,(\mathbf{i}+2\mathbf{j}+\mathbf{k})$

28. $\nabla f = \cos\pi z\mathbf{i}-\cos\pi z\mathbf{j}-\pi(x-y)\sin\pi z\mathbf{k}, \quad \nabla f(1,0,\frac{1}{2}) = -\pi\mathbf{k}$

29. $\nabla f = \mathbf{i}-\dfrac{y}{\sqrt{y^2+z^2}}\mathbf{j}-\dfrac{z}{\sqrt{y^2+z^2}}\mathbf{k};$ at $(2,-3,4)$, $\quad \nabla f = \mathbf{i}+\frac{3}{5}\mathbf{j}-\frac{4}{5}\mathbf{k}$

30. $\nabla f = -\sin(xyz^2)(yz^2\mathbf{i}+xz^2\mathbf{j}+2xyz\mathbf{k}), \quad \nabla f(\pi,\frac{1}{4},-1) = -\frac{\sqrt{2}}{2}\left(\frac{1}{4}\mathbf{i}+\pi\mathbf{j}-\frac{\pi}{2}\mathbf{k}\right)$

31. For the function $f(x,y)=3x^2-xy+y$, we have

$$\begin{aligned} f(\mathbf{x}+\mathbf{h})-f(\mathbf{x}) &= f(x+h_1,y+h_2)-f(x,y)\\ &= 3(x+h_1)^2-(x+h_1)(y+h_2)+(y+h_2)-\left[3x^3-xy+y\right]\\ &= [(6x-y)\,\mathbf{i}+(1-x)\,\mathbf{j}]\cdot(h_1\,\mathbf{i}+h_2\,\mathbf{j})+3h_1^2-h_1h_2\\ &= [(6x-y)\,\mathbf{i}+(1-x)\,\mathbf{j}]\cdot\mathbf{h}+3h_1^2-h_1h_2 \end{aligned}$$

The remainder $g(\mathbf{h})=3h_1^2-h_1h_2=(3h_1\,\mathbf{i}-h_1\,\mathbf{j})\cdot(h_1\,\mathbf{i}+h_2\,\mathbf{j})$, and

$$\frac{|g(\mathbf{h})|}{\|\mathbf{h}\|}=\frac{\|3h_1\,\mathbf{i}-h_1\,\mathbf{j}\|\cdot\|\mathbf{h}\|\cdot\cos\theta}{\|\mathbf{h}\|}\le\|3h_1\,\mathbf{i}-h_1\,\mathbf{j}\|$$

Since $\|3h_1\,\mathbf{i}-h_1\,\mathbf{j}\|\to 0$ as $\mathbf{h}\to\mathbf{0}$ it follows that

$$\nabla f=(6x-y)\,\mathbf{i}+(1-x)\,\mathbf{j}$$

32. $\nabla f=(x+2y)\mathbf{i}+(2x+2y)\mathbf{j}$

33. For the function $f(x,y,z)=x^2y+y^2z+z^2x$, we have

$$\begin{aligned} f(\mathbf{x}+\mathbf{h})-f(\mathbf{x}) &= f(x+h_1,y+h_2,z+h_3)-f(x,y,z)\\ &= (x+h_1)^2(y+h_2)+(y+h_2)^2(z+h_3)+(z+h_3)^2(x+h_1)-x^2y+y^2z+z^2x\\ &= \left(2xy+z^2\right)h_1+\left(2yz+x^2\right)h_2+\left(2xz+y^2\right)h_3+(2xh_2+yh_1+h_1h_2)\,h_1+\\ &\qquad (2yh_3+zh_2+h_2h_3)\,h_2+(2zh_1+xh_3+h_1h_3)\,h_3\\ &= \left[\left(2xy+z^2\right)\,\mathbf{i}+\left(2yz+x^2\right)\,\mathbf{j}+\left(2xz+y^2\right)\,\mathbf{k}\right]\cdot\mathbf{h}+g(\mathbf{h})\cdot\mathbf{h}, \end{aligned}$$

where $\quad g(\mathbf{h})=(2xh_2+yh_1+h_1h_2)\,\mathbf{i}+(2yh_3+zh_2+h_2h_3)\,\mathbf{j}+(2zh_1+xh_3+h_1h_3)\,\mathbf{k}$

Since $\dfrac{|g(\mathbf{h})|}{\|\mathbf{h}\|}\to 0$ as $\mathbf{h}\to\mathbf{0}$ $\quad$ it follows that

$$\nabla f=\left(2xy+z^2\right)\,\mathbf{i}+\left(2yz+x^2\right)\,\mathbf{j}+\left(2xz+y^2\right)\,\mathbf{k}$$

34. $\nabla f=4xy\mathbf{i}+2x^2\mathbf{j}+\dfrac{1}{z^2}\mathbf{k}$

35. $\nabla f = \mathbf{F}(x,y) = 2xy\,\mathbf{i} + (1+x^2)\,\mathbf{j} \quad \Rightarrow \quad \dfrac{\partial f}{\partial x} = 2xy \quad \Rightarrow \quad f(x,y) = x^2y + g(y)$ for some function g.

Now, $\dfrac{\partial f}{\partial y} = x^2 + g'(y) = 1 + x^2 \quad \Rightarrow \quad g'(y) = 1 \quad \Rightarrow \quad g(y) = y + C,\;\; C$ a constant.

Thus, $f(x,y) = x^2y + y$ (take $C = 0$) is a function whose gradient is $\mathbf{F}$.

36. $\nabla f = (2xy + x)\mathbf{i} + (x^2 + y)\mathbf{j} \implies f_x = 2xy + x, \quad f_y = x^2 + y$

Take $f(x,y) = x^2y + \dfrac{x^2}{2} + \dfrac{y^2}{2}$

37. $\nabla f = \mathbf{F}(x,y) = (x + \sin y)\,\mathbf{i} + (x\cos y - 2y)\,\mathbf{j} \quad \Rightarrow \quad \dfrac{\partial f}{\partial x} = x + \sin y \quad \Rightarrow \quad f(x,y) = \frac{1}{2}x^2 + x\sin y + g(y)$

for some function g.

Now, $\dfrac{\partial f}{\partial y} = x\cos y + g'(y) = x\cos y - 2y \quad \Rightarrow \quad g'(y) = -2y \quad \Rightarrow \quad g(y) = -y^2 + C,\;\; C$ a constant.

Thus, $f(x,y) = \frac{1}{2}x^2 + x\sin y - y^2$ (take $C = 0$) is a function whose gradient is $\mathbf{F}$.

38. $\nabla f = yz\mathbf{i} + (xz + 2yz)\mathbf{j} + (xy + y^2)\mathbf{k} \implies f_x = yz, \quad f_y = xz + 2yz, \quad f_z = xy + y^2$

Take $f(x,y,z) = xyz + y^2z$.

39. With $r = (x^2 + y^2 + z^2)^{1/2}$ we have

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

(a)

$$\begin{aligned}\nabla(\ln r) &= \frac{\partial}{\partial x}(\ln r)\,\mathbf{i} + \frac{\partial}{\partial y}(\ln r)\,\mathbf{j} + \frac{\partial}{\partial z}(\ln r)\mathbf{k}\\ &= \frac{1}{r}\frac{\partial r}{\partial x}\mathbf{i} + \frac{1}{r}\frac{\partial r}{\partial y}\mathbf{j} + \frac{1}{r}\frac{\partial r}{\partial z}\mathbf{k}\\ &= \frac{x}{r^2}\mathbf{i} + \frac{y}{r^2}\mathbf{j} + \frac{z}{r^2}\mathbf{k} = \frac{\mathbf{r}}{r^2}\end{aligned}$$

(b)

$$\begin{aligned}\nabla(\sin r) &= \frac{\partial}{\partial x}(\sin r)\,\mathbf{i} + \frac{\partial}{\partial y}(\sin r)\,\mathbf{j} + \frac{\partial}{\partial z}(\sin r)\mathbf{k}\\ &= \cos r\,\frac{\partial r}{\partial x}\mathbf{i} + \cos r\,\frac{\partial r}{\partial y}\mathbf{j} + \cos r\,\frac{\partial r}{\partial z}\mathbf{k}\\ &= (\cos r)\frac{x}{r}\mathbf{i} + (\cos r)\frac{y}{r}\mathbf{j} + (\cos r)\frac{z}{r}\mathbf{k}\\ &= \left(\frac{\cos r}{r}\right)\mathbf{r}\end{aligned}$$

(c) $\nabla e^r = \left(\dfrac{e^r}{r}\right)\mathbf{r}$ [same method as in (a) and (b)]

40. With $r^n = (x^2+y^2+z^2)^{n/2}$ we have

$$\frac{\partial r^n}{\partial x} = \frac{n}{2}(x^2+y^2+z^2)^{(n/2)-1}(2x) = n(x^2+y^2+z^2)^{(n-2)/2}x = nr^{n-2}x.$$

Similarly

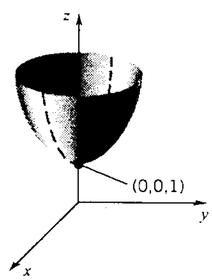
$$\frac{\partial r^n}{\partial y} = nr^{n-2}y \quad \text{and} \quad \frac{\partial r^n}{\partial z} = nr^{n-2}z.$$

Therefore

$$\nabla r^n = nr^{n-2}x\mathbf{i} + nr^{n-2}y\mathbf{j} + nr^{n-2}z\mathbf{k} = nr^{n-2}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}) = nr^{n-2}\mathbf{r}$$

41. (a) $\nabla f = 2x\,\mathbf{i} + 2y\,\mathbf{j} = \mathbf{0} \implies x = y = 0;\quad \nabla f = \mathbf{0}$ at $(0,0)$.

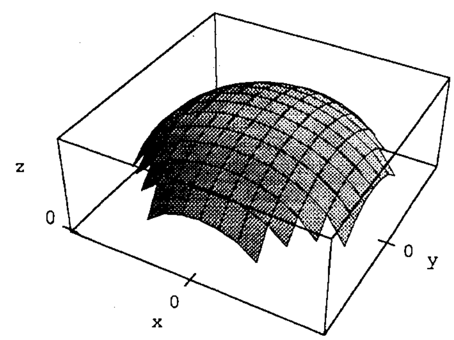
(b)



(c) f has an absolute minimum at $(0,0)$

42. (a) $\nabla f = \dfrac{-1}{\sqrt{4-x^2-y^2}}(x\mathbf{i}+y\mathbf{j}) = \mathbf{0}$ at $(0,0)$

(b)



(c) f has a maximum at $(0,0)$

43. (a) Let $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$. First, we take $\mathbf{h} = h\mathbf{i}$. Since $\mathbf{c}\cdot\mathbf{h}$ is $o(\mathbf{h})$,

$$0 = \lim_{\mathbf{h}\to\mathbf{0}} \frac{\mathbf{c}\cdot\mathbf{h}}{||\mathbf{h}||} = \lim_{h\to 0}\frac{c_1 h}{h} = c_1.$$

Similarly, $c_2 = 0$ and $c_3 = 0$.

(b) $(\mathbf{y}-\mathbf{z})\cdot\mathbf{h} = [f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - \mathbf{z}\cdot\mathbf{h}] + [\mathbf{y}\cdot\mathbf{h} - f(\mathbf{x}+\mathbf{h}) + f(\mathbf{x})] = o(\mathbf{h}) + o(\mathbf{h}) = o(\mathbf{h})$, so that, by part (a), $\mathbf{y}-\mathbf{z} = \mathbf{0}$.

44. $\displaystyle\lim_{\mathbf{h}\to\mathbf{0}} g(h) = \lim_{\mathbf{h}\to\mathbf{0}}\left(||\mathbf{h}||\frac{g(\mathbf{h})}{||\mathbf{h}||}\right) = \left(\lim_{h\to 0}||\mathbf{h}||\right)\left(\lim_{h\to 0}\frac{g(\mathbf{h})}{||\mathbf{h}||}\right) = (0)(0) = (0).$

45. (a) In Section 14.6 we showed that f was not continuous at $(0,0)$. It is therefore not differentiable at $(0,0)$.

(b) For $(x,y) \neq (0,0)$, $\dfrac{\partial f}{\partial x} = \dfrac{2y(y^2 - x^2)}{(x^2+y^2)^2}$. As (x,y) tends to $(0,0)$ along the positive y-axis,

$\dfrac{\partial f}{\partial x} = \dfrac{2y^3}{y^4} = \dfrac{2}{y}$ tends to ∞.

PROJECT 15.1

1. $f(x,y) = 2x^2 - y^2 - x^4 + 2$

(a) $\nabla f = (4x - 4x^3)\mathbf{i} + 2y\mathbf{j}$

Set $\nabla f = 0$:

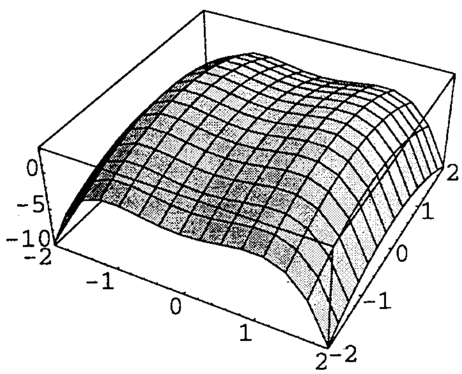
$4x - 4x^3 = 0 \qquad 2y = 0$

$x(1 - x^2) = 0$

$x = 0, 1, -1$

$\nabla f = 0$ at $(0,0)$, $(1,0)$, $(-1,0)$

(b)



2. $f(x,y) = 4xye^{-9x^2+y^2)}$

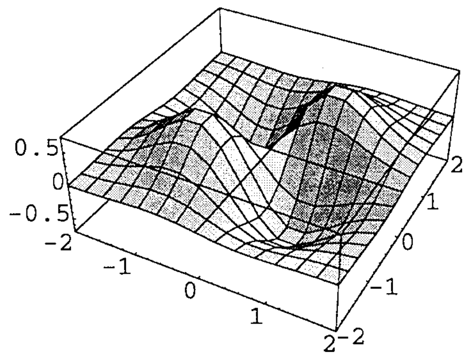
(a) $\nabla f = e^{-9x^2+y^2)}[4y - 8x^2y]\mathbf{i} + e^{-9x^2+y^2)}[4x - 8xy^2]\mathbf{j}$

Set $\nabla f = 0$:

$4y(1 - 2x^2) = 0 \qquad 4x(1 - 2y^2) = 0$

$\nabla f = 0$ at $(0,0)$, $(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2})$

(b)



3. $f(x,y) = (x^2+4y^2)e^{1-(x^2+y^2)}$

(a) $\nabla f = e^{1-(x^2+y^2)}[2x - 2x(x^2+4y^2)]\mathbf{i} + e^{1-(x^2+y^2)}[8y - 2y(x^2+4y^2)]\mathbf{j}$

Set $\nabla f = 0$:

$2x(1-x^2-4y^2) = 0 \qquad 2y(4-x^2-4y^2) = 0$

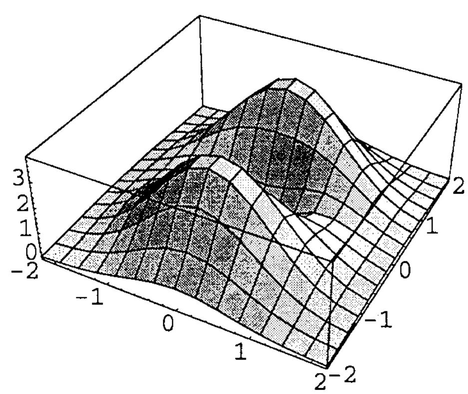
$x = 0 \text{ or } x^2+4y^2 = 1$

$x = 0: \quad 2y(4-4y^2) = 0 \Longrightarrow y = 0, \pm 1$

$x^2+4y^2 = 1: \quad 2y(3) = 0 \Longrightarrow y = 0, \; x = \pm 1$

$\nabla f = 0$ at $(0,0),\ (0,\pm 1),\ (\pm 1, 0)$

(b)



SECTION 15.2

1. $\nabla f = 2x\mathbf{i} + 6y\mathbf{j}, \quad \nabla f(1,1) = 2\mathbf{i} + 6\mathbf{j}, \quad \mathbf{u} = \frac{1}{2}\sqrt{2}\,(\mathbf{i} - \mathbf{j}), \qquad f'_{\mathbf{u}}(1,1) = \nabla f(1,1) \cdot \mathbf{u} = -2\sqrt{2}$

2. $\nabla f = [1 + \cos(x+y)]\mathbf{i} + \cos(x+y)\mathbf{j}, \quad \nabla f(0,0) = 2\mathbf{i} + \mathbf{j}, \quad \mathbf{u} = \dfrac{1}{\sqrt{5}}(2\mathbf{i} + \mathbf{j})$

$f'_{\mathbf{u}}(0,0) = \nabla f(0,0) \cdot \mathbf{u} = \sqrt{5}$

3. $\nabla f\,(e^y - ye^x)\,\mathbf{i} + (xe^y - e^x)\,\mathbf{j}, \quad \nabla f(1,0) = \mathbf{i} + (1-e)\mathbf{j}, \quad \mathbf{u} = \dfrac{1}{5}(3\mathbf{i} + 4\mathbf{j}),$

$f'_{\mathbf{u}}(1,0) = \nabla f(1,0) \cdot \mathbf{u} = \frac{1}{5}(7-4e)$

4. $\nabla f = \frac{1}{(x-y)^2}(-2y\mathbf{i} + 2x\mathbf{j}), \quad \nabla f(1,0) = 2\mathbf{j}, \quad \mathbf{u} = \frac{1}{2}(\mathbf{i} - \sqrt{3}\mathbf{j}),$

$f'_{\mathbf{u}}(1,0) = \nabla f(1,0) \cdot \mathbf{u} = -\sqrt{3}$

5. $\nabla f = \frac{(a-b)y}{(x+y)^2}\,\mathbf{i} + \frac{(b-a)x}{(x+y)^2}\,\mathbf{j}, \quad \nabla f(1,1) = \frac{a-b}{4}(\mathbf{i}-\mathbf{j}), \quad \mathbf{u} = \frac{1}{2}\sqrt{2}\,(\mathbf{i}-\mathbf{j}),$

$f'_{\mathbf{u}}(1,1) = \nabla f(1,1) \cdot \mathbf{u} = \frac{1}{4}\sqrt{2}\,(a-b)$

6. $\nabla f = \frac{1}{(cx+dy)^2}\left[(d-c)y\mathbf{i} + (c-d)x\mathbf{j}\right], \quad \nabla f(1,1) = \frac{d-c}{(c+d)^2}(\mathbf{i}, -\mathbf{j}), \quad \mathbf{u} = \frac{1}{\sqrt{c^2+d^2}}(c\mathbf{i} - d\mathbf{j})$

$f'_{\mathbf{u}}(1,1) = \nabla f(1,1) \cdot \mathbf{u} = \frac{d-c}{(c+d)\sqrt{c^2+d^2}}$

7. $\nabla f = \frac{2x}{x^2+y^2}\,\mathbf{i} + \frac{2y}{x^2+y^2}\,\mathbf{j}, \quad \nabla f(0,1) = 2\,\mathbf{j}, \quad \mathbf{u} = \frac{1}{\sqrt{65}}(8\,\mathbf{i} + \mathbf{j}),$

$f'_{\mathbf{u}}(0,1) = \nabla f \cdot \mathbf{u} = \frac{2}{\sqrt{65}}$

8. $\nabla f = 2xy\mathbf{i} + (x^2 + \sec^2 y)\mathbf{j}, \quad \nabla f(-1, \frac{\pi}{4}) = -\frac{\pi}{2}\mathbf{i} + 3\mathbf{j}, \quad \mathbf{u} = \frac{1}{\sqrt{5}}(\mathbf{i} - 2\mathbf{j})$

$f'_{\mathbf{u}}(-1, \frac{\pi}{4}) = \nabla f(-1, \frac{\pi}{4}) \cdot \mathbf{u} = -\frac{1}{\sqrt{5}}(\frac{\pi}{2} + 6)$

9. $\nabla f = (y+z)\mathbf{i} + (x+z)\mathbf{j} + (y+x)\mathbf{k}, \quad \nabla f(1,-1,1) = 2\mathbf{j}, \quad \mathbf{u} = \frac{1}{6}\sqrt{6}\,(\mathbf{i} + 2\mathbf{j} + \mathbf{k}),$

$f'_{\mathbf{u}}(1,-1,1) = \nabla f(1,-1,1) \cdot \mathbf{u} = \frac{2}{3}\sqrt{6}$

10. $\nabla f = (z^2 + 2xy)\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (y^2 + 2zx)\mathbf{k}, \quad \nabla f(1,0,1) = \mathbf{i} + \mathbf{j} + 2\mathbf{k}, \quad \mathbf{u} = \frac{1}{\sqrt{10}}(3\mathbf{i} - \mathbf{k})$

$f'_{\mathbf{u}}(1,0,1) = \nabla f(1,0,1) \cdot \mathbf{u} = \frac{\sqrt{10}}{10}$

11. $\nabla f = 2\left(x + y^2 + z^2\right)\left(\mathbf{i} + 2y\mathbf{j} + 3z^2\mathbf{k}\right), \quad \nabla f(1,-1,1) = 6(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}), \quad \mathbf{u} = \frac{1}{2}\sqrt{2}\,(\mathbf{i} + \mathbf{j}),$

$f'_{\mathbf{u}}(1,-1,1) = \nabla f(1,-1,1) \cdot \mathbf{u} = -3\sqrt{2}$

12. $\nabla f = (2Ax + Byz)\mathbf{i} + (Bxz + 2Cy)\mathbf{j} + Bxy\mathbf{k}, \quad \nabla f(1,2,1) = 2(A+B)\mathbf{i} + (B+4C)\mathbf{j} + 2B\mathbf{k}$

$\mathbf{u} = \frac{1}{\sqrt{A^2+B^2+C^2}}(A\mathbf{i} + B\mathbf{j} + C\mathbf{k}); \quad f'_{\mathbf{u}}(1,2,1) = \nabla f(1,2,1) \cdot \mathbf{u} = \frac{2A^2 + B^2 + 2AB + 6BC}{\sqrt{A^2+B^2+C^2}}$

13. $\nabla f = \tan^{-1}(y+z)\,\mathbf{i} + \frac{x}{1+(y+z)^2}\,\mathbf{j} + \frac{x}{1+(y+z)^2}\,\mathbf{k}, \quad \nabla f(1,0,1) = \frac{\pi}{4}\,\mathbf{i} + \frac{1}{2}\,\mathbf{j} + \frac{1}{2}\,\mathbf{k},$

$\mathbf{u} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} - \mathbf{k}), \qquad f'_{\mathbf{u}}(1,0,1) = \nabla f(1,0,1) \cdot \mathbf{u} = \frac{\pi}{4\sqrt{3}} = \frac{\sqrt{3}}{12}\,\pi$

14. $\nabla f = (y^2\cos z - 2\pi yz^2\cos\pi x + 6zx)\mathbf{i} + (2xy\cos z - 2z^2\sin\pi x)\mathbf{j} + (-xy^2\sin z - 4yz\sin\pi x + 3x^2)\mathbf{k}$

$\nabla f(0,-1,\pi) = (2\pi^3 - 1)\mathbf{i}; \quad \mathbf{u} = \frac{1}{3}(2\mathbf{i} - \mathbf{j} + 2\mathbf{k}), \quad f'_{\mathbf{u}}(0,-1,\pi) = \nabla f(0,-1,\pi) \cdot \mathbf{u} = \frac{2}{3}(2\pi^3 - 1).$

15. $\nabla f = \frac{x}{x^2+y^2}\,\mathbf{i} + \frac{y}{x^2+y^2}\,\mathbf{j}, \quad \mathbf{u} = \frac{1}{\sqrt{x^2+y^2}}\,(-x\mathbf{i} - y\mathbf{j}), \quad f'_{\mathbf{u}}(x,y) = \nabla f \cdot \mathbf{u} = -\frac{1}{\sqrt{x^2+y^2}}$

16. $\nabla f = e^{xy}\left[(y^2+xy^3-y^3)\mathbf{i}+(x-1)(2y+xy^2)\mathbf{j}\right], \quad \nabla f(0,1) = -2\mathbf{j}$

$\mathbf{u} = \dfrac{1}{\sqrt{5}}(-\mathbf{i}+2\mathbf{j}), \quad f'_{\mathbf{u}}(0,1) = \nabla f(0,1)\cdot\mathbf{u} = -\dfrac{4}{5}\sqrt{5}$

17. $\nabla f = (2Ax+2By)\,\mathbf{i}+(2Bx+2Cy)\,\mathbf{j}, \quad \nabla f(a,b) = (2aA+2bB)\mathbf{i}+(2aB+2bC)\,\mathbf{j}$

(a) $\mathbf{u} = \frac{1}{2}\sqrt{2}\,(-\mathbf{i}+\mathbf{j}), \quad f'_{\mathbf{u}}(a,b) = \nabla f(a,b)\cdot\mathbf{u} = \sqrt{2}\,[a(B-A)+b(C-B)]$

(b) $\mathbf{u} = \frac{1}{2}\sqrt{2}\,(\mathbf{i}-\mathbf{j}), \quad f'_{\mathbf{u}}(a,b) = \nabla f(a,b)\cdot\mathbf{u} = \sqrt{2}\,[a(A-B)+b(B-C)]$

18. $\nabla f = \dfrac{z}{x}\mathbf{i}-\dfrac{z}{y}\mathbf{j}+\ln\left(\dfrac{x}{y}\right)\mathbf{k}, \quad \nabla f(1,1,2) = 2\mathbf{i}-2\mathbf{j}$

$\mathbf{u} = \dfrac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}-\mathbf{k}); \quad f'_{\mathbf{u}}(1,1,2) = \nabla f(1,1,2)\cdot\mathbf{u} = 0$

19. $\nabla f = e^{y^2-z^2}(\mathbf{i}+2xy\mathbf{j}-2xz\mathbf{k}), \quad \nabla f(1,2,-2) = \mathbf{i}+4\mathbf{j}+4\mathbf{k}, \quad \mathbf{r}'(t) = \mathbf{i}-2\sin(t-1)\,\mathbf{j}-2e^{t-1}\mathbf{k},$

at $(1,2,-2)$ $\; t=1, \quad \mathbf{r}'(1) = \mathbf{i}-2\mathbf{k}, \quad \mathbf{u} = \frac{1}{5}\sqrt{5}\,(\mathbf{i}-2\mathbf{k}), \quad f'_{\mathbf{u}}(1,2,-2) = \nabla f(1,2,-2)\cdot\mathbf{u} = -\dfrac{7}{5}\sqrt{5}$

20. $\nabla f = 2x\mathbf{i}+z\mathbf{j}+y\mathbf{k}, \quad \nabla f(1,-3,2) = 2\mathbf{i}+2\mathbf{j}-3\mathbf{k}$

Direction: $\mathbf{r}'(-1) = -2\mathbf{i}+3\mathbf{j}-3\mathbf{k}, \quad \mathbf{u} = \dfrac{1}{\sqrt{22}}(-2\mathbf{i}+3\mathbf{j}-3\mathbf{k}), \quad f'_{\mathbf{u}}(1,-3,2) = \nabla f(1,-3,2)\cdot\mathbf{u} = \dfrac{1}{2}\sqrt{22}$

21. $\nabla f = (2x+2yz)\,\mathbf{i}+\left(2xz-z^2\right)\mathbf{j}+(2xy-2yz)\,\mathbf{k}, \quad \nabla f(1,1,2) = 6\,\mathbf{i}-2\,\mathbf{k}$

The vector $\mathbf{v} = 2\,\mathbf{i}+\mathbf{j}-3\,\mathbf{k}$ is a direction vector for the given line; $\mathbf{u} = \dfrac{1}{\sqrt{14}}(2\,\mathbf{i}+\mathbf{j}-3\,\mathbf{k})$

is a corresponding unit vector; $\quad f'_{\mathbf{u}}(1,1,2) = \nabla f(1,1,2)\cdot\mathbf{u} = \dfrac{18}{\sqrt{14}}$

22. $\nabla f = e^{x}(\cos\pi yz\,\mathbf{i}-\pi z\sin\pi yz\,\mathbf{j}-\pi y\sin\pi yz\,\mathbf{k}), \quad \nabla f(0,1,\frac{1}{2}) = -\dfrac{\pi}{2}\mathbf{j}-\pi\,\mathbf{k}$

The vector $\mathbf{v} = 2\,\mathbf{i}+3\,\mathbf{j}+5\,\mathbf{k}$ is a direction vector for the line; $\mathbf{u} = \dfrac{1}{\sqrt{38}}(2\,\mathbf{i}+3\,\mathbf{j}+5\,\mathbf{k})$

is a corresponding unit vector; $\quad f'_{\mathbf{u}}(0,1,\frac{1}{2}) = \nabla f(0,1,\frac{1}{2})\cdot\mathbf{u} = -\dfrac{13\pi}{2\sqrt{38}}$

23. $\nabla f = 2y^2e^{2x}\,\mathbf{i}+2ye^{2x}\,\mathbf{j}, \quad \nabla f(0,1) = 2\,\mathbf{i}+2\,\mathbf{j}, \quad \|\nabla f\| = 2\sqrt{2}, \quad \dfrac{\nabla f}{\|\nabla f\|} = \dfrac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$

f increases most rapidly in the direction $\mathbf{u} = \dfrac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$; the rate of change is $2\sqrt{2}$.

f decreases most rapidly in the direction $\mathbf{v} = -\dfrac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$; the rate of change is $-2\sqrt{2}$.

24. $\nabla f = [1+\cos(x+2y)]\mathbf{i}+2\cos(x+2y)\mathbf{j}, \quad \nabla f(0,0) = 2\mathbf{i}+2\mathbf{j}$

Fastest increase in direction $\mathbf{u} = \dfrac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$, rate of change $\|\nabla f(0,0)\| = 2\sqrt{2}$

Fastest decrease in direction $\mathbf{v} = -\dfrac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$, rate of change $-2\sqrt{2}$

25. $\nabla f = \dfrac{x}{\sqrt{x^2+y^2+z^2}}\,\mathbf{i}+\dfrac{y}{\sqrt{x^2+y^2+z^2}}\,\mathbf{j}+\dfrac{z}{\sqrt{x^2+y^2+z^2}}\,\mathbf{k},$

$\nabla f(1,-2,1) = \dfrac{1}{\sqrt{6}}(\mathbf{i}-2\,\mathbf{j}+\mathbf{k}), \quad \|\nabla f\| = 1$

f increases most rapidly in the direction $\mathbf{u} = \dfrac{1}{\sqrt{6}}(\mathbf{i} - 2\,\mathbf{j} + \mathbf{k})$; the rate of change is 1.

f decreases most rapidly in the direction $\mathbf{v} = -\dfrac{1}{\sqrt{6}}(\mathbf{i} - 2\,\mathbf{j} + \mathbf{k})$; the rate of change is -1.

26. $\nabla f = (2xze^y + z^2)\mathbf{i} + x^2ze^y\mathbf{j} + (x^2e^y + 2xz)\mathbf{k} \quad \nabla f(1, \ln 2, 2) = 8\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$

Fastest increase in direction $\mathbf{u} = \dfrac{1}{\sqrt{29}}(4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$, rate of change $\|\nabla f(1, \ln 2, 2)\| = 2\sqrt{29}$

Fastest decrease in direction $\mathbf{v} = -\dfrac{1}{\sqrt{29}}(4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$, rate of change $-2\sqrt{29}$

27. $\nabla f = f'(x_0)\,\mathbf{i}$. If $f'(x_0) \neq 0$, the gradient points in the direction in which f increases: to the right if $f'(x_0) > 0$, to the left if $f'(x_0) < 0$.

28. 0; the vector $c = \dfrac{\partial f}{\partial y}(x_0, y_0)\mathbf{i} - \dfrac{\partial f}{\partial x}(x_0, y_0)\mathbf{j}$ is perpendicular to the gradient $\nabla f(x_0, y_0)$ and points along the level curve of f at (x_0, y_0).

29. (a) $\displaystyle\lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\sqrt{h^2}}{h} = \lim_{h\to 0} \frac{|h|}{h}$ does not exist

(b) no; by Theorem 15.2.5 f cannot be differentiable at $(0, 0)$

30. (a) $\dfrac{g(\mathbf{x}+\mathbf{h})o(h)}{\|\mathbf{h}\|} = g(\mathbf{x}+\mathbf{h})\,\dfrac{o(\mathbf{h})}{\|\mathbf{h}\|} \to g(\mathbf{x}) \cdot 0 = 0$

(b) $\dfrac{|[g(\mathbf{x}+\mathbf{h}) - g(\mathbf{x})]\nabla f(x) \cdot \mathbf{h}|}{\|\mathbf{h}\|} \leq \dfrac{\|[g(\mathbf{x}+\mathbf{h}) - g(\mathbf{x})]\nabla f(\mathbf{x})\|\|\mathbf{h}\|}{\|\mathbf{h}\|}$

by Schwarz's inequality

$$= |g(\mathbf{x}+\mathbf{h}) - g(\mathbf{x})\| \cdot \|\nabla f(\mathbf{x})\| \to 0$$

31. $\nabla\lambda(x, y) = -\frac{8}{3}x\mathbf{i} - 6y\mathbf{j}$

(a) $\nabla\lambda(1,-1) = -\dfrac{8}{3}\mathbf{i} = 6\mathbf{j}$, $\quad \mathbf{u} = \dfrac{-\nabla\lambda(1,-1)}{\|\nabla\lambda(1,-1)\|} = \dfrac{\frac{8}{3}\mathbf{i} - 6\mathbf{j}}{\frac{2}{3}\sqrt{97}}$, $\quad \lambda'_{\mathbf{u}}(1,-1) = \nabla\lambda(1,-1) \cdot \mathbf{u} = -\dfrac{2}{3}\sqrt{97}$

(b) $\mathbf{u} = \mathbf{i}$, $\quad \lambda'_{\mathbf{u}}(1, 2) = \nabla\lambda(1, 2) \cdot \mathbf{u} = \left(-\frac{8}{3}\mathbf{i} - 12\mathbf{j}\right) \cdot \mathbf{i} = -\frac{8}{3}$

(c) $\mathbf{u} = \frac{1}{2}\sqrt{2}\,(\mathbf{i}+\mathbf{j})$, $\quad \lambda'_{\mathbf{u}}(2, 2) = \nabla\lambda(2, 2) \cdot \mathbf{u} = \left(-\frac{16}{3}\,\mathbf{i} - 12\,\mathbf{j}\right) \cdot \left[\frac{1}{2}\sqrt{2}\,(\mathbf{i}+\mathbf{j})\right] = -\frac{26}{3}\sqrt{2}$

32. $\nabla I = -4x\mathbf{i} - 2y\mathbf{j}$, so the path $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ starting at $(-2, 1)$ must satisfy $x'(t) = -4x(t)$, $y'(t) = -2y(t)$. With the initial conditions, this gives $x(t) = -2e^{-4t}$, $y(t) = e^{-2t}$: the particle will follow the parabolic path $x = -2y^2$ toward the origin.

33. (a) The projection of the path onto the xy-plane is the curve

$$C\colon\ \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

which begins at $(1, 1)$ and at each point has its tangent vector in the direction of $-\nabla f$.

Since

$$\nabla f = 2x\mathbf{i} + 6y\mathbf{j},$$

we have the initial-value problems

$$x'(t) = -2x(t), \quad x(0) = 1 \qquad \text{and} \qquad y'(t) = -6y(t), \quad y(0) = 1.$$

From Theorem 7.6.1 we find that

$$x(t) = e^{-2t} \qquad \text{and} \qquad y(t) = e^{-6t}.$$

Eliminating the parameter t, we find that C is the curve $y = x^3$ from $(1,1)$ to $(0,0)$.

(b) Here

$$x'(t) = -2x(t), \quad x(0) = 1 \qquad \text{and} \qquad y'(t) = -6y(t), \quad y(0) = -2$$

so that

$$x(t) = e^{-2t} \qquad \text{and} \qquad y(t) = -2e^{-6t}.$$

Eliminating the parameter t, we find that the projection of the path onto the xy-plane is the curve $y = -2x^3$ from $(1,-2)$ to $(0,0)$.

34. $z : f(x,y) = \dfrac{1}{2}x^2 - y^2$; $\nabla f = x\mathbf{i} - 2y\mathbf{j}$, so the projection $r(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ of the path onto the xy-plane must satisfy $x'(t) = x(t)$, $y(t) = -2y(t)$

(a) With initial point $(-1, 1, -\frac{1}{2})$,, we get $x(t) = -e^t$, $y(t) = e^{-2t}$, or $y = \dfrac{1}{x^2}$ from $(-1,1)$, in the direction of decreasing x.

(b) With initial point $(1, 0, \frac{1}{2})$, we get $x(t) = e^t$, $y(t) = 0$, or the x-axis from $(1,0)$, in the direction of increasing x.

35. The projection of the path onto the xy-plane is the curve

$$C\colon \ \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

which begins at (a^2, b^2) and at each point has its tangent vector in the direction of $-\nabla f = -\left(2a^2x\mathbf{i} + 2b^2y\mathbf{j}\right)$. Thus,

$$x'(t) = -2a^2x(t), \quad x(0) = a^2 \qquad \text{and} \qquad y'(t) = -2b^2y(t), \quad y(0) = b^2$$

so that

$$x(t) = a^2e^{-2a^2t} \qquad \text{and} \qquad y(t) = b^2e^{-2b^2t}.$$

Since

$$\left[\frac{x}{a^2}\right]^{b^2} = \left(e^{-2a^2t}\right)^{b^2} = \left[\frac{y}{b^2}\right]^{a^2},$$

C is the curve $\left(b^2\right)^{a^2} x^{b^2} = \left(a^2\right)^{b^2} y^{a^2}$ from (a^2, b^2) to $(0,0)$.

36. Must go in direction $-\nabla T = -e^y \cos x\mathbf{i} - e^y \sin x\mathbf{j}$, so must satisfy $x(t) = -e^{y(t)} \cos x(t)$,

$y'(t) = -e^{y(t)} \sin x(t)$. Dividing, we have $\dfrac{y'(t)}{x'(t)} = \dfrac{\sin x(t)}{\cos x(t)}$, or $\dfrac{dy}{dt} = \tan x$. With initial point

$(0,0)$, we get $y = \ln|\sec x|$, in the direction of decreasing x (since $x'(0) < 0$).

37. We want the curve

$$C\colon\ \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

which begins at $(\pi/4, 0)$ and at each point has its tangent vector in the direction of

$$\nabla T = -\sqrt{2}\,e^{-y}\sin x\,\mathbf{i} - \sqrt{2}\,e^{-y}\cos x\,\mathbf{j}.$$

From

$$x'(t) = -\sqrt{2}\,e^{-y}\sin x \qquad \text{and} \qquad y'(t) = -\sqrt{2}\,e^{-y}\cos x$$

we obtain

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \cot x$$

so that

$$y = \ln|\sin x| + C.$$

Since $y = 0$ when $x = \pi/4$, we get $C = \ln\sqrt{2}$ and $y = \ln|\sqrt{2}\sin x|$. As $\nabla T(\pi/4, 0) = -\mathbf{i} - \mathbf{j}$, the curve $y = \ln|\sqrt{2}\,\sin x|$ is followed in the direction of decreasing x.

38. $\nabla z = (1 - 2x)\mathbf{i} + (2 - 6y)\mathbf{j}$, so the projection of the path onto the xy-plane, satisfies $x'(t) = 1 - 2x(t)$, $y'(t) = 2 - 6y(t)$, or $\dfrac{dy}{dx} = \dfrac{2 - 6y}{1 - 2x}$. With initial point $(0,0)$, this gives the curve $3y = (2x - 1)^3 + 1$, in the direction of increasing x.

39. (a)
$$\lim_{h\to 0}\frac{f(2+h,\,(2+h)^2) - f(2,4)}{h} = \lim_{h\to 0}\frac{3(2+h)^2 + (2+h)^2 - 16}{h}$$
$$= \lim_{h\to 0} 4\left[\frac{4h + h^2}{h}\right] = \lim_{h\to 0} 4(4 + h) = 16$$

(b)
$$\lim_{h\to 0}\frac{f\left(\dfrac{h+8}{4},\, 4 + h\right) - f(2,4)}{h} = \lim_{h\to 0}\frac{3\left(\dfrac{h+8}{4}\right)^2 + (4+h) - 16}{h}$$
$$= \lim_{h\to 0}\frac{\frac{3}{16}h^2 + 3h + 12 + 4 + h - 16}{h}$$
$$= \lim_{h\to 0}\left(\tfrac{3}{16}h + 4\right) = 4$$

(c) $\mathbf{u} = \frac{1}{17}\sqrt{17}\,(\mathbf{i} + 4\mathbf{j})$, $\nabla f(2,4) = 12\mathbf{i} + \mathbf{j}$; $f'_{\mathbf{u}}(2,4) = \nabla f(2,4)\cdot\mathbf{u} = \frac{16}{17}\sqrt{17}$

(d) The limits computed in (a) and (b) are not directional derivatives. In (a) and (b) we have, in essence, computed $\nabla f(2,4)\cdot \mathbf{r}_0$ taking $\mathbf{r}_0 = \mathbf{i}+4\mathbf{j}$ in (a) and $\mathbf{r}_0 = \frac{1}{4}\mathbf{i}+\mathbf{j}$ in (b). In neither case is $\mathbf{r}_0$ a unit vector.

40. $\nabla f = \dfrac{-GMm}{(x^2+y^2+z^2)^{3/2}}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}) = \dfrac{-GMm}{\|\mathbf{r}\|^3}\mathbf{r}$

41. (a) $\mathbf{u} = \cos\theta\,\mathbf{i}+\sin\theta\,\mathbf{j}, \qquad \nabla f(x,y) = \dfrac{\partial f}{\partial x}\mathbf{i}+\dfrac{\partial f}{\partial y}\mathbf{j};$

$$f'_{\mathbf{u}}(x,y) = \nabla f\cdot\mathbf{u} = \left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}\right)\cdot(\cos\theta\,\mathbf{i}+\sin\theta\,\mathbf{j}) = \frac{\partial f}{\partial x}\cos\theta+\frac{\partial f}{\partial y}\sin\theta$$

(b) $\nabla f = \left(3x^2+2y-y^2\right)\mathbf{i}+(2x-2xy)\mathbf{j}, \qquad \nabla f(-1,2) = 3\,\mathbf{i}+2\,\mathbf{j}$

$$f'_{\mathbf{u}}(-1,2) = 3\cos(2\pi/3)+2\sin(2\pi/3) = \frac{2\sqrt{3}-3}{2}$$

42. $f'_{\mathbf{u}}(x,y) = \dfrac{\partial f}{\partial x}\cos\dfrac{5\pi}{4}+\dfrac{\partial f}{\partial y}\sin\dfrac{5\pi}{4} = 2xe^{2y}\left(-\dfrac{\sqrt{2}}{2}\right)+2x^2e^{2y}\left(-\dfrac{\sqrt{2}}{2}\right) = -\sqrt{2}xe^{2y}(1+x)$

$f'_{\mathbf{u}}(2,\ln 2) = -\sqrt{2}\cdot 2\cdot e^{2\ln 2}(1+2) = -24\sqrt{2}$

43.
$$\begin{aligned}\nabla(fg) &= \frac{\partial fg}{\partial x}\mathbf{i}+\frac{\partial fg}{\partial y}\mathbf{j} = \left(f\frac{\partial g}{\partial x}+g\frac{\partial f}{\partial x}\right)\mathbf{i}+\left(f\frac{\partial g}{\partial y}+g\frac{\partial f}{\partial y}\right)\mathbf{j}\\ &= f\left(\frac{\partial g}{\partial x}\mathbf{i}+\frac{\partial g}{\partial y}\mathbf{j}\right)+g\left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}\right) = f\,\nabla g+g\,\nabla f\end{aligned}$$

44.
$$\begin{aligned}\nabla\left(\frac{f}{g}\right) &= \frac{\partial}{\partial x}\left(\frac{f}{g}\right)\mathbf{i}+\frac{\partial}{\partial y}\left(\frac{f}{g}\right)\mathbf{j}+\frac{\partial}{\partial z}\left(\frac{f}{g}\right)\mathbf{k}\\ &= \frac{\frac{\partial f}{\partial x}g-f\frac{\partial g}{\partial x}}{g^2}\mathbf{i}+\frac{\frac{\partial f}{\partial y}g-f\frac{\partial g}{\partial y}}{g^2}\mathbf{j}+\frac{\frac{\partial f}{\partial z}g-f\frac{\partial g}{\partial z}}{g^2}\mathbf{k}\\ &= \frac{g(\mathbf{x})\nabla f(\mathbf{x})-f(\mathbf{x})\nabla g(\mathbf{x})}{g^2(\mathbf{x})}\end{aligned}$$

45. $\nabla f^n = \dfrac{\partial f^n}{\partial x}\mathbf{i}+\dfrac{\partial f^n}{\partial y}\mathbf{j} = nf^{n-1}\dfrac{\partial f}{\partial x}\mathbf{i}+nf^{n-1}\dfrac{\partial f}{\partial y}\mathbf{j} = nf^{n-1}\nabla f$

SECTION 15.3

1. $f(\mathbf{b}) = f(1,3) = -2;\quad f(\mathbf{a}) = f(0,1) = 0;\quad f(\mathbf{b}) - f(\mathbf{a}) = -2$

$\nabla f = (3x^2 - y)\,\mathbf{i} - x\mathbf{j};\quad \mathbf{b} - \mathbf{a} = \mathbf{i} + 2\mathbf{j}$ and $\nabla f \cdot (\mathbf{b} - \mathbf{a}) = 3x^2 - y - 2x$

The line segment joining **a** and **b** is parametrized by

$$x = t,\quad y = 1 + 2t,\quad 0 \le t \le 1$$

Thus, we need to solve the equation

$$3t^2 - (1 + 2t) - 2t = -2, \quad \text{which is the same as} \quad 3t^2 - 4t + 1 = 0,\ \ 0 \le t \le 1$$

The solutions are: $t = \frac{1}{3}, t = 1$. Thus, $\mathbf{c} = (\frac{1}{3}, \frac{5}{3})$ satisfies the equation.

Note that the endpoint **b** also satisfies the equation.

2. $\nabla f = 4z\mathbf{i} - 2y\mathbf{j} + (4x + 2z)\mathbf{k},\quad f(\mathbf{a}) = f(0,1,1) = 0,\quad f(\mathbf{b}) = f(1,3,2) = 3$

$\mathbf{b} - \mathbf{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$, so we want (x, y, z) such that

$\nabla f \cdot (\mathbf{b} - \mathbf{a}) = 4z - 4y + 4x + 2z = 6z - 4y + 4x = f(\mathbf{b}) - f(\mathbf{a}) = 3$

Parameterizing the line segment from **a** to **b** by $x(t) = t,\quad y(t) = 1 + 2t,\quad z(t) = 1 + t$,

we get $t = \frac{1}{2}$, or $\mathbf{c} = (\frac{1}{2}, 2, \frac{3}{2})$

3. (a) $f(x,y,z) = a_1x + a_2y + a_3z + C$ (b) $f(x,y,z) = g(x,y,z) + a_1x + a_2y + a_3z + C$

4. Using the mean-value theorem 15.3.1, there exists **c** such that $\nabla f(\mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) = 0$

5. (a) U is not connected

(b) (i) $g(\mathbf{x}) = f(\mathbf{x}) - 1$ (ii) $g(\mathbf{x}) = -f(\mathbf{x})$

6. Bt the mean-value theorem

$$f(\mathbf{x}_1) - f(\mathbf{x}_2) = \nabla f(c) \cdot (\mathbf{x}_1 - \mathbf{x}_2)$$

for some point **c** on the line segment $\mathbf{x}_1\mathbf{x}_2$. Since Ω is convex, **c** is in Ω. Thus

$$|f(\mathbf{x}_1) - f(\mathbf{x}_2)| = |\nabla f(c) \cdot (\mathbf{x}_1 - \mathbf{x}_2)| \le \|\nabla f(\mathbf{c})\|\|\mathbf{x}_1 - \mathbf{x}_2\| \le M\|\mathbf{x}_1 - \mathbf{x}_2\|.$$

by Schwarz's inequality $\qquad c \in \Omega$

7. $\nabla f = 2xy\mathbf{i} + x^2\mathbf{j}$;

$\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = (2\mathbf{i} + e^{2t}\mathbf{j}) \cdot (e^t\mathbf{i} - e^{-t}\mathbf{j}) = e^t$

8. $\nabla f = \mathbf{i} - \mathbf{j};\quad \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = (\mathbf{i} - \mathbf{j}) \cdot (a\mathbf{i} - ab\sin at\mathbf{j}) = a(1 + b\sin at)$

9. $\nabla f = \dfrac{-2x}{1 + (y^2 - x^2)^2}\mathbf{i} + \dfrac{2y}{1 + (y^2 - x^2)^2}\mathbf{j},\quad \nabla f(\mathbf{r}(t)) = \dfrac{-2\sin t}{1 + \cos^2 2t}\mathbf{i} + \dfrac{2\cos t}{1 + \cos^2 2t}\mathbf{j}$

$$\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \left(\frac{-2\sin t}{1 + \cos^2 2t}\mathbf{i} + \frac{2\cos t}{1 + \cos^2 2t}\mathbf{j}\right) \cdot (\cos t\,\mathbf{i} - \sin t\,\mathbf{j}) = \frac{-4\sin t\cos t}{1 + \cos^2 2t} = \frac{-2\sin 2t}{1 + \cos^2 2t}$$

10. $\nabla f = \dfrac{1}{2x^2 + y^3}(4x\mathbf{i} + 3y^2\mathbf{j})$

$$\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \frac{1}{2e^{4t} + t}(4e^{2t}\mathbf{i} + 3t^{2/3}\mathbf{j}) \cdot (2e^{2t}\mathbf{i} + \frac{1}{3}t^{-2/3}\mathbf{j}) = \frac{8e^{4t} + 1}{2e^{4t} + 1}$$

11. $\nabla f = (e^y - ye^{-x})\,\mathbf{i} + (xe^y + e^{-x})\,\mathbf{j}$; $\quad \nabla f(\mathbf{r}(t)) = (t^t - \ln t)\,\mathbf{i} + \left(t^t \ln t + \frac{1}{t}\right)\mathbf{j}$

$$\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \left((t^t - \ln t)\,\mathbf{i} + \left(t^t \ln t + \frac{1}{t}\right)\mathbf{j}\right) \cdot \left(\frac{1}{t}\mathbf{i} + [1 + \ln t]\mathbf{j}\right) = t^t\left(\frac{1}{t} + \ln t + [\ln t]^2\right) + \frac{1}{t}$$

12. $\nabla f = \dfrac{2}{x^2 + y^2 + z^2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

$$\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \frac{2}{1 + e^{4t}}(\sin t\mathbf{i} + \cos t\mathbf{j} + e^{2t}\mathbf{k}) \cdot (\cos t\mathbf{i} - \sin t\mathbf{j} + 2e^{2t}\mathbf{k}) = \frac{4e^{4t}}{1 + e^{4t}}$$

13. $\nabla f = y\mathbf{i} + (x - z)\mathbf{j} - y\mathbf{k}$;

$$\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \left(t^2\mathbf{i} + \left(t - t^3\right)\mathbf{j} - t^2\mathbf{k}\right) \cdot \left(\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}\right) = 3t^2 - 5t^4$$

14. $\nabla f = 2x\mathbf{i} + 2y\mathbf{k}$

$$\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = (2a\cos\omega t\mathbf{i} + 2b\sin\omega t\mathbf{j}) \cdot (-\omega a\sin\omega t\mathbf{i} + \omega b\cos\omega t\mathbf{j} + b\omega\mathbf{k}) = 2\omega(b^2 - a^2)\sin\omega t\cos\omega t$$

15. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$;

$$\begin{aligned}\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= (2a\cos\,\omega t\,\mathbf{i} + 2b\sin\,\omega t\,\mathbf{j} + \mathbf{k}) \cdot (-a\omega\,\sin\,\omega t\,\mathbf{i} + b\omega\,\cos\,\omega t\,\mathbf{j} + b\omega\mathbf{k}) \\ &= 2\omega\left(b^2 - a^2\right)\sin\,\omega t\,\cos\,\omega t + b\omega\end{aligned}$$

16. $\nabla f = y^2\cos(x + z)\mathbf{i} + 2y\sin(x + z)\mathbf{j} + y^2\cos(x + z)\mathbf{k}$

$$\begin{aligned}&\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \\ &= [\cos^2 t\cos(2t + t^3)\mathbf{i} + 2\cos t\sin(2t + t^3)\mathbf{j} + \cos^2 t\cos(2t + t^3)\mathbf{k}] \cdot (2\mathbf{i} - \sin t\mathbf{j} + 3t^2\mathbf{k}) \\ &= \cos t[(2 + 3t^2)\cos t\cos(2t + t^3) - 2\sin t\sin(2t + t^3)]\end{aligned}$$

17. $$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = (2x - 3y)(-\sin t) + (4y - 3x)(\cos t) \\ &= 2\cos t\sin t + 3\sin^2 t - 3\cos^2 t = \sin 2t - 3\cos 2t\end{aligned}$$

12. $$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} = \left(1 + 2\sqrt{\frac{y}{x}}\right)3t^2 + \left(2\sqrt{\frac{x}{y}} - 3\right)\left(-\frac{1}{t^2}\right) \\ &= \left(1 + \frac{2}{t^2}\right)3t^2 + (2t^2 - 3)\left(-\frac{1}{t^2}\right) = 3t^2 + 4 + \frac{3}{t^2}\end{aligned}$$

19. $$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \\ &= (e^x\sin y + e^y\cos x)\left(\tfrac{1}{2}\right) + (e^x\cos y + e^y\sin x)(2) \\ &= e^{t/2}\left(\tfrac{1}{2}\sin 2t + 2\cos 2t\right) + e^{2t}\left(\tfrac{1}{2}\cos\tfrac{1}{2}t + 2\sin\tfrac{1}{2}t\right)\end{aligned}$$

20. $$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} = (4x - y)(-2\sin 2t) + (2y - x)\cos t \\ &= 2\sin 2t(\sin t - 4\cos 2t) + \cos t(2\sin t - \cos 2t)\end{aligned}$$

21. $\dfrac{du}{dt}=\dfrac{\partial u}{\partial x}\dfrac{dx}{dt}+\dfrac{\partial u}{\partial y}\dfrac{dy}{dt}=(e^x\sin y)\,(2t)+(e^x\cos y)\,(\pi)$

$=e^{t^2}[2t\,\sin(\pi t)+\pi\,\cos(\pi t)]$

22. $\dfrac{du}{dt}=\dfrac{\partial u}{\partial x}\cdot\dfrac{dx}{dt}+\dfrac{\partial u}{\partial y}\cdot\dfrac{dy}{dt}+\dfrac{\partial u}{\partial z}\cdot\dfrac{dz}{dt}=-\dfrac{z}{x}2t+\dfrac{z}{y}\dfrac{1}{2\sqrt{t}}+\ln\left(\dfrac{y}{x}\right)e^t(1+t)$

$=-\dfrac{2t^2e^t}{t^2+1}+\dfrac{e^t}{2}+\ln\left(\dfrac{\sqrt{t}}{t^2+1}\right)e^t(1+t)$

23. $\dfrac{du}{dt}=\dfrac{\partial u}{\partial x}\dfrac{dx}{dt}+\dfrac{\partial u}{\partial y}\dfrac{dy}{dt}+\dfrac{\partial u}{\partial z}\dfrac{dz}{dt}$

$=(y+z)(2t)+(x+z)(1-2t)+(y+x)(2t-2)$

$=(1-t)(2t)+(2t^2-2t+1)(1-2t)+t(2t-2)$

$=1-4t+6t^2-4t^3$

24. $\dfrac{du}{dt}=\dfrac{\partial u}{\partial x}\cdot\dfrac{dx}{dt}+\dfrac{\partial u}{\partial y}\cdot\dfrac{dy}{dt}+\dfrac{\partial u}{\partial z}\cdot\dfrac{dz}{dt}=(\sin\pi y+2\pi\sin\pi x)2t+\pi x\cos\pi y(-1)-\cos\pi x(-2t)$

$=2t\left[\sin[\pi(1-t)]+\pi(1-t^2)\sin(\pi t^2)\right]-\pi t^2\cos[\pi(1-t)]+2t\cos(\pi t^2)$

25. $V=\dfrac{1}{3}\pi r^2h,\quad \dfrac{dV}{dt}=\dfrac{\partial V}{\partial r}\dfrac{dr}{dt}+\dfrac{\partial V}{\partial h}\dfrac{dh}{dt}=\left(\dfrac{2}{3}\pi rh\right)\dfrac{dr}{dt}+\left(\dfrac{1}{3}\pi r^2\right)\dfrac{dh}{dt}.$

At the given instant,

$$\frac{dV}{dt}=\frac{2}{3}\pi(280)(3)+\frac{1}{3}\pi(196)(-2)=\frac{1288}{3}\pi.$$

The volume is increasing at the rate of $\dfrac{1288}{3}\pi$ in.^{3}/ sec.

26. $v=\pi r^2h,\quad \dfrac{dv}{dt}=\dfrac{\partial v}{\partial r}\cdot\dfrac{dr}{dt}+\dfrac{\partial v}{\partial h}\cdot\dfrac{dh}{dt}=2\pi rh\dfrac{dr}{dt}+\pi r^2\dfrac{dh}{dt}$

$\dfrac{dr}{dt}=-2,\quad \dfrac{dh}{dt}=3,\quad r=13,\quad h=18,\implies \dfrac{dv}{dt}=-429\pi:$ decreasing at the rate of

$429\pi\ cm^3$/sec.

27. $A=\frac{1}{2}xy\,\sin\theta;\quad \dfrac{dA}{dt}=\dfrac{\partial A}{\partial x\dfrac{dx}{dt}}\dfrac{\partial A}{+\partial y\dfrac{dy}{dt}}+\dfrac{\partial A}{\partial\theta\dfrac{d\theta}{dt}}=\frac{1}{2}\left[(y\sin\theta)\dfrac{dx}{dt}+(x\sin\theta)\dfrac{dy}{dt}+(xy\cos\theta)\dfrac{d\theta}{dt}\right].$

At the given instant

$$\frac{dA}{dt}=\frac{1}{2}\left[(2\sin 1)\,(0.25)+(1.5\sin 1)\,(0.25)+(2(1.5)\cos 1)\,(0.1)\right]\cong 0.2871\,\text{ft}^2/s\cong 41.34\,\text{in}^2/s$$

28. $\dfrac{dz}{dx}=2x\dfrac{dx}{dt}+\dfrac{y}{2}\dfrac{dy}{dt}.$ But $x^2+y^2=13\implies 2x\dfrac{dx}{dt}+2y\dfrac{dy}{dt}=0\implies \dfrac{dy}{dt}=-\dfrac{x}{y}\dfrac{dx}{dt}$

$\implies \dfrac{dz}{dt}=2x\dfrac{dx}{dt}+\dfrac{y}{2}\left(-\dfrac{x}{y}\dfrac{dx}{dt}\right)=\dfrac{3x}{2}\dfrac{dx}{dt}=15.$ $\quad z$ is increasing 15 centimeters per second

29. $$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}=(2x-y)(\cos t)+(-x)(t\cos s)$$

$$=2s\cos^2 t-t\sin s\cos t-st\cos s\cos t$$

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}=(2x-y)(-s\sin t)+(-x)(\sin s)$$

$$=-2s^2\cos t\sin t+st\sin s\sin t-s\cos t\sin s$$

30. $$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$=[\cos(x-y)-\sin(x+y)]t+[-\cos(x-y)-\sin(x+y)]2s$$

$$=(t-2s)\cos(st-s^2+t^2)-(t+2s)\sin(st+s^2-t^2)$$

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$=[\cos(x-y)-\sin(x+y)]s+[-\cos(x-y)-\sin(x+y)](-2t)$$

$$=(s+2t)\cos(st-s^2+t^2)-(s-2t)\sin(st+s^2-t^2)$$

31. $$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}=(2x\tan y)(2st)+\left(x^2\sec^2 y\right)(1)$$

$$=4s^3t^2\tan\left(s+t^2\right)+s^4t^2\sec^2\left(s+t^2\right)$$

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}=(2x\tan y)\left(s^2\right)+\left(x^2\sec^2 y\right)(2t)$$

$$=2s^4t\tan\left(s+t^2\right)+2s^4t^3\sec^2\left(s+t^2\right)$$

32. $$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial s}$$
$$=z^2y\sec xy\tan xy(2t)+z^2x\sec xy\tan xy+2z\sec xy(2st)$$

$$=\sec[2st(s-t^2)]\left(2s^4t^3(s-t^2)\tan[2st(s-t^2)]+2s^3t^2\tan[2st(s-t^2)]+4s^3t^2\right)$$
$$\frac{\partial u}{\partial t}=z^2y\sec xy\tan xy(2s)+z^2x\sec xy\tan xy(-2t)+2z\sec xy(s^2)$$
$$=\sec[2st(s-t^2)]\left(2s^5t^2(s-t^2)\tan[2st(s-t^2)]-4s^5t^4\tan[2st(s-t^2)]+2s^4t\right)$$

33. $$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$
$$=(2x-y)(\cos t)+(-x)(-\cos(t-s))+2z(t\cos s)$$

$$=2s\cos^2 t-\sin(t-s)\cos t+s\cos t\cos(t-s)+2t^2\sin s\cos s$$
$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$$
$$=(2x-y)(-s\sin t)+(-x)(\cos(t-s))+2z(\sin s)$$
$$=-2s^2\cos t\sin t+s\sin(t-s)\sin t-s\cos t\cos(t-s)+2t\sin^2 s$$

34. $$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial s}$$
$$=e^{yz^2}\frac{1}{s}+xz^2e^{yz^2}\cdot 0+2xyze^{yz^2}2s$$
$$=\frac{1}{s}e^{t^3(s^2+t^2)^2}+4st^3(s^2+t^2)\ln(st)e^{t^3(s^2+t^2)^2}$$

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial t}$$
$$=e^{yz^2}\frac{1}{t}+xz^2e^{yz^2}3t^2+2xyze^{yz^2}2t$$
$$=\frac{1}{t}e^{t^3(s^2+t^2)^2}+t^2(s^2+t^2)(3s^2+7t^2)\ln(st)e^{t^3(s^2+t^2)^2}$$

35. $$\frac{d}{dt}[f(\mathbf{r}(t))]=\left[\nabla f(\mathbf{r}(t))\cdot\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}\right]\|\mathbf{r}'(t)\|$$
$$=f'_{\mathbf{u}(t)}(\mathbf{r}(t))\,\|\mathbf{r}'(t)\|\quad\text{where}\quad \mathbf{u}(t)=\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

36. $\frac{\partial}{\partial x}[f(r)]=\frac{d}{dr}[f(r)]\frac{\partial r}{\partial x}=f'(r)\frac{\partial r}{\partial x}=f'(r)\frac{x}{r};$ similarly.

$\frac{\partial}{\partial y}[f(r)]=f'(r)\frac{y}{r}$ and $\frac{\partial}{\partial z}[f(r)]=f'(r)\frac{z}{r}$.

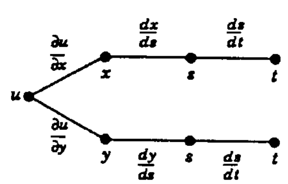
Therefore $\nabla f(r)=f'(r)\frac{x}{r}\mathbf{i}+f'(r)\frac{y}{r}\mathbf{j}+f'(r)\frac{z}{r}\mathbf{k}=f'(r)\frac{\mathbf{r}}{r}$.

37. (a) $(\cos r)\frac{\mathbf{r}}{r}$ (b) $(r\cos r+\sin r)\frac{\mathbf{r}}{r}$

38. (a) $\nabla(r\ln r)=(1+\ln r)\frac{\mathbf{r}}{r}$ (b) $\nabla(e^{1-r^2})=-2re^{1-r^2}\frac{\mathbf{r}}{r}=-2e^{1-r^2}\mathbf{r}$

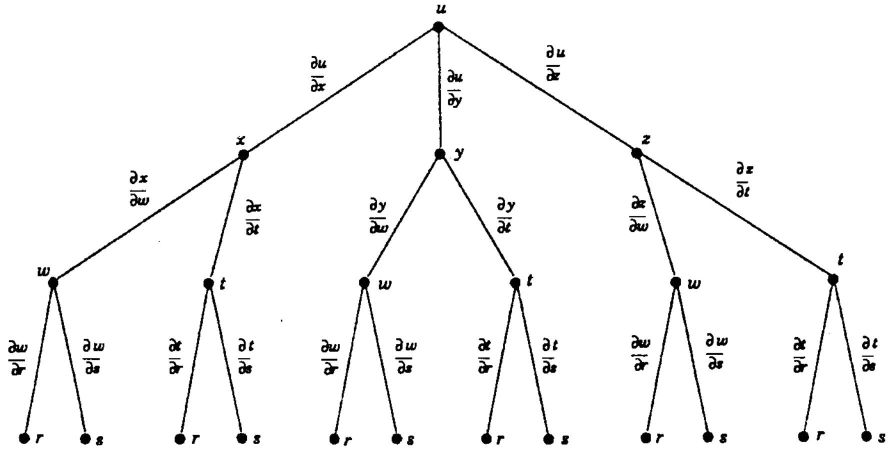
39. (a) $(r\cos r-\sin r)\frac{\mathbf{r}}{r^3}$ (b) $\left(\frac{\sin r-r\cos r}{\sin^2 r}\right)\frac{\mathbf{r}}{r}$

40. (a)



(b) $$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}\frac{ds}{dt}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}\frac{ds}{dt}$$

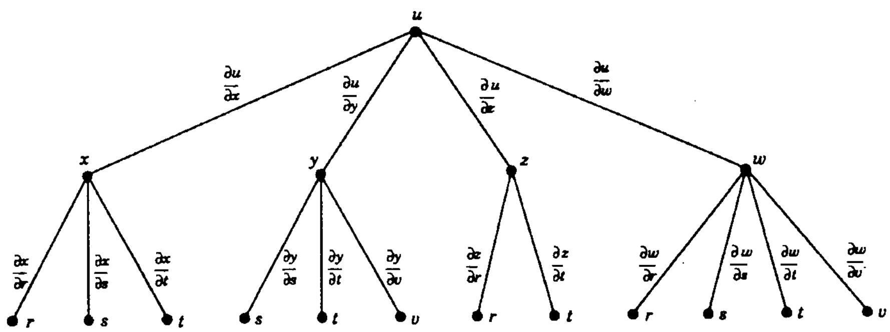
41. (a)



(b) $$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial w}\frac{\partial w}{\partial r}+\frac{\partial x}{\partial t}\frac{\partial t}{\partial r}\right)+\frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial w}\frac{\partial w}{\partial r}+\frac{\partial y}{\partial t}\frac{\partial t}{\partial r}\right)+\frac{\partial u}{\partial z}\left(\frac{\partial z}{\partial w}\frac{\partial w}{\partial r}+\frac{\partial z}{\partial t}\frac{\partial t}{\partial r}\right).$$

To obtain $\partial u/\partial s$, replace each r by s.

42. (a)



(b) $$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial r}+\frac{\partial u}{\partial w}\frac{\partial w}{\partial r},\qquad \frac{\partial u}{\partial v}=\frac{\partial u}{\partial y}\frac{\partial y}{\partial v}+\frac{\partial u}{\partial w}\frac{\partial w}{\partial v}$$

43. $$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$\frac{d^2u}{dt^2}=\frac{\partial u}{\partial x}\frac{d^2x}{dt^2}+\frac{dx}{dt}\left[\frac{\partial^2 u}{\partial x^2}\frac{dx}{dt}+\frac{\partial^2 u}{\partial y\partial x}\frac{dy}{dt}\right]+\frac{\partial u}{\partial y}\frac{d^2y}{dt^2}+\frac{dy}{dt}\left[\frac{\partial^2 u}{\partial x\,\partial y}\frac{dx}{dt}+\frac{\partial^2 u}{\partial y^2}\frac{dy}{dt}\right]$$

and the result follows.

44. Differentiate $\dfrac{\partial u}{\partial s}=\dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial s}+\dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial s}$ with respect to s:

$$\frac{\partial^2 u}{\partial s^2}=\left(\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial s}+\frac{\partial^2 u}{\partial y\partial x}\frac{\partial y}{\partial s}\right)\frac{\partial x}{\partial s}+\frac{\partial u}{\partial x}\frac{\partial^2 x}{\partial s^2}+\left(\frac{\partial^2 u}{\partial x\partial y}\frac{\partial x}{\partial s}+\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial s}\right)\frac{\partial y}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial^2 y}{\partial s^2}$$

$$=\frac{\partial^2 u}{\partial x^2}\left(\frac{\partial x}{\partial s}\right)^2+2\frac{\partial^2 u}{\partial x\partial y}\frac{\partial x}{\partial s}\frac{\partial y}{\partial s}+\frac{\partial^2 u}{\partial y^2}\left(\frac{\partial y}{\partial s}\right)^2+\frac{\partial u}{\partial x}\frac{\partial^2 x}{\partial s^2}+\frac{\partial u}{\partial y}\frac{\partial^2 y}{\partial s^2}$$

45. (a) $$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$$

(b) $$\left(\frac{\partial u}{\partial r}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2\cos^2\theta + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\cos\theta\sin\theta + \left(\frac{\partial u}{\partial y}\right)^2\sin^2\theta,$$

$$\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2\sin^2\theta - 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\cos\theta\sin\theta + \left(\frac{\partial u}{\partial y}\right)^2\cos^2\theta,$$

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2(\cos^2\theta + \sin^2\theta) + \left(\frac{\partial u}{\partial y}\right)^2(\sin^2\theta + \cos^2\theta) = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

46. (a) By Exercise 45 (a)

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cos\theta + \frac{\partial w}{\partial y}\sin\theta, \quad \frac{\partial w}{\partial \theta} = -\frac{\partial w}{\partial x}r\sin\theta + \frac{\partial w}{\partial y}r\cos\theta.$$

Solve these equations simultaneously for $\dfrac{\partial w}{\partial x}$ and $\dfrac{\partial w}{\partial y}$

(b) To obtain the first pair of equations set $w = r$;

to obtain the second pair of equations set $w = \theta$.

(c) θ is not independent of x; $r = \sqrt{x^2+y^2}$ gives

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta$$

47. Solve the equations in Exercise 45 (a) for $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta, \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial u}{\partial \theta}\cos\theta$$

Then $$\nabla u = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} = \frac{\partial u}{\partial r}(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) + \frac{1}{r}\frac{\partial u}{\partial \theta}(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})$$

48. $u(r,\theta) = r^2 \implies \nabla u = 2r\mathbf{e}_r$

49. $u(x,y) = x^2 - xy + y^2 = r^2 - r^2\cos\theta\,\sin\theta = r^2\left(1 - \frac{1}{2}\sin 2\theta\right)$

$$\frac{\partial u}{\partial r} = r(2 - \sin 2\theta), \quad \frac{\partial u}{\partial \theta} = -r^2\cos 2\theta$$

$$\nabla u = \frac{\partial u}{\partial r}\mathbf{e_r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\mathbf{e}_\theta = r(2 - \sin 2\theta)\mathbf{e_r} - r\,\cos 2\theta\,\mathbf{e}_\theta$$

50. $$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y}$$

$$\frac{\partial^2 u}{\partial r\partial\theta} = -\sin\theta\frac{\partial u}{\partial x} - r\sin\theta\left(\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y\partial x}\frac{\partial y}{\partial r}\right) + \cos\theta\frac{\partial u}{\partial y} + r\cos\theta\left(\frac{\partial^2 u}{\partial x\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial r}\right)$$

$$= -\sin\theta\frac{\partial u}{\partial x} + \cos\theta\frac{\partial u}{\partial y} + r\sin\theta\cos\theta\left(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2}\right) + r(\cos^2\theta - \sin^2\theta)\frac{\partial^2 u}{\partial x\partial y}$$

51. From Exercise 45 (a),

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial^2 u}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 u}{\partial y\,\partial x}\sin\theta\,\cos\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2}r^2\sin^2\theta - 2\frac{\partial^2 u}{\partial y\,\partial x}r^2\sin\theta\,\cos\theta + \frac{\partial^2 u}{\partial y^2}r^2\cos^2\theta - r\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right).$$

The term in parentheses is $\frac{\partial u}{\partial r}$. Now divide the second equation by r^2 and add the two equations. The result follows.

52. $u(x,y) = x^2 - 2xy + y^4$, $\quad \frac{\partial u}{\partial x} = 2x - 2y$, $\quad \frac{\partial u}{\partial y} = -2x + 4y^3$

$$\frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = \frac{2y-2x}{4y^3-2x} = \frac{y-x}{2y^3-x}$$

53. Set $u = xe^y + ye^x - 2x^2y$. Then

$$\frac{\partial u}{\partial x} = e^y + ye^x - 4xy, \qquad \frac{\partial u}{\partial y} = xe^y + e^x - 2x^2$$

$$\frac{dy}{dx} = -\frac{\partial u/\partial x}{\partial u/\partial y} = -\frac{e^y + ye^x - 4xy}{xe^y + e^x - 2x^2}.$$

54. $u(x,y) = x^{2/3} + y^{2/3}$, $\quad \frac{\partial u}{\partial x} = \frac{2}{3}x^{-1/3}$, $\quad \frac{\partial u}{\partial y} = \frac{2}{3}y^{-1/3}$

$$\Longrightarrow \frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = -\left(\frac{y}{x}\right)^{1/3}$$

55. Set $u = x\cos xy + y\cos x - 2$. Then

$$\frac{\partial u}{\partial x} = \cos xy - xy\sin xy - y\sin x, \qquad \frac{\partial u}{\partial y} = -x^2\sin xy + \cos x$$

$$\frac{dy}{dx} = -\frac{\partial u/\partial x}{\partial u/\partial y} = \frac{\cos xy - xy\sin xy - y\sin x}{x^2\sin xy - \cos x}.$$

56. $\frac{\partial u}{\partial x} = 2xz^3 + y$, $\quad \frac{\partial u}{\partial y} = 2y + x$, $\quad \frac{\partial u}{\partial z} = 4z^3 + 3x^2z^2$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial z}} = -\frac{2xz^3+y}{4z^3+3x^2z^2}, \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial z}} = -\frac{2y+x}{4z^3+3x^2z^2}$$

57. Set $u = \cos xyz + \ln\left(x^2+y^2+z^2\right)$. Then

$$\frac{\partial u}{\partial x} = -yz\sin xyz + \frac{2x}{x^2+y^2+z^2}, \qquad \frac{\partial u}{\partial y} = -xz\sin xyz + \frac{2y}{x^2+y^2+z^2}, \quad \text{and}$$

$$\frac{\partial u}{\partial z} = -xy\sin xyz + \frac{2z}{x^2+y^2+z^2}.$$

$$\frac{\partial z}{\partial x}=-\frac{\partial u/\partial x}{\partial u/\partial z}=-\frac{2x-yz\left(x^2+y^2+z^2\right)\sin xyz}{2z-xy\left(x^2+y^2+z^2\right)\sin xyz},$$

$$\frac{\partial z}{\partial y}=-\frac{\partial u/\partial y}{\partial u/\partial z}=-\frac{2y-xz\left(x^2+y^2+z^2\right)\sin xyz}{2z-xy\left(x^2+y^2+z^2\right)\sin xyz}.$$

58. (a) Use $\frac{d\mathbf{u}}{dt}=\frac{du_1}{dt}\mathbf{i}+\frac{du_2}{dt}\mathbf{j}$ and apply the chain rule to u_1, u_2.

(b) (i) $\frac{d\mathbf{u}}{dt}=t(e^x\cos y\mathbf{i}+e^y\sin y\mathbf{j})+\pi(-e^x\sin y\mathbf{i}+e^x\cos y\mathbf{j})$

$$=te^{t^2/2}(\cos\pi t\mathbf{i}+\sin\pi t\mathbf{j})+\pi e^{t^2/2}(-\sin\pi t\mathbf{i}+\cos\pi t\mathbf{j})$$

(ii) $\mathbf{u}(t)=e^{t^2/2}\cos\pi t\mathbf{i}+e^{t^2/2}\sin\pi t\mathbf{j}$

$$\frac{d\mathbf{u}}{dt}=(-\pi e^{t^2/2}\sin\pi t+te^{t^2/2}\cos\pi t)\mathbf{i}+(\pi e^{t^2/2}\cos\pi t+te^{t^2/2}\sin\pi t)\mathbf{j}$$

59. $\frac{\partial\mathbf{u}}{\partial s}=\frac{\partial\mathbf{u}}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial\mathbf{u}}{\partial y}\frac{\partial y}{\partial s},\quad \frac{\partial\mathbf{u}}{\partial t}=\frac{\partial\mathbf{u}}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial\mathbf{u}}{\partial y}\frac{\partial y}{\partial t}$

60. $\frac{d\mathbf{u}}{dt}=\frac{\partial\mathbf{u}}{\partial x}\frac{dx}{dt}+\frac{\partial\mathbf{u}}{\partial y}\frac{dy}{dt}+\frac{\partial\mathbf{u}}{\partial z}\frac{dz}{dt}$ where $\frac{\partial\mathbf{u}}{\partial x}=\frac{\partial u_1}{\partial x}\mathbf{i}+\frac{\partial u_2}{\partial x}\mathbf{j}+\frac{\partial u_3}{\partial x}\mathbf{k},$

$\frac{\partial\mathbf{u}}{\partial y}=\frac{\partial u_1}{\partial y}\mathbf{i}+\frac{\partial u_2}{\partial y}\mathbf{j}+\frac{\partial u_3}{\partial y}\mathbf{k},\quad \frac{\partial\mathbf{u}}{\partial z}=\frac{\partial u_1}{\partial z}\mathbf{i}+\frac{\partial u_2}{\partial z}\mathbf{j}+\frac{\partial u_3}{\partial z}\mathbf{k}.$

SECTION 15.4

1. Set $f(x,y)=x^2+xy+y^2$. Then,

$$\nabla f=(2x+y)\mathbf{i}+(x+2y)\mathbf{j},\quad \nabla f(-1,-1)=-3\mathbf{i}-3\mathbf{j}.$$

normal vector $\mathbf{i}+\mathbf{j}$; tangent vector $\mathbf{i}-\mathbf{j}$

tangent line $x+y+2=0$; normal line $x-y=0$

2. Set $f(x,y)=(y-x)^2-2x$, $\nabla f=-2(y-x+1)\mathbf{i}+2(y-x)\mathbf{j}$, $\nabla f(2,4)=-6\mathbf{i}+4\mathbf{j}$

normal vector $-3\mathbf{i}+2\mathbf{j}$; tangent vector $2\mathbf{i}+3\mathbf{j}$

tangent line $3x-2y+2=0$; normal line $2x+3y-16=0$

3. Set $f(x,y)=\left(x^2+y^2\right)^2-9\left(x^2-y^2\right)$. Then,

$$\nabla f=[4x(x^2+y^2)-18x]\mathbf{i}+\left[4y\left(x^2+y^2\right)+18y\right]\mathbf{j},\quad \nabla f\left(\sqrt{2},1\right)=-6\sqrt{2}\,\mathbf{i}+30\mathbf{j}.$$

normal vector $\sqrt{2}\,\mathbf{i}-5\,\mathbf{j}$; tangent vector $5\mathbf{i}+\sqrt{2}\,\mathbf{j}$

tangent line $\sqrt{2}x-5y+3=0$; normal line $5x+\sqrt{2}\,y-6\sqrt{2}=0$

4. Set $f(x,y) = x^3 + y^3$, $\nabla f = 3x^2\mathbf{i} + 3y^2\mathbf{j}$, $\nabla f(1,2) = 3\mathbf{i} + 12\mathbf{j}$

normal vector $\mathbf{i} + 4\mathbf{j}$; tangent vector $4\mathbf{i} - \mathbf{j}$

tangent line $x + 4y - 9 = 0$; normal line $4x - y - 2 = 0$

5. Set $f(x,y) = xy^2 - 2x^2 + y + 5x$. Then,

$$\nabla f = (y^2 - 4x + 5)\,\mathbf{i} + (2xy + 1)\,\mathbf{j}, \quad \nabla f(4,2) = -7\mathbf{i} + 17\mathbf{j}.$$

normal vector $7\mathbf{i} - 17\mathbf{j}$; tangent vector $17\mathbf{i} + 7\mathbf{j}$

tangent line $7x - 17y + 6 = 0$; normal line $17x + 7y - 82 = 0$

6. Set $f(x,y) = x^5 + y^5 - 2x^3$. $\nabla f = (5x^4 - 6x^2)\mathbf{i} + 5y^4\mathbf{j}$, $\nabla f(1,1) = -\mathbf{i} + 5\mathbf{j}$

normal vector $\mathbf{i} - 5\mathbf{j}$; tangent vector $5\mathbf{i} + \mathbf{j}$

tangent line $x - 5y + 4 = 0$; normal line $5x + y - 6 = 0$

7. Set $f(x,y) = 2x^3 - x^2y^2 - 3x + y$. Then,

$$\nabla f = (6x^2 - 2xy^2 - 3)\,\mathbf{i} + (-2x^2y + 1)\,\mathbf{j}, \quad \nabla f(1,-2) = -5\mathbf{i} + 5\mathbf{j}.$$

normal vector $\mathbf{i} - \mathbf{j}$; tangent vector $\mathbf{i} + \mathbf{j}$

tangent line $x - y - 3 = 0$; normal line $x + y + 1 = 0$

8. Set $f(x,y) = x^3 + y^2 + 2x$. $\nabla f = (3x^2 + 2)\mathbf{i} + 2y\mathbf{j}$, $\nabla f(-1,3) = 5\mathbf{i} + 6\mathbf{j}$

normal vector $5\mathbf{i} + 6\mathbf{j}$; tangent vector $6\mathbf{i} - 5\mathbf{j}$

tangent line $5x + 6y - 13 = 0$; normal line $6x - 5y + 21 = 0$

9. Set $f(x,y) = x^2y + a^2y$. By (15.4.4)

$$m = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{2xy}{x^2 + a^2}.$$

At $(0,a)$ the slope is 0.

10. Set $f(x,y,z) = (x^2 + y^2)^2 - z$. $\nabla f = 4x(x^2 + y^2)\mathbf{i} + 4y(x^2 + y^2)\mathbf{j} - \mathbf{k}$, $\nabla f(1,1,4) = 8\mathbf{i} + 8\mathbf{j} - \mathbf{k}$

Tangent plane: $8x + 8y - z - 12 = 0$

Normal: $x = 1 + 8t, \quad y = 1 + 8t, \quad z = 4 - t$

11. Set $f(x,y,z) = x^3 + y^3 - 3xyz$. Then,

$$\nabla f = (3x^2 - 3yz)\,\mathbf{i} + (3y^2 - 3xz)\,\mathbf{j} - 3xy\mathbf{k}, \quad \nabla f\left(1,2,\tfrac{3}{2}\right) = -6\mathbf{i} + \tfrac{15}{2}\mathbf{j} - 6\mathbf{k};$$

tangent plane at $\left(1,2,\frac{3}{2}\right)$: $-6(x-1) + \frac{15}{2}(y-2) - 6\left(z - \frac{3}{2}\right) = 0$, which reduces to $4x - 5y + 4z = 0$.

Normal: $x = 1 + 4t, \quad y = 2 - 5t, \quad z = \frac{3}{2} + 4t$

12. Set $f(x,y,z) = xy^2 + 2z^2$. $\nabla f = y^2\mathbf{i} + 2xy\mathbf{j} + 4z\mathbf{k}$, $\nabla f(1,2,2) = 4\mathbf{i} + 4\mathbf{j} + 8\mathbf{k}$

Tangent plane: $x + y + 2z - 7 = 0$

Normal: $x = 1 + t, \quad y = 2 + t, \quad z = 2 + 2t$

13. Set $z = g(x,y) = axy$. Then, $\nabla g = ay\mathbf{i} + ax\mathbf{j}$, $\nabla g\left(1, \dfrac{1}{a}\right) = \mathbf{i} + a\mathbf{j}$.

tangent plane at $\left(1, \frac{1}{a}, 1\right)$: $z - 1 = 1(x-1) + a\left(y - \frac{1}{a}\right)$, which reduces to $x + ay - z - 1 = 0$

Normal: $x = 1 + t, \quad y = \frac{1}{a} + at, \quad z = 1 - t$

14. Set $f(x,y,z) = \sqrt{x} + \sqrt{y} + \sqrt{z}$. $\nabla f = \frac{1}{2\sqrt{x}}\mathbf{i} + \frac{1}{2\sqrt{y}}\mathbf{j} + \frac{1}{2\sqrt{z}}\mathbf{k}$, $\nabla f(1,4,1) = \frac{1}{2}\mathbf{i} + \frac{1}{4}\mathbf{j} + \frac{1}{2}\mathbf{k}$

Tangent plane: $2x + y + 2z - 8 = 0$

Normal: $x = 1 + 2t, \quad y = 4 + t, \quad z = 1 + 2t$

15. Set $z = g(x,y) = \sin x + \sin y + \sin(x+y)$. Then,

$$\nabla g = [\cos x + \cos(x+y)]\,\mathbf{i} + [\cos y + \cos(x+y)]\,\mathbf{j}, \quad \nabla g(0,0) = 2\mathbf{i} + 2\mathbf{j};$$

tangent plane at $(0,0,0)$: $z - 0 = 2(x-0) + 2(y-0)$, $\quad 2x + 2y - z = 0$.

