GRAPHS & DIGRAPHS
SECOND EDITION

GARY CHARTRAND
LINDA LESNIAK

GRAPHS & DIGRAPHS

The Wadsworth & Brooks/Cole Mathematics Series

M. Adams, V. Guillemin, *Measure Theory and Probability*
W. Beckner, A. Calderón, R. Fefferman, P. Jones, *Conference on Harmonic Analysis in Honor of Antoni Zygmund*
G. Chartrand, L. Lesniak, *Graphs & Digraphs*, Second Edition
J. Cochran, *Applied Mathematics: Principles, Techniques, and Applications*
W. Derrick, *Complex Analysis and Applications*, Second Edition
J. Dieudonné, *History of Algebraic Geometry*
R. Durrett, *Brownian Motion and Martingales in Analysis*
S. Fisher, *Complex Variables*
A. Garsia, *Topics in Almost Everywhere Convergence*
K. Stromberg, *An Introduction to Classical Real Analysis*
R. Salem, *Algebraic Numbers and Fourier Analysis*, and L. Carleson, *Selected Problems on Exceptional Sets*

GRAPHS & DIGRAPHS

Second Edition

Gary Chartrand *Western Michigan University*

Linda Lesniak *Drew University*

Wadsworth & Brooks/Cole Advanced Books & Software
Monterey, California
A division of Wadsworth, Inc.

To our friend and colleague

Farrokh Saba

Printed in the United States of America
10 9 8 7 6 5 4 3 2 1

Library of Congress Cataloging-in-Publication Data

Chartrand, Gary.
Graphs & digraphs.

(The Wadsworth & Brooks/Cole mathematics series)
Rev. ed. of: Graphs & digraphs / Mehdi Behzad, Gary Chartrand, Linda Lesniak-Foster. c1979.
Bibliography: p. 336
Includes index.
1. Graph theory. 2. Directed graphs. I. Lesniak, Linda. Graphs & digraphs.
III. Title. IV. Title: Graphs and digraphs.
V. Series.
QA166.C4525 1986 511'.5 86-1519

ISBN 0-534-06324-1

Sponsoring Editor: John Kimmel
Editorial Assistant: Maria Alsadi
Production: Greg Hubit Bookworks
Typesetting: Permanent Typesetting and Printing Co., Ltd.
Printing and Binding: Maple-Vail Book Manufacturing Group

Preface

The exciting and rapidly growing area of graph theory is rich in theoretical results as well as applications to real-world problems. In this edition of *Graphs & Digraphs*, as in the first, our major emphasis is on the theoretical aspects of graph theory, and we have included what we believe to be the most interesting and important results in the field. In addition, however, we have introduced the reader to types of problems that can be modeled by graphs and we have indicated efficient algorithms for their solutions. In keeping with our belief that a background emphasizing theory and proof techniques is indispensable for the student of graph theory, we have included careful proofs that the algorithms do, in fact, accomplish what they claim. Exercises reflecting the addition of these algorithms as well as a substantial number of new exercises have been added.

A second major change in this edition is the integration of graph and digraph theory. The material on digraph theory, self-contained in the first edition, is now developed parallel to that of (undirected) graphs. This allows, for example, the max-flow min-cut theorem to be introduced early in the text and then used to establish results on connectivity and matching.

This text is intended for an introductory sequence in graph theory at the senior or beginning graduate level. However, a one-semester course could easily be designed by selecting those topics of major importance and interest to the students involved. To facilitate such a choice in this edition, we have judiciously chosen a number of topics to introduce and develop in the exercises rather than in the text itself. Three topics that are introduced early in the text can be omitted with little effect on the material that follows, namely Section 2.4 on the Reconstruction Problem, Section 3.2 on n-ary trees, and Sections 4.4–4.6 on embedding graphs on surfaces of positive genus.

It is a pleasure to thank a number of individuals who assisted us with this edition in a variety of ways. The discussions we had with Farhad Shahrokhi on graph algorithms were very useful to us, and we are most appreciative of the time and effort he spent on our behalf. We are grateful for the suggestions made by Garry Johns, Paresh J. Malde, Ortrud R. Oellermann, Robert Rieper, and Farrokh Saba. The advice given to us by reviewers of this edition was very helpful; we are delighted to thank Ruth A. Bari, Ralph Faudree, Ronald J. Gould, Jerrold R. Griggs, F. C. Holroyd, Gary T. Myers, and Richard D. Ringeisen. Our gratitude goes to Margo Johnson for her consistently excellent typing. Finally, we thank the staff of Wadsworth & Brooks/Cole Advanced Books & Software, particularly John Kimmel, for their interest in and assistance with this edition.

Gary Chartrand
Linda Lesniak

Contents

Chapter One

Graphs and Digraphs

In many disciplines we are faced with situations in which we want to find out how or whether a finite number of objects are related. If the relation is symmetric, we can model the situation by a graph. More generally, we can model the structure by a digraph. Hence, graphs and digraphs occur naturally and often. We begin our study with these two basic concepts.

1.1 Graphs

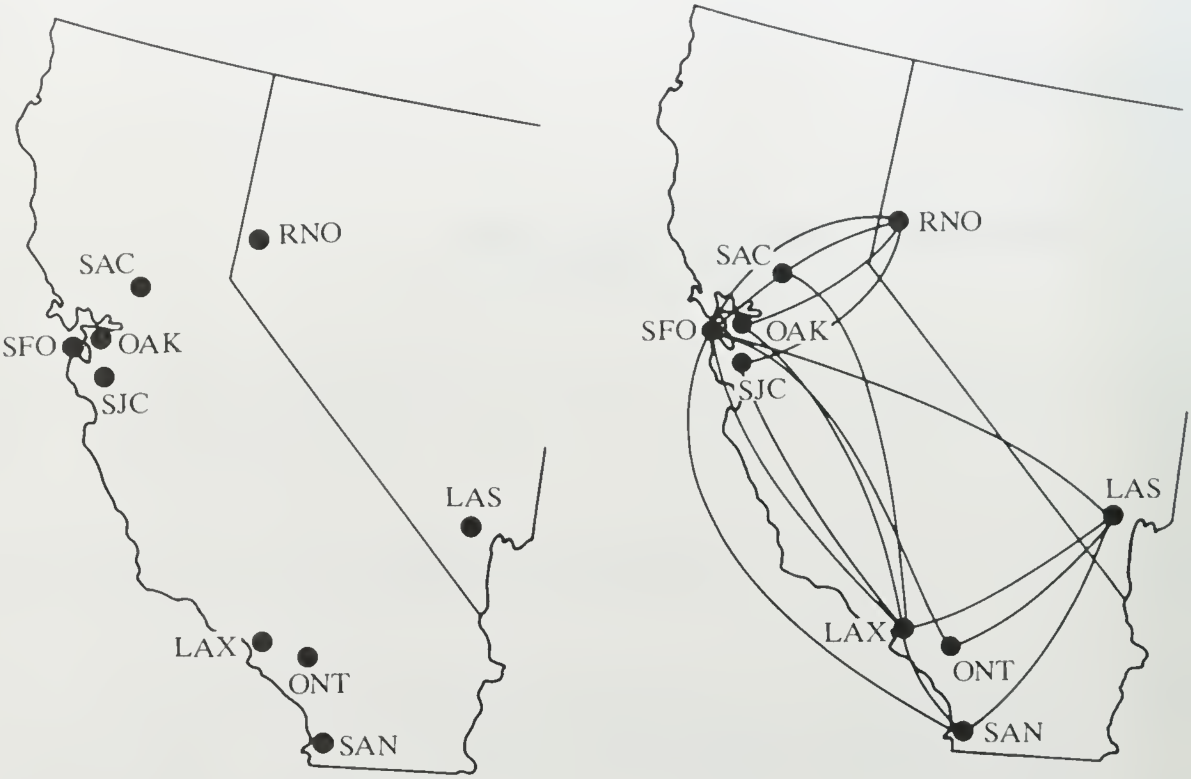
Many situations and structures give rise to graphs. Before we offer a precise definition of a graph, we present a few examples.

Assume that a California-based airline services several cities within California as well as Reno and Las Vegas, Nevada. These cities are indicated on the map shown in Figure 1.1(a).

This airline has several direct routes between certain pairs of these cities; the flying patterns are illustrated in Figure 1.1(b). The diagram in Figure 1.2(a) representing the cities serviced and the flying routes is a graph.

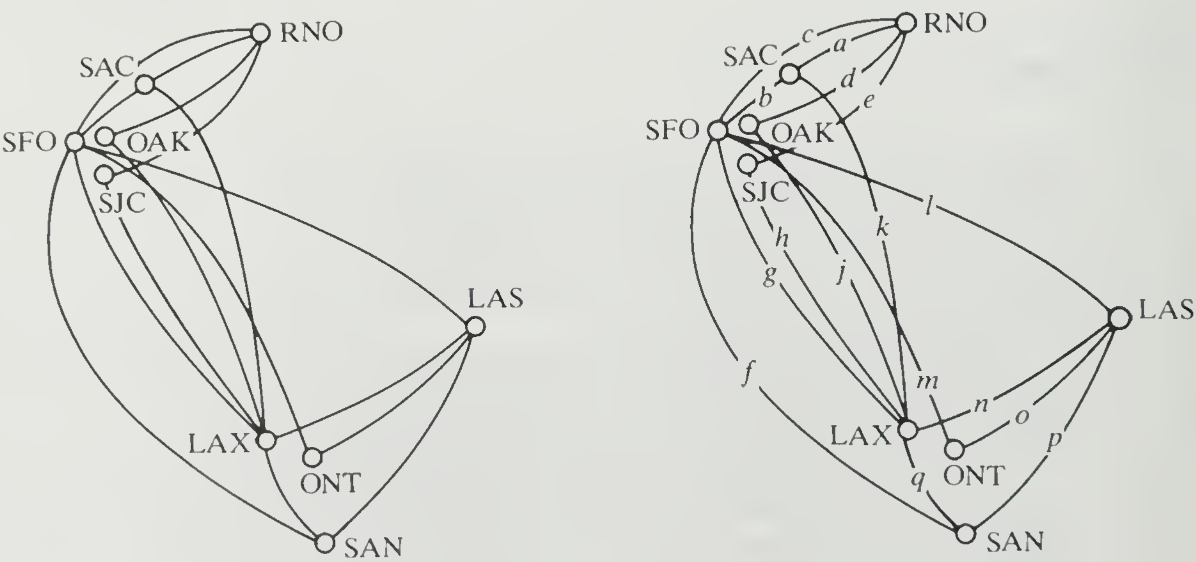
At times it is convenient to include additional information in a graph. For example, we might want to know the cost of each direct route. These costs (or weights) can be assigned to the edges of Figure 1.2(a), producing the network of Figure 1.2(b), where the labels a, b, and so on represent the costs.