Normal: $x = 2t, \quad y = 2t, \quad z = -t$

16. Set $f(x,y,z) = x^2 + xy + y^2 - 6x + 2 - z$. $\nabla f = (2x + y - 6)\mathbf{i} + (x + 2y)\mathbf{j} - \mathbf{k}$, $\nabla f(4,-2,-10) = -\mathbf{k}$

Tangent plane: $z = -10$

Normal: $x = 4, \quad y = -2, \quad z = t$

17. Set $f(x,y,z) = b^2c^2x^2 - a^2c^2y^2 - a^2b^2z^2$. Then,

$$\nabla f(x_0, y_0, z_0) = 2b^2c^2x_0\mathbf{i} - 2a^2c^2y_0\mathbf{j} - 2a^2b^2z_0\mathbf{k};$$

tangent plane at (x_0, y_0, z_0):

$$2b^2c^2x_0(x - x_0) - 2a^2c^2y_0(y - y_0) - 2a^2b^2z_0(z - z_0) = 0,$$

which can be rewritten as follows:

$$b^2c^2x_0x - a^2c^2y_0y - a^2b^2z_0z = b^2c^2{x_0}^2 - a^2c^2{y_0}^2 - a^2b^2{z_0}^2$$

$$= f(x_0, y_0, z_0) = a^2 + b^2 + c^2.$$

Normal: $x = x_0 + 2b^2c^2x_0t, \quad y = y_0 - 2a^2c^2y_0t, \quad z = z_0 - 2a^2b^2z_0t$

18. Set $f(x,y,z) = \sin(x\cos y) - z$. $\nabla f = \cos y\cos(x\cos y)\mathbf{i} - x\sin y\cos(x\cos y)\mathbf{j} - \mathbf{k}$,

$\nabla f(0, \frac{\pi}{2}, 0) = -\mathbf{k}$

Tangent plane: $z = 0$

Normal: $x = 0, \quad y = \frac{\pi}{2}, \quad z = t$

19. Set $z = g(x,y) = xy + a^3x^{-1} + b^3y^{-1}$.

$$\nabla g = (y - a^3x^{-2})\mathbf{i} + (x - b^3y^{-2})\mathbf{j}, \quad \nabla g = \mathbf{0} \implies y = a^3x^{-2} \text{ and } x = b^3y^{-2}.$$

Thus,

$y = a^3b^{-6}y^4, \quad y^3 = b^6a^{-3}, \quad y = b^2/a, \quad x = b^3y^{-2} = a^2/b$ and $g(a^2/b, b^2/a) = 3ab$.

The tangent plane is horizontal at $(a^2/b, b^2/a, 3ab)$.

20. $x = g(x,y) = 4x + 2y - x^2 + xy - y^2$. $\nabla g = (4 - 2x + y)\mathbf{i} + (2 + x - 2y)\mathbf{j}$

$\nabla g = \mathbf{0} \implies 4 - 2x + y = 0,\ 2 + x - 2y = 0 \implies x = \dfrac{10}{3},\ y = \dfrac{8}{3}$

The tangent plane is horizontal at $(\frac{10}{3}, \frac{8}{3}, \frac{28}{3})$

21. Set $z = g(x,y) = xy$. Then, $\nabla g = y\mathbf{i} + x\mathbf{j}$.

$$\nabla g = \mathbf{0} \implies x = y = 0.$$

The tangent plane is horizontal at $(0,0,0)$.

22. $z = g(x,y) = x^2 + y^2 - x - y - xy$. $\nabla g = (2x - 1 - y)\mathbf{i} + (2y - 1 - x)\mathbf{j}$

$\nabla g = 0 \implies 2x - 1 - y = 0 = 2y - 1 - x = 0 \implies x = 1,\ y = 1$

The tangent plane is horizontal at $(1, 1, -1)$

23. Set $z = g(x,y) = 2x^2 + 2xy - y^2 - 5x + 3y - 2$. Then,

$$\nabla g = (4x + 2y - 5)\,\mathbf{i} + (2x - 2y + 3)\,\mathbf{j}.$$

$$\nabla g = \mathbf{0} \implies 4x + 2y - 5 = 0 = 2x - 2y + 3 \implies x = \tfrac{1}{3}, \quad y = \tfrac{11}{6}.$$

The tangent plane is horizontal at $\left(\frac{1}{3}, \frac{11}{6}, -\frac{1}{12}\right)$.

24. (a) Set $f(x,y,z) = xy - z$. $\nabla f = y\mathbf{i} + x\mathbf{j} - \mathbf{k}$, $\nabla f(1,1,1) = \mathbf{i} + \mathbf{j} - \mathbf{k}$

upper unit normal $= \dfrac{\sqrt{3}}{3}(-\mathbf{i} - \mathbf{j} + \mathbf{k})$

(b) Set $f(x,y,z) = \dfrac{1}{x} - \dfrac{1}{y} - z$. $\nabla f = -\dfrac{1}{x^2}\mathbf{i} + \dfrac{1}{y^2}\mathbf{j} - \mathbf{k}$, $\nabla f(1,1,0) = -\mathbf{i} + \mathbf{j} - \mathbf{k}$

lower unit normal: $= \dfrac{\sqrt{3}}{3}(-\mathbf{i} + \mathbf{j} - \mathbf{k})$

25. $$\frac{x - x_0}{(\partial f/\partial x)(x_0, y_0, z_0)} = \frac{y - y_0}{(\partial f/\partial y)(x_0, y_0, z_0)} = \frac{z - z_0}{(\partial f/\partial z)(x_0, y_0, z_0)}$$

26. All the tangent planes pass through the origin. To see this, write the equation of the surface as $xf(x/y) - z = 0$. The tangent plane at (x_0, y_0, z_0) has equation

$$(x - x_0)\left[\frac{x_0}{y_0}f'\left(\frac{x_0}{y_0}\right) + f\left(\frac{x_0}{y_0}\right)\right] - (y - y_0)\left[\frac{{x_0}^2}{{y_0}^2}f'\left(\frac{x_0}{y_0}\right)\right] - (z - z_0) = 0.$$

The plane passes through the origin:

$$-\frac{{x_0}^2}{y_0}f'\left(\frac{x_0}{y_0}\right) - x_0 f\left(\frac{x_0}{y_0}\right) + \frac{{x_0}^2}{y_0}f'\left(\frac{x_0}{y_0}\right) + z_0 = z_0 - x_0 f\left(\frac{x_0}{y_0}\right) = 0.$$

27. Since the tangent planes meet at right angles, the normals ∇F and ∇G meet at right angles:

$$\frac{\partial F}{\partial x}\frac{\partial G}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial G}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial G}{\partial z} = 0.$$

28. The sum of the intercepts is a. To see this, note that the equation of the tangent plane at (x_0, y_0, z_0) can be written

$$\frac{x-x_0}{\sqrt{x_0}}+\frac{y-y_0}{\sqrt{y_0}}+\frac{z-z_0}{\sqrt{z_0}}=0.$$

Setting $y=z=0$ we have

$$\frac{x-x_0}{\sqrt{x_0}}=\sqrt{y_0}+\sqrt{z_0}.$$

Therefore the x-intercept is given by

$$x=x_0+\sqrt{x_0}(\sqrt{y_0}+\sqrt{z_0})=x_0+\sqrt{x_0}(\sqrt{a}-\sqrt{x_0})=\sqrt{x_0}\sqrt{a}.$$

Similarly the y-intercept is $\sqrt{y_0}\sqrt{a}$ and the z-intercept is $\sqrt{z_0}\sqrt{a}$. The sum of the intercepts is

$$(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})\sqrt{a}=\sqrt{a}\sqrt{a}=a.$$

29. The tangent plane at an arbitrary point (x_0, y_0, z_0) has equation

$$y_0z_0\,(x-x_0)+x_0z_0\,(y-y_0)+x_0y_0\,(z-z_0)=0,$$

which simplifies to

$$y_0z_0x+x_0z_0y+x_0y_0z=3x_0y_0z_0 \quad \text{and thus to} \quad \frac{x}{3x_0}+\frac{y}{3y_0}+\frac{z}{3z_0}=1.$$

The volume of the pyramid is

$$V=\frac{1}{3}Bh=\frac{1}{3}\left[\frac{(3x_0)\,(3y_0)}{2}\right](3z_0)=\frac{9}{2}x_0y_0z_0=\frac{9}{2}a^3.$$

30. The equation of the tangent plane at (x_0, y_0, z_0) can be written

$$x_0{}^{-1/3}(x-x_0)+y_0{}^{-1/3}(y-y_0)+z_0{}^{-1/3}(z-z_0)=0$$

Setting $y=z=0$, we get the x-intercept $x=x_0+x_0{}^{1/3}(y_0{}^{2/3}+z_0{}^{2/3})=x_0+x_0{}^{1/3}(a^{2/3}-x_0{}^{2/3})$

$\Longrightarrow\ x=x_0{}^{1/3}a^{2/3}$

Similarly, the y-intercept is $y_0{}^{1/3}a^{2/3}$ and the z-intercept is $z_0{}^{1/3}a^{2/3}$.

The sum of the squares of the intercepts is

$$(x_0{}^{2/3}+y_0{}^{2/3}+z_0{}^{2/3})a^{4/3}=a^{2/3}a^{4/3}=a^2.$$

31. The point $(2,3,-2)$ is the tip of $\mathbf{r}(1)$.

Since $\mathbf{r}'(t)=2\mathbf{i}-\dfrac{3}{t^2}\mathbf{j}-4t\mathbf{k}$, we have $\mathbf{r}'(1)=2\mathbf{i}-3\mathbf{j}-4\mathbf{k}$.

Now set $f(x,y,z)=x^2+y^2+3z^2-25$. The function has gradient $2x\mathbf{i}+2y\mathbf{j}+6z\mathbf{k}$.

At the point $(2,3,-2)$,

$$\nabla f=2(2\mathbf{i}+3\mathbf{j}-6\mathbf{k}).$$

The angle θ between $\mathbf{r}'(1)$ and the gradient gives

$$\cos\theta=\frac{(2\mathbf{i}-3\mathbf{j}-4\mathbf{k})}{\sqrt{29}}\cdot\frac{(2\mathbf{i}+3\mathbf{j}-6\mathbf{k})}{7}=\frac{19}{7\sqrt{29}}\cong 0.504.$$

Therefore $\theta\cong 1.043$ radians. The angle between the curve and the plane is

$$\frac{\pi}{2}-\theta\cong 1.571-1.043\cong 0.528 \text{ radians}.$$

32. The curve passes through the point $(3,2,1)$ at $t=1$, and its tangent vector is $\mathbf{r}'(1)=3\mathbf{i}+4\mathbf{j}+3\mathbf{k}$.

For the ellipsoid, set $f(x,y,z) = x^2 + 2y^2 + 3z^2$. $\nabla f = 2x\mathbf{i} + 4y\mathbf{j} + 6z\mathbf{k}$,

$\nabla f(3,2,1) = 6\mathbf{i} + 8\mathbf{j} + 6\mathbf{k}$, which is parallel to $\mathbf{r}'(1)$.

33. Set $f(x,y,z) = x^2y^2 + 2x + z^3$. Then,

$$\nabla f = (2xy^2 + 2)\,\mathbf{i} + 2x^2y\mathbf{j} + 3z^2\mathbf{k}, \quad \nabla f(2,1,2) = 6\mathbf{i} + 8\mathbf{j} + 12\mathbf{k}.$$

The plane tangent to $f(x,y,z) = 16$ at $(2,1,2)$ has equation

$$6(x-2) + 8(y-1) + 12(z-2) = 0, \quad \text{or} \quad 3x + 4y + 6z = 22.$$

Next, set $g(x,y,z) = 3x^2 + y^2 - 2z$. Then,

$$\nabla g = 6x\mathbf{i} + 2y\mathbf{j} - 2\mathbf{k}, \quad \nabla g(2,1,2) = 12\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}.$$

The plane tangent to $g(x,y,z) = 9$ at $(2,1,2)$ is

$$12(x-2) + 2(y-1) - 2(z-2) = 0, \quad \text{or} \quad 6x + y - z = 11.$$

34. Sphere: $f(x,y,z) = z^2 + y^2 + z^2 - 8x - 8y - 6z + 24$, $\nabla f = (2x-8)\mathbf{i} + (2y-8)\mathbf{j} + (2z-6)\mathbf{k}$

$\nabla f(2,1,1) = -4\mathbf{i} - 6\mathbf{j} - 4\mathbf{k}$

Ellipsoid: $g(x,y,z) = x^2 + 3y^2 + 2z^2$, $\nabla g = 2x\mathbf{i} + 6y\mathbf{j} + 4z\mathbf{k}$

$\nabla g(2,1,1) = 4\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}$

Since their normal vectors are parallel, the surfaces are tangent.

35. The gradient to the sphere at $(1,1,2)$ is

$$2x\mathbf{i} + (2y-4)\,\mathbf{j} + (2z-2)\mathbf{k} = 2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}.$$

The gradient to the paraboloid at $(1,1,2)$ is

$$6x\mathbf{i} + 4y\mathbf{j} - 2\mathbf{k} = 6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}.$$

Since

$$(2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) \cdot (6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = 0,$$

the surfaces intersect at right angles.

36. Surface A: Set $f(x,y,z) = xy - az^2$, $\nabla f = y\mathbf{i} + x\mathbf{j} - 2az\mathbf{k}$

Surface B: Set $g(x,y,z) = x^2 + y^2 + z^2$, $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

Surface C: Set $h(x,y,z) = z^2 + 2x^2 - c(z^2 + 2y^2)$. $\nabla h = 4x\mathbf{i} - 4cy\mathbf{j} + 2(1-c)z\mathbf{k}$

Where surface A and surface B intersect, $\nabla f \cdot \nabla g = 4(xy - az^2) = 0$

Where surface A and surface C intersect, $\nabla f \cdot \nabla h = 4(1-c)(xy - az^2) = 0$

Where surface B and surface C intersect, $\nabla g \cdot \nabla h = 4[2x^2 - 2cy^2 + 2(1-c)z^2] = 0$

37. (a) $3x + 4y + 6 = 0$ since plane p is vertical.

(b) $y = -\frac{1}{4}(3x + 6) = -\frac{1}{4}[3(4t-2) + 6] = -3t$

$z = x^2 + 3y^2 + 2 = (4t-2)^2 + 3(-3t)^2 + 2 = 43t^2 - 16t + 6$

$\mathbf{r}(t) = (4t-2)\mathbf{i} - 3t\mathbf{j} + (43t^2 - 16t + 6)\mathbf{k}$

(c) From part (b) the tip of $\mathbf{r}(1)$ is $(2,-3,33)$. We take

$\mathbf{r}'(1) = 4\mathbf{i} - 3\mathbf{j} + 70\mathbf{j}$ as $\mathbf{d}$ to write

$$\mathbf{R}(s) = (2\mathbf{i} - 3\mathbf{j} + 33\mathbf{k}) + s(4\mathbf{i} - 3\mathbf{j} + 70\mathbf{k}).$$

(d) Set $g(x,y) = x^2 + 3y^2 + 2$. Then,

$$\nabla g = 2x\mathbf{i} + 6y\mathbf{j} \quad \text{and} \quad \nabla g(2,-3) = 4\mathbf{i} - 18\mathbf{j}.$$

An equation for the plane tangent to $z = g(x,y)$ at $(2,-3,33)$ is

$$z - 33 = 4(x-2) - 18(y+3) \quad \text{which reduces to} \quad 4x - 18y - z = 29.$$

(e) Substituting t for x in the equations for p and p_1, we obtain

$$3t + 4y + 6 = 0 \quad \text{and} \quad 4t - 18y - z = 29.$$

From the first equation

$$y = -\tfrac{3}{4}(t+2)$$

and then from the second equation

$$z = 4t - 18\left[-\tfrac{3}{4}(t+2)\right] - 29 = \tfrac{35}{2}t - 2.$$

Thus,

$$(*) \qquad \mathbf{r}(t) = t\mathbf{i} - \left(\tfrac{3}{4}t + \tfrac{3}{2}\right)\mathbf{j} + \left(\tfrac{35}{2}t - 2\right)\mathbf{k}.$$

Lines l and l' are the same. To see this, consider how l and l' are formed; to assure yourself, replace t in $(*)$ by $4s + 2$ to obtain $\mathbf{R}(s)$ found in part (c).

SECTION 15.5

1. $\nabla f(x,y) = (2-2x)\,\mathbf{i} - 2y\mathbf{j} = \mathbf{0}$ only at $(1,0)$.

The difference

$$f(1+h,k) - f(1,0) = \left[2(1+h) - (1+h)^2 - k^2\right] - 1 = -h^2 - k^2$$

is negative for all small h and k; there is a local maximum of 1 at $(1,0)$.

2. $\nabla f(x,y) = (2-2x)\,\mathbf{i} + (2+2y)\,\mathbf{j} = \mathbf{0}$ only at $(1,-1)$.

The difference

$f(1+h,-1+k) - f(1,-1)$

$$= [2(1+h) + 2(-1+k) - (1+h)^2 + (-1+k)^2 + 5] - 5 = -h^2 + k^2$$

does not keep a constant sign for small h and k; $(1,-1)$ is a saddle point.

3. $\nabla f(x,y) = (2x+y+3)\,\mathbf{i} + (x+2y)\,\mathbf{j} = \mathbf{0}$ only at $(-2,1)$.

The difference

$$f(-2+h,\,1+k) - f(-2,1)$$
$$= [(-2+h)^2 + (-2+h)(1+k) + (1+k)^2 + 3(-2+h) + 1] - (-2) = h^2 + hk + k^2$$

is positive for all small h and k. To see this, note that

$$h^2 + hk + k^2 \geq h^2 + k^2 - |h|\,|k| > 0;$$

there is a local minimum of -2 at $(-2,1)$.

4. $\nabla f = (3x^2 - 3)\mathbf{i} + \mathbf{j}$ is never $\mathbf{0}$; there are no stationary points and no local extreme values.

5. $\nabla f = (2x+y-6)\,\mathbf{i} + (x+2y)\,\mathbf{j} = \mathbf{0}$ only at $(4,-2)$.

$f_{xx} = 2, \quad f_{xy} = 1, \quad f_{yy} = 2.$

At $(4,-2)$, $D = -3 < 0$ and $A = 2 > 0$ so we have a local min; the value is -10.

6. $\nabla f = (2x+2y+2)\mathbf{i} + (2x+6y+10)\mathbf{j} = \mathbf{0}$ only at $(1,-2)$

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y \partial x} = 2, \quad \frac{\partial^2 f}{\partial y^2} = 6; \quad D = 2^2 - 2\cdot 6 < 0, \ A = 2 \implies \text{local min of } -8.$$

7. $\nabla f = (3x^2 - 6y)\mathbf{i} + \left(3y^2 - 6x\right)\mathbf{j} = \mathbf{0}$ at $(2,2)$ and $(0,0)$.

$f_{xx} = 6x, \quad f_{xy} = -6, \quad f_{yy} = 6y, \quad D = 36 - 36xy.$

At $(2,2)$, $D = -108 < 0$ and $A = 12 > 0$ so we have a local min; the value is -8.

At $(0,0)$, $D = 36 > 0$ so we have a saddle point.

8. $\nabla f = (6x+y+5)\mathbf{i} + (x-2y-5)\mathbf{j} = \mathbf{0}$ at $\left(-\dfrac{5}{13}, -\dfrac{35}{13}\right)$

$$\frac{\partial^2 f}{\partial x^2} = 6, \quad \frac{\partial^2 f}{\partial y \partial x} = 1, \quad \frac{\partial^2 f}{\partial y^2} = -2; \quad D = 1^2 - 6\cdot(-2) > 0; \quad (-5/13,\ -35/13) \text{ is a saddle point.}$$

9. $\nabla f = (3x^2 - 6y + 6)\mathbf{i} + (2y - 6x + 3)\,\mathbf{j} = \mathbf{0}$ at $\left(5, \frac{27}{2}\right)$ and $\left(1, \frac{3}{2}\right)$.

$f_{xx} = 6x, \quad f_{xy} = -6, \quad f_{yy} = 2, \quad D = 36 - 12x.$

At $\left(5, \frac{27}{2}\right)$, $D = -24 < 0$ and $A = 30 > 0$ so we have a local min; the value is $-\frac{117}{4}$.

At $\left(1, \frac{3}{2}\right)$, $D = 24 > 0$ so we have a saddle point.

10. $\nabla f = (2x-2y-3)\mathbf{i} + (-2x+4y+5)\mathbf{j} = \mathbf{0}$ at $\left(\frac{1}{2}, -1\right)$

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y \partial x} = -2, \quad \frac{\partial^2 f}{\partial y^2} = 4; \quad D = (-2)^2 - 2\cdot 4 < 0, \ A = 2 \implies \text{local minimum of } -\tfrac{13}{4}.$$

11. $\nabla f = \sin y\,\mathbf{i} + x\cos y\,\mathbf{j} = \mathbf{0}$ at $(0, n\pi)$ for all integral n.

$f_{xx} = 0, \quad f_{xy} = \cos y, \quad f_{yy} = -x\sin y, \quad D = \cos^2 y.$

Since $D = \cos^2 n\pi = 1 > 0$, each stationary point is a saddle point.

12. $\nabla f = \sin y\,\mathbf{i} + (1 + x\cos y)\mathbf{j} = \mathbf{0}$ at $(-1, 2n\pi)$ and $(1, (2n+1)\pi)$ for all integral n.

$\dfrac{\partial^2 f}{\partial x^2} = 0, \quad \dfrac{\partial^2 f}{\partial y \partial x} = \cos y, \quad \dfrac{\partial^2 f}{\partial y^2} = -x\sin y; \quad D = \cos^2 y - 0\cdot(-x\sin y) > 0$ at the above points,

so they are all saddle points

13. $\nabla f = (2xy + 1 + y^2)\mathbf{i} + (x^2 + 2xy + 1)\,\mathbf{j} = \mathbf{0}$ at $(1,-1)$ and $(-1,1)$.

$f_{xx} = 2y, \quad f_{xy} = 2x + 2y, \quad f_{yy} = 2x, \quad D = 4(x+y)^2 - 4xy.$

At both $(1,-1)$ and $(-1,1)$ we have saddle points since $D = 4 > 0$.

14. $\nabla f = \left(\dfrac{1}{y} + \dfrac{y}{x^2}\right)\mathbf{i} + \left(-\dfrac{x}{y^2} - \dfrac{1}{x}\right)\mathbf{j} = \dfrac{x^2+y^2}{x^2 y}\mathbf{i} - \dfrac{x^2+y^2}{xy^2}\mathbf{j}$ is never $\mathbf{0}$;

no stationary points, no local extreme values.

15. $\nabla f = (y - x^{-2})\mathbf{i} + (x - 8y^{-2})\,\mathbf{j} = \mathbf{0}$ only at $\left(\frac{1}{2}, 4\right)$.

$f_{xx} = 2x^{-3}, \quad f_{xy} = 1, \quad f_{yy} = 16y^{-3}, \quad D = 1 - 32x^{-3}y^{-3}.$

At $\left(\frac{1}{2}, 4\right)$, $D = -3 < 0$ and $A = 16 > 0$ so we have a local min; the value is 6.

16. $\nabla f = (2x - 2y)\mathbf{i} + (-2x - 2y)\mathbf{j} = \mathbf{0}$ only at $(0,0)$

$\dfrac{\partial^2 f}{\partial x^2} = 2, \quad \dfrac{\partial^2 f}{\partial y \partial x} = -2, \quad \dfrac{\partial^2 f}{\partial y^2} = -2; \quad D = (-2)^2 - 2\cdot(-2) > 0;$ $(0,0)$ is a saddle point.

17. $\nabla f = (y - x^{-2})\mathbf{i} + (x - y^{-2})\,\mathbf{j} = 0$ only at $(1,1)$.

$f_{xx} = 2x^{-3}, \quad f_{xy} = 1, \quad f_{yy} = 2y^{-3}, \quad D = 1 - 4x^{-3}y^{-3}.$

At $(1,1)$, $D = -3 < 0$ and $A = 2 > 0$ so we have a local min; the value is 3.

18. $\nabla f = (2xy - y^2 - 1)\mathbf{i} + (x^2 - 2xy + 1)\mathbf{j} = 0$ at $(1,1)$, $(-1,1)$

$\dfrac{\partial^2 f}{\partial x^2} = 2y, \quad \dfrac{\partial^2 f}{\partial y \partial x} = 2(x-y), \quad \dfrac{\partial^2 f}{\partial y^2} = -2x; \quad D = 4(x-y)^2 + 4xy > 0$ at the above points;

$(1,1)$ and $(-1,1)$ are saddle points.

19. $\nabla f = \dfrac{2\left(x^2 - y^2 - 1\right)}{\left(x^2+y^2+1\right)^2}\mathbf{i} + \dfrac{4xy}{\left(x^2+y^2+1\right)^2}\mathbf{j} = \mathbf{0}$ at $(1,0)$ and $(-1,0)$.

$f_{xx} = \dfrac{-4x^3 + 12xy^2 + 12x}{\left(x^2+y^2+1\right)^3}, \quad f_{xy} = \dfrac{4y^3 + 4y - 12x^2 y}{\left(x^2+y^2+1\right)^3}, \quad f_{yy} = \dfrac{4x^3 + 4xy^2 + 4x - 16xy^2}{\left(x^2+y^2+1\right)^3}.$

At $(1,0)$,

$$A = f_{xx}(1,0) = 1 > 0, \quad B = f_{xy}(1,0) = 0, \quad C = f_{yy}(1,0) = 1, \quad D = -1 < 0.$$

Thus, $(1,0)$ is a local min; $f(1,0) = -1$.

At $(-1,0)$,

$$A = f_{xx}(-1,0) = -1 < 0, \quad B = f_{xy}(-1,0) = 0, \quad C = f_{yy}(-1,0) = -1, \quad D = -1 < 0.$$

Thus, $(1,0)$ is a local max; $f(-1,0) = 1$.

20. $\nabla f = \left(\ln xy + 1 - \dfrac{3}{x}\right)\mathbf{i} + \dfrac{x-3}{y}\mathbf{j} = \mathbf{0}$ at $(3,\ 1/3)$

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{x} + \frac{3}{x^2}, \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{1}{y}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{3-x}{y^2}$$

At $(3,\ 1/3)$, $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{2}{3}$, $\dfrac{\partial^2 f}{\partial y \partial x} = 3$, $\dfrac{\partial^2 f}{\partial y^2} = 0$ and $D = 9 > 0 \implies$ saddle point.

21. $\nabla f = (4x^3 - 4x)\,\mathbf{i} + 2y\,\mathbf{j} = \mathbf{0}$ at $(0,0)$, $(1,0)$, and $(-1,0)$.

$f_{xx} = 12x^2 - 4$, $f_{xy} = 0$, $f_{yy} = 2$, $D = 8 - 24x^2$.

At $(0,0)$, $D = 8 > 0$. Thus, $(0,0)$ is a saddle point.

At $(\pm 1, 0)$, $D = -16 < 0$ and $A = 8 > 0$. Thus, the points $(1,0)$ and $(-1,0)$ are local minima;

$f(\pm 1, 0) = -3$.

22. $\nabla f = 2xe^{x^2-y^2}(1 + x^2 + y^2)\mathbf{i} + 2ye^{x^2-y^2}(1 - x^2 - y^2)\mathbf{j} = \mathbf{0}$ at $(0,0)$, $(0,1)$, $(0,-1)$

$$A = \frac{\partial^2 f}{\partial x^2} = 2xe^{x^2-y^2}(2x + 2x^3 + 2xy^2) + e^{x^2-y^2}(2 + 6x^2)$$

$$B = \frac{\partial^2 f}{\partial y \partial x} = -2ye^{x^2-y^2}(2x + 2x^3 + 2xy^2) + e^{x^2-y^2}(4xy)$$

$$C = \frac{\partial^2 f}{\partial y^2} = -2ye^{x^2-y^2}(2y - 2yx^2 - 2y^3) + e^{x^2-y^2}(2 - 2x^2 - 6y^2)$$

At $(0,0)$, $B^2 - AC = 0 - (2)(2) = -4 < 0, A > 0$ local minimum of 0.

At $(0, \pm 1)$, $B^2 - AC = 0 - (2e^{-1})(-4e^{-1}) = 8e^{-2} > 0$, saddle points

23. (a) $\nabla f = (2x + ky)\,\mathbf{i} + (2y + kx)\,\mathbf{j}$ and $\nabla f(0,0) = \mathbf{0}$ independent of the value of k.

(b) $f_{xx} = 2$, $f_{xy} = k$, $f_{yy} = 2$, $D = k^2 - 4$. Thus, $D > 0$ for $|k| > 2$ and $(0,0)$ is a saddle point

(c) $D = k^2 - 4 < 0$ for $|k| < 2$. Since $A = f_{xx} = 2 > 0$, $(0,0)$ is a local minimum.

(d) The test is inconclusive when $D = k^2 - 4 = 0$ i.e., for $k = \pm 2$.

24. (a) $\nabla f = (2x + ky)\mathbf{i} + (kx + 8y)\mathbf{j} = \mathbf{0}$ at $(0,0)$

(b) $\dfrac{\partial^2 f}{\partial x^2} = 2$, $\dfrac{\partial^2 f}{\partial y \partial x} = k$, $\dfrac{\partial^2 f}{\partial y^2} = 8$; we want $k^2 - 16 > 0$, or $|k| > 4$

(c) We want $k^2 - 16 < 0$, or $|k| < 4$

(d) $k = \pm 4$

25. Let $P(x,y,z)$ be a point in the plane. We want to find the minimum of $f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$. However, it is sufficient to minimize the square of the distance: $F(x,y,z) = x^2 + y^2 + z^2$. It is clear that F has a minimum value, but no maximum value. Since P lies in the plane, $2x - y + 2z = 16$ which implies $y = 2x + 2z - 16 = 2(x + z - 8)$. Thus, we want to find the minimum value of

$$F(x,z) = x^2 + 4(x+z-8)^2 + z^2$$

Now,

$$\nabla F = [2x + 8(x+z-8)]\,wi + [8(x+z-8)\,\mathbf{k}$$

The gradient is **0** when

$$2x + 8(x+z-8) = 0 \quad \text{and} \quad 8(x+z-8) + 2z = 0$$

The only solution to this pair of equations is: $x = z = \dfrac{32}{9}$, from which it follows that $y = -\dfrac{16}{9}$.

The point in the plane that is closest to the origin is $P\left(\frac{32}{9}, -\frac{16}{9}, \frac{32}{9}\right)$.

The distance from the origin to the plane is: $F(P) = \frac{16}{3}$.

Check using (12.6.7): $d(P,0) = \dfrac{|2\cdot 0 - 0 + 2\cdot 0 - 16|}{\sqrt{2^2 + (-1)^2 + 2^2}} = \dfrac{16}{3}$.

26. We want to minimize $(x+1)^2 + (y-2)^2 + (z-4)^2$ on the plane. Since $z = -16 - \frac{3}{2}x + 2y$,

we need to minimize $f(x,y) = (x+1)^2 + (y-2)^2 + (-20 - \frac{3}{2}x + 2y)^2$;

$$\nabla f = \left(\tfrac{13}{2}x - 6y + 62\right)\mathbf{i} + (-84 - 6x + 10y)\,\mathbf{j} = \mathbf{0} \text{ at } (-4,6)$$

Closest point $(-4,6,2)$, distance$=\sqrt{(-1-(-4))^2 + (2-6)^2 + (4-2)^2} = \sqrt{29}$

27. Using the hint, we want to find the maximum value of $f(x,y) = 18xy - x^2y - xy^2$.

The gradient of f is:

$$\nabla D = \left(18y - 2xy - y^2\right)\mathbf{i} + \left(18x - x^2 - 2xy\right)\mathbf{j}$$

The gradient is **0** when

$$18y - 2xy - y^2 = 0 \quad \text{and} \quad 18x - x^2 - 2xy = 0$$

The solution set of this pair of equations is: $(0,0)$, $(18,0)$, $(0,18)$, $(6,6)$.

It is easy to verify that f is a maximum when $x = y = 6$. The three numbers that satisfy $x+y+z = 18$ and maximize the product xyz are: $x = 6$, $y = 6$, $z = 6$.

28. $f(y,z) = 30yz^2 - y^2z^2 - yz^3$, $\nabla f = (30z^2 - 2yz^2 - z^3)\mathbf{j} + (60yz - 2y^2z - 3yz^2)\mathbf{k} = \mathbf{0}$ at $(\frac{15}{2}, 15)$

(other points are not in the interior); $f\left(\dfrac{15}{2}, 15\right) = \dfrac{15^4}{4}$.

On the line $y + z = 30$, $f(y,z) = 0$ so the maximum of xyz^2 occurs at $x = y = \dfrac{15}{2}$, $z = 15$

29. $\nabla f = \dfrac{1}{(x^2+y^2)^{3/2}}(-x\mathbf{i} - y\mathbf{j})$ is never **0** on D. Note that $f(x,y)$ is the reciprocal of the distance of (x,y) from the origin. The point of D closest to the origin (draw a figure) is $(1,1)$. Therefore $f(1,1) = 1/\sqrt{2}$ is the maximum value of f. The point of D furthest from the origin is $(3,4)$. Therefore $f(3,4) = 1/5$ is the least value taken on by f.

30. Consider $g(t) = te^{-t}$, $0 \le t \le 25$. $g'(t) = e^{-t}(1-t)$; g has an absolute minimum of 0 at $t = 0$ and an absolute maximum of $\frac{1}{e}$ at $t = 1$. Therefore $(x^2+y^2)e^{-(x^2+y^2)}$ has an absolute minimum of 0 at $(0,0)$ and an absolute maximum of $\frac{1}{e}$ attained at all points of the unit circle $x^2+y^2 = 1$.

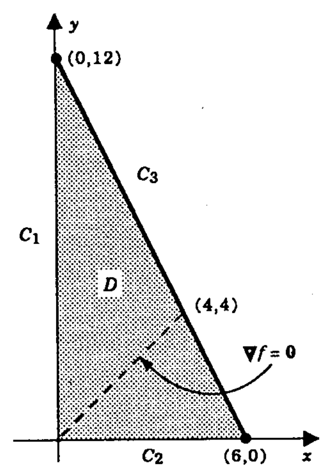
31. $\nabla f = 2(x-1)\mathbf{i} + 2(y-1)\mathbf{j} = \mathbf{0}$ only at $(1,1)$. As the sum of two squares, $f(x,y) \ge 0$. Thus, $f(1,1) = 0$ is a minimum. To examine the behavior of f on the boundary of D, we note that f represents the square of the distance between (x,y) and $(1,1)$. Thus, f is maximal at the point of the boundary furthest from $(1,1)$. This is the point $\left(-\sqrt{2}, -\sqrt{2}\right)$; the maximum value of f is $f\left(-\sqrt{2}, -\sqrt{2}\right) = 6 + 4\sqrt{2}$.

32. $\nabla f = y\mathbf{i} + (x-3)\mathbf{j} = \mathbf{0}$ at $(3,0)$, which is not in the interior of D. The boundary is

$$\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}. \quad f(\mathbf{r}(t)) = 3\sin t(3\cos t - 3) = 9\sin t(\cos t - 1).$$

$\dfrac{df}{dt} = 9(2\cos^2 t - \cos t - 1)$; $\dfrac{df}{dt} = 0 \implies \cos t = 1, -\frac{1}{2}$ which yields the points $A\,(3,0)$, $B\,(-\frac{3}{2}, \frac{3\sqrt{3}}{2})$, $C\,(-\frac{3}{2}, -\frac{3\sqrt{3}}{2})$: $f(A) = 0$, $f(B) = -\frac{27\sqrt{3}}{4}$ min, $f(C) = \frac{27\sqrt{3}}{4}$ max

33. $\nabla f = 2(x-y)\mathbf{i} - 2(x-y)\mathbf{j} = \mathbf{0}$ at each point of the line segment $y = x$ from $(0,0)$ to $(4,4)$. Since $f(x,x) = 0$ and $f(x,y) \ge 0$, f takes on its minimum of 0 at each of these points.



Next we consider the boundary of D. We parametrize each side of the triangle:

$$C_1: \mathbf{r}_1(t) = t\mathbf{j}, \quad t \in [0,12]$$

$$C_2: \mathbf{r}_2(t) = t\mathbf{i}, \quad t \in [0,6]$$

$$C_3: \mathbf{r}_3(t) = t\mathbf{i} + (12-2t)\mathbf{j}, \quad t \in [0,6]$$

and observe from

$$f(\mathbf{r}_1(t)) = t^2, \quad t \in [0,12]$$

$$f(\mathbf{r}_2(t)) = t^2, \quad t \in [0,6]$$

$$f(\mathbf{r}_3(t)) = (3t-12)^2, \quad t \in [0,6]$$

that f takes on its maximum of 144 at the point $(0,12)$.

34. $\nabla f = 3(x-3)\mathbf{i} + 2y\mathbf{j} = \mathbf{0}$ at $(3,0)$ is not in the interior of D.

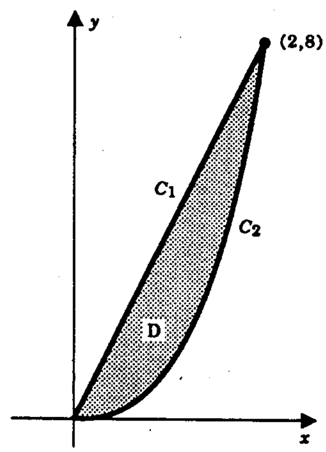
Boundary: On $y = x^2$, $f = (x-3)^2 + x^4$, $\dfrac{df}{dx} = 2(x-3) + 4x^3 = 0$ at $x = 1 \implies (1,1)$

On $y = 4x$, $f = (x-3)^2 + 16x^2$, $\dfrac{df}{dx} = 2(x-3) + 32x = 0$ at $x = \frac{3}{17} \implies (\frac{3}{17}, \frac{12}{17})$

So max and min occur at one or more of the following points: $(0,0)$, $(4,16)$, $(1,1)$, and $(\frac{3}{17}, \frac{12}{17})$.

Checking the value of f at these points, we find that $(1,1)$ gives an absolute min of 5, $(4,16)$ gives an absolute max of 257.

35. $\nabla f = 2(x-4)\mathbf{i} + 2y\mathbf{j}$ is never $\mathbf{0}$ at an interior point of D. Next we examine f on the boundary of D:



$$C_1 : \mathbf{r}_1(t) = t\mathbf{i} + 4t\mathbf{j}, \quad t \in [0, 2,],$$

$$C_2 : \mathbf{r}_2(t) = t\mathbf{i} + t^3\mathbf{j}, \quad t \in [0, 2].$$

Note that

$$f_1(t) = f(\mathbf{r}_1(t)) = 17t^2 - 8t + 16,$$

$$f_2(t) = f(\mathbf{r}_2(t)) = (t-4)^2 + t^6.$$

Next

$$f_1'(t) = 34t - 8 = 0 \quad \Longrightarrow \quad t = 4/17 \quad \text{and gives} \quad x = 4/17, \quad y = 16/17$$

and

$$f_2'(t) = 6t^5 + 2t - 8 = 0 \quad \Longrightarrow \quad t = 1 \quad \text{and gives} \quad x = 1, \quad y = 1.$$

The extreme values of f can be culled from the following list:

$$f(0,0) = 16, \quad f(2,8) = 68, \quad f\left(\tfrac{4}{17}, \tfrac{16}{17}\right) = \tfrac{256}{17}, \quad f(1,1) = 10.$$

We see that $f(1,1) = 10$ is the absolute minimum and $f(2,8)$ is the absolute maximum.

36. $\nabla f = 2(x+y-2)(\mathbf{i}+\mathbf{j}) = \mathbf{0}$ at all points $(x, 2-x)$, $0 \le x \le 1$, Note that since $f(x,y) \ge 0$, $f(x, 2-x) = 0$ is the absolute minimum. The absolute max is on the boundary; clearly $(x+y-2)^2$ is maximum at $(3,3)$; the maximum value is 16.

37. $f(x,y) = xy(1-x-y), \quad 0 \le x \le 1, \quad 0 \le y \le 1-x.$

[dom (f) is the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$.]

$\nabla f = (y - 2xy - y^2)\mathbf{i} + (x - 2xy - x^2)\,\mathbf{j} = \mathbf{0} \quad \Longrightarrow \quad x = y = \frac{1}{3}.$

(Note that $[0,0]$ is not an interior point of the domain of f.)

$f_{xx} = -2x, \quad f_{xy} = 1 - 2x - 2y, \quad f_{yy} = -2x, \quad D = (1-2x-2y)^2 - 4xy.$

At $\left(\frac{1}{3}, \frac{1}{3}\right)$, $D = -\frac{1}{3} < 0$ and $A > 0$ so we have a local max; the value is $1/27$.

Since $f(x,y) = 0$ at each point on the boundary of the domain, the local max of $1/27$ is also the absolute max.

38. $V = xyz, \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \implies V(x,y) = xyc\left(1 - \frac{x}{a} - \frac{y}{b}\right), \quad x > 0, \ y > 0, \ \frac{x}{a} + \frac{y}{b} < 1$

$\nabla V = yc\left(1 - \frac{2x}{a} - \frac{y}{b}\right)\mathbf{i} + xc\left(1 - \frac{x}{a} - \frac{2y}{b}\right)\mathbf{j} = \mathbf{0} \quad \text{at} \quad \left(\frac{a}{3}, \frac{b}{3}\right)$

Maximum volume $\quad V = \frac{a}{3} \cdot \frac{b}{3} \cdot \frac{c}{3} = \frac{abc}{27}$

39. $f(x,y) = (x-1)^2 + (y-2)^2 + z^2 = (x-1)^2 + (y-2)^2 + x^2 + 2y^2 \quad \left[\text{since } z = \sqrt{x^2 + 2y^2}\right]$

$\nabla f = [2(x-1) + 2x]\,\mathbf{i} + [2(y-2) + 4y]\,\mathbf{j} = \mathbf{0} \implies x = \frac{1}{2}, \ y = \frac{2}{3}.$

$f_{xx} = 4 > 0, \quad f_{xy} = 0, \quad f_{yy} = 6, \quad D = -24 < 0.$ Thus, f has a local minimum at $(1/2, 2/3)$.

The shortest distance from $(1,2,0)$ to the cone is $f\left(\frac{1}{2}, \frac{2}{3}\right) = \frac{1}{6}\sqrt{114}$

40. $C = 4xy + 3(2xz + 2yz) = 4xy + 6z(x+y).$

Since $xyz = 12$, we need to minimize $C(x,y) = 4xy + \frac{72}{xy}(x+y), \quad x > 0, \ y > 0.$

$$\nabla C = (4y - \frac{72}{x^2})\mathbf{i} + (4x - \frac{72}{y^2})\mathbf{j} = \mathbf{0} \quad \text{at} \quad (18^{1/3}, 18^{1/3})$$

dimensions $\sqrt[3]{18} \times \sqrt[3]{18} \times \frac{12}{18^{2/3}}.$

41. (a) $\nabla f = \frac{1}{2}x\,\mathbf{i} - \frac{2}{9}y\,\mathbf{j} = \mathbf{0}$ only at $(0,0)$.

(b) The difference

$$f(h,k) - f(0,0) = \tfrac{1}{4}h^2 - \tfrac{1}{9}k^2$$

does not keep a constant sign for all small h and k; $(0,0)$ is a saddle point. The function has no local extreme values.

(c) Being the difference of two squares, f can be maximized by maximizing $\frac{1}{4}x^2$ and minimizing $\frac{1}{9}y^2$; $(1,0)$ and $(-1,0)$ give absolute maximum value $\frac{1}{4}$. Similarly, $(0,1)$ and $(0,-1)$ give absolute minimum value $-\frac{1}{9}$.

42. (a) $\nabla f = anx^{n-1}\mathbf{i} + cny^{n-1}\mathbf{j} = \mathbf{0} \quad \text{at} \quad (0,0)$

(b) $\frac{\partial^2 f}{\partial x^2} = an(n-1)x^{n-2}, \quad \frac{\partial^2 f}{\partial y \partial x} = 0, \quad \frac{\partial^2 f}{\partial y^2} = cn(n-1)y^{n-2}; \quad$ at $(0,0)$, $D = 0$.

(c) (i) $(0,0)$ gives absolute min of 0 if n is even; no extreme value if n is odd.

(ii) $(0,0)$ gives absolute max of 0 if n is even; no extreme value if n is odd.

(iii) no extreme values.

43. (a) $f(x,y) = 0$ along the plane curve $y = x^{2/3}$. Since $f(x,y)$ is positive for points below the curve and is negative for points above the curve, there is a saddle point at the origin.

(b) $\nabla f = (ye^{xy} - 2\sin(x+y))\mathbf{i} + (xe^{xy} - 2\sin(x+y))\mathbf{j}$

$$f_{xx} = -2\cos(x+y) + y^2 e^{xy},$$

$$f_{xy} = -2\cos(x+y) + e^{xy}(1+xy),$$

$$f_{yy} = -2\cos(x+y) + x^2 e^{xy}.$$

At the origin $A = -2$, $B = -1$, $C = -2$, and $D = -3$. We have a local max; the value is 3.

44. $V = 8xyz, \quad x^2 + y^2 + z^2 = a^2 \implies V(x,y) = 8xy\sqrt{a^2 - x^2 - y^2}, \quad x > 0,\ y > 0,\ x^2 + y^2 < a^2$

$$\nabla V = \frac{8y(a^2 - x^2 - y^2) - 8x^2 y}{\sqrt{a^2 - x^2 - y^2}}\mathbf{i} + \frac{8x(a^2 - x^2 - y^2) - 8xy^2}{\sqrt{a^2 - x^2 - y^2}}\mathbf{j} = \mathbf{0} \quad \text{at} \quad \left(a/\sqrt{3},\, a/\sqrt{3}\right)$$

dimensions: $\dfrac{a}{\sqrt{3}} \times \dfrac{a}{\sqrt{3}} \times \dfrac{a}{\sqrt{3}}$, maximum volume: $\dfrac{8}{9}a^3\sqrt{3}$

45.
$$f(x,y) = \sum_{i=1}^{3}\left[(x - x_i)^2 + (y - y_i)^2\right]$$

$$\nabla f(x,y) = 2\left[(3x - x_1 - x_2 - x_3)\,\mathbf{i} + (3y - y_1 - y_2 - y_3)\,\mathbf{j}\right]$$

$$\nabla f = \mathbf{0} \quad \text{only at} \quad \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right) = (x_0, y_0).$$

The difference $\quad f(x_0 + h,\, y_0 + k) - f(x_0, y_0)$

$$= \sum_{i=1}^{3}\left[(x_0 + h - x_i)^2 + (y_0 + k - y_i)^2 - (x_0 - x_i)^2 - (y_0 - y_i)^2\right]$$

$$= \sum_{i=1}^{3}\left[2h(x_0 - x_i) + h^2 + 2k(y_0 - y_i) + k^2\right]$$

$$= 2h(3x_0 - x_1 - x_2 - x_3) + 2k(3y_0 - y_1 - y_2 - y_3) + h^2 + k^2$$

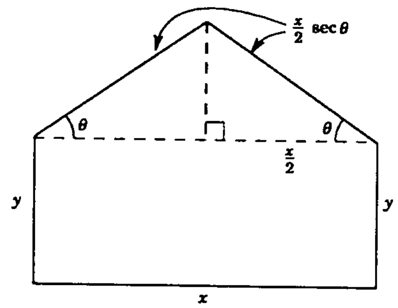
$$= h^2 + h^2$$

is positive for all small h and k. Thus, f has its absolute minimum at (x_0, y_0).

46. $\nabla f = \dfrac{y(ay - x^2)}{(a+x)^2(x+y)^2(b+y)}\mathbf{i} + \dfrac{x(bx - y^2)}{(a+x)(x+y)^2(b+y)^2}\mathbf{j} = \mathbf{0} \quad \text{at} \quad \left(a^{2/3}b^{1/3}, a^{1/3}b^{2/3}\right)$

Maximum is $f\left(a^{2/3}b^{1/3}, a^{1/3}b^{2/3}\right) = \left(a^{1/3} + b^{1/3}\right)^{-3}$.

47.



$$A = xy + \frac{1}{2}x\left(\frac{x}{2}\tan\theta\right),$$

$$P = x + 2y + 2\left(\frac{x}{2}\sec\theta\right),$$

$$0 < \theta < \frac{1}{2}\pi, \quad 0 < x < \frac{P}{1+\sec\theta}.$$

$$A(x,\theta) = \tfrac{1}{2}x(P - x - x\sec\theta) + \tfrac{1}{4}x^2\tan\theta,$$

$$\nabla A = \left(\frac{P}{2} - x - x\sec\theta + \frac{x}{2}\tan\theta\right)\mathbf{i} + \left(\frac{x^2}{4}\sec^2\theta - \frac{x^2}{2}\sec\theta\tan\theta\right)\mathbf{j},$$

$$\nabla A = \frac{1}{2}[P + x(\tan\theta - 2\sec\theta - 2)]\,\mathbf{i} + \frac{x^2}{4}\sec\theta\,(\sec\theta - 2\tan\theta)\,\mathbf{j}.$$

From $\dfrac{\partial A}{\partial\theta} = 0$ we get $\theta = \frac{1}{6}\pi$ and then from $\dfrac{\partial A}{\partial x} = 0$ we get

$$P + x\left(\tfrac{1}{3}\sqrt{3} - \tfrac{4}{3}\sqrt{3} - 2\right) = 0 \quad \text{so that} \quad x = (2-\sqrt{3})P.$$

Next,

$$A_{xx} = \tfrac{1}{2}(\tan\theta - 2\sec\theta - 2),$$

$$A_{x\theta} = \frac{x}{2}\sec\theta\,(\sec\theta - 2\tan\theta),$$

$$A_{\theta\theta} = \frac{x^2}{2}\sec\theta\left(\sec\theta\tan\theta - \sec^2\theta - \tan^2\theta\right).$$

By the second-partials test

$$A = -\tfrac{1}{2}(2+\sqrt{3}), \quad B = 0, \quad C = -\tfrac{1}{3}P^2\sqrt{3}\,(2-\sqrt{3})^2, \quad D < 0.$$

The area is a maximum when $\theta = \frac{1}{6}\pi$, $x = (2-\sqrt{3})\,P$ and $y = \frac{1}{6}(3-\sqrt{3})P$.

48. Profit $P(x,y) = N_1(x-50) + N_2(y-60) = 250(y-x)(x-50) + [32,000 + 250(x-2y)](y-60)$

$\nabla P = 250(2y - 2x - 10)\mathbf{i} + [32,000 + 250(2x + 70 - 4y)]\mathbf{j} = \mathbf{0}$

$\Longrightarrow\ x = 89, \quad y = 94$

49. From $\quad x = \frac{1}{2}y = \frac{1}{3}z = t \quad$ and $\quad x = y - 2 = z = s$

we take $\quad (t, 2t, 3t) \quad$ and $\quad (s, 2+s, s)$

as arbitrary points on the lines. It suffices to minimize the square of the distance between these points:

$$f(t,s) = (t-s)^2 + (2t-2-s)^2 + (3t-s)^2$$
$$= 14t^2 - 12ts + 3s^2 - 8t + 4s + 4, \qquad t, s \text{ real.}$$

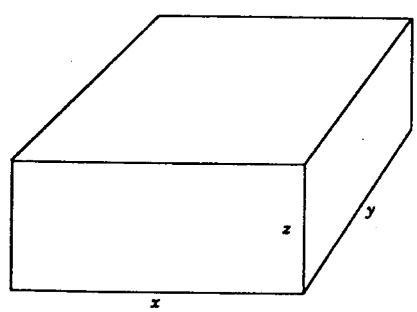
$$\nabla f = (28t - 12s - 8)\mathbf{i} + (-12t + 6s + 4)\mathbf{j}; \qquad \nabla f = \mathbf{0} \implies t = 0,\ s = -2/3.$$
$$f_{tt} = 28, \quad f_{ts} = -12, \quad f_{ss} = 6, \quad D = (-12)^2 - 6(28) = -24 < 0.$$

By the second-partials test, the distance is a minimum when $t = 0$, $s = -2/3$; the nature of the problem tells us the minimum is absolute. The distance is $\sqrt{f(0, 2/3)} = \frac{2}{3}\sqrt{6}$.

50. $\nabla f = \dfrac{2(ax+by+c)(ay^2+a-bxy-cx)}{(x^2+y^2+1)^2}\mathbf{i} + \dfrac{2(ax+by+c)(bx^2+b-axy-cy)}{(x^2+y^2+1)^2}\mathbf{j}$

Since we want a maximum and $f(x,y) \geq 0$ for all (x,y), we may assume that $ax + by + c \neq 0$.
Then $\nabla f = \mathbf{0} \implies x = \dfrac{a}{c},\ y = \dfrac{b}{c}.$ maximum value: $f(a/c,\ b/c) = a^2 + b^2 + c^2$.

51.



$$96 = xyz,$$
$$C = 30xy + 10(2xz + 2yz)$$
$$= 30xy + 20(x+y)\frac{96}{xy}.$$

$$C(x,y) = 30\left[xy + \frac{64}{x} + \frac{64}{y}\right],$$
$$\nabla C = 30(y - 64x^{-2})\mathbf{i} + 30(x - 64y^{-2})\mathbf{j} = \mathbf{0} \implies x = y = 4.$$

$$C_{xx} = 128x^{-3}, \quad C_{xy} = 1, \quad C_{yy} = 128y^{-3}.$$

When $x = y = 4$, we have $D = -3 < 0$ and $A = 2 > 0$ so the cost is minimized by making the dimensions of the crate $4 \times 4 \times 6$ meters.

52. (a) $\nabla f = (2ax + by)\mathbf{i} + (bx + 2cy)\mathbf{j}$

$$\frac{\partial^2 f}{\partial x^2} = 2a, \quad \frac{\partial^2 f}{\partial y \partial x} = b, \quad \frac{\partial^2 f}{\partial y^2} = 2c; \quad D = b^2 - 4ac.$$

(b) The point $(0,0)$ is the only stationary point. If $D > 0, (0,0)$ is a saddle point; if $D < 0$, $(0,0)$ is a local minimum if $a > 0$ and a local maximum if $a < 0$.

(c) (i) if $b > 0$, $f(x,y) = (\sqrt{a}x + \sqrt{c}y)^2$; every point on the line $\sqrt{a}x + \sqrt{c}y = 0$ is a stationary point and at each such point f takes on a local and absolute min of 0

if $b < 0$, $f(x,y) = (\sqrt{a}x - \sqrt{c}y)^2$; every point on the line $\sqrt{a}x - \sqrt{c}y = 0$ is a stationary point and at each such point f takes on a local and absolute min of 0

(ii) if $b > 0$, $f(x,y) = -(\sqrt{|a|}x - \sqrt{|c|}y)^2$; every point on the line $\sqrt{|a|}x - \sqrt{|c|}y = 0$ is a stationary point and at each such point f takes on a local and absolute max of 0

if $b < 0$, $f(x,y) = -(\sqrt{|a|}x + \sqrt{|c|}y)^2$; every point on the line $\sqrt{|a|}x + \sqrt{|c|}y = 0$ is a stationary point and at each such point f takes on a local and absolute max of 0

53. Let x, y and z be the length, width and height of the box. The surface area is given by

$$S = 2xy + 2xz + 2yz, \quad \text{so} \quad z = \frac{S - 2xy}{2(x+y)}, \quad \text{where } S \text{ is a constant, and } x, y, z > 0.$$

Now, the volume $V = xyz$ is given by:

$$V(x,y) = xy\left[\frac{S-2xy}{2(x+y)}\right]$$

and

$$\nabla V = y\left\{\left[\frac{S-2xy}{2(x+y)}\right] + xy\,\frac{2(x+y)(-2y) - (S-2xy)(2)}{4(x+y)^2}\right\}\mathbf{i}$$
$$+\left\{x\left[\frac{S-2xy}{2(x+y)}\right] + xy\,\frac{2(x+y)(-2x) - (S-2xy)(2)}{4(x+y)^2}\right\}\mathbf{j}$$

Setting $\dfrac{\partial V}{\partial x} = \dfrac{\partial V}{\partial y} = 0$ and simplifying, we get the pair of equations

$$2S - 4x^2 - 8xy = 0$$
$$2S - 4y^2 - 8xy = 0$$

from which it follows that $x = y = \sqrt{S/6}$. From practical considerations, we conclude that V has a maximum value at $(\sqrt{S/6}, \sqrt{S/6})$. Substituting these values into the equation for z, we get $z = \sqrt{S/6}$ and so the box of maximum volume is a cube.

54. $V = xyz, \quad S = xy + 2xz + 2yz \implies V(x,y) = xy\dfrac{(S-xy)}{2(x+y)}, \quad x > 0,\ y > 0,\ xy < S.$

$$\nabla V = \frac{y^2(S - x^2 - 2xy)}{2(x+y)^2}\mathbf{i} + \frac{x^2(S - y^2 - 2xy)}{2(x+y)^2}\mathbf{j}$$

$\nabla V = \mathbf{0} \implies x = \sqrt{\dfrac{S}{3}}, \quad y = \sqrt{\dfrac{S}{3}}$; dimensions for maximum volume: $\sqrt{\dfrac{S}{3}} \times \sqrt{\dfrac{S}{3}} \times \dfrac{1}{2}\sqrt{\dfrac{S}{3}}$

55. (a) $f(m,b) = [2-b]^2 + [-5-(m+b)]^2 + [4-(2m+b)]^2$.

$f_m = 10m + 6b - 6, \quad f_b = 6m + 6b - 2; \qquad f_m = f_b = 0 \implies m = 1, \quad b = -\frac{2}{3}.$

$f_{mm} = 10, \quad f_{mb} = 6, \quad f_{bb} = 6, \quad D = -24 < 0 \quad \Longrightarrow \quad$ a min.

Answer: the line $y = x - \frac{2}{3}$.

(b) $f(\alpha, \beta) = [2 - \beta]^2 + [-5 - (\alpha + \beta)]^2 + [4 - (4\alpha + \beta)]^2$.

$$f_\alpha = 34\alpha + 10\beta - 22, \quad f_\beta = 10\alpha + 6\beta - 2; \qquad f_\alpha = f_\beta = 0 \quad \Longrightarrow \quad \left[\begin{cases} \alpha = \frac{14}{13} \\ \beta = -\frac{19}{13} \end{cases} \right].$$

$f_{\alpha\alpha} = 34, \quad f_{\alpha\beta} = 10, \quad f_{\beta\beta} = 6, \quad D = -104 < 0 \quad \Longrightarrow \quad$ a min.

Answer: the parabola $y = \frac{1}{13}\left(14x^2 - 19\right)$.

56. (a) $f(m, b) = [2 - (-m + b)]^2 + [-1 - b]^2 + [1 - (m + b)]^2$

$f_m = 4m + 2, \quad f_b = 6b - 4, \quad f_m = f_b = 0 \Longrightarrow m = -\dfrac{1}{2}, \quad b = \dfrac{2}{3}$

$f_{mm} = 4, \quad f_{mb} = 0, \quad f_{bb} = 6, \quad D = -24 < 0 \Longrightarrow$ minimum

Answer: the line $y = -\dfrac{1}{2}x + \dfrac{2}{3}$

(b) $f(\alpha, \beta) = [2 - (\alpha + \beta)]^2 + [-1 - \beta]^2 + [1 - (\alpha + \beta)]^2$

$f_\alpha = 4\alpha + 4\beta - b, \quad f_\beta = 4\alpha + 6\beta - 4; \quad f_\alpha = f_\beta = 0 \Longrightarrow \alpha = \dfrac{5}{2}, \ \beta = -1$

$f_{\alpha\alpha} = 4, \quad f_{\alpha\beta} = 4, \quad f_{\beta\beta} = 6, \quad D = -8 < 0 \Longrightarrow$ minimum

Answer: the parabola $y = \dfrac{5}{2}x^2 - 1$

57. (a) Let x and y be the cross-sectional measurements of the box, and let l be its length.

Then

$$V = xyl, \quad \text{where} \quad 2x + 2y + l \leq 108, \quad x, y > 0$$

To maximize V we will obviously take $2x+2y+l = 108$. Therefore, $V(x, y) = xy(108-2x-2y)$ and

$$\nabla V = [y(108 - 2x - 2y) - 2xy]\,\mathbf{i} + [x(108 - 2x - 2y) - 2xy]\,\mathbf{j}$$

Setting $\dfrac{\partial V}{\partial x} = \dfrac{\partial V}{\partial y} = 0$, we get the pair of equations

$$\frac{\partial V}{\partial x} = 108y - 4xy - 2y^2 = 0$$
$$\frac{\partial V}{\partial y} = 108x - 4xy - 2x^2 = 0$$

from which it follows that $x = y = 18 \quad \Longrightarrow \quad l = 36$.

Now, at $(18, 18)$, we have

$$A = V_{xx} = -4y = -72 < 0, \quad B = V_{xy} = 108 - 4x - 4y = -36,$$
$$C = V_{yy} = -4x = -72, \quad \text{and} \quad D = (36)^2 - (72)^2 < 0.$$

Thus, V is a maximum when $x = y = 18$ and $l = 36$.

(b) Let r be the radius of the tube and let l be its length.

Then

$$V = \pi r^2 l, \quad \text{where} \quad 2\pi r + l \leq 108, \quad r > 0$$

To maximize V we take $2\pi r + l = 108$. Then $V(r) = \pi r^2(108 - 2\pi r) = 108\pi r^2 - 2\pi^2 r^3$. Now

$$\frac{dV}{dr} = 216\pi r - 6\pi^2 r^2$$

Setting $\frac{dV}{dr} = 0$, we get

$$216\pi r - 6\pi^2 r^2 = 0 \implies r = \frac{36}{\pi} \implies l = 36$$

Now, at $r = 36/\pi$, we have

$$\frac{d^2V}{dr^2} = 216\pi - 12\pi^2\,\frac{36}{\pi} = -216\pi < 0$$

Thus, V is a maximum when $r = 36/\pi$ and $l = 36$.

58. We want to minimize $S = 4\pi r^2 + 2\pi r h$ given that $V = \frac{4}{3}\pi r^3 + \pi r^2 h = 10,000$.

$$S(r) = 4\pi r^2 + 2\pi r\left(\frac{V}{\pi r^2} - \frac{4}{3}r\right) = \frac{4}{3}\pi r^2 + \frac{2V}{r}$$

$$S'(r) = \frac{8\pi r^3 - 6V}{3r^2} = 0 \implies r = \sqrt[3]{\frac{6V}{8\pi}}, \quad h = \frac{V}{\pi r^2} - \frac{4}{3}r = 0$$

The optimal container is a sphere of radius $r = \sqrt[3]{7500/\pi}$

59. Let S denote the cross-sectional area. Then

$$S = \frac{1}{2}\,(12 - 2x + 12 - 2x + 2x\cos\theta)\,x\sin\theta = 12x\sin\theta - 2x^2\sin\theta + \frac{1}{2}\,x^2\sin 2\theta,$$

where $0 < x < 6,\ \ 0 < \theta < \pi/2$

Now,

$$\nabla S = (12\sin\theta - 4x\sin\theta + x\sin 2\theta)\,\mathbf{i} + (12x\cos\theta - 2x^2\cos\theta + x^2\cos 2\theta)\,\mathbf{j}$$

Setting $\frac{\partial S}{\partial x} = \frac{\partial S}{\partial \theta} = 0$, we get the pair of equations

$$12\sin\theta - 4x\sin\theta + x\sin 2\theta = 0$$

$$12x\cos\theta - 2x^2\cos\theta + x^2\cos 2\theta = 0$$

from which it follows that $x = 4, \theta = \pi/3$.

Now, at $(4,\ \pi/3)$, we have

$$A = S_{xx} = -4\sin\theta + \sin 2\theta = -\frac{3}{2}\sqrt{3}, \quad B = S_{x\theta} = 12\cos\theta - 4x\cos\theta + 2x\cos 2\theta = -6,$$

$$C = S_{\theta\theta} = -12x\sin\theta + 2x^2\sin\theta - 2x^2\sin 2\theta = -24\sqrt{3} \quad \text{and} \quad D = 36 - 108 < 0.$$

Thus, S is a maximum when $x = 4$ and $\theta = \pi/3$.

60. Let (x, y, z) be on the ellipsoid, $x > 0,\ y > 0,\ z > 0.$ Then

$$V = 2x \cdot 2y \cdot 2z = 8xyz.$$

Note that V achieves its maximum $\iff$ $x^2y^2z^2$ achieves its maximum.

Let $s = x^2y^2z^2$, then

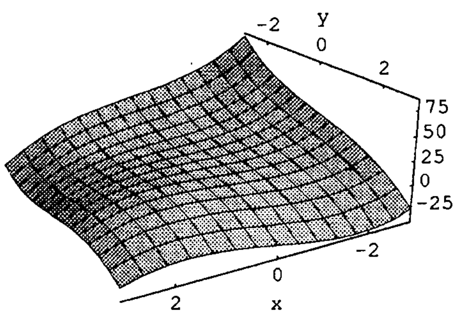
$$s = x^2y^2c^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right),$$
$$\frac{\partial s}{\partial x} = 2c^2xy^2\left(1 - \frac{2x^2}{a^2} - \frac{y^2}{b^2}\right) = 0$$
$$\frac{\partial s}{\partial y} = 2c^2x^2y\left(1 - \frac{x^2}{a^2} - \frac{2y^2}{b^2}\right) = 0$$

$$\implies \quad \frac{2x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \frac{x^2}{a^2} + \frac{2y^2}{b^2} = 1 \quad \implies \quad x = \frac{a}{\sqrt{3}}, \quad y = \frac{b}{\sqrt{3}}, \quad z = \frac{c}{\sqrt{3}}$$

Thus,

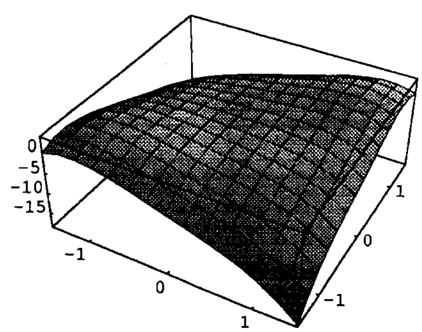
$$V_{\max} = 8xyz = 8 \cdot \frac{a}{\sqrt{3}} \cdot \frac{b}{\sqrt{3}} \cdot \frac{c}{\sqrt{3}} = \frac{8\sqrt{3}}{9}\,abc.$$

61. $(0,0)$ saddle point; $(1,1)$ local maximum



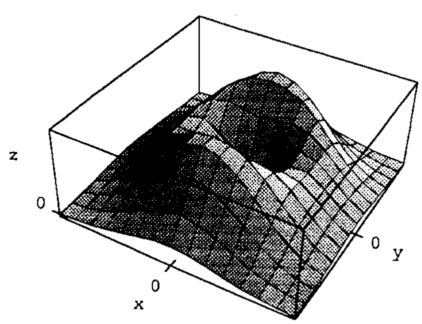
62. Saddle point at $(0,0)$,

local maxima at $(1,1)$, $(-1,-1)$

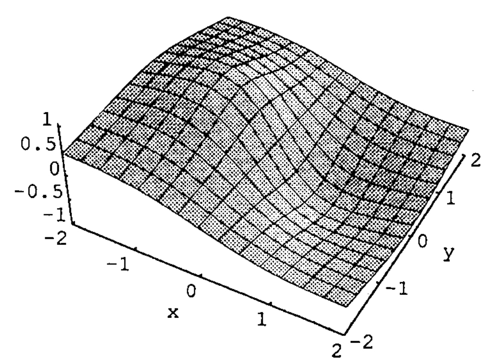


63. $(0,0)$ local min., $(0,-1)$, $(0,1)$ local max.

saddle points at $(1,0)$, $(-1,0)$.



64. $(1,0)$ local min; $(-1,0)$ local max



SECTION 15.6

1. $f(x,y) = x^2 + y^2, \qquad g(x,y) = xy - 1$

$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j}, \qquad \nabla g = y\mathbf{i} + x\mathbf{j}.$$

$$\nabla f = \lambda \nabla g \quad \implies \quad 2x = \lambda y \text{ and } 2y = \lambda x.$$

Multiplying the first equation by x and the second equation by y, we get

$$2x^2 = \lambda xy = 2y^2.$$

Thus, $x = \pm y$. From $g(x,y) = 0$ we conclude that $x = y = \pm 1$. The points $(1,1)$ and $(-1,-1)$ clearly give a minimum, since f represents the square of the distance of a point on the hyperbola from the origin. The minimum is 2.

2. $f(x,y) = xy, \quad g(x,y) = b^2x^2 + a^2y^2 - a^2b^2 = 0$

$$\nabla f = y\mathbf{i} + x\mathbf{j}, \quad \nabla g = 2b^2x\mathbf{i} + 2a^2y\mathbf{j}$$

$\nabla f = \lambda \nabla g \implies y = 2\lambda b^2 x, \quad x = 2\lambda a^2 y \implies a^2y^2 = b^2x^2$

From $g(x,y) = 0$ we get $2b^2x^2 = a^2b^2 \implies x = \pm\dfrac{a}{\sqrt{2}}, \quad y = \pm\dfrac{b}{\sqrt{2}}$

Clearly the maximum value of xy is $\frac{1}{2}ab$

3.

$$f(x,y) = xy, \qquad g(x,y) = b^2x^2 + a^2y^2 - a^2b^2$$

$$\nabla f = y\mathbf{i} + x\mathbf{j}, \qquad \nabla g = 2b^2x\mathbf{i} + 2a^2y\mathbf{j}.$$

$$\nabla f = \lambda \nabla g \implies y = 2\lambda b^2 x \text{ and } x = 2\lambda a^2 y.$$

Multiplying the first equation by a^2y and the second equation by b^2x, we get

$$a^2y^2 = 2\lambda a^2b^2xy = b^2x^2.$$

Thus, $ay = \pm bx$. From $g(x,y) = 0$ we conclude that $x = \pm\frac{1}{2}a\sqrt{2}$ and $y = \pm\frac{1}{2}b\sqrt{2}$.

Since f is continuous and the ellipse is closed and bounded, the minimum exists. It occurs at $\left(\frac{1}{2}a\sqrt{2}, -\frac{1}{2}b\sqrt{2}\right)$ and $\left(-\frac{1}{2}a\sqrt{2}, \frac{1}{2}b\sqrt{2}\right)$; the minimum is $-\frac{1}{2}ab$.

4. $f(x,y) = xy^2, \quad g(x,y) = x^2 + y^2 - 1$

$\nabla f = y^2\mathbf{i} + 2xy\mathbf{j}, \quad \nabla g = 2x\mathbf{i} + 2y\mathbf{j}$

$\nabla f = \lambda \nabla g \implies y^2 = 2\lambda x, \quad 2xy = 2\lambda y \implies y^2 = 2x^2$

From $g(x,y) = 0$ we get $3x^2 = 1, \implies x = \pm\dfrac{1}{\sqrt{3}}, \quad y = \pm\dfrac{\sqrt{2}}{\sqrt{3}}$

Minimum of xy^2 is: $-\dfrac{2}{9}\sqrt{3}$

5. Since f is continuous and the ellipse is closed and bounded, the maximum exists.

$$f(x,y) = xy^2, \qquad g(x,y) = b^2x^2 + a^2y^2 - a^2b^2$$

$$\nabla f = y^2\mathbf{i} + 2xy\mathbf{j}, \qquad \nabla g = 2b^2x\mathbf{i} + 2a^2y\mathbf{j}.$$

$$\nabla f = \lambda \nabla g \implies y^2 = 2\lambda b^2 x \text{ and } 2xy = 2\lambda a^2 y.$$

Multiplying the first equation by a^2y and the second equation by b^2x, we get

$$a^2y^3 = 2\lambda a^2b^2xy = 2b^2x^2y.$$

We can exclude $y = 0$; it clearly cannot produce the maximum. Thus,

$$a^2y^2 = 2b^2x^2 \text{ and, from } g(x,y) = 0,\ 3b^2x^2 = a^2b^2.$$

This gives us $x = \pm\frac{1}{3}\sqrt{3}\,a$ and $y = \pm\frac{1}{3}\sqrt{6}\,b$. This maximum occurs at $x = \frac{1}{3}\sqrt{3}\,a$, $y = \pm\frac{1}{3}\sqrt{6}\,b$; the value there is $\frac{2}{9}\sqrt{3}\,ab^2$.

6. $f(x,y) = x + y, \quad g(x,y) = x^4 + y^4 - 1$

$\nabla f = \mathbf{i} + \mathbf{j}, \quad \nabla g = 4x^3\mathbf{i} + 4y^3\mathbf{j}$

$\nabla f = \lambda\nabla g \implies 1 = 4\lambda x^3, \quad 1 = 4\lambda y^3 \implies x = y$

From $g(x,y) = 0$ we get $2x^4 = 1 \implies x = y = \pm 2^{-1/4}$

Maximum of $x + y$ is: $2 \cdot 2^{-1/4} = 2^{3/4}$

7. The given curve is closed and bounded. Since $x^2 + y^2$ represents the square of the distance from points on this curve to the origin, the maximum exists.

$$f(x,y) = x^2 + y^2, \qquad g(x,y) = x^4 + 7x^2y^2 + y^4 - 1$$

$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j}, \qquad \nabla g = \left(4x^3 + 14xy^2\right)\mathbf{i} + \left(4y^3 + 14x^2y\right)\mathbf{j}.$$

We use the cross-product equation (16.8.4):

$$2x(4y^3 + 14x^2y) - 2y(4x^3 + 14xy^2) = 0,$$

$$20x^3y - 20xy^3 = 0,$$

$$xy(x^2 - y^2) = 0.$$

Thus, $x = 0$, $y = 0$, or $x = \pm y$. From $g(x,y) = 0$ we conclude that the points to examine are

$$(0, \pm 1), \quad (\pm 1, 0), \quad \left(\pm\tfrac{1}{3}\sqrt{3}, \pm\tfrac{1}{3}\sqrt{3}\right).$$

The value of f at each of the first four points is 1; the value at the last four points is 2/3. The maximum is 1.

8. $f(x,y,z) = xyz, \quad g(x,y,z) = x^2 + y^2 + z^2 - 1$

$\nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}, \quad \nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

$\nabla f = \lambda\nabla g \implies yz = 2\lambda x, \quad xz = 2\lambda y, \quad xy = 2\lambda z \implies x^2 = y^2 = z^2.$

From $g(x,y,z) = 0$ we get $3x^2 = 1 \implies x = \pm\frac{1}{\sqrt{3}},\ y = \pm\frac{1}{\sqrt{3}},\ z = \pm\frac{1}{\sqrt{3}}$

Minimum of xyz is: $-\dfrac{1}{9}\sqrt{3}$

9. The maximum exists since xyz is continuous and the ellipsoid is closed and bounded.

$$f(x,y,z) = xyz, \qquad g(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

$$\nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}, \qquad \nabla g = \frac{2x}{a^2}\mathbf{i} + \frac{2y}{b^2}\mathbf{j} + \frac{2z}{c^2}\mathbf{k}.$$

$$\nabla f = \lambda \nabla g \implies yz = \frac{2x}{a^2}\lambda, \quad xz = \frac{2y}{b^2}\lambda, \quad xy = \frac{2z}{c^2}\lambda.$$

We can assume x, y, z are non-zero, for otherwise $f(x,y,z) = 0$, which is clearly not a maximum. Then from the first two equations

$$\frac{yza^2}{x} = 2\lambda = \frac{xzb^2}{y} \quad \text{so that} \quad a^2y^2 = b^2x^2 \quad \text{or} \quad x^2 = \frac{a^2y^2}{b^2}.$$

Similarly from the second and third equations we get

$$b^2z^2 = c^2y^2 \quad \text{or} \quad z^2 = \frac{c^2y^2}{b^2}.$$

Substituting these expressions for x^2 and z^2 in $g(x,y,z) = 0$, we obtain

$$\frac{1}{a^2}\left[\frac{a^2y^2}{b^2}\right] + \frac{y^2}{b^2} + \frac{1}{c^2}\left[\frac{c^2y^2}{b^2}\right] - 1 = 0, \quad \frac{3y^2}{b^2} = 1, \quad y = \pm\frac{1}{3}b\sqrt{3}.$$

Then, $x = \pm\frac{1}{3}a\sqrt{3}$ and $z = \pm\frac{1}{3}c\sqrt{3}$. The maximum value is $\frac{1}{9}\sqrt{3}\,abc$.

10. $f(x,y,z) = x + 2y + 4z, \quad g(x,y,z) = x^2 + y^2 + z^2 - 7$

$\nabla f = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}. \quad \nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

$\nabla f = \lambda \nabla g \implies 1 = 2\lambda x, \; 2 = 2\lambda y, \; 4 = 2\lambda z \implies y = 2x, \; z = 4x$

From $g(x,y,z) = 0$ we get $21x^2 = 7 \implies x = \pm\dfrac{1}{\sqrt{3}}$

Minimum of $x + 2y + 4z$ is: $-7\sqrt{3}$.

11. Since the sphere is closed and bounded and $2x + 3y + 5z$ is continuous, the maximum exists.

$$f(x,y,z) = 2x + 3y + 5z, \qquad g(x,y,z) = x^2 + y^2 + z^2 - 19$$

$$\nabla f = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}, \qquad \nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}.$$

$$\nabla f = \lambda \nabla g \implies 2 = 2\lambda x, \quad 3 = 2\lambda y, \quad 5 = 2\lambda z.$$

Since $\lambda \neq 0$ here, we solve the equations for x, y and z:

$$x = \frac{1}{\lambda}, \quad y = \frac{3}{2\lambda}, \quad z = \frac{5}{2\lambda},$$

and substitute these results in $g(x,y,z) = 0$ to obtain

$$\frac{1}{\lambda^2} + \frac{9}{4\lambda^2} + \frac{25}{4\lambda^2} - 19 = 0, \quad \frac{38}{4\lambda^2} - 19 = 0, \quad \lambda = \pm\frac{1}{2}\sqrt{2}.$$

The positive value of λ will produce positive values for x, y, z and thus the maximum for f. We get $x = \sqrt{2}$, $y = \frac{3}{2}\sqrt{2}$, $z = \frac{5}{2}\sqrt{2}$, and $2x + 3y + 5z = 19\sqrt{2}$.

12. $f(x,y,z) = x^4 + y^4 + z^4, \quad g(x,y,z) = x + y + z - 1$

$\nabla f = 4x^3\mathbf{i} + 4y^3\mathbf{j} + 4z^3\mathbf{k}, \quad \nabla g = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$\nabla f = \lambda \nabla g \implies 4x^3 = \lambda,\ 4y^3 = \lambda,\ 4z^3 = \lambda \implies x = y = z$

From $g(x,y,z) = 0$ we get $3x = 1, \implies x = \frac{1}{3} = y = z$

Minimum is: $\dfrac{1}{27}$

13.
$$f(x,y,z) = xyz, \qquad g(x,y,z) = \frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1$$

$$\nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}, \qquad \nabla g = \frac{1}{a}\mathbf{i} + \frac{1}{b}\mathbf{j} + \frac{1}{c}\mathbf{k}.$$

$$\nabla f = \lambda \nabla g \implies yz = \frac{\lambda}{a}, \quad xz = \frac{\lambda}{b}, \quad xy = \frac{\lambda}{c}.$$

Multiplying these equations by x, y, z respectively, we obtain

$$xyz = \frac{\lambda x}{a}, \quad xyz = \frac{\lambda y}{b}, \quad xyz = \frac{\lambda z}{c}.$$

Adding these equations and using the fact that $g(x,y,z) = 0$, we have

$$3xyz = \lambda\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = \lambda.$$

Since x, y, z are non-zero,

$$yz = \frac{\lambda}{a} = \frac{3xyz}{a}, \quad 1 = \frac{3x}{a}, \quad x = \frac{a}{3}.$$

Similarly, $y = \dfrac{b}{3}$ and $z = \dfrac{c}{3}$. The maximum is $\dfrac{1}{27}abc$.

14. Maximize area $A = xy$ given that the perimeter $P = 2x + 2y$

$f(x,y) = xy, \quad g(x,y) = 2x + 2y - P$

$\nabla f = y\mathbf{i} + x\mathbf{j}, \quad \nabla g = 2\mathbf{i} + 2\mathbf{j}; \qquad \nabla f = \lambda \nabla g \implies y = 2\lambda, \quad x = 2\lambda \implies x = y.$

The rectangle of maximum area is a square.

15. It suffices to minimize the square of the distance from $(0,1)$ to the parabola. Clearly, the minimum exists.

$$f(x,y) = x^2 + (y-1)^2, \qquad g(x,y) = x^2 - 4y$$

$$\nabla f = 2x\mathbf{i} + 2(y-1)\mathbf{j}, \qquad \nabla g = 2x\mathbf{i} - 4\mathbf{j}.$$

We use the cross-product equation (15.6.4):

$$2x(-4) - 2x(2y-2) = 0, \quad 4x + 4xy = 0, \quad x(y+1) = 0.$$

Since $y \geq 0$, we have $x = 0$ and thus $y = 0$. The minimum is 1.

16. Minimize $f(x,y) = (x-p)^2 + (y-4p)^2$ subject to $g(x,y) = 2px - y^2 = 0$

$\nabla f = 2(x-p)\mathbf{i} + 2(y-4p)\mathbf{j}, \quad \nabla g = 2p\mathbf{i} - 2y\mathbf{j}$

$\nabla f = \lambda \nabla g \implies 2(x-p) = 2\lambda p, \quad 2(y-4p) = -2\lambda y \implies x = \dfrac{4p^2}{y}$

From $g(x,y) = 0$ we get $\dfrac{8p^3}{y} = y^2 \implies y = 2p, \ x = 2p$

Distance to parabola is: $\sqrt{f(x,y)} = \sqrt{5}P$

17. It suffices to maximize and minimize the square of the distance from $(2,1,2)$ to the sphere. Clearly, these extreme values exist.

$$f(x,y,z) = (x-2)^2 + (y-1)^2 + (z-2)^2 \qquad g(x,y,z) = x^2 + y^2 + z^2 - 1$$

$$\nabla f = 2(x-2)\,\mathbf{i} + 2(y-1)\,\mathbf{j} + 2(z-2)\,\mathbf{k}, \qquad \nabla g = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}.$$

$$\nabla f = \lambda = \nabla g \quad \implies \quad 2(x-2) = 2x\lambda, \ \ 2(y-1) = 2y\lambda, \ \ 2(z-2) = 2z\lambda$$

Thus,

$$x = \frac{2}{1-\lambda}, \quad y = \frac{1}{1-\lambda}, \quad z = \frac{2}{1-\lambda}.$$

Using the fact that $x^2 + y^2 + z^2 = 1$, we have

$$\left(\frac{2}{1-\lambda}\right)^2 + \left(\frac{1}{1-\lambda}\right)^2 + \left(\frac{2}{1-\lambda}\right)^2 = 1 \quad \implies \quad \lambda = -2, 4$$

At $\lambda = -2$, $\quad (x,y,z) = (2/3,\ 1/3,\ 2/3)$ and $f(2/3, 1/3/2/3) = 4$

At $\lambda = 4$, $\quad (x,y,z) = (-2/3,\ -1/3,\ -2/3)$ and $f(-2/3, -1/3/-2/3) = 16$

Thus, $(2/3,\ 1/3,\ 2/3)$ is the closest point and $(-2/3,\ -1/3,\ -2/3)$ is the furthest point.

18. $f(x,y,z) = \sin x \sin y \sin z, \quad g(x,y,z) = x + y + z - \pi$

$\nabla f = \cos x \sin y \sin z\mathbf{i} + \sin x \cos y \sin z\mathbf{j} + \sin x \sin y \cos z\mathbf{k}, \quad \nabla g = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$\nabla f = \lambda \nabla g \implies \cos x \sin y \sin z = \lambda = \sin x \cos y \sin z = \sin z \sin y \cos z \implies \cos x = \cos y = \cos z$

$\implies x = y = z = \dfrac{\pi}{3}$

Maximum of $\sin x \sin y \sin z$ is: $\dfrac{3\sqrt{3}}{8}$

19.

$$f(x,y,z) = 3x - 2y + z, \qquad g(x,y,z) = x^2 + y^2 + z^2 - 14$$

$$\nabla f = 3\,\mathbf{i} - 2\,\mathbf{j} + \mathbf{k}, \qquad \nabla g = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}.$$

$$\nabla f = \lambda \nabla g \quad \implies \quad 3 = 2x\lambda, \quad -2 = 2y\lambda, \quad 1 = 2z\lambda.$$

Thus,

$$x = \frac{3}{2\lambda}, \quad y = -\frac{1}{\lambda}, \quad z = \frac{1}{2\lambda}.$$

Using the fact that $x^2+y^2+z^2=14$, we have

$$\left(\frac{3}{2\lambda}\right)^2+\left(-\frac{1}{\lambda}\right)^2+\left(\frac{1}{2\lambda}\right)^2=14 \implies \lambda=\pm\frac{1}{2}.$$

At $\lambda=-\frac{1}{2}$, $(x,y,z)=(3,-2,1)$ and $f(3,-2,1)=14$

At $\lambda=\frac{1}{2}$, $(x,y,z)=(-3,2,-1)$ and $f(-3,2,-1)=-14$

Thus, the maximum value of f on the sphere is 14.

20. $f(x,y,z)=xyz, \quad g(x,y,z)=x^2+y^2+z-4$

$\nabla f=yz\mathbf{i}+xz\mathbf{j}+xy\mathbf{k}, \quad \nabla g=2x\mathbf{i}+2y\mathbf{j}+\mathbf{k}$

$\nabla f=\lambda\nabla g \implies yz=2\lambda x,\ xz=2\lambda y,\ xy=\lambda \implies x^2=y^2=\frac{z}{2}$

From $g(x,y,z)=0$ we get $4x^2=4 \implies x=1,\ y=1,\ z=2.$

Maximum volume is 2

21. It's easier to work with the square of the distance; the minimum certainly exists.

$$f(x,y,z)=x^2+y^2+z^2, \qquad g(x,y,z)=Ax+By+Cz+D$$

$$\nabla f=2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}, \qquad \nabla g=A\mathbf{i}+B\mathbf{j}+C\mathbf{k}.$$

$$\nabla f=\lambda\nabla g \implies 2x=A\lambda,\ 2y=B\lambda,\ 2z=C\lambda.$$

Substituting these equations in $g(x,y,z)=0$, we have

$$\frac{1}{2}\lambda\left(A^2+B^2+C^2\right)+D=0, \quad \lambda\frac{-2D}{A^2+B^2+C^2}.$$

Thus, in turn,

$$x=\frac{-DA}{A^2+B^2+C^2}, \quad y=\frac{-DB}{A^2+B^2+C^2}, \quad z=\frac{-DC}{A^2+B^2+C^2}$$

so the minimum value of $\sqrt{x^2+y^2+z^2}$ is $|D|\left(A^2+B^2+C^2\right)^{-1/2}$.