By inspecting Figure 1.2, we can answer questions such as whether one can fly from San Diego to Reno and, if so, which route is least expensive. Of course, as graphs become more complex, solutions by inspection are no longer



(a) (b)

Figure 1.1 *Airline routes*



(a) (b)

Figure 1.2 *A graph and a network*

feasible. In Chapter 2 we will discuss an efficient means of finding a "shortest path" in any graph.

As a second example, assume that a business is expanding and plans to add several new positions; namely, a draftsman, an engineer, a computer programmer, a data analyst, and an assistant personnel manager. Seven individuals apply for these five positions, some of whom have the qualifications for two or more of the positions. This situation can be represented by the graph shown in Figure 1.3, where five points (denoted DRA, ENG, PRO, ANL, and PER) are used to indicate the positions and seven points (1, 2, . . . , 7) are used to indicate the applicants. Each point on the top of the graph represents an applicant, and each point on the bottom represents a position. A line is drawn between two points if the person is qualified for that position. A question that might be of interest is whether there are five individuals, from among the seven, who can be hired to fill all five positions. In graph theoretic terms, we are asking whether the set of jobs can be "matched" to a subset of the applicants. An algorithm that answers such questions will be discussed in Chapter 8.

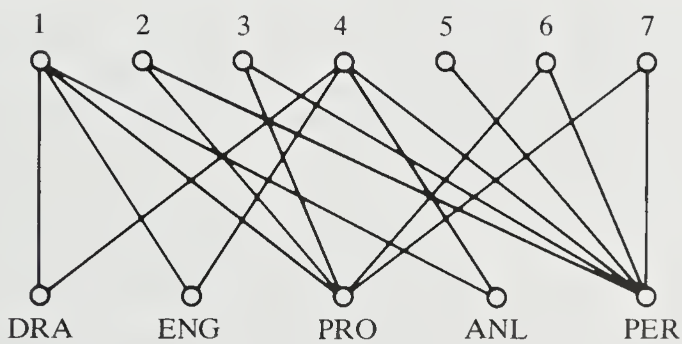


Figure 1.3 *A graph of jobs and applicants*

As a last example, let us suppose that eight experimental chemicals (A, B, . . . , H) are to be stored in large (expensive) storage bins. Some chemicals have the potential to interact with each other and, consequently, should not be stored in the same bin. This situation is illustrated in the graph of Figure 1.4, where each chemical is represented by a point and two points are joined by a line if the corresponding chemicals should not be stored together. We might ask: What is the least number of storage bins that are needed to store all eight chemicals? This type of question is of particular interest to graph theorists. At the present, the only known algorithms to solve problems of this type are very inefficient, and many mathematicians believe that no efficient solution exists. We will see in Chapter 10 an example of an efficient "heuristic" algorithm for this problem; that is, an algorithm that describes a *small*, but not the *least*, number of bins that will suffice.

Each of the examples discussed so far was based on a collection of objects (cities, people, jobs, chemicals), and relationships between certain pairs. These

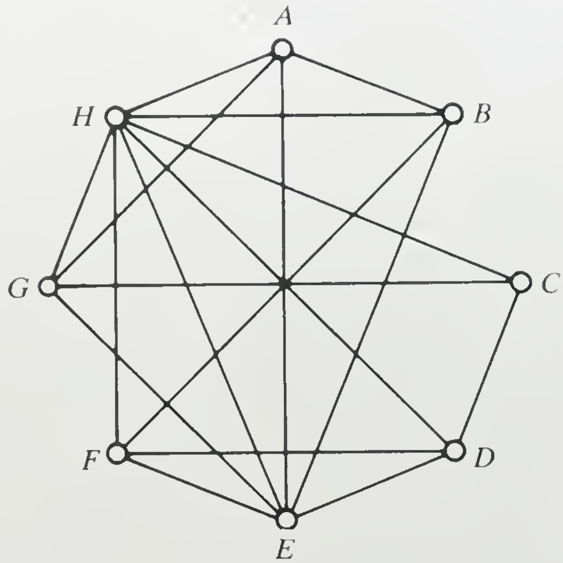


Figure 1.4 *A chemical interaction graph*

ideas are easily abstracted to produce the concept of a graph.

A *graph* G is a finite nonempty set of objects called *vertices* (the singular is *vertex*) together with a (possibly empty) set of unordered pairs of distinct vertices of G called *edges*. The *vertex set* of G is denoted by $V(G)$, while the *edge set* is denoted by $E(G)$.

The edge $e = \{u, v\}$ is said to *join* the vertices u and v. If $e = \{u, v\}$ is an edge of a graph G, then u and v are *adjacent vertices*, while u and e are *incident*, as are v and e. Furthermore, if e_1 and e_2 are distinct edges of G incident with a common vertex, then e_1 and e_2 are *adjacent edges*. It is convenient to henceforth denote an edge by uv or vu rather than by $\{u, v\}$.

The cardinality of the vertex set of a graph G is called the *order* of G and is denoted by $p(G)$, or more simply, p, while the cardinality of its edge set is the *size* of G and is denoted by $q(G)$ or q. A *(p, q) graph* has order p and size q.

It is customary to define or describe a graph by means of a diagram in which each vertex is represented by a point (which we draw as a small circle) and each edge $e = uv$ is represented by a line segment or curve joining the points corresponding to u and v.

A graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_p\}$ can also be described by means of matrices. One such matrix is the $p \times p$ *adjacency matrix* $A(G) = [a_{ij}]$, where

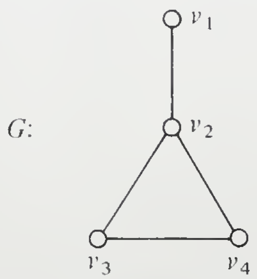
$$a_{ij} = \begin{cases} 1 & \text{if } v_iv_j \in E(G) \\ 0 & \text{if } v_iv_j \notin E(G) \end{cases}$$

Thus, the adjacency matrix of a graph G is a symmetric (0, 1) matrix having zero entries along the main diagonal.

For example, a graph G is defined by the sets

$$V(G)=\{v_1, v_2, v_3, v_4\} \quad \text{and} \quad E(G)=\{v_1v_2, v_2v_3, v_2v_4, v_3v_4\}.$$

A diagram of this graph and its adjacency matrix are shown in Figure 1.5.



$$A=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Figure 1.5 *A graph and its adjacency matrix*

The adjacency matrix representation of a graph is often convenient if one intends to use a computer to obtain some information or solve a problem concerning the graph. On the other hand, an adjacency matrix contains a great deal of extraneous data—often many 0's and twice as many 1's as needed. This unsatisfactory characteristic of the adjacency matrix is often alleviated by inputting the graph in a variety of other manners. For example, one could input the edge set and the order, or one could input adjacency arrays, where the vertices adjacent to a given vertex are listed. There are several other possibilities. The manner in which a graph is input normally depends on the problem to be solved and affects the algorithm and method chosen to solve the problem.

Two graphs often have the same structure, differing only in the way their vertices and edges are labeled or in the way they are drawn. To make this idea more exact, we introduce the concept of isomorphism. A graph G_1 is *isomorphic* to a graph G_2 if there exists a one-to-one mapping ϕ, called an *isomorphism*, from $V(G_1)$ onto $V(G_2)$ such that ϕ preserves adjacency; that is, $uv \in E(G_1)$ if and only if $\phi u \phi v \in E(G_2)$. It is easy to see that "is isomorphic to" is an equivalence relation on graphs; hence, this relation divides the collection of all graphs into equivalence classes, two graphs being *nonisomorphic* if they are in different equivalence classes. If G_1 is isomorphic to G_2, then we say G_1 and G_2 are *isomorphic* and write $G_1 \cong G_2$.

Each of the graphs G_i, $i=1, 2, 3$, of Figure 1.6 is a $(6, 9)$ graph. Here, G_1 and G_2 are isomorphic. For example, the mapping $\phi: V(G_1) \rightarrow V(G_2)$ defined by

$$\phi v_1 = v_1, \quad \phi v_2 = v_3, \quad \phi v_3 = v_5, \quad \phi v_4 = v_2, \quad \phi v_5 = v_4, \quad \phi v_6 = v_6$$

is an isomorphism. On the other hand, $G_1 \not\cong G_3$ since, for example, G_3 contains three pairwise adjacent vertices whereas G_1 does not. Of course, $G_2 \not\cong G_3$.