22. $f(x,y,z)=xyz, \quad g(x,y,z)=2xy+2xz+2yz-6a^2$

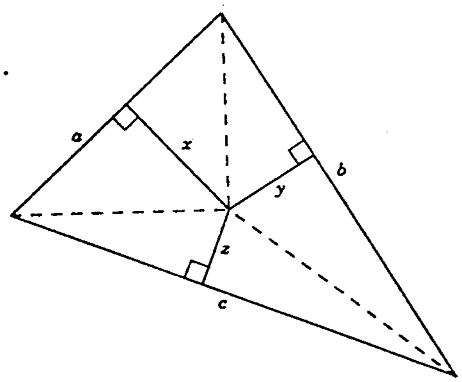
$\nabla f=yz\mathbf{i}+xz\mathbf{j}+xy\mathbf{k}, \quad \nabla g=2(y+z)\mathbf{i}+2(x+z)\mathbf{j}+2(x+y)\mathbf{k}$

$\nabla f=\lambda\nabla g \implies yz=2\lambda(y+z), \quad xz=2\lambda(x+z), \quad xy=2\lambda(x+y) \implies x=y=z$

From $g(x,y,z)=0$ we get $6x^2=6a^2 \implies x=y=z=a$

Maximum volume is a^3.

23.



area $A = \frac{1}{2}ax + \frac{1}{2}by + \frac{1}{2}cz.$

The geometry suggests that

$$x^2 + y^2 + z^2$$

has a minimum.

$$f(x,y,z) = x^2 + y^2 + z^2, \qquad g(x,y,z) = ax + by + cz - 2A$$

$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}, \qquad \nabla g = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

$$\nabla f = \lambda \nabla g \quad \Longrightarrow \quad 2x = a\lambda, \;\; 2y = b\lambda, \;\; 2z = c\lambda.$$

Solving these equations for x, y, z and substituting the results in $g(x,y,z) = 0$, we have

$$\frac{a^2\lambda}{2} + \frac{b^2\lambda}{2} + \frac{c^2\lambda}{2} - 2A = 0, \quad \lambda = \frac{4A}{a^2 + b^2 + c^2}$$

and thus

$$x = \frac{2aA}{a^2 + b^2 + c^2}, \quad y = \frac{2bA}{a^2 + b^2 + c^2}, \quad z = \frac{2cA}{a^2 + b^2 + c^2}.$$

The minimum is $4A^2(a^2 + b^2 + c^2)^{-1}$.

24. Use figure 15.6.4 and write the side condition as $x + y + z = 2\pi$.

For (a) maximize $f(x,y,z) = 8R^3 \sin \frac{1}{2}x \sin \frac{1}{2}y \sin \frac{1}{2}z$.

For (b) maximize $f(x,y,z) = 4R^2(\sin^2 \frac{1}{2}x + \sin^2 \frac{1}{2}y + \sin^2 \frac{1}{2}z)$.

Each maximum occurs with $x = y = z = \dfrac{2\pi}{3}$. This gives an equilateral triangle.

25. Since the curve is asymptotic to the line $y = x$ as $x \to -\infty$ and as $x \to \infty$, the maximum exists. The distance between the point (x, y) and the line $y - x = 0$ is given by

$$\frac{|y - x|}{\sqrt{1+1}} = \frac{1}{2}\sqrt{2}\,|y - x|. \qquad \text{(see Section 1.5)}$$

Since the points on the curve are below the line $y = x$, we can replace $|y - x|$ by $x - y$. To simplify the work we drop the constant factor $\frac{1}{2}\sqrt{2}$.

$$f(x,y) = x - y, \qquad g(x,y) = x^3 - y^3 - 1$$

$$\nabla f = \mathbf{i} - \mathbf{j}, \qquad \nabla g = 3x^2\mathbf{i} - 3y^2\mathbf{j}.$$

We use the cross-product equation (16.8.4):

$$1\left(-3y^2\right)-\left(3x^2\right)(-1)=0, \quad 3x^2-3y^2=0, \quad x=-y \quad (x\neq y).$$

Now $g(x,y)=0$ gives us

$$x^3-(-x)^3-1=0, \quad 2x^3=1, \quad x=2^{-1/3}.$$

The point is $\left(2^{-1/3},-2^{-1/3}\right)$.

26. Let r,s,t be the intercepts. We wish to minimize the volume

$$V=\frac{1}{6}rst \quad [\text{volume of pyramid} = \frac{1}{3}\text{base} \times \text{height}]$$

subject to the side condition $\frac{a}{r}+\frac{b}{s}+\frac{c}{t}=1$. The minimum occurs when all the intercepts are $a+b+c$.

27. It suffices to show that the square of the area is a maximum when $a=b=c$.

$$f(a,b,c)=s(s-a)(s-b)(s-c), \quad g(a,b,c)=a+b+c-2s$$

$$\nabla f=-s(s-b)(s-c)\mathbf{i}-s(s-a)(s-c)\mathbf{j}-s(s-a)(s-b)\mathbf{k}, \quad \nabla g=\mathbf{i}+\mathbf{j}+\mathbf{k}.$$

$$\nabla f=\lambda\nabla g \quad \Longrightarrow \quad -s(s-b)(s-c)=-s(s-a)(s-c)=-s(s-a)(s-b)=\lambda.$$

Thus, $s-b=s-a=s-c$ so that $a=b=c$. This gives us the maximum, as no minimum exists. [The area can be made arbitrarily small by taking a close to s.]

28. $f(x,y,z)=8xyz, \quad g(x,y,z)=a^2-x^2-y^2-z^2, \; x>0, \; y>0, \; z>0.$

$\nabla f=8yz\mathbf{i}+8xz\mathbf{j}+8xy\mathbf{k}, \quad \nabla g=-2x\,\mathbf{i}-2y\,\mathbf{j}-2z\,\mathbf{k}$

$\nabla f=\lambda\nabla g \implies 8yz=-2\lambda x, \quad 8xz=-2\lambda y, \quad 8xy=-2\lambda z \implies x=y=z$

The rectangular box of maximum volume inscribed in the sphere is a cube.

29. (a)

$f(x,y)=(xy)^{1/2}, \qquad g(x,y)=x+y-k, (x,y\geq 0, \; k$ a nonnegative constant

$$\nabla f=\frac{y^{1/2}}{2x^{1/2}}\mathbf{i}+\frac{x^{1/2}}{2y^{1/2}}\mathbf{j} \qquad \nabla g=\mathbf{i}+\mathbf{j}.$$

$$\nabla f=\lambda\nabla g \quad \Longrightarrow \quad \frac{y^{1/2}}{2x^{1/2}}=\lambda=\frac{x^{1/2}}{2y^{1/2}} \quad \Longrightarrow \quad x=y=\frac{k}{2}.$$

Thus, the maximum value of f is: $f(k/2,\,k/2)=\frac{k}{2}$.

(b) For all $x,y \;\; (x,y\geq 0)$ we have

$$(xy)^{1/2}=f(x,y)\leq f(k/2,\,k/2)=\frac{k}{2}=\frac{x+y}{2}.$$

30. (a) The maximum occurs when $x=y=z=\frac{k}{3}$, where $(xyz)^{1/3}=\frac{k}{3}$.

(b) If $x+y+z=k$, then, by (a), $(xyz)^{1/3}\leq\frac{k}{3}=\frac{x+y+z}{3}$

31. Simply extend the arguments used in Exercises 29 and 30.

32. $T(x,y,z) = 10xy^2z, \quad g(x,y,z) = x^2 + y^2 + z^2 - 1$

$\nabla T = 10y^2z\mathbf{i} + 20xyz\mathbf{j} + 10xy^2\mathbf{k}, \quad \nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

$\nabla T = \lambda\nabla g \implies 10y^2z = 2\lambda x, \quad 20xyz = 2\lambda y, \ 10xy^2 = 2\lambda z \implies x^2 = \dfrac{y^2}{2} = z^2$

From $g(x,y,z) = 0$ we get $4x^2 = 1 \implies x = \pm\frac{1}{2}, \ y = \pm\frac{1}{2}, z = \pm\frac{1}{2}$

Maximum temperature $\frac{5}{4}$ at $(\frac{1}{2}, \pm\frac{1}{2}, \frac{1}{2}), \quad (-\frac{1}{2}, \pm\frac{1}{2}, -\frac{1}{2})$

Minimum temperature $-\frac{5}{4}$ at $(\frac{1}{2}, \pm\frac{1}{2}, -\frac{1}{2}), \quad (-\frac{1}{2}, \pm\frac{1}{2}, \frac{1}{2})$

33. $S(r,h) = 2\pi r^2 + 2\pi rh, \qquad g(r,h) = \pi r^2 h, \quad (V$ constant

$\nabla S = (4\pi r + 2\pi h)\,\mathbf{i} + 2\pi r\,\mathbf{j}, \qquad \nabla g = 2\pi rh\,\mathbf{i} + \pi r^2\,\mathbf{j}.$

$$\nabla S = \lambda\nabla g \implies 4\pi r + 2\pi h = 2\pi rh\lambda, \quad 2\pi r = \pi r^2\lambda \implies r = \frac{2}{\lambda}, \quad h = \frac{4}{\lambda}.$$

Now $\pi r^2 h = V, \implies \lambda = \sqrt[3]{\dfrac{16\pi}{V}} \implies r = \sqrt[3]{\dfrac{V}{2\pi}}, \ h = \sqrt[3]{\dfrac{4V}{\pi}}.$

To minimize the surface area, take $r = \sqrt[3]{\dfrac{V}{2\pi}}$, and $h = \sqrt[3]{\dfrac{4V}{\pi}}$.

34. $f(x,y,z) = xyz^2, \quad g(x,y,z) = x + y + z - 30$

$\nabla f = yz^2\mathbf{i} + xz^2\mathbf{j} + 2xyz\mathbf{k}, \quad \nabla g = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$\nabla f = \lambda\nabla g \implies yz^2 = \lambda = xz^2 = 2xyz \implies x = y = \dfrac{z}{2} \implies x = y = \dfrac{15}{2}, \quad z = 15$

35. $S(l,w,h) = lw + 2(lh + wh), \qquad g(l,w,h) = lwh - 12, \quad (V$ constant)

$$\nabla S = (w + 2h)\,\mathbf{i} + (l + 2h)\,\mathbf{j} + 2(l + w)\,\mathbf{k}, \qquad \nabla g = wh\,\mathbf{i} + lh\,\mathbf{j} + lw\,\mathbf{k}.$$

$$\nabla S = \lambda\nabla g \implies w + 2h = \lambda wh, \quad l + 2h = \lambda lh, \quad 2(l + w) = \lambda lw \implies w = l = 2h.$$

Now $lwh = 12, \implies h = \sqrt[3]{3}, \ l = w = 2\sqrt[3]{3}.$

To minimize the surface area, take $l = w = 2\sqrt[3]{3}$ ft. and $h = \sqrt[3]{3}$ ft.

36. $f(x,y,z) = xyz, \quad g(x,y,z) = x + y + z - 1$

Maximum volume for $x = y = z = \frac{1}{3}, \quad xyz = \frac{1}{27}$

37. (a) $f(x,y,l) = xyl, \qquad g(x,y,l) = 2x + 2y + l - 108,$

$\nabla f = yl\,\mathbf{i} + xl\,\mathbf{j} + xy\,wk, \qquad \nabla g = 2\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}.$

$$\nabla f = \lambda\nabla g \implies yl = 2\lambda, \quad xl = 2\lambda, \quad xy = \lambda \implies y = x \text{ and } l = 2x.$$

Now $2x + 2y + l = 108, \implies x = 18$ and $l = 36$.

To maximize the volume, take $x = y = 18$ in. and $l = 36$ in.

(b) $f(r,l) = \pi r^2 l, \qquad g(r,l) = 2\pi r + l - 108,$

$\nabla f = 2\pi rl\,\mathbf{i} + \pi r^2\,\mathbf{j}, \qquad \nabla g = 2\pi\,\mathbf{i} + \mathbf{j}.$

$$\nabla f = \lambda \nabla g \quad \Longrightarrow \quad 2\pi r l = 2\pi\lambda, \quad \pi r^2 = \lambda, \quad l = \pi r.$$

Now $2\pi r + l = 108, \quad \Longrightarrow \quad r = \dfrac{36}{\pi}$ and $l = 36$.

To maximize the volume, take $r = 36/\pi$ in. and $l = 36$ in.

38. $S(r,h) = 4\pi r^2 + 2\pi r h, \quad g(r,h) = \dfrac{4}{3}\pi r^3 + \pi r^2 h - 10{,}000$

$\nabla S = (8\pi r + 2\pi h)\mathbf{i} + 2\pi r\mathbf{j}, \quad \nabla g = (4\pi r^2 + 2\pi r h)\mathbf{i} + \pi r^2\mathbf{j}$

$\nabla S = \lambda \nabla g \implies 2\pi(4r + h) = 2\pi r\lambda(2r + h), \quad 2\pi r = \lambda\pi r^2 \implies h = 0$

Maximum volume for sphere of radius $r = \sqrt[3]{7500/\pi}$ meters.

39. To simplify notation we set $x = Q_1, \quad y = Q_2, \quad z = Q_3$.

$$f(x,y,z) = 2x + 8y + 24z, \qquad g(x,y,z) = x^2 + 2y^2 + 4z^2 - 4{,}500{,}000{,}000$$

$$\nabla f = 2\mathbf{i} + 8\mathbf{j} + 24\mathbf{k}, \qquad \nabla g = 2x\mathbf{i} + 4y\mathbf{j} + 8z\mathbf{k}.$$

$$\nabla f = \lambda \nabla g \quad \Longrightarrow \quad 2 = 2\lambda x, \quad 8 = 4\lambda y, \quad 24 = 8\lambda z.$$

Since $\lambda \neq 0$ here, we solve the equations for x, y, z:

$$x = \frac{1}{\lambda}, \quad y = \frac{2}{\lambda}, \quad z = \frac{3}{\lambda},$$

and substitute these results in $g(x,y,z) = 0$ to obtain

$$\frac{1}{\lambda^2} + 2\left(\frac{4}{\lambda^2}\right) + 4\left(\frac{9}{\lambda^2}\right) - 45 \times 10^8 = 0, \quad \frac{45}{\lambda^2} = 45 \times 10^8, \quad \lambda = \pm 10^{-4}.$$

Since x, y, z are non-negative, $\lambda = 10^{-4}$ and

$$x = 10^4 = Q_1, \quad y = 2 \times 10^4 = Q_2, \quad z = 3 \times 10^4 = Q_3.$$

40. $f(x,y,z) = 8xyz, \qquad g(x,y,z) = 4x^2 + 9y^2 + 36z^2 - 36.$

$\nabla f(x,y,z) = 8yz\mathbf{i} + 8xz\mathbf{j} + 8xy\mathbf{k}. \qquad \nabla g(x,y,z) = 8\lambda x\mathbf{i} + 18\lambda y\mathbf{j} + 72\lambda z\mathbf{k}.$

$\nabla f = \lambda \nabla g$, gives

$$yz = \lambda x, \quad 4xz = 9\lambda y, \quad xy = 9\lambda z.$$

$$4\frac{xyz}{\lambda} = 4x^2, \quad 4\frac{xyz}{\lambda} = 9y^2, \quad 4\frac{xyz}{\lambda} = 36z^2.$$

Also notice

$$4x^2 + 9y^2 + 36z^2 - 36 = 0$$

We have

$$12\frac{xyz}{\lambda} = 36 \quad \Longrightarrow \quad x = \sqrt{3}, \quad y = \frac{2}{\sqrt{3}}, \quad z = \frac{1}{\sqrt{3}}.$$

Thus,

$$V = 8xyz = 8 \cdot \sqrt{3} \cdot \frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = \frac{16}{\sqrt{3}}.$$

PROJECT 15.6

1. $f(x,y,z) = xy^2z - x^2yz, \quad g(x,y,z) = x^2 + y^2 - 1, \quad h(x,y,z) = z^2 - x^2 - y^2, \; z \geq 0$

$\nabla f = (y^2z - 2xyz)\,\mathbf{i} + (2xyz - x^2z)\,\mathbf{j} + (xy^2 - x^2y)\,\mathbf{k}, \quad \nabla g = 2x\,\mathbf{i} + 2y\,\mathbf{j}, \quad \nabla h = -2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}.$

$\nabla f = \lambda \nabla g + \mu \nabla h \quad \Longrightarrow$

$$y^2z - 2xyz = 2x\lambda - 2x\mu 2x$$

$$2xyz - x^2z = 2y\lambda - 2y\mu$$

$$xy^2 - yx^2 = 2z\mu$$

Note first that $z = 1$. Setting $z = 1$ in the first two equations and adding, we get

$$y^2 - x^2 = 2x(\lambda - \mu) + 2y(\lambda - \mu) \quad \text{or} \quad (y + x)(y - x) = 2(y + x)(\lambda - \mu)$$

Therefore, either $y = -x$ or $\lambda - \mu = \dfrac{y - x}{2}$.

First, let $y = -x$. Then it follows that $x = \dfrac{1}{\sqrt{2}}, \quad y = -\dfrac{1}{\sqrt{2}}, \quad z = 1.$

If $y \neq -x$, then $\lambda - \mu = \dfrac{y - x}{2}$ and

$$y^2 - 2xy = x(y - x)$$

$$2xy - x^2 = y(y - x)$$

Solving these equations simultaneously, using the fact that $x^2 + y^2 = 1$, we obtain

$$9x^4 - 9x^2 + 1 = 0 \quad \Longrightarrow \quad x^2 = \frac{3 \pm \sqrt{5}}{6} \cong 0.873, \; 0.127$$

Now,

$$x^2 = 0.873 \quad \Longrightarrow \quad y^2 = 0.127 \quad \Longrightarrow \quad x = \pm 0.934, \; y = \pm 0.356;$$

$$x^2 = 0.127 \quad \Longrightarrow \quad y^2 = 0.873 \quad \Longrightarrow \quad x = \pm 0.356, \; y = \pm 0.934.$$

Clearly, the maximum value of f will occur when $x > 0$ and $y < 0$:

$$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1\right) = \frac{\sqrt{2}}{2} \cong 0.707,$$

$$f(0.934, -0.356, 1) = f(0.356, -0.934, 1) \cong 0.429.$$

Therefore, the maximum value of f subject to the constraints is $\dfrac{\sqrt{2}}{2}$.

2. $D(x,y,z) = x^2 + y^2 + z^2, \quad g(x,y,z) = x + 2y + 3z, \quad h(x,y,z) = 2x + 3y + z - 4$

$\nabla D = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}, \quad \nabla g = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \quad \nabla h = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

$\nabla D = \lambda \nabla g + \mu \nabla h \implies 2x = \lambda + 2\mu, 2 \quad y = 2\lambda + 3\mu, \quad 2z = 3\lambda + \mu \implies z = 5y - 7x$

Then $g(x,y,z)=0$ and $h(x,y,z)=0$ give $x=\frac{68}{75}, \quad y=\frac{16}{15}, \quad z=-\frac{76}{75}$

Closest point $\left(\frac{68}{75}, \frac{16}{15}, -\frac{76}{75}\right)$

3. $f(x,y,z)=x^2+y^2+z^2, \qquad g(x,y,z)=x+y-z+1=0, \qquad h(x,y,z)=x^2+y^2-z^2$

$$\nabla f = 2x\,\mathbf{i}+2y\,\mathbf{j}+2z\,wk, \qquad \nabla g=\mathbf{i}+\mathbf{j}-\mathbf{k}, \qquad \nabla h = 2x\,\mathbf{i}+2y\,\mathbf{j}-2z\,\mathbf{k}.$$

$$\nabla f=\lambda\nabla g+\mu\nabla h \implies 2x=\lambda+2x\mu, \quad 2y=\lambda+2y\mu, \quad 2z=-\lambda-2z\mu$$

Multiplying the first equation by y, the second equation by x and subtracting, yields

$$\lambda(y-x)=0.$$

Now $\lambda=0 \implies \mu=1 \implies x=y=z=0$. This is impossible since $x+y-z=-1$.

Therefore, we must have $y=x \implies z=\pm\sqrt{2}\,x$.

Substituting $y=x,\ z=\sqrt{2}\,x$ into the equation $x+y-z+1=0$, we get

$$x=-1-\frac{\sqrt{2}}{2} \implies y=-1-\frac{\sqrt{2}}{2},\ z=-1-\sqrt{2}$$

Substituting $y=x,\ z=-\sqrt{2}\,x$ into the equation $x+y-z+1=0$, we get

$$x=-1+\frac{\sqrt{2}}{2} \implies y=-1+\frac{\sqrt{2}}{2},\ z=-1+\sqrt{2}$$

Since

$$f\left(-1-\frac{\sqrt{2}}{2}, -1-\frac{\sqrt{2}}{2}, -1-\sqrt{2}\right)=6+4\sqrt{2} \text{ and}$$
$$f\left(-1+\frac{\sqrt{2}}{2}, -1+\frac{\sqrt{2}}{2}, -1+\sqrt{2}\right)=6-4\sqrt{2},$$

it follows that $\left(-1+\frac{\sqrt{2}}{2}, -1+\frac{\sqrt{2}}{2}, -1+\sqrt{2}\right)$ is closest to the origin and $\left(-1-\frac{\sqrt{2}}{2}, -1-\frac{\sqrt{2}}{2}, -1-\sqrt{2}\right)$ is furthest from the origin.

SECTION 15.7

1. $df=\left(3x^2y-2xy^2\right)\Delta x+\left(x^3-2x^2y\right)\Delta y$

2. $df=\frac{\partial f}{\partial x}\Delta x+\frac{\partial f}{\partial y}\Delta y+\frac{\partial f}{\partial z}\Delta z=(y+z)\Delta x+(x+z)\Delta y+(x+y)\Delta z$

3. $df=(\cos y+y\sin x)\,\Delta x-(x\sin y+\cos x)\,\Delta y$

4. $df=2xye^{2z}\Delta x+x^2e^{2z}\Delta y+2x^2ye^{2z}\Delta z$

5. $df=\Delta x-(\tan z)\,\Delta y-\left(y\sec^2 z\right)\Delta z$

6. $df = \left[\dfrac{x-y}{x+y} + \ln(x+y)\right]\Delta x + \left[\dfrac{x-y}{x+y} - \ln(x+y)\right]\Delta y$

7. $df = \dfrac{y(y^2+z^2-x^2)}{(x^2+y^2+z^2)^2}\,\Delta x + \dfrac{x(x^2+z^2-y^2)}{(x^2+y^2+z^2)^2}\,\Delta y - \dfrac{2xyz}{(x^2+y^2+z^2)^2}\,\Delta z$

8. $df = \left[\dfrac{2x}{x^2+y^2} + e^{xy}(x+y)\right]\Delta x + \left[\dfrac{2y}{x^2+y^2} + x^2e^{xy}\right]\Delta y$

9. $df = [\cos(x+y) + \cos(x-y)]\,\Delta x + [\cos(x+y) - \cos(x-y)]\,\Delta y$

10. $df = \ln\left(\dfrac{1+y}{1-y}\right)\Delta x + \dfrac{2x}{1-y^2}\Delta y$

11. $df = \left(y^2ze^{xz} + \ln z\right)\,\Delta x + 2ye^{xz}\,\Delta y + \left(xy^2e^{xz} + \dfrac{x}{z}\right)\,\Delta z$

12. $df = y(1-2x^2)e^{-(x^2+y^2)}\Delta x + x(1-2y^2)e^{-(x^2+y^2)}\Delta y$

13.
$$\begin{aligned}\Delta u &= \left[(x+\Delta x)^2 - 3(x+\Delta x)(y+\Delta y) + 2(y+\Delta y)^2\right] - \left(x^2 - 3xy + 2y^2\right)\\ &= \left[(1.7)^2 - 3(1.7)(-2.8) + 2(-2.8)^2\right] - \left(2^2 - 3(2)(-3) + 2(-3)^2\right)\\ &= (2.89 + 14.28 + 15.68) - 40 = -7.15\\ du &= (2x-3y)\,\Delta x + (-3x+4y)\,\Delta y\\ &= (4+9)(-0.3) + (-18)(0.2) = -7.50\end{aligned}$$

14. $du = \left(\sqrt{x-y} + \dfrac{x+y}{2\sqrt{x-y}}\right)\Delta x + \left(\sqrt{x-y} - \dfrac{x+y}{2\sqrt{x-y}}\right)\Delta y = 1$

15.
$$\begin{aligned}\Delta u &= \left[(x+\Delta x)^2(z+\Delta z) - 2(y+\Delta y)(z+\Delta z)^2 + 3(x+\Delta x)(y+\Delta y)(z+\Delta z)\right]\\ &\qquad - \left(x^2z - 2yz^2 + 3xyz\right)\\ &= \left[(2.1)^2(2.8) - 2(1.3)(2.8)^2 + 3(2.1)(1.3)(2.8)\right] - \left[(2)^2 3 - 2(1)(3)^2 + 3(2)(1)(3)\right] = 2.896\\ du &= (2xz+3yz)\,\Delta x + \left(-2z^2+3xz\right)\,\Delta y + \left(x^2-4yz+3xy\right)\,\Delta z\\ &= [2(2)(3)+3(1)(3)](0.1) + [-2(3)^2+3(2)(3)](0.3) + [2^2-4(1)(3)+3(2)(1)](-0.2) = 2.5\end{aligned}$$

16. $du = \dfrac{y^3+yz^2}{(x^2+y^2+z^2)^{3/2}}\Delta x + \dfrac{x^3+xz^2}{(x^2+y^2+z^2)^{3/2}}\Delta y - \dfrac{xyz}{(x^2+y^2+z^2)^{3/2}}\Delta z = \dfrac{77}{4(14)^{3/2}}$

17. $f(x,y) = x^{1/2}y^{1/4}; \quad x = 121, \quad y = 16, \quad \Delta x = 4, \quad \Delta y = 1$
$$\begin{aligned}f(x+\Delta x,\, y+\Delta y) &\cong f(x,y) + df\\ &= x^{1/2}\,y^{1/4} + \tfrac{1}{2}x^{-1/2}\,y^{1/4}\,\Delta x + \tfrac{1}{4}x^{1/2}\,y^{-3/4}\,\Delta y\\ \sqrt{125}\,\sqrt[4]{17} &\cong \sqrt{121}\,\sqrt[4]{16} + \tfrac{1}{2}(121)^{-1/2}\,(16)^{1/4}\,(4) + \tfrac{1}{4}(121)^{1/2}\,(16)^{-3/4}\,(1)\\ &= 11(2) + \tfrac{1}{2}\left(\tfrac{1}{11}\right)(2)(4) + \tfrac{1}{4}(11)\left(\tfrac{1}{8}\right)\\ &= 22 + \tfrac{4}{11} + \tfrac{11}{32} = 22\,\tfrac{249}{352} \cong 22.71\end{aligned}$$

18. $f(x,y) = (1-\sqrt{x})(1+\sqrt{y}), \quad x=9, \quad y=25, \quad \Delta x = 1, \quad \Delta y = -1$

$$df = -\frac{(1+\sqrt{y})}{2\sqrt{x}}\Delta x + \frac{(1-\sqrt{x})}{2\sqrt{y}}\Delta y = -\frac{4}{5}$$

$$f(10,24) \cong f(9,25) - \frac{4}{5} = -12\frac{4}{5}$$

19.
$$f(x,y) = \sin x \cos y; \quad x = \pi, \quad y = \frac{\pi}{4}, \quad \Delta x = -\frac{\pi}{7}, \quad \Delta y = -\frac{\pi}{20}$$

$$df = \cos x \cos y \, \Delta x - \sin x \sin y \, \Delta y$$

$$f(x+\Delta x, \, y + \Delta y) \cong f(x,y) + df$$

$$\sin\frac{6}{7}\pi \cos\frac{1}{5}\pi \cong \sin\pi\cos\frac{\pi}{4} + \left(\cos\pi\cos\frac{\pi}{4}\right)\left(-\frac{\pi}{7}\right) - \left(\sin\pi\sin\frac{\pi}{4}\right)\left(-\frac{\pi}{20}\right)$$

$$= 0 + \left(\frac{1}{2}\sqrt{2}\right)\left(\frac{\pi}{7}\right) + 0 = \frac{\pi\sqrt{2}}{14} \cong 0.32$$

20. $f(x,y) = \sqrt{x}\tan y, \quad x = 9, \; y = \frac{\pi}{4}, \; \Delta x = -1, \; \Delta y = \frac{1}{16}\pi$

$$df = \frac{1}{2\sqrt{x}}\tan y \Delta x + \sqrt{x}\sec^2 y \Delta y = -\frac{1}{6} + \frac{3\pi}{8}$$

$$f\left(8, \frac{5}{16}\pi\right) \cong f\left(9, \frac{\pi}{4}\right) - \frac{1}{6} + \frac{3\pi}{8} = \frac{17}{6} + \frac{3}{8}\pi \cong 4.01$$

21. $f(2.9, 0.01) \cong f(3,0) + df$, where df is to be evaluated at $x=3, \; y=0, \; \Delta x = -0.1, \; \Delta y = 0.01$.

$$df = \left(2xe^{xy} + x^2ye^{xy}\right)\Delta x + x^3e^{xy}\,\Delta y = \left[2(3)e^0 + (2)^2(0)e^0\right](-0.1) + 3^3e^0(0.01) = -0.33$$

Thus, $f(2.9, .01) \cong 3^2e^0 - 0.33 = 8.67$.

22. $x=2, \; y=3, \; z=3, \; \Delta x = 0.12, \; \Delta y = -0.08, \; \Delta z = 0.02$

$$df = 2xy\cos\pi z\Delta z + x^2\cos\pi z\Delta y - \pi x^2 y\sin\pi z\Delta z = -12(0.12) + 4(0.08) = -1.12$$

$$f(2.12, 2.92, 3.02) \cong f(2,3,3) - 1.12 = -13.12$$

23. $f(2.94, 1.1, 0.92) \cong f(3,1,1) + df$, where df is to be evaluated at $x=3, \; y=1, \; z=1$,

$\Delta x = -0.06, \; \Delta y = 0.1, \; \Delta z = -0.08$

$$df = \tan^{-1} yz\,\Delta x + \frac{xz}{1+y^2z^2}\,\Delta y + \frac{xy}{1+y^2z^2}\,\Delta z = \frac{\pi}{4}(-0.06) + (1.5)(0.1) + (1.5)(-0.08) \cong -0.441$$

Thus, $f(2.94, 1.1, 0.92) \cong 3 - 0.441 = 2.559$

24. $x=3, \; y=4, \; \Delta x = 0.06, \; \Delta y = -0.12$

$$df = \frac{x}{\sqrt{x^2+y^2}}\Delta x + \frac{y}{\sqrt{x^2+y^2}}\Delta y = \frac{3}{5}(0.06) + \frac{4}{5}(-0.12) = -0.06$$

$$f(3.06, 3.88) \cong f(3,4) - 0.06 = 4.94$$

25. $df = \frac{\partial z}{\partial x}\,\Delta x + \frac{\partial z}{\partial y}\,\Delta y = \frac{2y}{(x+y)^2}\,\Delta x - \frac{2x}{(x+y)^2}\,\Delta y$

With $x=4, \; y=2, \; \Delta x = 0.1, \; \Delta y = 0.1$, we get

$$df = \tfrac{4}{36}(0.1) - \tfrac{8}{36}(0.1) = -\tfrac{1}{90}.$$

The exact change is $\dfrac{4.1-2.1}{4.1+2.1} - \dfrac{4-2}{4+2} = \dfrac{2}{6.2} - \dfrac{1}{3} = -\dfrac{1}{93}.$

26. $f(r,h) = \pi r^2 h, \quad r = 8, \quad h = 12, \quad \Delta r = -0.3, \quad \Delta h = 0.2$

$df = 2\pi rh\Delta r + \pi r^2 \Delta h = 192\pi(-0.3) + 64\pi(0.2) = -44.8\pi$

decreases by approximately 44.8π cubic inches.

27. $S = 2\pi r^2 + 2\pi rh; \quad r = 8,\ h = 12,\ \Delta r = -0.3,\ \Delta h = 0.2$

$$dS = \frac{\partial S}{\partial r}\Delta r + \frac{\partial S}{\partial h}\Delta h = (4\pi r + 2\pi h)\Delta r + (2\pi r)\Delta h$$
$$= 56\pi(-0.3) + 16\pi(0.2) = -13.6\pi.$$

The area decreases about 13.6π in.^{2}.

28. $dT = 2x\cos\pi z\Delta x - 2y\sin\pi z\Delta y - (\pi x^2\sin\pi z + \pi y^2\cos\pi z)\Delta z$

$= 4(0.1) - 4\pi(0.2) = \dfrac{2}{5} - \dfrac{4}{5}\pi$

T decreases by about $\dfrac{4}{5}\pi - \dfrac{2}{5} \cong 2.11$

29. $S(9.98, 5.88, 4.08) \cong S(10,6,4) + dS = 248 + dS,$ where

$dS = (2w+2h)\Delta l + (2l+2h)\Delta w + (2l+2w)\Delta h = 20(-0.02) + 28(-0.12) + 32(0.08) = -1.20$

Thus, $S(9.98, 5.88, 4.08) \cong 248 - 1.20 = 246.80.$

30. $f(r,h) = \dfrac{\pi r^2 h}{3}, \quad r = 7,\ h = 10,\ \Delta r = 0.2,\ \Delta h = 0.15$

$df = \dfrac{2\pi rh}{3}\Delta r + \dfrac{\pi r^2}{3}\Delta h = \dfrac{140}{3}\pi(0.2) + \dfrac{49}{3}\pi(0.15) = \dfrac{\pi}{3}(35.35)$

$f(7.2, 10.15) \cong f(7,10) + \dfrac{\pi}{3}(35.35) = 525.35\dfrac{\pi}{3} \cong 550.15$

31. (a) $dV = yx\,\Delta x + xz\,\Delta y + xy\,\Delta z = (8)(6)(0.02) + (12)(6)(-0.05) + (12)(8)(0.03) = 0.24$

(b) $\Delta V = (12.02)(7.95)(6.03) - (12)(8)(6) = 0.22077$

32. (a) $S(x,y,z) = 2(xy + xz + yz), \quad x = 12,\ y = 8,\ z = 6,\ \Delta x = 0.02,\ \Delta y = -0.05,\ \Delta z = 0.03$

$dS = 2(y+z)\Delta x + 2(x+z)\Delta y + 2(x+y)\Delta z = 28(0.02) + 36(-0.05) + 40(0.03) = -0.04$

(b) $\Delta S = S(12.02, 7.95, 6.03) - S(12,8,6) = -0.0219$

33. $T(P) - T(Q) \cong dT = (-2x + 2yz)\,\Delta x + (-2y + 2xz)\,\Delta y + (-2z + 2xy)\,\Delta z$

Letting $x = 1,\ y = 3,\ z = 4,\ \Delta x = 0.15,\ \Delta y = -0.10,\ \Delta z = 0.10$, we have

$$dT = (22)(0.15) + (2)(-0.10) + (-2)(0.10) = 2.9$$

34. $k = \dfrac{PV}{T} = \dfrac{4.81}{300} = \dfrac{27}{25}.\quad P = \dfrac{27}{25}\dfrac{T}{V}\quad V = 81,\ T = 300,\ \Delta V = -5,\ \Delta T = 25$

$$\Delta P \cong dP = \frac{27}{25}\frac{1}{V}\Delta T - \frac{27}{25}\frac{T}{V^2}\Delta V = \frac{47}{81}$$

35. The area is given by $A = \frac{1}{2}x^2\tan\theta$. The change in area is approximated by:

$$dA = x\tan\theta\,\Delta x + \tfrac{1}{2}x^2\sec^2\theta\,\Delta\theta \cong 4(0.75)\,\Delta x + 8(1.5625)\,\Delta\theta = 3\,\Delta x + 12.5\,\Delta\theta$$

The area is more sensitive to a change in θ.

36. $A = \dfrac{1}{2}x^2\sin\theta;\quad \Delta A \cong dA = x\sin\theta\Delta x + \dfrac{x^2}{2}\cos\theta\Delta\theta$

The area is more sensitive to changes in θ.

37. (a) $\pi r^2 h = \pi(r+\Delta r)^2(h+\Delta h) \quad\Longrightarrow\quad \Delta h\,\dfrac{r^2h}{(r+\Delta r)^2} - h = -\dfrac{(2r+\Delta r)h}{(r+\Delta r)^2}\,\Delta r.$

$$df = (2\pi rh)\,\Delta r + \pi r^2\,\Delta h, \qquad df = 0 \quad\Longrightarrow\quad \Delta h = \frac{-2h}{r}\,\Delta r.$$

(b) $2\pi r^2 + 2\pi rh = 2\pi(r+\Delta r)^2 + 2\pi(r+\Delta r)(h+\Delta h).$

Solving for Δh,

$$\Delta h = \frac{r^2 + rh - (r+\Delta r)^2}{r+\Delta r} - h = -\frac{(2r+h+\Delta r)}{r+\Delta r}\,\Delta r.$$

$$df = (4\pi r + 2\pi h)\,\Delta r + 2\pi r\,\Delta h, \qquad df = 0 \quad\Longrightarrow\quad \Delta h = -\left(\frac{2r+h}{r}\right)\Delta r.$$

38. Amount of paint is increase in volume. $f(x,y,z) = xyz,\ x = 48$ in, $y = 24$ in, $z = 36$ in,

$\Delta x = \Delta y = \Delta z = \frac{2}{16}$ in. $\qquad \Delta f \cong df = yz\Delta x + xz\Delta y + xy\Delta z = 3774(\frac{2}{16}) = 468$

Use about 468 cubic inches of paint.

39. (a) $c(x,y) = \sqrt{x^2+y^2};\ x = 5, y = 12, \Delta x = \pm 1.5, \Delta y = \pm 1.5$

$$dc = \frac{\partial c}{\partial x}\,\Delta x + \frac{\partial c}{\partial y}\,\Delta y = \frac{x}{\sqrt{x^2+y^2}}\,\Delta x + \frac{y}{\sqrt{x^2+y^2}}\,\Delta y$$
$$= \frac{5}{13}(\pm 1.5) + \frac{12}{13}(\pm 1.5) \cong \pm 1.962$$

The maximum possible error in the value of the hypotenuse is 1.962 cm.

(b) $A(x,y) = \frac{1}{2}xy;\ x = 5, y = 12, \Delta x = \pm 1.5, \Delta y = \pm 1.5$

$$dA = \frac{\partial A}{\partial x}\,\Delta x + \frac{\partial A}{\partial y}\,\Delta y = \tfrac{1}{2}y\,\Delta x + \tfrac{1}{2}y\,\Delta y$$
$$= \tfrac{1}{2}(12)(\pm 1.5) + \tfrac{1}{2}(5)(\pm 1.5) \cong \pm 12.75$$

The maximum possible error in the value of the area is 12.75 cm^2.

40. (a) $dV = yz\Delta x + xz\Delta y + xy\Delta z, \quad x = 60 \text{ in}, \ y = 36 \text{ in}, \ z = 42 \text{ in}$

Maximum possible error $= 6192(\frac{1}{12}) = 516$ cubic inches.

(b) $dS = 2(y+z)\Delta x + 2(x+z)\Delta y + 2(x+y)\Delta z$

Maximum possible error $= 552(\frac{1}{12}) = 46$ square inches

41. $s = \dfrac{A}{A-W}; \ A = 9, \ W = 5, \ \Delta A = \pm 0.01, \ \Delta W = \pm 0.02$

$$\begin{aligned} ds = \frac{\partial s}{\partial A}\,\Delta A + \frac{\partial s}{\partial W}\,\Delta W &= \frac{-W}{(A-W)^2}\,\Delta A + \frac{A}{(A-W)^2}\,\Delta W \\ &= -\frac{5}{16}(\pm 0.01) + \frac{9}{16}(\pm 0.02) \cong \pm 0.014 \end{aligned}$$

The maximum possible error in the value of s is 0.014 lbs; $2.23 \le s + \Delta s \le 2.27$

42. (a) $V = \dfrac{\pi}{3} r^2 h, \quad r = 5, \ h = 12, \ \Delta r = 0.2, \ \Delta h = 0.3$

$dV = \frac{2}{3}\pi r h \Delta r + \frac{\pi r^2}{3}\Delta h$

Maximum error for $+\Delta r, \ +\Delta h \implies 40\pi(0.2) + \dfrac{25}{3}\pi(0.3) = 10.5\pi$ cubic inches.

(b) $S = \pi r \sqrt{r^2 + h^2}$

$dS = \dfrac{\pi(2r^2 + h^2)}{\sqrt{r^2+h^2}}\Delta r + \dfrac{\pi r h}{\sqrt{r^2+h^2}}\Delta h$

Maximum error for $+\Delta r, \ +\Delta h \implies \dfrac{194}{13}\pi(0.2) + \dfrac{60}{13}\pi(0.3) = \dfrac{284}{65}\pi$ square inches.

SECTION 15.8

1. $\dfrac{\partial f}{\partial x} = xy^2, \quad f(x,y) = \frac{1}{2}x^2y^2 + \phi(y), \quad \dfrac{\partial f}{\partial y} = x^2 y + \phi'(y) = x^2 y.$

Thus, $\phi'(y) = 0, \ \phi(y) = C,$ and $f(x,y) = \frac{1}{2}x^2y^2 + C.$

2. $\dfrac{\partial f}{\partial x} = x, \quad \dfrac{\partial f}{\partial y} = y \implies f(x,y) = \dfrac{1}{2}(x^2 + y^2) + C$

3. $\dfrac{\partial f}{\partial x} = y, \quad f(x,y) = xy + \phi(y), \quad \dfrac{\partial f}{\partial y} = x + \phi'(y) = x.$

Thus, $\phi'(y) = 0, \ \phi(y) = C,$ and $f(x,y) = xy + C.$

4. $\dfrac{\partial f}{\partial x} = x^2 + y \implies f(x,y) = \dfrac{x^3}{3} + xy + \phi(y); \quad \dfrac{\partial f}{\partial y} = x + \phi'(y) = y^3 + x \implies f(x,y) = \dfrac{1}{3}x^3 + \dfrac{1}{4}y^4 + xy + C$

5. No; $\dfrac{\partial}{\partial y}\left(y^3 + x\right) = 3y^2$ whereas $\dfrac{\partial}{\partial x}\left(x^2 + y\right) = 2x.$

6. $\dfrac{\partial f}{\partial x} = y^2 e^x - y \implies f(x,y) = y^2 e^x - xy + \phi(y);$

$$\frac{\partial f}{\partial y} = 2ye^x - x + \phi'(y) = 2ye^x - x \implies f(x,y) = y^2e^x - xy + C$$

7. $\dfrac{\partial f}{\partial x} = \cos x - y\sin x, \quad f(x,y) = \sin x + y\cos x + \phi(y), \quad \dfrac{\partial f}{\partial y} = \cos x + \phi'(y) = \cos x.$

Thus, $\phi'(y) = 0,\ \phi(y) = C,$ and $f(x,y) = \sin x + y\cos x + C.$

8. $\dfrac{\partial f}{\partial x} = 1 + e^y \implies f(x,y) = x + xe^y + \phi(y);$

$$\frac{\partial f}{\partial y} = xe^y + \phi'(y) = xe^y + y^2 \implies f(x,y) = x + xe^y + \frac{y^3}{3} + C$$

9. $\dfrac{\partial f}{\partial x} = e^x\cos y^2, \quad f(x,y) = e^x\cos y^2 + \phi(y), \quad \dfrac{\partial f}{\partial y} = -2ye^x\sin y^2 + \phi'(y) = -2ye^x\sin y^2.$

Thus, $\phi'(y) = 0,\ \phi(y) = C,$ and $f(x,y) = e^x\cos y^2 + C.$

10. $\dfrac{\partial^2 f}{\partial y\partial x} = -e^x\sin y, \quad \dfrac{\partial^2 f}{\partial x\partial y} = e^x\sin y \neq \dfrac{\partial^2 f}{\partial y\partial x};$ not a gradient.

11. $\dfrac{\partial f}{\partial y} = xe^x - e^{-y}, \quad f(x,y) = xye^x + e^{-y} + \phi(x), \quad \dfrac{\partial f}{\partial x} = ye^x + xye^x + \phi'(x) = ye^x(1+x).$

Thus, $\phi'(x) = 0,\ \phi(x) = C,$ and $f(x,y) = xye^x + e^{-y} + C.$

12. $\dfrac{\partial f}{\partial x} = e^x + 2xy \implies f(x,y) = e^x + x^2y + \phi(y); \quad \dfrac{\partial f}{\partial y} = x^2 + \phi'(y) = x^2 + \sin y$

$$\implies f(x,y) = e^x + x^2y - \cos y + C$$

13. No; $\dfrac{\partial}{\partial y}\left(xe^{xy} + x^2\right) = x^2e^{xy}$ whereas $\dfrac{\partial}{\partial x}\left(ye^{xy} - 2y\right) = y^2e^{xy}$

14. $\dfrac{\partial f}{\partial y} = x\sin x + 2y + 1 \implies f(x,y) = xy\sin x + y^2 + y + \phi(x)$

$$\frac{\partial f}{\partial x} = y\sin x + xy\cos x + \phi'(x) = y\sin x + xy\cos x \implies f(x,y) = xy\sin x + y^2 + y + C$$

15. $\dfrac{\partial}{\partial x} = 1 + y^2 + xy^2, \quad f(x,y) = x + xy^2 + \tfrac{1}{2}x^2y^2 + \phi(y), \quad \dfrac{\partial}{\partial y} = 2xy + x^2y + \phi'(y) = x^2y + y + 2xy + 1.$

Thus, $\phi'(y) = y + 1, \quad \phi(y) = \tfrac{1}{2}y^2 + y + C$ and $f(x,y) = x + xy^2 + \tfrac{1}{2}x^2y^2 + \tfrac{1}{2}y^2 + y + C.$

16. $\dfrac{\partial f}{\partial x} = 2\ln 3y + \dfrac{1}{x} \implies f(x,y) = 2x\ln 3y + \ln x + \phi(y); \quad \dfrac{\partial f}{\partial y} = \dfrac{2x}{y} + \phi'(y) = \dfrac{2x}{y} + y^2$

$$f(x,y) = 2x\ln 3y + \ln x + \frac{y^3}{3} + C$$

17. $\dfrac{\partial f}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}}, \quad f(x,y) = \sqrt{x^2+y^2} + \phi(y), \quad \dfrac{\partial f}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}} + \phi'(y) = \dfrac{y}{\sqrt{x^2+y^2}}.$

Thus, $\phi'(y) = 0,\ \phi(y) = C,$ and $f(x,y) = \sqrt{x^2+y^2} + C.$

18. $$\frac{\partial f}{\partial x} = x\tan y + \sec^2 x \implies f(x,y) = \frac{x^2}{2}\tan y + \tan x + \phi(y);$$
$$\frac{\partial f}{\partial y} = \frac{x^2}{2}\sec^2 y + \phi'(y) = \frac{x^2}{2}\sec^2 y + \pi y; \implies f(x,y) = \frac{x^2}{2}\tan y + \tan x + \frac{\pi}{2}y^2 + C.$$

19. $$\frac{\partial f}{\partial x} = x^2\sin^{-1} y, \quad f(x,y) = \tfrac{1}{3}x^3\sin^{-1} y + \phi(y), \quad \frac{\partial f}{\partial y} = \frac{x^3}{3\sqrt{1-y^2}} + \phi'(y) = \frac{x^3}{3\sqrt{1-y^2}} - \ln y.$$

Thus, $\phi'(y) = -\ln y,\ \phi(y) = y - y\ln y + C,$ and

$$f(x,y) = \frac{1}{3}\sin^{-1} y + y - y\ln y + C.$$

20. $$\frac{\partial f}{\partial x} = \frac{\tan^{-1} y}{\sqrt{1-x^2}} + \frac{x}{y} \implies f = \sin^{-1} x\tan^{-1} y + \frac{x^2}{2y} + \phi(y);$$
$$\frac{\partial f}{\partial y} = \frac{\sin^{-1} x}{1+y^2} - \frac{x^2}{2y^2} + \phi'(y) = \frac{\sin^{-1} x}{1+y^2} - \frac{x^2}{2y^2} + 1 \implies f(x,y) = \sin^{-1} x\tan^{-1} y + \frac{x^2}{2y} + y + C.$$

21. $$\frac{\partial f}{\partial x} = f(x,y), \quad \frac{\partial f/\partial x}{f(x,y)} = 1, \quad \ln f(x,y) = x + \phi(y), \quad \frac{\partial f/\partial y}{f(x,y)} = 0 + \phi'(y), \quad \frac{\partial f}{\partial y} = f(x,y).$$

Thus, $\phi'(y) = 1,\ \phi(y) = y + K,$ and $f(x,y) = e^{x+y+K} = Ce^{x+y}.$

22. $$\frac{\partial f}{\partial x} = e^{g(x,y)}g_x(x,y) \implies f(x,y) = e^{g(x,y)} + \phi(y);$$
$$\frac{\partial f}{\partial y} = e^{g(x,y)}g_y(x,y) + \phi'(y) = e^{g(x,y)}g_y(x,y) \implies f(x,y) = e^{g(x,y)} + C.$$

23. (a) $P = 2x,\ Q = z,\ R = y;$ $\quad \dfrac{\partial P}{\partial y} = 0 = \dfrac{\partial Q}{\partial x}, \quad \dfrac{\partial P}{\partial z} = 0 = \dfrac{\partial R}{\partial x}, \quad \dfrac{\partial Q}{\partial z} = 1 = \dfrac{\partial R}{\partial y}$

(b), (c), and (d)

$$\frac{\partial f}{\partial x} = 2x, \quad f(x,y,z) = x^2 + g(y,z).$$

$$\frac{\partial f}{\partial y} = 0 + \frac{\partial g}{\partial y} \quad \text{with} \quad \frac{\partial f}{\partial y} = z \implies \frac{\partial g}{\partial y} = z.$$

Then,

$$g(y,z) = yz + h(z),$$

$$f(x,y,z) = x^2 + yz + h(z),$$

$$\frac{\partial f}{\partial z} = 0 + y + h'(z) \quad \text{and} \quad \frac{\partial f}{\partial z} = y \implies h'(z) = 0.$$

Thus, $h(z) = C$ and $f(x,y,z) = x^2 + yz + C.$

24. $\dfrac{\partial f}{\partial x} = yz \implies f(x,y,z) = xyz + g(y,z); \quad \dfrac{\partial f}{\partial y} = xz + \dfrac{\partial g}{\partial y} = xz \implies f = xyz + h(z)$

$\dfrac{\partial f}{\partial z} = xy + h'(z) = xy \implies f(x,y,z) = xyz + C$

25. The function is a gradient by the test stated before Exercise 23.

Take $P = 2x + y, \quad Q = 2y + x + z, \quad R = y - 2z.$ Then

$$\frac{\partial P}{\partial y} = 1 = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = 0 = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = 1 = \frac{\partial R}{\partial y}.$$

Next, we find f where $\nabla f = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$.

$$\frac{\partial f}{\partial x} = 2x + y,$$

$$f(x,y,z) = x^2 + xy + g(y,z).$$

$$\frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} \quad \text{with} \quad \frac{\partial f}{\partial y} = 2y + x + z \quad \implies \quad \frac{\partial g}{\partial y} = 2y + z.$$

Then,

$$g(y,z) = y^2 + yz + h(z),$$

$$f(x,y,z) = x^2 + xy + y^2 + yz + h(z).$$

$$\frac{\partial f}{\partial z} = y + h'(z) = y - 2z \quad \implies \quad h'(z) = -2z.$$

Thus, $h(z) = -z^2 + C$ and $f(x,y,z) = x^2 + xy + y^2 + yz - z^2 + C.$

26. $\dfrac{\partial f}{\partial x} = 2x \sin 2y \cos z \implies f(x,y,z) = x^2 \sin 2y \cos z + g(y,z);$

$\dfrac{\partial f}{\partial y} = 2x^2 \cos 2y \cos z + \dfrac{\partial g}{\partial y} = 2x^2 \cos 2y \cos z \implies f(x,y,z) = x^2 \sin 2y \cos z + h(z)$

$\dfrac{\partial f}{\partial z} = -x^2 \sin 2y \sin z + h'(z) = -x^2 \sin 2y \sin z \implies f(x,y,z) = x^2 \sin 2y \cos z + C$

27. The function is a gradient by the test stated before Exercise 23.

Take $P = y^2z^3 + 1, \quad Q = 2xyz^3 + y, \quad R = 3xy^2z^2 + 1.$ Then

$$\frac{\partial P}{\partial y} = 2yz^3 = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = 3y^2z^2 = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = 6xyz^2 = \frac{\partial R}{\partial y}.$$

Next, we find f where $\nabla f = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$.

$$\frac{\partial f}{\partial x} = y^2z^3 + 1,$$

$$f(x,y,z) = xy^2z^3 + x + g(y,z).$$

$$\frac{\partial f}{\partial y} = 2xyz^3 \frac{\partial g}{\partial y} \quad \text{with} \quad \frac{\partial f}{\partial y} = 2xyz^3 + y \quad \implies \quad \frac{\partial g}{\partial y} = y.$$

Then,

$$g(y,z) = \tfrac{1}{2}\,y^2 + h(z),$$

$$f(x,y,z) = xy^2z^3 + x + \tfrac{1}{2}\,y^2 + h(z).$$

$$\frac{\partial f}{\partial z} = 3xy^2z^2 + h'(z) = 3xy^2z^2 + 1 \quad\Longrightarrow\quad h'(z) = 1.$$

Thus, $h(z) = z + C$ and $f(x,y,z) = xy^2z^3 + x + \tfrac{1}{2}\,y^2 + z + C.$

28. $\dfrac{\partial f}{\partial x} = \dfrac{y}{z} - e^z \implies f(x,y,z) = \dfrac{xy}{z} - xe^z + g(y,z)$

$\dfrac{\partial f}{\partial y} = \dfrac{x}{z} + \dfrac{\partial g}{\partial y} = \dfrac{x}{z} + 1 \implies f(x,y,z) = \dfrac{xy}{z} + y - xe^z + h(z)$

$\dfrac{\partial f}{\partial z} = -\dfrac{xy}{z^2} - xe^z + h'(z) = -xe^z - \dfrac{xy}{z^2} \implies f(x,y,z) = \dfrac{xy}{z} - xe^z + y + C$

29. $\mathbf{F}(\mathbf{r}) = \nabla\left(\dfrac{GmM}{r}\right)$

30.

$$\mathbf{h}(\mathbf{r}) = \begin{cases} \nabla\left(\dfrac{k}{n+2}r^{n+2}\right), & n \neq 2 \\ \nabla\,(k\ln r), & n = -2 \end{cases}$$

SECTION 15.9

1. $\dfrac{\partial P}{\partial y} = 2xy - 1 = \dfrac{\partial Q}{\partial x};$ the equation is exact.

$\dfrac{\partial f}{\partial x} = xy^2 - y \implies f(x,y) = \tfrac{1}{2}\,x^2y^2 - xy + \varphi(y)$

$\dfrac{\partial f}{\partial y} = x^2y - x + \varphi'(y) = x^2y - x \implies \varphi'(y) = 0 \implies \varphi(y) = 0$ (omit the constant)*

Therefore $f(x,y) = \tfrac{1}{2}\,x^2y^2 - xy,$ and a one-parameter family of solutions is:

$$\tfrac{1}{2}\,x^2y^2 - xy = C$$

* We will omit the constant at this step throughout this section.

2. $\dfrac{\partial}{\partial y}(e^x \sin y) = e^x\cos y = \dfrac{\partial}{\partial x}(e^x\cos y);$

$f(x,y) = e^x\sin y,$ and $e^x \sin y = C$ is a one-parameter family of solutions.

3. $\dfrac{\partial P}{\partial y} = e^y - e^x = \dfrac{\partial Q}{\partial x};$ the equation is exact.

$$\frac{\partial f}{\partial x} = e^y - ye^x \quad \Longrightarrow \quad f(x,y) = xe^y - ye^x + \varphi(y)$$

$$\frac{\partial f}{\partial y} = xe^y - e^x + \varphi'(y) = xe^y - e^x \quad \Longrightarrow \quad \varphi'(y) = 0 \quad \Longrightarrow \quad \varphi(y) = 0$$

Therefore $f(x,y) = xe^y - ye^x$, and a one-parameter family of solutions is:

$$xe^y - ye^x = C$$

4. $\dfrac{\partial}{\partial y}(\sin y) = \cos y = \dfrac{\partial}{\partial x}(x\cos y + 1)$;

$f(x,y) = x\sin y + y$, and $x\sin y + y = C$ is a one-parameter family of solutions.

5. $\dfrac{\partial P}{\partial y} = \dfrac{1}{y} + 2x = \dfrac{\partial Q}{\partial x}$; the equation is exact.

$$\frac{\partial f}{\partial x} = \ln y + 2xy \quad \Longrightarrow \quad f(x,y) = x\,\ln y + x^2 y + \varphi(y)$$

$$\frac{\partial f}{\partial y} = \frac{x}{y} + x^2 + \varphi'(y) = \frac{x}{y} + x^2 \quad \Longrightarrow \quad \varphi'(y) = 0 \quad \Longrightarrow \quad \varphi(y) = 0$$

Therefore $f(x,y) = x\,\ln y + x^2 y$, and a one-parameter family of solutions is:

$$x\,\ln y + x^2 y = C$$

6. $\dfrac{\partial}{\partial y}(2x\tan^{-1} y) = \dfrac{2x}{1+y^2} = \dfrac{\partial}{\partial x}\left(\dfrac{x^2}{1+y^2}\right)$;

$f(x,y) = x^2\tan^{-1} y$, and $x^2\tan^{-1} y = C$ is a one-parameter family of solutions.

7. $\dfrac{\partial P}{\partial y} = \dfrac{1}{x} = \dfrac{\partial Q}{\partial x}$; the equation is exact.

$$\frac{\partial f}{\partial x} = \frac{y}{x} + 6x \quad \Longrightarrow \quad f(x,y) = y\,\ln x + 3x^2 + \varphi(y)$$

$$\frac{\partial f}{\partial y} = \ln x + \varphi'(y) = \ln x - 2 \quad \Longrightarrow \quad \varphi'(y) = -2 \quad \Longrightarrow \quad \varphi(y) = -2y$$

Therefore $f(x,y) = y\,\ln x + 3x^2 - 2y$, and a one-parameter family of solutions is:

$$y\,\ln x + 3x^2 - 2y = C$$

8. $\dfrac{\partial}{\partial y}(e^x + \ln y + \dfrac{y}{x}) = \dfrac{1}{y} + \dfrac{1}{x} = \dfrac{\partial}{\partial x}(\dfrac{x}{y} + \ln x + \sin y)$;

$f(x,y) = e^x + x\ln y + y\ln x - \cos y$ and $e^x + x\ln y + y\ln x - \cos y = C$ is a one-parameter family of solutions.

9. $\dfrac{\partial P}{\partial y} = 3y^2 - 2y\,\sin x = \dfrac{\partial Q}{\partial x}$; the equation is exact.

$$\frac{\partial f}{\partial x} = y^3 - y^2\sin x - x \quad \Longrightarrow \quad f(x,y) = xy^3 + y^2\cos x - \tfrac{1}{2}x^2 + \varphi(y)$$

$$\frac{\partial f}{\partial y} = 3xy^2 + 2y\cos x + \varphi'(y) = 3xy^2 + 2y\cos x + e^{2y} \implies \varphi'(y) = e^{2y} \implies \varphi(y) = \tfrac{1}{2}e^{2y}$$

Therefore $f(x,y) = xy^3 + y^2\cos x - \frac{1}{2}x^2 + \frac{1}{2}e^{2y}$, and a one-parameter family of solutions is:

$$xy^3 + y^2\cos x - \tfrac{1}{2}x^2 + \tfrac{1}{2}e^{2y} = C$$

10. $\frac{\partial}{\partial y}(e^{2y} - y\cos xy) = 2e^{2y} - \cos xy + xy\sin xy = \frac{\partial}{\partial x}(2xe^{2y} - x\cos xy + 2y)$;
$f(x,y) = xe^{2y} - \sin xy + y^2$ and $xe^{2y} - \sin xy + y^2 = C$ is a one-parameter family of solutions.

11. (a) Yes: $\frac{\partial}{\partial y}[p(x)] = 0 = \frac{\partial}{\partial x}[q(y)]$.

(b) For all x, y such that $p(y)q(x) \neq 0$, $\frac{1}{p(y)q(x)}$ is an integrating factor.

Multiplying the differential equation by $\frac{1}{p(y)q(x)}$, we get

$$\frac{1}{q(x)} + \frac{1}{p(y)}y' = 0$$

which has the form of the differential equation in part (a).

12. Mimic the proof of the first part.

13. $\frac{\partial P}{\partial y} = e^{y-x} - 1$ and $\frac{\partial Q}{\partial x} = e^{y-x} - xe^{y-x}$; the equation is not exact.

Since $\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{xe^{y-x} - 1}\left(xe^{y-x} - 1\right) = 1$, $\mu(x) = e^{\int dx} = e^x$ is an

an integrating factor. Multiplying the given equation by e^x, we get

$$(e^y - ye^x) + (xe^y - e^x)\,y' = 0$$

This is the equation given in Exercise 3. A one-parameter family of solutions is:

$$xe^y - ye^x = C$$

14. $w = \frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{x + e^y}(e^y + x) = 1$ doesn't depend on x, so $e^{\int -dy} = e^{-y}$ is an integrating factor.

$(xe^{-y} + 1) - \frac{1}{2}x^2e^{-y}y' = 0$ is exact; $f(x,y) = \frac{1}{2}x^2e^{-y} + x$ and a one-parameter family of solutions is $\frac{1}{2}x^2e^{-y} + x = C$.

15. $\frac{\partial P}{\partial y} = 6x^2y + e^y = \frac{\partial Q}{\partial x}$; the equation is exact.

$$\frac{\partial f}{\partial x} = 3x^2y^2 + x + e^y \implies f(x,y) = x^3y^2 + \tfrac{1}{2}x^2 + xe^y + \varphi(y)$$

$$\frac{\partial f}{\partial y} = 2x^3y + xe^y + \varphi'(y) = 2x^3y + y + xe^y \quad \Longrightarrow \quad \varphi'(y) = y \quad \Longrightarrow \quad \varphi(y) = \tfrac{1}{2}\,y^2$$

Therefore $f(x,y) = x^3y^2 + \frac{1}{2}\,x^2 + xe^y + \frac{1}{2}\,y^2$, and a one-parameter family of solutions is:

$$x^3y^2 + \tfrac{1}{2}\,x^2 + xe^y + \tfrac{1}{2}\,y^2 = C$$

16. $\dfrac{\partial}{\partial y}(\sin 2x \cos y) = -\sin 2x \sin y = -2\sin x \cos x \sin y = \dfrac{\partial}{\partial x}(\sin 2x \sin y);$ exact.

$f(x,y) = \sin^2 x \cos y$ and $\sin^2 x \cos y = C$ is a one-parameter family of solutions.

17. $\dfrac{\partial P}{\partial y} = 3y^2$ and $\dfrac{\partial Q}{\partial x} = 0;$ the equation is not exact.

Since $\dfrac{1}{Q}\left(\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}\right) = \dfrac{1}{3y^2}\,(3y^2) = 1,\quad \mu(x) = e^{\int dx} = e^x$ is an

an integrating factor. Multiplying the given equation by e^x, we get

$$\left(y^3e^x + xe^x + e^x\right) + \left(3y^2xe^x\right)\,y' = 0$$

$$\frac{\partial f}{\partial x} = y^3e^x + xe^x + e^x \quad \Longrightarrow \quad f(x,y) = y^3e^x + xe^x + \varphi(y)$$

$$\frac{\partial f}{\partial y} = 3y^2e^x + \varphi'(y) = 3y^2e^x \quad \Longrightarrow \quad \varphi'(y) = 0 \quad \Longrightarrow \quad \varphi(y) = 0$$

Therefore $f(x,y) = y^3e^x + xe^x$, and a one-parameter family of solutions is:

$$y^3e^x + xe^x = C$$

18. $v = \dfrac{1}{Q}\left(\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}\right) = \dfrac{1}{xe^{2x+y}+1}\left(-2xe^{2x+y} - 2\right) = -2,$ independent of y, so $e^{\int -2\,dx} = e^{-2x}$ is an integrating factor. Thus $(e^y - 2ye^{-2x}) + (xe^y + e^{-2x})y' = 0$ is exact.

$f(x,y) = xe^y + ye^{-2x}$, and $xe^y + ye^{-2x} = C$ is a one-parameter family of solutions.

19. $\dfrac{\partial P}{\partial y} = 1 = \dfrac{\partial Q}{\partial x};$ the equation is exact.

$$\frac{\partial f}{\partial x} = x^2 + y \quad \Longrightarrow \quad f(x,y) = \tfrac{1}{3}\,x^3 + xy + \varphi(y)$$

$$\frac{\partial f}{\partial y} = x + \varphi'(y) = x + e^y \quad \Longrightarrow \quad \varphi'(y) = e^y \quad \Longrightarrow \quad \varphi(y) = e^y$$

Therefore $f(x,y) = \frac{1}{3}\,x^3 + xy + e^y$, and a one-parameter family of solutions is:

$$\tfrac{1}{3}\,x^3 + xy + e^y = C$$

Setting $x = 1,\ y = 0$, we get $C = \frac{4}{3}$ and

$$\tfrac{1}{3}\,x^3 + xy + e^y = \tfrac{4}{3} \qquad \text{or} \qquad x^3 + 3xy + 3e^y = 4$$

20. $\dfrac{\partial}{\partial y}(3x^2 - 2xy + y^3) = -2x + 3y^2 = \dfrac{\partial}{\partial x}\left(3xy^2 - x^2\right);$ exact

$f(x,y) = x^3 - x^2y + xy^3 \implies x^3 - x^2y + xy^3 = C.$

Substituting $x=-1,\ y=1$ we get $-1+1+1=C \implies x^3-x^2y+xy^3=1$

21. $\dfrac{\partial P}{\partial y}=4y$ and $\dfrac{\partial Q}{\partial x}=2y;$ the equation is not exact.

Since $\dfrac{1}{Q}\left(\dfrac{\partial P}{\partial y}-\dfrac{\partial Q}{\partial x}\right)=\dfrac{1}{2xy}(2y)=\dfrac{1}{x},\quad \mu(x)=e^{\int(1/x)\,dx}=e^{\ln x}=x$ is an integrating factor. Multiplying the given equation by x, we get

$$\left(2xy^2+x^3+2x\right)+\left(2x^2y\right)\,y'=0$$

$\dfrac{\partial f}{\partial y}=2x^2y \implies f(x,y)=x^2y^2+\varphi(x)$

$\dfrac{\partial f}{\partial x}=2xy^2+\varphi'(x)=2xy^2+x^3+2x \implies \varphi'(x)=x^3+2x \implies \varphi=\tfrac14x^4+x^2$

Therefore $f(x,y)=x^2y^2+\tfrac14x^4+x^2$, and a one-parameter family of solutions is:

$$x^2y^2+\tfrac14x^4+x^2=C$$

Setting $x=1,\ y=0$, we get $C=\tfrac54$ and

$$x^2y^2+\tfrac14x^4+x^2=\tfrac54 \qquad \text{or} \qquad 4x^2y^2+x^4+4x^2=5$$

22. $v=\dfrac{1}{Q}\left(\dfrac{\partial P}{\partial y}-\dfrac{\partial Q}{\partial x}\right)=\dfrac{2-6xy^2}{3x^2y^2-x}=-\dfrac{2}{x}$ doesn't depend on y, so $e^{\int-\frac{2}{x}\,dx}=x^{-2}$ is an integrating factor. Thus $(1+yx^{-2})+(3y^2-x^{-1})y'=0$ is exact.

$f(x,y)=x-\dfrac{y}{x}+y^3 \implies x-\dfrac{y}{x}+y^3=C$

Substituting $x=1,\ y=1$ we get $1-1+1=C \implies x-\dfrac{y}{x}+y=1.$

23. $\dfrac{\partial P}{\partial y}=3y^2$ and $\dfrac{\partial Q}{\partial x}=y^2;$ the equation is not exact.

Since $\dfrac{1}{P}\left(\dfrac{\partial P}{\partial y}-\dfrac{\partial Q}{\partial x}\right)=\dfrac{1}{y^3}(2y^2)=\dfrac{2}{y},\quad w(y)=e^{-\int(2/y)\,dy}=e^{-2\ln y}=y^{-2}$ is an integrating factor. Multiplying the given equation by y^{-2}, we get

$$y+\left(y^{-2}+x\right)\,y'=0$$

$\dfrac{\partial f}{\partial x}=y \implies f(x,y)=xy+\varphi(y)$

$\dfrac{\partial f}{\partial y}=x+\varphi'(y)=y^{-2}+x \implies \varphi'(y)=y^{-2} \implies \varphi(y)=-\dfrac{1}{y}$

Therefore $f(x,y)=xy-\dfrac{1}{y}$, and a one-parameter family of solutions is: $xy-\dfrac{1}{y}=C$

Setting $x=-2,\ y=-1$, we get $C=3$ and the solution $xy-\dfrac{1}{y}=3.$

24. $\dfrac{\partial}{\partial y}(x+y)^2=2(x+y)=\dfrac{\partial}{\partial x}(2xy+x^2-1);$ exact.

$$f(x,y) = \frac{x^3}{3} + x^2y + xy^2 - y \implies \frac{x^3}{3} + x^2y + xy^2 - y = C$$

Setting $x = 1,\ y = 1,$ we get $C = \frac{4}{3}$ and the solution $\frac{x^3}{3} + x^2y + xy^2 - y = \frac{4}{3}.$

25. $\frac{\partial P}{\partial y} = -2y \sinh(x - y^2) = \frac{\partial Q}{\partial x};$ the equation is exact.

$$\frac{\partial f}{\partial x} = \cosh(x - 2y^2) + e^{2x} \implies f(x,y) = \sinh(x - y^2) + \tfrac{1}{2}\,e^{2x} + \varphi(y)$$

$$\frac{\partial f}{\partial y} = -2y \cosh(x - y^2) + \varphi'(y) = y - 2y \cosh(x - y^2) \implies \varphi'(y) = y \implies \varphi(y) = \tfrac{1}{2}\,y^2$$

Therefore $f(x,y) = \sinh(x - y^2) + \frac{1}{2}\,e^{2x} + \frac{1}{2}\,y^2,$ and a one-parameter family of solutions is:

$$\sinh(x - y^2) + \tfrac{1}{2}\,e^{2x} + \tfrac{1}{2}\,y^2 = C$$

Setting $x = 2,\ y = \sqrt{2},$ we get $C = \frac{1}{2}e^4 + 1$ and the solution

$$\sinh(x - y^2) + \tfrac{1}{2}\,e^2x + \tfrac{1}{2}\,y^2 = \tfrac{1}{2}e^4 + 1$$

26. Write the linear equation as $p(x)y - q(x) + y' = 0.$ Then $P(x) = p(x)y - q(x),\quad Q(x) = 1$ and $v = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = p(x)$ depends only on x. Therefore $v = e^{\int p(x)\,dx}$ is an integrating factor.

27. (a) $\frac{\partial P}{\partial y} = 2xy + kx^2$ and $\frac{\partial Q}{\partial x} = 2xy + 3x^2 \implies k = 3.$

(b) $\frac{\partial P}{\partial y} = e^{2xy} + 2xye^{2xy}$ and $\frac{\partial Q}{\partial x} = ke^{2xy} + 2kxye^{2xy} \implies k = 1.$

28. (a) We need $g'(y)\sin x = y^2 f'(x).$ Take $g(y) = \frac{1}{3}y^3$ and $f(x) = -\cos x$

(b) We need $g'(y)e^y + g(y)e^y = y,$ that is $\frac{d}{dy}\,[g(y)e^y] = y.$

It follows that $g(y)e^y = \frac{1}{2}y^2 + C, \implies g(y) = e^{-y}(\frac{1}{2}y^2 + C).$

29. $y' = y^2x^3;$ the equation is separable.

$$y^{-2}\,dy = x^3\,dx \implies -\frac{1}{y} = \tfrac{1}{4}\,x^4 + C \implies y = \frac{-4}{x^4 + C}$$

30. $y' = \frac{y}{x + \sqrt{xy}} = \frac{y/x}{1 + \sqrt{y/x}} \implies$ the equation is homogeneous.

Set $y = vx.$ Then $y' = v + xv'$ and

$$v + xv' = \frac{v}{1 + \sqrt{v}} \implies \int \frac{dx}{x} + \int \frac{1 + \sqrt{v}}{v^{3/2}}\,dv = C \implies \ln|x| + \ln|v| - \frac{2}{\sqrt{v}} = C \implies \ln|y| - 2\sqrt{\frac{x}{y}} = C$$

31. $y' + \frac{4}{x}\,y = x^4;$ the equation is linear.

$H(x) = \int (4/x)\,dx = 4\ln x = \ln x^4$, integrating factor: $e^{\ln x^4} = x^4$

$$x^4y' + 4x^3y = x^8$$
$$\frac{d}{dx}\left[x^4y\right] = x^8$$
$$x^4y = \tfrac{1}{9}x^9 + C$$
$$y = \tfrac{1}{9}x^5 + Cx^{-4}$$

32. $y' + 2xy = 2x^3$; the equation is linear with integrating factor $e^{\int 2x\,dx} = e^{x^2}$
$\Longrightarrow \dfrac{d}{dx}(e^{x^2}y) = 2x^3e^{x^2} \implies e^{x^2}y = e^{x^2}(x^2-1) + C \Longrightarrow y = x^2 - 1 + Ce^{-x^2}$.

33. $\dfrac{\partial P}{\partial y} = e^{xy} + xye^{xy} = \dfrac{\partial Q}{\partial x}$; the equation is exact.

$\dfrac{\partial f}{\partial x} = ye^{xy} - 2x \implies f(x,y) = e^{xy} - x^2 + \varphi(y)$

$\dfrac{\partial f}{\partial y} = xe^{xy} + \varphi'(y) = \dfrac{2}{y} + xe^{xy} \implies \varphi'(y) = \dfrac{2}{y} \implies \varphi(y) = 2\ln|y|$

Therefore $f(x,y) = e^{xy} - x^2 + 2\ln|y|$, and a one-parameter family of solutions is:

$$e^{xy} - x^2 + 2\ln|y| = C$$

34. $w = \dfrac{1}{p}\left(\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}\right) = \dfrac{1}{y}(1-2y)$ depends only on y, so an integrating factor is

$$e^{\int -w(y)\,dy} = e^{\int [2-(1/y)]\,dy} = e^{2y-\ln y} = \frac{1}{y}e^{2y}.$$

Then $e^{2y}\,dx + \left(2xe^{2y} - \dfrac{1}{y}\right)dy = 0$ is exact.

$f(x,y) = xe^{2y} - \ln y$, and a one-parameter family of solutions is $xe^{2y} - \ln y = C$.

CHAPTER 16

SECTION 16.1

1. $\sum_{i=1}^{3}\sum_{j=1}^{3} 2^{i-1}3^{j+1} = \left(\sum_{i=1}^{3} 2^{i-1}\right)\left(\sum_{j=1}^{3} 3^{j+1}\right) = (1+2+4)(9+27+81) = 819$

2. $2+2^2+3+3^2+4+4^2+5+5^2 = 68$

3. $\sum_{i=1}^{4}\sum_{j=1}^{3}(i^2+3i)(j-2) = \left[\sum_{i=1}^{4}(i^2+31)\right]\left[\sum_{j=1}^{3}(j-2)\right] = (4+10+18+28)(-1+0+1) = 0$

4. $\frac{2}{2}+\frac{2}{3}+\frac{2}{4}+\frac{2}{5}+\frac{2}{6}+\frac{2}{7}+\frac{4}{2}+\frac{4}{3}+\frac{4}{4}+\frac{4}{5}+\frac{4}{6}+\frac{4}{7}+\frac{6}{2}+\frac{6}{3}+\frac{6}{4}+\frac{6}{5}+\frac{6}{6}+\frac{6}{7} = 19\frac{4}{35}.$

5. $\sum_{i=1}^{m}\Delta x_i = \Delta x_1 + \Delta x_2 + \cdots + \Delta x_n = (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1})$

$$= x_n - x_0 = a_2 - a_1$$

6. $(y - y_0) + (y_2 - y_1) + \cdots + (y_n - y_{n-1}) = y_n - y_0 = b_2 - b_1$

7. $\sum_{i=1}^{m}\sum_{j=1}^{n}\Delta x_i\,\Delta y_j = \left(\sum_{i=1}^{m}\Delta x_i\right)\left(\sum_{j=1}^{n}\Delta y_j\right) = (a_2 - a_1)(b_2 - b_1)$

8. $\sum_{j=1}^{n}\sum_{k=1}^{q}\Delta y_j \Delta z_k = \left(\sum_{j=1}^{n}\Delta y_j\right)\left(\sum_{k=1}^{q}\Delta z_k\right) = (b_2 - b_1)(c_2 - c_1)$

9. $\sum_{i=1}^{m}(x_i + x_{i-1})\,\Delta x_i = \sum_{i=1}^{m}(x_i + x_{i-1})(x_i - x_{i-1}) = \sum_{i=1}^{m}\left({x_i}^2 - x_{i-1}^2\right)$

$$= {x_m}^2 - {x_0}^2 = {a_2}^2 - {a_1}^2$$

10. $\sum_{j=1}^{n}\frac{1}{2}\left({y_j}^2 + y_j y_{j-1} + {y_{j-1}}^2\right)\Delta y_j = \frac{1}{2}\sum_{j=1}^{n}\left({y_j}^3 - {y_{j-1}}^3\right) = \frac{1}{2}\left({b_2}^3 - {b_1}^3\right)$

11. $\sum_{i=1}^{m}\sum_{j=1}^{n}(x_i + x_{i-1})\,\Delta x_i\,\Delta y_j = \left(\sum_{i=1}^{m}(x_i + x_{i-1})\,\Delta x_i\right)\left(\sum_{j=1}^{n}\Delta y_j\right)$

(Exercise 9)

$$= \left({a_2}^2 - {a_1}^2\right)(b_2 - b_1)$$

12. $\sum_{i=1}^{m}\sum_{j=1}^{n}(y_i + y_{j-1})\Delta x_i \Delta y_j = \left(\sum_{i=1}^{m}\Delta x_i\right)\left[\sum_{j=1}^{n}\left({y_j}^2 - {y_{j-1}}^2\right)\right] = (a_2 - a_1)({b_2}^2 - {b_1}^2)$

13. $$\sum_{i=1}^{m}\sum_{j=1}^{n}(2\Delta x_i - 3\Delta y_j) = 2\left(\sum_{i=1}^{m}\Delta x_i\right)\left(\sum_{j=1}^{n}1\right) - 3\left(\sum_{i=1}^{m}1\right)\left(\sum_{j=1}^{n}\Delta y_j\right)$$
$$= 2n\,(a_2 - a_1) - 3m\,(b_2 - b_1)$$

14. $$\sum_{i=1}^{m}\sum_{j=1}^{n}(3\Delta x_i - 2\Delta y_j) = 3\sum_{i=1}^{m}\sum_{j=1}^{n}\Delta x_i - 2\sum_{i=1}^{m}\sum_{j=1}^{n}\Delta y_j = 3n(a_2 - a_1) - 2m(b_2 - b_1).$$

15. $$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{q}\Delta x_i\,\Delta y_j\,\Delta z_k = \left(\sum_{i=1}^{m}\Delta x_i\right)\left(\sum_{j=1}^{n}\Delta y_j\right)\left(\sum_{k=1}^{q}\Delta z_k\right)$$
$$= (a_2 - a_1)\,(b_2 - b_1)\,(c_2 - c_1)$$

16. $$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{q}(x_i + x_{i-1})\Delta x_i\Delta y_j\Delta z_k = \left[\sum_{i=1}^{m}({x_i}^2 - {x_{i-1}}^2)\right]\left(\sum_{j=1}^{n}\Delta y_j\right)\left(\sum_{k=1}^{q}\Delta z_k\right)$$
$$= ({a_2}^2 - {a_1}^2)(b_2 - b_1)(c_2 - c_1)$$

17. $$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\delta_{ijk}a_{ijk} = a_{111} + a_{222} + \cdots + a_{nnn} = \sum_{p=1}^{n}a_{ppp}$$

18. Start with $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}$. Take all the a_{ij} (there are only a finite number of them) and order them in any order you chose. Call the first one b_1, the second b_2, and so on. Then
$$\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij} = \sum_{p=1}^{r}b_p \quad \text{where} \quad r = m \times n.$$

19. $$\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha a_{ij} = \sum_{i=1}^{m}\alpha\left(\sum_{j=1}^{n}a_{ij}\right) = \alpha\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}$$

20. Follows from the additive property of single-sigma summations.

SECTION 16.2

1. $L_f(P) = 2\frac{1}{4}$, $U_f(P) = 5\frac{3}{4}$

2. $L_f(P) = 3$, $U_f(P) = 5$

3. (a) $\displaystyle L_f(P) = \sum_{i=1}^{m}\sum_{j=1}^{n}(x_{i-1} + 2y_{j-1})\,\Delta x_i\,\Delta y_j, \quad U_f(P) = \sum_{i=1}^{m}\sum_{j=1}^{n}(x_i + 2y_j)\,\Delta x_i\,\Delta y_j$

(b) $L_f(P) \le \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{x_{i-1}+x_i}{2}+2\left(\frac{y_{j-1}+y_j}{2}\right)\right]\Delta x_i\,\Delta y_j \le U_f(P).$

The middle expression can be written

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{2}\left(x_i{}^2-x_{i-1}^2\right)\Delta y_j+\sum_{i=1}^{m}\sum_{j=1}^{n}\left(y_j{}^2-y_{j-1}^2\right)\Delta x_i.$$

The first double sum reduces to

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{2}\left(x_i{}^2-x_{i-1}^2\right)\Delta y_j=\frac{1}{2}\left(\sum_{i=1}^{m}\left(x_i{}^2-x_{i-1}^2\right)\right)\left(\sum_{j=1}^{n}\Delta y_j\right)=\frac{1}{2}\,(4-0)\,(1-0)=2.$$

In like manner the second double sum also reduces to 2. Thus, $I=4$; the volume of the prism bounded above by the plane $z=x+2y$ and below by R.

4. $L_f(P)=-7/16,\quad U_f(P)=7/16$

5. $L_f(P)=-7/24,\quad U_f(P)=7/24$

6. (a) $L_f(p)=\sum_{i=1}^{m}\sum_{j=1}^{n}(x_{i-1}-y_j)\Delta x_i\Delta y_j,\quad U_f(P)=\sum_{i=1}^{m}\sum_{j=1}^{n}(x_i-y_{j-1})\Delta x_i\Delta y_j$

(b) $L_f(P)\le\sum_{i=1}^{m}\sum_{j=1}^{n}\left(\frac{x_i+x_{i-1}}{2}-\frac{y_j+y_{j-1}}{2}\right)\Delta x_i\Delta y_j\le U_f(P)$

The middle expression can be written

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{2}(x_i{}^2-x_{i-1}{}^2)\Delta y_j-\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{2}(y_j{}^2-y_{j-1}{}^2)\Delta x_i.$$

The first sum reduces to

$$\frac{1}{2}\left(\sum_{i=1}^{m}(x_i{}^2-x_{i-1}{}^2)\right)\left(\sum_{j=1}^{n}\Delta y_j\right)=\frac{1}{2}(1-0)(1-0)=\frac{1}{2}.$$

In like manner the second sum also reduces to $\frac{1}{2}$. Thus $I=\frac{1}{2}-\frac{1}{2}=0.$

7. (a) $L_f(P)=\sum_{i=1}^{m}\sum_{j=1}^{n}(4x_{i-1}\,y_{j-1})\,\Delta x_i\,\Delta y_j,\quad U_f(P)=\sum_{i=1}^{m}\sum_{j=1}^{n}(4x_i\,y_j)\,\Delta x_i\,\Delta y_j$

(b) $L_f(P)\le\sum_{i=1}^{m}\sum_{j=1}^{n}(x_i+x_{i-1})\,(y_j+y_{j-1})\,\Delta x_1\,\Delta y_j\le U_f(P).$

The middle expression can be written

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left(x_i{}^2 - x_{i-1}^2\right)\left(y_j{}^2 - y_{j-1}^2\right) = \left(\sum_{i=1}^{m} x_i{}^2 - x_{i-1}^2\right)\left(\sum_{j=1}^{n} y_j{}^2 - y_{j-1}^2\right)$$

by (16.1.5)

$$= \left(b^2 - 0^2\right)\left(d^2 - 0^2\right) = b^2d^2.$$

It follows that $I = b^2d^2$.

8. (a) $L_f(P) = \sum_{i=1}^{m}\sum_{j=1}^{n} 3({x_{i-1}}^2 + {y_{j-1}}^2)\Delta x_i \Delta y_j, \quad U_f(P) = \sum_{i=1}^{m}\sum_{j=1}^{n} 3({x_i}^2 + {y_j}^2)\Delta x_i \Delta y_j$

(b) $L_f(P) \le \sum_{i=1}^{m}\sum_{j=1}^{n} \left[({x_i}^2 + x_i x_{i-1} + {x_{i-1}}^2) + ({y_j}^2 + y_j y_{j-1} + {y_{j-1}}^2)\right]\Delta x_i \Delta y_j \le U_f(P)$

Since in general $(A^2 + AB + B^2)(A - B) = A^3 - B^3$, the middle expression can be written

$$\sum_{i=1}^{m}\sum_{j=1}^{n}({x_i}^3 - {x_{i-1}}^3)\Delta y_j + \sum_{i=1}^{m}\sum_{j=1}^{n}({y_j}^3 - {y_{j-1}}^3)\Delta x_i,$$

which reduces to

$$\left(\sum_{i=1}^{m} {x_i}^3 - {x_{i-1}}^3\right)\left(\sum_{j=1}^{n}\Delta y_j\right) + \left(\sum_{i=1}^{m}\Delta x_i\right)\left(\sum_{j=1}^{n} {y_j}^3 - {y_{j-1}}^3\right).$$

This can be evaluated as $b^3d + bd^3 = bd(b^2 + d^2)$. It follows that $I = bd(b^2 + d^2)$.