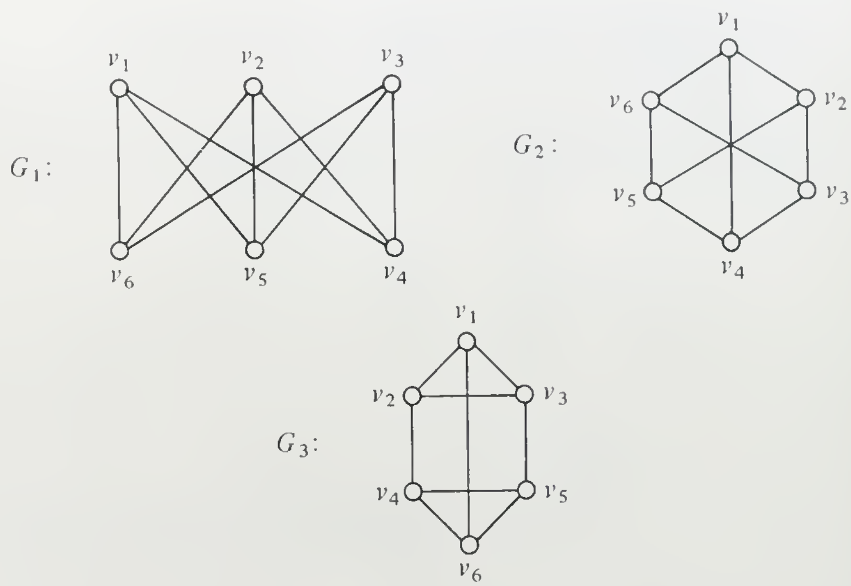


Figure 1.6 *Isomorphic and nonisomorphic graphs*

If G is a (p, q) graph, then $p \geq 1$ and $0 \leq q \leq \binom{p}{2} = p(p-1)/2$. There is only one (1, 0) graph (up to isomorphism), and this is referred to as the *trivial graph*. A *nontrivial graph* then has $p \geq 2$.

Two graphs G_1 and G_2 are *identical*, denoted $G_1 = G_2$, if $V(G_1) = V(G_2)$ and $E(G_1) = E(G_2)$. Clearly, two graphs may be isomorphic yet not identical. The graphs G_1 and G_2 of Figure 1.6 are not identical (even though $V(G_1) = V(G_2)$ and $G_1 \cong G_2$) since, for example, $v_1v_5 \in E(G_1)$ and $v_1v_5 \notin E(G_2)$.

All 20 nonidentical graphs of order 4 and size 3, having vertex set {1, 2, 3,4}, are shown in Figure 1.7. Among these graphs, there are only three

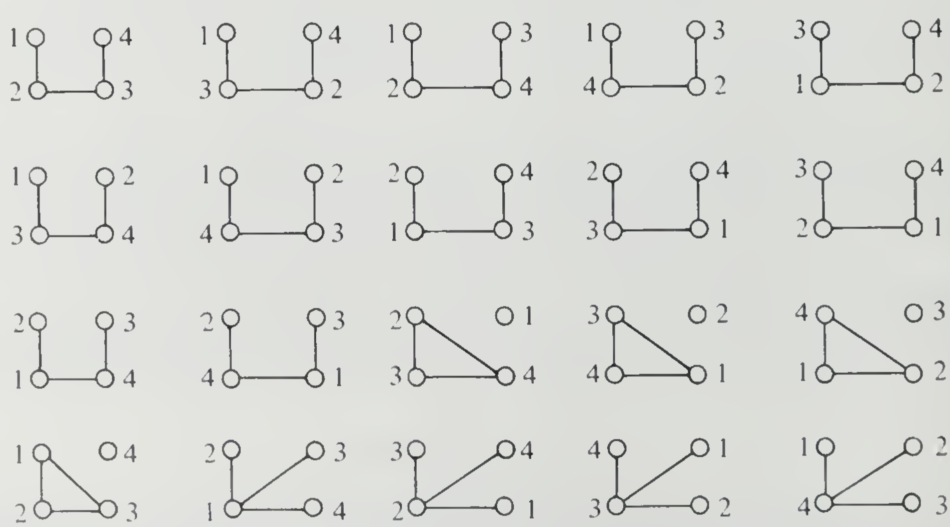


Figure 1.7 *The nonidentical* (4, 3) *graphs having vertex set* {1, 2, 3, 4}

nonisomorphic classes of graphs. The total number of nonidentical graphs having vertex set {1, 2, 3, 4} is 64; in fact, the total number of nonidentical graphs of order p with the same vertex set V is $2^{p(p-1)/2}$. This is obvious for $p=1$. If $p \geq 2$ and G is a graph with vertex set $V(G)$, then for each pair u, v of distinct vertices, there are two possibilities depending on whether uv is or is not an edge of G. Since there are $p(p-1)/2$ distinct pairs of vertices, there are $2^{p(p-1)/2}$ such nonidentical graphs G.

With the exception of the order and the size, the numbers that one encounters most frequently in the study of graphs are the degrees of its vertices. The *degree of a vertex* v in a graph G is the number of edges of G incident with v. The degree of a vertex v in G is denoted $\deg_G v$ or simply $\deg v$ if G is clear from the context. A vertex is called *odd* or *even* depending on whether its degree is odd or even. A vertex of degree 0 in G is called an *isolated vertex* and a vertex of degree 1 is an *end-vertex* of G. In Figure 1.8, a graph G is shown together with the degrees of its vertices.

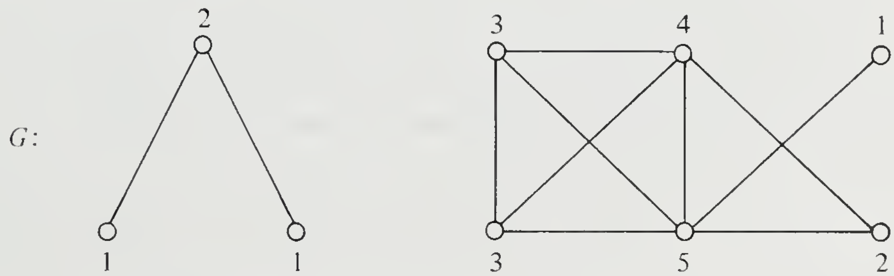


Figure 1.8 *The degrees of the vertices of a graph*

Observe that for the graph G in Figure 1.8, $p=9$ and $q=11$, while the sum of the degrees of its nine vertices is 22. The fact that this last number equals $2q$ for the graph G is not merely a coincidence. Every edge is incident with two vertices; hence, when the degrees of the vertices are summed, each edge is counted twice. We state this as our first theorem, which, not so coincidentally, is sometimes called "The First Theorem of Graph Theory".

Theorem 1.1 *Let G be a (p, q) graph where $V(G) = \{v_1, v_2, \ldots, v_p\}$. Then*

$$\sum_{i=1}^{p} \deg v_i = 2q.$$

This result has an interesting consequence.

Corollary 1.1 *In any graph, there is an even number of odd vertices.*

Proof Let G be a graph of size q. Also, let W be the set of odd vertices of G and let U be the set of even vertices of G. By Theorem 1.1,

$$\sum_{v \in V(G)} \deg v = \sum_{v \in W} \deg v + \sum_{v \in U} \deg v = 2q.$$

Certainly, $\Sigma_{v \in U} \deg v$ is even; hence $\Sigma_{v \in W} \deg v$ is even, implying that $|W|$ is even and thereby proving the corollary. ■

Frequently, a graph under study is contained within some larger graph also being investigated. We consider several instances of this now. A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; in such a case, we also say that G is a *supergraph* of H. If G and H are graphs, not all of whose vertices are labeled, then H is also considered to be a subgraph of G if any unlabeled vertices can be labeled so that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If H is a subgraph of G, then we write $H \subset G$.

The simplest type of subgraph of a graph G is that obtained by deleting a vertex or edge. If $v \in V(G)$ and $|V(G)| \geq 2$, then $G - v$ denotes the subgraph with vertex set $V(G) - \{v\}$ and whose edges are all those of G not incident with v; if $e \in E(G)$, then $G - e$ is the subgraph having vertex set $V(G)$ and edge set $E(G) - \{e\}$. The deletion of a set of vertices or set of edges is defined analogously. These concepts are illustrated in Figure 1.9.

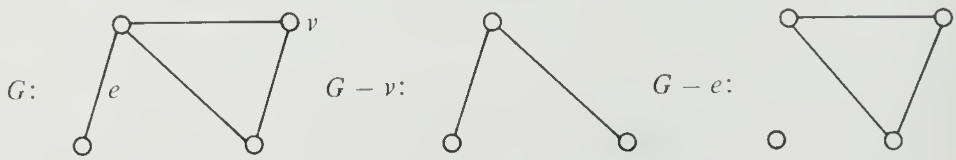


Figure 1.9 *The deletion of an element of a graph*

If u and v are nonadjacent vertices of a graph G, then $G + f$, where $f = uv$, denotes the graph with vertex set $V(G)$ and edge set $E(G) \cup \{f\}$. Clearly, $G \subset G + f$.

We have seen that $G - e$ has the same vertex set as G and that G has the same vertex set as $G + f$. Whenever a subgraph H of a graph G has the same order as that of G, then H is called a *spanning subgraph* of G.