9. (a) $L_f(P) = \sum_{i=1}^{m}\sum_{j=1}^{n} 3\left(x_{i-1}^2 - {y_j}^2\right)\Delta x_i\,\Delta y_j, \quad U_f(P) = \sum_{i=1}^{m}\sum_{j=1}^{n} 3\left({x_i}^2 - y_{j-1}^2\right)\Delta x_i\,\Delta y_j$

(b) $L_f(P) \le \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\left({x_i}^2 + x_i x_{i-1} + x_{i-1}^2\right) - \left({y_j}^2 + y_j y_{j-1} + y_{j-1}^2\right)\right]\Delta x_i\,\Delta y_j \le U_f(P).$

Since in general $\left(A^2 + AB + B^2\right)(A - B) = A^3 - B^3$, the middle expression can be written

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left({x_i}^3 - x_{i-1}^3\right)\Delta y_j - \sum_{i=1}^{m}\sum_{j=1}^{n}\left({y_j}^3 - y_{j-1}^3\right)\Delta x_i,$$

which reduces to

$$\left(\sum_{i=1}^{m} {x_i}^3 - x_{i-1}^3\right)\left(\sum_{j=1}^{n}\Delta y_j\right) - \left(\sum_{i=1}^{m}\Delta x_i\right)\left(\sum_{j=1}^{n} {y_j}^3 - y_{j-1}^3\right).$$

This can be evaluated as $b^3d - bd^3 = bd\left(b^2 - d^2\right)$. It follows that $I = bd\,(b^2 - d^2)$.

10. On each subrectangle, the minimum and the maximum of f are equal, so f is constant on each subrectangle and therefore (since f is continuous) on the entire rectangle R. Then

$$\iint_R f(x,y)\,dxdy = f(a,c)(b-a)(d-c).$$

11. $\displaystyle\iint_\Omega dxdy = \int_a^b \phi(x)\,dx$

12. Suppose that there is a point (x_0, y_0) on the boundary of Ω at which f is not zero. As (x, y) tends to (x_0, y_0) through that part of R which is outside Ω, $f(x, y)$, being zero, tends to zero. Since $f(x_0, y_0)$ is not zero, $f(x, y)$ does not tend to $f(x_0, y_0)$. Thus the extended function f can not be continuous at (x_0, y_0).

13. Suppose $f(x_0, y_0) \neq 0$. Assume $f(x_0, y_0) > 0$. Since f is continuous, there exists a disc Ω_ϵ with radius ϵ centered at (x_0, y_0) such that $f(x, y) > 0$ on Ω_ϵ. Let R be a rectangle contained in Ω_ϵ. Then $\displaystyle\iint_R f(x,y)\,dxdy > 0$, which contradicts the hypothesis.

14. $\displaystyle\iint_R (x+2y)\,dx\,dy = 2;\quad \text{area}(R) = (2)(1) = 2, \quad \text{so average value} = \frac{2}{2} = 1$

15. By Exercise 7, Section 16.2, $\displaystyle\iint_R 4xy\,dxdy = 2^2 3^2 = 36$. Thus

$$f_{avg} = \frac{1}{\text{area}\,(R)} \iint_R 4xy\,dxdy = \frac{1}{6}\,(36) = 6$$

16. $\displaystyle\iint_R (x^2+y^2)\,dxdy = \frac{bd(b^2+d^2)}{3};\quad \text{area}(R) = bd, \quad \text{so average value} = \frac{b^2+d^2}{3}$

17. By Theorem 16.2.10, there exists a point $(x_1, y_1) \in D_r$ such that

$$\iint_{D_r} f(x,y)\,dxdy = f(x_1,y_1) \iint_R dxdy = f(x_1,y_1)\pi r^2 \quad\Longrightarrow\quad f(x_1,y_1) = \frac{1}{\pi r^2}\iint_{D_r} f(x,y)\,dxdy$$

As $r \to 0$, $(x_1, y_1) \to (x_0, y_0)$ and $f(x_1, y_1) \to f(x_0, y_0)$ since f is continuous.

The result follows.

18. $0 \le \sin(x+y) \le 1$ for all $(x,y) \in R$. Thus, $0 \le \iint_R \sin(x+y)\,dxdy \le \iint_R dxdy = 1$

SECTION 16.3

1. $\int_0^1 \int_0^3 x^2\, dy\, dx = \int_0^1 3x^2\, dx = 1$

2. $\int_0^3 \int_0^1 e^{x+y}\, dxdy = \int_0^3 (e^{1+y} - e^y)\, dy = \left[e^{1+y} - e^y\right]_0^3 = e^4 - e^3 - e + 1$

3. $\int_0^1 \int_0^3 xy^2\, dy\, dx = \int_0^1 x \left[\frac{1}{3}y^3\right]_0^3 dx = \int_0^1 9x\, dx = \frac{9}{2}$

4. $\int_0^1 \int_0^x x^3 y\, dydx = \int_0^1 x^3 \frac{x^2}{2}\, dx = \frac{1}{12}$

5. $\int_0^1 \int_0^x xy^3\, dy\, dx = \int_0^1 x \left[\frac{1}{4}y^4\right]_0^x dx = \int_0^1 \frac{1}{4}x^5\, dx = \frac{1}{24}$

6. $\int_0^1 \int_0^x x^2y^2\, dydx = \int_0^1 x^2 \frac{x^3}{3}\, dx = \frac{1}{18}$

7. $\int_0^{\pi/2} \int_0^{\pi/2} \sin(x+y)\, dy\, dx = \int_0^{\pi/2} [-\cos(x+y)]_0^{\pi/2}\, dx = \int_0^{\pi/2} \left[\cos x - \cos\left(x + \frac{\pi}{2}\right)\right] dx = 2$

8. $\int_0^{\pi/2} \int_0^{\pi/2} \cos(x+y)\, dxdy = \int_0^{\pi/2} \left[\sin y - \sin\left(\frac{\pi}{2} + y\right)\right] dy = \left[\cos y - \cos\left(\frac{\pi}{2} + y\right)\right]_0^{\pi/2} = 0$

9. $\int_0^{\pi/2} \int_0^{\pi/2} (1 + xy)\, dy\, dx = \int_0^{\pi/2} \left[y + \frac{1}{2}xy^2\right]_0^{\pi/2} dx = \int_0^{\pi/2} \left(\frac{1}{2}\pi + \frac{1}{8}\pi^2 x\right) dx = \frac{1}{4}\pi^2 + \frac{1}{64}\pi^4$

10. $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x + 3y^3)\, dx\, dy = \int_{-1}^1 6y^3\sqrt{1-y^2}\, dy = 0$ (integrand is odd)

11. $\int_0^1 \int_{y^2}^y \sqrt{xy}\, dx\, dy = \int_0^1 \sqrt{y} \left[\frac{2}{3}x^{3/2}\right]_{y^2}^y dy = \int_0^1 \frac{2}{3}\left(y^2 - y^{7/2}\right) dy = \frac{2}{27}$

12. $\int_0^1 \int_0^{y^2} ye^x\, dxdy = \int_0^1 y(e^{y^2} - 1)\, dy = \left[\frac{e^{y^2}}{2} - \frac{y^2}{2}\right]_0^1 = \frac{1}{2}(e - 2)$

13. $$\int_{-2}^2 \int_{\frac{1}{2}y^2}^{4-\frac{1}{2}y^2} (4 - y^2)\, dx\, dy = \int_{-2}^2 (4 - y^2)\left[\left(4 - \frac{1}{2}y^2\right) - \left(\frac{1}{2}y^2\right)\right] dy$$
$$= 2\int_0^2 (16 - 8y^2 + y^4)\, dy = \frac{512}{15}$$

14. $$I = \int_0^1 \int_{x^3}^{x^2} (x^4 + y^2)\, dydx = \int_0^1 \left[x^4 y + \frac{y^3}{3}\right]_{x^3}^{x^2} dx = \int_0^1 \left(\frac{4x^6}{3} - x^7 - \frac{x^9}{3}\right) dx$$
$$= \left[\frac{4x^7}{21} - \frac{x^8}{8} - \frac{x^{10}}{30}\right]_0^1 = \frac{9}{280}$$

15. 0 by symmetry (integrand odd in y, Ω symmetric about x-axis)

16. $\int_0^1 \int_0^{2y} e^{-y^2/2}\, dxdy = \int_0^1 2ye^{-y^2/2}\, dy = \left[-2e^{-y^2/2}\right]_0^1 = 2\left(1 - \frac{1}{\sqrt{e}}\right)$

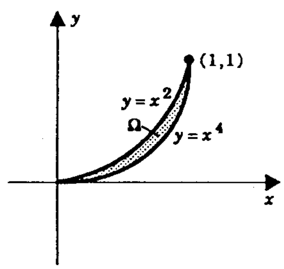
17. $\displaystyle\int_0^2\int_0^{x/2} e^{x^2}\,dy\,dx = \int_0^2 \frac{1}{2}xe^{x^2}\,dx = \left[\frac{1}{4}e^{x^2}\right]_0^2 = \frac{1}{4}\left(e^4-1\right)$

18. $\displaystyle\int_{-1}^0\int_{x^3}^{x^4}(x+y)\,dydx + \int_0^1\int_{x^4}^{x^3}(x+y)\,dydx = \int_{-1}^0\left[xy+\frac{y^2}{2}\right]_{x^3}^{x^4}dx + \int_0^1\left[xy+\frac{y^2}{2}\right]_{x^4}^{x^3}dx$

$$= \int_{-1}^0\left(x^5+\frac{x^8}{2}-x^4-\frac{x^6}{2}\right)dx + \int_0^1\left(x^4+\frac{x^6}{2}-x^5-\frac{x^8}{2}\right)dx$$

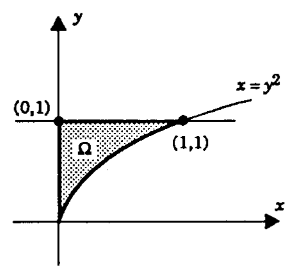
$$= \left[\frac{x^6}{6}+\frac{x^9}{18}-\frac{x^5}{5}-\frac{x^7}{14}\right]_{-1}^0 + \left[\frac{x^5}{5}+\frac{x^7}{14}-\frac{x^6}{6}-\frac{x^9}{18}\right]_0^1 = -\frac{1}{3}$$

19.



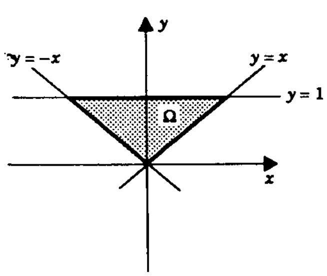
$\displaystyle\int_0^1\int_{y^{1/2}}^{y^{1/4}} f(x,y)\,dx\,dy$

20.



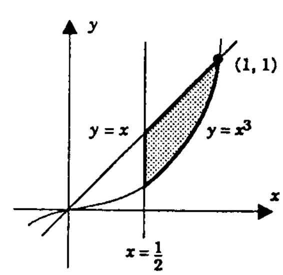
$\displaystyle\int_0^1\int_{\sqrt{x}}^1 f(x,y)\,dydx$

21.



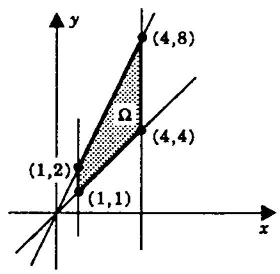
$\displaystyle\int_{-1}^0\int_{-x}^1 f(x,y)\,dy\,dx + \int_0^1\int_x^1 f(x,y)\,dy\,dx$

22.



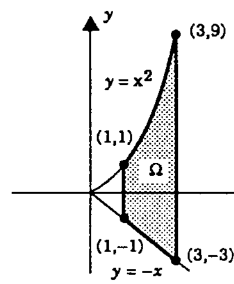
$\displaystyle\int_{\sqrt[3]{1/2}}^{1/2}\int_{1/2}^{y^{1/3}} f(x,y)\,dxdy + \int_{1/2}^1\int_y^{y^{1/3}} f(x,y)\,dxdy$

23.



$$\int_1^2\int_1^y f(x,y)\,dx\,dy+\int_2^4\int_{y/2}^y f(x,y)\,dx\,dy$$
$$+\int_4^8\int_{y/2}^4 f(x,y)\,dx\,dy$$

24.



$$\int_{-3}^{-1}\int_{-y}^3 f(x,y)\,dx\,dy+\int_{-1}^1\int_1^3 f(x,y)\,dx\,dy$$
$$+\int_1^9\int_{\sqrt{y}}^3 f(x,y)\,dx\,dy$$

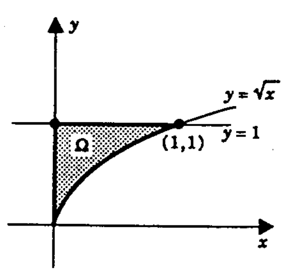
25. $\displaystyle\int_{-2}^4\int_{1/4x^2}^{\frac{1}{2}x+2} dy\,dx=\int_{-2}^4\left[\frac{1}{2}x+2-\frac{1}{4}x^2\right]dx=9$

26. $\displaystyle\int_0^3\int_y^{4y-y^2} dx\,dy=\int_0^3(3y-y^2)\,dy=\frac{9}{2}$

27. $\displaystyle\int_0^{1/4}\int_{2y^{3/2}}^y dx\,dy=\int_0^{1/4}\left[y-2y^{3/2}\right]dy=\frac{1}{160}$

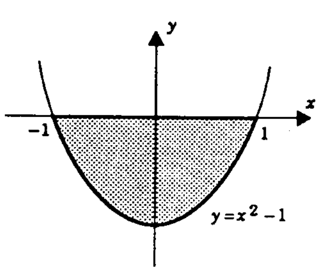
28. $\displaystyle\int_2^3\int_{6/x}^{5-x} dy\,dx=\int_2^3(5-x-6/x)\,dx=\frac{5}{2}+6\ln\frac{2}{3}$

29.



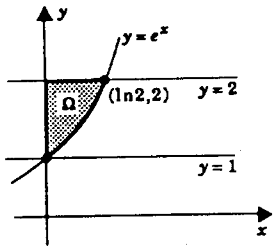
$$\int_0^1\int_0^{y^2}\sin\left(\frac{y^3+1}{2}\right)dx\,dy=\int_0^1 y^2\sin\left(\frac{y^3+1}{2}\right)dy$$
$$=\left[-\frac{2}{3}\cos\left(\frac{y^3+1}{2}\right)\right]_0^1$$
$$=\frac{2}{3}\left(\cos\frac{1}{2}-\cos 1\right)$$

30.



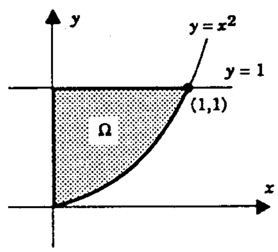
$$\int_{-1}^1\int_{x^2-1}^0 x^2\,dy\,dx=\int_{-1}^1 x^2(1-x^2)\,dx$$
$$=\frac{4}{15}$$

31.



$$\int_0^{\ln 2}\int_{e^x}^{2} e^{-x}\,dy\,dx = \int_0^{\ln 2} e^{-x}\,(2-e^x)\,dx$$
$$= \left[-2e^{-x} - x\right]_0^{\ln 2} = 1 - \ln 2$$

32.



$$\int_0^1\int_0^{\sqrt{y}} \frac{x^3}{\sqrt{x^4+y^2}}\,dx\,dy = \int_0^1 \frac{1}{2}(\sqrt{2}-1)y\,dy$$
$$= \frac{1}{4}(\sqrt{2}-1)$$

33. $\displaystyle\int_1^2\int_{y-1}^{2/y} dx\,dy = \int_1^2 \left[\frac{2}{y} - (y-1)\right] dy = \ln 4 - \frac{1}{2}$

34. $\displaystyle\int_0^1\int_0^{1-x} (x+y)\,dy\,dx = \int_0^1 \left[x(1-x) + \frac{(1-x)^2}{2}\right] dx = \frac{1}{3}$

35. $\displaystyle\int_0^2\int_0^{3-\frac{3}{2}x} \left(4-2x-\frac{4}{3}y\right) dy\,dx = \int_0^3\int_0^{2-\frac{2}{3}y} \left(4-2x-\frac{4}{3}y\right) dx\,dy = 4$

36. $\displaystyle\int_0^1\int_0^1 (2x+3y)\,dy\,dx = \int_0^1 (2x+\frac{3}{2})\,dx = \frac{5}{2}$

37. $\displaystyle\int_0^2\int_0^{1-\frac{1}{2}x} x^3y\,dy\,dx = \int_0^2\int_0^{2-2y} x^3y\,dx\,dy = \frac{2}{15}$

38. $\displaystyle\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx = 2\int_{-1}^1 \left[x^2\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{3/2}\right] dx = \frac{\pi}{2}$

39.
$$\int_0^2\int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} (2x+1)\,dy\,dx = \int_{-1}^1\int_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} (2x+1)\,dx\,dy$$
$$= \int_{-1}^1 \left[x^2+x\right]_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} dy$$
$$= 6\int_{-1}^1 \sqrt{1-y^2}\,dy = 6\left(\frac{\pi}{2}\right) = 3\pi$$

40. $\displaystyle\int_{-1}^1\int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} (4-y^2-\frac{1}{4}x^2)\,dy\,dx = \int_{-1}^1 \left[6\sqrt{1-x^2} - \frac{1}{2}x^2\sqrt{1-x^2} - \frac{2}{3}(1-x^2)^{3/2}\right] dx = \frac{43}{16}\pi.$

41. $\displaystyle\int_0^1\int_0^{1-x} (x^2+y^2)\,dy\,dx = \int_0^1 \left(2x^2 - \frac{4}{3}x^3 - x + \frac{1}{3}\right) dx = \frac{1}{6}$

42. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(1-x)\,dy\,dx = \int_{-1}^{1}(2\sqrt{1-x^2}-2x\sqrt{1-x^2})\,dx = \pi$

43. $\int_0^1\int_{x^2}^{x}(x^2+3y^2)\,dy\,dx = \int_0^1\left(2x^3-x^4-x^6\right)\,dx = \dfrac{11}{70}$

44. $\int_1^2\int_{2y-1}^{5-y}(1+xy)\,dx\,dy = \int_1^2(6+9y-3y^2-\frac{3}{2}y^3)\,dy = \dfrac{55}{8}$

45. $\int_0^a\int_0^{\sqrt{a^2-x^2}}\sqrt{a^2-x^2}\,dy\,dx = \int_0^a(a^2-x^2)\,dx\,dy = \dfrac{2}{3}a^2$

46. $\int_0^a\int_0^{b(1-x/a)}c\left(1-\dfrac{x}{a}-\dfrac{y}{b}\right)\,dy\,dx = \int_0^a\dfrac{bc}{2}\left(1-\dfrac{x}{a}\right)^2\,dx = \dfrac{abc}{6}$

47. $\int_0^1\int_y^1 e^{y/x}dx\,dy = \int_0^1\int_0^x e^{y/x}dy\,dx = \int_0^1\left[xe^{y/x}\right]_0^x dx = \int_0^1 x(e-1)\,dx = \dfrac{1}{2}(e-1)$

48. $\int_0^1\int_0^{\cos^{-1}y}e^{\sin x}\,dx\,dy = \int_0^{\pi/2}\int_0^{\cos x}e^{\sin x}\,dy\,dx = \int_0^{\pi/2}\cos x e^{\sin x}\,dx = e-1$

49. $\int_0^1\int_x^1 x^2e^{y^4}dy\,dx = \int_0^1\int_0^y x^2e^{y^4}dx\,dy = \int_0^1\left[\dfrac{1}{3}x^3e^{y^4}\right]_0^y dy = \dfrac{1}{3}\int_0^1 y^3e^{y^4}dy = \dfrac{1}{12}(e-1)$

50. $\int_0^1\int_0^y e^{y^2}\,dx\,dy = \int_0^1 ye^{y^2}\,dy = \dfrac{1}{2}(e-1)$

51. $f_{avg} = \dfrac{1}{8}\int_{-1}^1\int_0^4 x^2y\,dy\,dx = \dfrac{1}{8}\int_{-1}^1 8x^2\,dx = \int_{-1}^1 x^2\,dx = \dfrac{2}{3}$

52. area of $\Omega = \frac{1}{4}\pi$ (quarter circle of radius 1)

Average value $= \dfrac{4}{\pi}\int_0^1\int_0^{\sqrt{1-x^2}}xy\,dy\,dx = \dfrac{2}{\pi}\int_0^1 x(1-x^2)\,dx = \dfrac{1}{2\pi}$

53. $f_{avg} = \dfrac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ln 2}\int_{\ln 2}^{2\ln 2}\dfrac{1}{xy}\,dy\,dx = \dfrac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ln 2}\dfrac{1}{x}\ln 2\,dx = 1$

54. area of $\Omega = 1$ (parallelogram)

Average value $= \int_0^1\int_{x-1}^{x+1}e^{x+y}\,dy\,dx = \int_0^1\left(e^{2x+1}-e^{2x-1}\right)\,dx = \dfrac{1}{2}(e^3-2e+e^{-1})$.

55. $\iint_R f(x)g(y)\,dxdy = \int_c^d\int_a^b f(x)g(y)\,dx\,dy = \int_c^d\left(\int_a^b f(x)g(y)\,dx\right)\,dy$

$= \int_c^d g(y)\left(\int_a^b f(x)\,dx\right)\,dy = \left(\int_a^b f(x)\,dx\right)\left(\int_c^d g(y)\,dy\right)$

56. We have $R: \ a \le x \le b, \quad -c \le y \le c$. Set $g_x(y) = f(x,y)$. The function g_x is odd and thus

$$\int_{-c}^{c} g_x(y)\,dy = 0.$$

It follows that

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \left(\int_{-c}^{c} f(x,y)\,dy\right)\,dx$$

$$= \int_a^b \left(\int_{-c}^{c} g_x(y)\,dy\right)\,dx = \int_a^b 0\,dx = 0$$

57. We have $R: \ -a \le x \le a, \quad c \le y \le d$. Set $f(x,y) = g_y(x)$. For each fixed $y \in [\,c,d\,]$, g_y is an odd function. Thus

$$\int_{-a}^{a} g_y(x)\,dx = 0. \qquad (5.8.8)$$

Therefore

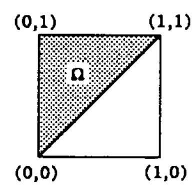
$$\iint_R f(x,y)\,dxdy = \int_c^d \int_{-a}^{a} f(x,y)\,dx\,dy$$

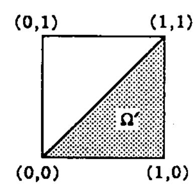
$$= \int_c^d \int_{-a}^{a} g_y(x)\,dx\,dy = \int_c^d 0\,dy = 0.$$

58. Symmetry about the origin [we want Ω to contain $(-x,-y)$ whenever it contains (x,y)].

59. Note that $\Omega = \{\,(x,y): \ 0 \le x \le y, \quad 0 \le y \le 1\}$.

Set $\Omega' = \{\,(x,y): \ 0 \le y \le x, \quad 0 \le x \le 1\}$.





$$\iint_\Omega f(x)f(y)\,dxdy = \int_0^1 \int_0^y f(x)f(y)\,dx\,dy$$

$$= \int_0^1 \int_0^x f(y)f(x)\,dy\,dx$$

x and y are dummy variables

$$= \int_0^1 \int_0^x f(x)f(y)\,dy\,dx = \iint_{\Omega'} f(x)f(y)\,dxdy.$$

Note that Ω and Ω' don't overlap and their union is the unit square

$$R = \{\,(x,y): \ 0 \le x \le 1, \quad 0 \le y \le 1\}.$$

If $\displaystyle\int_0^1 f(x)\,dx = 0$, then

$$0 = \left(\int_0^1 f(x)\,dx\right)\left(\int_0^1 f(y)\,dy\right) = \iint_R f(x)f(y)\,dxdy$$

by Exercise 55

$$= \iint_\Omega f(x)f(y)\,dxdy + \iint_{\Omega'} f(x)f(y)\,dxdy$$

$$= 2\iint_\Omega f(x)f(y)\,dxdy$$

and therefore $\displaystyle\iint_{\Omega} f(x)f(y)\,dxdy = 0.$

60.
$$\int_0^x \int_a^b \frac{\partial f}{\partial x}(t,y)\,dy\,dt = \int_a^b \int_0^x \frac{\partial f}{\partial x}(t,y)\,dt\,dy$$
$$= \int_a^b [f(x,y) - f(0,y)]\,dy$$
$$= \int_a^b f(x,y)\,dy - \int_a^b f(0,y)\,dy.$$

Thus
$$\int_a^b f(x,y)\,dy = \int_0^x \int_a^b \frac{\partial f}{\partial x}(t,y)\,dy\,dt + \int_a^b f(0,y)\,dy$$
and
$$\frac{d}{dx}\left[\int_a^b f(x,y)\,dy\right] = \frac{d}{dx}\left[\int_0^x \int_a^b \frac{\partial f}{\partial x}(t,y)\,dy\,dt\right] + \frac{d}{dx}\left[\int_a^b f(0,y)\,dy\right]$$
$$= \frac{d}{dx}\left[\int_0^x H(t)\,dt\right] + 0 = H(x) = \int_a^b \frac{\partial f}{\partial x}(x,y)\,dy.$$

by Theorem 5.3.5

61. Let M be the maximum value of $|\,f(x,y)\,|$ on Ω.
$$\int_{\phi_1(x+h)}^{\phi_2(x+h)} = \int_{\phi_1(x+h)}^{\phi_1(x)} + \int_{\phi_1(x)}^{\phi_2(x)} + \int_{\phi_2(x)}^{\phi_2(x+h)}$$
$$|\,F(x+h) - F(x)\,| = \left| \int_{\phi_1(x+h)}^{\phi_2(x+h)} f(x,y)\,dy - \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \right|$$
$$= \left| \int_{\phi_1(x+h)}^{\phi_1(x)} f(x,y)\,dy + \int_{\phi_2(x)}^{\phi_2(x+h)} f(x,y)\,dy \right|$$
$$\le \left| \int_{\phi_1(x+h)}^{\phi_1(x)} f(x,y)\,dy \right| + \left| \int_{\phi_2(x)}^{\phi_2(x+h)} f(x,y)\,dy \right|$$
$$\le |\,\phi_1(x) - \phi_1(x+h)\,|\,M + |\,\phi_2(x+h) - \phi_2(x)\,|\,M.$$

The expression on the right tends to 0 as h tends to 0 since ϕ_1 and ϕ_2 are continuous.

PROJECT 16.3

1. (a) $M_{32} = \frac{2-0}{3} \cdot \frac{3-0}{2}\,[f(\frac{1}{3},\frac{3}{4}) + f(1,\frac{3}{4}) + f(\frac{5}{3},\frac{3}{4}) + f(\frac{1}{3},\frac{9}{4}) + f(1,\frac{9}{4}) + f(\frac{5}{3},\frac{9}{4})] = \frac{144}{6} = 24.$

(c) $\displaystyle\int_0^3 \int_0^2 (x+2y)\,dx\,dy = 24$

2. (a) $M_{32} = \frac{2-0}{2} \cdot \frac{2-0}{2}\,[f(\frac{1}{2},\frac{1}{2}) + f(\frac{1}{2},\frac{3}{2}) + f(\frac{1}{2},\frac{3}{2}) + f(\frac{3}{2},\frac{3}{2})] = 16.$

(c) $\int_0^2\int_0^2 4xy\,dx\,dy = 16$

3. (a) $M_{32} = \frac{\frac{\pi}{2}-0}{2}\cdot\frac{\frac{\pi}{2}-0}{2}\left[f(\frac{\pi}{8},\frac{\pi}{8}) + f(\frac{3\pi}{8},\frac{\pi}{8}) + f(\frac{\pi}{8},\frac{3\pi}{8}) + f(\frac{3\pi}{8},\frac{3\pi}{8})\right] = \frac{\pi^2}{16}(1.70710) \simeq 1.05303.$

(c) $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin x \sin y\,dx\,dy = 1$

4. The trapezoidal rule $T_n = \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1} + f(x_n)\right]$ (see Section 8.7)

can be written $T_n = \sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2}\Delta x, \quad \Delta x = \frac{b-a}{n}.$

In this form, $\frac{f(x_{i-1}) + f(x_i)}{2}$ is the average of the values of f at the endpoints of the subinterval $[x_{i-1}, x_i]$ and Δx is the length of the subinterval.

A generalization of this to a double integral over a rectangle $R:\ a \le x \le b, c \le y \le d,$ is:

$$T_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{f(x_{i-1},y_{j-1}) + f(x_i,y_{j-1}) + f(x_{i-1},y_j) + f(x_i,y_j)}{4}\Delta x\Delta y$$

where $\Delta x = \frac{b-a}{m}$, and $\Delta y = \frac{d-c}{n}$.

5. (a) Applying the formula in Problem 4, we get

$T_{32} = \frac{2-0}{3}\cdot\frac{3-0}{2}\cdot\frac{1}{4}\left[f(0,0) + 2f(\frac{2}{3},0) + 2f(\frac{4}{3},0) + f(2,0) + 2f(0,\frac{3}{2}) + 4f(\frac{2}{3},\frac{3}{2}) + 4f(\frac{4}{3},\frac{3}{2}) + 2f(2,\frac{3}{2}) +\right.$
$\left.f(0,3) + 2f(\frac{2}{3},3) + 2f(\frac{4}{3},3) + f(2,3)\right] = \frac{1}{4}(96) = 24$

SECTION 16.4

1. $\int_0^{\pi/2}\int_0^{\sin\theta} r\,\cos\theta\,dr\,d\theta = \int_0^{\pi/2}\frac{1}{2}\sin^2\theta\cos\theta\,d\theta = \left[\frac{1}{6}\sin^3\theta\right]_0^{\pi/2} = \frac{1}{6}$

2. $\int_0^{\pi/4}\int_0^{\cos 2\theta} r\,dr\,d\theta = \int_0^{\pi/4}\frac{\cos^2 2\theta}{2}d\theta = \frac{\pi}{16}$

3. $\int_0^{\pi/2}\int_0^{3\sin\theta} r^2\,dr\,d\theta = \int_0^{\pi/2} 9\sin^3\theta\,d\theta = 9\int_0^{\pi/2}(1-\cos^2\theta)\sin\theta\,d\theta = 9\left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi/2} = 6$

4. $\int_{-\pi/3}^{2\pi/3}\int_0^{2\cos\theta} r\sin\theta\,dr\,d\theta = \int_{-\pi/3}^{2\pi/3} 2\cos^2\theta\sin\theta\,d\theta = \frac{1}{6}$

5. (a) $\Gamma:\ 0 \le \theta \le 2\pi, \quad 0 \le r \le 1$

$$\iint_{\Gamma}(\cos r^2)r\,dr d\theta = \int_0^{2\pi}\int_0^1(\cos r^2)r\,dr\,d\theta$$
$$= 2\pi\int_0^1 r\,\cos r^2\,dr = \pi\,\sin 1 \cong 0.84\,\pi$$

(b) $\Gamma: \ 0 \le \theta \le 2\pi, \quad 1 \le r \le 2$

$$\iint_\Gamma (\cos r^2) r \, dr d\theta = \int_0^{2\pi} \int_1^2 (\cos r^2) r \, dr \, d\theta$$

$$= 2\pi \int_1^2 r \ \cos r^2 \, dr = \pi(\sin 2 - \sin 1) \cong 0.07\pi$$

6. (a) $\Gamma: \ 0 \le \theta \le 2\pi, \quad 0 \le r \le 1.$

$$\iint_\Gamma (\sin r) r \, dr \, d\theta = \int_0^{2\pi} \int_0^1 (\sin r) r \, dr \, d\theta = 2\pi \int_0^1 r \sin r \, dr = 2\pi(\sin 1 - \cos 1) \cong 0.60\pi$$

(b) $\Gamma: \ 0 \le \theta \le 2\pi, \quad 1 \le r \le 2.$

$$\iint_\Gamma (\sin r) r \, dr \, d\theta = \int_0^{2\pi} \int_1^2 (\sin r) r \, dr \, d\theta = 2\pi \int_1^2 r \sin r \, dr = 2\pi[\cos 1 - 2\cos 2 + \sin 2 - \sin 1] \cong 2.88\pi$$

7. (a) $\Gamma: \ 0 \le \theta \le \pi/2, \quad 0 \le r \le 1$

$$\iint_\Gamma (r\cos\theta + r\sin\theta) r \, dr d\theta = \int_0^{\pi/2} \int_0^1 r^2(\cos\theta + \sin\theta) \, dr \, d\theta$$

$$= \left(\int_0^{\pi/2} (\cos\theta + \sin\theta) \, d\theta\right) \left(\int_0^1 r^2 \, dr\right) = 2\left(\frac{1}{3}\right) = \frac{2}{3}$$

(b) $\Gamma: \ 0 \le \theta \le \pi/2, \quad 1 \le r \le 2$

$$\iint_\Gamma (r\cos\theta + r \ \sin\theta) r \, dr d\theta = \int_0^{\pi/2} \int_1^2 r^2(\cos\theta + \sin\theta) \, dr \, d\theta$$

$$= \left(\int_0^{\pi/2} (\cos\theta + \sin\theta) \, d\theta\right) \left(\int_1^2 r^2 \, dr\right) = 2\left(\frac{7}{3}\right) = \frac{14}{3}$$

8. $\Gamma: \ 0 \le \theta \le \frac{\pi}{3}, \quad 0 \le r \le \frac{1}{\cos\theta}$

$$\iint_\Gamma \sqrt{x^2 + y^2} \, dx \, dy = \int_0^{\pi/3} \int_0^{1/\cos\theta} r \cdot r \, dr \, d\theta = \int_0^{\pi/3} \frac{d\theta}{3\cos^3\theta} = \frac{1}{3} \int_0^{\pi/3} \sec^3\theta \, d\theta$$

$$= \left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right]_0^{\pi/3} = \frac{1}{3}\sqrt{3} + \frac{1}{6}\ln(2 + \sqrt{3})$$

9. $\displaystyle\int_{-\pi/2}^{\pi/2} \int_0^1 r^2 \, dr \, d\theta = \frac{1}{3}\pi$

10. $\displaystyle\int_0^{\pi/2} \int_0^2 r^2 \, dr \, d\theta = \frac{\pi}{2} \int_0^2 r^2 \, dr = \frac{4}{3}\pi$

11. $\displaystyle\int_0^{\pi/3} \int_0^1 r^4 \, dr \, d\theta = \frac{1}{15}\pi$

12. $\displaystyle\int_0^{\pi/3}\int_0^{1/2\cos\theta} r^4\cos\theta\sin\theta\,dr\,d\theta + \int_{\pi/3}^{\pi/2}\int_0^1 r^4\cos\theta\sin\theta\,dr\,d\theta$

$\displaystyle= \int_0^{\pi/3}\frac{\sin\theta}{5(32)\cos^4\theta}\,d\theta + \int_{\pi/3}^{\pi/2}\frac{1}{5}\cos\theta\sin\theta\,d\theta = \frac{7}{480}+\frac{1}{40}=\frac{19}{480}$

13. $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\sin(x^2+y^2)\,dx\,dy = \int_0^{\pi/2}\int_0^1 \sin(r^2)\,r\,dr\,d\theta = \int_0^{\pi/2}\frac{1}{2}(1-\cos 1)\,d\theta = \frac{\pi}{4}(1-\cos 1)$

14. $\displaystyle\int_0^{2\pi}\int_0^1 e^{-r^2}r\,dr\,d\theta = 2\pi\int_0^1 re^{-r^2}\,dr = \pi(1-\frac{1}{e})$

15. $\displaystyle\int_0^2\int_0^{\sqrt{2x-x^2}} x\,dx\,dy = \int_0^{\pi/2}\int_0^{2\cos\theta} r\,\cos\theta\, r\,dr\,d\theta = \frac{8}{3}\int_0^{\pi/2}\cos^4\theta\,d\theta = \frac{8}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{2}$

(See Exercise 62, Section 8.3)

16. The region is the inside of the circle $(x-1/2)^2+y^2=1/4$, which has polar equation $r=\cos\theta$. So the integral becomes

$$\int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta} r^3\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\frac{1}{4}\cos^4\theta\,d\theta = \frac{3\pi}{32}$$

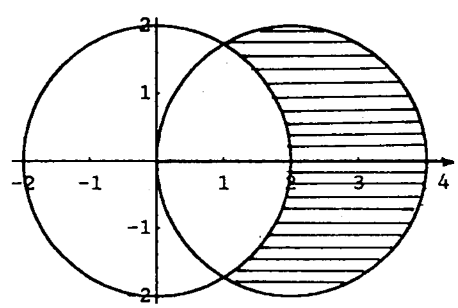
17. $\displaystyle A = \int_0^{\pi/3}\int_0^{3\sin 3\theta} r\,dr\,d\theta = \frac{9}{2}\int_0^{\pi/3}\sin^2 3\theta\,d\theta = \frac{9}{4}\int_0^{\pi/3}(1-6\cos\theta)\,d\theta = \frac{3\pi}{4}$

18. $\displaystyle A = \int_0^{2\pi}\int_0^{2(1-\cos\theta)} r\,dr\,d\theta = \int_0^{2\pi} 2(1-\cos\theta)^2\,d\theta = 6\pi$

19. First we find the points of intersection:

$r = 4\cos\theta = 2 \implies \cos\theta = \dfrac{1}{2}$

$\implies \theta = \pm\dfrac{\pi}{3}.$



$$A = \int_{-\pi/3}^{\pi/3}\int_2^{4\cos\theta} r\,dr\,d\theta = \int_{-\pi/3}^{\pi/3}(8-\cos^2\theta-2)\,d\theta = \int_{-\pi/3}^{\pi/3}(2+4\cos 2\theta)\,d\theta = \frac{4\pi}{3}+2\sqrt{3}$$

20. $\displaystyle A = \int_0^{2\pi}\int_0^{1+2\cos\theta} r\,dr\,d\theta = \int_0^{2\pi}\frac{1}{2}(1+2\cos\theta)^2\,d\theta = 3\pi$

21. $\displaystyle A = 4\int_0^{\pi/4}\int_0^{2\sqrt{\cos 2\theta}} r\,dr\,d\theta = 8\int_0^{\pi/4}\cos 2\theta\,d\theta = 4$

22. $\displaystyle A = \int_{-\pi/3}^{\pi/3}\int_{1+\cos\theta}^{3\cos\theta} r\,dr\,d\theta = \int_{-\pi/3}^{\pi/3}\frac{1}{2}\left[9\cos^2\theta-(1+\cos\theta)^2\right]d\theta = \left[\frac{3\theta}{2}+\sin 2\theta-\sin\theta\right]_{-\pi/3}^{\pi/3} = \pi$

23. $$\int_0^{2\pi}\int_0^b \left(r^2\sin\theta + br\right)dr\,d\theta = \int_0^{2\pi}\left[\frac{1}{3}r^3\sin\theta + \frac{b}{2}r^2\right]_0^b d\theta$$
$$= b^3\int_0^{2\pi}\left(\frac{1}{3}\sin\theta + \frac{1}{2}\right)d\theta = b^3\pi$$

24. $V = \int_0^{2\pi}\int_0^1 (1-r^2)r\,dr\,d\theta = 2\pi\int_0^1 (r-r^3)\,dr = \frac{\pi}{2}$

25. $$8\int_0^{\pi/2}\int_0^2 \frac{r}{2}\sqrt{12-3r^2}\,dr\,d\theta = 8\int_0^{\pi/2}\left[-\frac{1}{18}\left(12-3r^2\right)^{3/2}\right]_0^2 d\theta$$
$$= 8\int_0^{\pi/2}\frac{4}{3}\sqrt{3}\,d\theta = \frac{16}{3}\sqrt{3}\,\pi$$

26. $V = \int_0^{2\pi}\int_0^{\sqrt{5}} \sqrt[6]{5-r^2}\;r\,dr\,d\theta = 2\pi\int_0^{\sqrt{5}} r\sqrt[6]{5-r^2}\,dr = \frac{30}{7}(5)^{1/6}\pi$

27. $$\int_0^{2\pi}\int_0^1 r\sqrt{4-r^2}\,dr\,d\theta = \int_0^{2\pi}\left[-\frac{1}{3}\left(4-r^2\right)^{3/2}\right]_0^1 d\theta$$
$$= \int_0^{2\pi}\left(\frac{8}{3}-\sqrt{3}\right)d\theta = \frac{2}{3}(8-3\sqrt{3})\pi$$

28. $V = \int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta}(1-r^2)r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left(\frac{\cos^2\theta}{2}-\frac{\cos^4\theta}{2}\right)d\theta = \frac{5\pi}{32}$

29.
$$\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} 2r^2\cos\theta\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left[\frac{2}{3}r^3\cos\theta\right]_0^{2\cos\theta} d\theta$$
$$= \int_{-\pi/2}^{\pi/2}\frac{16}{3}\cos^4\theta\,d\theta = \frac{32}{3}\int_0^{\pi/2}\cos^4\theta\,d\theta = \frac{32}{3}\left(\frac{3}{16}\pi\right) = 2\pi$$

Ex. 46, Sect. 8.3

30. $\int_{-\pi/2}^{\pi/2}\int_0^{2a\cos\theta} r^2\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\frac{8a^3}{3}\cos^3\theta d\theta = \frac{32}{9}a^3$

31. $$\frac{b}{a}\int_0^{\pi}\int_0^{a\sin\theta} r\sqrt{a^2-r^2}\,dr\,d\theta = \frac{b}{a}\int_0^{\pi}\left[-\frac{1}{3}\left(a^2-r^2\right)^{3/2}\right]_0^{a\sin\theta} d\theta$$
$$= \frac{1}{3}a^2b\int_0^{\pi}\left(1-\cos^3\theta\right)d\theta = \frac{1}{3}\pi a^2 b$$

32. (a) $V = 2\int_0^{2\pi}\int_0^r \sqrt{R^2 - s^2}\; s\,ds\,d\theta = 4\pi\int_0^r S\sqrt{R^2 - S^2}\,dS = \frac{4\pi}{3}\left[R^3 - (R^2 - r^2)^{3/2}\right].$

(b) $\frac{4}{3}\pi R^3 - V = \frac{4}{3}\pi(R^2 - r^2)^{3/2}$

SECTION 16.5

1. $M = \int_{-1}^1\int_0^1 x^2\,dy\,dx = \frac{2}{3}$

$x_M\,M = \int_{-1}^1\int_0^1 x^3\,dy\,dx = 0 \quad\Longrightarrow\quad x_M = 0$

$y_M\,M = \int_{-1}^1\int_0^1 x^2y\,dy\,dx = \int_{-1}^1 \frac{1}{2}x^2\,dx = \frac{1}{3} \quad\Longrightarrow\quad y_M = \frac{1/3}{1/2} = \frac{1}{2}$

2. $M = \int_0^1\int_0^{\sqrt{x}} (x+y)\,dy\,dx = \int_0^1 \left(x^{3/2} + \frac{x}{2}\right)\,dx = \frac{13}{20}$

$x_M M = \int_0^1\int_0^{\sqrt{x}} x(x+y)\,dy\,dx = \int_0^1 \left(x^{5/2} + \frac{x^2}{2}\right)\,dx = \frac{19}{42} \Longrightarrow x_M = \frac{190}{273}$

$y_M M = \int_0^1\int_0^{\sqrt{x}} y(x+y)\,dy\,dx = \int_0^1 \left(\frac{x^2}{2} + \frac{x^{3/2}}{3}\right)\,dx = \frac{23}{60} \Longrightarrow y_M = \frac{23}{39}$

3. $M = \int_0^1\int_{x^2}^1 xy\,dy\,dx = \frac{1}{2}\int_0^1 (x - x^5)\,dx = \frac{1}{6}$

$x_M\,M = \int_0^1\int_{x^2}^1 x^2y\,dy\,dx\frac{1}{2}\int_0^1 (x^2 - x^6)\,dx = \frac{2}{21} \quad\Longrightarrow\quad x_M = \frac{2/21}{1/6} = \frac{12}{21}$

$y_M\,M = \int_0^1\int_{x^2}^1 xy^2\,dy\,dx = \frac{1}{3}\int_0^1 (x - x^7)\,dx = \frac{1}{8} \quad\Longrightarrow\quad y_M = \frac{1/8}{1/6} = \frac{3}{4}$

4. $M = \int_0^\pi\int_0^{\sin x} y\,dy\,dx = \int_0^\pi \frac{\sin^2 x}{2}\,dx = \frac{\pi}{4}$

$x_M M = \int_0^\pi\int_0^{\sin x} xy\,dy\,dx = \int_0^\pi x\frac{\sin^2 x}{2}\,dx = \frac{\pi^2}{8} \Longrightarrow x_M = \frac{\pi}{2}$

$y_M M = \int_0^\pi\int_0^{\sin x} y^2\,dy\,dx = \int_0^\pi \frac{\sin^3 x}{3}\,dx = \frac{4}{9} \Longrightarrow y_M = \frac{16}{9\pi}$

5. $M = \int_0^8\int_0^{x^{1/3}} y^2\,dy\,dx = \frac{1}{3}\int_0^8 x\,dx = \frac{32}{3}$

$x_M\,M = \int_0^8\int_0^{x^{1/3}} xy^2\,dy\,dx\frac{1}{3}\int_0^8 x^2\,dx = \frac{512}{9} \quad\Longrightarrow\quad x_M = \frac{512/9}{32/3} = \frac{16}{3}$

$$y_M\, M = \int_0^8 \int_0^{x^{1/3}} y^3\, dy\, dx = \frac{1}{4}\int_0^8 x^{4/3}\, dx = \frac{96}{7} \implies y_M = \frac{96/7}{32/3} = \frac{9}{7}$$

6. $$M = \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\, dy\, dx = \int_0^a \frac{x}{2}(a^2-x^2)\, dx = \frac{a^4}{8}$$

$$x_M M = \int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 y\, dy\, dx = \int_0^a \frac{x^2}{2}(a^2-x^2)\, dx = \frac{a^5}{15} \implies x_M = \frac{8}{15}a$$

$$y_M M = \int_0^a \int_0^{\sqrt{a^2-x^2}} xy^2\, dy\, dx = \int_0^a \frac{x}{3}(a^2-x^2)^{3/2}\, dx = \frac{a^5}{15} \implies y_M = \frac{8}{15}a$$

7. $$M = \int_0^1 \int_{2x}^{3x} xy\, dy\, dx = \frac{5}{2}\int_0^1 x^3\, dx = \frac{5}{8}$$

$$x_M\, M = \int_0^1 \int_{2x}^{3x} x^2 y\, dy\, dx = \frac{5}{2}\int_0^1 x^4\, dx = \frac{1}{2} \implies x_M = \frac{1/2}{5/8} = \frac{4}{5}$$

$$y_M\, M = \int_0^1 \int_{2x}^{3x} xy^2\, dy\, dx = \frac{19}{3}\int_0^1 x^4\, dx = \frac{19}{15} \implies y_M = \frac{19/15}{5/8} = \frac{152}{75}$$

8. $$M = \int_0^2 \int_0^{3-\frac{3}{2}x} (x+y)\, dy\, dx = \int_0^2 \left[x(3-\frac{3}{2}x) + \frac{1}{2}(3-\frac{3}{2}x)^2\right] dx = 5$$

$$x_M M = \int_0^2 \int_0^{3-\frac{3}{2}x} x(x+y)\, dy\, dx = \int_0^2 \left[x^2(3-\frac{3}{2}x) + \frac{x}{2}(3-\frac{3}{2}x)^2\right] dx = \frac{7}{2} \implies x_M = \frac{7}{10}$$

$$y_M M = \int_0^2 \int_0^{3-\frac{3}{2}x} y(x+y)\, dy\, dx = \int_0^2 \left[\frac{x}{2}(3-\frac{3}{2}x)^2 + \frac{(3-\frac{3}{2}x)^3}{3}\right] dx = \frac{9}{2} \implies y_M = \frac{9}{10}$$

9. $$M = \int_0^{2\pi} \int_0^{1+\cos\theta} r^2\, dr\, d\theta = \frac{1}{3}\int_0^{2\pi} (1 + 3\cos\theta + 3\cos^2\theta + \cos^3\theta)\, d\theta = \frac{5\pi}{3}$$

$$x_M\, M = \int_0^{2\pi} \int_0^{1+\cos\theta} r^3 \cos\theta\, dr\, d\theta = \frac{1}{4}\int_0^{2\pi} (1+\cos\theta)^4 \cos\theta\, d\theta$$

$$= \frac{1}{4}\int_0^{2\pi} \left[\cos\theta + 4\cos^2\theta + 6\cos^3\theta + 4\cos^4\theta + \cos^5\theta\right] d\theta$$

$$= \frac{7\pi}{4}$$

Therefore, $x_M = \dfrac{7\pi/4}{5\pi/3} = \dfrac{21}{20}$.

$$y_M\, M = \int_0^{2\pi} \int_0^{1+\cos\theta} r^3 \sin\theta\, dr\, d\theta = \frac{1}{4}\int_0^{2\pi} (1+\cos\theta)^4 \sin\theta\, d\theta = \frac{1}{4}\left[\frac{1}{5}(1+\cos\theta)^5\right]_0^{2\pi} = 0$$

Therefore, $y_M = 0$.

10. $$M = \iint_{\Omega} y\,dx\,dy = \int_{\pi/6}^{5\pi/6} \int_1^{2\sin\theta} r\sin\theta\, r\,dr\,d\theta = \int_{\pi/6}^{5\pi/6} \left(\frac{8}{3}\sin^4\theta - \frac{1}{3}\sin\theta\right)\,d\theta = \frac{2\pi}{3} - \frac{5\sqrt{3}}{12}$$

$x_M = 0$ by symmetry

$$y_M\,M = \int_{\pi/6}^{5\pi/6} \int_1^{2\sin\theta} r^2\sin^2\theta\, r\,dr\,d\theta = \int_{\pi/6}^{5\pi/6} \left(4\sin^6\theta - \frac{1}{4}\sin^2\theta\right)\,d\theta = \frac{11\pi}{12} - \frac{23\sqrt{3}}{72}$$

Therefore, $y_M = \dfrac{66\pi - 23\sqrt{3}}{48\pi - 30\sqrt{3}}$

11. $\Omega:\ -L/2 \le x \le L/2, \quad -W/2 \le y \le W/2$

$$I_x = \iint_{\Omega} \frac{M}{LW}\,y^2\,dxdy = \frac{4M}{LW}\int_0^{W/2}\int_0^{L/2} y^2 dx\,dy = \frac{1}{12}MW^2$$

symmetry

$$I_y = \iint_{\Omega} \frac{M}{LW}x^2\,dxdy = \frac{1}{12}ML^2, \quad I_z = \iint_{\Omega} \frac{M}{LW}\left(x^2+y^2\right)dxdy = \frac{1}{12}M\left(L^2+W^2\right)$$

$$K_x = \sqrt{I_x/M} = W/2\sqrt{3}, \quad K_y = \sqrt{I_y/M} = L/2\sqrt{3}$$

$$K_z = \sqrt{I_z/M} = \sqrt{L^2+W^2}\Big/2\sqrt{3}$$

12. $\lambda(x,y) = k\left(x + \dfrac{L}{2}\right) \qquad M = \displaystyle\int_{-W/2}^{W/2}\int_{-L/2}^{L/2} k\left(x+\frac{L}{2}\right)dx\,dy = \frac{kWL^2}{2}$

$$I_x = \int_{-W/2}^{W/2}\int_{-L/2}^{L/2} k\left(x+\frac{L}{2}\right)y^2\,dx\,dy = \left(\int_{-W/2}^{W/2} y^2\,dy\right)\left(\int_{-L/2}^{L/2} k\left(x+\frac{L}{2}\right)dx\right)$$

$$= \left(\frac{W^3}{12}\right)\left(\frac{M}{W}\right) = \frac{1}{12}MW^2$$

$$I_y = \int_{-W/2}^{W/2}\int_{-L/2}^{L/2} k\left(x+\frac{L}{2}\right)x^2\,dx\,dy = \frac{kL^4W}{24} = \frac{1}{12}ML^2$$

$$I_z = \int_{-W/2}^{W/2}\int_{-L/2}^{L/2} k\left(x+\frac{L}{2}\right)(x^2+y^2)\,dx\,dy = I_x + I_y = \frac{1}{12}M(L^2+W^2).$$

13.

$$M = \iint_\Omega k\left(x+\frac{L}{2}\right)dxdy = \iint_\Omega \frac{1}{2}kL\,dxdy = \frac{1}{2}kL(\text{ area of }\Omega) = \frac{1}{2}kL^2W$$

symmetry

$$x_M M = \iint_\Omega x\left[k\left(x+\frac{L}{2}\right)\right]dxdy = \iint_\Omega \left(kx^2+\frac{1}{2}Lx\right)dxdy$$

$$= \iint_\Omega kx^2\,dxdy = 4k\int_0^{W/2}\int_0^{L/2} x^2\,dx\,dy = \frac{1}{12}kWL^3$$

symmetry symmetry

$$= \tfrac{1}{6}\left(\tfrac{1}{2}kL^2W\right)L = \tfrac{1}{6}ML; \quad x_M = \tfrac{1}{6}L$$

$$y_M M = \iint_\Omega y\left[k\left(x+\frac{L}{2}\right)\right]dxdy = 0; \quad y_M = 0$$

by symmetry

14. $I_z = \iint_\Omega \lambda(x,y)[x^2+y^2]\,dx\,dy = \iint_\Omega \lambda(x,y)x^2\,dx\,dy + \iint_\Omega \lambda(x,y)y^2\,dx\,dy = I_x + I_y.$

Since $I_z = I_x + I_y$, we have $MK_z{}^2 = MK_x{}^2 + MK_y{}^2$ therefore $K_z{}^2 = K_x{}^2 + K_y{}^2$.

15.

$$I_x = \iint_\Omega \frac{4M}{\pi R^2}y^2\,dxdy = \frac{4M}{\pi R^2}\int_0^{\pi/2}\int_0^R r^3\sin^2\theta\,dr\,d\theta$$

$$= \frac{4M}{\pi R^2}\left(\int_0^{\pi/2}\sin^2\theta\,d\theta\right)\left(\int_0^R r^3\,dr\right) = \frac{4M}{\pi R^2}\left(\frac{\pi}{4}\right)\left(\frac{1}{4}R^4\right) = \frac{1}{4}MR^2$$

$$I_y = \tfrac{1}{4}MR^2, \quad I_z = \tfrac{1}{2}MR^2$$

$$K_x = K_y = \tfrac{1}{2}R, \quad K_z = R/\sqrt{2}$$

16. $I_z = I_M + d^2M$. Rotation doesn't change d, doesn't change M, and doesn't change I_M.

17. I_M, the moment of inertia about the vertical line through the center of mass, is

$$\iint_\Omega \frac{M}{\pi R^2}\left(x^2+y^2\right)dxdy$$

where Ω is the disc of radius R centered at the origin. Therefore

$$I_M = \frac{M}{\pi R^2}\int_0^{2\pi}\int_0^R r^3\,dr\,d\theta = \frac{1}{2}MR^2.$$

We need $I_0 = \frac{1}{2}MR^2 + d^2M$ where d is the distance from the center of the disc to the origin. Solving this equation for d, we have $d = \sqrt{I_0 - \frac{1}{2}MR^2}\Big/\sqrt{M}$.

18. $$I_x = \int_a^b \int_0^{f(x)} \lambda y^2\,dy\,dx = \frac{\lambda}{3}\int_a^b [f(x)]^3\,dx$$

$$I_y = \int_a^b \int_0^{f(x)} \lambda x^2\,dy\,dx = \lambda \int_a^b x^2 f(x)\,dx.$$

19. $\Omega:\ 0 \le x \le a, \quad 0 \le y \le b$

$$I_x = \iint_\Omega \frac{4M}{\pi ab}y^2\,dxdy = \frac{4M}{\pi ab}\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} y^2\,dy\,dx = \frac{1}{4}Mb^2$$

$$I_y = \iint_\Omega \frac{4M}{\pi ab}x^2\,dxdy = \frac{4M}{\pi ab}\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} x^2\,dy\,dx = \frac{1}{4}Ma^2$$

$$I_z = \tfrac{1}{4}M\left(a^2+b^2\right)$$

20. $$I_x = \int_0^1 \int_0^{\sqrt{x}} (x+y)y^2\,dy\,dx = \int_0^1 \left(\frac{x^{5/2}}{3} + \frac{x^2}{4}\right)dx = \frac{15}{84}$$

$$I_y = \int_0^1 \int_0^{\sqrt{x}} (x+y)x^2\,dy\,dx = \int_0^1 \left(x^{7/2} + \frac{x^3}{2}\right)dx = \frac{25}{72}; \qquad I_z = I_x + I_y.$$

21. $$I_x = \int_0^1 \int_{x^2}^1 xy^3\,dy\,dx = \frac{1}{4}\int_0^1 (x - x^9)\,dx = \frac{1}{10}$$

$$I_y = \int_0^1 \int_{x^2}^1 x^3y\,dy\,dx \frac{1}{2}\int_0^1 (x^3 - x^7)\,dx = \frac{1}{16}$$

$$I_z = \int_0^1 \int_{x^2}^1 xy(x^2+y^2)\,dy\,dx = I_x + I_y = \frac{13}{80}$$

22. $$I_x = \int_0^8 \int_0^{\sqrt[3]{x}} y^2 \cdot y^2\,dy\,dx = \int_0^8 \frac{x^{5/3}}{5}\,dx = \frac{96}{5}$$

$$I_y = \int_0^8 \int_0^{\sqrt[3]{x}} y^2 \cdot x^2\,dy\,dx = \int_0^8 \frac{x^3}{3}\,dx = \frac{1024}{3}; \qquad I_z = I_x + I_y$$

23. $$I_x = \int_0^{2\pi} \int_0^{1+\cos\theta} r^4 \sin^2\theta\,dr\,d\theta = \frac{1}{5}\int_0^{2\pi} (1+\cos\theta)^5 \sin^2\theta\,d\theta = \frac{33\pi}{40}$$

$$I_y = \int_0^{2\pi} \int_0^{1+\cos\theta} r^4 \cos^2\theta\,dr\,d\theta = \frac{1}{5}\int_0^{2\pi} (1+\cos\theta)^5 \cos^2\theta\,d\theta = \frac{93\pi}{40}$$

$$I_z = \int_0^{2\pi} \int_0^{1+\cos\theta} r^4\,dr\,d\theta = I_x + I_y = \frac{63\pi}{20}$$

24. $$x_M = \frac{x_1M_1 + x_2M_2}{M_1 + M_2}, \quad y_M = \frac{y_1M_1 + y_2M_2}{M_1 + M_2}$$

25. $\Omega: \ r_1^2 \le x^2+y^2 \le r_2^2, \quad A = \pi\left(r_2^2 - r_1^2\right)$

(a) Place the diameter on the x-axis.

$$I_x = \iint_\Omega \frac{M}{A} y^2\,dxdy = \frac{M}{A}\int_0^{2\pi}\int_{r_1}^{r_2}\left(r^2\sin^2\theta\right) r\,dr\,d\theta = \frac{1}{4}M\left(r_2^2 + r_1^2\right)$$

(b) $\frac{1}{4}M\left(r_2^2+r_1^2\right) + Mr_1^2 = \frac{1}{4}M\left(r_2^2 + 5r_1^2\right)$ (parallel axis theorem)

(c) $\frac{1}{4}M\left(r_2^2+r_1^2\right) + Mr_2^2 = \frac{1}{4}M\left(5r_2^2 + r_1^2\right)$

26. Set $r_1 = r_2 = r$ in the proceeding problem. Then the required moments of inertia are

(a) $\frac{1}{2}Mr^2$ (b) $\frac{3}{2}Mr^2$.

27. $\Omega: \ r_1^2 \le x^2+y^2 \le r_2^2, \quad A = \pi\left(r_2^2 - r_1^2\right)$

$$I = \iint_\Omega \frac{M}{A}\left(x^2+y^2\right)dxdy = \frac{M}{A}\int_0^{2\pi}\int_{r_1}^{r_2} r^3\,dr\,d\theta = \frac{1}{2}M(r_2^2+r_1^2)$$

28. Let l be the x-axis and let the plane of the plate be the xy-plane. Then

$$\begin{aligned} I - I_M &= \iint_\Omega \lambda(x,y)y^2\,dx\,dy - \iint_\Omega \lambda(x,y)(y-y_M)^2\,dx\,dy \\ &= \iint_\Omega \lambda(x,y)[2y_My - y_M^2]\,dx\,dy \\ &= 2y_M\iint_\Omega y\lambda(x,y)\,dx\,dy - y_M^2\iint_\Omega \lambda(x,y)\,dx\,dy \\ &= 2y_M^2M - y_M^2M = y_M^2M = d^2M. \end{aligned}$$

29. $$M = \iint_\Omega k\left(R - \sqrt{x^2+y^2}\right)dxdy = k\int_0^{\pi}\int_0^R \left(Rr - r^2\right)dr\,d\theta = \frac{1}{6}k\pi R^3$$

$x_M = 0$ by symmetry

$$y_MM = \iint_\Omega y\left[k\left(R-\sqrt{x^2+y^2}\right)\right]dxdy = k\int_0^{\pi}\int_0^R \left(Rr^2 - r^3\right)\sin\theta\,dr\,d\theta = \frac{1}{6}kR^4$$

$y_M = R/\pi$

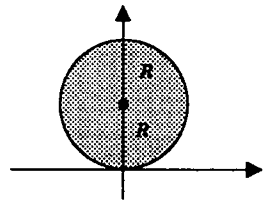
30. $$I_x = \iint_\Omega k(R-\sqrt{x^2+y^2})y^2\,dx\,dy = k\int_0^{\pi}\int_0^R (R-r)\,r^2\sin^2\theta\,r\,dr\,d\theta = \frac{k\pi R^5}{40} = \frac{3MR^2}{20}.$$

$$I_y = k\int_0^{\pi}\int_0^R (R-r)r^2\cos^2\theta r\,dr\,d\theta = \frac{k\pi R^5}{40} = \frac{3MR^2}{20}$$

$$I_z = I_x + I_y = \frac{3MR^2}{10}.$$

31. Place P at the origin.

$$M = \iint_\Omega k\sqrt{x^2+y^2}\,dxdy$$
$$= k\int_0^\pi \int_0^{2R\sin\theta} r^2\,dr\,d\theta = \frac{32}{9}kR^3$$



$$x_M = 0 \quad \text{by symmetry}$$

$$y_M M = \iint_\Omega y\left(k\sqrt{x^2+y^2}\right)dxdy = k\int_0^\pi \int_0^{2R\sin\theta} r^3\sin\theta\,dr\,d\theta = \frac{64}{15}kR^4$$

$$y_M = 6R/5$$

Answer: the center of mass lies on the diameter through P at a distance $6R/5$ from P.

32. Putting the right angle at the origin, we have $\lambda(x,y) = k(x^2+y^2)$.

$$M = \int_0^b \int_0^{h-\frac{h}{b}x} k(x^2+y^2)\,dy\,dx = \frac{1}{12}kbh(b^2+h^2)$$
$$x_M M = \int_0^b \int_0^{h-\frac{h}{b}x} kx(x^2+y^2)\,dy\,dx = \frac{kb^2h(3b^2+h^2)}{60} \implies x_M = \frac{b(3b^2+h^2)}{5(b^2+h^2)}$$
$$y_M M = \int_0^b \int_0^{h-\frac{h}{b}x} ky(x^2+y^2)\,dy\,dx = \frac{kbh^2(b^2+3h^2)}{60} \implies y_M = \frac{h(b^2+3h^2)}{5(b^2+h^2)}$$

33. Suppose Ω, a basic region of area A, is broken up into n basic regions $\Omega_1, \cdots, \Omega_n$ with areas $A_1, \cdots, A_n$. Then

$$\overline{x}A = \iint_\Omega x\,dxdy = \sum_{i=1}^n \left(\iint_{\Omega_i} x\,dxdy\right) = \sum_{i=1}^n \overline{x}_i\,A_i = \overline{x}_1\,A_1 + \cdots + \overline{x}_n\,A_n.$$

The second formula can be derived in a similar manner.

SECTION 16.6

1. They are equal; they both give the volume of T.

2. (a) $L_f(P) = \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^q x_{i-1}y_{j-1}z_{k-1}\Delta x_i \Delta y_j \Delta z_k, \quad U_f(P) = \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^q x_i y_j z_k \Delta x_i \Delta y_j \Delta z_k$

(b) $x_{i-1}y_{j-1}z_{k-1} \le \left(\frac{x_i+x_{i-1}}{2}\right)\left(\frac{y_j+y_{j-1}}{2}\right)\left(\frac{z_k+z_{k-1}}{2}\right) \le x_i y_j z_k$

$$x_{i-1}y_{j-1}z_{k-1}\Delta x_i \Delta y_j \Delta z_k \le \frac{1}{8}\left(x_i{}^2 - x_{i-1}{}^2\right)\left(y_j{}^2 - y_{j-1}{}^2\right)\left(z_k{}^2 - z_{k-1}{}^2\right) \le x_i y_j z_k \Delta x_i \Delta y_j \Delta z_k$$

$$L_f(P) \le \frac{1}{8}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{q}\left({x_i}^2 - {x_{i-1}}^2\right)\left({y_j}^2 - {y_{j-1}}^2\right)\left({z_k}^2 - {z_{k-1}}^2\right) \le U_f(P).$$

The middle term can be written

$$\frac{1}{8}\left(\sum_{i=1}^{m}{x_i}^2 - {x_{i-1}}^2\right)\left(\sum_{j=1}^{n}{y_j}^2 - {y_{j-1}}^2\right)\left(\sum_{k=1}^{q}{z_k}^2 - {z_{k-1}}^2\right) = \frac{1}{8}(1)(1)(1) = \frac{1}{8}.$$

Therefore $I = \dfrac{1}{8}$.

3. $\displaystyle\iiint_{\Pi} \alpha\, dx\, dy\, dz = \alpha \iiint_{\Pi} dx\, dy\, dz = \alpha\,(\text{volume of }\Pi) = \alpha(a_2 - a_1)(b_2 - b_1)(c_2 - c_1)$

4. Since the volume is 1, the average value is $\displaystyle\iiint_{\Omega} xyz\, dx\, dy\, dz = \frac{1}{8}$.

5. Let $P_1 = \{x_0, \cdots, x_m\}$, $P_2 = \{y_0, \cdots, y_n\}$, $P_3 = \{z_0, \cdots, z_q\}$ be partitions of $[0,a]$, $[0,b]$, $[0,c]$ respectively and let $P = P_1 \times P_2 \times P_3$. Note that

$$x_{i-1}y_{j-1} \le \left(\frac{x_i + x_{i-1}}{2}\right)\left(\frac{y_j + y_{j-1}}{2}\right) \le x_i y_j$$

and therefore

$$x_{i-1}y_{j-1}\,\Delta x_i\,\Delta y_j\,\Delta z_k \le \tfrac{1}{4}\left({x_i}^2 - x_{i-1}^2\right)\left({y_j}^2 - y_{j-1}^2\right)\Delta z_k \le x_i y_j\,\Delta x_i\,\Delta y_j \Delta z_k.$$

It follows that

$$L_f(P) \le \frac{1}{4}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{q}\left({x_i}^2 - x_{i-1}^2\right)\left({y_j}^2 - y_{j-1}^2\right)\Delta z_k \le U_f(P).$$

The middle term can be written

$$\frac{1}{4}\left(\sum_{i=1}^{m}{x_i}^2 - x_{i-1}^2\right)\left(\sum_{j=1}^{n}{y_j}^2 - y_{j-1}^2\right)\left(\sum_{k=1}^{q}\Delta z_k\right) = \frac{1}{4}\,a^2b^2c.$$

6. $I_x = I_{xy} + I_{xz}, \quad I_y = I_{xy} + I_{yz}, \quad I_z = I_{xz} + I_{yz}$

7. $\overline{x}_1 = a, \quad \overline{y}_1 = b, \quad \overline{z}_1 = c; \qquad \overline{x}_0 = A, \quad \overline{y}_0 = B, \quad \overline{z}_0 = C$

$$\begin{aligned}\overline{x}_1 V_1 + \overline{x}V = \overline{x}_0 V_0 \quad &\Longrightarrow \quad a^2bc + (ABC - abc)\,\overline{x} = A^2BC\\ &\Longrightarrow \quad \overline{x} = \frac{A^2BC - a^2bc}{ABC - abc}\end{aligned}$$

similarly

$$\overline{y} = \frac{AB^2C - ab^2c}{ABC - abc}, \quad \overline{z} = \frac{ABC^2 - abc^2}{ABC - abc}$$

8. Encase T in a box Π. A partition P of Π breaks up Π into little boxes Π_{ijk}. Since f is nonnegative on Π, all the m_{ijk} are nonnegative. Therefore

$$0 \le L_f(P) \le \iiint_T f(x,y,z)\,dx\,dy\,dz.$$

9. $M = \iiint_\Pi Kz\,dxdydz$

Let $P_1 = \{x_0, \cdots, x_m\}$, $P_2 = \{y_0, \cdots, y_n\}$, $P_3 = \{z_0, \cdots, z_q\}$ be partitions of $[0,a]$ and let $P = P_1 \times P_2 \times P_3$. Note that

$$z_{k-1} \le \tfrac{1}{2}(z_k + z_{k-1}) \le z_k$$

and therefore

$$Kz_{k-1}\,\Delta x_i\,\Delta y_j\,\Delta z_k \le \tfrac{1}{2}K\,\Delta x_i\,\Delta y_j\left(z_k{}^2 - z_{k-1}^2\right) \le Kz_k\,\Delta x_i\,\Delta y_j \Delta z_k.$$

It follows that

$$L_f(P) \le \frac{1}{2}K\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{q}\Delta x_i\,\Delta y_j\,\left(z_k{}^2 - z_{k-1}^2\right) \le U_f(P).$$

The middle term can be written

$$\frac{1}{2}K\left(\sum_{i=1}^{m}\Delta x_i\right)\left(\sum_{j=1}^{n}\Delta y_j\right)\left(\sum_{k=1}^{q} z_k{}^2 - z_{k-1}^2\right) = \frac{1}{2}K(a)\,(a)\,(a^2) = \frac{1}{2}Ka^4.$$

$M = \tfrac{1}{2}Ka^4$ where K is the constant of proportionality for the density function.

10. $x_M M = \iiint_\Pi Kzx\,dx\,dy\,dz = \dfrac{Ka^5}{4} \implies x_M = \dfrac{1}{2}a$

$y_M M = \iiint_\Pi Kzy\,dx\,dy\,dz = \dfrac{Ka^5}{4} \implies y_M = \dfrac{1}{2}a$

$z_M M = \iiint_\Pi Kz^2\,dx\,dy\,dz = \dfrac{Ka^5}{3} \implies z_M = \dfrac{2}{3}a.$

11.

$$I_z = \iiint_\Pi Kz\left(x^2+y^2\right)\,dxdydz$$

$$= \underbrace{\iiint_\Pi Kx^2z\,dxdydz}_{I_1} + \underbrace{\iiint_\Pi Ky^2z\,dxdydz}_{I_2}.$$

We will calculate I_1 using the partitions we used in doing Exercise 9. Note that

$$x_{i-1}^2 z_{k-1} \le \left(\frac{x_i{}^2 + x_i x_{i-1} + x_{i-1}^2}{3}\right)\left(\frac{z_k + z_{k-1}}{2}\right) \le x_i{}^2 z_k$$

and therefore

$$Kx_{i-1}^2 z_{k-1}\,\Delta x_i\,\Delta y_j\,\Delta z_k \le \tfrac{1}{6}\,K\big(x_i{}^3 - x_{i-1}^3\big)\;\Delta y_j\;\big(z_k{}^2 - z_{k-1}^2\big) \le Kx_i{}^2 z_k{}^2\,\Delta x_i\,\Delta y_j\,\Delta z_k.$$

It follows that

$$L_f(P) \le \frac{1}{6}\,K\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{q}\big(x_i{}^3 - x_{i-1}^3\big)\;\Delta y_j\;\big(z_k{}^2 - z_{k-1}^2\big) \le U_f(P).$$

The middle term can be written

$$\frac{1}{6}\,K\left(\sum_{i=1}^{m} x_i{}^3 - x_{i-1}^3\right)\left(\sum_{j=1}^{n}\Delta y_j\right)\left(\sum_{k=1}^{q} z_k{}^2 - z_{k-1}^2\right) = \frac{1}{6}\,Ka^3\,(a)(a^2) = \frac{1}{6}\,Ka^6.$$

Similarly $I_2 = \frac{1}{6}\,Ka^6$ and therefore

by Exercise 9

$$I_z = \tfrac{1}{3}\,Ka^6 = \tfrac{2}{3}\left(\tfrac{1}{2}\,Ka^4\right)a^2 = \tfrac{2}{3}\,Ma^2.$$

SECTION 16.7

1. $\displaystyle\int_0^a\int_0^b\int_0^c dx\,dy\,dz = \int_0^a\int_0^b c\,dy\,dz = \int_0^a bc\,dz = abc$

2. $\displaystyle\int_0^1\int_0^x\int_0^y y\,dz\,dy\,dx = \int_0^1\int_0^x y^2\,dy\,dx = \int_0^1 \frac{x^3}{3}\,dx = \frac{1}{12}.$

3. $$\int_0^1\int_1^{2y}\int_0^x (x+2z)\,dz\,dx\,dy = \int_0^1\int_1^{2y}\left[xz+z^2\right]_0^x\,dx\,dy = \int_0^1\int_1^{2y} 2x^2\,dx\,dy$$
$$= \int_0^1\left[\frac{2}{3}\,x^3\right]_1^{2y}\,dy = \int_0^1\left(\frac{16}{3}\,y^3 - \frac{2}{3}\right)dy = \frac{2}{3}$$

4. $\displaystyle\int_0^1\int_{1-x}^{1+x}\int_0^{xy} 4z\,dz\,dy\,dx = \int_0^1\int_{1-x}^{1+x} 2x^2y^2\,dy\,dx = \int_0^1 \frac{2x^2}{3}\left[(1+x)^3 - (1-x)^3\right]\,dx = \frac{11}{9}$

5. $\displaystyle\int_0^2\int_{-1}^1\int_0^3 (z-xy)\,dz\,dy\,dx = \int_0^2\int_{-1}^1\left[\frac{1}{2}z^2 - xyz\right]_1^3\,dy\,dx$

$\displaystyle= \int_0^2\int_{-1}^1 (4-2xy)\,dy\,dx = \int_0^2\left[2y - xy^2\right]_{-1}^1\,dx = \int_0^2 8\,dy = 16$

6. $\displaystyle\int_0^2\int_{-1}^1\int_1^3 (z-xy)\,dy\,dx\,dz = \int_0^2\int_{-1}^1 (2z-4x)\,dx\,dz = \int_0^2 4z\,dz = 8$

7. $\displaystyle\int_0^{\pi/2}\int_0^1\int_0^{\sqrt{1-x^2}} x\cos z\,dy\,dx\,dz = \int_0^{\pi/2}\int_0^1\left[xy\cos z\right]_0^{\sqrt{1-x^2}}\,dx\,dz$

$\displaystyle= \int_0^{\pi/2}\int_0^1 x\sqrt{1-x^2}\cos z\,dx\,dz = \int_0^{\pi/2}\left[-\frac{1}{3}\,(1-x^2)^{3/2}\cos z\right]_0^1\,dz = \frac{1}{3}\int_0^{\pi/2}\cos z\,dz = \frac{1}{3}$

8. $\displaystyle\int_{-1}^{2}\int_{1}^{y+2}\int_{e}^{e^2}\frac{x+y}{z}\,dz\,dx\,dy=\int_{-1}^{2}\int_{1}^{y+2}(x+y)\,dx\,dy=\int_{-1}^{2}\left[\frac{(y+2)^2-1}{2}+(y+1)y\right]dy=\frac{27}{2}$

9. $\displaystyle\int_{1}^{2}\int_{y}^{y^2}\int_{0}^{\ln x}ye^z\,dz\,dx\,dy=\int_{1}^{2}\int_{y}^{y^2}\left[ye^z\right]_0^{\ln x}\,dx\,dy$

$$=\int_{1}^{2}\int_{y}^{y^2}y(x-1)\,dx\,dy=\int_{1}^{2}\left[\frac{1}{2}x^2y-xy\right]_y^{y^2}dy=\int_{1}^{2}\left(\frac{1}{2}y^5-\frac{3}{2}y^3+y^2\right)dy=\frac{47}{24}$$

10. $\displaystyle\int_{0}^{\pi/2}\int_{0}^{\pi/2}\int_{0}^{1}e^z\cos x\sin y\,dz\,dy\,dx=\int_{0}^{\pi/2}\int_{0}^{\pi/2}(e-1)\cos x\sin y\,dy\,dx=\int_{0}^{\pi/2}(e-1)\cos x\,dx=e-1$

11.

$$\iiint_{\Pi} f(x)g(y)h(z)\,dxdydz=\int_{c_1}^{c_2}\left[\int_{b_1}^{b_2}\left(\int_{a_1}^{a_2}f(x)g(y)h(z)\,dx\right)dy\right]dz$$

$$=\int_{c_1}^{c_2}\left[\int_{b_1}^{b_2}g(y)h(z)\left(\int_{a_1}^{a_2}f(x)\,dx\right)dy\right]dz$$

$$=\int_{c_1}^{c_2}\left[h(z)\left(\int_{a_1}^{a_2}f(x)\,dx\right)\left(\int_{b_1}^{b_2}g(y)\,dy\right)dz\right]$$

$$=\left(\int_{a_1}^{a_2}f(x)\,dx\right)\left(\int_{b_1}^{b^2}g(y)\,dy\right)\left(\int_{c_1}^{c_2}h(z)\,dz\right)$$

12. $\displaystyle\left(\int_0^1 x^3\,dx\right)\left(\int_0^2 y^2\,dy\right)\left(\int_0^3 z\,dz\right)=\left(\frac{1}{4}\right)\left(\frac{8}{3}\right)\left(\frac{9}{2}\right)=3$

13. $\displaystyle\left(\int_0^1 x^2\,dx\right)\left(\int_0^2 y^2\,dy\right)\left(\int_0^3 z^2\,dz\right)=\left(\frac{1}{3}\right)\left(\frac{8}{3}\right)\left(\frac{27}{3}\right)=8$

14. $\displaystyle M=\iiint_{\Pi} kxyz\,dx\,dy\,dz=k\left(\int_0^a x\,dx\right)\left(\int_0^b y\,dy\right)\left(\int_0^c z\,dz\right)=\frac{1}{8}ka^2b^2c^2$

15.

$$x_M M=\iiint_{\Pi} kx^2yz\,dxdydz=k\left(\int_0^a x^2\,dx\right)\left(\int_0^b y\,dy\right)\left(\int_0^c z\,dz\right)$$

$$=k\left(\tfrac{1}{3}a^3\right)\left(\tfrac{1}{2}b^2\right)\left(\tfrac{1}{2}c^2\right)=\tfrac{1}{12}ka^3b^2c^2.$$

By Exercise 14, $M=\frac{1}{8}ka^2b^2c^2$. Therefore $\overline{x}=\frac{2}{3}a$. Similarly, $\overline{y}=\frac{2}{3}b$ and $\overline{z}=\frac{2}{3}c$.

16. (a)

$$I=\iiint_{\Pi} kxyz\left[(x-a)^2+(y-b)^2\right]dx\,dy\,dz$$

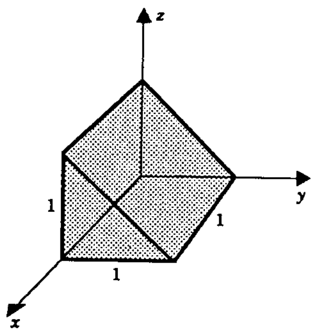
$$=k\left(\int_0^a x(x-a)^2\,dx\right)\left(\int_0^b y\,dy\right)\left(\int_0^c z\,dz\right)+k\left(\int_0^a x\,dx\right)\left(\int_0^b y(y-b)^2\,dy\right)\left(\int_0^c z\,dz\right)$$

$$=\frac{1}{48}ka^2b^2c^2(a^2+b^2).$$

Since $M = \frac{1}{8}ka^2b^2c^2$, $I = \frac{1}{6}M(a^2+b^2)$.

(b) $I_M - \dfrac{1}{18}(a^2+b^2)$ by parallel axis theorem

17.



18. $V = \iiint\limits_T dx\,dy\,dz = \int_0^1\int_0^1\int_0^{1-y} dz\,dy\,dx = \dfrac{1}{2}$

19. center of mass is the centroid

$$\overline{x} = \tfrac{1}{2} \quad \text{by symmetry}$$

$$\overline{y}V = \iiint\limits_T y\,dxdydz = \int_0^1\int_0^1\int_0^{1-y} y\,dz\,dy\,dx = \int_0^1\int_0^1 \left(y-y^2\right)dy\,dx$$
$$= \int_0^1\left[\frac{1}{2}y^2 - \frac{1}{3}y^3\right]_0^1 dx = \int_0^1 \frac{1}{6}\,dx = \frac{1}{6}$$

$$\overline{z}V = \iiint\limits_T z\,dxdydz = \int_0^1\int_0^1\int_0^{1-y} z\,dz\,dy\,dx = \int_0^1\int_0^1 \frac{1}{2}(1-y)^2\,dy\,dx$$
$$= \frac{1}{2}\int_0^1\int_0^1 \left(1-2y+y^2\right)dy\,dx = \frac{1}{2}\int_0^1\left[y - y^2\frac{1}{3}y^3\right]_0^1 dx = \frac{1}{2}\int_0^1 \frac{1}{3}\,dx = \frac{1}{6}$$

$V = \frac{1}{2}$ (by Exercise 18); $\overline{y} = \frac{1}{3}$, $\overline{z} = \frac{1}{3}$

20. $I_x = \iiint\limits_T \dfrac{M}{V}(y^2+z^2)\,dx\,dy\,dz = \dfrac{1}{3}M$

$I_y = \iiint\limits_T \dfrac{M}{V}(x^2+z^2)\,dx\,dy\,dz = \dfrac{1}{2}M$

$I_x = \iiint\limits_T \dfrac{M}{V}(x^2+y^2)\,dx\,dy\,dz = \dfrac{1}{2}M$

21. $\displaystyle\int_{-r}^{r}\int_{-\phi(x)}^{\phi(x)}\int_{-\psi(x,y)}^{\psi(x,y)} k\left(r - \sqrt{x^2+y^2+z^2}\right)dz\,dy\,dx$ with $\phi(x) = \sqrt{r^2-x^2}$,

$\psi(x,y) = \sqrt{r^2-(x^2+y^2)}$, k the constant of proportionality

22. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} k\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$

23. $\displaystyle\int_{0}^{1}\int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}}\int_{-2x-3y-10}^{1-y^2} dz\,dy\,dx$

24. $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{1-x^2/2}}^{\sqrt{1-x^2/2}}\int_{2+y^2}^{4-x^2-y^2} dz\,dy\,dx$

25. $\displaystyle\int_{-1}^{1}\int_{-2\sqrt{2-2x^2}}^{2\sqrt{2-2x^2}}\int_{3x^2+y^2/4}^{4-x^2-y^2/4} k\left(z-3x^2-\frac{1}{4}y^2\right)\,dz\,dy\,dx$

26. $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}}\int_{z^2+2y^2}^{4-z^2} k\sqrt{x^2+y^2}\,dx\,dz\,dy$

27. $\displaystyle\iiint_T (x^2z+y)\,dx\,dy\,dz = \int_0^2\int_1^3\int_0^1 (x^2z+y)\,dx\,dy\,dz = \int_0^2\int_1^3\left[\frac{1}{3}x^3z+xy\right]_0^1 dy\,dz$

$$= \int_0^2\int_1^3\left(\frac{1}{3}z+y\right)dy\,dz = \int_0^2\left[\frac{1}{3}zy+\frac{1}{2}y^2\right]_1^3 dz = \int_0^2\left(\frac{2}{3}z+4\right)dz = \frac{28}{3}$$

28. $\displaystyle\int_0^1\int_0^y\int_0^{x+y} 2ye^x\,dz\,dx\,dy = \int_0^1\int_0^y 2y(x+y)e^x\,dx\,dy = \int_0^1 (4y^2e^y-2ye^y+2y-2y^2)\,dy = 4e-\frac{29}{3}$

29. $\displaystyle\iiint_T x^2y^2z^2\,dx\,dy\,dz = \int_{-1}^0\int_0^{y+1}\int_0^1 x^2y^2z^2\,dx\,dz\,dy + \int_0^1\int_0^{1-y}\int_0^1 x^2y^2z^2\,dx\,dz\,dy$

$$= \int_{-1}^0\int_0^{y+1}\left[\frac{1}{3}x^3y^2z^2\right]_0^1 dz\,dy + \int_0^1\int_0^{1-y}\left[\frac{1}{3}x^3y^2z^2\right]_0^1 dz\,dy$$

$$= \frac{1}{3}\int_{-1}^0\int_0^{y+1} y^2z^2\,dz\,dy + \frac{1}{3}\int_0^1\int_0^{1-y}\left[y^2z^2\right]_0^1 dz\,dy$$

$$= \frac{1}{3}\int_{-1}^0\left[\frac{1}{3}y^2z^3\right]_0^{y+1} dy + \frac{1}{3}\int_0^1\left[\frac{1}{3}y^2z^3\right]_0^{1-y} dy$$

$$= \frac{1}{9}\int_{-1}^0 (y^5+3y^4+3y^3+y^2)\,dy + \frac{1}{9}\int_0^1 (y^2-3y^3+3y^4-y^5)\,dy = \frac{1}{270}$$

30. $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} xy\,dz\,dy\,dx = \int_0^2\int_0^{\sqrt{4-x^2}} xy\sqrt{4-x^2-y^2}\,dy\,dx$

$$= \int_0^{\pi/2}\int_0^2 r^2\cos\theta\sin\theta\sqrt{4-r^2}\,r\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^2 r^3\sqrt{4-r^2}\,dr = \frac{1}{4}\int_0^4 (4\sqrt{u}-u^{3/2})\,du = \frac{32}{15}$$

31. $$\iiint_T y^2\,dx\,dy\,dz = \int_0^3 \int_0^{2-2x/3} \int_0^{6-2x-3y} y^2\,dz\,dy\,dx = \int_0^3 \int_0^{2-2x/3} \left[y^2 z\right]_0^{6-2x-3y} dy\,dx$$

$$= \int_0^3 \int_0^{2-2x/3} (6y^2 - 2xy^2 - 3y^3)\,dy\,dx$$

$$= \int_0^3 \left[2y^3 - \frac{2}{3}xy^3 - \frac{3}{4}y^4\right]_0^{2-2x/3} dx$$

$$= \frac{1}{4}\int_0^3 \left(2 - \frac{2}{3}x\right) dx = \frac{12}{5}$$

32. $$\int_0^1 \int_0^{1-x^2} \int_0^{\sqrt{1-y}} y^2\,dz\,dy\,dx = \int_0^1 \int_0^{1-x^2} y^2\sqrt{1-y}\,dy\,dx$$

$$= \int_0^1 \left[-\frac{2}{3}x^3 + \frac{4}{5}x^5 - \frac{2}{7}x^7 + \frac{16}{105}\right] dx = \frac{1}{12}$$

33. $$V = \int_0^2 \int_{x^2}^{x+2} \int_0^x dz\,dy\,dx = \int_0^2 \int_{x^2}^{x+2} x\,dy\,dx = \int_0^2 \left(x^2 + 2x - x^3\right) dx = \frac{8}{3}$$

$$\overline{x}V = \int_0^2 \int_{x^2}^{x+2} \int_0^x x\,dz\,dy\,dx = \int_0^2 \int_{x^2}^{x+2} x^2\,dy\,dx = \int_0^2 \left(x^3 + 2x^2 - x^4\right) dx = \frac{44}{15}$$

$$\overline{y}V = \int_0^2 \int_{x^2}^{x+2} \int_0^x y\,dz\,dy\,dx = \int_0^2 \int_{x^2}^{x+2} xy\,dy\,dx = \int_0^2 \frac{1}{2}\left(x^3 + 4x^2 + 4x - x^5\right) dx = 6$$

$$\overline{z}V = \int_0^2 \int_{x^2}^{x+2} \int_0^x z\,dz\,dy\,dx = \int_0^2 \int_{x^2}^{x+2} \frac{1}{2}x^2\,dy\,dx = \int_0^2 \frac{1}{2}\left(x^3 + 2x^2 - x^4\right) dx = \frac{22}{15}$$

$$\overline{x} = \frac{11}{10}, \quad \overline{y} = \frac{9}{4}, \quad \overline{z} = \frac{11}{20}$$

34. (a) $M = \int_0^1 \int_0^1 \int_0^1 kz\,dx\,dy\,dz = \frac{1}{2}k$

(b) $M = \int_0^1 \int_0^1 \int_0^1 k(x^2 + y^2 + z^2)\,dz\,dy\,dx = k$

35. $V = \int_{-1}^2 \int_0^3 \int_{2-x}^{4-x^2} dz\,dy\,dx = \frac{27}{2}; \quad (\overline{x}, \overline{y}, \overline{z}) = \left(\frac{1}{2}, \frac{3}{2}, \frac{12}{5}\right)$

36. $$\iiint_T (x - \overline{x})\,dx\,dy\,dz = \iiint_T x\,dx\,dy\,dz - \overline{x}\iiint_T dx\,dy\,dz = \overline{x}V - \overline{x}V = 0$$

similarly the other two integrals are zero.