Among the most important subgraphs we shall encounter are the "induced subgraphs". If U is a nonempty subset of the vertex set $V(G)$ of a graph G, then the subgraph $\langle U \rangle$ of G *induced* by U is the graph having vertex set U and whose edge set consists of those edges of G incident with two elements of U. A subgraph H of G is called *vertex-induced* or *induced*, denoted $H < G$, if $H \cong \langle U \rangle$ for some subset U of $V(G)$. Similarly, if F is a nonempty subset of $E(G)$, then the subgraph $\langle F \rangle$ *induced* by F is the graph whose vertex set consists of those vertices of G incident with at least one edge of F and whose edge set is F. A subgraph H of G is *edge-induced* if $H \cong \langle F \rangle$ for some subset F of $E(G)$. It is a simple consequence of the definitions that every induced subgraph of a graph G can be obtained by removing vertices from G

while every subgraph of G can be obtained by deleting vertices and edges. These concepts are illustrated in Figure 1.10 for the graph G, where

$$V(G)=\{v_1, v_2, v_3, v_4, v_5, v_6\}, \quad U=\{v_1, v_2, v_5\}, \quad \text{and} \quad F=\{v_1v_4, v_2v_5\}.$$

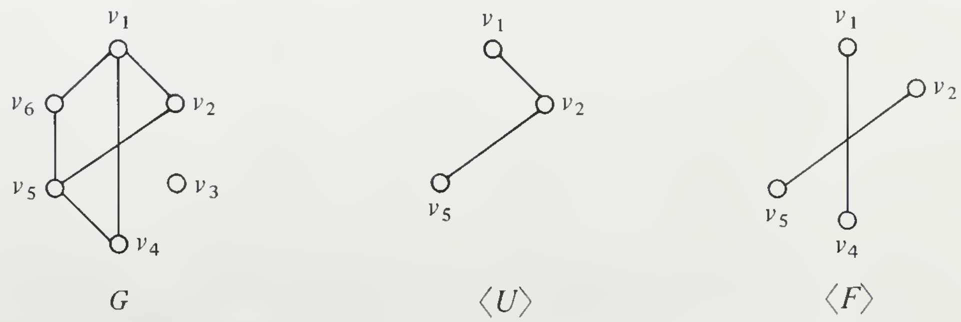


Figure 1.10 *Vertex-induced and edge-induced subgraphs*

The reader should be aware of possible confusion between nonisomorphic and nonidentical subgraphs. For example, in graph G_3 of Figure 1.6, how many subgraphs of G_3 have three vertices and three edges? The answer is obviously "two", since what is certainly desired here is the number of nonidentical such subgraphs. The reader could incorrectly give an answer of "one" here, interpreting the question as the number of nonisomorphic such subgraphs. Hence the reader must consider carefully the context in which the question is posed.

There are certain classes of graphs that occur so often that they deserve special mention and in some cases, special notation. We describe the most prominent of these in this section.

A graph G is *regular of degree r* if for each vertex v of G, $\deg v = r$; such graphs are also called *r-regular*. The 3-regular graphs are referred to as *cubic* graphs. A graph is *complete* if every two of its vertices are adjacent. A complete (p, q) graph is therefore a regular graph of degree $p-1$ having $q=p(p-1)/2$; we denote this graph by K_p. In Figure 1.11 are shown all (nonisomorphic) regular graphs with $p=4$, including the complete graph $G_3 \cong K_4$.

The *complement* $\bar{G}$ of a graph G is the graph with vertex set $V(G)$ such that two vertices are adjacent in $\bar{G}$ if and only if these vertices are not adjacent in G. Hence, if G is a (p, q) graph, then $\bar{G}$ is a $(p, \bar{q})$ graph, where $q+\bar{q}=\binom{p}{2}$. In Figure 1.11, the graphs G_0 and G_3 are complementary, as are G_1 and G_2. The complement $\bar{K}_p$ of the complete graph K_p has p vertices and no edges and is referred to as the *empty graph* of order p. A graph G is *self-complementary* if $G \cong \bar{G}$.

A graph G is *n-partite*, $n \geq 1$, if it is possible to partition $V(G)$ into n subsets $V_1, V_2, \ldots, V_n$ (called *partite sets*) such that every element of $E(G)$ joins a vertex of V_i to a vertex of V_j, $i \neq j$. If G is a 1-partite graph of order p, then $G \cong \bar{K}_p$. For $n=2$, such graphs are called *bipartite graphs*; this class of

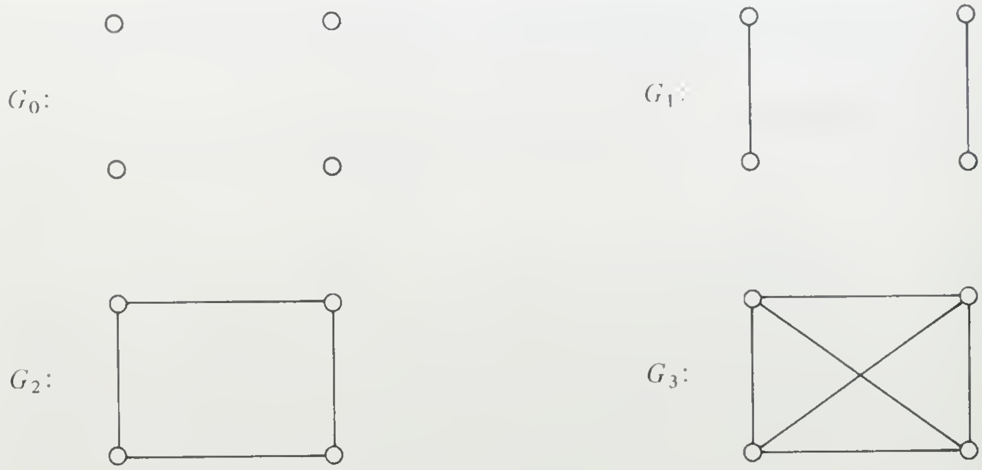


Figure 1.11 *The regular graphs of order 4*

graphs is particularly important and will be encountered many times. In Figure 1.12, a bipartite graph G_1 is shown; a second graph G_2, identical to G_1, is also given to emphasize the bipartite character of G_1. If G is a regular bipartite graph with partite sets V_1 and V_2, then $|V_1| = |V_2|$ (see Exercise 1.10; also see [ACLO1]).

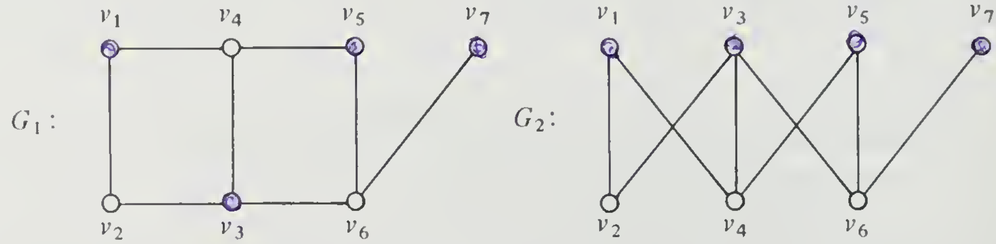


Figure 1.12 *A bipartite graph*

A *complete n-partite graph* G is an n-partite graph with partite sets V_1, V_2, $\ldots$, V_n having the added property that if $u \in V_i$ and $v \in V_j$, $i \neq j$, then $uv \in E(G)$. If $|V_i| = p_i$, then this graph is denoted by $K(p_1, p_2, \ldots, p_n)$. (The order of the numbers $p_1, p_2, \ldots, p_n$ is not important.) Note that a complete n-partite graph is complete if and only if $p_i = 1$ for all i, in which case it is K_n. If $p_i = t$ for all i, then the complete n-partite graph is regular and is also denoted by $K_{n(t)}$. Thus, $K_{n(1)} \cong K_n$.

A complete bipartite graph with partite sets V_1 and V_2, where $|V_1| = m$ and $|V_2| = n$, is then denoted by $K(m, n)$. The graph $K(1, n)$ is called a *star*.

There are many ways of combining graphs to produce new graphs. We next describe some binary operations defined on graphs. This discussion introduces notation that will prove very useful in giving examples. In the following definitions, we assume that G_1 and G_2 are two graphs with disjoint vertex sets.

The *union* $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. If a graph G consists of n (≥ 2) disjoint copies of a graph H, then we write $G = nH$. The graph $2K_1 \cup 3K_2 \cup K(1, 3)$ is shown in Figure 1.13.

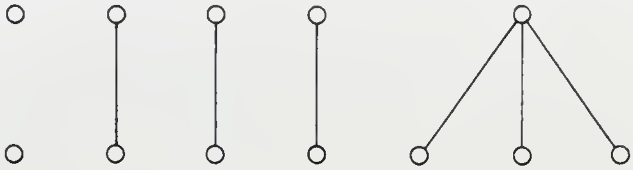


Figure 1.13 *The union of graphs*

The *join* $G = G_1 + G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and

$$E(G) = E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1) \quad \text{and} \quad v \in V(G_2)\}.$$

Using the join operation, we can see that $K(m, n) \cong \bar{K}_m + \bar{K}_n$. Another illustration is given in Figure 1.14.

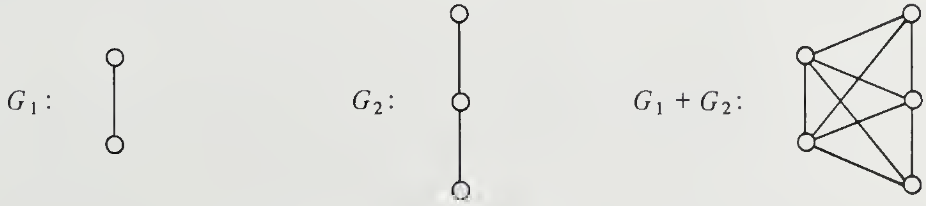


Figure 1.14 *The join of two graphs*

The *cartesian product* $G = G_1 \times G_2$ has $V(G) = V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either

$$u_1 = v_1 \quad \text{and} \quad u_2v_2 \in E(G_2)$$

or

$$u_2 = v_2 \quad \text{and} \quad u_1v_1 \in E(G_1).$$

An example is shown in Figure 1.15.