37. $V=\int_0^a\int_0^{\phi(x)}\int_0^{\psi(x,y)}dz\,dy\,dx=\frac{1}{6}abc$ with $\phi(x)=b\left(1-\frac{x}{a}\right),\quad \psi(x,y)=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

$(\overline{x},\,\overline{y},\,\overline{z})=\left(\frac{1}{4}a,\,\frac{1}{4}b,\,\frac{1}{4}c\right)$

38. $I_z=\iiint_T\frac{M}{V}(x^2+y^2)\,dx\,dy\,dz=\frac{1}{30}\left(\frac{M}{V}\right)=\frac{1}{5}M$

39. $\Pi: 0\le x\le a,\quad 0\le y\le b,\quad 0\le z\le c$

(a) $I_z=\int_0^a\int_0^b\int_0^c\frac{M}{abc}\left(x^2+y^2\right)\,dz\,dy\,dx=\frac{1}{3}M\left(a^2+b^2\right)$

(b) $I_M=I_z-d^2M=\frac{1}{3}M\left(a^2+b^2\right)-\frac{1}{4}\left(a^2+b^2\right)M=\frac{1}{12}M\left(a^2+b^2\right)$

parallel axis theorem (16.5.7)

(c) $I=I_M+d^2M=\frac{1}{12}M\left(a^2+b^2\right)+\frac{1}{4}a^2M=\frac{1}{3}Ma^2+\frac{1}{12}Mb^2$

parallel axis theorem (16.5.7)

40. $V=\int_1^2\int_1^2\int_{-2}^{1+x+y}dz\,dy\,dx=\int_1^2\int_1^2(3+x+y)\,dy\,dx=6$

$\overline{x}V=\int_1^2\int_1^2\int_{-2}^{1+x+y}x\,dz\,dy\,dx=\frac{109}{12}\implies\overline{x}=\frac{109}{72}=\overline{y}$ by symmetry

$\overline{z}V=\int_1^2\int_1^2\int_{-2}^{1+x+y}z\,dz\,dy\,dx=\frac{73}{12}\implies\overline{z}=\frac{73}{72}.$

41.
$$M=\int_0^1\int_0^1\int_0^y k\left(x^2+y^2+z^2\right)\,dz\,dy\,dx=\int_0^1\int_0^1 k\left(x^2y+y^3+\frac{1}{3}y^3\right)dy\,dx$$
$$=\int_0^1 k\left(\frac{1}{2}x^2+\frac{1}{3}\right)dx=\frac{1}{2}k$$

$(x_M,\,y_M,\,z_M)=\left(\frac{7}{12},\,\frac{34}{45},\,\frac{37}{90}\right)$

42. T is symmetric (a) about the yz-plane, (b) about the xz-plane, (c) about the xy-plane,

(d) about the origin.

43. (a) 0 by symmetry

(b) $\iiint\limits_T (a_1 x + a_2 y + a_3 z + a_4)\, dxdydz = \iiint\limits_T a_4\, dxdydz = a_4$ (volume of ball) $= \frac{4}{3}\pi a_4$

by symmetry

44. $\displaystyle\int_0^2 \int_0^2 \int_{2-y}^{4-y^2} x^2y^2\, dz\, dy\, dx = \frac{352}{45}$

45. $\displaystyle V = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\, dy\, dx = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx$

polar coordinates

$$= 8\int_0^{\pi/2} \int_0^a \sqrt{a^2-r^2}\, r\, dr\, d\theta$$

$$= -4\int_0^{\pi/2} \left[\frac{2}{3}(a^2-r^2)^{3/2}\right]_0^a d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2} d\theta = \frac{4}{3}\pi a^3$$

46. $\displaystyle 8\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} dz\, dy\, dx = \frac{4}{3}\pi abc.$

47. $\displaystyle M = \int_{-2}^2 \int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2} \int_{x^2+3y^2}^{4-y^2} k|x|\, dz\, dy\, dx = 4\int_0^2 \int_0^{\sqrt{4-x^2}/2} \int_{x^2+3y^2}^{4-y^2} kx\, dz\, dy\, dx$

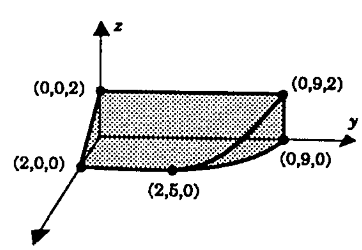
$$= 4\,k\int_0^2 \int_0^{\sqrt{4-x^2}/2} \left(4x - x^3 - 4xy^2\right) dy\, dx = \frac{4}{3}\,k\int_0^2 x\left(4-x^2\right)^{3/2} dx = \frac{128}{15}\,k$$

48. using polar coordinates $\displaystyle V = 2\int_0^{2\pi} \int_0^1 (r - r^3)\, dr\, d\theta = \pi$

49. $\displaystyle M = \int_{-1}^2 \int_0^3 \int_{2-x}^{4-x^2} k(1+y)\, dz\, dy\, dx = \frac{135}{4}k;\quad (x_M, y_M, z_M) = \left(\frac{1}{2}, \frac{9}{5}, \frac{12}{5}\right)$

50. (a) $\displaystyle V = \int_0^2 \int_0^{2-z} \int_0^{9-x^2} dy\, dx\, dz$

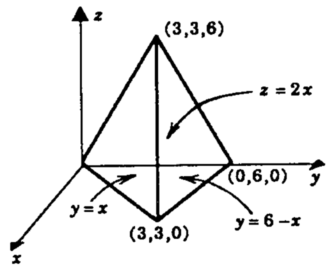
(b) $\displaystyle V = \int_0^2 \int_0^{2-x} \int_0^{9-x^2} dy\, dz\, dx$



(c) $\displaystyle V = \int_0^5 \int_0^2 \int_0^{2-x} dz\, dx\, dy + \int_5^9 \int_0^{\sqrt{9-y}} \int_0^{2-x} dz\, dx\, dy$

51. (a) $V = \int_0^6 \int_{z/2}^3 \int_x^{6-x} dy\, dx\, dz$

(b) $V = \int_0^3 \int_0^{2x} \int_x^{6-x} dy\, dz\, dx$



(c) $V = \int_0^6 \int_{z/2}^3 \int_{z/2}^y dx\, dy\, dz + \int_0^6 \int_3^{(12-z)/2} \int_{z/2}^{6-y} dx\, dy\, dz$

52. (a) $V = \iint_{\Omega_{xy}} 2\sqrt{4-y}\, dx\, dy$

(b) $V = \iint_{\Omega_{xy}} \left(\int_{-\sqrt{4-y}}^{\sqrt{4-y}} dz \right) dx\, dy$

(c) $V = \int_{-4}^4 \int_{|x|}^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} dz\, dy\, dx$

(d) $V = \int_0^4 \int_{-y}^y \int_{-\sqrt{4-y}}^{\sqrt{4-y}} dz\, dx\, dy$

53. (a) $V = \iint_{\Omega_{yz}} 2y\, dydz$

(b) $V = \iint_{\Omega_{yz}} \left(\int_{-y}^y dx \right) dydz$

(c) $V = \int_0^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} \int_{-y}^y dx\, dz\, dy$

(d) $V = \int_{-2}^2 \int_0^{4-z^2} \int_{-y}^y dx\, dy\, dz$

54. (a) $V = \iint_{\Omega_{xz}} (4 - z^2 - |x|)\, dx\, dz$

(b) $V = \iint_{\Omega_{xz}} \left(\int_{|x|}^{4-z^2} dy \right) dx\, dz$

(c) $V = \int_{-2}^2 \int_{z^2-4}^{4-z^2} \int_{|x|}^{4-z^2} dy\, dx\, dz$

(d) $V = \int_{-2}^0 \int_{-\sqrt{4+x}}^{\sqrt{4+x}} \int_{|x|}^{4-z^2} dy\, dz\, dx + \int_0^2 \int_{-\sqrt{4-x}}^{\sqrt{4-x}} \int_{|x|}^{4-z^2} dy\, dz\, dx$

SECTION 16.8

1. $r^2 + z^2 = 9$

2. $r = 2$

3. $z = 2r$

4. $r\cos\theta = 4z$

5. $4r^2 = z^2$

6. $r^2 \sin^2\theta + z^2 = 8$

7. $\int_0^\pi \int_0^2 \int_0^{4-r^2} r\, dz\, dr\, d\theta = \int_0^\pi \int_0^2 (4r - r^2)\, dr\, d\theta = \int_0^\pi 4\, d\theta = 4\pi$

8. $\int_0^{\pi/4} \int_0^1 \int_0^{\sqrt{1-r^2}} r\cos\theta\, dz\, dr\, d\theta = \int_0^{\pi/4} \int_0^1 r\sqrt{1-r^2}\cos\theta\, dr\, d\theta = \frac{\sqrt{2}}{2} \int_0^1 r\sqrt{1-r^2}\, dr = \frac{\sqrt{2}}{6}$

9. $$\int_0^{\pi/4}\int_0^1\int_0^{r\cos\theta} r\sec^3\theta\,dz\,dr\,d\theta = \int_0^{\pi/4}\int_0^1 r^2\sec^2\theta\,dr\,d\theta = \frac{1}{3}\int_0^{\pi/4}\sec^2\theta\,d\theta = \frac{1}{3}$$

10. $$\int_0^{\pi}\int_0^{4\cos\theta}\int_0^{\sqrt{16-r^2}} r\,dz\,dr\,d\theta = \int_0^{\pi}\int_0^{4\cos\theta} r\sqrt{16-r^2}\,dr\,d\theta = \int_0^{\pi}\frac{64}{3}(1-\sin^3\theta)\,d\theta = \frac{64}{3}\pi - \frac{4}{3}$$

11. Set the lower base of the cylinder on the xy-plane so that the axis of the cylinder coincides with the z-axis. Assume that the density varies directly as the distance from the lower base.

$$M = \int_0^{2\pi}\int_0^R\int_0^h kzr\,dz\,dr\,d\theta = \frac{1}{2}k\pi R^2h^2$$

12. $x_M = y_M = 0$ by symmetry

$$z_M M = \int_0^{2\pi}\int_0^R\int_o^h kz^2r\,dz\,dr\,d\theta = \frac{1}{3}k\pi R^2h^3$$

$$M = \frac{1}{2}k\pi R^2h^2, \quad z_M = \frac{2}{3}h$$

The center of mass lies on the axis of the cylinder at a distance $\frac{2}{3}h$ from the base of zero mass density.

13. $$I = I_z = k\int_0^{2\pi}\int_0^R\int_0^h zr^3\,dr\,d\theta\,dz$$

$$= \tfrac{1}{4}k\pi R^4h^2 = \tfrac{1}{2}\left(\tfrac{1}{2}k\pi R^2h^2\right)R^2 = \tfrac{1}{2}MR^2$$

from Exercise 11

14. (a) $$I = \frac{M}{\pi r^2 h}\int_0^{2\pi}\int_0^R\int_0^h r^3\,dz\,dr\,d\theta = \frac{1}{2}MR^2$$

(b) $$I = \frac{M}{\pi r^2 h}\int_0^{2\pi}\int_0^R\int_0^h (r^2\sin^2\theta + z^2)r\,dz\,dr\,d\theta = \frac{1}{4}MR^2 + \frac{1}{3}Mh^2$$

(c) $$I = \tfrac{1}{4}MR^2 + \tfrac{1}{3}Mh^2 - M(\tfrac{1}{2}h)^2 = \tfrac{1}{4}MR^2 + \tfrac{1}{12}Mh^2$$

15. Inverting the cone and placing the vertex at the origin, we have

$$V = \int_0^h\int_0^{2\pi}\int_0^{(R/h)z} r\,dr\,d\theta\,dz = \frac{1}{3}\pi R^2h.$$

16. $x_M = y_M = 0$ by symmetry

$$z_M M = \int_0^h\int_0^{2\pi}\int_0^{(R/h)^2}\left(\frac{M}{V}\right)zr\,dr\,d\theta\,dz = \left(\frac{M}{V}\right)\frac{\pi R^2h^2}{4} \implies z_M = \frac{\pi R^2h^2}{4V} = \frac{3}{4}h$$

On the axis of the cone at a distance $\frac{3}{4}h$ from the vertex.

17. $I = \frac{M}{V}\int_0^h \int_0^{2\pi} \int_0^{(R/h)z} r^3\,dr\,d\theta\,dz = \frac{3}{10}MR^2$

18. $I = \frac{M}{V}\int_0^h \int_0^{2\pi} \int_0^{(R/h)/z} (r^2\sin^2\theta + z^2)\,r\,dr\,d\theta\,dz = \frac{3}{20}MR^2 + \frac{3}{5}Mh^2.$

19. $V = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} r\,dz\,dr\,d\theta = \frac{1}{2}\pi$

20. $M = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} kzr\,dz\,dr\,d\theta = \frac{1}{6}\pi k$

21. $M = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} k(r^2+z^2)r\,dz\,dr\,d\theta = \frac{1}{4}k\pi$

22. $\int_0^{\pi}\int_0^1\int_r^1 z^3 r\,dz\,dr\,d\theta = \int_0^{\pi}\int_0^1 \frac{1}{4}(1-r^4)\,r\,dr\,d\theta = \frac{\pi}{12}$

23. $\int_0^{\pi/2}\int_0^1\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 r\sqrt{4-r^2}\,dr\,d\theta = \int_0^{\pi/2}\left(\frac{8}{3}-\sqrt{3}\right)d\theta = \frac{1}{6}\left(8-3\sqrt{3}\right)\pi$

24. $\int_0^{\pi/2}\int_0^1\int_0^{\sqrt{1-r^2}} zr\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 \frac{r}{2}(1-r^2)\,dr\,d\theta = \frac{\pi}{16}$

25.
$$\int_0^3\int_0^{\sqrt{9-y^2}}\int_0^{\sqrt{9-x^2y^2}} \frac{1}{\sqrt{x^2+y^2}}\,dz\,dx\,dy = \int_0^{\pi/2}\int_0^3\int_0^{\sqrt{9-r^2}} \frac{1}{r}\cdot r\,dz\,dr\,d\theta$$
$$= \int_0^{\pi/2}\int_0^3 \sqrt{9-r^2}\,dr\,d\theta$$
$$= \int_0^{\pi/2}\left[\frac{r}{2}\sqrt{9-r^2} + \frac{9}{2}\sin^{-1}\frac{r}{3}\right]_0^3 d\theta$$
$$= \frac{9\pi}{4}\int_0^{\pi/2} d\theta = \frac{9}{8}\pi^2$$

26. $\int_0^{2\pi}\int_0^1\int_{r^2}^{2-r^2} r^2 dz\,dr\,d\theta = 4\pi\int_0^1 (1-r^2)r^2\,dr = \frac{8\pi}{15}$

27. $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^2 \sin(x^2+y^2)\,dz\,dy\,dx = \int_0^{\pi/2}\int_0^1\int_0^2 \sin(r^2)r\,dz\,dr\,d\theta = 2\int_0^{\pi/2}\int_0^1 r\,\sin(r^2)\,dr\,d\theta$

$$= 2\int_0^{\pi/2}\left[-\frac{1}{2}\cos(r^2)\right]_0^1 d\theta = (1-\cos 1)\int_0^{\pi/2} d\theta = \frac{\pi}{2}(1-\cos 1) \cong 0.7221$$

28. $V = \int_{-\pi/2}^{\pi/2}\int_0^{2a\cos\theta}\int_0^{r^2/a} r\,dz\,dr\,d\theta = \frac{3}{2}\pi a^3$

29. $$V = \int_{-\pi/2}^{\pi/2} \int_0^{2a\cos\theta} \int_0^r r\,dz\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \int_0^{2a\cos\theta} r^2\,dr\,d\theta$$
$$= \int_{-\pi/2}^{\pi/2} \frac{8}{3}a^3\cos^3\theta\,d\theta = \frac{32}{9}a^3$$

30. $$V = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \int_0^{2+\frac{1}{2}r\cos\theta} r\,dz\,dr\,d\theta = \frac{5}{2}\pi$$

31. $$V = \int_{-\pi/2}^{\pi/2} \int_0^{a\cos\theta} \int_0^{a-r} r\,dz\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \int_0^{a\cos\theta} r(a-r)\,dr\,d\theta$$
$$= \int_{-\pi/2}^{\pi/2} a^3\left(\frac{1}{2}\cos^2\theta - \frac{1}{3}\cos^3\theta\right)d\theta = \frac{1}{36}a^2(9\pi - 16)$$

32. $$V = \int_0^{2\pi} \int_0^3 \int_{r+1}^{\sqrt{25-r^2}} r\,dz\,dr\,d\theta = \frac{41}{3}\pi$$

33. $$V = \int_{-\pi/2}^{\pi/2} \int_0^{\cos\theta} \int_{r^2}^{r\cos\theta} r\,dz\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \int_0^{\cos\theta} \left(r^2\cos\theta - r^3\right)dr\,d\theta$$
$$= \int_{-\pi/2}^{\pi/2} \frac{1}{12}\cos^4\theta\,d\theta = \frac{1}{32}\pi$$

34. $$V = \int_0^{2\pi} \int_0^a \int_{\sqrt{2}r}^{\sqrt{a^2+r^2}} r\,dz\,dr\,d\theta = \frac{2\pi}{3}a^3(\sqrt{2}-1)$$

35. $$V = \int_0^{2\pi} \int_0^{1/2} \int_{r\sqrt{3}}^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^{1/2} \left(r\sqrt{1-r^2} - r^2\sqrt{3}\right)dr\,d\theta = \frac{1}{3}\pi\left(2-\sqrt{3}\right)$$

36. $$V = \int_0^{2\pi} \int_1^2 \int_0^{\frac{1}{2}\sqrt{36-r^2}} r\,dz\,dr\,d\theta = \frac{1}{3}\pi(35\sqrt{35} - 128\sqrt{2}).$$

SECTION 16.9

1. $\left(\sqrt{3},\ \frac{1}{4}\pi,\ \cos^{-1}\left[\frac{1}{3}\sqrt{3}\right]\right)$

2. $\left(\frac{1}{2}\sqrt{6},\ \frac{1}{2}\sqrt{2},\ \sqrt{2}\right)$

3. $\left(\frac{3}{4},\ \frac{3}{4}\sqrt{3},\ \frac{3}{2}\sqrt{3}\right)$

4. $\left(2\sqrt{10},\ \frac{2}{3}\pi,\ \cos^{-1}\left[\frac{3}{10}\sqrt{10}\right]\right)$

5. $\rho = \sqrt{2^2 + 2^2 + (2\sqrt{6}/3)^2} = \dfrac{4\sqrt{6}}{3}$

$\phi = \cos^{-1}\left(\dfrac{2\sqrt{6}/3}{4\sqrt{6}/3}\right) = \cos^{-1}(1/2) = \dfrac{\pi}{3}$

$\theta = \tan^{-1}(1) = \dfrac{\pi}{4}$

$(\rho, \theta, \phi) = \left(\dfrac{4\sqrt{6}}{3}, \dfrac{\pi}{4}, \dfrac{\pi}{3}\right)$

6. $\left(8, -\dfrac{\pi}{4}, \dfrac{2\pi}{3}\right)$

7. $x = \rho \sin\phi \cos\theta = 3 \sin 0 \cos(\pi/2) = 0$

$y = \rho \sin\phi \sin\theta = 3 \sin 0 \sin(\pi/2) = 0$

$z = \rho \cos\phi = 3\cos 0 = 3$

$(x, y, z) = (0, 0, 3)$

8. $(1, \pi/4, \pi/4)$

9. The circular cylinder $x^2 + y^2 = 1$; the radius of the cylinder is 1 and the axis is the z-axis.

10. The xy-plane.

11. The lower nappe of the circular cone $z^2 = x^2 + y^2$.

12. Vertical plane which bisects the first and third quadrants of the xy-plane.

13. Horizontal plane one unit above the xy-plane.

14. sphere $x^2 + y^2 + (z - \frac{1}{2})^2 = \frac{1}{4}$ of radius $\frac{1}{2}$ and center $(0, 0, \frac{1}{2})$

15. $\displaystyle\int_0^{\pi/3}\int_0^{2\pi}\int_0^1 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = \frac{1}{3}\int_0^{\pi/3}\int_0^{2\pi} \sin\phi \, d\theta \, d\phi = \frac{2\pi}{3}\int_0^{\pi/3} \sin\phi \, d\phi = \frac{\pi}{3}$

16. $\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^2 \rho^3 \sin\phi \, d\rho \, d\phi \, d\theta = 8\pi \int_0^{\pi} \sin\phi d\phi = 16\pi$

17.
$$\begin{aligned}\int_0^{\pi/4}\int_0^{\pi}\int_0^{2\cos\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi &= \frac{8}{3}\int_0^{\pi/4}\int_0^{\pi} \cos^3\phi \sin\phi \, d\theta \, d\phi \\ &= \frac{8}{3}\pi \int_0^{\pi/4} \cos^3\phi \sin\phi \, d\phi \\ &= \frac{8}{3}\pi \left[-\frac{1}{4}\cos^4\phi\right]_0^{\pi/4} = \frac{\pi}{2}\end{aligned}$$

18. $\displaystyle\int_0^{\pi/4}\int_0^{2\pi}\int_0^{\sec\phi} \rho^3 \cos\phi \sin\phi \, d\rho \, d\theta \, d\phi = \frac{\pi}{2}\int_0^{\pi/4} \sec^4\phi \cos\phi \sin\phi \, d\phi = \frac{3}{4}\pi$

19.
$$\begin{aligned}\int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} dz \, dy \, dx &= \int_{\pi/4}^{\pi/2}\int_0^{\pi/2}\int_0^{\sqrt{2}} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\ &= \frac{2}{3}\sqrt{2}\int_{\pi/4}^{\pi/2}\int_0^{\pi/2} \sin\phi \, d\theta \, d\phi \\ &= \frac{\sqrt{2}}{3}\pi \int_{\pi/4}^{\pi/2} \sin\phi \, d\phi = \frac{\pi}{3}\end{aligned}$$

20. $\displaystyle\int_0^{\pi/4}\int_0^{\pi/2}\int_0^2 \rho^4 \sin\phi\, d\rho\, d\theta\, d\phi = \frac{16\pi}{5}\int_0^{\pi/4}\sin\phi\, d\phi = \frac{8\pi}{5}(2-\sqrt{2})$

21. $\displaystyle\int_0^3\int_0^{\sqrt{9-y^2}}\int_0^{\sqrt{9-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\, dz\, dx\, dy$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 \rho\cos\phi\cdot\rho\cdot\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi/2}\frac{1}{2}\sin 2\phi\, d\phi\int_0^{\pi/2}d\theta\int_0^3\rho^4\, d\rho = \left[-\frac{1}{4}\cos 2\phi\right]_0^{\pi/2}\left(\frac{\pi}{2}\right)\left[\frac{1}{5}\rho^5\right]_0^3$$

$$= \frac{243\pi}{20}$$

22. $\displaystyle\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \frac{1}{\rho^2}\rho^2\sin\phi\, d\rho\, d\theta\, d\phi = \frac{\pi}{2}\int_0^{\pi/2}\sin\phi\, d\phi = \frac{\pi}{2}$

23. $\displaystyle V = \int_0^{2\pi}\int_0^{\pi}\int_0^R \rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \frac{4}{3}\pi R^3$

24. $r = \rho\sin\phi, \quad \theta = \theta, \quad z = \rho\cos\phi$

25. $\displaystyle V = \int_0^{\alpha}\int_0^{\pi}\int_0^R \rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \frac{2}{3}\alpha R^3$

26. $\displaystyle M = \int_0^{2\pi}\int_0^{\pi}\int_0^R k(R-\rho)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \frac{1}{3}k\pi R^4$

27. $\displaystyle M = \int_0^{2\pi}\int_0^{\tan^{-1}(r/h)}\int_0^{h\sec\phi} k\rho^3\sin\phi\, d\rho\, d\phi\, d\theta$

$$= \int_0^{2\pi}\int_0^{\tan^{-1}(r/h)}\frac{kh^4}{4}\tan\phi\sec^3\phi\, d\phi\, d\theta$$

$$= \frac{kh^4}{4}\int_0^{2\pi}\frac{1}{3}\left[\sec^3\phi\right]_0^{\tan^{-1}(r/h)}d\theta = \frac{kh^4}{4}\int_0^{2\pi}\frac{1}{3}\left[\left(\frac{\sqrt{r^2+h^2}}{h}\right)^3 - 1\right]d\theta$$

$$= \frac{1}{6}k\pi h\left[\left(r^2+h^2\right)^{3/2} - h^3\right]$$

28. $\displaystyle V = \int_0^{2\pi}\int_0^{\tan^{-1} r/h}\int_0^{h\sec\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \frac{2\pi}{3}\int_0^{\tan^{-1}(r/h)} h^3\tan\phi\sec^2\phi\, d\phi$

$$= \frac{\pi}{3}h^3\tan^2(\tan^{-1}\left(\frac{r}{h}\right)) = \frac{1}{3}\pi r^2 h$$

29. center ball at origin; density $= \dfrac{M}{V} = \dfrac{3M}{4\pi R^3}$

(a) $\displaystyle I = \frac{3M}{4\pi R^3}\int_0^{2\pi}\int_0^{\pi}\int_0^R \rho^4\sin^3\phi\, d\rho\, d\phi\, d\theta = \frac{2}{5}MR^2$

(b) $I = \frac{2}{5}MR^2 + R^2M = \frac{7}{5}MR^2$ (parallel axis theorem)

30. The center of mass is the centroid; $V = \frac{2}{3}\pi R^3$

$$\bar{z}V = \int_0^{2\pi}\int_0^{\pi/2}\int_0^R (\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \tfrac{1}{4}\pi R^4$$

$\bar{z} = \frac{3}{8}R;\qquad (\bar{x},\bar{y},\bar{z}) = (0,0,\frac{3}{8}R)$

31. center balls at origin; density $= \dfrac{M}{V} = \dfrac{3M}{4\pi\left(R_2^{\,3} - R_1^{\,3}\right)}$

(a) $$I = \frac{3M}{4\pi\left(R_2^{\,3}-R_1^{\,3}\right)}\int_0^{2\pi}\int_0^{\pi}\int_{R_1}^{R_2}\rho^4\sin^3\phi\,d\rho\,d\phi\,d\theta = \frac{2}{5}M\left(\frac{R_2^{\,5}-R_1^{\,5}}{R_2^{\,3}-R_1^{\,3}}\right)$$

This result can be derived from Exercise 29 without further integration. View the solid as a ball of mass M_2 from which is cut out a core of mass M_1.

$$M_2 = \frac{M}{V}V_2 = \frac{3M}{4\pi\left(R_2^{\,3}-R_1^{\,3}\right)}\left(\frac{4}{3}\pi R_2^{\,3}\right) = \frac{MR_2^{\,3}}{R_2^{\,3}-R_1^{\,3}};\quad \text{similarly}\quad M_1 = \frac{MR_1^{\,3}}{R_2^{\,3}-R_1^{\,3}}.$$

Then

$$I = I_2 - I_1 = \frac{2}{5}M_2R_2^{\,2} - \frac{2}{5}M_1R_1^{\,2} = \frac{2}{5}\left(\frac{MR_2^{\,3}}{R_2^{\,3}-R_1^{\,3}}\right)R_2^{\,2} - \frac{2}{5}\left(\frac{MR_1^{\,3}}{R_2^{\,3}-R_1^{\,3}}\right)R_1^{\,2}$$

$$= \frac{2}{5}M\left(\frac{R_2^{\,5}-R_1^{\,5}}{R_2^{\,3}-R_1^{\,3}}\right).$$

(b) Outer radius R and inner radius R_1 gives

$$\text{moment of inertia} = \frac{2}{5}M\left(\frac{R^5-R_1^{\,5}}{R^3-R_1^{\,3}}\right). \qquad [\text{part } (a)]$$

As $R_1 \to R$,

$$\frac{R^5-R_1^{\,5}}{R^3-R_1^{\,3}} = \frac{R^4+R^3R_1+R^2R_1^{\,2}+RR_1^{\,3}+R_1^{\,4}}{R^2+RR_1+R_1^{\,2}} \longrightarrow \frac{5R^4}{3R^2} = \frac{5}{3}R^2.$$

Thus the moment of inertia of spherical shell of radius R is

$$\tfrac{2}{5}M\left(\tfrac{5}{3}R^2\right) = \tfrac{2}{3}MR^2.$$

(c) $I = \frac{2}{3}MR^2 + R^2M = \frac{5}{3}MR^2$ (parallel axis theorem)

32. (a) The center of mass is the centroid; using the result of Exercise 30,

$\bar{x} = \bar{y} = 0$

$$\bar{z} = \frac{\bar{z}_2V_2 - \bar{z}_1V_1}{V} = \frac{\frac{3}{8}R_2\frac{4}{3}\pi R_2^{\,3} - \frac{3}{8}R_1\frac{4}{3}\pi R_1^{\,3}}{\frac{4}{3}\pi(R_2^{\,3}-R_1^{\,3})} = \frac{3(R_2^{\,2}+R_1^{\,2})(R_2+R_1)}{8(R_2^{\,2}+R_2R_1+R_1^{\,2})}$$

(b) Setting $R_1 = R_2 = R$ in (a), we get $\bar{x} = \bar{y} = 0,\quad \bar{z} = \frac{1}{2}R$

33. $$V = \int_0^{2\pi}\int_0^{\alpha}\int_0^{a}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{2}{3}\pi\,(1-\cos\alpha)\,a^3$$

34. $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^1 e^{\rho^3}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{1}{3}\pi(e-1)\left(2-\sqrt{2}\right)$

35. (a) Substituting $x=\rho\sin\phi\cos\theta,\quad y=\rho\sin\phi\sin\theta,\quad z=\rho\cos\phi$

into $x^2+y^2+(z-R)^2=R^2$

we have $\rho^2\sin^2\phi+(\rho\cos\phi-R)^2=R^2,$

which simplifies to $\rho=2R\cos\phi.$

(b) $0\le\theta\le 2\pi,\quad 0\le\phi\le\pi/4,\quad R\sec\phi\le\rho\le 2R\cos\phi$

36. (a) $\displaystyle M=\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2R\cos\phi} k\rho^3\sin\phi\,d\rho\,d\phi\,d\theta=\frac{8}{5}k\pi R^4$

(b) $\displaystyle M=\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2R\cos\phi} k\rho^3\sin^2\phi\,d\rho\,d\phi\,d\theta=\frac{1}{4}k\pi^2R^4$

(c) $\displaystyle M=\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2R\cos\phi} k\rho^3\cos^2\theta\sin^2\phi\,d\rho\,d\phi\,d\theta=\frac{1}{8}k\pi^2R^4$

37.
$$\begin{aligned}V&=\int_0^{2\pi}\int_0^{\pi/4}\int_0^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta+\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{2\sqrt{2}\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta\\&=\frac{1}{3}\left(16-6\sqrt{2}\right)\pi\end{aligned}$$

38. $\displaystyle V=\int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\frac{8}{3}\pi$

39. Encase T in a spherical wedge W. W has spherical coordinates in a box Π that contains S. Define f to be zero outside of T. Then

$$F(\rho,\theta,\phi)=f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)$$

is zero outside of S and

$$\begin{aligned}\iiint_T f(x,y,z)\,dxdydz&=\iiint_W f(x,y,z)\,dxdydz\\&=\iiint_\Pi F(\rho,\theta,\phi)\,\rho^2\sin\phi\,d\rho d\theta d\phi\\&=\iiint_S F(\rho,\theta,\phi)\,\rho^2\sin\phi\,d\rho d\theta d\phi.\end{aligned}$$

40. Break up T into little basic solids $T_1,\dots,T_N$. Choose a point (x_i^*,y_i^*,z_i^*) from each T_i and view all the mass as concentrated there. Now T_i attracts m with a force

$$F_i\cong-\frac{Gm\lambda(x_i^*,y_i^*,z_i^*)(\text{Volume of }T_i)}{{r_i}^3}\mathbf{r}_i$$

where $\mathbf{r}_i$ is the vector from (x_i^*,y_i^*,z_i^*) to (a,b,c). We therefore have

$$F_i \cong \frac{Gm\lambda(x_i^*, y_i^*, z_i^*)[(x_i^* - a)\mathbf{i} + (y_i^* - b)\mathbf{j} + (z_i^* - c)\mathbf{k}]}{[(x_i^* - a)^2 + (y_i^* - b)^2 + (z_i^* - c)^2]^{3/2}} \quad (\text{Volume of } T_i).$$

The sum of these approximations is a Riemann sum for the triple integral given and tends to that triple integral as the maximum diameter of the T_i tends to zero.

41. T is the set of all (x, y, z) with spherical coordinates (ρ, θ, ϕ) in the set

$$S: \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi/4, \quad R \sec\phi \le \rho \le 2R\cos\phi.$$

T has volume $V = \frac{2}{3}\pi R^3$. By symmetry the $\mathbf{i}, \mathbf{j}$ components of force are zero and

$$\mathbf{F} = \left\{ \frac{3GmM}{2\pi R^3} \iiint_T \frac{z}{(x^2 + y^2 + z^2)^{3/2}}\, dxdydz \right\} \mathbf{k}$$

$$= \left\{ \frac{3GmM}{2\pi R^3} \iiint_S \left(\frac{\rho\cos\phi}{\rho^3}\right) \rho^2 \sin\phi\, d\rho d\theta d\phi \right\} \mathbf{k}$$

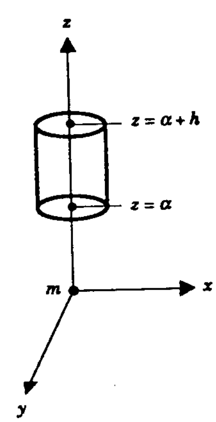
$$= \left\{ \frac{3GmM}{2\pi R^3} \int_0^{2\pi} \int_0^{\pi/4} \int_{R\sec\phi}^{2R\cos\phi} \cos\phi\, \sin\phi\, d\rho\, d\phi\, d\theta \right\} \mathbf{k}$$

$$= \frac{GmM}{R^2}\left(\sqrt{2} - 1\right)\mathbf{k}.$$

42. With the coordinate system shown in the figure, T is the set of all points (x, y, z) with cylindrical coordinates (r, θ, z) in the set

$$S: \quad 0 \le r \le R, \quad 0 \le \theta \le 2\pi, \quad \alpha \le z \le \alpha + h.$$

The gravitational force is



$$F = \left[\iiint_T \left(\frac{GmM}{V}\right) \frac{z}{(x^2 + y^2 + z^2)^{3/2}}\, dxdydz \right] \mathbf{k}$$

$$= \left[\frac{GmM}{\pi R^2 h} \iiint_S \frac{zr}{(r^2 + z^2)^{3/2}}\, dr\, d\theta\, dz \right] \mathbf{k}$$

$$= \left[\frac{GmM}{\pi R^2 h} \int_0^{2\pi} \int_0^R \int_\alpha^{\alpha+h} \frac{zr}{(r^2 + z^2)^{3/2}}\, dzdrd\theta \right] \mathbf{k}$$

$$= \frac{2GmM}{R^2 h} \left(\sqrt{R^2 + \alpha^2} - \sqrt{R^2 + (\alpha + h)^2} + h\right) \mathbf{k}$$

SECTION 16.10

1. $ad - bc$ **2.** 1 **3.** $2\left(v^2 - u^2\right)$

4. $u\ln v - u$ **5.** $u^2v^2 - 4uv$ **6.** $1 + \dfrac{1}{uv}$

7. abc **8.** 2 **9.** r

10. $\rho^2 \sin\phi$ **11.** $w\,(1 + w\cos v)$

12. (a) $dx - by = (ad - bc)u_0$ (b) $cx - ay = (bc - ad)v_0$

13. Set $u = x + y, \quad v = x - y.$ Then

$$x = \frac{u+v}{2}, \quad y = \frac{u-v}{2} \quad \text{and} \quad J(u,\,v) = -\frac{1}{2}.$$

Ω is the set of all (x, y) with uv-coordinates in

$$\Gamma: \quad 0 \le u \le 1, \quad 0 \le v \le 2.$$

Then

$$\iint_{\Omega} \left(x^2 - y^2\right)\,dxdy = \iint_{\Gamma} \frac{1}{2}uv\,dudv = \frac{1}{2}\int_0^1 \int_0^2 uv\,dv\,du$$
$$= \frac{1}{2}\left(\int_0^1 u\,du\right)\left(\int_0^2 v\,dv\right) = \frac{1}{2}\left(\frac{1}{2}\right)(2) = \frac{1}{2}.$$

14. Using the changes of variables from Exercise 13,

$$\iint_{\Omega} 4xy\,dx\,dy = \int_0^1 \int_0^2 4\left(\frac{u^2 - v^2}{4}\right)\frac{1}{2}\,dv\,du = \frac{1}{2}\int_0^1 \int_0^2 \left(u^2 - v^2\right)\,dv\,du = -1$$

15. $$\frac{1}{2}\int_0^1 \int_0^2 u\,\cos(\pi v)\,dv\,du = \frac{1}{2}\left(\int_0^1 u\,du\right)\left(\int_0^2 \cos(\pi v)\,dv\right) = \frac{1}{2}\left(\frac{1}{2}\right)(0) = 0$$

16. Set $u = x - y, \quad v = x + 2y.$ Then

$$x = \frac{2u+v}{3}, \quad y = \frac{v-u}{3}, \quad \text{and} \quad J(u,v) = \frac{1}{3}$$

Ω is the set of all (x, y) with uv-coordinates in the set

$$\Gamma: \quad 0 \le u \le \pi, \quad 0 \le v \le \pi/2.$$

Therefore

$$\iint_{\Omega} (x+y)\,dxdy = \iint_{\Gamma} \frac{1}{9}(u + 2v)\,du\,dv = \frac{1}{9}\int_0^{\pi} \int_0^{\pi/2} (u + 2v)\,dv\,du = \frac{1}{18}\pi^3.$$

17. Set $u = x - y, \quad v = x + 2y.$ Then

$$x = \frac{2u+v}{3}, \quad y = \frac{v-u}{3}, \quad \text{and} \quad J(u,v) = \frac{1}{3}.$$

Ω is the set of all (x, y) with uv-coordinates in the set

$$\Gamma:\ 0 \le u \le \pi, \quad 0 \le v \le \pi/2.$$

Therefore

$$\iint_{\Omega} \sin(x-y)\cos(x+2y)\,dxdy = \iint_{\Gamma} \frac{1}{3}\sin u\cos v\,dudv = \frac{1}{3}\int_0^{\pi}\int_0^{\pi/2} \sin u\ \cos v\,dv\,du$$

$$= \frac{1}{3}\left(\int_0^{\pi}\sin u\,du\right)\left(\int_0^{\pi/2}\cos v\,dv\right) = \frac{1}{3}(2)(1) = \frac{2}{3}.$$

18. Using the change of variables from Exercise 16,

$$\iint_{\Omega} \sin 3x\,dx\,dy = \int_0^{\pi}\int_0^{\pi/2} \sin(2u+v)\frac{1}{3}\,du\,dv = 0.$$

19. Set $u = xy, \quad v = y.$ Then

$$x = u/v, \quad y = v \quad \text{and} \quad J(u,v) = 1/v.$$

$$xy = 1, \qquad xy = 4 \implies u = 1, \qquad u = 4$$
$$y = x, \qquad y = 4x \implies u/v = v, \qquad 4u/v = v \implies v^2 = u, \qquad v^2 = 4u$$

Ω is the set of all (x,y) with uv-coordinates in the set

$$\Gamma:\quad 1 \le u \le 4, \quad \sqrt{u} \le v \le 2\sqrt{u}.$$

(a) $$A = \iint_{\Gamma} \frac{1}{v}\,dudv = \int_1^4\int_{\sqrt{u}}^{2\sqrt{u}} \frac{1}{v}\,dv\,du = \int_1^4 \ln 2\,du = 3\ln 2$$

(b) $$\overline{x}A = \int_1^4\int_{\sqrt{u}}^{2\sqrt{u}} \frac{u}{v^2}\,dv\,du = \frac{7}{3}; \quad \overline{x} = \frac{7}{9\ln 2}$$

$$\overline{y}A = \int_1^4\int_{\sqrt{u}}^{2\sqrt{u}} dv\,du = \frac{14}{3}; \quad \overline{y} = \frac{14}{9\ln 2}$$

20. $J(r,\theta) = abr, \quad \Gamma:\quad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi$

$$A = \iint_{\Gamma} abr\,dr\,d\theta = ab\int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \pi ab$$

21. Set $u = x + y, \quad v = 3x - 2y.$ Then

$$x = \frac{2u+v}{5}, \quad y = \frac{3u-v}{5} \quad \text{and} \quad J(u,v) = -\frac{1}{5}.$$

With $\Gamma:\quad 0 \le u \le 1, \quad 0 \le v \le 2$

$$M = \int_0^1\int_0^2 \frac{1}{5}\lambda\,dv\,du = \frac{2}{5}\lambda \quad \text{where} \quad \lambda \text{ is the density.}$$

Then

$$I_x = \int_0^1 \int_0^2 \left(\frac{3u-v}{5}\right)^2 \frac{1}{5}\lambda\, dv\, du = \frac{8\lambda}{375} = \frac{4}{75}\left(\frac{2}{5}\lambda\right) = \frac{4}{75}M,$$

$$I_y = \int_0^1 \int_0^2 \left(\frac{2u+v}{5}\right)^2 \frac{1}{5}\lambda\, dv\, du = \frac{28\lambda}{375} = \frac{14}{75}\left(\frac{2}{5}\lambda\right) = \frac{14}{75}M,$$

$$I_z = I_x + I_y = \frac{18}{75}M.$$

22. $x = \dfrac{u+v}{2}, \quad y = \dfrac{v-u}{2}, \quad J(u,v) = \dfrac{1}{2} \qquad \Gamma: \quad -2 \le u \le 2, \quad -4 \le v \le -u^2$

$$A = \iint_\Gamma \frac{1}{2}\, du\, dv = \frac{1}{2}\int_{-2}^2 \int_{-4}^{-u^2} dv\, du = \frac{16}{3}$$

23. Set $u = x - 2y, \quad v = 2x + y.$ Then

$$x = \frac{u+2v}{5}, \quad y = \frac{v-2u}{5} \quad \text{and} \quad J(u,v) = \frac{1}{5}.$$

Γ is the region between the parabola $v = u^2 - 1$ and the line $v = 2u + 2$. A sketch of the curves shows that

$$\Gamma: \quad -1 \le u \le 3, \quad u^2 - 1 \le v \le 2u + 2.$$

Then

$$A = \frac{1}{5}\,(\text{area of } \Gamma) = \frac{1}{5}\int_{-1}^3 \left[(2u+2) - (u^2 - 1)\right] du = \frac{32}{15}.$$

24. $\overline{x}A = \dfrac{1}{2}\displaystyle\int_{-2}^2 \int_{-4}^{-u^2} \dfrac{u+v}{2}\, dvdu = -\dfrac{32}{5} \quad \overline{y}A = \dfrac{1}{2}\displaystyle\int_{-2}^2 \int_{-4}^{-u^2} \dfrac{v-u}{2}\, dv\, du = -\dfrac{32}{5}$

$A = \dfrac{16}{3} \implies \overline{x} = \overline{y} = -\dfrac{6}{5}$

25. The choice $\theta = \pi/6$ reduces the equation to $13u^2 + 5v^2 = 1$. This is an ellipse in the uv-plane with area $\pi ab = \pi/\sqrt{65}$. Since $J(u,v) = 1$, the area of Ω is also $\pi/\sqrt{65}$.

26. $\displaystyle\iint_{S_a} \frac{e^{-(x-y)^2}}{1+(x+y)^2}\, dxdy = \frac{1}{2}\iint_\Gamma \frac{e^{-u^2}}{1+v^2}\, dudv$

where Γ is the square in the uv-plane with vertices $(-2a, 0)$, $(0, -2a)$, $(2a, 0)$, $(0, 2a)$.

Γ contains the square $-a \le u \le a, \quad -a \le v \le a$ and is contained in the square

$-2a \le u \le 2a, \quad -2a \le v \le 2a.$. Therefore

$$\frac{1}{2}\int_{-a}^a \int_{-a}^a \frac{e^{-u^2}}{1+v^2}\, du\, dv \le \frac{1}{2}\iint_\Gamma \frac{e^{-u^2}}{1+v^2}\, dudv \le \frac{1}{2}\int_{-2a}^{2a} \int_{-2a}^{2a} \frac{e^{-u^2}}{1+v^2}\, du\, dv.$$

The two extremes can be written

$$\frac{1}{2}\left(\int_{-a}^a e^{-u^2} du\right)\left(\int_{-a}^a \frac{1}{1+v^2} dv\right) \quad \text{and} \quad \frac{1}{2}\left(\int_{-2a}^{2a} e^{-u^2} du\right)\left(\int_{-2a}^{2a} \frac{1}{1+v^2} dv\right).$$

As $a \to \infty$ both expressions tend to $\frac{1}{2}(\sqrt{\pi})(\pi) = \frac{1}{2}\pi^{3/2}$. It follows that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-(x-y)^2}}{1+(x+y)^2}\,dx\,dy = \frac{1}{2}\pi^{3/2}.$$

27. $J = abc\rho^2 \sin\phi; \quad V = \int_0^{2\pi}\int_0^{\pi}\int_0^1 abc\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{4}{3}\pi abc$

28. $\overline{x} = \overline{y} = 0$

$$\overline{z}V = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 (c\rho\cos\phi)abc\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{\pi abc^2}{4} \implies \overline{z} = \frac{3}{8}c\,.$$

29. $V = \frac{2}{3}\pi abc, \quad \lambda = \frac{M}{V} = \frac{3M}{2\pi abc}$

$$I_x = \frac{3M}{2\pi abc}\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 (b^2\rho^2\,\sin^2\phi\sin^2\theta + c^2\rho^2\,\cos^2\phi)abc\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \tfrac{1}{5}M\left(b^2+c^2\right)$$

$$I_y = \tfrac{1}{5}M\left(a^2+c^2\right), \quad I_z = \tfrac{1}{5}M\left(a^2+b^2\right)$$

30. $I = \int_0^{2\pi}\int_0^1\int_0^{\pi} \rho^2(abc\rho^2\sin\phi)\,d\phi\,d\rho\,d\theta = \frac{4}{5}\pi abc$

PROJECT 16.10

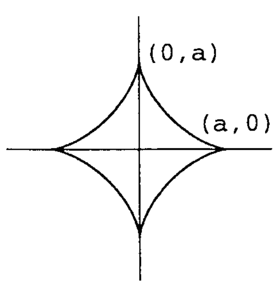
1. (a) $\theta = \tan^{-1}\left[\left(\frac{ay}{bx}\right)^{1/\alpha}\right], \quad r = \left[\left(\frac{x}{a}\right)^{2/\alpha} + \left(\frac{y}{b}\right)^{2/\alpha}\right]^{\alpha/2}$

(b)

$$\left.\begin{aligned} ar_1(\cos\theta_1)^\alpha &= ar_2(\cos\theta_2)^\alpha \\ br_1(\sin\theta_1)^\alpha &= br_2(\sin\theta_2)^\alpha \\ r_1 > 0, \quad 0 < \theta &< \frac{1}{2}\pi \end{aligned}\right\} \implies r_1 = r_2, \quad \theta_1 = \theta_2$$

2. $J = ab\alpha r\cos^{\alpha-1}\theta\sin^{\alpha-1}\theta$

3. (a)



(b) $x = ar\cos^3\theta,\ y = ar\sin^3\theta; \quad x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \implies r = 1$ and $x = a\cos^3\theta,\ y = a\sin^3\theta$

$$A = \int_{\frac{\pi}{2}}^0 y(\theta)x'(\theta)\,d\theta = \int_{\frac{\pi}{2}}^0 a\sin^3\theta(3a\cos^2\theta[-\sin\theta])\,d\theta$$

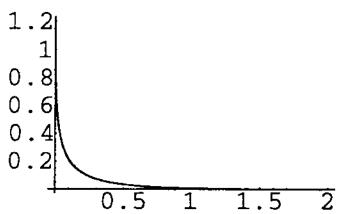
$$= 3a^2 \int_0^{\frac{\pi}{2}} \sin^4\theta cos^2\theta\, d\theta = 3a^2 \int_0^{\frac{\pi}{2}} (\sin^4\theta - \sin^6\theta)\, d\theta$$

$$= 3a^2 \left[\frac{3\cdot 1}{4\cdot 2}\frac{\pi}{2} - \frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\frac{\pi}{2}\right] \quad \text{(See Exercise 62(b) in 8.3)}$$

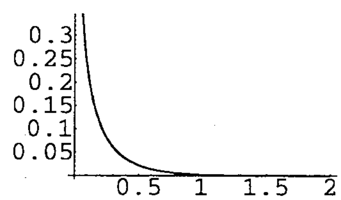
$$= \frac{3a^2\pi}{32}$$

(c) Entire area enclosed: $4\cdot\dfrac{3a^2\pi}{32} = \dfrac{3a^2\pi}{8}$

4. (a)



$a = 3$, $b = 2$



$a = 2$, $b = 3$

(b) From Problem 2, Jacobian $J = 8abr\cos^7\theta\sin^7\theta$

$$A = \int_0^{\frac{\pi}{2}}\int_0^1 8abr\cos^7\theta\sin^7\theta\, d\theta = 4ab\int_0^{\frac{\pi}{2}}\cos^7\theta\sin^7\theta\, d\theta = \frac{ab}{70}$$

CHAPTER 17

SECTION 17.1

1. (a) $\mathbf{h}(x,y) = y\,\mathbf{i} + x\,\mathbf{j}; \quad \mathbf{r}(u) = u\,\mathbf{i} + u^2\,\mathbf{j}, \quad u \in [0,1]$

$$x(u) = u, \quad y(u) = u^2; \quad x'(u) = 1, \quad y'(u) = 2u$$

$$\mathbf{h}(\mathbf{r}(u)) \cdot \mathbf{r}'(u) = y(u)\,x'(u) + x(u)\,y'(u) = u^2(1) + u(2u) = 3u^2$$

$$\int_C \mathbf{h}(\mathbf{r}) \cdot \mathbf{dr} = \int_0^1 3u^2\,du = 1$$

(b) $h(x,y) = y\,\mathbf{i} + x\,\mathbf{j}; \quad \mathbf{r}(u) = u^3\,\mathbf{i} - 2u\,\mathbf{j}, \quad u \in [0,1]$

$$x(u) = u^3, \quad y(u) = -2u; \quad x'(u) = 3u^2, \quad y'(u) = -2$$

$$\mathbf{h}(\mathbf{r}(u)) \cdot \mathbf{r}'(u) = y(u)\,x'(u) + x(u)\,y'(u) = (-2u)(3u^2) + u^3(-2) = -8u^3$$

$$\int_C \mathbf{h}(\mathbf{r}) \cdot \mathbf{dr} = \int_0^1 -8u^3\,du = -2$$

2. (a) $\displaystyle\int_C \mathbf{h} \cdot \mathbf{dr} = \int_0^1 (u\mathbf{i} + u^2\mathbf{j}) \cdot (\mathbf{i} + 2u\mathbf{j})\,du = \int_0^1 (u + 2u^3)\,du = 1$

(b) $\displaystyle\int_C \mathbf{h} \cdot \mathbf{dr} = \int_0^1 (u^3\mathbf{i} - 2u\mathbf{j}) \cdot (3u^2\mathbf{i} - 2\mathbf{j})\,du = \int_0^1 (3u^5 + 4u)\,du = \frac{5}{2}$

3. $h(x,y) = y\,\mathbf{i} + x\,\mathbf{j}; \quad \mathbf{r}(u) = \cos u\,\mathbf{i} - \sin u\,\mathbf{j}, \quad u \in [0, 2\pi]$

$$x(u) = \cos u, \quad y(u) = -\sin u; \quad x'(u) = -\sin u, \quad y'(u) = -\cos u$$

$$\mathbf{h}(\mathbf{r}(u)) \cdot \mathbf{r}'(u) = y(u)\,x'(u) + x(u)\,y'(u) = \sin^2 u - \cos^2 u$$

$$\int_C \mathbf{h}(\mathbf{r}) \cdot \mathbf{dr} = \int_0^{2\pi} (\sin^2 u - \cos^2 u)\,du = 0$$

4. (a) $\displaystyle\int_C \mathbf{h} \cdot \mathbf{dr} = \int_0^1 (e^{-u}\mathbf{i} + 2\mathbf{j}) \cdot (e^u\mathbf{i} - e^{-u}\mathbf{j})\,du = \int_0^1 (1 - 2e^{-u})\,du = 2e^{-1} - 1$

(b) $\displaystyle\int_C \mathbf{h} \cdot \mathbf{dr} = \int_0^2 2\mathbf{j} \cdot (1-u)\mathbf{i}\,du = \int_0^2 0\,du = 0$

5. (a) $\mathbf{r}(u) = (2-u)\,\mathbf{i} + (3-u)\,\mathbf{j}, \quad u \in [0,1]$

$$\int_C \mathbf{h}(\mathbf{r}) \cdot \mathbf{dr} = \int_0^1 (-5 + 5u - u^2)\,du = -\frac{17}{6}$$

(b) $\mathbf{r}(u) = (1+u)\,\mathbf{i} + (2+u)\,\mathbf{j}, \quad u \in [0,1]$

$$\int_C \mathbf{h}(\mathbf{r}) \cdot \mathbf{dr} = \int_0^1 (1 + 3u + u^2)\,du = \frac{17}{6}$$

6. (a) $\int_C \mathbf{h}\cdot d\mathbf{r} = \int_1^4 \left(\frac{1}{\sqrt{u}(1+u)}\mathbf{i} + \frac{1}{u\sqrt{1+u}}\mathbf{j}\right)\cdot\left(\frac{1}{2\sqrt{u}}\mathbf{i} + \frac{1}{2\sqrt{1+u}}\mathbf{j}\right) du = \int_1^4 \frac{1}{u(1+u)}\, du = \ln\frac{8}{5}$

(b) $\int_C \mathbf{h}\cdot d\mathbf{r} = \int_0^1 \frac{1}{(1+u)^3}(\mathbf{i}+\mathbf{j})\cdot(\mathbf{i}+\mathbf{j})\, du = \int_0^1 \frac{2}{(1+u)^3}\, du = \frac{3}{4}$

7. $C = C_1 \cup C_2 \cup C_3$ where,

$C_1: \mathbf{r}(u) = (1-u)(-2\,\mathbf{i}) + u(2\,\mathbf{i}) = (4u-2)\,\mathbf{i}, \quad u \in [0,1]$

$C_2: \mathbf{r}(u) = (1-u)(2\,\mathbf{i}) + u(2\,wj) = (2-2u)\,\mathbf{i} + 2u\,\mathbf{j}, \quad u \in [0,1]$

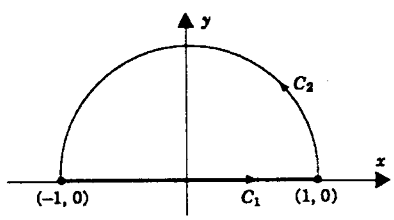
$C_3: \mathbf{r}(u) = (1-u)(2\,\mathbf{j}) + u(-2\,\mathbf{i}) = -2u\,\mathbf{i} + (2-2u)\,\mathbf{j}, \quad u \in [0,1]$

$$\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = 0 + (-4) + (-4) = -8$$

8. $\mathbf{r}(u) = (-1+2u)\mathbf{i} + (1+u)\mathbf{j}, \quad u \in [0,1]$

$$\int_C \mathbf{h}\cdot d\mathbf{r} = \int_0^1 (e^{-2+u}\mathbf{i} + e^{3u}\mathbf{j})\cdot(2\mathbf{i}+\mathbf{j})\, du = \int_0^1 (2e^{-2+u} + e^{3u})\, du = \frac{e^5 - e^2 + 6e - 6}{3e^2}$$

9.



$C_1: \mathbf{r}(u) = (-1+2u)\,\mathbf{i}, \quad u \in [0,1]$

$C_2: \mathbf{r}(u) = \cos u\,\mathbf{i} + \sin u\,\mathbf{j}, \quad u \in [0,\pi]$

$$\int_C = \int_{C_1} + \int_{C_2} = 0 + (-\pi) = -\pi$$

10. Bottom: $\mathbf{r}(u) = u\mathbf{i}$; $\int_0^1 u^3\mathbf{j}\cdot\mathbf{i}\, du = \int_0^1 0\, du = 0$

Right side: $\mathbf{r}(u) = \mathbf{i} + u\mathbf{j}$; $\int_0^1 [3u\mathbf{i} + (1+2u)\mathbf{j}]\cdot\mathbf{j}\, du = \int_0^1 (1+2u)\, du = 2$

Top: $\mathbf{r}(u) = (1-u)\mathbf{i} + \mathbf{j}$; $\int_0^1 3(1-u)^2\mathbf{i}\cdot(-\mathbf{i})\, du = \int_0^1 -3(1-u)^2\, du = -1$

Left: $\mathbf{r}(u) = (1-u)\mathbf{j}$; $\int_0^1 2(1-u)\mathbf{j}\cdot(-\mathbf{j})\, du = \int_0^1 -2(1-u)\, du = -1$

$\int_C \mathbf{h}\cdot d\mathbf{r} =$ sum of the above $= 0$

11. (a) $\mathbf{r}(u) = u\,\mathbf{i} + u\,\mathbf{j} + u\mathbf{k}, \quad u \in [0,1]$

$$\int_C \mathbf{h}(\mathbf{r})\cdot d\mathbf{r} = \int_0^1 3u^2\, du = 1$$

(b) $\displaystyle\int_C \mathbf{h}(\mathbf{r})\cdot d\mathbf{r} = \int_0^1 (2u^3+u^5+3u^6)\,du = \frac{23}{21}$

12. (a) $\displaystyle\int_C \mathbf{h}\cdot d\mathbf{r} = \int_0^1 e^u(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})\,du = \int_0^1 3e^u\,du = 3(e-1)$

(b) $\displaystyle\int_C \mathbf{h}\cdot d\mathbf{r} = \int_0^1 (e^u\mathbf{i}+e^{u^2}\mathbf{j}+e^{u^3}\mathbf{k})\cdot(\mathbf{i}+2u\mathbf{j}+3u^2\mathbf{k})\,du = \int_0^1 (e^u+2ue^{u^2}+3u^2e^{u^3})\,du = 3(e-1).$

13. (a) $\mathbf{r}(u) = 2u\,\mathbf{i}+3u\,\mathbf{j}-u\,\mathbf{k},\quad u\in[0,1]$

$$\int_C \mathbf{h}(\mathbf{r})\cdot d\mathbf{r} = \int_0^1 (2\cos 2u+3\sin 3u+3u^2)\,du = \left[\sin 2u-\cos 3u+u^3\right]_0^1 = 2+\sin 2-\cos 3$$

(b) $\displaystyle\int_C \mathbf{h}(\mathbf{r})\cdot d\mathbf{r} = \int_0^1 \left(2u\cos u^2+3u^2\sin u^3-u^4\right)\,du = \left[\sin u^2-\cos u^3-\frac{1}{5}u^5\right]_0^1 = \frac{4}{5}+\sin 1-\cos 1$

14. (a) $\displaystyle\int_C \mathbf{h}\cdot d\mathbf{r} = \int_0^1 (-2u^2\mathbf{i}+4u^3\mathbf{j}-2u^3\mathbf{k})\cdot(2\mathbf{i}-\mathbf{j}+\mathbf{k})\,du = \int_0^1 (-4u^2-6u^3)\,du = -\frac{17}{6}$

(b) $\displaystyle\int_C \mathbf{h}\cdot d\mathbf{r} = \int_0^1 (\mathbf{i}+ue^{2u}\mathbf{j}+u\mathbf{k})\cdot(e^u\mathbf{i}-e^{-u}\mathbf{j}+\mathbf{k})\,du = \int_0^1 (e^u-ue^u+u)\,du = e-\frac{3}{2}$

15. $\mathbf{r}(u) = (1-u)(\mathbf{j}+4\mathbf{k})+u(\mathbf{i}-4\mathbf{k})$

$= u\mathbf{i}+(1-u)\mathbf{j}+(4-8u)\mathbf{k},\quad u\in[0,1]$

$$\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = \int_0^1 (-32u+97u^2-64u^3)\,du = \frac{1}{3}$$

16. Place the origin at the center of the circular path C and use the time parameter t. Motion along C at constant speed is uniform circular motion

$$\mathbf{r}(t) = \mathbf{r}(\cos\omega t\mathbf{i}+\sin\omega t\mathbf{j}).$$

Differentiation gives

$$\mathbf{r}'(t) = r\omega(-\sin\omega t\mathbf{i}+\cos\omega t\mathbf{j}),\quad \mathbf{r}''(t) = -r\omega^2(\cos\omega t\mathbf{i}+\sin\omega t\mathbf{j}).$$

The force on the object is

$$\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t).$$

Note that $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = 0$ for all t, and therefore W is 0 on every time integral.

Physical explanation: At each instant the force on the object is perpendicular to the path of the object. Thus the component of force in the direction of the motion is always zero.

17. (a) $\mathbf{r}(u) = u\,\mathbf{i}+u^2\,\mathbf{j},\quad u\in[0,2]$

$$\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = \int_0^2 \left[(u+2)u^2+(2u+u^2)2u\right]\,du = \int_0^2 \left(3u^3+6u^2\right)\,du = 28$$

(b) $\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = \int_0^{\pi/2} \left[-(\cos u + 2)\sin^2 u + \cos u(2\cos u + \sin u)\right] du$

$= \int_0^{\pi/2} \left[-\sin^2 u \cos u + \cos 2u + \sin u \cos u\right] du = \frac{1}{6}$

18. $C_1:\quad \mathbf{r}(u) = u\mathbf{i};\quad \int_0^1 u\mathbf{i}\cdot\mathbf{i}\,du = \int_0^1 u\,du = \frac{1}{2}$

$C_2:\quad \mathbf{r}(u) = \mathbf{i} + u\mathbf{j};\quad \int_0^1 (\cos u\mathbf{i} - u\sin 1\mathbf{j})\cdot\mathbf{j}\,du = \int_0^1 -u\sin 1\,du = -\frac{1}{2}\sin 1$

$C_3:\quad \mathbf{r}(u) = (1-u)\mathbf{i}+\mathbf{j};\quad \int_0^1 [(1-u)\cos 1\mathbf{i} - \sin(1-u)\mathbf{j}]\cdot(-\mathbf{i})\,du = \int_0^1 (u-1)\cos 1\,du = -\frac{1}{2}\cos 1$

$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \frac{1}{2} - \frac{1}{2}\sin 1 - \frac{1}{2}\cos 1 = \frac{1}{2}(1 - \sin 1 - \cos 1)$

19. $\mathbf{r}(u) = \cos u\,\mathbf{i} + \sin u\,\mathbf{j} + u\,\mathbf{k},\quad u \in [0, 2\pi]$

$\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = \int_0^{2\pi} \left[-\cos^2 u \sin u + \cos^2 u \sin u + u^2\right] du = \int_0^{2\pi} u^2\,du = \frac{8\pi^3}{3}$

20. $C_1:\quad \mathbf{r}(u) = u\mathbf{i};\quad \mathbf{F}(\mathbf{r}(u)) = \mathbf{0},\quad \int_{C_1} \mathbf{F}\cdot d\mathbf{r} = 0$

$C_2:\quad \mathbf{r}(u) = \mathbf{i} + u\mathbf{j};\quad \int_0^1 u\mathbf{k}\cdot\mathbf{j}\,du = \int_0^1 0\,du = 0$

$C_3:\quad \mathbf{r}(u) = \mathbf{i} + \mathbf{j} + u\,\mathbf{k};\quad \int_0^1 (u\mathbf{i} + u\mathbf{j} + \mathbf{k})\cdot\mathbf{k}\,du = \int_0^1 du = 1$

$W = 1$

21.

$$\int_C \mathbf{q}\cdot d\mathbf{r} = \int_a^b [\mathbf{q}\cdot\mathbf{r}'(u)]\,du = \int_a^b \frac{d}{du}[\mathbf{q}\cdot\mathbf{r}(u)]\,du$$

$$= [\mathbf{q}\cdot\mathbf{r}(b)] - [\mathbf{q}\cdot\mathbf{r}(a)]$$

$$= \mathbf{q}\cdot[\mathbf{r}(b) - \mathbf{r}(a)]$$

$$\int_C \mathbf{r}\cdot d\mathbf{r} = \int_a^b [\mathbf{r}(u)\cdot\mathbf{r}'(u)]\,du$$

$$= \frac{1}{2}\int_a^b \|\mathbf{r}\|\,d\|\mathbf{r}\| \quad \text{(see Exercise 53, Section 13.1)}$$

$$= \frac{1}{2}\left(\|\mathbf{r}(b)\|^2 - \|\mathbf{r}(a)\|^2\right)$$

22. (a) $\mathbf{r}(u) = (1-2u)\mathbf{i};\quad \int_{C_1} \mathbf{h}\cdot d\mathbf{r} = \int_0^1 (1-2u)^2\mathbf{i}\cdot(-2\mathbf{i})\,du = \int_0^1 -2(1-2u)^2\,du = -\frac{2}{3}$

(b) $\int_{C_2} \mathbf{h}\cdot d\mathbf{r} = \int_0^1 (\mathbf{i} + u\mathbf{j})\cdot\mathbf{j}\,du + \int_0^1 [(1-2u)^2\mathbf{i} + \mathbf{j}]\cdot(-2\mathbf{i})\,du + \int_0^1 [\mathbf{i} + (1-u)\mathbf{j}]\cdot(-\mathbf{j})\,du$

$= \int_0^1 u\,du + \int_0^1 -2(1-2u)^2\,du + \int_0^1 -(1-u)\,du = -\frac{2}{3}$

(c) $\mathbf{r}(u) = \cos u\mathbf{i} + \sin u\mathbf{j}, \quad u \in [0, \pi]$

$$\int_{C_3} \mathbf{h} \cdot d\mathbf{r} = \int_0^\pi (\cos^2 u\mathbf{i} + \sin u\mathbf{j}) \cdot (-\sin u\mathbf{i} + \cos u\mathbf{j})\, du = \int_0^\pi (-\sin u \cos^2 u + \sin u \cos u)\, du = -\frac{2}{3}$$

23. $\int_C \mathbf{f}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b [\mathbf{f}(\mathbf{r}(u)) \cdot \mathbf{r}'(u)]\, du = \int_a^b [f(u)\mathbf{i} \cdot \mathbf{i}]\, du = \int_a^b f(u)\, du$

24. Follows from the linearity of the dot product and of ordinary integrals.

25. $E : \mathbf{r}(u) = a\cos u\,\mathbf{i} + b\sin u\,\mathbf{j}, \quad u \in [0, 2\pi]$

$$W = \int_0^{2\pi} \left[\left(-\frac{1}{2}b\sin u\right)(-a\sin u) + \left(\frac{1}{2}a\cos u\right)(b\cos u)\right] du = \int_0^{2\pi} ab\, du = \pi ab$$

If the ellipse is traversed in the opposite direction, then $W = -\pi ab$. In both cases $|W| = \pi ab =$ area of the ellipse.

26. force at time t: $\quad m\mathbf{r}''(t) = 2m\beta\mathbf{j}$

work during time interval: $\quad W = \int_0^1 4m\beta^2 t\, dt = 2m\beta^2$

27. $\mathbf{r}(t) = \alpha t\mathbf{i} + \beta t^2\mathbf{j} + \gamma t^3\mathbf{k}$

$\mathbf{r}'(t) = \alpha\mathbf{i} + 2\beta t\mathbf{j} + 3\gamma t^2\mathbf{k}$

force at time $t = m\mathbf{r}''(t) = m(2\beta\mathbf{j} + 6\gamma t\mathbf{k})$

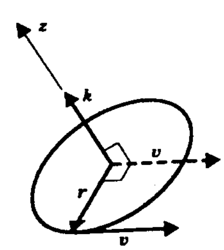
$$W = \int_0^1 [m(2\beta\mathbf{j} + 6\gamma t\mathbf{k}) \cdot (\alpha\mathbf{i} + 2\beta t\mathbf{j} + 3\gamma t^2\mathbf{k})]\, dt$$

$$= m\int_0^1 (4\beta^2 t + 18\gamma^2 t^3)\, dt = \left(2\beta^2 + \frac{9}{2}\gamma^2\right) m$$

28. (a) $\mathbf{v} \perp \mathbf{k}, \quad \mathbf{v} \perp \mathbf{r}, \quad \|\mathbf{v}\| = \omega \quad$ and $\quad \omega\mathbf{k},\ \mathbf{r},\ \mathbf{v},$ form a right-handed triple

(b) We can parametrize C counterclockwise by

$$\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j}, \quad 0 \le t \le 2\pi.$$



Then

$$\mathbf{r}'(t) = -a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j}$$

and

$$\int_C (\omega\mathbf{k} \times \mathbf{r}) \cdot d\mathbf{r} = \int_0^{2\pi} (\omega\mathbf{k} \times \mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt.$$

Now

$$\mathbf{k} \times \mathbf{r}(t) = a\cos t\,\mathbf{j} - a\sin t\,\mathbf{i}.$$

So

$$(\mathbf{k} \times \mathbf{r}(t)) \cdot \mathbf{r}'(t) = a^2(\cos^2 t + \sin^2 t) = a^2.$$

Thus

$$\int_C (\omega\mathbf{k}\times\mathbf{r})\cdot d\mathbf{r} + \int_0^{2\pi} \omega a^2\,dt = \omega a^2(2\pi) = 2\omega(\pi a^2) = 2\omega A.$$

If C is parametrized clockwise, the circulation is $-2\omega A$.

29. Take $C\colon \mathbf{r}(t) = r\cos t\,\mathbf{i} + r\sin t\,\mathbf{j}, \quad t\in[0,2\pi]$

$$\begin{aligned}\int_C \mathbf{v}(\mathbf{r})\cdot d\mathbf{r} &= \int_0^{2\pi} [\mathbf{v}(\mathbf{r}(t))\cdot\mathbf{r}'(t)]\,dt\\ &= \int_0^{2\pi} [f(x(t),y(t))\,\mathbf{r}(t)\cdot\mathbf{r}'(t)]\,dt\\ &= \int_0^{2\pi} f(x(t),y(t))\,[\mathbf{r}(t)\cdot\mathbf{r}'(t)]\,dt = 0\end{aligned}$$

since for the circle $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$ identically. The circulation is zero.

30. (a) $\mathbf{r}(u) = \mathbf{i} + u\,\mathbf{j};\quad W = \displaystyle\int_0^2 \frac{k}{1+u^2}(\mathbf{i}+u\,\mathbf{j})\cdot\mathbf{j}\,du = \int_0^2 \frac{ku}{1+u^2}\,du = \frac{k}{2}\ln 5$

(b) $\mathbf{r}(u) = u\mathbf{i} + \mathbf{j};\quad W = \displaystyle\int_0^1 \frac{k}{u^2+1}(u\mathbf{i}+\mathbf{j})\cdot\mathbf{i}\,du = \int_0^1 \frac{ku}{u^2+1}\,du = \frac{k}{2}\ln 2.$

31. (a) $\mathbf{r}(u) = (1-u)(\mathbf{i}+2\,\mathbf{k}) + u(\mathbf{i}+3\,\mathbf{j}+2\,\mathbf{k}) = \mathbf{i}+3u\,\mathbf{j}+2\,\mathbf{k}, \quad u\in[0,1].$

$$\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = \int_0^1 \frac{9uk}{5+9u^2)^{3/2}}\,du = \left[\frac{-k}{\sqrt{5+9u^2}}\right]_0^1 = \frac{k}{\sqrt{5}} - \frac{k}{\sqrt{14}}$$

(b) $C = C_1\cup C_2$, where

$C_1:\ \mathbf{r}(u) = (1-u)\,\mathbf{i} + 5u\,\mathbf{i} = (1+4u)\,\mathbf{i}, \quad u\in[0,1],$

$C_2:\ x^2+y^2+z^2 = 25 \Longrightarrow \quad \|\mathbf{r}\| = 5$

$$\int_{C_1} \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = \int_0^1 \frac{4k(1+4u)}{(1+4u)^3}\,du = \int_0^1 \frac{4k}{(1+4u)^2}\,du \left[\frac{-k}{1+4u}\right]_0^1 = \frac{4}{5}\,k$$

$$\begin{aligned}\int_{C_2} \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} &= \int_{C_2} \frac{k\mathbf{r}}{\|\mathbf{r}\|^3}\cdot d\mathbf{r}\\ &= \frac{k}{5^3}\int_{C_2} \mathbf{r}\cdot d\mathbf{r} = \frac{k}{5^3}\int_{C_2} \|\mathbf{r}\|\,d\|\mathbf{r}\| \quad \text{(see Exercise 53, Section 13.1)}\\ &= \frac{k}{5^3}\left[\frac{1}{2}\|\mathbf{r}\|^2\right]_{(5,0,0)}^{(0,5/\sqrt{2},5/\sqrt{2})} = 0\end{aligned}$$

Therefore, $\displaystyle\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = \frac{4}{5}\,k.$

32. (a) $$W = \int_0^{\pi/2} \frac{-k}{(9\cos^2 u + 16\sin^2 u)^{3/2}}\,(3\cos u\mathbf{i} + 4\sin u\mathbf{j})\cdot(-3\sin u\mathbf{i} + 4\cos u\mathbf{j})\,du$$

$$= \int_0^{\pi/2} \frac{-7k\sin u\cos u}{(9\cos^2 u + 16\sin^2 u)^{3/2}}\,du = -\frac{k}{12}$$

(b) $$W = \int_0^1 \frac{-k}{[(3-3u)^2+(4u)^2]^{3/2}}\,[(3-3u)\mathbf{i}+4u\mathbf{j}]\cdot(-3\mathbf{i}+4\mathbf{j})\,du$$
$$= \int_0^1 \frac{-k(25u-9)}{[(3-3u)^2+(4u)^2]^{3/2}}\,du = -\frac{k}{12}$$

33. $\mathbf{r}(u) = u\,\mathbf{i} + \alpha\,u(1-u)\,\mathbf{j}, \quad \mathbf{r}'(u) = \mathbf{i} + \alpha(1-2u)\,\mathbf{j}, \quad u \in [0,1]$

$$W(\alpha) = \int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = \int_0^1 \left[(\alpha^2u^2(1-u)^2+1\right] + [u+\alpha\,u(1-u)]\alpha(1-2u)]\,dx$$
$$= \int_0^1 \left[1+(\alpha+\alpha^2)u-(2\alpha+2\alpha^2)u^2+\alpha^2u^4\right]\,du = 1-\frac{1}{6}\alpha+\frac{1}{30}\alpha^2$$

$$W'(\alpha) = -\frac{1}{6}+\frac{1}{15}\alpha \quad \Longrightarrow \quad \alpha = \frac{15}{6}$$

The work done by $\mathbf{F}$ is a minimum when $\alpha = 15/6$.

SECTION 17.2

1. $\mathbf{h}(x,y) = \nabla f(x,y)$ where $f(x,y) = \frac{1}{2}(x^2+y^2)$

C is closed $\Longrightarrow \int_C \mathbf{h}(\mathbf{r})\cdot d\mathbf{r} = 0$

2. $x\mathbf{i}+y\mathbf{j}$ is a gradient (Exercise 1); we need integrate only $y\mathbf{i}$.

$$\int_C \mathbf{h}(\mathbf{r})\cdot d\mathbf{r} = \int_0^{2\pi} y(t)x'(t)\,dt = \int_0^{2\pi}(b\sin t)(-a\sin t)\,dt = -\pi ab$$

3. $\mathbf{h}(x,y) = \nabla f(x,y)$ where $f(x,y) = x\cos\pi y$; $\quad \mathbf{r}(0) = \mathbf{0}, \quad \mathbf{r}(1) = \mathbf{i}-\mathbf{j}$

$$\int_C \mathbf{h}(\mathbf{r})\cdot d\mathbf{r} = \int_C \nabla f(\mathbf{r})\cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(1,-1)-f(0,0) = -1$$

4. $\mathbf{h} = \nabla f$ with $f(x,y) = \dfrac{x^3}{3}+\dfrac{y^3}{3}-xy$, and C is closed, so $\int_C \mathbf{h}\cdot d\mathbf{r} = 0$

5. $\mathbf{h}(x,y) = \nabla f(x,y)$ where $f(x,y) = \frac{1}{2}x^2y^2$; $\quad \mathbf{r}(0) = \mathbf{j}, \quad \mathbf{r}(1) = -\mathbf{j}$

$$\int_C \mathbf{h}(\mathbf{r})\cdot d\mathbf{r} = \int_C \nabla f(\mathbf{r})\cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(0,-1)-f(0,1) = 0-0 = 0$$

6. $e^y\mathbf{i}+xe^y\mathbf{j}$ is a gradient; we need integrate only $\mathbf{i}-x\mathbf{j}$

$C = C_1 \cup C_2 \cup C_3 \cup C_4$ where

$C_1 : \mathbf{r}(u) = (2u-1)\mathbf{i}-\mathbf{j}, \quad u \in [0,1]$

$C_2 : \mathbf{r}(u) = \mathbf{i}+(2u-1)\mathbf{j}, \quad u \in [0,1]$

$C_3 : \mathbf{r}(u) = (1-2u)\mathbf{i}+\mathbf{j}, \quad u \in [0,1]$

$C_4 : \mathbf{r}(u) = -\mathbf{i}+(1-2u)\mathbf{j}, \quad u \in [0,1]$

$$\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = 2 + (-2) + (-2) + (-2) = -4$$

7. $\mathbf{h}(x,y) = \nabla f(x,y)$ where $f(x,y) = x^2y - xy^2$; $\mathbf{r}(0) = \mathbf{i}$, $\mathbf{r}(\pi) = -\mathbf{i}$

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \int_C \nabla f(\mathbf{r}) \cdot d\mathbf{r} = f(\mathbf{r}(\pi)) - f(\mathbf{r}(0)) = f(-1,0) - f(1,0) = 0 - 0 = 0$$

8. $\mathbf{h}(x,y) = \nabla f(x,y)$ where $f(x,y) = (x^2y^4)^{3/2}$

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \int_C \nabla f(\mathbf{r}) \cdot d\mathbf{r} = f(-1,0) - f(1,0) = 1 - 1 = 0$$

9. $\mathbf{h}(x,y) = \nabla f(x,y)$ where $f(x,y) = (x^2 + y^4)^{3/2}$

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \int_C \nabla f(\mathbf{r}) \cdot d\mathbf{r} = f(1,0) - f(-1,0) = 1 - 1 = 0$$

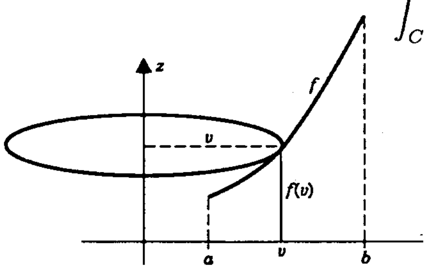
10. $\mathbf{h} = \nabla f$ with $f(x,y) = \cosh x^2 y$; and C is closed, so $\int_C \mathbf{h} \cdot d\mathbf{r} = 0$

11. $\mathbf{h}(x,y)$ is not a gradient, but part of it,

$$2x \cosh y\, \mathbf{i} + (x^2 \sinh y - y)\mathbf{j},$$

is a gradient. Since we are integrating over a closed curve, the contribution of the gradient part is 0. Thus

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \int_C (-y\mathbf{i}) \cdot d\mathbf{r}.$$



$C_1 : \mathbf{r}(u) = \mathbf{i} + (-1 + 2u)\mathbf{j}, \quad u \in [0,1]$

$C_2 : \mathbf{r}(u) = (1 - 2u)\mathbf{i} + \mathbf{j}, \quad u \in [0,1]$

$C_3 : \mathbf{r}(u) = -\mathbf{i} + (1 - 2u)\mathbf{j}, \quad u \in [0,1]$

$C_4 : \mathbf{r}(u) = (-1 + 2u)\mathbf{i} - \mathbf{j}, \quad u \in [0,1]$

$$\begin{aligned}
\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} &= \int_{C_1} (-y\mathbf{i}) \cdot d\mathbf{r} + \int_{C_2} (-y\mathbf{i}) \cdot d\mathbf{r} + \int_{C_3} (-y\mathbf{i}) \cdot d\mathbf{r} + \int_{C_4} (-y\mathbf{i}) \cdot d\mathbf{r} \\
&= 0 + \int_0^1 -\mathbf{i} \cdot (-2\mathbf{i})\, du + 0 + \int_0^1 \mathbf{i} \cdot (2\mathbf{i})\, du \\
&= 0 + \int_0^1 2\, du + 0 + \int_0^1 2\, du \\
&= 4
\end{aligned}$$

12. $\mathbf{h}(x,y) = \nabla \left(\dfrac{x^2y^2}{2} \right)$

(a) $\displaystyle\int_0^2 (u^5\mathbf{i} + u^4\mathbf{j}) \cdot (\mathbf{i} + 2u\mathbf{j})\, du = \int_0^2 3u^5\, du = 32$

(b) $f(2,4) - f(0,0) = 32 - 0 = 32$

13. $\mathbf{h}(x,y) = (3x^2y^3 + 2x)\,\mathbf{i} + (3x^3y^2 - 4y)\,\mathbf{j}$; $\dfrac{\partial P}{\partial y} = 9x^2y^2 = \dfrac{\partial Q}{\partial x}$. Thus $\mathbf{h}$ is a gradient.

(a) $\mathbf{r}(u) = u\,\mathbf{i} + e^u\,\mathbf{j}$, $\mathbf{r}'(u) = \mathbf{i} + e^u\,\mathbf{j}$, $u \in [0,1]$

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^1 \left[(3u^2e^{3u} + 2u) + 3u^3e^3u - 4e^{2u})\right] du = \left[u^3e^{3u} + u^2 - 2e^{2u}\right]_0^1 = e^3 - 2e^2 + 3$$

(b) $\dfrac{\partial f}{\partial x} = 3x^2y^3 + 2x \implies f(x,y) = x^3y^3 + x^2 + g(y)$;

$$\frac{\partial f}{\partial y} = 3x^3y^2 + g'(y) = 3x^3 - 4y \implies g'(y) = -4y \implies g(y) = -2y^2$$

Therefore, $f(x,y) = x^3y^3 + x^2 - 2y^2$.

Now, at $u = 0$, $r(0) = 0\,\mathbf{i} + \mathbf{j} = (0,1)$; at $u = 1$, $r(1) = \mathbf{i} + e\,\mathbf{j} = (1,e)$ and

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \left[x^3y^3 + x^2 - 2y^2\right]_{(0,1)}^{(1,e)} = e^3 - 2e^2 + 3$$

14. $\mathbf{h}(x,y) = \nabla(x^2 \sin y - e^x)$

(a) $\displaystyle\int_0^{\pi} \left[(2\cos u \sin u - e^{\cos u})\,\mathbf{i} + (\cos^2 u \cos u)\mathbf{j}\right] \cdot (-\sin u\mathbf{i} + \mathbf{j})\,du = e - e^{-1}$

(b) $f(-1,\pi) - f(1,0) = e - e^{-1}$

15. $\mathbf{h}(x,y) = (e^{2y} - 2xy)\,\mathbf{i} + (2xe^{2y} - x^2 + 1)\,\mathbf{j}$; $\dfrac{\partial P}{\partial y} = 2e^{2y} - 2x = \dfrac{\partial Q}{\partial x}$. Thus $\mathbf{h}$ is a gradient.

(a) $\mathbf{r}(u) = ue^u\,\mathbf{i} + (1+u)\,\mathbf{j}$, $\mathbf{r}'(u) = (1 = u)e^u\,\mathbf{i} + \mathbf{j}$, $u \in [0,1]$

$$\begin{aligned}\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} &= \int_0^1 \left[e^2(3ue^{3u} + e^{3u} - 2u^3e^{2u} - 5u^2e^{2u} - 2ue^{2u} + 1\right] du \\ &= \left[e^2ue^{3u} - u^3e^{2u} - u^2e^{2u} + u\right]_0^1 = e^5 - 2e^2 + 1\end{aligned}$$

(b) $\dfrac{\partial f}{\partial x} = e^{2y} - 2xy \implies f(x,y) = xe^{2y} - x^2y + g(y)$.

$$\frac{\partial f}{\partial y} = 2xe^{2y} - x^2 + g'(y) = 3x^3 - 4y \implies g'(y) = 1 \implies g(y) = y$$

Therefore, $f(x,y) = xe^{2y} - x^2y + y$.

Now, at $u = 0$, $r(0) = 0\,\mathbf{i} + \mathbf{j} = (0,1)$; at $u = 1$, $r(1) = e\,\mathbf{i} + 2\,\mathbf{j} = (e,2)$ and

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \left[xe^{2y} - x^2y + y\right]_{(0,1)}^{(e,2)} = e^5 - 2e^2 + 1$$

16. $\mathbf{h}(x,y,z) = \nabla f$ with $f(x,y,z) = xy^2z^3$ $\displaystyle\int_C \mathbf{h} \cdot d\mathbf{r} = f(1,1,1) - f(0,0,0) = 1$

17. $\mathbf{h}(x,y,z) = (2xz + \sin y)\,\mathbf{i} + x\cos y\,\mathbf{j} + x^2\,\mathbf{k}$;

$\dfrac{\partial P}{\partial y} = \cos y = \dfrac{\partial Q}{\partial x}$, $\dfrac{\partial P}{\partial z} = 2x = \dfrac{\partial R}{\partial x}$, $\dfrac{\partial Q}{\partial z} = 0 = \dfrac{\partial R}{\partial y}$. Thus $\mathbf{h}$ is a gradient.

$$\frac{\partial f}{\partial x} = 2xz + \sin y, \implies f(x,y,z) = x^2z + x\sin y + g(y,z)$$

$$\frac{\partial f}{\partial y} = x\cos y + \frac{\partial g}{\partial y} = x\cos y, \implies g(y,z) = h(z) \implies f(x,y,z) = x^2 z + x\sin y + h(z)$$

$$\frac{\partial f}{\partial z} = x^2 + h'(z) = x^2 \implies h'(z) = 0 \implies h(z) = C$$

Therefore, $f(x,y,z) = x^2 z + x\sin y$ (take $C = 0$)

$$\int_C \mathbf{h}(\mathbf{r}) \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = \left[x^2 z + x\sin y\right]_{\mathbf{r}(0)}^{\mathbf{r}(2\pi)} = \left[x^2 z + x\sin y\right]_{(1,0,0)}^{(1,0,2\pi)} = 2\pi$$

18. $\mathbf{h}(x,y,z) = \nabla f$ with $f(x,y,z) = z^3 - e^{-x}\ln y$

$$\int_C \mathbf{h} \cdot d\mathbf{r} = f(2,e^2,2) - f(1,1,1) = 7 - e^{-2} + e^{-1}$$

19. $\mathbf{F}(x,y) = (x + e^{2y})\,\mathbf{i} + (2y + 2xe^{2y})\,\mathbf{j};\quad \dfrac{\partial P}{\partial y} = 2e^{2y} = \dfrac{\partial Q}{\partial x}.$ Thus $\mathbf{F}$ is a gradient.

$$\frac{\partial f}{\partial x} = x + e^{2y} \implies f(x,y) = \frac{1}{2}x^2 + xe^{2y} + g(y);$$

$$\frac{\partial f}{\partial y} = 2xe^{2y} + g'(y) = 2y + 2xe^{2y} \implies g'(y) = 2y \implies g(y) = y^2 \quad (\text{take } C = 0)$$

Therefore, $f(x,y) = \frac{1}{2}x^2 + xe^{2y} + y^2$.

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = \left[\frac{1}{2}x^2 + xe^{2y} + y^2\right]_{\mathbf{r}(0)}^{\mathbf{r}(2\pi)} = \left[\frac{1}{2}x^2 + xe^{2y} + y^2\right]_{(3,0)}^{(3,0)} = 0$$

20. $\mathbf{F} = \nabla f$ with $f(x,y,z) = x^2\ln y - xyz$ $\quad W = f(3,2,2) - f(1,2,1) = 8\ln 2 - 10$

21. Set $f(x,y,z) = g(x)$ and $C: \mathbf{r}(u) = u\mathbf{i},\quad u \in [a,b]$.

In this case

$$\nabla f(\mathbf{r}(u)) = g'(x(u))\mathbf{i} = g'(u)\mathbf{i} \quad \text{and} \quad \mathbf{r}'(u) = \mathbf{i},$$

so that

$$\int_C \nabla f(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b [\nabla f(\mathbf{r}(u)) \cdot \mathbf{r}'(u)]\,du = \int_a^b g'(u)\,du.$$

Since $\quad f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = g(b) - g(a)$,

$$\int_C \nabla f(\mathbf{r}) \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \quad \text{gives} \quad \int_a^b g'(u)\,du = g(b) - g(a).$$

22. $\mathbf{F}(x,y,z) = \dfrac{k}{(x^2+y^2+z^2)^{n/2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{j}) = \nabla f$

(a) $n = 2:\quad f(\mathbf{r}) = K\ln r + C$ $\qquad$ (b) $n \neq 2:\quad f(\mathbf{r}) = -\left(\dfrac{K}{n-2}\right)\dfrac{1}{r^{n-2}} + C$

23. $\mathbf{F}(\mathbf{r}) = \nabla\left(\dfrac{mG}{r}\right);\quad W = \displaystyle\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = mG\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$

24. (a) Since the denominator is never 0 in Ω, P and Q are continuously differentiable on Ω.

$$\frac{\partial P}{\partial y} = \frac{x^2 - y^2}{(x^2 + y^2)^2} = \frac{\partial Q}{\partial x}.$$

(b) Take $\mathbf{r}(u) = \frac{1}{2}\cos u\,\mathbf{i} + \frac{1}{2}\sin u\,\mathbf{j}$.

$$\int_C \mathbf{h} \cdot d\mathbf{r} = \int_0^{2\pi} \left(\frac{\frac{1}{2}\sin u}{1/4}\mathbf{i} - \frac{\frac{1}{2}\cos u}{1/4}\mathbf{j}\right) \cdot \left(-\frac{1}{2}\sin u\,\mathbf{i} + \frac{1}{2}\cos u\,\mathbf{j}\right) du = \int_0^{2\pi} - du = -2\pi$$

Therefore $\mathbf{h}$ is not a gradient since the integral over C (a closed curve) is not zero.

(c) $\Omega : 0 < x^2 + y^2 < 1$ is an open plane region but is not simply connected.

25. $\mathbf{F}(x,y,z) = 0\,\mathbf{i} + 0\,\mathbf{j} + \dfrac{-mGr_0^2}{(r_0+z)^2}\mathbf{k}$; $\dfrac{\partial P}{\partial y} = 0 = \dfrac{\partial Q}{\partial x}$, $\dfrac{\partial P}{\partial z} = 0 = \dfrac{\partial R}{\partial x}$, $\dfrac{\partial Q}{\partial z} = 0 = \dfrac{\partial R}{\partial y}$.

Therefore, $\mathbf{F}(x,y,z)$ is a gradient.

$$\frac{\partial f}{\partial x} = 0 \implies f(x,y,z) = g(y,z); \quad \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 0 \implies g(y,z) = h(z).$$

Therefore $f(x,y,z) = h(z)$.

Now $\dfrac{\partial f}{\partial z} = h'(z) = \dfrac{-mGr_0^2}{(r_0+z)^2} \implies f(x,y,z) = h(z) = \dfrac{mGr_0^2}{r_0+z}$

26. $W = f(x,y,0) - f(x,y,300) = mGr_0 - \dfrac{mG{r_0}^2}{r_0+300} = 13.21\,G.$

27. By Exercise 25, the work required to lift an object of mass m a distance of h miles above the surface of the earth is:

$$W = \int_0^h -\mathbf{F} \cdot d\mathbf{r} = -f(0,0,h) + f(0,0,0) = \frac{mGr_0^2 h}{r_0(r_0+h)}$$

where $\mathbf{F} = \dfrac{-mGr_0^2}{(r_0+z)^2}\mathbf{k}$, $\mathbf{r} = 0\,\mathbf{i} + 0\,\mathbf{j} + u\,\mathbf{k}$, $u \in [0,h]$, and $f = \dfrac{mGr_0^2}{r_0+z}$.

In this particular case, put $r_0 = 4000$, $h = 500/5280$ and $m = 8000/(32 \cdot 5280)$.

Then $W \cong 0.0045\,G$.

SECTION 17.3

1. If f is continuous, then $-f$ is continuous and has antiderivatives u. The scalar fields $U(x,y,z) = u(x)$ are potential functions for $\mathbf{F}$:

$$\nabla U = \frac{\partial U}{\partial x}\mathbf{i} + \frac{\partial U}{\partial y}\mathbf{j} + \frac{\partial U}{\partial z}\mathbf{k} = \frac{du}{dx}\mathbf{i} = -f\mathbf{i} = -\mathbf{F}.$$

2.

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \frac{d}{dt}\left[\frac{1}{2}m(\mathbf{v} \cdot \mathbf{v})\right] = m(\mathbf{v} \cdot \mathbf{a}) = \mathbf{v} \cdot m\mathbf{a}$$
$$= \mathbf{v} \cdot \mathbf{F} = \mathbf{v} \cdot \frac{e}{c}[\mathbf{v} \times \mathbf{B}] = 0$$

3. The scalar field $U(x,y,z) = cz + d$ is a potential energy function for $\mathbf{F}$. We know that the total mechanical energy remains constant. Thus, for any times t_1 and t_2,

$$\tfrac{1}{2}m[v(t_1)]^2 + U(\mathbf{r}(t_1)) = \tfrac{1}{2}m[v(t_2)]^2 + U(\mathbf{r}(t_2)).$$

This gives

$$\tfrac{1}{2}m[v(t_1)]^2 + cz(t_1) + d = \tfrac{1}{2}m[v(t_2)]^2 + cz(t_2) + d.$$

Solve this equation for $v(t_2)$ and you have the desired formula.

4. Throughout the motion, the total mechanical energy of the object remains constant:

$$\frac{1}{2}mv^2 - \frac{GmM}{r} = E.$$

At firing $v = v_0$, $r = R_e =$ the radius of the earth and we have

$$\frac{1}{2}mv_0{}^2 - \frac{GmM}{R_e} = E.$$

As $r \to \infty$, $v \to 0$ (by assumption) and also $-GmM/r \to 0$.

Thus $E = 0$ and we have

$$\frac{1}{2}mv_0{}^2 = \frac{GmM}{R_e} \quad \text{and} \quad v_0 = \sqrt{\frac{2GM}{R_e}}.$$

(Note that v_0 is independent of the mass of the projectile.)

5. (a) We know that $-\nabla U$ points in the direction of maximum decrease of U. Thus $\mathbf{F} = -\nabla U$ attempts to drive objects toward a region where U has lower values.

(b) At a point where u has a minimum, $\nabla U = \mathbf{0}$ and therefore $\mathbf{F} = \mathbf{0}$.

6. We have $x(0) = 2$, $x'(0) = v(0) = 1$. Inserting these values in the formula for E we have

$$E = \frac{1}{2}m + 2\lambda.$$

Since $E = \frac{1}{2}mv^2 + \frac{1}{2}\lambda x^2$ is constant, the maximum value of v comes when $x = 0$. Then

$$E = \frac{1}{2}mv^2 + \frac{1}{2}m + 2\lambda \quad \text{and} \quad v = \sqrt{1 + 4\lambda/m}.$$

The maximum value of x comes when $v = 0$ (at the endpoints of the oscillation). Then

$$E = \frac{1}{2}\lambda x^2 = \frac{1}{2}m + 2\lambda \quad \text{and} \quad x = \sqrt{m/2 + 4}.$$

7. (a) By conservation of energy $\frac{1}{2}mv^2 + U = E$. Since E is constant and U is constant, v is constant.

(b) ∇U is perpendicular to any surface where U is constant. Obviously so is $\mathbf{F} = -\nabla U$.

8. $\mathbf{F}(\mathbf{r}) = \dfrac{k}{r^2}\mathbf{r} = \nabla f \quad \text{where} \quad f(\mathbf{r}) = k \ln r$

9. $f(x,y,z) = -\dfrac{k}{\sqrt{x^2 + y^2 + z^2}}$ is a potential function for $\mathbf{F}$. The work done by $\mathbf{F}$ moving an object along C is:

$$W = \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \nabla f \cdot d\mathbf{r} = f[\mathbf{r}(b)] - f[\mathbf{r}(a)].$$

Since $\mathbf{r}(a) = (x_0, y_0, z_0)$ and $\mathbf{r}(b) = (x_1, y_1, z_1)$ are points on the unit sphere,

$$f[\mathbf{r}(b)] - f[\mathbf{r}(a)] = -k \quad \text{and so} \quad W = 0$$

SECTION 17.4

1. $\mathbf{r}(u) = u\,\mathbf{i} + 2u\,\mathbf{j}, \quad u \in [0,1]$

$$\int_C (x - 2y)\,dx + 2x\,dy = \int_0^1 \{[x(u) - 2y(u)]x'(u) + 2x(u)\,y'(u)\}\,du = \int_0^1 u\,du = \frac{1}{2}$$

2. $\mathbf{r}(u) = u\mathbf{i} + 2u^2\mathbf{j}, \quad u \in [0,1]$

$$\int_C (x - 2y)\,dx + 2x\,dy = \int_0^1 \{[x(u) - 2y(u)]\,x'(u) + 2x(u)y'(u)\}\,du$$
$$= \int_0^1 (u + 4u^2)\,du = \frac{11}{6}$$

3. $C = C_1 \cup C_2$

$C_1 : \mathbf{r}(u) = u\,\mathbf{i}, \quad u \in [0,1]; \qquad C_2 : \mathbf{r}(u) = \mathbf{i} + 2u\mathbf{j}, \quad u \in [0,1]$

$$\int_{C_1} (x - 2y)\,dx + 2x\,dy = \int_{C_1} x\,dx = \int_0^1 x(u)\,x'(u)\,du = \int_0^1 u\,du = \frac{1}{2}$$

$$\int_{C_2} (x - 2y)\,dx + 2x\,dy = \int_{C_2} 2x\,dy = \int_0^1 4\,du = 4$$

$$\int_C = \int_{C_1} + \int_{C_2} = \frac{9}{2}$$

4. $C = C_1 \cup C_2$

$C_1 : \mathbf{r}(u) = 2u\mathbf{j}, \quad u \in [0,1]; \quad C_2 : \quad \mathbf{r}(u) = u\mathbf{i} + 2\mathbf{j}, \quad u \in [0,1]$

$$\int_{C_1} (x - 2y)\,dx + 2x\,dy = \int_{C_1} 0\,dy = 0$$

$$\int_{C_2} (x - 2y)\,dx + 2x\,dy = \int_{C_2} (x - 4)\,dx = \int_0^1 (x(u) - 4)x'(u)\,du = \int_0^1 (u - 4)\,du = -\frac{7}{2}$$

$$\int_C = \int_{C_1} + \int_{C_2} = -\frac{7}{2}$$

5. $\mathbf{r}(u) = 2u^2\,\mathbf{i} + u\,\mathbf{j}, \quad u \in [0,1]$

$$\int_C y\,dx + xy\,dy = \int_0^1 [y(u)\,x'(u) + x(u)\,y(u)\,y'(u)]\,du = \int_0^1 (4u^2 + 2u^3)\,du = \frac{11}{6}$$

6. $\mathbf{r}(u) = 2u\mathbf{i} + u\mathbf{j}, \quad u \in [0,1]$

$$\int_C y\,dx + xy\,dy = \int_0^1 [y(u)x'(u) + x(u)y(u)y'(u)]\,du$$

$$= \int_0^1 (2u + 2u^2)\,du = \frac{5}{3}$$

7. $C = C_1 \cup C_2 \qquad C_1: \mathbf{r}(u) = u\,\mathbf{j}, \quad u \in [0,1]; \qquad C_2: \mathbf{r}(u) = 2u\,\mathbf{i} + \mathbf{j}, \quad u \in [0,1]$

$$\int_{C_1} y\,dx + xy\,dy = 0$$

$$\int_{C_2} y\,dx + xy\,dy = \int_{C_2} y\,dx = \int_0^1 y(u)\,x'(u)\,du = \int_0^1 2\,du = 2$$

$$\int_C = \int_{C_1} + \int_{C_2} = 2$$

8. $\mathbf{r}(u) = 2u^3\mathbf{i} + u\mathbf{j}, \quad u \in [0,1]$

$$\int_C y\,dx + xy\,dy = \int_0^1 [y(u)x'(u) + x(u)y(u)y'(u)]\,du$$

$$= \int_0^1 (6u^3 + 2u^4)\,du = \frac{19}{10}$$

9. $\mathbf{r}(u) = 2u\,\mathbf{i} + 4u\,\mathbf{j}, \quad u \in [0,1]$

$$\int_C y^2\,dx + (xy - x^2)\,dy = \int_0^1 \left\{y^2(u)x'(u) + \left[x(u)y(u) - x^2(u)\right]y'(u)\right\}\,du$$

$$= \int_0^1 \left[(4u)^2(2) + (8u^2 - 4u^2)(4)\right]\,du = \int_0^1 48u^2\,du = 16$$

10. $\mathbf{r}(u) = u\mathbf{i} + u^2\mathbf{j}, \quad u \in [0,2]$

$$\int_C y^2\,dx + (xy - x^2)\,dy = \int_0^2 [y^2(u)x'(u) + (x(u)y(u) - x^2(u))y'(u)]\,du$$

$$= \int_0^2 (3u^4 - 2u^3)\,du = \frac{56}{5}$$

11. $\mathbf{r}(u) = \frac{1}{8}u^2\,\mathbf{i} + u\,\mathbf{j}, \quad u \in [0,4]$

$$\int_C y^2\,dx + (xy - x^2)\,dy = \int_0^4 \left\{y^2(u)x'(u) + \left[x(u)y(u) - x^2(u)\right]y'(u)\right\}\,du$$

$$= \int_0^4 \left[u^2\left(\frac{u}{4}\right) + \left(\frac{u^2}{8}(u) - \left(\frac{u^2}{8}\right)^2(1)\right)\right]\,du$$

$$= \int_0^1 \left[\frac{3}{8}u^3 - \frac{1}{64}u^4\right]\,du = \frac{104}{5}$$

12. $C = C_1 \cup C_2 \qquad C_1 : \mathbf{r}(u) = 2u\mathbf{i}, \quad u \in [0,1]; \quad C_2 : \mathbf{r}(u) = 2\mathbf{i} + 4u\mathbf{j}, \quad u \in [0,1]$

$$\int_{C_1} y^2\,dx + (xy - x^2)\,dy = \int_{C_1} 0\,dx = 0$$

$$\int_{C_2} y^2\,dx + (xy - x^2)\,dy = \int_{C_2} (2y-4)\,dy = \int_0^1 [2y(u) - 4]y'(u)\,du = \int_0^1 16(2u-1)\,du = 0$$

$$\int_C = \int_{C_1} + \int_{C_2} = 0$$

13. $\mathbf{r}(u) = u\,\mathbf{i} + u\,\mathbf{j}, \quad u \in [\,0,1\,]$

$$\int_C (y^2 + 2x + 1)\,dx + (2xy + 4y - 1)\,dy$$

$$= \int_0^1 \left\{[y^2(u) + 2x(u) + 1]x'(u) + [2x(u)y(u) + 4y(u) - 1]y'(u)\right\}\,du$$

$$\int_0^1 \left[(u^2 + 2u + 1) + (2u^2 + 4u - 1)\right]\,du = \int_0^1 \left(3u^2 + 6u\right)\,du = 4$$

14. $\mathbf{r}(u) = u\mathbf{i} + u^2\mathbf{j}, \quad u \in [0,1].$

$$\int_C (y^2 + 2x + 1)\,dx + (2xy + 4y - 1)\,dy$$

$$= \int_0^1 \left[(y^2(u) + 2x(u) + 1)x'(u) + (2x(u)y(u) + 4y(u) - 1)y'(u)\right]\,du$$

$$= \int_0^1 (5u^4 + 8u^3 + 1)\,du = 4$$

15. $\mathbf{r}(u) = u\,\mathbf{i} + u^3\,\mathbf{j}, \quad u \in [\,0,1\,]$

$$\int_C (y^2 + 2x + 1)\,dx + (2xy + 4y - 1)\,dy$$

$$= \int_0^1 \left\{[y^2(u) + 2x(u) + 1]x'(u) + [2x(u)y(u) + 4y(u) - 1]y'(u)\right\}\,du$$

$$= \int_0^1 \left[(u^6 + 2u + 1) + (2u^4 + 4u^3 - 1)3u^2\right]\,du = \int_0^1 \left(7u^6 + 12u^5 - 3u^2 + 2u + 1\right)\,du = 4$$

16. $C = C_1 \cup C_2 \cup C_3$

$C_1 : \mathbf{r}(u) = 4u\mathbf{i},\ u \in [0,1]; \quad C_2 : \mathbf{r}(u) = 4\mathbf{i} + 2u\mathbf{j},\ u \in [0,1]; \quad C_3 : \mathbf{r}(u) = (4-3u)\mathbf{i} + (2-u)\mathbf{j},\ u \in [0,1]$

$$\int_{C_1} = \int_{C_1} (y^2 + 2x + 1)\,dx = \int_0^1 4(8u+1)\,du = 20$$

$$\int_{C_2} = \int_{C_2} (8y + 4y - 1)\,dy = \int_0^1 2(24u - 1)\,du = 22$$

$$\int_{C_3} = \int_0^1 \left\{-3\left[(2-u)^2 + 2(4-3u) + 1\right] - \left[2(4-3u)(2-u) + 4(2-u) - 1\right]\right\}\,du$$

$$= \int_0^1 (-9u^2 + 54u - 62)\, du = -38.$$

$$\int_C = \int_{C_1} + \int_{C_2} + \int_{c_3} = 20 + 22 - 38 = 4.$$

17. $\mathbf{r}(u) = u\mathbf{i} + u\mathbf{j} + u\mathbf{k}, \quad u \in [0,1]$

$$\int_C y\,dx + 2z\,dy + x\,dz = \int_0^1 [y(u)\,x'(u) + 2z(u)\,y'(u) + x(u)\,z'(u)]\,du = \int_0^1 4u\,du = 2$$

22.

$$\int_C y\,dx + 2z\,dy + x\,dz = \int_0^1 [y(u)x'(u) + 2z(u)y'(u) + x(u)z'(u)]\,du$$

$$= \int_0^1 (u^2 + 3u^3 + 4u^4)\,du = \frac{113}{60}$$

19. $C = C_1 \cup C_2 \cup C_3$

$C_1: \mathbf{r}(u) = u\mathbf{k}, \quad u \in [0,1]; \quad C_2: \mathbf{r}(u) = u\mathbf{j} + \mathbf{k}, \quad u \in [0,1]; \quad C_3: \mathbf{r}(u) = u\mathbf{i} + \mathbf{j} + \mathbf{k}, \quad u \in [0,1]$

$$\int_{C_1} y\,dx + 2z\,dy + x\,dz = 0$$

$$\int_{C_2} y\,dx + 2z\,dy + x\,dz = \int_{C_2} 2z\,dy = \int_0^1 2z(u)\,y'(u)\,du = \int_0^1 2\,du = 2$$

$$\int_{C_3} y\,dx + 2z\,dy + x\,dz = \int_{C_3} y\,dx = \int_0^1 y(u)\,x'(u)\,du = \int_0^1 du = 1$$

$$\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = 3$$

20. $C = C_1 \cup C_2 \cup C_3$

$C_1 : \mathbf{r}(u) = u\mathbf{i}, \quad u \in [0,1]; \quad C_2 : \mathbf{r}(u) = \mathbf{i} + u\mathbf{j}, \quad u \in [0,1]; \quad C_3 : \mathbf{r}(u) = \mathbf{i} + \mathbf{j} + u\mathbf{k}, \quad u \in [0,1]$

$$\int_{C_1} y\,dx + 2z\,dy + x\,dz = 0$$

$$\int_{C_2} y\,dx + 2z\,dy + x\,dz = 0$$

$$\int_{C_3} y\,dx + 2z\,dy + x\,dz = \int_{C_3} x\,dz = \int_0^1 du = 1$$

21. $\mathbf{r}(u) = 2u\,\mathbf{i} + 2u\,\mathbf{j} + 8u\,\mathbf{k}, \quad u \in [0,1]$

$$\begin{aligned}
&\int_C xy\,dx + 2z\,dy + (y+z)\,dz \\
&= \int_0^1 \{x(u)y(u)x'(u) + 2z(u)y'(u) + [y(u)+z(u)]z'(u)\}\,du \\
&= \int_0^1 [(2u)(2u)(2) + 2(8u)(2) + (2u+8u)(8)]\,du \\
&= \int_0^1 \left(8u^2 + 112u\right)\,du = \frac{176}{3}
\end{aligned}$$

22. $C = C_1 \cup C_2 \cup C_3$

$C_1 : \mathbf{r}(u) = 2u\mathbf{i}, \quad u \in [0,1]; \quad C_2 : \mathbf{r}(u) = 2\mathbf{i} + 2u\mathbf{j}, \quad u \in [0,1]; \quad C_3 : \mathbf{r}(u) = 2\mathbf{i} + 2\mathbf{j} + 2u\mathbf{k}, \quad u \in [0,1]$

$$\int_{C_1} xy\,dx + 2z\,dy + (y+z)\,dz = 0$$

$$\int_{C_2} xy\,dx + 2z\,dy + (y+z)\,dz = 0$$

$$\int_{C_3} xy\,dx + 2z\,dy + (y+z)\,dz = \int_{C_2} (y+z)\,dz = \int_0^1 2(2+2u)\,du = 6$$

23. $\mathbf{r}(u) = u\,\mathbf{i} + u\,\mathbf{j} + 2u^2\,\mathbf{k}, \quad u \in [0,2]$

$$\begin{aligned}
&\int_C xy\,dx + 2z\,dy + (y+z)\,dz \\
&= \int_0^2 \{x(u)y(u)x'(u) + 2z(u)y'(u) + [y(u)+z(u)]z'(u)\}\,du \\
&= \int_0^2 \left[(u)(u)(1) + 2(2u^2)(1) + (u+2u^2)(4u)\right]\,du \\
&= \int_0^2 \left(8u^3 + 9u^2\right)\,du = 56
\end{aligned}$$

24. $C = C_1 \cup C_2$

$C_1 : \mathbf{r}(u) = 2u\mathbf{i} + 2u\mathbf{j} + 2u\mathbf{k},\ u \in [0,1]; \quad C_2 : \mathbf{r}(u) = 2\mathbf{i} + 2\mathbf{j} + (2+6u)\mathbf{k},\ u \in [0,1].$

$$\int_{C_1} xy\,dx + 2z\,dy + (y+z)\,dz = \int_0^1 8(u^2+2u)\,du = \frac{32}{3}$$

$$\int_{C_2} xy\,dx + 2z\,dy + (y+z)\,dz = \int_{C_2} (y+z)\,dz = \int_0^1 6(4+6u)\,du = 42$$

$$\int_C = \int_{C_1} + \int_{C_2} = \frac{158}{3}$$

25. $\mathbf{r}(u) = (u-1)\,\mathbf{i} + (1+2u^2)\,\mathbf{j} + u\,\mathbf{k}, \quad u \in [1,2]$

$$\begin{aligned}&\int_C x^2y\,dx + y\,dy + xz\,dz \\ &= \int_1^2 \left[x^2(u)y(u)x'(u) + y(u)y'(u) + x(u)z(u)z'(u)\right]\,du \\ &= \int_1^2 \left[(u-1)^2(1+2u^2)(1) + (1+2u^2)(4u) + (u-1)u\right]\,du \\ &= \int_1^2 \left(2u^4 + 4u^3 + 4u^2 + u + 1\right)\,du = \frac{1177}{30}\end{aligned}$$

26. $\mathbf{r}(u) = \left(2 - \dfrac{u^2}{2}\right)\mathbf{i} + \sqrt{1 - \dfrac{u^2}{4}}\,\mathbf{j} + u\,\mathbf{k}, \quad u \in [0,2]$

$$\int_C y\,dx + yz\,dy + z(x-1)\,dz = \int_0^1 \left[-u^2\sqrt{1-\frac{u^2}{4}} + \frac{u^2}{2}(2-u^2) + u(1-\frac{u^2}{2})\right]\,du = -\frac{\pi}{2} + \frac{73}{120}$$

27. (a) $\dfrac{\partial P}{\partial y} = 6x - 4y = \dfrac{\partial Q}{\partial x}$

$$\frac{\partial f}{\partial x} = x^2 + 6xy - 2y^2 \quad \Longrightarrow \quad f(x,y) = \frac{1}{3}x^3 + 3x^2y - 2xy^2 + g(y)$$

$$\frac{\partial f}{\partial y} = 3x^2 - 4xy + g'(y) = 3x^2 - 4xy + 2y \quad \Longrightarrow \quad g'(y) = 2y \quad \Longrightarrow g(y) = y^2 + C$$

Therefore, $f(x,y) = \dfrac{1}{3}x^3 + 3x^2y - 2xy^2 + y^2$ (take $C = 0$)

(b) $\displaystyle\int_C (x^2 + 6xy - 2y^2)\,dx + (3x^2 - 4xy + 2y)\,dy = [f(x,y)]_{(3,0)}^{(0,4)} = 7$

(c) $\displaystyle\int_C' (x^2 + 6xy - 2y^2)\,dx + (3x^2 - 4xy + 2y)\,dy = [f(x,y)]_{(4,0)}^{(0,3)} = -\frac{37}{3}$

28. (a) $\mathbf{F} = \nabla f$ where $f(x,y,z) = x^2y + xz^2 - y^2z$

(b) $\displaystyle\int_C (2xy + z^2)\,dx + (x^2 - 2yz)\,dy + (2xz - y^2)\,dz = f(3,2,-1) - f(1,0,1) = 25 - 1 = 24$

(c) $\displaystyle\int_{C'} (2xy + z^2)\,dx + (x^2 - 2yz)\,dy + (2xz - y^2)\,dz = f(1,0,1) - f(3,2-1) = -24$

29. $s'(u) = \sqrt{[x'(u)]^2 + [y'(u)]^2} = a$

(a) $\displaystyle M = \int_C k(x+y)\,ds = k\int_0^{\pi/2} [x(u) + y(u)]\,s'(u)\,du = ka^2\int_0^{\pi/2} (\cos u + \sin u)\,du = 2ka^2$

(b)
$$\begin{aligned}x_M M &= \int_C kx(x+y)\,ds = k\int_0^{\pi/2} x(u)\,[x(u) + y(u)]\,s'(u)\,du \\ &= ka^3\int_0^{\pi/2} (\cos^2 u + \cos u \sin u)\,du = \frac{1}{4}ka^3(\pi + 2)\end{aligned}$$

$$y_M M = \int_C ky(x+y)\,ds = k\int_0^{\pi/2} y(u)\,[x(u)+y(u)]\,s'(u)\,du$$
$$= ka^3\int_0^{\pi/2}(\sin u\cos u + \sin^2 u)\,du = \frac{1}{4}ka^3(\pi+2)$$

$x_M = y_M = \frac{1}{8}a(\pi+2)$

30. (a) $I = \int_C \frac{M}{L}x^2\,ds = \frac{Ma^2}{2\pi}\int_0^{2\pi}\cos^2 u\,du = \frac{1}{2}Ma^2$

(b) $I = \int_C \frac{M}{L}a^2\,ds = \frac{Ma}{2\pi}\int_C ds = Ma^2$

31. (a) $I_z = \int_C k(x+y)a^2\,ds = a^2\int_C k(x+y)\,ds = a^2M = Ma^2$

(b) The distance from the point (x,y) to the line $y=x$ is $|y-x|/\sqrt{2}$. Therefore

$$I = \int_C k(x+y)\left[\frac{1}{2}(y-x)^2\right]ds = \frac{1}{2}k\int_0^{\pi/2}(a\cos u + a\sin u)(a\sin u - a\cos u)^2 a\,du$$
$$= \frac{1}{2}ka^4\int_0^{\pi/2}(\sin u - \cos u)^2\frac{d}{du}(\sin u - \cos u)\,du$$
$$= \frac{1}{2}ka^4\left[\frac{1}{3}(\sin u - \cos u)^3\right]_0^{\pi/2} = \frac{1}{3}ka^4.$$

From Exercise 29, $M = 2ka^2$. Therefore

$$I = \tfrac{1}{6}(2ka^2)a^2 = \tfrac{1}{6}Ma^2.$$

32. (a) $M = \int_C k\,ds = \int_0^{2\pi} k\sqrt{\sin^2 u + (1-\cos u)^2}\,du = \int_0^{2\pi} 2k\sin\frac{1}{2}u\,du = 8k$

(b)
$$x_M M = \int_C kx\,ds = \int_0^{2\pi}[(1-\cos u)(2k\sin\frac{1}{2}u)]\,du$$
$$= 4k\int_0^{2\pi}\sin^3\frac{1}{2}u\,du = \frac{32}{3}k;\quad x_M = \frac{4}{3}$$

$$y_M M = \int_C ky\,ds = \int_0^{2\pi}[u-\sin u)(2k\sin\frac{1}{2}u)]\,du$$
$$= 2k\int_0^{2\pi}(u\sin\frac{1}{2}u - 2\sin^2\frac{1}{2}u\cos\frac{1}{2}u)\,du$$
$$= 8\pi k;\quad y_M = \pi$$

33. (a) $s'(u) = \sqrt{a^2+b^2}$

$$L = \int_C ds = \int_0^{2\pi} \sqrt{a^2+b^2}\,du = 2\pi\sqrt{a^2+b^2}$$

(b) $x_M = 0, \quad y_M = 0$ (by symmetry)

$$z_M = \frac{1}{L}\int_C z\,ds = \frac{1}{2\pi\sqrt{a^2+b^2}}\int_0^{2\pi} bu\sqrt{a^2+b^2}\,du = b\pi$$

(c) $$I_x = \int_C \frac{M}{L}(y^2+z^2)\,ds = \frac{M}{2\pi}\int_0^{2\pi}(a^2\sin^2 u + b^2u^2)\,du = \frac{1}{6}M(3a^2\pi + 8b^2\pi^2)$$

$I_y = \frac{1}{6}M(3a^2\pi + 8b^2\pi^2)$ similarly

$I_z = Ma^2$ (all the mass is at distance a from the z-axis)

34. (a) $s'(u) = 2u^2+1$

$$L = \int_C ds = \int_0^a (2u^2+1)\,du = \frac{2}{3}a^3 + a = \frac{a(2a^2+3)}{3}$$

(b) $$x_M = \frac{1}{L}\int_C x\,ds = \frac{3}{a(2a^2+3)}\int_0^a (2u^3+u)\,du = \frac{3a(a^2+1)}{2(2a^2+3)}$$

$$y_M = \frac{1}{L}\int_C y\,ds = \frac{3}{a(2a^2+3)}\int_0^a (2u^4+u^2)\,du = \frac{a^2(6a^2+5)}{5(2a^2+3)}$$

$$z_M = \frac{1}{L}\int_C z\,ds = \frac{3}{a(2a^3+3)}\int_0^a \left(\frac{4}{3}u^5 + \frac{2}{3}u^3\right)du = \frac{a^3(4a^2+3)}{6(2a^3+3)}$$

(c) $$I_z = \frac{M}{L}\int_C (x^2+y^2)\,ds = \frac{3M}{a(2a^3+3)}\int_0^a [(u^2+u^4)(2u^2+1)]\,du$$
$$= \frac{Ma^2(30a^4+63a^2+35)}{35(2a^2+3)}$$

35. $$M = \int_C k(x^2+y^2+z^2)\,ds$$
$$= k\sqrt{a^2+b^2}\int_0^{2\pi}(a^2+b^2u^2)\,du = \frac{2}{3}\pi k\sqrt{a^2+b^2}\,(3a^2+4\pi^2b^2)$$

36. $C: \mathbf{r} = \mathbf{r}(u), \quad u \in [a,b]$

$$\int_C \mathbf{h}(\mathbf{r})\cdot d\mathbf{r} = \int_a^b [\mathbf{h}(\mathbf{r}(u))\cdot\mathbf{r}'(u)]\,du$$
$$= \int_a^b \left[\mathbf{h}(\mathbf{r}(u))\cdot\frac{\mathbf{r}'(u)}{\|\mathbf{r}'(u)\|}\right]\|\mathbf{r}'(u)\|\,du$$
$$= \int_a^b [\mathbf{h}(\mathbf{r}(u))\cdot\mathbf{T}(\mathbf{r}(u))]s'(u)\,du$$
$$= \int_C [\mathbf{h}(\mathbf{r})\cdot\mathbf{T}(\mathbf{r})]\,ds$$

SECTION 17.5

1. (a) $\oint_C xy\,dx + x^2\,dy = \int_{C_1} xy\,dx + x^2\,dy + \int_{C_2} xy\,dx + x^2\,dy + \int_{C_3} xy\,dx + x^2\,dy,$ where

$C_1: \mathbf{r}(u) = u\,\mathbf{i} + u\,\mathbf{j},\quad u \in [0,1];\quad C_2: \mathbf{r}(u) = (1-u\,\mathbf{i} + \mathbf{j},\quad u \in [0,1]$

$C_3: \mathbf{r}(u) = (1-u)\,\mathbf{j},\quad u \in [0,1].$

$$\int_{C_1} xy\,dx + x^2\,dy = \int_0^1 (u^2+u^2)\,du = \frac{2}{3}$$

$$\int_{C_2} xy\,dx + x^2\,dy = \int_0^1 -(1-u)\,du = -\frac{1}{2}$$

$$\int_{C_3} xy\,dx + x^2\,dy = \int_0^1 0^2(-1)\,du = 0$$

Therefore, $\oint_C xy\,dx + x^2\,dy = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}.$

(b) $\oint_C xy\,dx + x^2\,dy = \iint_\Omega x\,dx\,dy = \int_0^1 \int_0^y x\,dx\,dy = \int_0^1 \left[\frac{1}{2}x^2\right]_0^y du = \frac{1}{2}\int_0^1 y^2\,dy = \frac{1}{6}$

2. (a) $C = C_1 \cup C_3 \cup C_3 \cup C_4$

$C_1: \mathbf{r}(u) = u\mathbf{i},\ u \in [0,1];\quad C_2: \mathbf{r}(u) = \mathbf{i} + u\mathbf{j},\ u \in [0,1]$

$C_3: \mathbf{r}(u) = (1-u)\mathbf{i} + \mathbf{j},\ u \in [0,1];\quad C_4: \mathbf{r}(u) = (1-u)\mathbf{j},\quad u \in [0,1]$

$$\int_{C_1} x^2y\,dx + 2y^2\,dy = 0$$

$$\int_{C_2} x^2y\,dx + 2y^2\,dy = \int_{C_2} 2y^2\,dy = \int_0^1 2u^2\,du = \frac{2}{3}$$

$$\int_{C_3} x^2y\,dx + 2y^2\,dy = \int_{C_3} x^2\,dx = \int_0^1 -(1-u)^2\,du = -\frac{1}{3}$$

$$\int_{C_4} x^2y\,dx + 2y^2\,dy = \int_{C_4} 2y^2\,dy = \int_0^1 -2(1-u)^2\,du = -\frac{2}{3}$$

$$\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = -\frac{1}{3}$$

(b) $\oint_C x^2y\,dx + 2y^2\,dy = \iint_\Omega \left[\frac{\partial}{\partial x}(2y^2) - \frac{\partial}{\partial y}(x^2y)\right] dx\,dy = \int_0^1 \int_0^1 -x^2\,dx\,dy = -\frac{1}{3}$

3. (a) $C:\ \mathbf{r}(u) = \cos u\,\mathbf{i} + \sin u\,\mathbf{j},\quad u \in [\,0, 2\pi\,]$

$$\oint_C (3x^2 + y)\,dx + (2x + y^3)\,dy \int_0^{2\pi} \left[(3\cos^2 u + \sin u)(-\,\sin u) + (2\cos u + \sin^3 u)\cos u\right]\,du$$

$$= \int_0^{2\pi} \left[3\cos^2 u(-\sin u) - \sin^2 u + 2\cos^2 u + \sin^3 u\,\cos u\right]\,du$$

$$= \left[\cos^3 u - \frac{1}{2}u + \frac{1}{4}\,sin2u + u + \frac{1}{2}\,\sin 2u + \frac{1}{4}\,sin^4 u\right]_0^{2\pi} = \pi$$

(b) $\oint_C (3x^2 + y)\,dx + (2x + y^3)\,dy = \iint_\Omega 1\,dx\,dy = \text{area } \Omega = \pi$

4. (a) $C = C_1 \cup C_2$

$C_1 : \mathbf{r}(u) = u\mathbf{i} + u^2\mathbf{j},\ u \in [0,1];\quad C_2 : \mathbf{r}(u) = (1-u)\mathbf{i} + (1-u)\mathbf{j},\ u \in [0,1]$

$$\int_{C_1} y^2\,dx + x^2\,dy = \int_0^1 (u^4 + 2u^3)\,du = \frac{7}{10}$$

$$\int_{C_2} y^2\,dx + x^2\,dy = \int_0^1 -2(1-u)^2\,du = -\frac{2}{3};\quad \oint_C = \int_{C_1} + \int_{C_2} = \frac{1}{30}$$

(b) $\oint_C y^2\,dx + x^2\,dy = \iint\limits_\Omega \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(y^2)\right]\,dx\,dy = \int_0^1 \int_{x^2}^{x} 2(x-y)\,dy\,dx = \frac{1}{30}$

5. $\oint_C 3y\,dx + 5x\,dy = \iint\limits_\Omega (5-3)\,dxdy = 2A = 2\pi$

6. $\oint_C 5x\,dx + 3y\,dy = \iint\limits_\Omega 0\,dx\,dy = 0$

7. $\oint_C x^2\,dy = \iint\limits_\Omega 2x\,dxdy = 2\overline{x}A = 2\left(\frac{a}{2}\right)(ab) = a^2 b$

8. $\oint_C y^2\,dx = \iint\limits_\Omega -2y\,dx\,dy = -ab^2$

9.
$$\oint_C (3xy + y^2)\,dx + (2xy + 5x^2)\,dy = \iint\limits_\Omega [(2y + 10x) - (3x + 2y)]\,dxdy$$

$$= \iint\limits_\Omega 7x\,dxdy = 7\,\overline{x}A = 7(1)(\pi) = 7\pi$$

10. $\oint_C (xy + 3y^2)\,dx + (5xy + 2x^2)\,dy = \iint\limits_\Omega (3x - y)\,dx\,dy = (3\overline{x} - \overline{y})A = (3+2)\pi = 5\pi.$

11. $$\oint_C (2x^2+xy-y^2)\,dx+(3x^2-xy+2y^2)\,dy=\iint_\Omega [(6x-y)-(x-2y)]\,dxdy$$
$$=\iint_\Omega (5x+y)\,dxdy=(5\overline{x}+\overline{y})A=(5a+0)(\pi r^2)=5a\pi r^2$$

12. $$\oint_C (x^2-2xy+3y^2)\,dx+(5x+1)\,dy=\iint_\Omega (5+2x-6y)\,dx\,dy=(5+2\overline{x}-6\overline{y})A=(5-6b)\pi r^2$$

13. $$\oint_C e^x\sin y\,dx+e^x\cos y\,dy=\iint_\Omega [e^x\cos y-e^x\cos y]\,dxdy=0$$

14. $$\oint_C e^x\cos y\,dx+e^x\sin y\,dy=\iint_\Omega 2e^x\sin y\,dx\,dy=\int_0^1\int_0^\pi 2e^x\sin y\,dy\,dx=4(e-1)$$

15. $$\oint_C 2xy\,dx+x^2\,dy=\iint_\Omega [2x-2x]\,dxdy=0$$

16. $$\oint_C y^2\,dx+2xy\,dy=\iint_\Omega 0\,dx\,dy=0$$

17. $C:\ \mathbf{r}(u)=a\cos u\,\mathbf{i}+a\sin u\,\mathbf{j};\quad u\in[0,2\pi]$
$$A=\oint_C -y\,dx=\int_0^{2\pi}(-a\sin u)(-\sin u)\,du=a^2\int_0^{2\pi}\sin^2 u\,du=a^2\left[\frac{1}{2}u-\frac{1}{4}\sin 2u\right]_0^{2\pi}=\pi a^2$$

18. $C:\mathbf{r}(u)=a\cos^3 u\mathbf{i}+a\sin^3 u\mathbf{j},\quad u\in[0,2\pi]$
$$A=\oint_C -y\,dx=\int_0^{2\pi}(-a\sin^3 u)(-3a\cos^2 u\sin u)\,du=3a^2\int_0^{2\pi}\sin^4 u\cos^2 u\,du=\frac{3}{8}\pi a^2$$

19. $A=\oint_C x\,dy$, where $C=C_1\cup C_2\cup C_3$;
$$C_1:\ \mathbf{r}(u)=au\,\mathbf{i},\ \ u\in[0,1];\quad C_2:\ \mathbf{r}(u)=a(1-u)\,\mathbf{i}+bu\,\mathbf{j},\ \ u\in[0,1];$$
$$C_3:\ \mathbf{r}(u)=b(1-u)\,\mathbf{j},\ \ u\in[0,1].$$
$$\int_{C_1}x\,dy=0;\quad \int_{C_2}x\,dy=\int_0^1 ab(1-u)\,du=\frac{1}{2}ab;\quad \int_{C_3}x\,dy=0$$
Therefore, $A=\frac{1}{2}ab$.

20. $$A=\int_{-2}^{2}(4-x^2)\,dx=\frac{32}{3}$$

21. $$\oint_C (ay+b)\,dx+(cx+d)\,dy=\iint_\Omega (c-a)\,dxdy=(c-a)A$$

22. $\oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \iint_\Omega (-5)\,dx\,dy = -5A = -\frac{15}{8}\pi a^2$

Exercise 18

23. We take the arch from $x = 0$ to $x = 2\pi R$. (Figure 9.11.1) Let C_1 be the line segment from $(0,0)$ to $(2\pi R, 0)$ and let C_2 be the cycloidal arch from $(2\pi R, 0)$ back to $(0,0)$. Letting $C = C_1 \cup C_2$, we have

$$\begin{aligned} A &= \oint_C x\,dy = \int_{C_1} x\,dy + \int_{C_2} x\,dy = 0 + \int_{C_2} x\,dy \\ &= \int_{2\pi}^{0} R(\theta - \sin\theta)(R\sin\theta)\,d\theta \\ &= R^2 \int_0^{2\pi} (\sin^2\theta - \theta\sin\theta)\,d\theta \\ &= R^2 \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} + \theta\cos\theta - \sin\theta\right]_0^{2\pi} = 3\pi R^2. \end{aligned}$$

24. $\oint_C y^3\,dx + (3x - x^3)\,dy = \iint_\Omega (3 - 3x^2 - 3y^2)\,dx\,dy = 3\iint_\Omega (1 - x^2 - y^2)\,dx\,dy$

The double integral is maximized by

$$\Omega : 0 \le x^2 + y^2 \le 1.$$

(This is the maximal region on which the integral is nonnegative.) The line integral is maximized by the unit circle traversed counterclockwise.

25. $\mathbf{F}(x,y) = (x^2 - y^3)\,\mathbf{i} + (x^2 + y^2)\,\mathbf{j}; \quad C:\ \mathbf{r} = \cos u\,\mathbf{i} + \sin u\,\mathbf{j}, \quad u \in [0, 2\pi]$

$$\begin{aligned} W = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_\Omega (2x + 3y^2)\,dx\,dy &= \int_0^{2\pi}\int_0^1 (2r\cos\theta + 3r^2\sin^2\theta)\,r\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1 (2r^2\cos\theta + 3r^3\sin^2\theta)\,dr\,d\theta \\ &= \frac{2}{3}\int_0^{2\pi}\cos\theta\,d\theta + \frac{3}{4}\int_0^{2\pi}\sin^2\theta\,d\theta \\ &= \frac{2}{3}\left[\sin\theta\right]_0^{2\pi} + \frac{3}{4}\left[\frac{1}{2}u - \frac{1}{4}\sin 2u\right]_0^{2\pi} = \frac{3}{4}\pi \end{aligned}$$

26. $W = \oint_C (x^3 - x^2y)\,dx + xy^2\,dy = \int_0^1 \int_{x^2}^{\sqrt{x}} (y^2 + x^2)\,dy\,dx = \frac{6}{35}$

27. Taking Ω to be of type II (see Figure 17.5.2), we have

$$\begin{aligned} \iint_\Omega \frac{\partial Q}{\partial x}(x,y)\,dxdy &= \int_c^d \int_{\phi_3(y)}^{\phi_4(y)} \frac{\partial Q}{\partial x}(x,y)\,dx\,dy \\ &= \int_c^d \{Q[\phi_4(y), y] - Q[\phi_3(y), y]\}\,dy \\ (*) &= \int_c^d Q[\phi_4(y), y]\,dy - \int_c^d Q[\phi_3(y), y]\,dy. \end{aligned}$$

The graph of $x=\phi_4(y)$ from $x=c$ to $x=d$ is the curve

$$C_4: \mathbf{r}_4(u)=\phi_4(u)\mathbf{i}+u\mathbf{j}, \qquad u\in[c,d].$$

The graph of $x=\phi_3(y)$ from $x=c$ to $x=d$ is the curve

$$C_3: \mathbf{r}_3(u)=\phi_3(u)\mathbf{i}+u\mathbf{j}, \qquad u\in[c,d].$$

Then

$$\oint_C Q(x,y)\,dy=\int_{C_4} Q(x,y)\,dy-\int_{C_3} Q(x,y)\,dy$$
$$=\int_c^d Q[\phi_4(u),u]\,du-\int_c^d Q[\phi_3(u),u]\,du.$$

Since u is a dummy variable, it can be replaced by y. Comparison with $(*)$ gives the result.

28. $-\frac{\lambda}{3}\oint_{y^3\,dx}=-\frac{\lambda}{3}\iint\limits_{\Omega}(-3y^2)\,dx\,dy=\iint\limits_{\Omega}\lambda y^2\,dx\,dy=I_x$

$\frac{\lambda}{3}\oint_{x^3}dy=-\frac{\lambda}{3}\iint\limits_{\Omega}3x^2\,dx\,dy=\iint\limits_{\Omega}\lambda x^2\,dx\,dy=I_y$

29. $\oint_{C_1}=\oint_{C_2}+\oint_{C_3}$

30. Let Ω be the region enclosed by C. Then

$$\int_C f(x)\,dx+g(y)\,dy=\pm\oint_C f(x)\,dx+g(y)\,dy$$
$$=\pm\iint\limits_{\Omega}\overbrace{\left(\frac{\partial}{\partial x}[g(y)]-\frac{\partial}{\partial y}[f(x)]\right)}^{0}dx\,dy=0$$

31. $\dfrac{\partial P}{\partial y}=\dfrac{-2xy}{(x^2+y^2)^2}=\dfrac{\partial Q}{\partial x}$ except at $(0,0)$

(a) If C does not enclose the origin, and Ω is the region enclosed by C, then

$$\oint_C \frac{x}{x^2+y^2}\,dx+\frac{y}{x^2+y^2}\,dy=\iint\limits_{\Omega}0\,dxdy=0.$$

(b) If C does enclose the origin, then

$$\oint_C=\oint_{C_a}$$

where $C_a: \mathbf{r}(u)=a\cos u\,\mathbf{i}+a\sin u\,\mathbf{j}$, $u\in[0,2\pi]$ is a small circle in the inner region of C.

In this case

$$\oint_C=\int_0^{2\pi}\left[\frac{a\cos u}{a^2}(-a\sin u)+\frac{a\sin u}{a^2}(a\cos u)\right]du=\int_0^{2\pi}0\,du=0.$$

The integral is still 0.

32. (a) $\oint_C -\dfrac{y^3}{(x^2+y^2)^2}\,dx + \dfrac{xy^2}{(x^2+y^2)^2}\,dy = \iint_\Omega 0\,dy\,dx = 0$

(b) By Green's theorem, $\oint_C = \oint_{C'}$, where C' is a circle about the origin. $\mathbf{r}(u) = a\cos u\mathbf{i} + a\sin u\mathbf{j}$.

$$\oint_{C'} = \int_0^{2\pi} (\sin^4 u + \sin^2 u\cos^2 u)\,du = \int_0^{2\pi} \sin^2 u\,du = \pi$$

33. If Ω is the region enclosed by C, then

$$\oint_C \mathbf{v}\cdot d\mathbf{r} = \oint_C \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy = \iint_\Omega \left\{\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right)\right\} dxdy$$
$$= \iint_\Omega 0\,dxdy = 0.$$

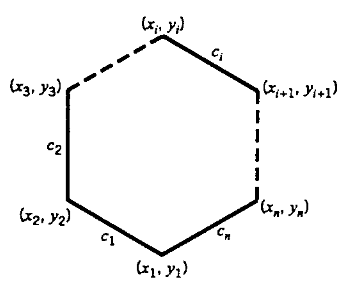
equality of mixed partials

34. $\mathbf{r}(u) = [x_1 + (x_2 - x_1)u]\mathbf{i} + [y_1 + (y_2 - y_1)u]\mathbf{j}, \quad u \in [0,1]$

$$\int_C -y\,dx + x\,dy = \int_0^1 \{[-y_1 - (y_2 - y_1)u]\,(x_2 - x_1) + [x_1 + (x_2 - x_1)u]\,(y_2 - y_1)\}\,du$$
$$= \int_0^1 (x_1y_2 - x_2y_1)\,du = x_1y_2 - x_2y_1.$$

35. $A = \dfrac{1}{2}\oint_C (-y\,dx + x\,dy)$

$= \left[\int_{C_1} + \int_{C_2} + \cdots \int_{C_n}\right]$



Now

$$\int_{C_i} (-y\,dx + x\,dy) = \int_0^1 \{[y_i + u(y_{i+1} - y_i)]\,(x_{i+1} - x_i) + [x_i + u(x_{i+1} - x_i)]\,(y_{i+1} - y_i)\}\,du$$
$$= x_iy_{i+1} - x_{i+1}y_i, \quad i = 1, 2, \ldots, n; \; x_{n+1} = x_1, \; y_{n+1} = y_1$$

Thus, $A = \dfrac{1}{2}\,[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_ny_1 - x_1y_n)]$

36. (a) $A = \dfrac{1}{2}[0 + (8 - 1) + 0] = \dfrac{7}{2}$

(b) $A = \dfrac{1}{2}[0 + (12 - 2) + (12 - 0) + (0 + 6) + 0] = 14$

PROJECT 17.5

1. Replace y by xt and solve for x to get $x = \dfrac{3at}{1+t^3}$

Then $y = xt = \dfrac{3at^2}{1+t^3}$

2. $\lim_{t\to -1^-} x(t) = \lim_{t\to -1^-} \dfrac{3at}{1+t^3} = \infty$

$\lim_{t\to -1^-} y(t) = \lim_{t\to -1^-} \dfrac{3at^2}{1+t^3} = -\infty$

$\lim_{t\to -1^+} x(t) = \lim_{t\to -1^+} \dfrac{3at}{1+t^3} = -\infty$

$\lim_{t\to -1^+} y(t) = \lim_{t\to -1^+} \dfrac{3at^2}{1+t^3} = \infty$

$x(t)+y(t) = \dfrac{3at}{1+t^3} + \dfrac{3at^2}{1+t^3} = \dfrac{3at(1+t)}{(1+t)(1-t+t^2)} = \dfrac{3at}{1-t+t^2}$ and $\lim_{t\to -1}[x(t)+y(t)] = -a$,

so $x(t)+y(t)+a \to 0$ as $t \to -1$.

3. The curve is symmetric with respect to the line $y = x$ since interchanging x and y leaves the equation unchanged.

As t varies from $-\infty$ to -1, the point $(x(t), y(t))$ moves along the curve from $(0,0)$ to "$(\infty, -\infty)$"; as t varies from -1 to ∞, the point $(x(t), y(t))$ moves along the curve from "$(-\infty, \infty)$" to $(0,0)$. The loop is traced out in the counter-clockwise direction at t increases from 0 to ∞.

4. The bottom half C of the loop is formed by the curves

$C_1:\quad x(t) = \dfrac{3at}{1+t^3},\ y(t) = \dfrac{3at^2}{1+t^3},\quad 0 \le t \le 1$ and

$C_2:\quad x(t) = 1-t,\ y(t) = 1-t,\quad 0 \le t \le 1.$

Area enclosed by the loop:

$$A = 2\left[\frac{1}{2}\oint_C -y\,dx + x\,dy\right] = \oint_C -y\,dx + x\,dy = \oint_{C_1} -y\,dx + x\,dy + \oint_{C_2} -y\,dx + x\,dy$$

$$= 3a^2\int_0^1 \frac{3t^2}{(1+t^3} dt = \frac{3}{2}a^2.$$

5. Rotate the axes through an angle of $\frac{\pi}{4}$ in the clockwise direction and then translate the origin to the point $(0, -\frac{a}{\sqrt{2}})$. In the new coordinate system, the parametric equations of the curve are:

$$u = \frac{3a}{\sqrt{2}}\left(\frac{t-t^2}{1+t^3}\right),\quad v = \frac{3a}{\sqrt{2}}\left(\frac{t+t^2}{1+t^3}\right) + \frac{a}{\sqrt{2}}$$

and the u-axis is the asymptote. Using the third equation in (17.5.2), it can be shown that the area between the curve and the asymptote is given by:

$$A = \int_{-1}^{0} [v(t)u'(t) - u(t)v'(t)]\,dt = \frac{3}{2}a^2$$

6. (b) As an alternative solution in Problem 4, the whole loop is formed by

$$x(t) = \frac{(2n+1)at^n}{1+t^{2n+1}}, \quad y(t) = \frac{(2n+1)at^{n+1}}{1+t^{2n+1}}, \quad 0 \le t \le \infty.$$

Thus, the area enclosed by the loop is given by :

$$A = \frac{1}{2}\oint_C -y\,dx + x\,dy = a^2(2n+1)^2 \int_0^\infty \left[\frac{(2n+1)t^{2n}}{(1+t^{2n+1})^3} - \frac{nt^{2n}}{(1+t^{2n+1})^2}\right] dt = a^2(n+\frac{1}{2}).$$

SECTION 17.6

1. $4[(u^2-v^2)\mathbf{i} - (u^2+v^2)\mathbf{j} + 2uv\mathbf{k}]$ **2.** $u\mathbf{k}$ **3.** $2(\mathbf{j}-\mathbf{i})$

4. $\sin u \sin v\mathbf{i} + \cos u \cos v\mathbf{j} + (\sin^2 u \sin^2 v - \cos^2 u \cos^2 v)\mathbf{k}$

5. $\mathbf{r}(u,v) = 3\cos u\,\cos v\,\mathbf{i} + 2\sin u\,\cos v\,\mathbf{j} + 6\sin v\,\mathbf{k}, \quad u \in [0\,2\pi],\ v \in [0, \pi/2]$

6. $\mathbf{r}(\theta, z) = 2\cos\theta\mathbf{i} + 2\sin\theta\mathbf{j} + z\mathbf{k}, \quad \theta \in [0, 2\pi],\ z \in [1,4].$

7. $\mathbf{r}(u,v) = 2\cos u\,\cos v\,\mathbf{i} + 2\sin u\,\cos v\,\mathbf{j} + 2\sin v\,\mathbf{k}, \quad u \in [0\,2\pi],\ v \in (\pi/4, \pi/2]$

8. $\mathbf{r}(s,\theta) = s\cos\theta\mathbf{i} + s\sin\theta\mathbf{j} + (s\cos\theta + 2)\mathbf{k}, \quad s \in [0,1],\ \theta \in [0, 2\pi].$

9. The surface consists of all points of the form $(x, g(x,z), z)$ with $(x,z) \in \Omega$. This set of points is given by

$$\mathbf{r}(u,v) = u\mathbf{i} + g(u,v)\mathbf{j} + v\mathbf{k}, \quad (u,v) \in \Omega.$$

10. $\mathbf{r}(y,z) = h(y,z)\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \quad (y,z) \in \Gamma$

11. $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$; ellipsoid

12. $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$; elliptic paraboloid.

13. $x^2/a^2 - y^2/b^2 = z$; hyperbolic paraboloid

14. Choose a real number v. The points of the surface at height $z = c\sinh v$ form an ellipse E_v. The projection of E_v onto the xy-plane has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \sinh^2 v = \cosh^2 v,$$

which can be written

$$\frac{x^2}{a^2\cosh^2 v} + \frac{y^2}{b^2\cosh^2 v} = 1.$$

This ellipse can be parametrized by

$$\mathbf{r}(u) = (a\cosh v)\cos u\,\mathbf{i} + (b\cosh v)\sin u\,\mathbf{j}, \quad u \in [0, 2\pi].$$

Thus E_v can be parametrized by

$$\mathbf{R}(u) = (a\cosh v)\cos u\,\mathbf{i} + (b\cosh v)\sin u\,\mathbf{j} + c\sinh v\,\mathbf{k}, \quad u \in [0, 2\pi].$$

Letting v range over the set of real numbers we obtain the entire surface.

15. For each $v \in [a, b]$, the points on the surface at level $z = f(v)$ form a circle of radius v.

That circle can be parametrized:

$$\mathbf{R}(u) = v\cos u\,\mathbf{i} + v\sin u\,\mathbf{j} + f(v)\mathbf{k}, \quad u \in [0, 2\pi].$$

Letting v range over $[a, b]$, we obtain the entire surface:

$$\mathbf{r}(u, v) = v\cos u\,\mathbf{i} + v\sin u\,\mathbf{j} + f(v)\mathbf{k}; \quad 0 \le u \le 2\pi, \quad a \le v \le b.$$

16. For the parametrization given in the answer to Exercise 15

$N(u, v) = -vf'(v)\cos u\mathbf{i} + v\mathbf{j} - vf'(v)\sin u\mathbf{k}, \quad \| \mathbf{N}(u, v) \| = v\sqrt{1 + [f'(v)]^2}.$

Therefore

$$A = \int_0^{2\pi} \left\{ \int_a^b v\sqrt{1 + [f'(v)]^2}\,dv \right\} du$$

$$= 2\pi \int_a^b v\sqrt{1 + [f'(v)]^2}\,dv = \int_a^b 2\pi x\sqrt{1 + [f'(v)]^2}\,dx$$

17. Since γ is the angle between p and the xy-plane, γ is the angle between the upper normal to p and $\mathbf{k}$. (Draw a figure.) Therefore, by 17.6.5,

$$\text{area of } \Gamma = \iint_{\Omega} \sec\gamma\,dxdy = (\sec\gamma)A_{\Omega} = A_{\Omega}\sec\gamma.$$

γ is constant

18. $\mathbf{n} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ is an upper normal.

$$\cos\gamma = \frac{\mathbf{n} \cdot \mathbf{k}}{\sqrt{3}} = \frac{1}{\sqrt{3}}, \quad \sec\gamma = \sqrt{3} \quad A = \sqrt{3}\pi b^2$$

19. The surface is the graph of the function

$$f(x, y) = c\left(1 - \frac{x}{a} - \frac{y}{b}\right) = \frac{c}{ab}(ab - bx - ay)$$

defined over the triangle $\Omega : 0 \le x \le a, \quad 0 \le y \le b(1 - x/a)$. Note that Ω has area $\frac{1}{2}ab$.

$$A = \iint_{\Omega} \sqrt{[f_x'(x, y)]^2 + [f_y'(x, y)]^2 + 1}\,dxdy$$

$$= \iint_{\Omega} \sqrt{c^2/a^2 + c^2/b^2 + 1}\,dxdy$$

$$= \frac{1}{ab}\sqrt{a^2b^2 + a^2c^2 + b^2c^2} \iint_{\Omega} dxdy = \frac{1}{2}\sqrt{a^2b^2 + a^2c^2 + b^2c^2}.$$

20. $f(x,y)=\sqrt{x^2+y^2},\quad \Omega: 0\le x^2+y^2\le 1$

$$A=\iint_{\Omega}\sqrt{[f_x'(x,y)]^2+[f_y'(x,y)]^2+1}\,dx\,dy=\iint_{\Omega}\sqrt{2}\,dx\,dy=\sqrt{2}\pi$$

21. $f(x,y)=x^2+y^2,\quad \Omega: 0\le x^2+y^2\le 4$

$$A=\iint_{\Omega}\sqrt{4x^2+4y^2+1}\;dxdy \qquad \text{[change to polar coordinates]}$$

$$=\int_0^{2\pi}\int_0^2\sqrt{4r^2+1}\,r\,dr\,d\theta$$

$$=2\pi\left[\tfrac{1}{12}(4r^2+1)^{3/2}\right]_0^2=\tfrac{1}{6}\pi(17\sqrt{17}-1)$$

22. $f(x,y)=\sqrt{2xy},\quad \Omega: 0\le x\le a,\quad 0\le y\le b$

$$A=\iint_{\Omega}\frac{x+y}{\sqrt{2xy}}\,dx\,dy=\frac{1}{\sqrt{2}}\iint_{\Omega}(\sqrt{x/y}+\sqrt{y/x})\,dx\,dy$$

$$=\frac{1}{\sqrt{2}}\int_0^a\int_0^b\left(\sqrt{x/y}+\sqrt{y/x}\right)\,dy\,dx$$

$$=\frac{2}{3}\sqrt{2}(a+b)\sqrt{ab}$$

23. $f(x,y)=a^2-(x^2+y^2),\quad \Omega: \tfrac{1}{4}a^2\le x^2+y^2\le a^2$

$$A=\iint_{\Omega}\sqrt{4x^2+4y^2+1}\;dxdy \qquad \text{[change to polar coordinates]}$$

$$=\int_0^{2\pi}\int_{a/2}^{a}r\sqrt{4r^2+1}\;dr\,d\theta=2\pi\left[\frac{1}{12}(4r^2+1)^{3/2}\right]_{a/2}^{a}$$

$$=\frac{\pi}{6}\left[(4a^2+1)^{3/2}-(a^2+1)^{3/2}\right]$$

24. $f(x,y)=\dfrac{1}{\sqrt{3}}(x+y)^{3/2},\quad \Omega: 0\le x\le 2,\quad 0\le y\le 2-x$

$$A=\iint\frac{1}{\sqrt{2}}\sqrt{3x+3y+2}\,dx\,dy=\frac{1}{\sqrt{2}}\int_0^2\int_0^{2-x}\sqrt{3x+3y+2}\,dx\,dy=\frac{464}{135}$$

25. $f(x,y)=\tfrac{1}{3}(x^{3/2}+y^{3/2}),\quad \Omega: 0\le x\le 1,\quad 0\le y\le x$

$$A=\iint_{\Omega}\frac{1}{2}\sqrt{x+y+4}\;dxdy$$

$$=\int_0^1\int_0^x\frac{1}{2}\sqrt{x+y+4}\;dy\,dx=\int_0^1\left[\frac{1}{3}(x+y+4)^{3/2}\right]_0^x\,dx$$

$$=\int_0^1\frac{1}{3}\left[(2x+4)^{3/2}-(x+4)^{3/2}\right]\,dx=\frac{1}{3}\left[\frac{1}{5}(2x+4)^{5/2}-\frac{2}{5}(x+4)^{5/2}\right]_0^1$$

$$=\frac{1}{15}(36\sqrt{6}-50\sqrt{5}+32)$$

26. $f(x,y) = y^2, \quad \Omega : 0 \le x \le 1, \quad 0 \le y \le 1$

$$A = \iint_{\Omega} \sqrt{4y^2+1}\,dx\,dy = \int_0^1 \int_0^1 \sqrt{4y^2+1}\,dy\,dx = \frac{1}{4}\left[2\sqrt{5} + \ln(2+\sqrt{5})\right]$$

27. The surface $x^2 + y^2 + z^2 - 4z = 0$ is a sphere of radius 2 centered at $(0,0,2)$:

$$x^2 + y^2 + z^2 - 4z = 0 \iff x^2 + y^2 + (z-2)^2 = 4.$$

The quadric cone $z^2 = 3(x^2+y^2)$ intersects the sphere at height $z = 3$:

$$\left.\begin{array}{r} x^2 + y^2 + z^2 - 4z = 0 \\ z^2 = 3(x^2+y^2) \end{array}\right\} \implies \begin{array}{r} 3(x^2+y^2) + 3z^2 - 12z = 0 \\ 4z^2 - 12z = 0 \\ z = 3. \quad (\text{since } z \ge 2) \end{array}$$

The surface of which we are asked to find the area is a spherical segment of width 1 (from $z = 3$ to $z = 4$) in a sphere of radius 2. The area of the segment is 4π. (Exercise 27, Section 9.9.)

A more conventional solution. The spherical segment is the graph of the function

$$f(x,y) = 2 + \sqrt{4-(x^2+y^2)}, \quad \Omega : 0 \le x^2 + y^2 \le 3.$$

Therefore

$$\begin{aligned} A &= \iint_{\Omega} \sqrt{\left(\frac{-x}{\sqrt{4-x^2-y^2}}\right)^2 + \left(\frac{-y}{\sqrt{4-x^2-y^2}}\right)^2 + 1}\; dxdy \\ &= \iint_{\Omega} \frac{2}{\sqrt{4-(x^2+y^2)}}\, dxdy \\ &= \int_0^{2\pi}\int_0^{\sqrt{3}} \frac{2r}{\sqrt{4-r^2}}\, dr\, d\theta \qquad [\text{changed to polar coordinates}] \\ &= 2\pi\left[-2\sqrt{4-r^2}\right]_0^{\sqrt{3}} = 4\pi \end{aligned}$$

28. The spherical segment is the graph of the function

$$f(x,y) = a + \sqrt{a^2 - (x^2+y^2)}, \quad \Omega \le x^2 + y^2 \le 2ab - b^2.$$

Therefore

$$\begin{aligned} A &= \iint_{\Omega} \frac{a}{\sqrt{a^2-(x^2+y^2)}}\, dx\, dy \quad [\text{change to polar coordinate}] \\ &= \int_0^{2\pi}\int_0^{\sqrt{2ab-b^2}} \frac{ar}{\sqrt{a^2-r^2}}\, dr\, d\theta \\ &= 2\pi ab. \end{aligned}$$

29. (a) $$\iint_\Omega \sqrt{\left[\frac{\partial g}{\partial y}(y,z)\right]^2+\left[\frac{\partial g}{\partial z}(y,z)\right]^2+1}\,dydz=\iint_\Omega \sec\left[\alpha(y,z)\right]\,dydz$$

where α is the angle between the unit normal with positive **i** component and the positive x-axis

(b) $$\iint_\Omega \sqrt{\left[\frac{\partial h}{\partial x}(x,z)\right]^2+\left[\frac{\partial h}{\partial z}(x,z)\right]^2+1}\,dxdz=\iint_\Omega \sec\left[\beta(x,z)\right]\,dxdz$$

where β is the angle between the unit normal with positive **j** component and the positive y-axis

30. (a) $\mathbf{r}_u{}'=-a\sin u\mathbf{i}+a\cos u\mathbf{j};\quad \mathbf{r}'_u=\mathbf{k}\qquad \mathbf{N}(u,v)=\mathbf{r}'_u\times\mathbf{r}'_v=a\cos u\mathbf{i}+a\sin u\mathbf{j}$

(b) $$A=\iint_\Omega \|\,\mathbf{N}(u,v)\,\|\;du\,dv=\iint_\Omega a\,du\,dv=\int_0^{2\pi}\int_0^{l}a\,dv\,du=2\pi la$$

31. (a) $\mathbf{N}(u,v)=v\cos u\sin\alpha\cos\alpha\,\mathbf{i}+v\sin u\sin\alpha\cos\alpha\,\mathbf{j}-v\sin^2\alpha\,\mathbf{k}$

(b) $$A=\iint_\Omega \|\mathbf{N}(u,v)\|\,dudv=\iint_\Omega v\sin\alpha\,dudv$$
$$=\int_0^{2\pi}\int_0^{s}v\sin\alpha\,dv\,du=\pi s^2\sin\alpha$$

32. Choose a number $v\in[0,\pi]$. The points of the surface at height $z=b\cos v$ form a circle C_v. The projection of C_v onto the xy-plane had equation
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\cos^2 v=\sin^2 v$$
which can be written
$$x^2+y^2=a^2\sin^2 v.$$
This circle can be parametrized by
$$\mathbf{r}(u)=(a\sin v)\cos u\mathbf{i}+(\sin v)\sin u\mathbf{j},\quad u\in[0,\pi].$$
Thus C_v can be parametrized by
$$\mathbf{R}(u)=(a\sin v)\cos u\mathbf{i}+(a\sin v)\sin u\mathbf{j}+b\cos v\mathbf{k},\quad u\in[0,\pi].$$
Letting v range from 0 to π we obtain the entire surface, so the parametrization is
$$\mathbf{r}(u,v)=a\cos u\sin v\mathbf{i}+a\sin u\sin v\mathbf{j}+b\cos v\mathbf{k}.$$
Next, $\mathbf{N}(u,v)=-ab\cos u\sin^2 v\mathbf{i}-ab\sin u\sin^2 v\mathbf{j}-a^2\sin v\cos v\mathbf{k}$,

so
$$A=\iint_\Omega \|\,\mathbf{N}(u,v)\,\|\;du\,dv=\int_0^{2\pi}\int_0^{\pi}\sqrt{a^2b^2\sin^4 v+a^4\sin^2 v\cos^2 v}\,dv\,du$$
$$=2\pi a\int_0^{\pi}\sin v\sqrt{b^2\sin^2 v+a^2\cos^2 v}\,dv$$

33. $A = \sqrt{A_1{}^2 + A_2{}^2 + A_3{}^2}$; the unit normal to the plane of Ω is a vector of the form

$$\cos\gamma_1\,\mathbf{i} + \cos\gamma_2\,\mathbf{j} + \cos\gamma_3\,\mathbf{k}.$$

Note that

$$A_1 = A\cos\gamma_1, \quad A_2 = A\cos\gamma_2, \quad A_3 = A\cos\gamma_3.$$

Therefore

$$A_1{}^2 + A_2{}^2 + A_3{}^2 = A^2[\cos^2\gamma_1 + \cos^2\gamma_2 + \cos^2\gamma_3] = A^2.$$

34. We can parametrize the surface by setting

$$\mathbf{R}(r\,\theta) = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} + f(r,\theta)\mathbf{k}, \quad (r,\theta) \in \Omega.$$

The integrand is $\|\,\mathbf{N}(r,\theta)\,\|$.

35. (a) (We use Exercise 34.) $f(r,\,\theta) = r + \theta$; $\Omega : 0 \le r \le 1, \quad 0 \le \theta\pi$

$$A = \iint_\Omega \sqrt{r^2\left[f_r'(r,\,\theta)\right]^2 + \left[f_\theta'(r,\,\theta)\right]^2 + r^2}\;drd\theta = \iint_\Omega \sqrt{2r^2+1}\;drd\theta$$

$$= \int_0^\pi \int_0^1 \sqrt{2r^2+1}\;dr\,d\theta = \frac{1}{4}\sqrt{2}\pi\left[\sqrt{6} + \ln\left(\sqrt{2}+\sqrt{3}\right)\right]$$

(b) $f(r,\,\theta) = re^\theta$; $\Omega : 0 \le r \le a, \quad 0 \le \theta \le 2\pi$

$$A = \iint_\Omega r\sqrt{2e^{2\theta}+1}\;drd\theta = \left(\int_0^{2\pi} \sqrt{2e^{2\theta}+1}\;d\theta\right)\left(\int_0^a r\,dr\right)$$

$$= \tfrac{1}{2}a^2[\sqrt{2e^{4\pi}+1} - \sqrt{3} + \ln\left(1+\sqrt{3}\right) - \ln\left(1+\sqrt{2e^{4\pi}+1}\right)]$$

36. $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j}, \quad (u,v) \in \Omega$

Straightforward calculation shows that $\|\,\mathbf{N}(u,v)\,\| = |J(u,v)|$.

SECTION 17.7

For Exercises 1–6 we have $\sec[\gamma(x,y)] = \sqrt{y^2+1}$. $\mathbf{N}(x,y) = -y\mathbf{j} + \mathbf{k}$, so $\|\,N(x,y)\,\| = \sqrt{y^2+1}$.

1. $$\iint_S d\sigma = \int_0^1\int_0^1 \sqrt{y^2+1}\;dx\,dy = \int_0^1 \sqrt{y^2+1}\;dy = \frac{1}{2}[\sqrt{2} + \ln\left(1+\sqrt{2}\right)]$$

2. $$\iint_S x^2\,d\sigma \int_0^1\int_0^1 x^2\sqrt{y^2+1}\,dy\,dx$$

$$= \left(\int_0^1 x^2\,dx\right)\left(\int_0^1 \sqrt{y^2+1}\,dy\right) = \frac{1}{6}\left[\sqrt{2} + \ln(1+\sqrt{2})\right]$$

3. $\iint_S 3y\,d\sigma = \int_0^1\int_0^1 3y\sqrt{y^2+1}\,dy\,dx = \int_0^1 3y\sqrt{y^2+1}\,dy = \left[(y^2+1)^{3/2}\right]_0^1 = 2\sqrt{2}-1$

4.
$$\begin{aligned}\iint_S (x-y)\,d\sigma &= \int_0^1\int_0^1 x\sqrt{y^2+1}\,dy\,dx - \int_0^1\int_0^1 y\sqrt{y^2+1}\,dy\,dx\\ &= \left(\int_0^1 x\,dx\right)\left(\int_0^1 \sqrt{y^2+1}\,dy\right) - \int_0^1 y\sqrt{y^2+1}\,dy\\ &= \frac{1}{4}[\sqrt{2}+\ln(1+\sqrt{2})] - \frac{1}{3}(2\sqrt{2}-1)\\ &= \frac{1}{3} - \frac{5}{12}\sqrt{2} + \frac{1}{4}\ln(1+\sqrt{2})\end{aligned}$$

5. $\iint_S \sqrt{2z}\,d\sigma = \iint_S y\,d\sigma = \frac{1}{3}(2\sqrt{2}-1)$ (Exercise 3)

6. $\iint_S \sqrt{1+y^2}\,d\sigma = \int_0^1\int_0^1 (1+y^2)\,dy\,dx = \int_0^1 (1+y^2)\,dy = \frac{4}{3}$

7. $\iint_S xy\,d\sigma;\quad S:\mathbf{r}(u,v) = (6-2u-3v)\,\mathbf{i} + u\,\mathbf{j} + v\,\mathbf{k},\quad 0\le u\le 3-\frac{3}{2}v,\quad 0\le v\le 2$

$$\|\mathbf{N}(u,v)\| = \|(-2\,\mathbf{i}+\mathbf{j})\times(-3\,\mathbf{i}+\mathbf{k})\| = \sqrt{14}$$

$$\begin{aligned}\iint_S xy\,d\sigma &= \sqrt{14}\iint_\Omega x(u,v)y(u,v)\,du\,dv\\ &= \sqrt{14}\iint_\Omega (6-2u-3v)u\,du\,dv\\ &= \sqrt{14}\int_0^2\int_0^{3-3v/2}(6u-2u^2-3uv)\,du\,dv\\ &= \sqrt{14}\left[3\left(3-\tfrac{3}{2}v\right)^2 - \tfrac{2}{3}\left(3-\tfrac{3}{2}v\right)^3 - \tfrac{3}{2}v\left(3-\tfrac{3}{2}v\right)^2\right]dv = \frac{9}{2}\sqrt{14}\end{aligned}$$

8. S is given by $z = f(x,y) = 1-x-y,\quad 0\le x\le 1,\quad 0\le y\le 1-x$

$$\begin{aligned}\iint_S xyz\,d\sigma &= \int_0^1\int_0^{1-x} xy(1-x-y)\sqrt{(-1)^2+(-1)^2+1}\,dy\,dx\\ &= \sqrt{3}\int_0^1\int_0^{1-x} xy(1-x-y)\,dy\,dx = \frac{\sqrt{3}}{120}\end{aligned}$$

9. $\iint_S x^2z\,d\sigma;\quad S:\mathbf{r}(u,v) = (\cos u\,\mathbf{i} + v\,\mathbf{j} + \sin u\,\mathbf{k},\quad 0\le u\le\pi,\quad 0\le v\le 2.$

$$\mathbf{N}(u,v)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -\sin u & 0 & \cos u\\ 0 & 1 & 0\end{vmatrix}=-\cos u\,\mathbf{i}-\sin u\,\mathbf{k}\quad\text{and}\quad \|\mathbf{N}(u,v)\|=1.$$

$$\iint_S x^2 z\,d\sigma=\iint_\Omega \cos^2 u\,\sin u\,du=\int_0^2\int_0^\pi \cos^2 u\,\sin u\,du=\frac{4}{3}$$

10.
$$\iint_S (x^2+y^2+z^2)\,d\sigma=\iint_{x^2+y^2\le 1}[x^2+y^2+(x+2)^2]\sqrt{2}\,dx\,dy$$
$$=\int_0^{2\pi}\int_0^1 [r^2+(r\cos\theta+2)^2]\sqrt{2}r\,dr\,d\theta$$
$$=\int_0^{2\pi}\int_0^1 (r^3+r^3\cos^2\theta+4r^2\cos\theta+4)\,dr\,d\theta=\frac{19}{4}\pi$$

11. $\iint_S (x^2+y^2)\,d\sigma;\quad S:\mathbf{r}(u,v)=(\cos u\,\cos v\,\mathbf{i}+\cos u\,\sin v\,\mathbf{j}+\sin u\,\mathbf{k},\quad 0\le u\le \pi/2,\quad 0\le v\le 2\pi.$

$$\mathbf{N}(u,v)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -\sin u\,\cos v & -\sin u\,\sin v & \cos u\\ -\cos u\,\cos v & \cos u\,\cos v1 & 0\end{vmatrix}=-\cos^2 u\,\cos v\,\mathbf{i}+\cos^2 u\,\sin v\,wj-\sin u\,\cos u\,\mathbf{k};$$

$\|\mathbf{N}(u,v)\|=\cos u.$

$$\iint_S (x^2+y^2)\,d\sigma=\iint_\Omega \cos^2 u\,\cos u\,du=\int_0^{2\pi}\int_0^{\pi/2}\cos^3 u\,du=\frac{4}{3}\pi$$

12.
$$\iint_S (x^2+y^2)\,d\sigma=\iint_{x^2+y^2\le 1}(x^2+y^2)\sqrt{4x^2+4y^2+1}\,dx\,dy=\int_0^{2\pi}\int_0^1 r^2\sqrt{4r^2+1}\,r\,dr\,d\theta$$
$$=2\pi\int_0^1 r^3\sqrt{4r^2+1}\,dr=\frac{25\sqrt{5}+1}{60}\pi$$

For Exercises 13–16 the surface S is given by:

$$f(x,y)=a-x-y;\quad 0\le x\le a,\ 0\le y\le a-x\quad\text{and}\quad \sec[\gamma(x,y)]=\sqrt{3}.$$

13. $M=\iint_S \lambda(x,y,x)\,d\sigma=\int_0^a\int_0^{a-x} k\sqrt{3}\,dy\,dx=\int_0^a k\sqrt{3}\,(a-x)\,dx=\frac{1}{2}a^2k\sqrt{3}$

14.
$$M=\iint_S k(x+y)\,d\sigma=\int_0^a\int_0^{a-x} k(x+y)\sqrt{3}\,dy\,dx$$
$$=\frac{1}{2}k\sqrt{3}\int_0^a (a^2-x^2)\,dx=\frac{1}{3}\sqrt{3}a^3k$$

15. $M=\iint_S \lambda(x,y,z)\,d\sigma=\int_0^a\int_0^{a-x} kx^2\sqrt{3}\,dy\,dx=\int_0^a k\sqrt{3}x^2(a-x)\,dx=\frac{1}{12}a^4k\sqrt{3}$

16.

$$\overline{x}A = \iint_S x\,d\sigma = \int_0^a \int_0^{a-x} x\sqrt{3}\,dy\,dx$$

$$= \sqrt{3}\int_0^a (ax - x^2)\,dy = \frac{1}{6}\sqrt{3}a^3$$

$$A = \iint_S d\sigma = \int_0^a \int_0^{a-x} \sqrt{3}\,dy\,dx$$

$$= \sqrt{3}\int_0^a (a-x)\,dx = \frac{1}{2}\sqrt{3}a^2$$

$$\overline{x} = \overline{x}A/A = \frac{1}{3}a; \quad \text{similarly} \quad \overline{y} = \overline{z} = \frac{1}{3}a$$

17. $S: \mathbf{r}(u,v) = a\cos u\cos v\,\mathbf{i} + a\sin u\cos v\,\mathbf{j} + a\sin v\,\mathbf{k}$ with $0 \le u \le 2\pi$, $0 \le v \le \frac{1}{2}\pi$. By a previous calculation $||\mathbf{N}(u,v)|| = a^2\cos v$.