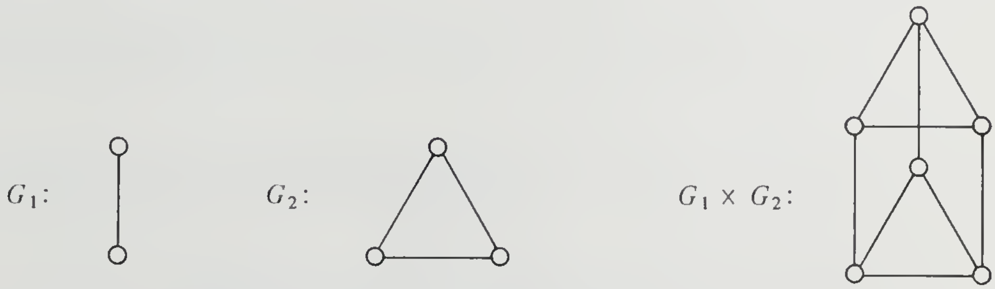


Figure 1.15 *The cartesian product of two graphs*

An important class of graphs is defined in terms of cartesian products; these are the "cubes". The *n-cube* Q_n is the graph K_2 if $n=1$, while for $n>1$, Q_n is defined inductively as $Q_{n-1} \times K_2$. The cube Q_n can also be considered as that graph whose vertices are labeled by the binary n-tuples $(a_1, a_2, \ldots, a_n)$ (that is, a_i is 0 or 1 for $1 \leq i \leq n$) and such that two vertices are adjacent if and only if their corresponding n-tuples differ in precisely one position. It is easily observed that Q_n is an n-regular graph of order 2^n. The n-cubes, $n = 1, 2$, and 3, are shown in Figure 1.16 with appropriate labelings.

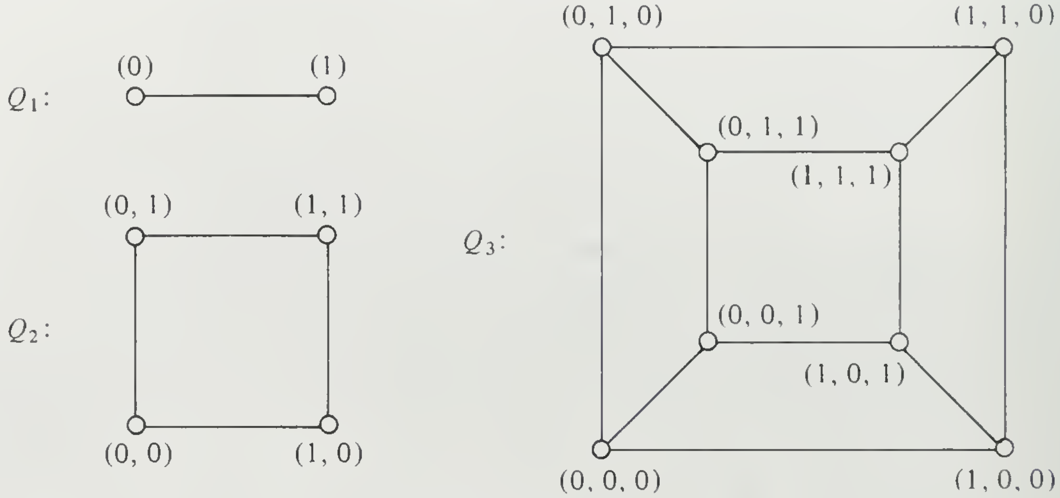


Figure 1.16 *Cubes*

Exercises 1.1

1.1 Figure 1.6 shows two nonisomorphic (6, 9) graphs. Give another example of two nonisomorphic graphs H_1 and H_2 such that $p(H_1) = p(H_2)$ and $q(H_1) = q(H_2)$.

1.2 Determine all nonisomorphic graphs of order 5.

1.3 Determine all nonidentical (4, 4) graphs having vertex set $\{1, 2, 3, 4\}$.

1.4 Prove or disprove: Let $V = \{v_1, v_2, \ldots, v_p\}$. The number of nonidentical (p, q) graphs having vertex set V equals the number of nonidentical $(p, \binom{p}{2} - q)$ graphs having vertex set V.

1.5 Let p be a given positive integer, and let m and n be nonnegative integers such that $m + n = p$ and n is even. Show that there exists a graph G of order p having m even vertices and n odd vertices.

1.6 Determine all nonisomorphic subgraphs of the graph G of Figure 1.9. How many of these are induced? How many are edge-induced?

1.7 For a graph G, let $V_1, V_2 \subseteq V(G)$, where V_1, V_2 and $V_1 \cap V_2$ are nonempty. Prove that

(a) $E(\langle V_1 \rangle) \cup E(\langle V_2 \rangle) \subseteq E(\langle V_1 \cup V_2 \rangle)$, and

(b) $E(\langle V_1 \rangle) \cap E(\langle V_2 \rangle) = E(\langle V_1 \cap V_2 \rangle)$.

1.8 Show that it is *not* always the case that every edge-induced subgraph of a graph G can be obtained by removing edges from G.

1.9 How many subgraphs of the graph G of Figure 1.10 contain four vertices and four edges?

1.10 Prove that if G is a regular bipartite graph with partite sets V_1 and V_2, then $|V_1| = |V_2|$.

1.11 Let G be a graph such that n is the largest degree of its vertices. Prove that there exists a supergraph H of G such that $G < H$ and H is n-regular.

1.12 If $H < G$, does it follow that $\bar{H} < \bar{G}$?

1.13 Prove that there exists a self-complementary graph of order p if and only if $p \equiv 0 \pmod 4$ or $p \equiv 1 \pmod 4$.

1.14 Determine all self-complementary graphs of order 5 or less.

1.15 Let G be a self-complementary graph of order p, where $p \equiv 1 \pmod 4$. Prove that G contains at least one vertex of degree $(p-1)/2$. (*Hint:* Prove the stronger result that G contains an odd number of vertices of degree $(p-1)/2$.)

1.16 The *eigenvalues* of a graph G of order p are the eigenvalues of its adjacency matrix; that is, if G has adjacency matrix A, then the eigenvalues of G are the p (not necessarily distinct) numbers satisfying the determinant equation

$$\det(\lambda I_p - A) = 0$$

where I_p is the $p \times p$ identity matrix.
Determine the eigenvalues of:

(a) K_3, (b) $K(1, 2)$, and (c) $K(1, 3)$.

1.17 Let G be a nonempty graph with the property that whenever $uv \notin E(G)$ and $vw \notin E(G)$, then $uw \notin E(G)$. Prove that G has this property if and only if G is a complete n-partite graph for some $n \geq 2$.

1.2 Digraphs

A *directed graph* or *digraph* D is a finite nonempty set of objects called *vertices* together with a (possibly empty) set of ordered pairs of distinct vertices of D called *arcs* or *directed edges.* As with graphs, the vertex set of D is denoted by $V(D)$ and the arc set is denoted by $E(D)$. A digraph D with $V(D) = \{u, v, w\}$ and $E(D) = \{(u, w), (w, u), (u, v)\}$ is illustrated in Figure 1.17. Observe that when a digraph is described by means of a diagram, the "direction" of each arc is indicated by an arrowhead.

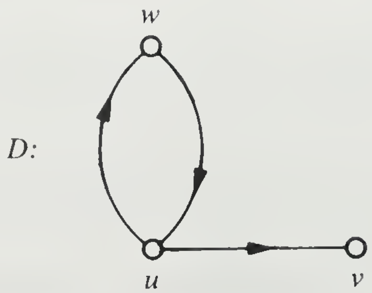


Figure 1.17 *A digraph*

Digraphs are more natural (and useful) than graphs for representing situations in which order or direction is involved in the relationships between pairs of objects. For example, Figure 1.18(a) indicates the street system of a small town, including one-way streets (unmarked streets are two-way). In Figure 1.18(b), the vertices correspond to street intersections and there is an arc from u to v if it is legal to drive from the intersection associated with u to

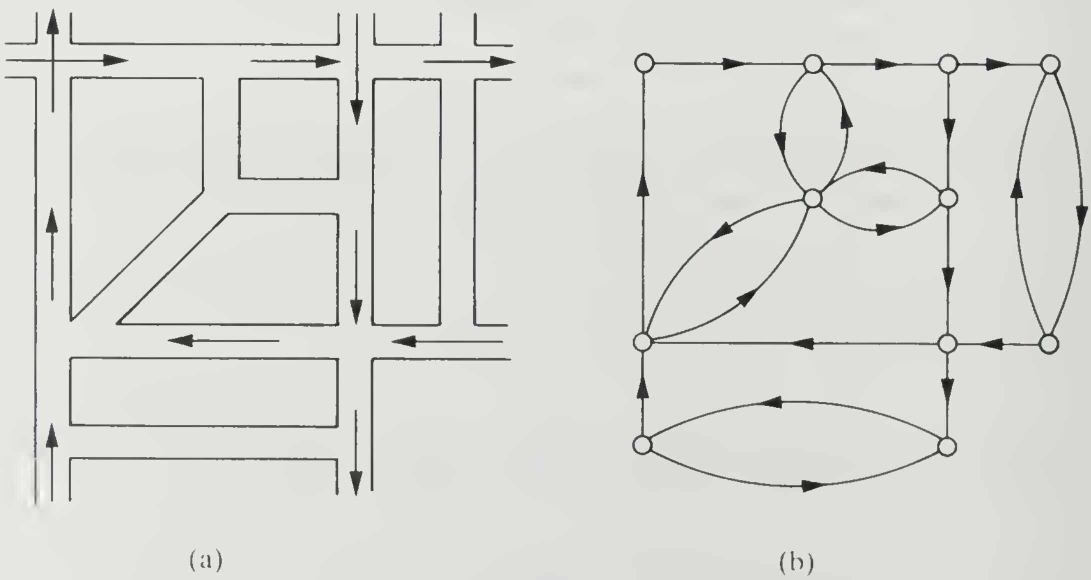


Figure 1.18 *A street system digraph*

the intersection associated with v (without passing through any other intersection).