$\overline{x} = 0, \quad \overline{y} = 0 \qquad$ (by symmetry)

$$\overline{z}A = \iint_S z\,d\sigma = \iint_\Omega z(u,v)\,||\mathbf{N}(u,v)||\,dudv = \int_0^{2\pi}\int_0^{\pi/2} a^3\sin v\cos v\,dv\,du = \pi a^3$$

$\overline{z} = \frac{1}{2}a \quad$ since $\quad A = 2\pi a^2$

18. $\mathbf{N}(u,v) = 2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

$$A = \iint_S d\sigma = \iint_\Omega ||\,\mathbf{N}(u,v)\,||\,du\,dv = \int_0^1\int_0^1 2\sqrt{3}\,du\,dv = 2\sqrt{3}$$

19. $\mathbf{N}(u,v) = (\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \cdot (\mathbf{i} - \mathbf{j}) = 2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

$$\text{flux in the direction of } \mathbf{N} = \iint_S \left(\mathbf{v}\cdot\frac{\mathbf{N}}{||\mathbf{N}||}\right) d\sigma = \iint_\Omega [\mathbf{v}(x(u),\,y(u),\,z(u))\cdot\mathbf{N}(u,v)]\,dudv$$

$$= \iint_\Omega [(u+v)\mathbf{i} - (u-v)\mathbf{j}] \cdot [2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}]\,dudv.$$

$$= \iint_\Omega 4v\,dudv = 4\int_0^1\int_0^1 v\,dv\,du = 2$$

20. $\sec[\gamma(x,y)] = \sqrt{x^2 + y^2 + 1}$

$$M = \iint_S kxy\,d\sigma = k\int_0^1\int_0^1 xy\sqrt{x^2+y^2+1}\,dy\,dx$$

$$= \frac{1}{3}k\int_0^1 \left[x(x^2+2)^{3/2} - x(x^2+1)^{3/2}\right] dx$$

$$= \frac{1}{15}(9\sqrt{3} - 8\sqrt{2} + 1)k$$

For Exercises 21–23: $\mathbf{n} = \dfrac{1}{a}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

S : $\mathbf{r}(u,v) = a\cos u\cos v\,\mathbf{i} + a\sin u\cos v\,\mathbf{j} + a\sin v\,\mathbf{k}$ with $0 \le u \le 2\pi, \quad -\frac{1}{2}\pi \le v \le \frac{1}{2}\pi$

$\|\mathbf{N}(u,v)\| = a^2\cos v$

21. With $\mathbf{v} = z\mathbf{k}$

$$\text{flux} = \iint_S (\mathbf{v}\cdot\mathbf{n})\,d\sigma = \frac{1}{a}\iint_S z^2\,d\sigma = \frac{1}{a}\iint_\Omega (a^2\sin^2 v)(a^2\cos v)\,dudv$$
$$= a^3\int_0^{2\pi}\int_{-\pi/2}^{\pi/2}(\sin^2 v\cos v)\,d\sigma = \frac{4}{3}\pi a^3$$

22. With $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$\text{flux} = \iint_S (\mathbf{v}\cdot\mathbf{n})\,d\sigma = a\iint_S d\sigma = aA = 4\pi a^3$$

23. With $\mathbf{v} = y\mathbf{i} - x\mathbf{j}$

$$\text{flux} = \iint_S (\mathbf{v}\cdot\mathbf{n})\,d\sigma = \frac{1}{a}\iint_S \underbrace{(yx - xy)}_{0}\,d\sigma = 0$$

24. $$I_x = \iint_S (y^2 + z^2)\,d\sigma = 2\sqrt{3}\int_0^1\int_0^1 (5u^2 - 2uv + v^2)\,dv\,du = 3\sqrt{3}$$
$$I_y = \iint_S (x^2 + z^2)\,d\sigma = 2\sqrt{3}\int_0^1\int_0^1 (5u^2 + 2uv + v^2)\,dv\,du = 5\sqrt{3}$$
$$I_z = \iint_S (x^2 + z^2)\,d\sigma = 4\sqrt{3}\int_0^1\int_0^1 (u^2 + v^2)\,d\sigma = \frac{8}{3}\sqrt{3}$$

For Exercises 25–27 the triangle S is the graph of the function

$$f(x,y) = a - x - y \quad \text{on} \quad \Omega : 0 \le x \le a, \quad 0 \le y \le a - x.$$

The triangle has area $A = \frac{1}{2}\sqrt{3}a^2$.

25. With $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$\text{flux} = \iint_S (\mathbf{v}\cdot\mathbf{n})\,d\sigma = \iint_\Omega (-v_1 f'_x - v_2 f'_y + v_3)\,dxdy$$
$$= \iint_\Omega [-x(-1) - y(-1) + (a - x - y)]\,dxdy = a\iint_\Omega dxdy = aA = \frac{1}{2}\sqrt{3}a^3$$

26. With $v = (x + z)k$

$$\text{flux} = \iint_S (\mathbf{v}\cdot\mathbf{n})\,d\sigma = \iint_\Omega (-v_1 f'_x - v_2 f'_y + v_3)\,dx\,dy$$
$$= \iint_\Omega (a - y)\,dx\,dy = \int_0^a\int_0^{a-x}(a - y)\,dx\,dy = \frac{1}{3}a^3$$

27. With $v = x^2\mathbf{i} - y^2\mathbf{j}$

$$\text{flux} = \iint_S (\mathbf{v}\cdot\mathbf{n})\,d\sigma = \iint_\Omega (-v_1 f_x' - v_2 f_y' + v_3)\,dxdy$$

$$= \iint_\Omega [-x^2(-1) - (-y^2)(-1) + 0]\,dxdy = \int_0^a \int_0^{a-x} (x^2 - y^2)\,dy\,dx$$

$$= \int_0^a \left[ax^2 - x^3 - \frac{1}{3}(a-x)^3\right] dx = \left[\frac{1}{3}ax^3 - \frac{1}{4}x^4 + \frac{1}{12}(a-x)^4\right]_0^a = 0$$

28. With $\mathbf{v} = -xy^2\mathbf{i} + z\,\mathbf{j}$

$$\text{flux} = \iint_S (\mathbf{v}\cdot\mathbf{n})\,d\sigma = \iint_\Omega (-v_1 f_x{}' - v_2 f_y{}' + v_3)\,dx\,dy$$

$$= \int_0^1 \int_0^2 (xy^3 - x^2y)\,dy\,dx = \frac{4}{3}$$

29. With $\mathbf{v} = xz\,\mathbf{j} - xy\,\mathbf{k}$

$$\text{flux} = \iint_S (\mathbf{v}\cdot\mathbf{n})\,d\sigma = \iint_\Omega (-v_1 f_x' - v_2 f_y' + v_3)\,dxdy$$

$$= \iint_\Omega (-x^3y - xy)\,dxdy = \int_0^1 \int_0^2 (-x^3y - xy)\,dy\,dx$$

$$= \int_0^1 -2(x^3 + x)\,dx = -\frac{3}{2}$$

30. With $\mathbf{v} = x^2y\,\mathbf{i} + Z62\,\mathbf{k}$

$$\text{flux} = \iint_S (\mathbf{v}\cdot\mathbf{n})\,d\sigma = \iint_\Omega (-v_1 f_x' - v_2 f_y' + v_3)\,dx\,dy$$

$$= \int_0^1 \int_0^2 (-x^2y^2 + x^2y^2)\,dy\,dx = 0$$

31. $\mathbf{n} = \frac{1}{a}(x\mathbf{i} + y\mathbf{j})$

$$\text{flux} = \iint_S (\mathbf{v}\cdot\mathbf{n})\,d\sigma = \frac{1}{a}\iint_S [(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\cdot(x\mathbf{i} + y\mathbf{j})]\,d\sigma$$

$$= \frac{1}{a}\iint_S (x^2 + y^2)\,d\sigma = a\iint_S d\sigma = a\,(\text{area of } S) = a\,(2\pi a l) = 2\pi a^2 l$$

32.

$$\text{flux} = \iint_S \left(GmM\frac{\mathbf{r}}{r^3}\cdot\frac{\mathbf{r}}{r}\right) d\sigma = GmM\iint_S \frac{1}{r^2}\,d\sigma$$

$$= GmM\iint_S \frac{1}{a^2}\,d\sigma = \frac{GmM}{a^2}\iint_S d\sigma = \frac{GmM}{a^2}(4\pi a^2) = 4\pi GmM$$

In Exercises 33–36, S is the graph of $f(x,y)=\frac{2}{3}(x^{3/2}+y^{3/2}),\quad 0\le x\le 1,\ 0\le y\le 1-x$

We use

$$\text{flux}=\iint_S(\mathbf{v}\cdot\mathbf{n})\,d\sigma=\iint_\Omega(-v_1f_x'-v_2f_y'+v_3)\,dx\,dy.$$

33. With $\mathbf{v}=x\,\mathbf{i}-y\,\mathbf{j}+\frac{3}{2}z\,\mathbf{k}$

$$\text{flux}=\iint_S(\mathbf{v}\cdot\mathbf{n})\,d\sigma=\iint_\Omega(-v_1f_x'-v_2f_y'+v_3)\,dxdy=\iint_\Omega 2y^{3/2}\,dxdy$$

$$=\int_0^1\int_0^{1-x}2y^{3/2}\,dy\,dx=\int_0^1\frac{4}{5}(1-x)^{5/2}\,dx=\frac{8}{35}$$

34. With $\mathbf{v}=x^2\mathbf{i}$,

$$\text{flux}=\int_0^1\int_0^{1-x}-x^{5/2}\,dy\,dx=-\frac{4}{63}.$$

35. With $\mathbf{v}=y^2\,\mathbf{j}$

$$\text{flux}=\iint_S(\mathbf{v}\cdot\mathbf{n})\,d\sigma=\iint_\Omega(-v_1f_x'-v_2f_y'+v_3)\,dxdy=\iint_\Omega -y^{5/2}\,d\sigma$$

$$=\int_0^1\int_0^{1-x}-y^{5/2}\,dy\,dx=\int_0^1-\frac{2}{7}(1-x)^{7/2}\,dx=-\frac{4}{63}$$

36. With $\mathbf{v}=y\mathbf{i}-\sqrt{xy}\mathbf{j}$,

$$\text{flux}=\int_0^1\int_0^{1-x}(-yx^{1/2}+\sqrt{xy}\,y^{1/2})\,dy\,dx=0.$$

37. $\bar{x}=0,\quad \bar{y}=0$ by symmetry. You can verify that $\|\mathbf{N}(u,v)\|=v\sin\alpha$.

$$\bar{z}A=\iint_S z\,d\sigma=\iint_\Omega(v\cos\alpha)(v\sin\alpha)\,dudv=\sin\alpha\cos\alpha\int_0^{2\pi}\int_0^s v^2\,dv\,du=\tfrac{2}{3}\pi\sin\alpha\cos\alpha\,s^3$$

$\bar{z}=\frac{2}{3}s\cos\alpha$ since $A=\pi s^2\sin\alpha$

38.

$$M=\iint_\Omega k\sqrt{x^2+y^2}\,d\sigma=\iint_\Omega k\sqrt{x^2+y^2}\sqrt{2}\,dx\,dy$$

$$=k\sqrt{2}\int_0^{2\pi}\int_0^1 r^2\,dr\,d\theta=\frac{2}{3}\sqrt{2}\pi k$$

39. $f(x,y)=\sqrt{x^2+y^2}$ on $\Omega:0\le x^2+y^2\le 1;\quad \lambda(x,y,z)=k\sqrt{x^2+y^2}$

$x_M=0,\quad y_M=0$ (by symmetry)

$$z_MM=\iint_S z\lambda(x,y,z)\,d\sigma=\iint_\Omega k(x^2+y^2)\sec[\gamma(x,y)]\,dxdy$$

$$=k\sqrt{2}\iint_\Omega(x^2+y^2)\,dxdy$$

$$=k\sqrt{2}\int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta=\frac{1}{2}\sqrt{2}\pi k$$

$z_M=\frac{3}{4}$ since $M=\frac{2}{3}\sqrt{2}\pi k$ (Exercise 38)

40. (a)

$$I_x = \iint_{\Omega} k\sqrt{x^2+y^2}(y^2+z^2)\,d\sigma = \iint_{\Omega} k\sqrt{x^2+y^2}(y^2+x^2+y^2)\,d\sigma$$

$$= k\sqrt{2}\iint_{\Omega}\left[y^2(x^2+y^2)^{1/2}+(x^2+y^2)^{3/2}\right]\,dx\,dy$$

$$= k\sqrt{2}\int_0^{2\pi}\int_0^1 (r^4\sin^2\theta + r^4)\,dr\,d\theta = \frac{3\sqrt{2}}{5}\pi k$$

(b) $I_y = I_x$ by symmetry

(c)

$$I_z = \iint_{\Omega} k\sqrt{x^2+y^2}(x^2+y^2)\,d\sigma = \iint_{S} k(x^2+y^2)^{3/2}\,d\sigma$$

$$= k\sqrt{2}\iint_{\Omega}(x^2+y^2)^{3/2}\,dx\,dy = k\sqrt{2}\int_0^{2\pi}\int_0^1 r^4\,dr\,d\theta = \frac{2}{5}\sqrt{2}\pi k$$

41. no answer required

42.

$$M = \iint_S k(y^2+z^2)\,d\sigma = 2\sqrt{3}k\iint_S [(u-v)^2+4u^2]\,du\,dv$$

$$= 2\sqrt{3}k\int_0^1\int_0^1 (5u^2-2uv+v^2)\,dv\,du = 3\sqrt{3}k$$

43.

$$x_M M = \iint_S x\lambda(x,y,z)\,d\sigma = \iint_S kx(x^2+y^2)\,d\sigma$$

$$= 2\sqrt{3}\,k\iint_{\Omega}(u+v)\left[(u-v)^2+4u^2\right]\,dudv$$

$$= 2\sqrt{3}\,k\int_0^1\int_0^1 (5u^3-2u^2v+uv^2+5u^2v-2uv^2+v^3)\,dv\,du$$

$$= 2\sqrt{3}\,k\int_0^1\left(5u^3-u^2+\frac{1}{3}u+\frac{5}{2}u^2-\frac{2}{3}u+\frac{1}{4}\right)\,du = \frac{11}{3}\sqrt{3}k$$

$x_M = \dfrac{11}{9}$ since $M = 3\sqrt{3}k$ (Exercise 42)

44.

$$I_z = \iint_S \lambda(x,y,z)(x^2+y^2)\,d\sigma = \iint_S k(y^2+z^2)(x^2+y^2)\,d\sigma$$

$$= 2\sqrt{3}k\iint_{\Omega}\left[(u-v)^2+4u^2\right]\left[(u+v)^2+(u-v)^2\right]\,du\,dv$$

$$= 4\sqrt{3}k\int_0^1\int_0^1 (5u^4-2u^3v+5u^2v^2-2uv^3+v^4)\,du\,dv = \frac{226}{45}\sqrt{3}k.$$

45. Total flux out of the solid is 0. It is clear from a diagram that the outer unit normal to the cylindrical side of the solid is given by $\mathbf{n} = x\mathbf{i} + y\mathbf{j}$ in which case $\mathbf{v} \cdot \mathbf{n} = 0$. The outer unit normals to the top and bottom of the solid are $\mathbf{k}$ and $-\mathbf{k}$ respectively. So, here as well, $\mathbf{v} \cdot \mathbf{n} = 0$ and the total flux is 0.

46. The flux through the upper boundary is 0:

$$(y\mathbf{i} - x\mathbf{j}) \cdot \mathbf{k} = 0.$$

The flux through the lower boundary is 0:

$$\mathbf{v} \cdot \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} - \mathbf{k}\right) = (y\mathbf{i} - x\mathbf{j}) \cdot (2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}) = (2xy - 2xy) = 0.$$

Thus the total flux out of the solid is 0.

47. The surface $z = \sqrt{2 - (x^2 + y^2)}$ is the upper half of the sphere $x^2 + y^2 + z^2 = 2$. The surface intersects the surface $z = x^2 + y^2$ in a circle of radius 1 at height $z = 1$. Thus the upper boundary of the solid, call it S_1, is a segment of width $\sqrt{2} - 1$ on a sphere of radius $\sqrt{2}$. The area of S_1 is therefore $2\pi\sqrt{2}(\sqrt{2} - 1)$. (Exercise 25, Section 10.10.) The upper unit normal to S_1 is the vector

$$\mathbf{n} = \frac{1}{\sqrt{2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).$$

Therefore

$$\text{flux through } S_1 = \iint_{S_1} (\mathbf{v} \cdot \mathbf{n})\, d\sigma = \frac{1}{\sqrt{2}} \iint_{S_1} \overbrace{(x^2 + y^2 + z^2)}^{2}\, d\sigma$$

$$= \sqrt{2} \iint_{S_1} d\sigma = \sqrt{2}\ (\text{area of } S_1) = 4\pi(\sqrt{2} - 1).$$

The lower boundary of the solid, call it S_2, is the graph of the function

$$f(x, y) = x^2 + y^2 \quad \text{on} \quad \Omega : 0 \le x^2 + y^2 \le 1.$$

Taking $\mathbf{n}$ as the lower unit normal, we have

$$\text{flux through } S_2 = \iint_{S_2} (\mathbf{v} \cdot \mathbf{n})\, d\sigma = \iint_{\Omega} \left(v_1 f_x' + v_2 f_y' - v^3\right)\, dxdy$$

$$= \iint_{\Omega} (x^2 + y^2)\, dxdy = \int_0^{2\pi} \int_0^1 r^3\, dr\, d\theta = \frac{1}{2}\pi.$$

The total flux out of the solid is $4\pi(\sqrt{2} - 1) + \frac{1}{2}\pi = (4\sqrt{2} - \frac{7}{2})\pi$.

48.

face	$\mathbf{n}$	$\mathbf{v}\cdot\mathbf{n}$	flux
$x=0$	$-\mathbf{i}$	$-xz=0$	0
$x=1$	$\mathbf{i}$	$xz=1$	$\frac{1}{2}$
$y=0$	$-\mathbf{j}$	$-4xyz^2=0$	0
$y=1$	$\mathbf{j}$	$4xyz^2=4xz^2$	$\frac{2}{3}$
$z=0$	$-\mathbf{k}$	$-2z=0$	0
$z=1$	$\mathbf{k}$	$2z=2$	2

total flux $= \frac{1}{2}+\frac{1}{3}+2=\frac{19}{6}$

SECTION 17.8

1. $\nabla\cdot\mathbf{v}=2, \quad \nabla\times\mathbf{v}=\mathbf{0}$

2. $\nabla\cdot\mathbf{v}=0, \quad \nabla\times\mathbf{v}=\mathbf{0}$

3. $\nabla\cdot\mathbf{v}=0, \quad \nabla\times\mathbf{v}=\mathbf{0}$

4. $\nabla\cdot\mathbf{v}=-\dfrac{4xy}{(x^2+y^2)^2}, \quad \nabla\times\mathbf{v}=\dfrac{2(y^2-x^2)}{(x^2+y^2)^2}\,\mathbf{k}$

5. $\nabla\cdot\mathbf{v}=6, \quad \nabla\times\mathbf{v}=\mathbf{0}$

6. $\nabla\cdot\mathbf{v}=0, \quad \nabla\times\mathbf{v}=\mathbf{0}$

7. $\nabla\cdot\mathbf{v}=yz+1, \quad \nabla\times\mathbf{v}=-x\mathbf{i}+xy\mathbf{j}+(1-x)z\mathbf{k}$

8. $\nabla\cdot\mathbf{v}=2y(x+z), \quad \nabla\times\mathbf{v}=(2xy-y^2)\mathbf{i}-y^2\mathbf{j}-x^2\mathbf{k}$

9. $\nabla\cdot\mathbf{v}=1/r^2, \quad \nabla\times\mathbf{v}=\mathbf{0}$

10. $\nabla\cdot\mathbf{v}=e^x(3+x), \quad \nabla\times\mathbf{v}=-e^xz\mathbf{j}+e^xy\mathbf{k}$

11. $\nabla\cdot\mathbf{v}=2(x+y+z)e^{r^2}, \quad \nabla\times\mathbf{v}=2e^{r^2}\left[(y-z)\mathbf{i}-(x-z)\mathbf{j}+(x-y)\mathbf{k}\right]$

12. $\nabla\cdot\mathbf{v}=0, \quad \nabla\times\mathbf{v}=-2\left[ze^{z^2}\mathbf{i}+xe^{x^2}\mathbf{j}+ye^{y^2}\mathbf{k}\right]$

13. $\nabla\cdot\mathbf{v}=f'(x), \quad \nabla\times\mathbf{v}=\mathbf{0}$

14. each partial derivative that appears in the curl is 0

15. use components.

16. $\nabla\cdot\mathbf{F}=-GmM\left[\nabla\cdot(r^{-3}\mathbf{r})\right]=-GmM(0)=0$

linearity (Exercise 15) (17.8.8)

$\nabla\times\mathbf{F}=-GmM[\nabla\times(r^{-3}\mathbf{r})]=-GmM(\mathbf{0})=\mathbf{0}0$

linearity (Exercise 15) (17.8.8)

17. $\nabla\cdot\mathbf{F}=\dfrac{\partial P}{\partial x}+\dfrac{\partial Q}{\partial y}+\dfrac{\partial R}{\partial z}=2+4-6=0$

18. $\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(2yz - 6xz) = 6x - 2y + (2y - 6x) = 0$

19. $$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & -2z \end{vmatrix} = \mathbf{0}$$

20. $\mathbf{F}(x, y, z) = (2x + y + 2z)\mathbf{i} + (x + 4y - 3z)\mathbf{j} + (2x - 3y - 6z)\mathbf{k}$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + y + 2z & x + 4y - 3z & 2x - 3y - 6z \end{vmatrix}$$

$$= (-3 + 3)\mathbf{i} - (2 - 2)\mathbf{j} + (1 - 1)\mathbf{k} = \mathbf{0}$$

21. $\nabla^2 f = 12(x^2 + y^2 + z^2)$

22. $\nabla^2 f = \nabla \cdot \nabla f = \nabla \cdot (yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}) = 0$

23. $\nabla^2 f = 2y^3 z^4 + 6x^2 y z^4 + 12x^2 y^3 z^2$

24. Note that for $r = \sqrt{x^2 + y^2 + z^2}$, $\frac{\partial r}{\partial x} = \frac{x}{r}$

Then $\frac{\partial^2}{\partial x^2}(\cos r) = \frac{\partial}{\partial x}\left(\frac{-x \sin r}{r}\right) = \frac{-r^2 \sin r - x^2 r \cos r + x^2 \sin r}{r^3}$,

With similar formulas for y and z. Therefore

$$\nabla^2 f = \frac{\partial^2}{\partial x^2}\cos r + \frac{\partial^2}{\partial y^2}\cos r + \frac{\partial^2}{\partial z^2}\cos r$$

$$= \frac{-3r^2 \sin r - (x^2 + y^2 + z^2) r \cos r + (x^2 + y^2 + z^2)\sin r}{r^3}$$

$$= -\cos r - 2r^{-1}\sin r$$

25. $\nabla^2 f = e^r(1 + 2r^{-1})$

26. $\frac{\partial^2}{\partial x^2}\ln r = \frac{\partial}{\partial x}\left(\frac{x}{r^2}\right) = \frac{r^2 - 2x^2}{r^4}$, with similar formula for y and z.

Then $\nabla^2 f = \frac{3r^2 - 2(x^2 + y^2 + z^2)}{r^4} = \frac{1}{r^2}$

27. (a) $2r^2$ (b) $-1/r$

28. (a)

$$(\mathbf{u}\cdot\nabla)\mathbf{r} = (\mathbf{u}\cdot\nabla x)\mathbf{i} + (\mathbf{u}\cdot\nabla y)\mathbf{j} + (\mathbf{u}\cdot\nabla_z)\mathbf{k}$$
$$= (\mathbf{u}\cdot\mathbf{i})\mathbf{i} + (\mathbf{u}\cdot\mathbf{j})\mathbf{j} + (\mathbf{u}\cdot\mathbf{k})\mathbf{k} = \mathbf{u}$$

(b)

$$(\mathbf{r}\cdot\nabla)\mathbf{u} = (\mathbf{r}\cdot\nabla yz)\mathbf{i} + (\mathbf{r}\cdot\nabla xz)\mathbf{j} + (\mathbf{r}\cdot\nabla xy)\mathbf{k}$$
$$= [\mathbf{r}\cdot(z\mathbf{j}+y\mathbf{k})]\mathbf{i} + [\mathbf{r}\cdot(z\mathbf{i}+x\mathbf{k})]\mathbf{j} + [\mathbf{r}\cdot(y\mathbf{i}+x\mathbf{j})]\mathbf{k}$$
$$= (yz+zy)\mathbf{i} + (xz+zx)\mathbf{j} + (xy+yx)\mathbf{k}$$
$$= 2(yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}) = 2\mathbf{u}$$

29.

$$\nabla^2 f = \nabla^2 g(r) = \nabla\cdot(\nabla g(r)) = \nabla\cdot\left(g'(r)r^{-1}\mathbf{r}\right)$$
$$= \left[(\nabla g'(r))\cdot r^{-1}\mathbf{r}\right] + g'(r)\left(\nabla\cdot r^{-1}\mathbf{r}\right)$$
$$= \left\{\left[g''(r)r^{-1}\mathbf{r}\right]\cdot r^{-1}\mathbf{r}\right\} + g'(r)(2r^{-1})$$
$$= g''(r) + 2r^{-1}g'(r)$$

30. (a)

$$\nabla\cdot(f\mathbf{v}) = \frac{\partial}{\partial x}(fv_1) + \frac{\partial}{\partial y}(fv_2) + \frac{\partial}{\partial z}(fv_3)$$
$$= \left(f\frac{\partial v_1}{\partial x} + \frac{\partial f}{\partial x}v_1\right) + \left(f\frac{\partial v_2}{\partial y} + \frac{\partial f}{\partial y}v_2\right) + \left(f\frac{\partial v_3}{\partial z} + \frac{\partial f}{\partial z}v_3\right)$$
$$= \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right)\cdot\mathbf{v} + f\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right)$$
$$= (\nabla f)\cdot\mathbf{v} + f(\nabla\cdot\mathbf{v})$$

(b)

$$\nabla\times(f\mathbf{v})$$
$$= \left[\frac{\partial}{\partial y}(fv_3) - \frac{\partial}{\partial z}(fv_2)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(fv_1) - \frac{\partial}{\partial x}(fv_3)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(fv_2) - \frac{\partial}{\partial y}(fv_1)\right]\mathbf{k}$$
$$= \left[f\frac{\partial v_3}{\partial y} - \frac{\partial f}{\partial y}v_3 - f\frac{\partial v_2}{\partial z} + \frac{\partial f}{\partial z}v_2\right]\mathbf{i} + \text{etc.}$$

(c)

$$\nabla\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}$$

$$\mathbf{i}\text{-component of } \nabla\times(\nabla\times\mathbf{v}) = \frac{\partial}{\partial y}\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)$$
$$= \frac{\partial^2 v_2}{\partial y\partial x}-\frac{\partial^2 v_1}{\partial y^2}-\frac{\partial^2 v_1}{\partial z^2}+\frac{\partial^2 v_3}{\partial z\partial x}$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}\right)-\left(\frac{\partial^2 v_1}{\partial y^2}+\frac{\partial^2 v_1}{\partial z^2}\right)$$

Adding and subtracting $\frac{\partial^2 v_1}{\partial x^2}$, we get

$$\frac{\partial}{\partial x}\left(\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}\right)-\left(\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 v_1}{\partial y^2}+\frac{\partial^2 v_1}{\partial z^2}\right)=\frac{\partial}{\partial x}(\nabla\cdot\mathbf{v})-\nabla^2 v_1 = \text{the } \mathbf{i}\text{-component of } \nabla^2\mathbf{v}.$$

Equality of the other components can be obtained in a similar manner.

31. $\frac{\partial f}{\partial x}=2x+y+2z,\ \frac{\partial^2 f}{\partial x^2}=2;\ \frac{\partial f}{\partial y}=4y+x-3z,\ \frac{\partial^2 f}{\partial y^2}=4;$

$\frac{\partial f}{\partial z}=-6z+2x-3y,\ \frac{\partial^2 f}{\partial z^2}=-6;$

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=2+4-6=0$$

32. $f(\mathbf{r})=\frac{1}{r}.\ \frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)=\frac{\partial}{\partial x}\left(\frac{-x}{r^3}\right)=\frac{-r^2+3x^2}{r^5},$ with similar formulas for y and z

Then $\nabla^2 f=\frac{-3r^2+3(x^2+y^2+z^2)}{r^5}=0.$

33. $n=-1$

34. Since $\nabla\cdot(\nabla f)=\nabla^2 f=0$, the gradient field ∇f is solenoidal.

∇f is irrotational by Theorem 17.8.4

SECTION 17.9

1. $\iint_S(\mathbf{v}\cdot\mathbf{n})\,d\sigma=\iiint_T(\nabla\cdot\mathbf{v})\,dxdydz=\iiint_T 3\,dxdydz=3V=4\pi$

2. $\iint_S(\mathbf{v}\cdot\mathbf{n})\,d\sigma=\iiint_T(\nabla\cdot\mathbf{v})\,dx\,dy\,dz=\iiint_T(-3)\,dx\,dy\,dz=-3V=-4\pi$

3. $\iint_S(\mathbf{v}\cdot\mathbf{n})\,d\sigma=\iiint_T(\nabla\cdot\mathbf{v})\,dxdydz=\iiint_T 2(x+y+z)\,dxdydz.$

The flux is zero since the function $f(x,y,z)=2(x+y+z)$ satisfies the relation $f(-x,-y,-z)=-f(x,y,z)$ and T is symmetric about the origin.

4. $$\iint_S (\mathbf{v}\cdot\mathbf{n})\,d\sigma=\iiint_T(\nabla\cdot\mathbf{v})\,dx\,dy\,dz=\iiint_T(-2x-2y+1)\,dx\,dy\,dz=\underbrace{\iiint_T dx\,dy\,dz}_{\text{by symmetry}}=V=\frac{4}{3}\pi$$

5.

face	$\mathbf{n}$	$\mathbf{v}\cdot\mathbf{n}$	flux
$x=0$	$-\mathbf{i}$	0	0
$x=1$	$\mathbf{i}$	1	1
$y=0$	$-\mathbf{j}$	0	0
$y=1$	$\mathbf{j}$	1	1
$z=0$	$-\mathbf{k}$	0	0
$z=1$	$\mathbf{k}$	1	1

total flux = 3

$$\iiint_T(\nabla\cdot\mathbf{v})\,dxdydz=\iiint_T 3\,dxdydz=3V=3$$

6.

face	$\mathbf{n}$	$\mathbf{v}\cdot\mathbf{n}$	flux
$x=0$	$-\mathbf{i}$	$-xy=0$	0
$x=1$	$\mathbf{i}$	$xy=y$	1/2
$y=0$	$-\mathbf{j}$	$-yz=0$	0
$y=1$	$\mathbf{j}$	$yz=z$	1/2
$z=0$	$-\mathbf{k}$	$-xz=0$	0
$z=1$	$\mathbf{k}$	$xz=x$	1/2

total flux $=\frac{3}{2}$

$$\iiint_T(\nabla\cdot\mathbf{v})\,dxdydz=\iiint_T(y+z+x)\,dxdydz=(\bar{y}+\bar{z}+\bar{x})V=(\frac{1}{2}+\frac{1}{2}+\frac{1}{2})(1)=\frac{3}{2}.$$

7.

face	$\mathbf{n}$	$\mathbf{v}\cdot\mathbf{n}$	flux
$x=0$	$-\mathbf{i}$	0	0
$x=1$	$\mathbf{i}$	1	1
$y=0$	$-\mathbf{j}$	xz	fluxes add up to 0
$y=1$	$\mathbf{j}$	$-xz$	
$z=0$	$-\mathbf{k}$	0	0
$z=1$	$\mathbf{k}$	1	1

total flux $= 2$

$$\iiint_T (\nabla\cdot\mathbf{v})\,dxdydz = \iiint_T 2\,(x+z)\,dxdydz = 2\,(\overline{x}+\overline{z})V = 2\,(\tfrac{1}{2}+\tfrac{1}{2})1 = 2$$

8.

face	$\mathbf{n}$	$\mathbf{v}\cdot\mathbf{n}$	flux
$x=0$	$-\mathbf{i}$	$-x=0$	0
$x=1$	$\mathbf{i}$	$x=1$	1
$y=0$	$-\mathbf{j}$	$-xy=0$	0
$y=1$	$\mathbf{j}$	$xy=x$	$1/2$
$z=0$	$-\mathbf{k}$	$-xyz=0$	0
$z=1$	$\mathbf{k}$	$xyz=xy$	$1/4$

total flux $= \dfrac{7}{4}$

$$\iiint_T (\nabla\cdot\mathbf{v})\;dxdydz = \iiint_T (1+x+xy)\,dxdydz = \int_0^1\int_0^1\int_0^1 (1+x+xy)\,dx\,dy\,dz = \frac{7}{4}.$$

9. flux $= \displaystyle\iiint_T (1+4y+6z)\,dxdydz = (1+4\overline{y}+6\overline{z})V = (1+0+3)\,9\pi = 36\pi$

10.

$$\text{flux} = \iiint_T (\nabla\cdot\mathbf{v})\;dxdydz = \int_0^1\int_0^{1-x}\int_0^{1-x-y} (y+z+x)\,dz\,dy\,dx$$

$$= (\overline{x}+\overline{y}+\overline{z})V = \left(\frac{3}{4}\right)\left(\frac{1}{6}\right) = \frac{1}{8}$$

11.
$$\text{flux} = \iiint_T (2x + x - 2x)\, dxdydz \iiint_T x\, dxdydz$$
$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x\, dz\, dy\, dx$$
$$= \int_0^1 \int_0^{1-x} \left(x - x^2 - xy\right)\, dy\, dx$$
$$= \int_0^1 \left[xy - x^2y - \frac{1}{2}xy^2\right]_0^{1-x} dx$$
$$= \int_0^1 \left(\frac{1}{2}x - x^2 + \frac{1}{2}x^3\right) dx = \frac{1}{24}$$

12. $$\text{flux} = \iiint_T (\nabla \cdot \mathbf{v})\, dxdydz = \iiint_T 4y\, dx\, dy\, dz = 4\overline{y}V = 4(1)\left(\frac{32}{3}\right) = \frac{128}{3}$$

13.
$$\text{flux} = \iiint_T 2(x + y + z)\, dxdydz = \int_0^4 \int_0^2 \int_0^{2\pi} 2(r\,\cos\theta + r\,\sin\theta + z)r\, dr\, d\theta\, dz$$
$$= \int_0^4 \int_0^2 4\pi\, rz\, dr\, dz$$
$$= \int_0^4 8\pi\, z\, dz = 64\pi$$

14. $$\text{flux} = \iiint_T (\nabla \cdot \mathbf{v})\, dxdydz = \iiint_T (2 + 2x)\, dy\, dy\, dz = (2 + 2\overline{x})V = \frac{32}{3}\pi$$

15. $$\text{flux} = \iiint_T (2y + 2y + 3y)\, dxdydz = 7\overline{y}V = 0$$

16. $$\text{flux} = \iiint_T (\nabla \cdot \mathbf{v})\, dxdydz = \iiint_T 7y\, dx\, dy\, dz = 7\,\overline{y}\,V = 7\left(\frac{a}{2}\right)a^3 = \frac{7}{2}a^4$$

17. $$\text{flux} = \iiint_T (A + B + C)\, dxdydz = (A + B + C)V$$

18.
$$\iint_S (\nabla f \cdot \mathbf{n})\, d\sigma = \iiint_T [\nabla \cdot (\nabla f)]\, dxdydz$$
$$= \iiint_T (\nabla^2 f)\, dxdydz = \iiint_T 0\, dxdydz = 0$$

19. Let T be the solid enclosed by S and set $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$.

$$\iint_S n_1\, d\sigma = \iint_S (\mathbf{i} \cdot \mathbf{n})\, d\sigma = \iiint_T (\nabla \cdot \mathbf{i})\, dxdydz = \iiint_T 0\, dxdydz = 0.$$

Similarly

$$\iint_S n_2\, d\sigma = 0 \quad \text{and} \quad \iint_S n_3\, d\sigma = 0.$$

20. (a) The identity follows from setting $\mathbf{v} = \nabla f$ in (17.8.6).

$$\begin{aligned}\iint_S (f f_n')\, d\sigma = \iint_S (f\nabla f \cdot \mathbf{n})\, d\sigma &= \iiint_T [\nabla \cdot (f\nabla f)]\, dxdydz \\ &= \iiint_T [\| \nabla f \|^2 + f(\nabla^2 f)]\, dxdydz \\ &= \iiint_T \| \nabla f \|^2\, dxdydz \quad \text{since} \quad \nabla^2 f = 0\end{aligned}$$

(b)

$$\begin{aligned}\iint_S (g f_n')\, d\sigma = \iint_S (g\nabla f \cdot \mathbf{n})\, d\sigma &= \iiint_T [\nabla \cdot (g\nabla f)]\, dxdydz \\ &= \iiint_T \{(\nabla g \cdot \nabla f) + g[\nabla \cdot (\nabla f)]\}\, dxdydz \\ &= \iiint_T [(\nabla g \cdot \nabla f) + g(\nabla^2 f)]\, dxdydz\end{aligned}$$

21. A routine computation shows that $\nabla \cdot (\nabla f \times \nabla g) = 0.$ Therefore

$$\iint_S [(\nabla f \times \nabla g) \cdot \mathbf{n}]\, d\sigma = \iiint_T [\nabla \cdot (\nabla f \times \nabla g)]\, dxdydz = 0.$$

22. Since $\nabla \cdot \mathbf{r} = 3,$ we can write

$$V = \iiint_T dxdydz = \iiint_T \left(\nabla \cdot \frac{\mathbf{r}}{3}\right) dxdydz = \iint_S \left(\frac{1}{3}\mathbf{r} \cdot \mathbf{n}\right) d\sigma, \quad \text{by the divergence theorem.}$$

23. Set $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}.$

$$\begin{aligned}F_1 = \iint_S [\rho(z-c)\mathbf{i} \cdot \mathbf{n}]\, d\sigma &= \iiint_T [\nabla \cdot \rho(z-c)\mathbf{i}]\, dxdydz \\ &= \iiint_T \underbrace{\frac{\partial}{\partial x}[\rho(z-c)]}_{0}\, dxdydz = 0.\end{aligned}$$

Similarly $F_2 = 0.$

$$F_3 = \iint_S [\rho(z-c)\mathbf{k} \cdot \mathbf{n}]\, d\sigma = \iiint_T [\nabla \cdot \rho(z-c)\mathbf{k}]\, dxdydz$$

$$= \iiint_T \frac{\partial}{\partial z}[\rho(z-c)]\, dxdydz$$

$$= \iiint_T \rho\, dxdydz = W.$$

24.

$$\tau_{Tot} \cdot \mathbf{i} = \iint_S \{[\mathbf{r} \times \rho(c-z)\mathbf{n}] \cdot \mathbf{i}\}\, d\sigma$$

(12.5.6)

$$= -\iint_S [(\mathbf{i} \times \mathbf{r}) \cdot \rho(c-z)\mathbf{n}]\, d\sigma$$

$$= \rho \iint_S (z-c)[(\mathbf{i} \times \mathbf{r}) \cdot \mathbf{n}]\, d\sigma$$

divergence theorem

$$= \rho \iiint_T [\nabla \cdot (z-c)(\mathbf{i} \times \mathbf{r})]\, dxdydz$$

$$\mathbf{i} \times \mathbf{r} = \mathbf{i} \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = -z\mathbf{j} + y\mathbf{k}$$

$$(z-c)(\mathbf{i} \times \mathbf{r}) = (z-c)(-z\mathbf{j} + y\mathbf{k}) = (cz - z^2)\mathbf{j} + (yz - cy)\mathbf{k}$$

$$\nabla \cdot (z-c)(\mathbf{i} \times \mathbf{r}) = y$$

$$\tau_{Tot} \cdot \mathbf{i} = \rho \iiint_T y\, dxdydz = \rho\overline{y}V = \overline{y}(\rho V) = \overline{y}W$$

$$(\mathbf{r} \times \mathbf{F}) \cdot \mathbf{i} = [(\overline{x}\mathbf{i} + \overline{y}\mathbf{j} + \overline{z}\mathbf{k}) \times W\mathbf{k}] \cdot \mathbf{i}$$

$$= (-\overline{x}W\mathbf{j} + \overline{y}W\mathbf{i}) \cdot \mathbf{i} = \overline{y}W = \tau_{Tot} \cdot \mathbf{i}$$

Equality of the other components can be shown in a similar manner.

PROJECT 17.9

1. For $\mathbf{r} \neq 0$, $\nabla \cdot \mathbf{E} = \nabla \cdot qr^{-3}\mathbf{r} = q(-3+3)r^{-3} = 0$ by (17.8.8)

2. By the divergence theorem, flux of $\mathbf{E}$ out of $S = \iiint_T (\nabla \cdot \mathbf{E})\, dx\, dy\, dz = \iiint_T 0\, dx\, dy\, dz = 0$

3. On $S_a, \mathbf{n} = \dfrac{\mathbf{r}}{r}$, and thus $\mathbf{E} \cdot \mathbf{n} = q\dfrac{\mathbf{r}}{r^3} \cdot \dfrac{\mathbf{r}}{r} = \dfrac{q}{r^2} = \dfrac{q}{a^2}$

Thus flux of $\mathbf{E}$ out of $S_a = \displaystyle\iint_{S_a} (\mathbf{E} \cdot \mathbf{n})\, d\sigma = \iint_{S_a} \frac{q}{a^2}\, d\sigma = \frac{q}{a^2}(\text{area of } S_a) = \frac{q}{a^2}(4\pi a^2) = 4\pi q.$

SECTION 17.10

For Exercises 1–4: $\mathbf{n} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $C: \mathbf{r}(u) = \cos u\,\mathbf{i} + \sin u\,\mathbf{j}, \quad u \in [0, 2\pi]$.

1. (a) $\displaystyle\iint_S [(\nabla \times \mathbf{v}) \cdot \mathbf{n}]\, d\sigma = \iint_S (\mathbf{0} \cdot \mathbf{n})\, d\sigma = 0$

(b) S is bounded by the unit circle $C: \mathbf{r}(u) = \cos u\,\mathbf{i} + \sin u\,\mathbf{j}, \quad u \in [0, 2\pi]$.

$\displaystyle\oint_C \mathbf{v}(\mathbf{r}) \cdot d\mathbf{r} = 0$ since $\mathbf{v}$ is a gradient.

2. (a) $\displaystyle\iint_S [(\nabla \times \mathbf{v}) \cdot \mathbf{n}]\, d\sigma = \iint_S (-2\mathbf{k} \cdot \mathbf{n})\, d\sigma = -2\iint_S z\, d\sigma = -2\bar{z}A = -2\left(\tfrac{1}{2}\right) 2\pi = -2\pi$

Exercise 17, Section 17.7

(b) $\displaystyle\oint_C \mathbf{v}(\mathbf{r}) \cdot d\mathbf{r} = \oint_C y\, dx - x\, dy = \int_0^{2\pi} (-\sin^2 u - \cos^2 u)\, du = -2\pi$

3. (a) $\displaystyle\iint_S [(\nabla \times \mathbf{v}) \cdot \mathbf{n}]\, d\sigma = \iint_S \left[(-3y^2\mathbf{i} + 2z\mathbf{j} + 2\mathbf{k}) \cdot \mathbf{n}\right]\, d\sigma$

$$= \iint_S (-3xy^2 + 2yz + 2z)\, d\sigma$$

$$= \underbrace{\iint_S (-3xy^2)\, d\sigma}_{0} + \underbrace{\iint_S 2yz\, d\sigma}_{0} + 2\iint_S z\, d\sigma = 2\bar{z}V = 2(\frac{1}{2})2\pi = 2\pi$$

Exercise 17, Section 17.7

(b) $\displaystyle\oint_C \mathbf{v}(\mathbf{r}) \cdot d\mathbf{r} = \oint_C z^2\, dx + 2x\, dy = \oint_C 2x\, dy = \int_0^{2\pi} 2\cos^2 u\, du = 2\pi$

4. **(a)**

$$\iint_S [(\nabla \times \mathbf{v}) \cdot \mathbf{n}]\, d\sigma = \iint_S [(-6y\mathbf{i} + 6x\mathbf{j} - 2x\mathbf{k}) \cdot \mathbf{n}]\, d\sigma$$

$$= \iint_S (-2xz)\, d\sigma = 0 \quad \text{by symmetry}$$

$$\oint_C \mathbf{v}(\mathbf{r}) \cdot \mathbf{dr} = \int_C 6xz\,dx - x^2, dy = -\int_C x^2\,dy$$
$$= -\int_0^{2\pi} \cos^3 u\,du = -\int_0^{2\pi} (\cos u - \sin^2 u \cos u)\,du = 0$$

For Exercises 5–7 take $S\colon z = 2 - x - y$ with $0 \le x \le 2,\ 0 \le y \le 2 - x$ and C as the triangle $(2,0,0)$, $(0,2,0)$, $(0,0,2)$. Then $C = C_1 \cup C_2 \cup C_3$ with

$$C_1\colon \mathbf{r}_1(u) = 2(1-u)\,\mathbf{i} + 2u\mathbf{j}, \quad u \in [0,1],$$
$$C_2\colon \mathbf{r}_2(u) = 2(1-u)\,\mathbf{j} + 2u\mathbf{k}, \quad u \in [0,1],$$
$$C_3\colon \mathbf{r}_3(u) = 2(1-u)\,\mathbf{k} + 2u\mathbf{i}, \quad u \in [0,1].$$

$n = \frac{1}{3}\sqrt{3}(\mathbf{i} + \mathbf{j} + \mathbf{k})$ area of $S\colon A = 2\sqrt{3}$ centroid: $\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$

5. (a) $$\iint_S [(\nabla \times \mathbf{v}) \cdot \mathbf{n}]\ d\sigma = \iint_S \frac{1}{3}\sqrt{3}\,d\sigma = \frac{1}{3}\sqrt{3}A = 2$$

(b) $$\oint_C \mathbf{v}(\mathbf{r}) \cdot \mathbf{dr} = \left(\int_{C_1} + \int_{C_2} + \int_{C_3}\right) \mathbf{v}(\mathbf{r}) \cdot \mathbf{dr} = -2 + 2 + 2 = 2$$

6. (a)
$$\iint_S [(\nabla \times \mathbf{v}) \cdot \mathbf{n}]\,d\sigma = \iint_S [(-2x\mathbf{j} - 2y\mathbf{k}) \cdot \mathbf{n}]\,d\sigma$$
$$= -\frac{2}{3}\sqrt{3}\iint (x+y)\,d\sigma = -\frac{2}{3}\sqrt{3}(\overline{x} + \overline{y})A = -\frac{16}{3}$$

(b) $$\oint_C \mathbf{v}(\mathbf{r}) \cdot \mathbf{dr} = \left(\int_{C_1} + \int_{C_2} + \int_{C_3}\right) \mathbf{v}(\mathbf{r}) \cdot dr = -\frac{8}{3} + 0 - \frac{8}{3} = -\frac{16}{3}$$

7. (a) $$\iint_S [(\nabla \times \mathbf{v}) \cdot \mathbf{n}]\ d\sigma = \iint_S (y\mathbf{k} \cdot \mathbf{n})\,d\sigma = \frac{1}{3}\sqrt{3}\iint_S y\,d\sigma = \frac{1}{3}\sqrt{3}\overline{y}A = \frac{4}{3}$$

(b) $$\oint_C \mathbf{v}(\mathbf{r}) \cdot \mathbf{dr} = \left(\int_{C_1} + \int_{C_2} + \int_{C_3}\right) \mathbf{v}(\mathbf{r}) \cdot \mathbf{dr} = \left(\frac{4}{3} - \frac{32}{5}\right) + \frac{32}{5} + 0 = \frac{4}{3}$$

8. By (17.10.2) $\mathbf{v}$ is a gradient: $\mathbf{v} = \nabla\phi$. Therefore
$$\int_C \mathbf{v}(\mathbf{r}) \cdot \mathbf{dr} = \int_C [(\nabla\phi) \cdot dr] = 0 \quad \text{by (17.2.2).}$$

9. The bounding curve is the set of all (x, y, z) with
$$x^2 + y^2 = 4 \quad \text{and} \quad z = 4.$$

Traversed in the positive sense with respect to $\mathbf{n}$, it is the curve $-C$ where
$$C\ :\ \mathbf{r}(u) = 2\cos u\,\mathbf{i} + 2\sin u\,\mathbf{j} + 4\mathbf{k}, \qquad u \in [0, 2\pi].$$

By Stokes's theorem the flux we want is

$$-\int_C \mathbf{v}(\mathbf{r})\cdot d\mathbf{r} = -\int_C y\,dx + z\,dy + x^2z^2\,dz$$

$$= -\int_0^{2\pi} \left(-4\sin^2 u + 8\cos u\right)\,du = 4\pi.$$

10. The bounding curve is the set of all (x, y, z) with

$$x^2 + z^2 = 9, \quad y = -8.$$

Traversed in the positive direction with respect to $\mathbf{n}$, it is the curve $-C$ where

$$C:\ \mathbf{r}(u) = 3\cos u\, wi - 8\,\mathbf{j} + 3\sin u\,\mathbf{k}, \qquad u \in [0, 2\pi].$$

By Stokes's theorem the flux we want is

$$-\int_C \mathbf{v}(\mathbf{r})\cdot d\mathbf{r} = -\int_C \frac{1}{2}y\,dx + 2xz\,dy - 3x\,dz$$

$$= -\int_0^{2\pi} (12\sin u - 27\cos^2 u)\,du = -27\pi.$$

11. The bounding curve C for S is the bounding curve of the elliptical region $\Omega : \frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$. Since

$$\nabla\times\mathbf{v} = 2x^2yz^2\mathbf{i} - 2xy^2z^2\mathbf{j}$$

is zero on the xy-plane, the flux of $\nabla\times\mathbf{v}$ through Ω is zero, the circulation of $\mathbf{v}$ about C is zero, and therefore the flux of $\nabla\times\mathbf{v}$ through S is zero.

12. Let T be the solid enclosed by S. By our condition on $\mathbf{v}$, $\nabla\times\mathbf{v}$ is continuously differentiable on T. Therefore by the divergence theorem

$$\iint_S [(\nabla\times\mathbf{v})\cdot\mathbf{n}]\,d\sigma = \iiint_T [\nabla\cdot(\nabla\times\mathbf{v})]\,dxdydz.$$

This is zero since the divergence of a curl is zero.

13. C bounds the surface

$$S\colon z = \sqrt{1 - \tfrac{1}{2}(x^2+y^2)}, \qquad (x, y) \in \Omega$$

with $\Omega : x^2 + (y - \frac{1}{2})^2 \leq \frac{1}{4}$. Routine calculation shows that $\nabla\times\mathbf{v} = y\mathbf{k}$. The circulation of $\mathbf{v}$ with respect to the upper unit normal $\mathbf{n}$ is given by

$$\iint_S (y\mathbf{k}\cdot\mathbf{n})\,d\sigma \underset{(17.7.9)}{=} \iint_\Omega y\,dxdy = \bar{y}A = \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{1}{8}\pi.$$

If $-\mathbf{n}$ is used, the circulation is $-\frac{1}{8}\pi$. Answer: $\pm\frac{1}{8}\pi$.

14. $\nabla\times\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$. Since the plane $x + 2y + z = 0$ passes through the origin, it intersects the sphere in a circle of radius a. The surface S bounded by this circle is a disc of radius a with upper unit normal

$$\mathbf{n} = \frac{1}{6}\sqrt{6}(\mathbf{i} + 2\mathbf{j} + \mathbf{k}).$$

The circulation of $\mathbf{v}$ with respect to $\mathbf{n}$ is given by

$$\iint_S [(\nabla \times \mathbf{v}) \cdot \mathbf{n}]\, d\sigma = \iint_S \left(-\frac{5}{6}\sqrt{6}\right) d\sigma = -\frac{5}{6}\sqrt{6}A = -\frac{5}{6}\sqrt{6}\pi a^2.$$

If $-\mathbf{n}$ is used, the circulation is $\frac{5}{6}\sqrt{6}\pi a^2$. Answer: $\pm\frac{5}{6}\sqrt{6}\pi a^2$.

15. $\nabla \times \mathbf{v} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$. The paraboloid intersects the plane in a curve C that bounds a flat surface S that projects onto the disc $x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$ in the xy-plane. The upper unit normal to S is the vector $\mathbf{n} = \frac{1}{2}\sqrt{2}\,(-\mathbf{j} + \mathbf{k})$. The area of the base disc is $\frac{1}{4}\pi$. Letting γ be the angle between $\mathbf{n}$ and $\mathbf{k}$, we have $\cos\gamma = \mathbf{n} \cdot \mathbf{k} = \frac{1}{2}\sqrt{2}$ and $\sec\gamma = \sqrt{2}$. Therefore the area of S is $\frac{1}{4}\sqrt{2}\pi$. The circulation of $\mathbf{v}$ with respect to $\mathbf{n}$ is given by

$$\iint_S [(\nabla \times \mathbf{v}) \cdot \mathbf{n}]\, d\sigma = \iint_S -\frac{1}{2}\sqrt{2}\, d\sigma = \left(-\frac{1}{2}\sqrt{2}\right)(\text{area of } S) = -\frac{1}{4}\pi.$$

If $-\mathbf{n}$ is used, the circulation is $\frac{1}{4}\pi$. Answer: $\pm\frac{1}{4}\pi$.

16. $\nabla \times \mathbf{v} = -y\mathbf{i} - z\mathbf{j} - x\mathbf{k}$. The curve C bounds a flat surface S that projects onto the disc $x^2 + y^2 = b^2$ in the xy-plane. The upper unit normal to S is the vector $\mathbf{n} = \frac{1}{2}\sqrt{2}(\mathbf{j} + \mathbf{k})$. The area of the base disc is πb^2. Letting γ be the angle between $\mathbf{n}$ and $\mathbf{k}$, we have $\cos\gamma = \mathbf{n} \cdot \mathbf{k} = \frac{1}{2}\sqrt{2}$ and $\sec\gamma = \sqrt{2}$. Therefore the area of S is $\pi b^2\sqrt{2}$. The circulation of $\mathbf{v}$ with respect to $\mathbf{n}$ is given by

$$\iint_S [(\nabla \times \mathbf{v}) \cdot \mathbf{n}]\, d\sigma = -\frac{1}{2}\sqrt{2}\iint_S (x + z)\, d\sigma \underset{\text{by symmetry}}{=} -\frac{1}{2}\sqrt{2}\iint_S z\, d\sigma = -\frac{1}{2}\sqrt{2}\bar{z}A.$$

It's clear by symmetry that $\bar{z} = a^2$, the height at which S intersects the xz-plane. Since $A = \pi b^2\sqrt{2}$, the circulation is $-\pi a^2 b^2$. If $-\mathbf{n}$ is used, the circulation becomes $\pi a^2 b^2$, Answer: $\pm\pi a^2 b^2$.

17. Straightforward calculation shows that

$$\nabla \times (\mathbf{a} \times \mathbf{r}) = \nabla \times [(a_2 z - a_3 y)\,\mathbf{i} + (a_3 x - a_1 z)\,\mathbf{j} + (a_1 y - a_2 x)\mathbf{k}] = 2\mathbf{a}.$$

18. $\nabla \times (\phi\nabla\psi) = (\nabla\phi \times \nabla\psi) + \phi[\nabla \times \nabla\psi] = \nabla\phi \times \nabla\psi$

(17.8.7)

since the curl of a gradient is zero. Therefore the result follows from Stokes's theorem.

19. In the plane of C, the curve C bounds some Jordan region that we call Ω. The surface $S \cup \Omega$ is a piecewise–smooth surface that bounds a solid T. Note that $\nabla \times \mathbf{v}$ is continuously differentiable on T. Thus, by the divergence theorem,

$$\iiint_T [\nabla \cdot (\nabla \times \mathbf{v})]\, dxdydz = \iint_{S\cup\Omega} [(\nabla \times \mathbf{v}) \cdot \mathbf{n}]\, d\sigma$$

where $\mathbf{n}$ is the outer unit normal. Since the divergence of a curl is identically zero, we have

$$\iint_{S\cup\Omega} [(\nabla \times \mathbf{v}) \cdot \mathbf{n}]\, d\sigma = 0.$$

Now $\mathbf{n}$ is $\mathbf{n}_1$ on S and $\mathbf{n}_2$ on Ω. Thus

$$\iint_S [(\nabla \times \mathbf{v}) \cdot \mathbf{n}_1]\, d\sigma + \iint_\Omega [(\nabla \times \mathbf{v}) \cdot \mathbf{n}_2]\, d\sigma = 0.$$

This gives

$$\iint_S [(\nabla \times \mathbf{v}) \cdot \mathbf{n}_1]\, d\sigma = \iint_\Omega [(\nabla \times \mathbf{v}) \cdot (-\mathbf{n}_2)]\, d\sigma = \oint_C \mathbf{v}(\mathbf{r}) \cdot \mathbf{dr}$$

where C is traversed in a positive sense with respect to $-\mathbf{n}_2$ and therefore in a positive sense with respect to $\mathbf{n}_1$. ($-\mathbf{n}_2$ points toward S.)

20. By the chain rule $\quad \dfrac{dx}{dt} = \dfrac{d}{dt}[x(u(t), v(t))] = \dfrac{\partial x}{\partial u}u'(t) + \dfrac{\partial x}{\partial v}v'(t).$

Thus

$$\int_{C_1} v_1\, dx = \int_a^b \left(v_1 \frac{dx}{dt}\right)\, dt = \int_a^b \left[v_1 \frac{\partial x}{\partial u}u'(t) + v_1 \frac{\partial x}{\partial v}v'(t)\right]\, dt$$
$$= \int_{C_r} v_1 \frac{\partial x}{\partial u}\, du + v_1 \frac{\partial x}{\partial v}\, dv$$

by Green's theorem

$$= \iint_\Gamma \left[\frac{\partial}{\partial u}\left(v_1 \frac{\partial x}{\partial v}\right) - \frac{\partial}{\partial v}\left(v_1 \frac{\partial x}{\partial u}\right)\right]\, du\, dv.$$

The integrand can be written

$$\frac{\partial v_1}{\partial u}\frac{\partial x}{\partial v} + v_1 \frac{\partial^2 x}{\partial u \partial v} - \frac{\partial v_1}{\partial v}\frac{\partial x}{\partial v} - v_1 \frac{\partial^2 x}{\partial v \partial u} = \frac{\partial v_1}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial v_1}{\partial v}\frac{\partial x}{\partial u}.$$

equality of partials

Thus we have

$$\int_C v_1\, dx = \iint_\Gamma \left[\frac{\partial v_1}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial v_1}{\partial v}\frac{\partial x}{\partial u}\right]\, du\, dv.$$

By our previous choice of unit normal, $\quad \mathbf{n} = \mathbf{N}/\,\|\,\mathbf{N}\,\|$. $\quad$ Therefore

$$\iint_S [(\nabla \times v_1\mathbf{i}) \cdot \mathbf{n}]\, d\sigma = \iint_\Gamma [(\nabla \times v_1\mathbf{i}) \cdot \mathbf{N}]\, du\, dv.$$

Note that

$$(\nabla \times v_1\mathbf{i}) \cdot \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix}$$

$$= \left(\frac{\partial v_1}{\partial z}\mathbf{j} - \frac{\partial v_1}{\partial y}\mathbf{k}\right) \cdot \left[\left(\frac{\partial x}{\partial v}\frac{\partial z}{\partial u} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right)\mathbf{i} + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\mathbf{k}\right]$$

$$= \frac{\partial v_1}{\partial z}\left(\frac{\partial x}{\partial v}\frac{\partial z}{\partial u} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right) + \frac{\partial v_1}{\partial y}\left(\frac{\partial x}{\partial v}\frac{\partial y}{\partial u} - \frac{\partial x}{\partial u}\frac{\partial y}{\partial v}\right)$$

$$= \left(\frac{\partial v_1}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial v_1}{\partial y}\frac{\partial y}{\partial u}\right)\frac{\partial x}{\partial v} - \left(\frac{\partial v_1}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial v_1}{\partial y}\frac{\partial y}{\partial v}\right)\frac{\partial x}{\partial u}.$$

Now, by the chain rule,

$$\frac{\partial v_1}{\partial u} = \frac{\partial v_1}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v_1}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial v_1}{\partial z}\frac{\partial z}{\partial u}, \qquad \frac{\partial v_1}{\partial v} = \frac{\partial v_1}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v_1}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial v_1}{\partial z}\frac{\partial z}{\partial v}.$$

Therefore

$$(\nabla \times v_1\mathbf{i}) \cdot \mathbf{N} = \left(\frac{\partial v_1}{\partial u} - \frac{\partial v_1}{\partial x}\frac{\partial x}{\partial u}\right)\frac{\partial x}{\partial v} - \left(\frac{\partial v_1}{\partial v} - \frac{\partial v_1}{\partial x}\frac{\partial x}{\partial v}\right)\frac{\partial x}{\partial u}$$
$$= \frac{\partial v_1}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial v_1}{\partial v}\frac{\partial x}{\partial u}$$

and, as asserted,

$$\iint_S [(\nabla \times v_1\mathbf{i}) \cdot \mathbf{n}]\, d\sigma = \iint_\Gamma \left[\frac{\partial v_1}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial v_1}{\partial v}\frac{\partial x}{\partial u}\right] du\, dv.$$

CHAPTER 18

SECTION 18.1

1. first order, ordinary

2. ordinary, second order

3. first order, partial

4. ordinary, third order

5. second order, ordinary

6. partial, second order

7. second order, partial

8. ordinary, fourth order

9. $y_1'(x) = \frac{1}{2}\,e^{x/2}; \qquad 2y_1' - y_1 = 2\left(\frac{1}{2}\right)e^{x/2} - e^{x/2} = 0; \quad y_1$ is a solution.

$y_2'(x) = 2x + e^{x/2}; \qquad 2y_2' - y_2 = 2\left(2x + e^{x/2}\right) - \left(x^2 + 2e^{x/2}\right) = 4x - x^2 \neq 0;$

y_2 is not a solution.

10. $y_1' + xy_1 = -xe^{-x^2/2} + xe^{-x^2/2} = 0;$ not a solution

$y_2' + xy_2 = -Cxe^{-x^2/2} + x + Cxe^{-x^2/2} = x; \quad y_2$ is a solution.

11. $y_1'(x) = \dfrac{-e^x}{(e^x+1)^2}; \qquad y_1' + y_1 = \dfrac{-e^x}{(e^x+1)^2} + \dfrac{1}{e^x+1} = \dfrac{1}{(e^x+1)^2} = y_1^2; \quad y_1$ is a solution.

$y_2'(x) = \dfrac{-Ce^x}{(Ce^x+1)^2}; \qquad y_2' + y_2 = \dfrac{-Ce^x}{(Ce^x+1)^2} + \dfrac{1}{Ce^x+1} = \dfrac{1}{(Ce^x+1)^2} = y_2^2;$

y_2 is a solution.

12. $y_1'' + 4y_1 = -8\sin 2x + 8\sin 2x = 0; \quad y_1$ is a solution.

$y_2'' + 4y_2 = -2\cos x + 8\cos x = 6\cos x;$ not a solution.

13. $y_1'(x) = 2e^{2x}, \quad y_1'' = 4e^{2x}; \qquad y_1'' - 4y_1 = 4e^{2x} - 4e^{2x} = 0; \quad y_1$ is a solution.

$y_2'(x) = 2C\cosh 2x, \quad y_2'' = 4C\sinh 2x; \qquad y_2'' - 4y_2 = 4C\sinh 2x - 4C\sinh 2x = 0;$

y_2 is a solution.

14. $y_1'' - 2y_1' - 3y_1 = e^{-x} + 18e^{3x} - 2(-e^{-x} + 6e^{3x}) - 3(e^{-x} + 2e^{3x}) = 0;$ not a solution

$y_2'' - 2y_2' - 3y_2 = \dfrac{7}{4}\left[(6+9x)e^{3x} - 2(1+3x)e^{3x} - 3xe^{3x}\right] = 7e^{3x}; \quad y_2$ is a solution.

15. $\dfrac{\partial u_1}{\partial x} = -\lambda\sin\lambda x\sin\lambda at, \qquad \dfrac{\partial^2 u_1}{\partial x^2} = -\lambda^2\cos\lambda x\sin\lambda at$

$\dfrac{\partial u_1}{\partial t} = -\lambda a\cos\lambda x\cos\lambda at, \qquad \dfrac{\partial^2 u_1}{\partial t^2} = -(\lambda a)^2\cos\lambda x\sin\lambda at$

$$a^2\frac{\partial^2 u_1}{\partial x^2} = -a^2\lambda^2\cos\lambda x\sin\lambda at = \frac{\partial^2 u_1}{\partial t^2}; \qquad u_1 \text{ is a solution.}$$

$\dfrac{\partial u_2}{\partial x} = \cos(x - at), \qquad \dfrac{\partial^2 u_2}{\partial x^2} = -\sin(x - at)$

$\dfrac{\partial u_2}{\partial t} = -a\cos(x - at), \qquad \dfrac{\partial^2 u_2}{\partial t^2} = a^2\sin(x - at)$

$$a^2\frac{\partial^2 u_2}{\partial x^2} = a^2\sin(x - at) = \frac{\partial^2 u_2}{\partial t^2}; \qquad u_2 \text{ is a solution.}$$

16. $\dfrac{\partial^2 u_1}{\partial x^2}+\dfrac{\partial^2 u_1}{\partial y^2}=-\sin x\cosh y+\sin x\cosh y=0;\quad u_1$ is a solution.

$\dfrac{\partial^2 u_2}{\partial x^2}+\dfrac{\partial^2 u_2}{\partial y^2}=\dfrac{2(y^2-x^2)}{(x^2+y^2)^2}+\dfrac{2(x^2-y^2)}{(x^2+y^2)^2}=0;\quad u_2$ is a solution.

17. $y_1'(x)=-\frac{1}{2}x^{-3/2},\quad y_1''=\frac{3}{4}x^{-5/2};$

$$4x^2y_1''-12x\,y_1'-9y_1=4x^2\left(\tfrac{3}{4}x^{-5/2}\right)-12x\left(-\tfrac{1}{2}x^{-3/2}\right)-9x^{-1/2}$$
$$=3x^{-1/2}+6x^{-1/2}-9x^{-1/2}=0;$$

y_1 is a solution.

$y_2'(x)=-\frac{1}{2}C_1x^{-3/2}+\frac{9}{2}C_2x^{7/2},\quad y_2''=\frac{3}{4}C_1x^{-5/2}+\frac{63}{4}C_2x^{5/2};$

$4x^2y_2''-12x\,y_2'-9y_2$

$=4x^2\left(\frac{3}{4}C_1x^{-5/2}+\frac{63}{4}C_2x^{5/2}\right)-12x\left(-\frac{1}{2}C_1x^{-3/2}+\frac{9}{2}C_2x^{7/2}\right)-9\left(C_1x^{-1/2}+C_2x^{9/2}\right)$

$=C_1\left(3x^{-1/2}+6x^{-1/2}-9x^{-1/2}\right)+C_2\left(63x^{9/2}-54x^{9/2}-9x^{9/2}\right)=0;$

y_2 is a solution.

18. $a^2\dfrac{\partial^2 u_1}{\partial x^2}=a^2\left(-e^{-a^2t}\sin x\right)=-a^2e^{-a^2t}\sin x=\dfrac{\partial u_1}{\partial t};\quad u_1$ is a solution.

$a^2\dfrac{\partial^2 u_2}{\partial x^2}=a^2\left(-\lambda^2e^{-a^2t}\sin\lambda x\right)\neq-a^2e^{-a^2t}\sin\lambda x=\dfrac{\partial u_2}{\partial t};\quad u_2$ is not a solution

19. $y'(x)=5Ce^{5x}=5y$ Thus, $y=Ce^{5x}$ is a solution.

$y(0)=Ce^{5\cdot 0}=2\implies C=2;\quad y=2e^{5x}$ satisfies the side condition.

20. $xy'+y=x\left(\dfrac{2x}{3}-\dfrac{C}{x^2}\right)+\left(\dfrac{x^2}{3}+\dfrac{C}{x}\right)=\dfrac{3x^2}{3}=x^2.$

$2=y(3)=\dfrac{3^2}{3}+\dfrac{C}{3}=3+\dfrac{C}{3}\implies C=-3;\quad y=\dfrac{x^2}{3}-\dfrac{3}{x}.$

21. It was shown in Exercise 11 that $y=\dfrac{1}{Ce^x+1}$ is a one-parameter family of solutions of $y'+y=y^2$.

$y(1)=\dfrac{1}{Ce+1}=-1\implies C=-\dfrac{2}{e};\quad y=\dfrac{1}{-2e^{x-1}+1}$ satisfies the side condition.

22. $y'=\ln\dfrac{C}{x}-1=\dfrac{x\ln(C/x)-x}{x}=\dfrac{y-x}{x}.$

$4=y(2)=2\ln\dfrac{C}{2}\implies C=2e^2\implies y=x\ln\dfrac{2e^2}{x}$

23. $y'=C_1+\frac{1}{2}C_2x^{-1/2},\quad y''=-\frac{1}{4}C_2x^{-3/2};$

$$2x^2y''-xy'+y=2x^2\left(-\tfrac{1}{4}C_2x^{-3/2}\right)-x\left(C_1+\tfrac{1}{2}C_2x^{-1/2}\right)+C_1x+C_2x^{1/2}$$
$$=-\tfrac{1}{2}C_2x^{1/2}-C_1x-\tfrac{1}{2}C_2x^{1/2}+C_1x+C_2x^{1/2}=0$$

Therefore, $y=C_1x+C_2x^{1/2}$ is a two-parameter family of solutions.

$$y(4) = C_1(4) + C_2(4)^{1/2} = 1 = 4C_1 + 2C_2$$
$$y'(4) = C_1 + \tfrac{1}{2}C_2(4)^{-1/2} = -2 = C_1 + \tfrac{1}{4}C_2$$

implies $C_1 = -\frac{17}{4}$, $C_2 = 9$; $y = -\frac{17}{4}x + 9x^{1/2}$ satisfies the side conditions.

24. $y'' + 9y = -9C_1 \sin 3x - 9C_2 \cos 3x + 9(C_1 \sin 3x + C_2 \cos 3x) = 0.$
$1 = y(\frac{\pi}{2}) = -C_1 \implies C_1 = -1$; $1 = y'(\frac{\pi}{2}) = 3C_2 \implies C_2 = \frac{1}{3}$
$\implies y = -\sin 3x + \dfrac{1}{3}\cos 3x.$

25. $y' = 2C_1x + 2C_2x \ln x + C_2x$, $y'' = 2C_1 + 3C_2 + 2C_2 \ln x$;

$$\begin{aligned} &x^2y'' - 3xy' + 4y \\ &\quad = x^2(2C_1 + 3C_2 + 2C_2 \ln x) - 3x(2C_1x + 2C_2x \ln x + C_2x) + 4\left(C_1x^2 + C_2x^2 \ln x\right) \\ &\quad = x^2(2C_1 + 3C_2 - 6C_1 - 3C_2 + 4C_1) + x^2 \ln x\,(2C_2 - 6C_2 + 4C_2) = 0 \end{aligned}$$

Therefore, $y = C_1x^2 + C_2x^2 \ln x$ is a two parameter family of solutions.
$y(1) = C_1 = 0$, $y'(1) = 2C_1 + C_2 = 1 \implies C_2 = 1$; $y = x^2 \ln x$ satisfies the side conditions.

26. $y' = C_2e^x + 2C_3e^{2x} + \dfrac{1}{2}x + \dfrac{3}{4} - (1+x)e^x$

$y'' = C_2e^x + 4C_3e^{2x} + \dfrac{1}{2} - (2+x)e^x$

$y''' = C_2e^x + 8C_3e^{2x} - (3+x)e^x$

$y''' - 3y'' + 2y' = -(3+x)e^x - \dfrac{3}{2} + 3(2+x)e^x + x + \dfrac{3}{2} - 2(1+x)e^x = x + e^x.$

$1 = y(0) = C_1 + C_2 + C_3$; $-\dfrac{1}{4} = y'(0) = C_2 + 2C_3 - \dfrac{1}{4}$

$-\dfrac{3}{2} = y''(0) = C_2 + 4C_3 - \dfrac{3}{2} \implies C_2 = C_3 = 0$, $C_1 = 1 \implies y = 1 + \dfrac{1}{4}x^2 + \dfrac{3}{4}x - xe^x.$

27. $y = e^{rx}$, $y' = re^{rx}$; $y' - 3y = re^{rx} - 3e^{rx} = 0 \implies r = 3.$

28. $y'' - 5y' + 6y = e^{rx}(r^2 - 5r + 6) = 0$ if $r = 2, 3.$

29. $y = e^{rx}$, $y' = re^{rx}$, $y'' = r^2e^{rx}$;
$$y'' + 6y' + 9y = r^2e^{rx} + 6re^{rx} + 9e^{rx} = e^{rx}(r+3)^2 = 0 \implies r = -3$$

30. $y''' - 3y' + 2y = e^{rx}(r^3 - 3r + 2) = e^{rx}(r-1)^2(r+2) = 0$ if $r = 1, -2$

31. $y = x^r$, $y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$;
$$xy'' + y' = x\left[r(r-1)x^{r-2}\right] + rx^{r-1} = \left[r^2 - r + r\right]x^{r-1} = 0 \implies r = 0$$

32. $x^2y'' + xy' - y = x^2r(r-1)x^{r-2} + xrx^{r-1} - x^r = x^r(r^2 - 1) = 0 \implies r = 1, -1$

33. $y = x^r, \quad y' = rx^{r-1}, \quad y'' = r(r-1)x^{r-2};$

$$\begin{aligned} 4x^2y'' - 4xy' + 3y &= 4x^2r(r-1)x^{r-2} - 4x\,r\,x^{r-1} + 3\,x^r \\ &= \left(4r^2 - 8r + 3\right)x^r = 0 \quad \Longrightarrow \quad 4r^2 - 8r + 3 = 0 \\ &\Longrightarrow \quad (2r-1)(2r-3) = 0 \quad \Longrightarrow \quad r = \tfrac{1}{2}, \ \tfrac{3}{2} \end{aligned}$$

34. $x^3y''' - 2x^2y'' - 2xy' + 8y = x^3r(r-1)(r-2)x^{r-3} - 2x^2r(r-1)x^{r-2} - 2xrx^{r-1} + 8x^r$
$= x^r(r^3 - 5r^2 + 2r + 8) = x^r(r-2)(r-4)(r+1) = 0 \quad \text{if} \quad r = 2, 4, -1$

35. (a) $y(0) = C_1\sin(0) + C_2\cos(0) = 0 \quad \Longrightarrow \quad C_2 = 0;$

Since $y = C_1\sin(4\cdot\pi/2) = C_1\sin(2\pi) = 0, \quad y = C_1\sin 4x$ satisfies the boundary conditions

$$y(0) = 0, \qquad y(\pi/2)$$

for all values of C_1.

(b) As shown above, $y(0) = 0 \quad \Longrightarrow \quad C_2 = 0.$

Now $y(\pi/8) = C_1\sin(4\cdot\pi/8) = C_1\sin(\pi/2) = 0 \quad \Longrightarrow \quad C_1 = 0$. Therefore, $y = 0$ is the only member of the family that satisfies the boundary conditions

$$y(0) = 0, \qquad y(\pi/8) = 0.$$

36. Assume $r \neq 0$.

(a) $0 = y(0) = C_2 \Longrightarrow y = C_1\sin rx.$

$0 = y(\pi) = C_1\sin r\pi.$ Since we want $C_1 \neq 0$, we need $\sin r\pi = 0$.

Therefore, $r = n$, n any integer except 0.

(b) Again, $y(0) = 0 \Longrightarrow y = C_1\sin rx$

$0 = y(\frac{\pi}{2}) = C_1\sin\frac{r\pi}{2}$. Since we want $C_1 \neq 0$, we need $\sin\frac{r\pi}{2} = 0$,

so $r = 2n, \quad n = \pm 1, \pm 2, \cdots$

SECTION 18.2

1. $y' + xy = xy^3 \quad \Longrightarrow \quad y^{-3}y' + xy^{-2} = x.$ Let $v = y^{-2}, \quad v' = -2y^{-3}y'.$

$$\begin{aligned} -\frac{1}{2}v' + xv &= x \\ v' - 2xv &= -2x \\ e^{-x^2}v' - 2xe^{-x^2}v &= -2xe^{-x^2} \\ e^{-x^2}v &= e^{-x^2} + C \\ v &= 1 + Ce^{x^2} \\ y^2 &= \frac{1}{1 + Ce^{x^2}}. \end{aligned}$$

2. $y' - y = -(x^2 + x + 1)y^2 \implies y^{-2}y' - y^{-1} = -(x^2 + x + 1)$. Let $v = y^{-1}$, $v' = -y^{-2}y'$.

$$-v' - v = -(x^2 + x + 1)$$
$$v' + v = x^2 + x + 1$$
$$e^x v = \int e^x(x^2 + x + 1)\,dx = x^2e^x - xe^x + 2e^x + C$$
$$v = x^2 - x + 2 + Ce^{-x}$$
$$y = \frac{1}{x^2 - x + 2 + Ce^{-x}}.$$

3. $y' - 4y = 2e^x y^{\frac{1}{2}} \implies y^{-\frac{1}{2}}y' - 4y^{\frac{1}{2}} = 2e^x$. Let $v = y^{\frac{1}{2}}$, $v' = \frac{1}{2}y^{-\frac{1}{2}}y'$.

$$2v' - 4v = 2e^x$$
$$v' - 2v = e^x$$
$$e^{-2x}v' - 2e^{-2x}v = e^{-x}$$
$$e^{-2x}v = -e^{-x} + C$$
$$v = -e^x + Ce^{2x}$$
$$y = (Ce^{2x} - e^x)^2.$$

4. $y' = \frac{1}{2xy} + \frac{y}{2x} \implies yy' - \frac{1}{2x}y^2 = \frac{1}{2x}$. Let $v = y^2$, $v' = 2yy'$.

$$\frac{1}{2}v' - \frac{1}{2x}v = \frac{1}{2x}$$
$$v' - \frac{1}{x}v = \frac{1}{x}$$
$$\frac{1}{x}v' - \frac{1}{x^2}v = \frac{1}{x^2}$$
$$\frac{1}{x}v = -\frac{1}{x} + C$$
$$v = Cx - 1$$
$$y^2 = Cx - 1.$$

5. $(x-2)y' + y = 5(x-2)^2 y^{\frac{1}{2}} \implies y^{-\frac{1}{2}}y' + \frac{1}{x-2}y^{\frac{1}{2}} = 5(x-2)$. Let $v = y^{\frac{1}{2}}, quad v' = \frac{1}{2}y^{-\frac{1}{2}}y'$.

$$2v' + \frac{1}{x-2}v = 5(x-2)$$
$$v' + \frac{1}{2(x-2)}v = \frac{5}{2}(x-2)$$
$$\sqrt{x-2}v' + \frac{1}{2\sqrt{x-2}}v = \frac{5}{2}(x-2)^{\frac{3}{2}}$$
$$\sqrt{x-2}v = (x-2)^{\frac{5}{2}} + C$$
$$v = (x-2)^2 + \frac{C}{\sqrt{x-2}}$$
$$y = \left[(x-2)^2 + \frac{C}{\sqrt{x-2}}\right]^2.$$

6. $yy' - xy^2 + x = 0.$ Let $v = y^2$, $v' = 2yy'$.

$$\begin{aligned}
\frac{1}{2}v' - xv &= -x \\
v' - 2xv &= -2x \\
e^{-x^2}v' - 2xe^{-x^2}v &= -2xe^{-x^2} \\
e^{-x^2}v &= e^{-x^2} + C \\
v &= 1 + Ce^{x^2} \\
y &= \sqrt{1 + Ce^{x^2}}.
\end{aligned}$$

7. $y' + xy = y^3 e^{x^2} \implies y^{-3}y' + xy^{-2} = e^{x^2}.$ Let $v = y^{-2}$, $v' = -2y^{-3}y'$.

$$\begin{aligned}
-\frac{1}{2}v' + xv &= e^{x^2} \\
v' - 2xv &= -2e^{x^2} \\
e^{-x^2}v' - 2xe^{-x^2}v &= -2 \\
e^{-x^2}v &= -2x + C \\
v &= -2xe^{x^2} + Ce^{x^2} \\
y^{-2} &= Ce^{x^2} - 2xe^{x^2}.
\end{aligned}$$

$C = 4 \implies y^{-2} = 4e^{x^2} - 2xe^{x^2}.$

8. $y' + \frac{1}{x}y = \frac{\ln x}{x}y^2 \implies y^{-2}y' + \frac{1}{x}y^{-1} = \frac{\ln x}{x}.$ Let $v = y^{-1}$, $v' = -y^{-2}y'$.

$$\begin{aligned}
-v' + \frac{1}{x}v &= \frac{\ln x}{x} \\
v' - \frac{1}{x}v &= -\frac{\ln x}{x} \\
\frac{1}{x}v' - \frac{1}{x^2}v &= -\frac{\ln x}{x^2} \\
\frac{1}{x}v &= -\int \frac{\ln x}{x^2}\,dx = \frac{1}{x}(\ln x + 1) + C \\
v &= \ln x + 1 + Cx \\
y &= \frac{1}{\ln x + 1 + Cx}.
\end{aligned}$$

$1 = \frac{1}{\ln 1 + 1 + C} \implies C = 0 \implies y = \frac{1}{\ln x + 1}.$

9. $2x^3y' - 3x^2y = y^3 \implies y^{-3}y' - \frac{3}{2x}y^{-2} = \frac{1}{2x^3}.$ Let $v = y^{-2}$, $v' = -2y^{-3}y'$.

$$-\frac{1}{2}v' - \frac{3}{2x}v = \frac{1}{2x^3}$$
$$v' + \frac{3}{x}v = -\frac{1}{x^3}$$
$$x^3v' + 3x^2v = -1$$
$$x^3v = -x + C$$
$$v = \frac{C-x}{x^3}$$
$$y^2 = \frac{x^3}{C-x}$$

$$1 = \frac{1}{C-x} \implies C = 2 \implies y^2 = \frac{x^3}{2-x}.$$

10. $y' + \tan x y = \sec^3 x y^2 \implies y^{-2}y' + \tan x y^{-1} = \sec^3 x.$ Let $v = y^{-1}$, $v' = -y^{-2}y'$.

$$-v' + \tan x v = \sec^3 x$$
$$v' - \tan x v = -\sec^3 x$$
$$\cos x v' - \sin x v = -\sec^2 x$$
$$\cos x v = -\tan x + C$$
$$\frac{\cos x}{y} = -\tan x + C$$

$$\frac{\cos 0}{3} = -\tan 0 + C \implies C = \frac{1}{3} \implies \frac{\cos x}{y} = \frac{1}{3} - \tan x.$$

11. $y' - \frac{y}{x}\ln y = xy \implies \frac{y'}{y} - \frac{1}{X}\ln y = x.$ Let $\mu = \ln y$ $, \mu' = \frac{y'}{y}.$

$$\mu' - \frac{1}{x}\mu = x$$
$$\frac{1}{x}\mu' - \frac{1}{x^2}\mu = 1$$
$$\frac{1}{x}\mu = x + C$$
$$\mu = x^2 + Cx$$

$$\ln y = x^2 + Cx.$$

12. (a)

$$y' + yf(x)\ln y = g(x)y$$
$$\frac{y'}{y} + f(x)\ln y = g(x)$$
$$\mu' + f(x)\mu = g(x).$$

(b) $\cos y y' + g(x)\sin y = f(x)$. Let $\mu = \sin y$, $\mu' = \cos y y'$.

Thus we have $\mu' + g(x)\mu = f(x)$.