Flowcharts that display the logical flow possibilities in computer programs are another example of the use of digraphs to represent a situation.

The terminology used in discussing digraphs is quite similar to that used for graphs. The cardinality of the vertex set of a digraph D is called the *order* of D and is denoted by $p(D)$, or simply p. The *size* $q(D)$ (or q) of D is the cardinality of its arc set. A *(p, q) digraph* is a digraph D with order p and size q.

If $a = (u, v)$ is an arc of a digraph D, then a is said to *join u and v*. We further say that a is *incident from u* and *incident to v*, while u is *incident to a* and v is *incident from a*. Moreover, u is said to be *adjacent to v* and v is *adjacent from u*. In the digraph D of Figure 1.17, vertex u is adjacent to vertex v, but v is *not* adjacent to u. Two vertices u and v of a digraph D are *nonadjacent* if u is neither adjacent to nor adjacent from v in D.

The *outdegree* od v of a vertex v of a digraph D is the number of vertices of D that are adjacent from v. The *indegree* id v of v is the number of vertices of D adjacent to v. The *degree* deg v of a vertex v of D is defined by

$$\deg v = \text{od } v + \text{id } v.$$

In the digraph D of Figure 1.17, od $u = 2$, id $u =$ id $v =$ id $w =$ od $w = 1$, while od $v = 0$. For the same digraph, deg $u = 3$, deg $w = 2$, and deg $v = 1$.

We now present the "First Theorem of Digraph Theory".

Theorem 1.2 *If D is a digraph of order p and size q with $V(D) = \{v_1, v_2, \ldots, v_p\}$, then*

$$\sum_{i=1}^{p} \text{od } v_i = \sum_{i=1}^{p} \text{id } v_i = q.$$

Proof When the outdegrees of the vertices are summed, each arc is counted once, since every arc is incident *from* exactly one vertex. Similarly, when the indegrees are summed, an arc is counted just once since every arc is incident *to* a single vertex. ■

A digraph D_1 is *isomorphic* to a digraph D_2 if there exists a one-to-one mapping ϕ, called an *isomorphism*, from $V(D_1)$ onto $V(D_2)$ such that $(u, v) \in E(D_1)$ if and only if $(\phi u, \phi v) \in E(D_2)$. The relation "is isomorphic to" is an equivalence relation on digraphs. Thus, this relation partitions the set of all digraphs into equivalence classes; two digraphs are *nonisomorphic* if they belong to different equivalence classes. If D_1 is isomorphic to D_2, then we say D_1 and D_2 are *isomorphic* and write $D_1 \cong D_2$.

There is only one (1, 0) digraph (up to isomorphism); this is the *trivial digraph*. Also, there is only one (2, 0), (2, 1), and (2, 2) digraph (up to

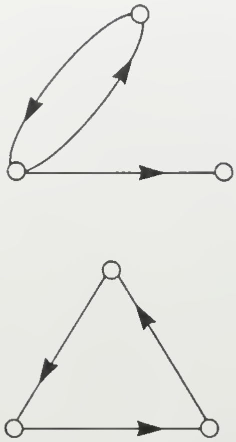


Figure 1.19 *The* (3, 3) *digraphs*

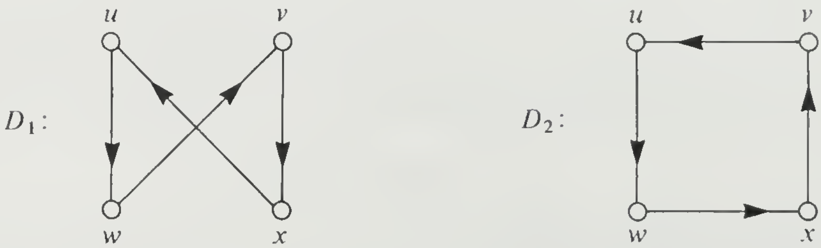


Figure 1.20 *Isomorphic, nonidentical digraphs*

isomorphism). There are four (3, 3) digraphs, and they are shown in Figure 1.19.

Two digraphs D_1 and D_2 are *identical*, written $D_1 = D_2$, if $V(D_1) = V(D_2)$ and $E(D_1) = E(D_2)$. Two identical digraphs are necessarily isomorphic, but not conversely. The digraphs of Figure 1.20 are isomorphic, but not identical.

A digraph D_1 is a *subdigraph* of a digraph D if $V(D_1) \subseteq V(D)$ and $E(D_1) \subseteq E(D)$. If D_1 and D are digraphs, not all of whose vertices are labeled, then D_1 is also considered to be a subdigraph of D if any unlabeled vertices can be labeled so that $V(D_1) \subseteq V(D)$ and $E(D_1) \subseteq E(D)$. We write $D_1 \subset D$ to indicate that D_1 is a subdigraph of D. A subdigraph D_1 of D is a *spanning subdigraph* if D_1 has the same order as D.

If D is a nontrivial digraph and $v \in V(D)$, then $D - v$ is that digraph with vertex set $V(D) - \{v\}$ and whose arcs are all those of D that are neither incident to nor from v. If $a \in E(D)$, then $D - a$ is the subdigraph with vertex set $V(D)$ and arc set $E(D) - \{a\}$. The deletion of a set of vertices or set of arcs is defined analogously.

If D is a digraph such that $u, v \in V(D)$ and $(u, v) \notin E(D)$, then the digraph $D + a$, where $a = (u, v)$, has vertex set $V(D)$ and arc set $E(D) \cup \{a\}$.

If U is a nonempty subset of the vertex set of a digraph D, then the subdigraph $\langle U \rangle$ of D *induced* by U is that digraph having vertex set U and whose arc set consists of all those arcs of D joining vertices of U. A subdigraph D_1 is said to be *induced* and denoted $D_1 < D$ if $D_1 \cong \langle U \rangle$ for some subset U of $V(D)$. If F is a nonempty subset of $E(D)$, then the subdigraph $\langle F \rangle$ *induced* by F is that digraph whose vertex set consists of those vertices of D incident to or from at least one arc of F and whose arc set is F. A subdigraph D_1 of D is *arc-induced* if $D_1 \cong \langle F \rangle$ for some subset F of $E(D)$. As with graphs, every induced subdigraph of a digraph D can be obtained by removing vertices from D and every subdigraph of D can be produced by removing vertices and arcs. We illustrate these ideas in Figure 1.21 for the digraph D, where

$$V(D) = \{v_1, v_2, v_3, v_4\}, \quad U = \{v_1, v_2, v_3\}, \quad \text{and}$$
$$F = \{(v_1, v_2), (v_2, v_4)\}.$$

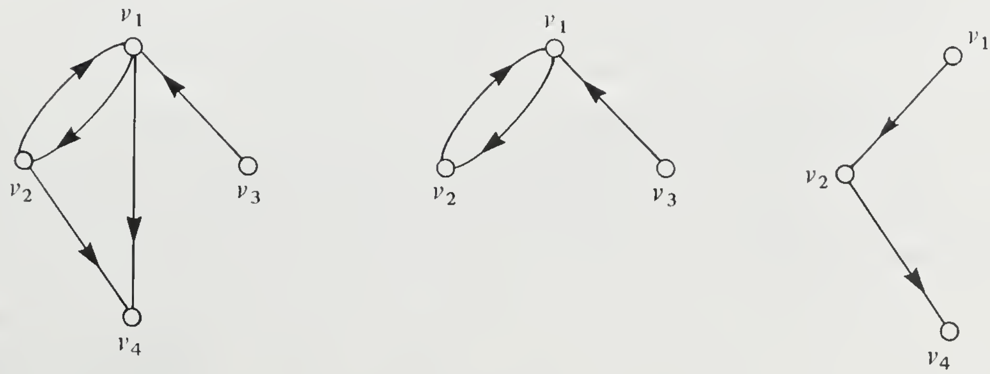


Figure 1.21 *Induced and arc-induced subdigraphs*

We now consider certain types of digraphs that occur regularly in our discussions. A digraph D is called *symmetric* if, whenever (u, v) is an arc of D, then (v, u) is as well. There is a natural one-to-one correspondence between the set of symmetric digraphs and the set of graphs. A digraph D is called an *asymmetric digraph* or an *oriented graph* if whenever (u, v) is an arc of D, then (v, u) is *not* an arc of D. Thus, an oriented graph D can be obtained from a graph G by assigning a direction to (or by "orienting") each edge of G, thereby transforming each edge of G into an arc and transforming G itself into an asymmetric digraph: D is also called an *orientation* of G. The digraph D_1 of Figure 1.22 is symmetric while D_2 is asymmetric; the digraph D_3 has neither property.

A digraph D is called *complete* if for every two distinct vertices u and v of D, at least one of the arcs (u, v) and (v, u) is present in D. The *complete symmetric digraph* of order p has both arcs (u, v) and (v, u) for every two distinct vertices u and v and is denoted by K_p^*. Indeed, if G is a graph, then G^* denotes the symmetric digraph obtained by replacing each edge of G by a

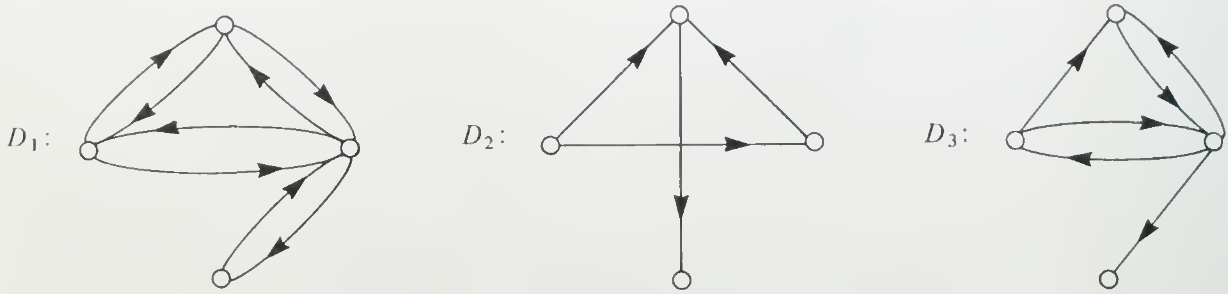


Figure 1.22 *Symmetric and asymmetric digraphs*

symmetric pair of arcs. The digraph K_p^* has size $p(p-1)$ and od $v =$ id $v = p-1$ for every vertex v of D. The digraphs K_1^*, K_2^*, K_3^*, and K_4^* are shown in Figure 1.23.

A complete asymmetric digraph is called a *tournament* and will be studied in some detail in Chapter 7.

A digraph D is called *regular of degree r* or *r-regular* if od $v =$ id $v = r$ for every vertex v of D. The digraph K_p^* is $(p-1)$-regular. A 1-regular digraph D_1 and 2-regular digraph D_2 are shown in Figure 1.24. The digraph D_2 is a tournament.

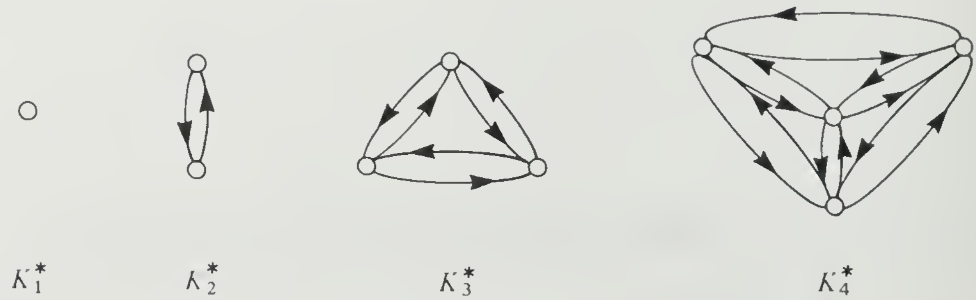


Figure 1.23 *Complete symmetric digraphs*

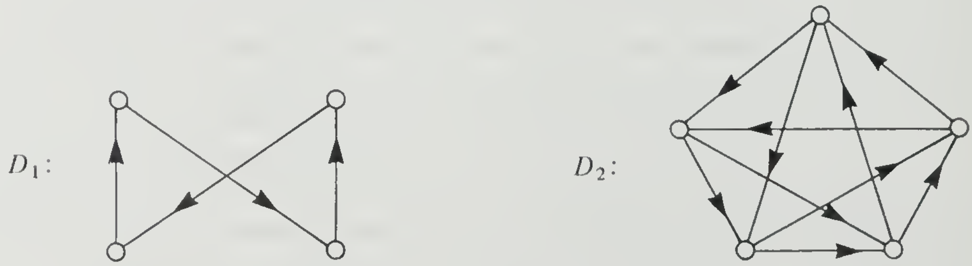


Figure 1.24 *Regular digraphs*

Exercises 1.2

1.18 Determine all (pairwise nonisomorphic) digraphs of order 4 and size 4.

1.19 Prove or disprove: For every integer $p \geq 2$, there exists a digraph D of order p such that for every two distinct vertices u and v of D, od $u \neq$ od v and id $u \neq$ id v.

1.20 Prove or disprove: No digraph contains an odd number of vertices of odd outdegree or an odd number of vertices of odd indegree.

1.21 Let D_1 and D_2 be isomorphic digraphs.

(a) Prove that D_1 and D_2 have the same order and the same size.

(b) If ϕ is an isomorphism from $V(D_1)$ onto $V(D_2)$, then prove that for every $v \in V(D_1)$, $\mathrm{od}_{D_1} v = \mathrm{od}_{D_2} \phi v$ and $\mathrm{id}_{D_1} v = \mathrm{id}_{D_2} \phi v$.

1.22 Prove or disprove: If D_1 and D_2 are two digraphs with $V(D_1) = \{u_1, u_2, \ldots, u_p\}$ and $V(D_2) = \{v_1, v_2, \ldots, v_p\}$ such that $\mathrm{id}_{D_1} u_i = \mathrm{id}_{D_2} v_i$ and $\mathrm{od}_{D_1} u_i = \mathrm{od}_{D_2} v_i$ for $i = 1, 2, \ldots, p$, then $D_1 \cong D_2$.

1.23 Prove that if every proper induced subdigraph of a digraph D of order $p \geq 4$ is regular, then $E(D) = \varnothing$ or $D = K_p^*$.

1.24 Prove or disprove: If r and p are integers with $0 \leq r < p$, then there exists an r-regular digraph of order p.

1.25 Let D be a digraph, and let $r = \max(\{\mathrm{od}\, v \mid v \in V(D)\} \cup \{\mathrm{id}\, w \mid w \in V(D)\})$. Prove that there exists an r-regular digraph H such that $D < H$.

1.26 Prove that there exist regular tournaments of every odd order but there are no regular tournaments of even order.

1.27 The *adjacency matrix* $A(D)$ of a digraph D with $V(D) = \{v_1, v_2, \ldots, v_p\}$ is the $p \times p$ matrix $[a_{ij}]$ defined by $a_{ij} = 1$ if $(v_i, v_j) \in E(D)$ and $a_{ij} = 0$ otherwise.

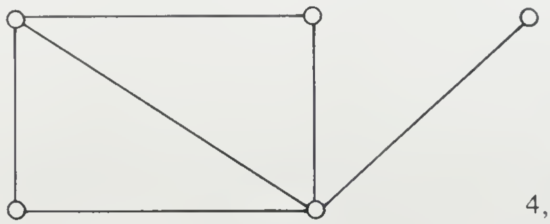
(a) What information do the row sums and column sums of the adjacency matrix of a digraph provide?

(b) Characterize matrices that are adjacency matrices of digraphs.

1.3 Degree Sequences

In this section, we investigate the concept of degree in graphs and digraphs in more detail.

A sequence $d_1, d_2, \ldots, d_p$ of nonnegative integers is called a *degree* *sequence* of a graph G if the vertices of G can be labeled $v_1, v_2, \ldots, v_p$ so that $\deg v_i = d_i$ for all i. For example, a degree sequence of the graph of Figure 1.25 is 4, 3, 2, 2, 1 (or 1, 2, 2, 3, 4, or 2, 1, 4, 2, 3).



4, 3, 2, 2, 1

Figure 1.25 *A degree sequence of a graph*

Given a graph G, a degree sequence of G can be easily determined, of course. On the other hand, if a sequence s: $d_1, d_2, \ldots, d_p$ of nonnegative integers is given, then under what conditions is s a degree sequence of some graph? If such a graph exists, then s is called a *graphical sequence*. Certainly the conditions $d_i \leq p-1$ for all i and $\sum_{i=1}^{p} d_i$ is even are necessary for a sequence to be graphical, but these conditions are not sufficient. The sequence 3, 3, 3, 1 is not graphical, for example. A necessary and sufficient condition for a sequence to be graphical was found by Havel [H8] and later rediscovered by Hakimi [H2].

Theorem 1.3 (Havel-Hakimi) *A sequence s: $d_1, d_2, \ldots, d_p$ of nonnegative integers with $d_1 \geq d_2 \geq \ldots \geq d_p$, $p \geq 2$, $d_1 \geq 1$, is graphical if and only if the sequence s_1: $d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_p$ is graphical.*

Proof Assume that s_1 is a graphical sequence. Then there exists a graph G_1 of order $p-1$ such that s_1 is a degree sequence of G_1. Thus, the vertices of G_1 can be labeled as $v_2, v_3, \ldots, v_p$ so that

$$\deg v_i = \begin{cases} d_i - 1 & 2 \leq i \leq d_1 + 1 \\ d_i & d_1 + 2 \leq i \leq p \end{cases}.$$

A new graph G can now be constructed by adding a new vertex v_1 and the d_1 edges $v_1 v_i$, $2 \leq i \leq d_1 + 1$. Then in G, $\deg v_i = d_i$ for $1 \leq i \leq p$, and so s: $d_1, d_2, \ldots, d_p$ is graphical.

Conversely, let s be a graphical sequence. Hence there exist graphs of order p with degree sequence s. Among all such graphs let G be one such that $V(G) = \{v_1, v_2, \ldots, v_p\}$, $\deg v_i = d_i$ for $i = 1, 2, \ldots, p$, and the sum of the degrees of the vertices adjacent with v_1 is maximum. We show first that v_1 is adjacent with vertices having degrees $d_2, d_3, \ldots, d_{d_1+1}$.

Suppose, to the contrary, that v_1 is not adjacent with vertices having degrees $d_2, d_3, \ldots, d_{d_1+1}$. Then there exist vertices v_j and v_k with $d_j > d_k$ such that v_1 is adjacent to v_k but not to v_j. Since the degree of v_j exceeds that of v_k, there exists a vertex v_n such that v_n is adjacent to v_j but not to v_k. Removing the edges v_1v_k and v_jv_n and adding the edges v_1v_j and v_kv_n results in a graph G' having the same degree sequence as G. However, in G' the sum of the degrees of the vertices adjacent with v_1 is larger than that in G, contradicting the choice of G.

Thus, v_1 is adjacent with vertices having degrees $d_2, d_3, \ldots, d_{d_1+1}$, and the graph $G - v_1$ has degree sequence s_1 so that s_1 is graphical. ■

With the aid of Theorem 1.3, we may now present an algorithm that allows us to determine whether a finite sequence of nonnegative integers is graphical.