13. $f(x,y)=\dfrac{x^2+y^2}{2xy};\qquad f(tx,ty)=\dfrac{(tx)^2+(ty)^2}{2(tx)(ty)}=\dfrac{t^2(x^2+y^2)}{t^2(2xy)}=\dfrac{x^2+y^2}{2xy}=f(x,y)$

Set $vx=y$. Then, $v+xv'=y'$ and

$$v+xv'=\frac{x^2+v^2x^2}{2vx^2}=\frac{1+v^2}{2v}$$
$$v-\frac{1+v^2}{2v}+xv'=0$$
$$v^2-1+2xvv'=0$$
$$\frac{1}{x}\,dx+\frac{2v}{v^2-1}\,dv=0$$
$$\int\frac{1}{x}\,dx+\int\frac{2v}{v^2-1}\,dv=C$$
$$\ln|x|+\ln|v^2-1|=K\qquad\text{or}\qquad x(v^2-1)=C$$

Replacing v by y/x, we get

$$x\left(\frac{y^2}{x^2}-1\right)=C\qquad\text{or}\qquad y^2-x^2=Cx$$

14. $f(tx,ty)=\dfrac{(ty)^2}{(tx)(ty)+(tx)^2}=\dfrac{y^2}{xy+x^2}=f(x,y).$

Set $vx=y$. Then, $v+xv'=y'$ and

$$v+xv'=\frac{v^2}{1+v^2}$$
$$\int\frac{dx}{x}+\int\frac{v+1}{v}\,dv=C$$
$$\ln|x|+v+\ln|v|=C$$
$$v+\ln|xv|=C$$
$$\frac{y}{x}+\ln|y|=C$$

15. $f(x,y)=\dfrac{x-y}{x+y};\qquad f(tx,ty)=\dfrac{(tx)-(ty)}{tx+ty}=\dfrac{t(x-y}{t(x+y)}=\dfrac{x-y}{x+y}=f(x,y)$

Set $vx=y$. Then, $v+xv'=y'$ and

$$v+xv'=\frac{x-vx}{x+vx}=\frac{1-v}{1+v}$$
$$v^2+2v-1+x(1+v)v'=0$$
$$\frac{1}{x}\,dx+\frac{1+v}{v^2+2v-1}\,dv=0$$
$$\int\frac{1}{x}\,dx+\int\frac{1+v}{v^2+2v-1}\,dv=C$$
$$\ln|x|+\tfrac{1}{2}\ln|v^2+2v-1|=K\qquad\text{or}\qquad x\sqrt{v^2+2v-1}=C$$

Replacing v by y/x, we get

$$x\sqrt{\frac{y^2}{x^2}+2\frac{y}{x}-1}=C\qquad\text{or}\qquad y^2+2xy-x^2=C$$

16. $f(tx,ty)=\dfrac{tx+ty}{tx-ty}=\dfrac{x+y}{x-y}=f(x,y).$

Set $vx = y$. Then, $v + xv' = y'$ and

$$v + xv' = \frac{x + vx}{x - vx} = \frac{1+v}{1-v}$$

$$\int \frac{dx}{x} + \int \frac{v-1}{v^2+1}\,dv = C_1$$

$$\ln|x| + \frac{1}{2}\ln|v^2+1| - \tan^{-1} v = C_1$$

$$\ln x^2 + \ln(v^2+1) - 2\tan^{-1} v = C \quad (= 2C_1)$$

$$\ln[x^2(v^2+1)] - 2\tan^{-1} v = C$$

$$\ln[x^2 - y^2] - 2\tan^{-1}\left(\frac{y}{x}\right) = C$$

17. $f(x,y) = \dfrac{x^2e^{y/x} + y^2}{xy}$; $\quad f(tx,ty) = \dfrac{(tx)^2 - e^{(ty)/(tx)} + (ty)^2}{(tx)(ty)} = \dfrac{t^2\left(x^2e^{y/x} + y^2\right)}{t^2(xy)} = f(x,y)$

Set $vx = y$. Then, $v + xv' = y'$ and

$$v + xv' = \frac{x^2e^v + v^2x^2}{vx^2} = \frac{e^v + v^2}{v}$$

$$v^2 + 2v - 1 + x(1+v)v' = 0$$

$$v^2 + xvv' = e^v + v^2$$

$$-e^v + xvv' = 0$$

$$\frac{1}{x}\,dx = ve^{-v}\,dv$$

$$\int \frac{1}{x}\,dx = \int ve^{-v}\,dv$$

$$\ln|x| = -ve^{-v} - e^{-v} + C$$

Replacing v by y/x, and simplifying, we get

$$y + x = xe^{y/x}(C - \ln|x|)$$

18. $f(tx,ty) = \dfrac{(tx)^2 + 3(ty)^2}{4(tx)(ty)} = \dfrac{x^2+3y^2}{4xy} = f(x,y).$

Set $vx = y$. Then, $v + xv' = y'$ and

$$v + xv' = \frac{x^2 + 3x^2v^2}{4x^2v}$$

$$\int \frac{dx}{x} + \int \frac{4v}{v^2-1}\,dv = C_1$$

$$\ln|x| + 2\ln|v^2-1| = C_1$$

$$x(v^2-1)^2 = C \ (= e^{C_1})$$

$$(y^2 - x^2)^2 = Cx^3$$

19. $f(x,y) = \dfrac{y}{x} + \sin(y/x)$; $\quad f(tx,ty) = \dfrac{(ty)}{tx} + \sin[(ty/tx)] = \dfrac{y}{x} + \sin(y/x) = f(x,y)$

Set $vx = y$. Then, $v + xv' = y'$ and

$$v + xv' = \frac{vx}{x} + \sin[(vx)/x] = v + \sin v$$
$$xv' = \sin v$$
$$\csc v\, dv = \frac{1}{x}\, dx$$
$$\int \csc v\, dv = \int \frac{1}{x}\, dx$$
$$\ln|\csc v - \cot v| = \ln|x| + K \qquad \text{or} \qquad \csc v - \cot v = Cx$$

Replacing v by y/x, and simplifying, we get

$$1 - \cos(y/x) = Cx\sin(y/x)$$

20. $f(x,y) = \frac{y}{x}\left(1 + \ln\left(\frac{y}{x}\right)\right)$; $\quad f(tx,ty) = \frac{ty}{tx}\left(1 + \ln\left(\frac{ty}{tx}\right)\right) = \frac{y}{x}\left(1 + \ln\left(\frac{y}{x}\right)\right) = f(x,y)$

Set $vx = y$. Then, $v + xv' = y'$ and

$$v + xv' = \frac{vx}{x}\left(1 + \ln\left(\frac{vx}{x}\right)\right) = v(1 + \ln v)$$
$$xv' = v\ln v$$
$$\frac{1}{v\ln v}\, dv = \frac{1}{x}\, dx$$
$$\int \frac{1}{v\ln v}\, dv = \int \frac{1}{x}\, dx$$
$$\ln|\ln v| = \ln|x| + K$$
$$\ln\left(\frac{y}{x}\right) = Cx$$
$$\frac{y}{x} = e^{Cx} \quad \text{or} \quad y = xe^{Cx}$$

21. The differential equation is homogeneous since

$$f(x,y) = \frac{y^3 - x^3}{xy^2}; \qquad f(tx,ty) = \frac{(ty)^3 - (tx)^3}{(tx)(ty)^2} = \frac{t^3(y^3 - x^3)}{t^3(xy^2)} = \frac{y^3 - x^3}{xy^2} = f(x,y)$$

Set $vx = y$. Then, $v + xv' = y'$ and

$$v + xv' = \frac{(vx)^3 - x^3}{v^2x^3} = \frac{v^3 - 1}{v^2}$$
$$1 + xv^2v' = 0$$
$$\frac{1}{x}\, dx + v^2\, dv = 0$$
$$\int \frac{1}{x}\, dx + \int v^2\, dv = 0$$
$$\ln|x| + \frac{1}{3}v^3 = C$$

Replacing v by y/x, we get

$$y^3 + 3x^3 \ln|x| = Cx^3$$

Applying the side condition $y(1) = 2$, we have

$$8 + 3\ln 1 = C \implies C = 8 \quad \text{and} \quad y^3 + 3x^3 \ln|x| = 8x^3$$

22. $\dfrac{dy}{dx} = \dfrac{1}{\sin(y/x)} + \dfrac{y}{x}$. Set $y = vx$. Then $y' = v + xv'v$ and

$$v + xv' = \frac{1}{\sin v} + v$$

$$\int \frac{dx}{x} - \int \sin v\, dv = C$$

$$\ln|x| + \cos v = C$$

$$\ln|x| + \cos\left(\frac{y}{x}\right) = C$$

$$y(1) = 0 \implies 0 + \cos 0 = C \implies C = 1 \implies \ln|x| + \cos\left(\frac{y}{x}\right) = 1$$

23. $y' = y \implies y = Ce^x$. Also, $y(0) = 1 \implies C = 1$

Thus $y = e^x$ and $y(1) = 2.71828$

(a) 2.48832, relative error= 8.46%.

(b) 2.71825, relative error= 0.001%.

24. $y' = x + \implies y = Ce^x - x - 1, y(0) = 2 \implies C = 3$

Thus $y = 3e^x - x - 1$ and $y(1) \simeq 6.15485$

(a) 5.46496, relative error= 11.2%.

(b) 6.15474, relative error= 0%.

25. (a) 2.59374, relative error= 4.58%.

(b) 2.71828, relative error= 0%.

26. (a) 5.78124, relative error= 11.21%.

(b) 6.15482, relative error= 0%.

27. $y' = 2x \implies y = x^2 + C$. Also, $y(2) = 5 \implies C = 1$

Thus $y = x^2 + 1$ and $y(1) = 2$.

(a) 1.9, relative error= 5.0%.

(b) 2.0, relative error= 0%.

28. $y' = 3x^2 \implies y = x^3 + C$. Also, $y(1) = 2 \implies C = 1$

Thus $y = x^3 + 1$ and $y(0) = 1$.

(a) 0.84500, relative error= 15.5%.

(b) 1.0, relative error= 0%.

29. $y' = \dfrac{1}{2y}$

Thus $y = \sqrt{x}$ and $y(2) = \sqrt{2} \simeq 1.41421$.

(a) 1.42052, relative error $= -0.45\%$.

(b) 1.41421, relative error $= 0\%$.

30. $y' = \dfrac{1}{3y^2}$

Thus $y = x^{\frac{1}{3}}$ and $y(2) \simeq 1.25992$.

(a) 1.26494, relative error $= -0.4\%$.

(b) 1.25992, relative error $= 0\%$.

31. (a) 2.65330, relative error $= 2.39\%$.

(b) 2.71828, relative error $= 0\%$.

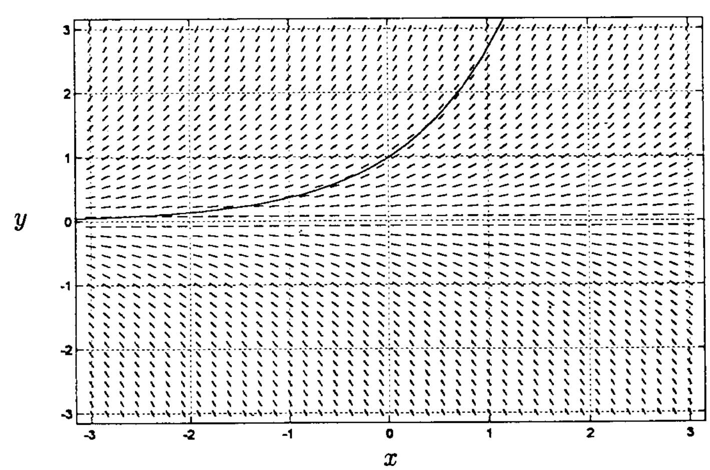
32. (a) 5.95989, relative error $= 3.17\%$.

(b) 6.15487, relative error $= 0\%$.

PROJECT 18.2

1. (a) and (b)

$y' = y$, $y(0) = 1$



(c) $y - y' = 0$ $\quad H(x) = \int -dx = -x$; integrating factor: e^{-x}

$$e^{-x}y' - e^{-x}y = 0$$

$$\frac{d}{dx}(e^{-x}y) = 0$$

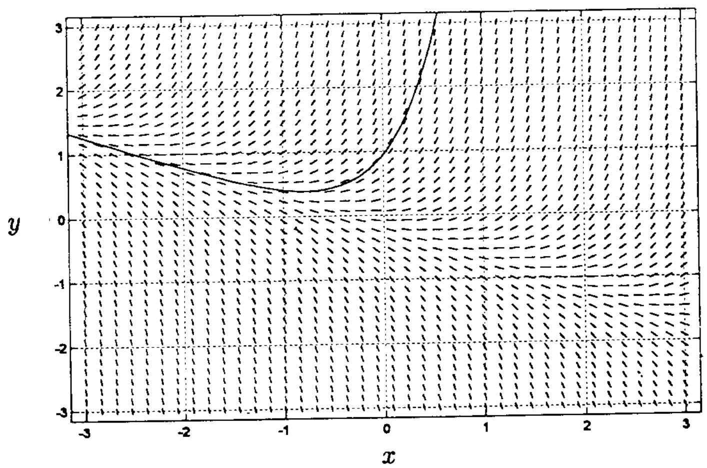
$$e^{-x}y = C$$

$$y = Ce^{x}$$

$y(0) = 1 \implies C = 1.$ Thus $y = e^{x}$.

2. (a) and (b)

$y' = x + 2y$



(c) $y' - 2y = x \qquad H(x) = \int -2\,dx = -2x; \quad$ integrating factor: e^{-2x}

$$e^{-2x}y' - 2e^{-2x}y = xe^{-2x}$$

$$\frac{d}{dx}(e^{-2x}y) = xe^{-2x}$$

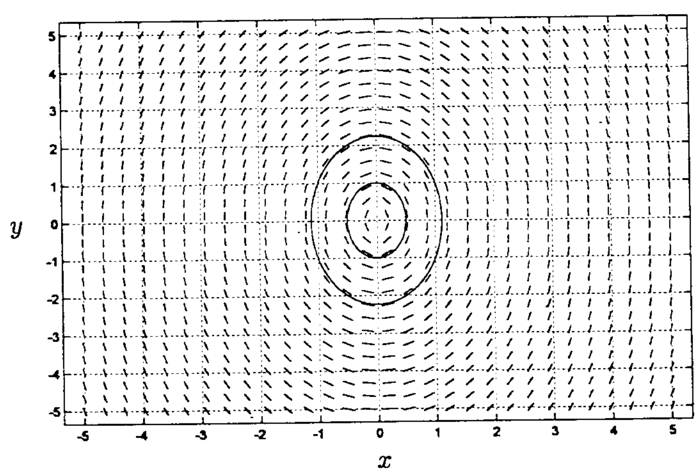
$$e^{-2x}y = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$$

$$y = Ce^{2x} - \frac{1}{2}x - \frac{1}{4}$$

$y(0) = 1 \implies C = \frac{5}{4}. \qquad$ Thus $y = \frac{5}{4}e^{2x} - \frac{1}{2}x - \frac{1}{4}$.

3. (a) and (b)

$y' = -4x/y$



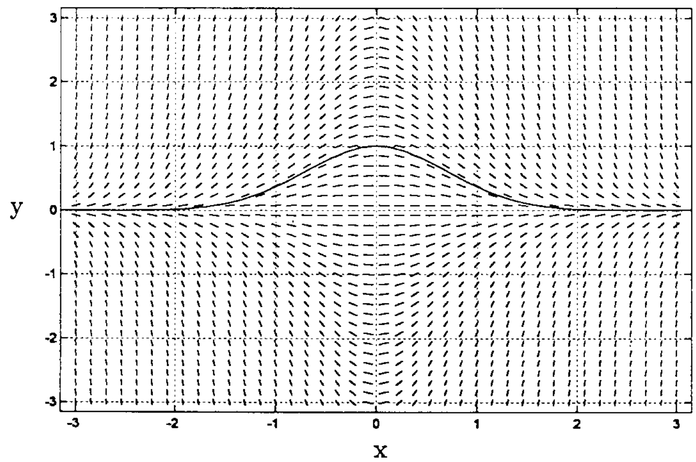
(c) $yy' = -\frac{4x}{y}$

$$\frac{1}{2}y^2 = -2x^2 + C \quad \text{or} \quad x^2 + \frac{1}{4}y^2 = C$$

$y(0) = 1 \implies C = \frac{1}{4}. \qquad$ Thus $4x^2 + y^2 = 1$.

4. (a) and (b)

$y' = -2xy$



(c) $y' + 2xy = 0 \qquad H(x) = \int 2x\,dx = x^2;$ integrating factor: e^{x^2}

$$e^{x^2}y' + 2xe^{x^2}y = 0$$
$$\frac{d}{dx}(e^{x^2}y) = 0$$
$$e^{x^2}y = C$$
$$y = Ce^{-x^2}$$

$y(0) = 1 \implies C = 1.$ Thus $y = e^{-x^2}$.

SECTION 18.3

1. The characteristic equation is:

$$r^2 + 2r - 15 = 0 \qquad \text{or} \qquad (r+5)(r-3) = 0.$$

The roots are: $r = 3,\ -5$. The general solution is:

$$y = C_1e^{3x} + C_2e^{-5x}.$$

2. $r^2 - 13r + 42 = 0 \implies r = -6,\ -7; \quad y = C_1e^{-6x} + C_2e^{-7x}$

3. The characteristic equation is:

$$r^2 + 8r + 16 = 0 \qquad \text{or} \qquad (r+4)^2 = 0.$$

There is only one root: $r = -4$. By Theorem 18.3.6 II, the general solution is:

$$y = C_1e^{-4x} + C_2xe^{-4x}.$$

4. $r^2 + 7r + 3 = 0 \implies r = -\frac{7}{2} \pm \frac{\sqrt{37}}{2}; \quad y = C_1e^{\frac{-7+\sqrt{37}}{2}x} + C_2e^{\frac{-7-\sqrt{37}}{2}x}.$

5. The characteristic equation is: $r^2+2r+5=0$.

The roots are complex: $r=-1\pm 2i$. By Theorem 18.3.6 III, the general solution is:

$$y=e^{-x}\left(C_1\cos 2x+C_2\sin 2x\right).$$

6. $r^2-3r+8=0 \implies r=\frac{3}{2}\pm\frac{\sqrt{23}}{2}i;\quad y=e^{3/2}x\left(C_1\cos\frac{\sqrt{23}}{2}x+C_2\sin\frac{\sqrt{23}}{2}x\right)$

7. The characteristic equation is:

$$2r^2+5r-3=0 \qquad \text{or} \qquad (2r-1)(r+3)=0.$$

The roots are: $r=\frac{1}{2},\ -3$. The general solution is:

$$y=C_1e^{x/2}+C_2e^{-3x}.$$

8. $r^2-12=0 \implies r=\pm 2\sqrt{3};\quad y=C_1e^{2\sqrt{3}x}+C_2e^{-2\sqrt{3}x}$.

9. The characteristic equation is:

$$r^2+12=0.$$

The roots are complex: $r=\pm 2\sqrt{3}\,i$. The general solution is:

$$y=C_1\cos 2\sqrt{3}\,x+C_2\sin 2\sqrt{3}\,x.$$

10. $r^2-3r+\frac{9}{4}=0 \implies r=\frac{3}{2};\quad y=C_1e^{\frac{3}{2}x}+C_2xe^{\frac{3}{2}x}$.

11. The characteristic equation is:

$$5r^2+\tfrac{11}{4}r-\tfrac{3}{4}=0 \qquad \text{or} \qquad 20r^2+11r-3=(5r-1)(4r+3)=0.$$

The roots are: $r=\frac{1}{5},\ -\frac{3}{4}$. The general solution is:

$$y=C_1e^{x/5}+C_2e^{-3x/4}.$$

12. $2r^2+3r=0 \implies r=0,-\frac{3}{2};\quad y=C_1+C_2e^{-\frac{3}{2}x}$.

13. The characteristic equation is:

$$r^2+9=0.$$

The roots are complex: $r=\pm 3i$. The general solution is:

$$y=C_1\cos 3x+C_2\sin 3x.$$

14. $r^2-r-30=0 \implies r=6,-5;\quad y=C_1e^{6x}+C_2e^{-5x}$.

15. The characteristic equation is:

$$2r^2+2r+1=0.$$

The roots are complex: $r=-\frac{1}{2}\pm\frac{1}{2}i$. The general solution is:

$$y=e^{-x/2}\left[C_1\cos(x/2)+C_2\sin(x/2)\right].$$

16. $r^2 - 4r + 4 = 0 \implies r = 2; \quad y = C_1 e^{2x} + C_2 x e^{2x}.$

17. The characteristic equation is:

$$8r^2 + 2r - 1 = 0 \qquad \text{or} \qquad (4r-1)(2r+1) = 0.$$

The roots are: $r = \frac{1}{4},\ -\frac{1}{2}$. The general solution is:

$$y = C_1 e^{x/4} + C_2 e^{-x/2}.$$

18. $5r^2 - 2r + 1 = 0 \implies r = \dfrac{1}{10} \pm \dfrac{1}{5} i; \quad y = e^{x/10}\left(C_1 \cos\dfrac{x}{5} + C_2 \sin\dfrac{x}{5}\right).$

19. The characteristic equation is:

$$r^2 - 5r + 6 = 0 \qquad \text{or} \qquad (r-3)(r-2) = 0.$$

The roots are: $r = 3,\ 2$. The general solution and its derivative are:

$$y = C_1 e^{3x} + C_2 e^{2x}, \qquad y' = 3C_1 e^{3x} + 2C_2 e^{2x}.$$

The conditions: $y(0) = 1, \quad y'(0) = 1$ require that

$$C_1 + C_2 = 1 \qquad \text{and} \qquad 3C_1 + 2C_2 = 1.$$

Solving these equations simultaneously gives $C_1 = -1, \quad C_2 = 2$.

The solution of the initial value problem is: $y = 2e^{2x} - e^{3x}$.

20. $r^2 + 2r + 1 = 0 \implies r = -1; \quad y = C_1 e^{-x} + C_2 x e^{-x}$

$1 = y(2) = C_1 e^{-2} + 2C_2 e^{-2}, \quad 2 = y'(2) = -C_1 e^{-2} - C_2 e^{-2}$

$\implies C_1 = -5e^2,\ C_2 = 3e^2 \implies y = -5e^{2-x} + 3xe^{2-x}.$

21. The characteristic equation is:

$$r^2 + \tfrac{1}{4} = 0.$$

The roots are: $r = \pm\frac{1}{2} i$. The general solution and its derivative are:

$$y = C_1 \cos(x/2) + C_2 \sin(x/2) \qquad y' = -\tfrac{1}{2} C_1 \sin(x/2) + \tfrac{1}{2} C_2 \cos(x/2).$$

The conditions: $y(\pi) = 1, \quad y'(\pi) = -1$ require that

$$C_2 = 1 \qquad \text{and} \qquad C_1 = 2.$$

The solution of the initial value problem is: $y = 2\cos(x/2) + \sin(x/2)$.

22. $r^2 - 2r + 2 = 0 \implies r = 1 \pm i; \quad y = e^x(C_1 \cos x + C_2 \sin x).$

$-1 = y(0) = C_1, \quad -1 = y'(0) = C_1 - C_2 \implies C_2 = 0, \quad y = -e^x \cos x$

23. The characteristic equation is:

$$r^2 + 4r + 4 = 0 \qquad \text{or} \qquad (r+2)^2 = 0.$$

There is only one root: $r = -2$. The general solution and its derivative are:

$$y = C_1 e^{-2x} + C_2 x e^{-2x} \qquad y' = -2C_1 e^{-2x} + C_2 e^{-2x} - 2C_2 x e^{-2x}.$$

The conditions: $y(-1) = 2, \quad y'(-1) = 1$ require that

$$C_1e^2 - C_2e^2 = 2 \qquad \text{and} \qquad -2C_1e^2 + 3C_2e^2 = 1.$$

Solving these equations simultaneously gives $C_1 = 7e^{-2}$, $C_2 = 5e^{-2}$.
The solution of the initial value problem is: $y = 7e^{-2}e^{-2x} + 5e^{-2}xe^{-2x} = 7e^{-2(x+1)} + 5xe^{-2(x+1)}$.

24. $r^2 - 2r + 5 = 0 \implies r = 1 \pm 2i$; $y = e^x(C_1 \cos 2x + C_2 \sin 2x)$.
$0 = y(\pi/2) = e^{\pi/2}(-C_1) \implies C_1 = 0$; $2 = y'(\pi/2) = e^{\pi/2}(-2C_2) \implies C_2 = -e^{-\pi/2}$
$\implies y = -e^{x-\pi/2}\sin 2x$.

25. The characteristic equation is:

$$r^2 - r - 2 = 0 \qquad \text{or} \qquad (r-2)(r+1) = 0.$$

The roots are: $r = 2, -1$. The general solution and its derivative are:

$$y = C_1e^{2x} + C_2e^{-x} \qquad y' = 2C_1e^{2x} - C_2e^{-x}.$$

(a) $y(0) = 1 \implies C_1 + C_2 = 1 \implies C_2 = 1 - C_1$.

Thus, the solutions that satisfy $y(0) = 1$ are: $y = Ce^{2x} + (1-C)e^{-x}$.

(b) $y'(0) = 1 \implies 2C_1 - C_2 = 1 \implies C_2 = 2C_1 - 1$.

Thus, the solutions that satisfy $y'(0) = 1$ are: $y = Ce^{2x} + (2C-1)e^{-x}$.

(c) To satisfy both conditions, we must have $2C - 1 = 1 - C \implies C = \frac{2}{3}$.

The solution that satisfies $y(0) = 1$, $y'(0) = 1$ is:

$$y = \tfrac{2}{3}\,e^{2x} + \tfrac{1}{3}\,e^{-x}.$$

26. $r^2 - \omega^2 = 0 \implies r = \pm\omega$; $y = A_1e^{\omega x} + A_2e^{-\omega x}$
Since $e^{\omega x} = \cosh \omega x + \sinh \omega x$ and $e^{-\omega x} = \cosh \omega x - \sinh \omega x$, we can write
$y = C_1 \cosh \omega x + C_2 \sinh \omega x$ (with $C_1 = A_1 + A_2$, $C_2 = A_1 - A_2$).

27. $\alpha = \dfrac{r_1 + r_2}{2}$, $\beta = \dfrac{r_1 - r_2}{2}$;
$y = k_1e^{r_1x} + k_2e^{r_2x} = e^{\alpha x}\left(C_1 \cosh \beta x + C_2 \sinh \beta x\right)$, where $k_1 = \dfrac{C_1 + C_2}{2}$, $k_2 = \dfrac{C_1 - C_2}{2}$.

28. $r^2 + \omega^2 = 0 \implies r = \pm\omega i$; $y = C_1 \cos \omega x + C_2 \sin \omega x$.
Assuming that $C_1^2 + C_2^2 > 0$, we have

$$C_1 \cos \omega t + C_2 \sin \omega t = \sqrt{{C_1}^2 + {C_2}^2}\left(\frac{C_1}{\sqrt{{C_1}^2 + {C_2}^2}}\cos \omega t + \frac{C_2}{\sqrt{{C_1}^2 + {C_2}^2}}\sin \omega t\right)$$
$$= A\left(\sin \phi_0 \cos \omega t + \cos \phi_0 \sin \omega t\right),$$

where $A = \sqrt{{C_1}^2 + {C_2}^2}$ and ϕ_0, $\phi_0 \in [0, 2\pi)$, is the angle such that

$$\sin \phi_0 = \frac{C_1}{\sqrt{{C_1}^2 + {C_2}^2}} \qquad \text{and} \qquad \cos \phi_0 = \frac{C_2}{\sqrt{{C_1}^2 + {C_2}^2}}$$

29. (a) Let $y_1 = e^{rx}$, $y_2 = xe^{rx}$. Then

$$W(x) = y_1y_2' - y_2y_1' = e^{rx}\left[e^{rx} + rxe^{rx}\right] - xe^{rx}\left[re^{rx}\right] = e^{2rx} \neq 0$$

(b) Let $y_1 = e^{\alpha x}\cos\beta x$, $y_2 = e^{\alpha x}\sin\beta x$, $\beta \neq 0$. Then

$$\begin{aligned} W(x) &= y_1y_2' - y_2y_1' \\ &= e^{\alpha x}\cos\beta x\left[\alpha e^{\alpha x}\sin\beta x + \beta e^{\alpha x}\cos\beta x\right] - e^{\alpha x}\sin\beta x\left[\alpha e^{\alpha x}\cos\beta x - \beta e^{\alpha x}\sin\beta x\right] \\ &= \beta e^{2\alpha x} \neq 0 \end{aligned}$$

30. Characteristic equation: $r^2 + 10^3 r + \dfrac{1}{C} = 0$; roots: $r = \dfrac{-10^3 \pm \sqrt{10^6 - 4/C}}{2}$.

(a) $r = 100(-5 \pm \sqrt{5})$; $y = C_1e^{100(-5+\sqrt{5})x} + C_2e^{100(-5-\sqrt{5})x}$

(b) $r = -500$; $y = C_1e^{-500x} + C_2xe^{-500x}$

(c) $r = 500(-1 \pm i)$; $y = e^{-500x}\left(C_1\cos 500x + C_2\sin 500x\right)$

31. (a) The solutions $y_1 = e^{2x}$, $y_2 = e^{-4x}$ imply that the roots of the characteristic equation are $r_1 = 2$ $r_2 = -4$. Therefore, the characteristic equation is:

$$(r-2)(r+4) = r^2 + 2r - 8 = 0$$

and the differential equation is: $y'' + 2y' - 8y = 0$.

(b) The solutions $y_1 = 3e^{-x}$, $y_2 = 4e^{5x}$ imply that the roots of the characteristic equation are $r_1 = -1$ $r_2 = 5$. Therefore, the characteristic equation is

$$(r+1)(r-5) = r^2 - 4r - 5 = 0$$

and the differential equation is: $y'' - 4y' - 5y = 0$.

(c) The solutions $y_1 = 2e^{3x}$, $y_2 = xe^{3x}$ imply that 3 is the only root of the characteristic equation. Therefore, the characteristic equation is

$$(r-3)^2 = r^2 - 6r + 9 = 0$$

and the differential equation is: $y'' - 6y' + 9y = 0$.

32. (a) We want $r = \pm 2i$, so $r^2 = -4$. Differential equation: $y'' + 4y = 0$

(b) We want $r = -2 \pm 3i$, so $(r+2)^2 = -9$. Differential equation: $y'' + 4y' + 13y = 0$

33. Suppose that $y_1(x) = ky_2(x)$, where k is a constant. Then

$$W(x) = y_1y_2' - y_2y_1' = ky_2y_2' - ky_2y_2' = 0.$$

Now suppose that $y_1y_2' - y_2y_1' = 0$. Then

$$\left[\frac{y_1(x)}{y_2(x)}\right]' = \frac{y_2y_1' - y_1y_2'}{y_2^2} = -\frac{y_1y_2' - y_2y_1'}{y_2^2} = 0.$$

This implies that $\dfrac{y_1}{y_2} = k$, for some constant k, that is $y_1 = ky_2$.

34. $r^2 + ar + b = 0 \implies r_1,\ r_2 = \dfrac{-a \pm \sqrt{a^2 - 4b}}{2}$.

If $a^2 - 4b > 0$, then $\sqrt{a^2 - 4b} < a$, so $\dfrac{-a \pm \sqrt{a^2 - 4b}}{2}$ is negative, and the solutions:
$y = C_1e^{r_1x} + C_2e^{r_2x} \to 0$ as $x \to \infty$.
If $a^2 - 4b = 0$, then $r = r_1 = r_2 = -a/2 < 0$, and the solutions:
$y = C_1e^{rx} + C_2xe^{rx} \to 0$ as $x \to \infty$.
If $a^2 - 4b < 0$, then $y = e^{-\frac{a}{2}x}(C_1 \cos\sqrt{b^2 - 4a}\ x + C_2 \sin\sqrt{b^2 - 4a}\ x)$ satisfies
$|y| < e^{-\frac{a}{2}x} \implies y \to 0$ as $x \to \infty$.

35. (a) If $a = 0$, $b > 0$, then the general solution of the differential equation is:

$$y = C_1 \cos\sqrt{b}\,x + C_2 \sin\sqrt{b}\,x = A \cos\left(\sqrt{b}\,x + \phi\right)$$

where A and ϕ are constants. Clearly $|\,y(x)\,| \le |\,A\,|$ for all x.

(b) If $a > 0$, $b = 0$, then the general solution of the differential equation is:

$$y = C_1 + C_2e^{-ax} \qquad \text{and} \qquad \lim_{x\to\infty} y(x) = C_1.$$

The solution which satisfies the conditions: $y(0) = y_0$, $y'(0) = y_1$ is:

$$y = y_0 + \frac{y_1}{a} - \frac{y_1}{a}\,e^{-ax} \quad \text{and} \quad \lim_{x\to\infty} y(x) = y_0 + \frac{y_1}{a}; \qquad k = y_0 + \frac{y_1}{a}.$$

36. From the hint, $\dfrac{dy}{dx} = \dfrac{dy}{dz}\dfrac{1}{x}$. Differentiating with respect to x again, we have
$\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dz^2}\dfrac{dz}{dx}\dfrac{1}{x} + \dfrac{dy}{dz}\left(-\dfrac{1}{x^2}\right) = \dfrac{1}{x^2}\left(\dfrac{d^2y}{dz^2} - \dfrac{dy}{dz}\right)$.

Substituting into the differential equation $x^2y'' + \alpha xy' + \beta y = 0$, we get

$$\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) + \alpha\frac{dy}{dz} + \beta y = 0, \quad \text{or} \quad \frac{d^2y}{dz^2} + a\frac{dy}{dz} + by = 0,$$

where $a = \alpha - 1$, $b = \beta$.

37. From Exercise 36, the change of variable $z = \ln x$ transforms the equation

$$x^2y'' - xy' - 8y = 0$$

into the differential equation with constant coefficients

$$\frac{d^2y}{dz^2} - 2\frac{dy}{dz} - 8y = 0.$$

The characteristic equation is:

$$r^2 - 2r - 8 = 0 \qquad \text{or} \qquad (r-4)(r+2) = 0$$

The roots are: $r = 4,\ r = -2,$ and the general solution (in terms of z) is:

$$y = C_1 e^{4z} + C_2 e^{-2z}.$$

Replacing z by $\ln x$ we get

$$y = C_1 e^{4 \ln x} + C_2 e^{-2 \ln x} = C_1 x^4 + C_2 x^{-2}.$$

38. Using the result of Exercise 36, we get

$\dfrac{d^2y}{dz^2} - 3\dfrac{dy}{dz} + 2y = 0,$ so $r^2 - 3r + 2 = 0 \implies r = 1, 2.$

$\implies y = C_1 e^z + C_2 e^{2z}.$ Substituting $z = \ln x,$ we get $y = C_1 x + C_2 x^2.$

39. From Exercise 36, the change of variable $z = \ln x$ transforms the equation

$$x^2 y'' - 3xy' + 4y = 0$$

into the differential equation with constant coefficients

$$\frac{d^2y}{dz^2} - 4\frac{dy}{dz} + 4y = 0.$$

The characteristic equation is:

$$r^2 - 4r + 4 = 0 \qquad \text{or} \qquad (r-2)^2 = 0.$$

The only root is: $r = 2,$ and the general solution (in terms of z) is:

$$y = C_1 e^{2z} + C_2 z e^{-2z}.$$

Replacing z by $\ln x$ we get

$$y = C_1 e^{2 \ln x} + C_2 \ln x\, e^{2 \ln x} = C_1 x^2 + C_2 x^2 \ln x.$$

40. From Exercise 36, we get $\dfrac{d^2y}{dz^2} - 2\dfrac{dy}{dz} + 5y = 0$

$r^2 - 2r + 5 = 0 \implies r = 1 \pm 2i;$ and $y = e^z\,(C_1 \cos 2z + C_2 \sin 2z)\,.$

Substituting $z = \ln x$ we get: $y = x\,[C_1 \cos(2 \ln x) + C_2 \sin(2 \ln x)]\,.$

41. (a) $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!} \cdots = \displaystyle\sum_{k=0}^{\text{inf}} \frac{1}{k!}\, x^k$

(b) $e^{i\theta} = 1 + (i\theta) + \dfrac{(i\theta)^2}{2!} + \dfrac{(i\theta)^3}{3!} + \cdots + \dfrac{(i\theta)^n}{n!} + \cdots$

$$= 1 + i\theta - \frac{1}{2!}\theta^2 - i\,\frac{1}{3!}\theta^3 + \cdots + (i)^n \frac{1}{n!}\theta^n + \cdots = \cos\theta + i\,\sin\theta$$

(c) $e^{-i\theta} = 1 + (-i\theta) + \dfrac{(-i\theta)^2}{2!} + \dfrac{(-i\theta)^3}{3!} + \cdots + \dfrac{(-i\theta)^n}{n!} + \cdots$

$$= 1 - i\theta - \frac{1}{2!}\theta^2 + i\,\frac{1}{3!}\theta^3 + \cdots + (-i)^n \frac{1}{n!}\theta^n + \cdots = \cos\theta - i\,\sin\theta$$

SECTION 18.4

1. First consider the reduced equation. The characteristic equation is:

$$r^2 + 5r + 6 = (r+2)(r+3) = 0$$

and $u_1(x) = e^{-2x}$, $u_2(x) = e^{-3x}$ are fundamental solutions. Therefore, a particular solution of the given equation has the form

$$y = Ax + B.$$

The derivatives of y are: $y' = A$, $y'' = 0$.

Substituting y and its derivatives into the given equation gives

$$0 + 5A + 6(Ax + B) = 3x + 4.$$

Thus,

$$6A = 3$$

$$5A + 6B = 4$$

The solution of this pair of equations is: $A = \frac{1}{2}$, $B = \frac{1}{4}$, and $y = \frac{1}{2}x + \frac{1}{4}$.

2. The constant $y_p = -\frac{1}{2}$ is a solution.

3. First consider the reduced equation. The characteristic equation is:

$$r^2 + 2r + 5 = 0$$

and $u_1(x) = e^{-x}\cos 2x$, $u_2(x) = e^{-x}\sin 2x$ are fundamental solutions. Therefore, a particular solution of the given equation has the form

$$y = Ax^2 + Bx + C.$$

The derivatives of y are: $y' = 2Ax + B$, $y'' = 2A$.

Substituting y and its derivatives into the given equation gives

$$2A + 2(2Ax + B) + 5(Ax^2 + Bx + C) = x^2 - 1.$$

Thus,

$$5A = 1$$

$$4A + 5B = 0$$

$$2A + 2B + 5C = -1$$

The solution of this system of equations is: $A = \frac{1}{5}$, $B = -\frac{4}{25}$, $C = -\frac{27}{125}$, and

$$y = \frac{1}{5}x^2 - \frac{4}{25}x - \frac{27}{125}.$$

4. We try $y = Ax^3 + Bx^2 + Cx + D$:

$y'' + y' - 2y = (6Ax + 2B) + (3Ax^2 + 2Bx + C) - 2(Ax^3 + Bx^2 + Cx + D) = x^3 + x.$

$\implies -2A = 1, \quad 3A - 2B = 0, \quad 6A + 2B - 2C = 1, \quad 2B + C - 2D = 0$

$\implies A = -\frac{1}{2},\ B = -\frac{3}{4},\ C = -\frac{9}{4},\ D = -\frac{15}{8}; \qquad y_p = -\frac{1}{2}x^3 - \frac{3}{4}x^2 - \frac{9}{4}x - \frac{15}{8}.$

5. First consider the reduced equation. The characteristic equation is:

$$r^2 + 6r + 9 = (r + 3)^2 = 0$$

and $u_1(x) = e^{-3x}$, $u_2(x) = xe^{-3x}$ are fundamental solutions. Therefore, a particular solution of the given equation has the form

$$y = Ae^{3x}.$$

The derivatives of y are: $y' = 3Ae^{3x}$, $y'' = 9Ae^{3x}0.$

Substituting y and its derivatives into the given equation gives

$$9Ae^{3x} + 18Ae^{3x} + 9Ae^{3x} = e^{3x}.$$

Thus, $36A = 1 \implies A = \dfrac{1}{36}$, and $y = \frac{1}{36}e^{3x}$.

6. Since -3 is a double root of the characteristic equation $r^2 + 6r + 9 = 0$, we try

$y = Ax^2e^{-3x}$. Then $y' = A(-3x^2 + 2x)e^{-3x}$, $y'' = A(9x^2 - 12x + 2)e^{-3x}$, and

$$[A(9x^2 - 12x + 2) + 6A(-3x^2 + 2x) + 9Ax^2]e^{-3x} = e^{-3x}, \quad \text{or} \quad 2Ae^{-3x} = e^{-3x}$$

Thus $A = \frac{1}{2}$ and $y_p = \frac{1}{2}x^2e^{-3x}$.

7. First consider the reduced equation. The characteristic equation is:

$$r^2 + 2r + 2 = 0$$

and $u_1(x) = e^{-x}\cos x$, $u_2(x) = e^{-x}\sin x$ are fundamental solutions. Therefore, a particular solution of the given equation has the form

$$y = Ae^{x}.$$

The derivatives of y are: $y' = Ae^{x}$, $y'' = Ae^{x}$.

Substituting y and its derivatives into the given equation gives

$$Ae^{x} + 2Ae^{x} + 2Ae^{x} = e^{x}.$$

Thus, $5A = 1 \implies A = \frac{1}{5}$ and $y = \frac{1}{5}e^{x}$.

8. Try $y = (A + Bx)e^{-x}$. Substituting into $y'' + 4y' + 4y = xe^{-x}$ gives

$A = -2,\ B = 1; \qquad y_p = (x - 2)e^{-x}$

9. First consider the reduced equation. The characteristic equation is:

$$r^2 - r - 12 = (r-4)(r+3) = 0$$

and $u_1(x) = e^{4x}$, $u_2(x) = e^{-3x}$ are fundamental solutions. Therefore, a particular solution of the given equation has the form

$$y = A\cos x + B\sin x.$$

The derivatives of y are: $y' = -A\sin x + B\cos x$, $y'' = -A\cos x - B\sin x$.

Substituting y and its derivatives into the given equation gives

$$-A\cos x - B\sin x - (-A\sin x + B\cos x) - 12(A\cos x + B\sin x) = \cos x.$$

Thus,

$$-13A - B = 1$$

$$A - 13B = 0$$

The solution of this system of equations is: $A = -\frac{13}{170}$, $B = -\frac{1}{170}$, and

$$y = -\tfrac{13}{170}\cos x - \tfrac{1}{170}\sin x.$$

10. Try $y = A\cos x + B\sin x$. Substituting into $y'' - y' - 12y = \sin x$ gives

$A = \frac{1}{170}$, $B = \frac{-13}{170}$; $y_p = \frac{1}{170}\cos x - \frac{13}{170}\sin x$.

11. First consider the reduced equation. The characteristic equation is:

$$r^2 + 7r + 6 = (r+6)(r+1) = 0$$

and $u_1(x) = e^{-6x}$, $u_2(x) = e^{-x}$ are fundamental solutions. Therefore, a particular solution of the given equation has the form

$$y = A\cos 2x + B\sin 2x.$$

The derivatives of y are: $y' = -2A\sin 2x + 2B\cos 2x$, $y'' = -4A\cos 2x - 4B\sin 2x$.

Substituting y and its derivatives into the given equation gives

$$-4A\cos 2x - 4B\sin 2x + 7(-2A\sin 2x + 2B\cos 2x) + 6(A\cos 2x + B\sin 2x) = 3\cos 2x.$$

Thus,

$$2A + 14B = 3$$

$$-14A + 2B = 0$$

The solution of this system of equations is: $A = \frac{3}{100}$, $B = \frac{21}{100}$ and

$$y = \tfrac{3}{10}\cos 2x + \tfrac{21}{100}\sin 2x.$$

12. Try $y = A\cos 3x + B\sin 3x$. Substituting into $y'' + y' + 3y = \sin 3x$ gives

$A = -\frac{1}{15}$, $B = -\frac{2}{15}$; $y_p = -\frac{1}{15}\cos 3x - \frac{2}{15}\sin 3x$.

13. First consider the reduced equation. The characteristic equation is:

$$r^2 - 2r + 5 = 0$$

and $u_1(x) = e^x \cos 2x$, $u_2(x) = e^x \sin 2x$ are fundamental solutions. Therefore, a particular solution of the given equation has the form

$$y = Ae^{-x}\cos 2x + Be^{-x}\sin 2x$$

The derivatives of y are: $y' = -Ae^{-x}\cos 2x - 2Ae^{-x}\sin 2x - Be^{-x}\sin 2x + 2be^{-x}\cos 2x$,

$y'' = 4Ae^{-x}\sin -3Ae^{-x}\cos 2x - 4be^{-x}\cos 2x - 3Be^{-x}\sin 2x$.

Substituting y and its derivatives into the given equation gives

$$4Ae^{-x}\sin -3Ae^{-x}\cos 2x - 4be^{-x}\cos 2x - 3Be^{-x}\sin 2x -$$

$$2\left(-Ae^{-x}\cos 2x - 2Ae^{-x}\sin 2x - Be^{-x}\sin 2x + 2be^{-x}\cos 2x\right) +$$

$$5\left(Ae^{-x}\cos 2x + Be^{-x}\sin 2x\right) = e^{-x}\sin 2x.$$

Equating the coefficients of $e^{-x}\cos 2x$ and $e^{-x}\sin 2x$ we get,

$$8A + 4B = 1$$

$$4A - 8B = 0$$

The solution of this system of equations is: $A = \frac{1}{10}$, $B = \frac{1}{20}$ and

$$y = \frac{1}{10}e^{-x}\cos 2x + \frac{1}{20}e^{-x}\sin 2x.$$

14. Try $y = e^{2x}(A\cos x + B\sin x)$. Substituting into $y'' + 4y' + 5y = e^{2x}\cos x$ gives

$A = \frac{1}{20}$, $B = \frac{1}{40}$; $y_p = e^{2x}\left(\frac{1}{20}\cos x + \frac{1}{40}\sin x\right)$.

15. First consider the reduced equation. The characteristic equation is:

$$r^2 + 6r + 8 = (r + 4)(r + 2) = 0$$

and $u_1(x) = e^{-4x}$, $u_2(x) = e^{-2x}$ are fundamental solutions. Therefore, a particular solution of the given equation has the form

$$y = Axe^{-2x}.$$

The derivatives of y are: $y' = Ae^{-2x} - 2Axe^{-2x}$, $y'' = -4Ae^{-2x} + 4Axe^{-2x}$.

Substituting y and its derivatives into the given equation gives

$$-4Ae^{-2x} + 4Axe^{-2x} + 6\left(Ae^{-2x} - 2Axe^{-2x}\right) + 8Axe^{-2x} = 3e^{-2x}$$

Thus, $2A = 3 \implies A = \frac{3}{2}$ and $y = \frac{3}{2}xe^{-2x}$.

16. Try $y = e^x(A\cos x + B\sin x)$. Substituting into $y'' - 2y' + 5y = e^x \sin x$ gives $A = 0,\ B = \frac{1}{3}$; $y_p = \frac{1}{3}e^x \sin x$.

17. First consider the reduced equation: $y'' + y = 0$. The characteristic equation is:

$$r^2 + 1 = 0$$

and $u_1(x) = \cos x$, $u_2(x) = \sin x$ are fundamental solutions. A particular solution of the given equation has the form

$$y = Ae^x.$$

The derivatives of y are: $y' = y'' = Ae^x$.

Substitute y and its derivatives into the given equation:

$$Ae^x + Ae^x = e^x \implies A = \frac{1}{2} \text{ and } y = \tfrac{1}{2}e^x.$$

The general solution of the given equation is: $y = C_1\cos x + C_2\sin x + \frac{1}{2}e^x$.

18. $r^2 - 2r + 1 = 0 \implies r = 1 \implies y = C_1e^x + C_2xe^x$ is the general solution of the reduced equation. To find a particular solution, we try $y = A\cos 2x + B\sin 2x$. Substituting into $y'' - 2y' + y = -25\sin 2x$ gives $A = -4,\ B = 3$, so $y_p = 3\sin 2x - 4\cos 2x$.

Therefore the general solution is: $y = C_1e^x + C_2xe^x + 3\sin 2x - 4\cos 2x$.

19. First consider the reduced equation: $y'' - 3y' - 10y = 0$. The characteristic equation is:

$$r^2 - 3r - 10 = (r-5)(r+2) = 0$$

and $u_1(x) = e^{5x}$, $u_2(x) = e^{-2x}$ are fundamental solutions. A particular solution of the given equation has the form

$$y = Ax + B.$$

The derivatives of y are: $y' = A$, $y'' = 0$.

Substitute y and its derivatives into the given equation:

$$-3A - 10(Ax + B) = -x - 1 \implies A = \tfrac{1}{10},\ B = \tfrac{7}{100} \text{and } y = \tfrac{1}{10}x + \tfrac{7}{100}$$

The general solution of the given equation is:

$$y = C_1e^{5x} + C_2e^{-2x} + \cos x + C_2\sin x + \tfrac{1}{10}x + \tfrac{7}{100}$$

20. $r^2 + 4 = 0 \implies r = \pm 2i \implies y = C_1\cos 2x + C_2\sin 2x$, general solution of reduced equation.

Particular solution: try $y = x(A + Bx)\cos 2x + x(C + Dx)\sin 2x$.

Substituting into $y'' + 4y = x\cos 2x$ gives $A = 0,\ B = -\frac{1}{8},\ C = \frac{1}{16},\ D = 0$;
$y_p = -\dfrac{1}{8}x^2\cos 2x + \dfrac{1}{16}x\sin 2x$. General solution: $y = C_1\cos 2x + C_2\cos 2x - \frac{1}{8}x^2\cos 2x + \frac{1}{16}x\sin 2x$.

21. First consider the reduced equation: $y'' + 3y' - 4y = 0$. The characteristic equation is:

$$r^2 + 3r - 4 = (r+4)(r-1) = 0$$

and $u_1(x) = e^x$, $u_2(x) = e^{-4x}$ are fundamental solutions. A particular solution of the given equation has the form

$$y = Axe^{-4x}.$$

The derivatives of y are: $y' = Ae^{-4x} - 4Axe^{-4x}$, $y'' = -8Ae^{-4x} + 16Axe^{-4x}$.

Substitute y and its derivatives into the given equation:

$$-8Ae^{-4x} + 16Axe^{-4x} + 3\left(Ae^{-4x} - 4Axe^{-4x}\right) - 4Axe^{-4x} = e^{-4x}.$$

This implies $-5A = 1$,so $A = -\frac{1}{5}$ and $y = -\frac{1}{5}xe^{-4x}$.

The general solution of the given equation is: $y = C_1e^x + C_2e^{-4x} - \frac{1}{5}xe^{-4x}$.

22. $r^2 + 2r = 0 \implies r = 0, -2 \implies y = C_1 + C_2e^{-2x}$, general solution of reduced equation.

Particular solution: try $y = A\cos 2x + B\sin 2x$.

Substituting into $y'' + 2y' = 4\sin 2x$ gives $A = B = -\frac{1}{2}$; $y_p = -\frac{1}{2}\cos 2x - \frac{1}{2}\sin 2x$.

General solution: $y = C_1 + C_2e^{-2x} - \frac{1}{2}\cos 2x - \frac{1}{2}\sin 2x$.

23. First consider the reduced equation: $y'' + y' - 2y = 0$. The characteristic equation is:

$$r^2 + r - 2 = (r+2)(r-1) = 0$$

and $u_1(x) = e^{-2x}$, $u_2(x) = e^x$ are fundamental solutions. A particular solution of the given equation has the form

$$y = x(A + Bx)e^x.$$

The derivatives of y are:

$$y' = (A + (2B + A)x + Bx^2)e^x, \quad y'' = (2A + 2B + (4B + A)x + Bx^2)e^x.$$

Substitute y and its derivatives into the given equation:

$$(2A + 2B + (4B + A)x + Bx^2 + A + (2B + A)x + Bx^2 - 2Ax - 2Bx^2)e^x = 3xe^x.$$

This implies $A = -\frac{1}{3}$ $,B = \frac{1}{2}$ so $y = x(-\frac{1}{3} + \frac{1}{2}x)e^x$.

The general solution of the given equation is: $y = C_1e^{-2x} + C_2e^x - \frac{1}{3}xe^x + \frac{1}{2}x^2e^x$.

24. $r^2+4r+4=0 \implies r=-2 \implies y=C_1e^{-2x}+C_2xe^{-2x}$, general solution of reduced equation.

Particular solution: try $y=x^2(A+Bx)e^{-2x}$.

Substituting into $y''+4y'+4y=xe^{-2x}$ gives $A=0,\ B=\frac{1}{6};\quad y_p=\frac{1}{6}x^3e^{-2x}$.

General solution: $\quad y=C_1e^{-2x}+C_2xe^{-2x}+\frac{1}{6}x^3e^{-2x}$.

25. Let $y_1(x)$ be a solution of $y''+ay'+by=\phi_1(x)$, let $y_2(x)$ be a solution of $y''+ay'+by=\phi_2(x)$, and let $z=y_1+y_2$. Then

$$\begin{aligned} z''+az'+bz &= (y_1''+y_2'')+a(y_1'+y_2')+b(y_1+y_2) \\ &= (y_1''+ay_1'+by_1)+(y_2''+ay_2'+y_2)=\phi_1+\phi_2. \end{aligned}$$

26. (a) $y=-\frac{1}{15}x$ is a particular solution of $y''+2y'-15y=x$

$y=-\frac{1}{7}e^{2x}$ is a particular solution of $y''+2y'-15y=e^{2x}$

Therefore $\quad y=-\frac{1}{15}x-\frac{1}{7}e^{2x}$ is a particular solution of $y''+2y'-15y=x+e^{2x}$.

(b) $y=\frac{1}{20}e^{-x}$ is a particular solution of $y''-7y'+12y=e^{-x}$.

$y=\frac{7}{130}\cos 2x+\frac{2}{65}\sin 2x$ is a particular solution of $y''-7y'+12y=\sin 2x$.

Therefore $y=\frac{1}{20}e^{-x}+\frac{7}{130}\cos 2x+\frac{2}{65}\sin 2x$ is a particular solution of

$y''-7y'+12y=e^{-x}+\sin 2x$.

27. First consider the reduced equation: $y''+4y'+3y=0$. The characteristic equation is:

$$r^2+4r+3=(r+3)(r+1)=0$$

and $u_1(x)=e^{-3x}$, $u_2(x)=e^{-x}$ are fundamental solutions. Since $\cosh x=\frac{1}{2}\left(e^x+e^{-x}\right)$, a particular solution of the given equation has the form

$$y=Ae^x+Bxe^{-x}$$

The derivatives of y are: $\quad y'=Ae^x+Be^{-x}-Bxe^{-x} \qquad y''=Ae^x-2Be^{-x}+Bxe^{-x}$.

Substitute y and its derivatives into the given equation:

$$Ae^x-2Be^{-x}+Bxe^{-x}+4\left(Ae^x+Be^{-x}-Bxe^{-x}\right)+3\left(Ae^x+Bxe^{-x}\right)=\frac{1}{2}\left(e^x+e^{-x}\right)e^{-4x}.$$

Equating coefficients, we get $\quad A=\frac{1}{16},\quad B=\frac{1}{4},\quad$ and so $\ y=\frac{1}{16}e^x+\frac{1}{4}xe^{-x}$.

The general solution of the given equation is: $\quad y=C_1e^{-3x}+C_2e^{-x}+\frac{1}{16}e^x+\frac{1}{4}xe^{-x}$.

28. $r^2+1=0 \implies r=\pm i$. Fundamental solutions: $u_1=\cos x,\quad u_2=\sin x$.

Wronskian: $\quad W=u_1u_2'-u_1'u_2=1;\qquad \phi(x)=3\sin x\sin 2x$

$$z_1=-\int\frac{u_2\phi}{W}\,dx=-\int 3\sin^2 x\sin 2x\,dx=-6\int\sin^3 x\cos x\,dx=-\frac{3}{2}\sin^4 x,$$

$$z_2=\int\frac{u_1\phi}{W}\,dx=\int 3\cos x\sin x\sin 2x\,dx=\frac{3}{2}\int\sin^2 2x\,dx=\frac{3}{16}(4x-\sin 4x).$$

Therefore $y_p = z_1u_1 + z_2u_2 = -\frac{3}{2}\sin^4 x\cos x + \frac{3}{16}(4x - \sin 4x)\sin x.$

29. First consider the reduced equation $y'' - 2y' + y = 0$. The characteristic equation is:

$$r^2 - 2r + 1 = (r-1)^2 = 0$$

and $u_1(x) = e^x$, $u_2(x) = xe^x$ are fundamental solutions. Their Wronskian is given by

$$W = u_1u_2' - u_2u_1' = e^x(e^x + xe^x) - xe^x(e^x) = e^{2x}$$

Using variation of parameters, a particular solution of the given equation will have the form

$$y = u_1z_1 + u_2z_2,$$

where

$$z_1 = -\int \frac{xe^x(xe^x\cos x)}{e^{2x}}\,dx = -\int x^2\cos x\,dx = -x^2\sin x - 2x\cos x + 2\sin x,$$

$$z_2 = \int \frac{e^x(xe^x\cos x}{e^{2x}}\,dx = \int x\cos x\,dx = x\sin x + \cos x$$

Therefore,

$$y = e^x\left(-x^2\sin x - 2x\cos x + 2\sin x\right) + xe^x\left(x\sin x + \cos x\right) = 2e^x\sin x - xe^x\cos x.$$

30. $r^2 + 1 = 0.$ Fundamental solutions: $u_1 = \cos x,\ u_2 = \sin x.$

Wronskian: $W = u_1u_2' - u_1'u_2 = 1;$ $\phi(x)\phi(x) = \csc x.$

$$z_1 = -\int \frac{u_2\phi}{W}\,dx = -\int \sin x\csc x\,dx = -x,$$

$$z_2 = \int \frac{u_1\phi}{W}\,dx = \int \cos x\csc x\,dx = \int \cot x\,dx = \ln(\sin x) \quad [\sin x > 0 \text{ since } 0 < x < \pi].$$

Therefore $y_p = z_1u_1 + z_2u_2 = -x\cos x + \ln(\sin x)\sin x.$

31. First consider the reduced equation $y'' - 4y' + 4y = 0$. The characteristic equation is:

$$r^2 - 4r + 4 = (r-2)^2 = 0$$

and $u_1(x) = e^{2x}$, $u_2(x) = xe^{2x}$ are fundamental solutions. Their Wronskian is given by

$$W = u_1u_2' - u_2u_1' = e^{2x}\left(e^{2x} + 2xe^{2x}\right) - xe^{2x}(2e^{2x}) = e^{4x}.$$

Using variation of parameters, a particular solution of the given equation will have the form

$$y = u_1z_1 + u_2z_2,$$

where

$$z_1 = -\int \frac{xe^{2x}\left(\frac{1}{3}x^{-1}e^{2x}\right)}{e^{4x}}\,dx = -\frac{1}{3}\int dx = -\frac{1}{3}x,$$

$$z_2 = \int \frac{e^{2x}\left(\frac{1}{3}x^{-1}e^{2x}\right)}{e^{4x}}\,dx = -\frac{1}{3}\int \frac{1}{x}\,dx = \frac{1}{3}\ln|x|.$$

Therefore,

$$y = e^{2x}\left(-\tfrac{1}{3}x\right) + xe^{2x}\left(\tfrac{1}{3}\ln|x|\right) = -\tfrac{1}{3}xe^{2x} + \tfrac{1}{3}x\ln|x|\,e^{2x}.$$

Note: Since $u = -\tfrac{1}{3}xe^{2x}$ is a solution of the reduced equation,

$$y = \tfrac{1}{3}x\ln|x|\,e^{2x}$$

is also a particular solution of the given equation.

32. $r^2 + 4 = 0 \implies r = \pm 2i$. Fundamental solutions: $u_1 = \cos 2x, u_2 = \sin 2x$.

Wronskian: $W = u_1u_2' - u_1'u_2 = 2\cos^2 2x + 2\sin^2 2x = 2;$ $\quad \phi(x) = \sec^2 2x$

$$z_1 = -\int \frac{u_2\phi}{W}\,dx = -\int \frac{\sin 2x}{2\cos^2 2x}\,dx = \sec 2x,$$

$$z_2 = \int \frac{u_1\phi}{W}\,dx = \int \frac{\cos 2x}{2\cos^2 2x}\,dx = \frac{1}{2}\int \sec 2x\,dx = \ln|\sec 2x + \tan 2x|.$$

Therefore $y_p = z_1u_1 + z_2u_2 = \sec 2x\cos 2x + \ln|\sec 2x + \tan 2x|\sin 2x = 1 + \ln|\sec 2x + \tan 2x|\sin 2x.$

33. First consider the reduced equation $y'' + 4y' + 4y = 0$. The characteristic equation is:

$$r^2 + 4r + 4 = (r+2)^2 = 0$$

and $u_1(x) = e^{-2x}$, $u_2(x) = xe^{-2x}$ are fundamental solutions. Their Wronskian is given by

$$W = u_1u_2' - u_2u_1' = e^{-2x}\left(e^{-2x} - 2xe^{2x}\right) - xe^{-2x}(-2e^{-2x}) = e^{-4x}.$$

Using variation of parameters, a particular solution of the given equation will have the form

$$y = u_1z_1 + u_2z_2,$$

where

$$z_1 = -\int \frac{xe^{-2x}\left(x^{-2}e^{-2x}\right)}{e^{-4x}}\,dx = -\int \frac{1}{x}\,dx = -\ln|x|$$

$$z_2 = \int \frac{e^{-2x}\left(x^{-2}e^{-2x}\right)}{e^{-4x}}\,dx = \int \frac{1}{x^2}\,dx = -\frac{1}{x}$$

Therefore,

$$y = e^{-2x}(-\ln|x|) + xe^{-2x}\left(-\frac{1}{x}\right) = -e^{-2x}\ln|x| - e^{-2x}.$$

Note: Since $u = -e^{-2x}$ is a solution of the reduced equation, we can take

$$y = \tfrac{1}{3}x\ln|x|e^{2x}$$

34. $r^2 + 2r + 1 = 0 \implies r = -1$. Fundamental solutions: $u_1 = e^{-x},\ u_2 = xe^{-x}$.

Wronskian: $W = u_1u_2' - u_1'u_2 = (1-x)e^{-2x} + xe^{-2x} = e^{-2x};$ $\quad \phi(x) = e^{-x}\ln x.$

$$z_1 = -\int \frac{u_2\phi}{W}\,dx = -\int \frac{xe^{-x}e^{-x}\ln x}{e^{-2x}}\,dx = -\int x\ln x\,dx = -\frac{1}{2}x^2\ln x + \frac{x^2}{4},$$

$$z_2 = \int \frac{u_1\phi}{W}\,dx = \int \frac{e^{-x}e^{-x}\ln x}{e^{-2x}}\,dx = \int \ln x\,dx = x\ln x - x.$$

Therefore
$$\begin{aligned} y_p = z_1u_1 + z_2u_2 &= e^{-x}(x\ln x - x) + xe^{-x}\left(\frac{x^2}{4} - \frac{1}{2}x^2\ln x\right) \\ &= e^{-x}\left(x\ln x - x + \frac{x^3}{4} - \frac{1}{2}x^3\ln x\right). \end{aligned}$$

35. First consider the reduced equation $y'' - 2y' + 2y = 0$. The characteristic equation is:

$$r^2 - 2r + 2 = 0$$

and $u_1(x) = e^x\cos x$, $u_2(x) = e^x\sin x$ are fundamental solutions. Their Wronskian is given by

$$W = e^x\cos x\,[e^x\sin x + e^x\cos x] - e^x\sin x\,[e^x\cos x - e^x\sin x] = e^{2x}$$

Using variation of parameters, a particular solution of the given equation will have the form

$$y = u_1z_1 + u_2z_2,$$

where

$$z_1 = -\int \frac{e^x\sin x\cdot e^x\sec x}{e^{2x}}\,dx = -\int \tan x\,dx = -\ln|\sec x| = \ln|\cos x|$$

$$z_2 = \int \frac{e^x\cos x\cdot e^x\sec x}{e^{2x}}\,dx = \int dx = x$$

Therefore,

$$y = e^x\cos x\,(\ln|\cos x|) + e^x\sin x(x) = e^x\cos x\,\ln|\cos x| + xe^x\sin x.$$

36. Follows directly from

$$(ve^{kx})'' + a(ve^{kx})' + b(ve^{kx}) = e^{kx}[v'' + (2k+a)v' + (k^2 + ak + b)v].$$

37. Assume that the forcing function $F(t) = F_0$ (constant). Then the differential equation has a particular solution of the form $i = A$. The derivatives of i are: $i' = i'' = 0$. Substituting i and its derivatives into the equation, we get

$$\frac{1}{C}A = F_0 \implies A = CF_0 \implies i = CF_0.$$

The characteristic equation for the reduced equation is:

$$Lr^2 + Rr + \frac{1}{C} = 0 \implies r_1, r_2 = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L} = \frac{-R \pm \sqrt{CR^2 - 4L}}{2L\sqrt{C}}$$

(a) If $CR^2 = 4L$, then the characteristic equation has only one root: $r = -R/2L$, and $u_1 = e^{-(R/2L)t}$, $u_2 = t\,e^{-(R/2L)t}$ are fundamental solutions.

The general solution of the given equation is:

$$i(t) = C_1e^{-(R/2L)t} + C_2t\,e^{-(R/2L)t} + CF_0$$

and its derivative is:

$$i'(t) = -C_1(R/2L)e^{-(R/2L)t} + C_2e^{-(R/2L)t} - C_2(R/2L)t\,e^{-(R/2L)t}.$$

Applying the side conditions $i(0) = 0, \quad i'(0) = F_0/L,$ we get

$$C_1 + CF_0 = 0$$

$$(-R/2L)C_1 + C_2 = F_0/L$$

The solution is $C_1 = -CF_0, \quad C_2 = \dfrac{F_0}{2L}(2 - RC).$

The current in this case is:

$$i(t) = -CF_0e^{-(R/2L)t} + \frac{F_0}{2L}(2 - RC)\,t\,e^{-(R/2L)t} + CF_0.$$

(b) If $CR^2 - 4L < 0$ then the characteristic equation has complex roots:

$$r_1 = -R/2L \pm i\beta, \quad \text{where} \quad \beta = \sqrt{\frac{4L - CR^2}{4CL^2}} \quad (\text{here } i^2 = -1)$$

and fundamental solutions are: $u_1 = e^{-(R/2L)t}\cos\beta t, \quad u_2 = e^{-(R/2L)t}\sin\beta t.$

The general solution of the given differential equation is:

$$i(t) = e^{-(R/2L)t}\,(C_1\cos\beta t + C_2\sin\beta t) + CF_0$$

and its derivative is:

$$i'(t) = (-R/2L)e^{-(R/2L)t}\,(C_1\cos\beta t + C_2\sin\beta t) + \beta e^{-(R/2L)t}\,(-\,C_1\sin\beta t + C_2\cos\beta t)\,.$$

Applying the side conditions $i(0) = 0, \quad i'(0) = F_0/L,$ we get

$$C_1 + CF_0 = 0$$

$$(-R/2L)C_1 + \beta C_2 = F_0/L$$

The solution is $C_1 = -CF_0, \quad C_2 = \dfrac{F_0}{2L\beta}(2 - RC).$

The current in this case is:

$$i(t) = e^{-(R/2L)t}\left(\frac{F_0}{2L\beta}(2 - RC)\,\sin\beta t - CF_0\cos\beta t\right) + CF_0.$$

38. (a) $x^2y_1'' - xy_1' + y_1 = x^2\cdot 0 - x\cdot 1 + x = 0:$ y_1 is a solution.

$x^2y_2'' - xy_2' + y_2 = x^2\left(\frac{1}{x}\right) - x(\ln x + 1) + x\ln x = 0:$ y_2 is a solution.

$W = y_1y_2' - y_1'y_2 = x(\ln x + 1) - 1(x\ln x) = x$ is nonzero on $(0, \infty)$.

(b) To use the method of variation of parameters as described in the text, we first re-write the equation in the form

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = \frac{4}{x}\ln x.$$

Then, a particular solution of the equation will have the form $y_p = y_1z_1 + y_2z_2,$ where

$$z_1 = -\int \frac{x\ln x\cdot[(4/x)\,\ln x]}{x}\,dx = -4\int \frac{1}{x}(\ln x)^2\,dx = -\frac{4}{3}(\ln x)^3$$

and

$$z_2 = \int \frac{x \cdot [(4/x)\ \ln x]}{x}\, dx = 4 \int \frac{\ln x}{x}\, dx = 2(\ln x)^2$$

Thus, $y_p = -\frac{4}{3}x\,(\ln x)^3 + x\,\ln x \cdot 2(\ln x)^2$ which simplifies to: $y_p = \frac{2}{3}x\,(\ln x)^3$.

39. (a) Let $y_1(x) = \sin\left(\ln x^2\right)$. Then

$$y_1' = \left(\frac{2}{x}\right) \cos\left(\ln x^2\right) \quad \text{and} \quad y_1'' = -\left(\frac{4}{x^2}\right) \sin\left(\ln x^2\right) - \left(\frac{2}{x^2}\right) \cos\left(\ln x^2\right)$$

Substituting y_1 and its derivatives into the differential equation, we have

$$x^2 \left[-\left(\frac{4}{x^2}\right) \sin\left(\ln x^2\right) - \left(\frac{2}{x^2}\right) \cos\left(\ln x^2\right)\right] - x\left[\left(\frac{2}{x}\right) \cos\left(\ln x^2\right)\right] + 4\sin\left(\ln x^2\right) = 0$$

The verification that y_2 is a solution is done in exactly the same way.

The Wronskian of y_1 and y_2 is:

$$\begin{aligned} W(x) &= y_1 y_2' - y_2 y_1' \\ &= \sin\left(\ln x^2\right)\left[-\left(\frac{2}{x}\right) \sin\left(\ln x^2\right)\right] - \cos\left(\ln x^2\right)\left[\left(\frac{2}{x}\right) \cos\left(\ln x^2\right)\right] \\ &= -\frac{2}{x} \neq 0 \text{ on } (0, \infty) \end{aligned}$$

(b) To use the method of variation of parameters as described in the text, we first re-write the equation in the form

$$y'' + x^{-1}\,y' + 4x^{-2}\,y = x^{-2}\sin(\ln x).$$

Then, a particular solution of the equation will have the form $y = y_1 z_1 + y_2 z_2$, where

$$\begin{aligned} z_1 &= -\int \frac{\cos(\ln x^2)x^{-2}\sin(\ln x)}{-2/x}\, dx \\ &= \tfrac{1}{2}\textstyle\int \cos(2\ \ln x)x^{-1}\sin(\ln x)\, dx \\ &= \tfrac{1}{2}\textstyle\int \cos 2u\ \sin u\, du \qquad (u = \ln x) \\ &= \tfrac{1}{2}\textstyle\int (2\cos^2 u - 1)\sin u\, du \\ &= -\tfrac{1}{3}\cos^3 u + \tfrac{1}{2}\sin u \end{aligned}$$

and

$$\begin{aligned} z_2 &= \int \frac{\sin(\ln x^2)x^{-2}\sin(\ln x)}{-2/x}\, dx \\ &= -\tfrac{1}{2}\textstyle\int \sin(2\ \ln x)x^{-1}\sin(\ln x)\, dx \\ &= -\tfrac{1}{2}\textstyle\int \sin 2u\ \sin u\, du \qquad (u = \ln x) \\ &= -\int (\sin^2 u\ \cos u\, du \\ &= -\tfrac{1}{3}\sin^3 u \end{aligned}$$

Thus, $y = \sin 2u\left(-\frac{1}{3}\cos^3 u + \frac{1}{2}\sin u)\right) - \cos 2u\left(\frac{1}{3}\sin^3 u\right)$ which simplifies to:

$$y = \tfrac{1}{3}\sin u = \tfrac{1}{3}\sin(\ln x).$$

SECTION 18.5

1. The equation of motion is of the form

$$x(t) = A\sin(\omega t + \phi_0).$$

The period is $T = 2\pi/\omega = \pi/4$. Therefore $\omega = 8$. Thus

$$x(t) = A\sin(8t + \phi_0) \text{ and } v(t) = 8A\cos(8t + \phi_0).$$

Since $x(0) = 1$ and $v(0) = 0$, we have

$$1 = A\sin\phi_0 \quad \text{and} \quad 0 = 8A\cos\phi_0.$$

These equations are satisfied by taking $A = 1$ and $\phi_0 = \pi/2$.

Therefore the equation of motion reads

$$x(t) = \sin\left(8t + \tfrac{1}{2}\pi\right).$$

The amplitude is 1 and the frequency is $8/2\pi = 4/\pi$.

2. $x(t) = A\sin(\omega t + \phi_0)$. $\omega = 2\pi f = 2\pi\left(\dfrac{1}{\pi}\right) = 2$

$0 = x(0) = A\sin\phi_0, \quad -2 = x'(0) = \omega A\cos\phi_0 \implies A = 1, \quad \phi_0 = \pi.$

Amplitude 1, period $T = \frac{1}{f} = \pi$.

3. We can write the equation of motion as

$$x(t) = A\sin\left(\frac{2\pi}{T}t\right).$$

Differentiation gives

$$v(t) = \frac{2\pi A}{T}\cos\left(\frac{2\pi}{T}t\right).$$

The object passes through the origin whenever $\sin[(2\pi/T)] = 0$.

Then $\cos[(2\pi/T)t] = \pm 1$ and $v = \pm 2\pi A/T$.

4. $x(t) = A\sin\left(\frac{2\pi}{T}t + \phi_0\right), \quad v = x'(t) = \frac{2\pi}{T}A\cos\left(\frac{2\pi}{T}t + \phi_0\right).$

Note that $x^2 + \left(\frac{T}{2\pi}v\right)^2 = A^2$.

At $x = x_0,\ v = \pm v_0$, so $A = \sqrt{x_0{}^2 + \left(\frac{T}{2\pi}v_0\right)^2} = (1/2\pi)\sqrt{4\pi^2 x_0^2 + T^2 v_0^2}$.

5. In this case $\phi_0 = 0$ and, measuring t_2 in seconds, $T = 6$.

Therefore $\omega = 2\pi/6 = \pi/3$ and we have

$$x(t) = A\sin\left(\frac{\pi}{3}t\right), \quad v(t) = \frac{\pi A}{3}\cos\left(\frac{\pi}{3}t\right).$$

Since $v(0) = 5$, we have $\pi A/3 = 5$ and therefore $A = 15/\pi$.

The equation of motion can be written

$$x(t) = (15/\pi)\sin\left(\tfrac{1}{3}\pi t\right)$$

6. (a) $A\sin(\omega t + \phi_0) = A\cos(\omega t + \phi_0 - \frac{\pi}{2})$; take $\phi_1 = \phi_0 - \frac{1}{2}\pi$.

   (b) $A\sin(\omega t + \phi_0) = A\cos\phi_0 \sin\omega t + A\sin\phi_0\cos\omega t = B\sin\omega t + C\cos\omega t.$

7. $x(t) = x_0\sin\left(\sqrt{k/m}\,t + \frac{1}{2}\pi\right)$

8. (a) maximum speed at $x = 0$.

   (b) zero speed at $x = \pm x_0$.

   (c) maximum acceleration (in absolute value) at $x = \pm x_0$.

   (d) zero acceleration at $x = 0$ (when total force is zero).

9. The equation of motion for the bob reads

$$x(t) = x_0\sin\left(\sqrt{k/m}\;t + \tfrac{1}{2}\pi\right). \qquad \text{(Exercise 7)}$$

Since $v(t) = \sqrt{k/m}\;x_0\cos\left(\sqrt{k/m}\;t + \frac{1}{2}\pi\right)$, the maximum speed is $\sqrt{k/m}\;x_0$.
The bob takes on half of that speed where $\left|\cos\left(\sqrt{k/m}\;t + \frac{1}{2}\pi\right)\right| = \frac{1}{2}$. Therefore

$$\left|\sin\left(\sqrt{k/m}\;t + \tfrac{1}{2}\pi\right)\right| = \sqrt{1 - \tfrac{1}{4}} = \tfrac{1}{2}\sqrt{3} \quad\text{and}\quad x(t) = \pm\tfrac{1}{2}\sqrt{3}\,x_0.$$

10. $v = -\sqrt{\dfrac{k}{m}}x_0\sin\left(\sqrt{\dfrac{k}{m}}t\right)$ has maximum value $\sqrt{\dfrac{k}{m}}\;x_0$, so the maximum kinetic energy is

$$\frac{1}{2}mv^2 = \frac{1}{2}m\frac{k}{m}{x_0}^2 = \frac{1}{2}k{x_0}^2.$$

11.
$$\begin{aligned}\text{KE} &= \tfrac{1}{2}m[v(t)]^2 = \tfrac{1}{2}m(k/m){x_0}^2\cos^2\left(\sqrt{k/m}\;t + \tfrac{1}{2}\pi\right)\\ &= \tfrac{1}{4}k{x_0}^2\left[1 + \cos\left(2\sqrt{k/m}\;t + \tfrac{1}{2}\pi\right)\right].\\ \text{Average KE} &= \frac{1}{2\pi\sqrt{m/k}}\int_0^{2\pi\sqrt{m/k}}\frac{1}{4}k{x_0}^2\left[1 + \cos\left(2\sqrt{k/m}\;t + \frac{1}{2}\pi\right)\right]dt\\ &= \tfrac{1}{4}k{x_0}^2.\end{aligned}$$

12. $v(t) = -\sqrt{\dfrac{k}{m}}x_0\sin\left(\sqrt{\dfrac{k}{m}}\;t\right) = \pm\sqrt{\dfrac{k}{m}}\sqrt{{x_0}^2 - [x(t)]^2}.$

13. Setting $y(t) = x(t) - 2$, we can write $x''(t) = 8 - 4x(t)$ as $y''(t) + 4y(t) = 0$.

This is simple harmonic motion about the point $y = 0$; that is, about the point $x = 2$. The equation of motion is of the form

$$y(t) = A\sin(2t + \phi_0).$$

Since $y(0) = x(0) - 2 = -2$, the amplitude A is 2. Since $\omega = 2$, the period T is $2\pi/2 = \pi$.

14. (a) Since $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$, $\sin\theta \cong \theta$ for small θ.

(b) The general solution is $\theta(t) = A\sin\left(\sqrt{g/L}\,t + \phi_0\right)$

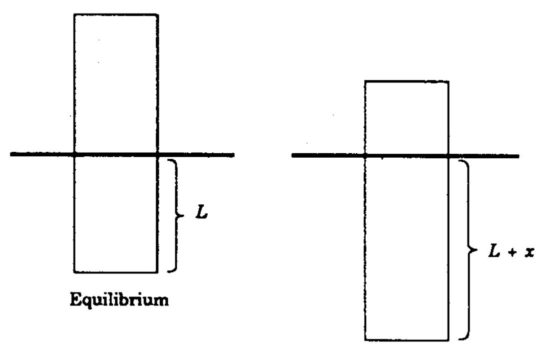
(i) Here $A = \theta_0$ and $\phi_0 = \frac{\pi}{2}$, so $\theta(t) = \theta_0 \sin\left(\sqrt{g/L}\,t + \frac{\pi}{2}\right) = \theta_0 \cos\left(\sqrt{g/L}\,t\right)$.

(ii) $0 = \theta(0) = A\sin\phi_0$, $-\sqrt{\frac{g}{L}}\,\theta_0 = \theta'(0) = A\sqrt{\frac{g}{L}}\cos\phi_0 \implies A = \theta_0,\ \phi_0 = \pi$.

Therefore, the equation of motion becomes $\theta(t) = -\theta_0 \sin\left(\sqrt{g/L}\,t\right)$

(c) $\omega = \sqrt{\frac{g}{L}}$, $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} = 2 \implies L = \frac{g}{\pi^2} \cong 3.24$ feet or 0.993 meters.

15. (a) Take the downward direction as positive. We begin by analyzing the forces on the buoy at a general position x cm beyond equilibrium. First there is the weight of the buoy: $F_1 = mg$. This is a downward force. Next there is the buoyancy force equal to the weight of the fluid displaced; this force is in the opposite direction: $F_2 = -\pi r^2 (L + x)\rho$. We are neglecting friction so the total force is



$$F = F_1 + F_2 = mg - \pi r^2 (L + x)\rho = \left(mg - \pi r^2 L\rho\right) - \pi r^2 x\rho.$$

We are assuming at the equilibrium point that the forces (weight of buoy and buoyant force of fluid) are in balance:

$$mg - \pi r^2 L\rho = 0.$$

Thus,

$$F = -\pi r^2 x\rho.$$

By Newton's

$$F = ma \qquad \text{(force = mass} \times \text{acceleration)}$$

we have

$$ma = -\pi r^2 x\rho \qquad \text{and thus} \qquad a + \frac{\pi r^2 \rho}{m}x = 0.$$

Thus, at each time t,

$$x''(t) + \frac{\pi r^2 \rho}{m}x(t) = 0.$$

(b) The usual procedure shows that

$$x(t) = x_0 \sin\left(r\sqrt{\pi\rho/m}\,t + \tfrac{1}{2}\pi\right).$$

The amplitude A is x_0 and the period T is $(2/r)\sqrt{m\pi/\rho}$.

16. Uniform circular motion consists of simple harmonic motion in both x and y, the two being out of phase by $\frac{\pi}{2}$.

17. From (18.5.4), we have

$$x(t) = Ae^{(-c/2m)t}\sin(\omega t + \phi_0) = \frac{A}{e^{(c/2m)t}}\sin(\omega t + \phi_0) \quad \text{where} \quad \omega = \frac{\sqrt{4}km - \omega^2}{2m}$$

If c increases, then both the amplitude, $\left|\dfrac{A}{e^{(c/2m)t}}\right|$ and the frequency $\dfrac{\omega}{2\pi}$ decrease.

18. Assume that $r_1 > r_2$. If $C_1 = 0$ or $C_2 = 0$, then $x = C_1e^{r_1t} + C_2e^{r_2t}$ can never be zero. If both C_1 and C_2 are nonzero, then $C_1e^{r_1t} + C_2e^{r_2t} = 0$ implies $e^{(r_1-r_2)t} = -\dfrac{C_2}{C_1}$. Since $e^{(r_1-r_2)t}$ is an increasing function $(r_1 > r_2)$, it can take the value $-\dfrac{C_2}{C_1}$ at most once.
By the same reasoning, $x'(t) = C_1r_1e^{r_1t} + C_2r_2e^{r_2t}$ can be zero at most once. Therefore the motion can change direction at most once.

19. Set $x(t) = 0$ in (18.5.6). The result is:

$$C_1e^{(-c/2m)t} + C_2te^{(-c/2m)t} = 0 \quad \Longrightarrow \quad C_1 + C_2t = 0 \quad \Longrightarrow \quad t = -C_1/C_2$$

Thus, there is at most one value of t at which $x(t) = 0$.

The motion changes directions when $x'(t) = 0$:

$$x'(t) = -C_1(c/2m)e^{(-c/2m)t} + C_2e^{(-c/2m)t} - C_2(c/2m)te^{(-c/2m)t}.$$

Now,

$$x'(t) = 0 \quad \Longrightarrow \quad -C_1(c/2m) + C_2 - C_2t(c/2m) = 0 \quad \Longrightarrow \quad t = \frac{C_2 - C_1(c/2m)}{C_2(c/2m)}$$

and again we conclude that there is at most one value of t at which $x'(t) = 0$.

20. If $\gamma \neq \omega$, we try $x_p = A\cos\gamma t + B\sin\gamma t$ as a particular solution of $x'' + \omega^2x = \dfrac{F_0}{m}\cos\gamma t$.

Substituting x_p into the equation, we get $-\gamma^2x_p + \omega^2x_p = \frac{F_0}{m}\cos\gamma t$,

giving $x_p = \dfrac{F_0/m}{\omega^2 - \gamma^2}\cos\gamma t.$

21. $x(t) = A\sin(\omega t + \phi_0) + \dfrac{F_0/m}{\omega^2 - \gamma^2}\cos(\gamma t)$

If $\omega/\gamma = m/n$ is rational, then $m/\omega = n/\gamma$ is a period.

22. If $\gamma = \omega$, we try $x_p = At\cos\omega t + Bt\sin\omega t$ as a particular solution of $x'' + \omega^2 x = \dfrac{F_0}{m}\cos\omega t$.

Substituting x_p into the equation, we have

$$(2B\omega - A\omega^2 t)\cos\omega t - (2A\omega + B\omega^2 t)\sin\omega t + \omega^2(At\cos\omega t + Bt\sin\omega t) = \frac{F_0}{m}\cos\omega t,$$

which gives $A = 0,\ B = \dfrac{F_0}{2\omega m}$, as required.

23. The characteristic equation is

$$r^2 + 2\alpha r + \omega^2 = 0; \qquad \text{the roots are } r_1,\ r_2 = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$$

Since $0 < \alpha < \omega$, $\alpha^2 < \omega^2$ and the roots are complex. Thus, $u_1(t) = e^{-\alpha t}\cos\beta t$, $u_2(t) = e^{-\alpha t}\sin\beta t$, where $\beta = \sqrt{\alpha^2 - \omega^2}$ are fundamental solutions, and the general solution is:

$$x(t) = e^{-\alpha t}(C_1\cos\beta t + C_2\sin\beta t); \quad \beta = \sqrt{\alpha}^2 - \omega^2$$

24. Straightforward computation.

25. Set $\omega = \gamma$ in the particular solution x_p given in Exercise 24. Then we have

$$x_p = \frac{F_0}{2\alpha\gamma m}\sin\gamma t$$

As $c = 2\alpha m \to 0^+$, the amplitude $\left|\frac{F_0}{2\alpha\gamma m}\right| \to \infty$

26. $\dfrac{F_0/m}{(\omega^2 - \gamma^2)^2 + 4\alpha^2\gamma^2}\left[(\omega^2 - \gamma^2)\cos\gamma t + 2\alpha\gamma\sin\gamma t\right]$

$$= \frac{F_0/m}{\sqrt{(\omega^2 - \gamma^2)^2 + 4\alpha^2\gamma^2}}\left[\frac{\omega^2 - \gamma^2}{\sqrt{(\omega^2 - \gamma^2)^2 + 4\alpha^2\gamma^2}}\cos\gamma t + \frac{2\alpha\gamma}{\sqrt{(\omega^2 - \gamma^2)^2 + 4\alpha^2\gamma^2}}\sin\gamma t\right]$$

Setting $\phi = \tan^{-1}\left(\dfrac{\omega^2 - \gamma^2}{2\alpha\gamma}\right)$, this expression becomes.

$$\frac{F_0/m}{\sqrt{(\omega^2 - \gamma^2)^2 + 4\alpha^2\gamma^2}}(\sin\phi\cos\gamma t + \cos\phi\sin\gamma t) = \frac{F_0/m}{\sqrt{(\omega^2 - \gamma^2)^2 + 4\alpha^2\gamma^2}}\sin(\gamma t + \phi)$$

27. $\left(\omega^2 - \gamma^2\right)^2 + 4\alpha^2\gamma^2 = \omega^4 + \gamma^4 + 2\gamma^2(2\alpha^2 - \omega^2)$ increases as γ increases.

28. (a) To maximize the amplitude, we need to minimize

$$(\omega^2-\gamma^2)^2+4\alpha^2\gamma^2=\omega^4-2(\omega^2-2\alpha^2)\gamma^2+\gamma^4.$$

This is a parabola in γ^2, and the minimum occurs when $\gamma^2=\omega^2-2\alpha^2$.

Therefore the maximum amplitude occurs when $\gamma=\sqrt{\omega^2-2\alpha^2}$

(b) $f=\dfrac{2\pi}{\gamma}=\dfrac{2\pi}{\sqrt{\omega^2-2\alpha^2}}$

(c) When $\gamma^2=\omega^2-2\alpha^2$, the amplitude is: $\dfrac{F_0/m}{\sqrt{(\omega^2-\gamma^2)^2+4\alpha^2\gamma^2}}=\dfrac{F_0/m}{2\alpha\sqrt{\omega^2-\alpha^2}}$

(d) $\sin 2\alpha=c$, the resonant amplitude in (c) can be rewritten $\dfrac{F_0}{c\sqrt{\omega^2-c^2/4m^2}}$.

This gets large as c gets small.