Algorithm 1A *Given a sequence of $p(\geq 1)$ nonnegative integers:*

1. *If some integer in the sequence exceeds $p-1$, then the sequence is not graphical. Otherwise, continue to Step 2.*
2. *If all integers in the sequence are 0, then the sequence is graphical. If the sequence contains a negative integer, then the sequence is not graphical. Otherwise, continue to Step 3.*
3. *Reorder the numbers in the current sequence, if necessary, so that it is a nonincreasing sequence.*
4. *Delete the first number, say n, from the sequence, and subtract 1 from the next n numbers in the sequence. Return to Step 2.*

Theorem 1A *Algorithm 1A determines whether a given sequence of $p(\geq 1)$ nonnegative integers is graphical.*

Proof If the algorithm stops before Step 4, then the result is immediate. We may thus assume that the algorithm has proceeded through Step 4 at least once.

The proof will therefore be complete once we show that by repeating Step 4, we eventually arrive at a sequence every term of which is 0 or at a sequence containing a negative integer. Since the algorithm has proceeded through Step 1, every term in the original sequence is at most $p-1$. If every term of this sequence is at most $p-2$, then certainly applying Step 4 produces a sequence with $p-1$ terms, each of which is at most $p-2$. If some term of the original sequence equals $p-1$, then since 1 is subtracted from each of the remaining terms, every term of the resulting sequence is at most $p-2$. Hence, if Step 4 is applied a second time, a sequence with $p-2$ terms is produced in which each term is at most $p-3$, and, in general, if Step 4 is applied k times, $1 \leq k \leq p-1$, a sequence with $p-k$ terms is produced in which each term is at most $p-k-1$. If Step 4 were applied $p-1$ times, the resulting sequence contains one term that is at most 0. Therefore, we must eventually arrive at a sequence with the desired property. ■

We now illustrate Algorithm 1A with the sequence

$$s:\quad 5, 3, 3, 3, 3, 2, 2, 2, 1, 1, 1, 0, 0.$$

We immediately proceed to Step 4, getting

$$s_1':\quad 2, 2, 2, 2, 1, 2, 2, 1, 1, 1.$$

Reordering this sequence, we obtain

$$s_1:\quad 2, 2, 2, 2, 2, 2, 1, 1, 1, 1.$$

Continuing to apply Algorithm 1A, we have

$$\begin{aligned} s_2'&:\quad 1, 1, 2, 2, 2, 1, 1, 1, 1\\ s_2&:\quad 2, 2, 2, 1, 1, 1, 1, 1, 1\\ s_3'=s_3&:\quad 1, 1, 1, 1, 1, 1, 1, 1\\ s_4'&:\quad 0, 1, 1, 1, 1, 1, 1\\ s_4&:\quad 1, 1, 1, 1, 1, 1, 0\\ s_5'&:\quad 0, 1, 1, 1, 1, 0\\ s_5&:\quad 1, 1, 1, 1, 0, 0\\ s_6'&:\quad 0, 1, 1, 0, 0\\ s_6&:\quad 1, 1, 0, 0, 0\\ s_7'=s_7&:\quad 0, 0, 0, 0. \end{aligned}$$

Algorithm 1A therefore shows that s is graphical. If we can observe that some sequence prior to s_7 is graphical, then we can conclude by Theorem 1.3 that s is graphical. For example, the sequence s_3 is easily seen to be graphical since it is the degree sequence of the graph G_3 of Figure 1.26. By Theorem 1.3, each of the sequences s_2, s_1, and s is in turn graphical. To construct a graph with degree sequence s_2, we proceed in reverse from s_3' to s_2, observing that a vertex should be added to G_3 so that it is adjacent to two vertices of degree 1. We thus obtain a graph G_2 with degree sequence s_2 (or s_2'). Proceeding from s_2' to s_1, we again add a new vertex joining it to two vertices of degree 1 in G_2. This gives a graph G_1 with degree sequence s_1 (or s_1'). Finally, we obtain a graph G with degree sequence s by considering s_1'; that is, a new vertex is added to G_1, joining it to vertices of degrees 2, 2, 2, 2, 1. Graph G is then completed by inserting two isolated vertices.

It should be pointed out that graph G in Figure 1.26 is not the only graph with degree sequence s. Indeed, there are graphs that cannot be produced by the method used to construct graph G of Figure 1.26. For example, graph H of Figure 1.27 is such a graph.

A *good* graph-theoretic algorithm is one in which the number of computational steps required for its implementation on any graph (or digraph) of order p is bounded above by a polynomial in p. For example, it can be verified that Algorithm 1A is a good algorithm.

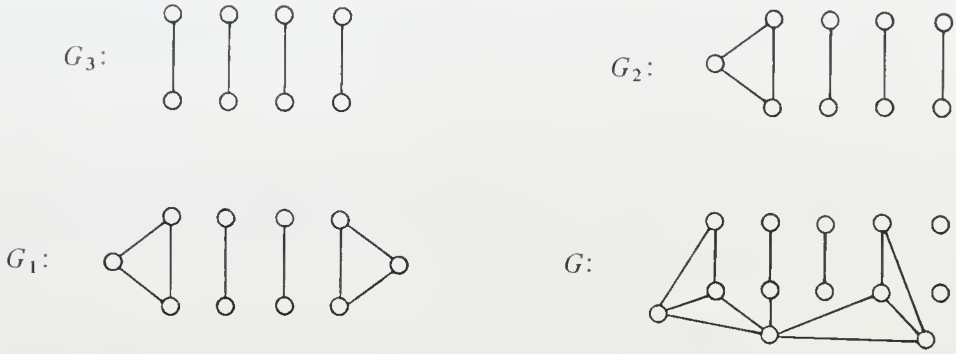


Figure 1.26 *Construction of a graph G with given degree sequence*

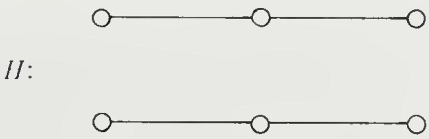


Figure 1.27 *A graph that cannot be constructed by the method following Theorem 1.3.*

Another result that determines which sequences are graphical comes from Erdös and Gallai [EG2].

Theorem 1.4 (Erdös and Gallai) *A sequence $d_1, d_2, \ldots, d_p$ of nonnegative integers with $d_1 \geq d_2 \geq \ldots \geq d_p$ is graphical if and only if $\Sigma_{i=1}^{p} d_i$ is even and for each integer n, $1 \leq n \leq p-1$,*

$$\sum_{i=1}^{n} d_i \leq n(n-1) + \sum_{i=n+1}^{p} \min\{n, d_i\}.$$

When considering degree sequences, we are interested not only in degrees but also in their frequencies. We now delete this last requirement. Denote the *degree set* of a graph G (that is, the set of degrees of the vertices of G) by $\mathcal{D}_G$. For example, if $G \cong K(1, 2, 4)$, then $\mathcal{D}_G = \{3, 5, 6\}$. We now investigate the question of which sets of positive integers are the degree sets of graphs. This question is completely answered by a result of Kapoor, Polimeni, and Wall [KPW1].

Theorem 1.5 *For every set $S = \{a_1, a_2, \ldots, a_n\}$, $n \geq 1$, of positive integers, with $a_1 < a_2 < \ldots < a_n$, there exists a graph G such that $\mathcal{D}_G = S$. Furthermore, the minimum order $\mu(S) = \mu(a_1, a_2, \ldots, a_n)$ of such a graph G is $\mu(S) = a_n + 1$.*

Proof If G is a graph such that $\mathcal{D}_G = S$, then G has order at least $a_n + 1$. Thus we must show that such a graph G having order exactly $a_n + 1$ exists. We proceed by induction on n. For $n = 1$, we observe that every vertex of the complete graph K_{a_1+1} has degree a_1 so that $\mu(a_1) = a_1 + 1$. For $n = 2$, the vertices of the graph $F = K_{a_1} + (\overline{K}_{a_2-a_1+1})$ have degrees a_1 and a_2, and since F has order $a_2 + 1$, we conclude that $\mu(a_1, a_2) = a_2 + 1$.

Let $n \geq 2$. Assume for every set S containing m positive integers, where $1 \leq m \leq n$, that $\mu(S) = a_m + 1$, where a_m is the largest element of S. Let $S_1 = \{b_1, b_2, \ldots, b_{n+1}\}$ be a set of $n+1$ positive integers such that $b_1 < b_2 < \ldots < b_{n+1}$. By the inductive hypothesis,

$$\mu(b_2 - b_1, b_3 - b_1, \ldots, b_n - b_1) = (b_n - b_1) + 1.$$

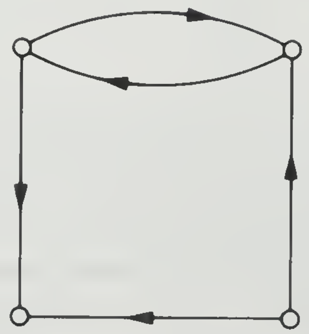
Hence, there exists a graph H of order $(b_n - b_1) + 1$ such that

$$\mathcal{D}_H = \{b_2 - b_1, b_3 - b_1, \ldots, b_n - b_1\}.$$

$$G = K_{b_1} + (\overline{K}_{b_{n+1}-b_n} \cup H)$$

has order $b_{n+1} + 1$, and $\mathcal{D}_G = \{b_1, b_2, \ldots, b_{n+1}\}$; hence, $\mu(b_1, b_2, \ldots, b_{n+1}) = b_{n+1} + 1$, which completes the proof. ■

When considering degree sequences for digraphs, it is necessary to account for both the outdegree and indegree of each vertex. A sequence $(s_1, t_1), (s_2, t_2), \ldots, (s_p, t_p)$ of ordered pairs of nonnegative integers is called a *degree sequence* of a digraph D if the vertices of D can be labeled $v_1, v_2, \ldots, v_p$ so that od $v_i = s_i$ and id $v_i = t_i$ for all i. For example, a degree sequence of the digraph of Figure 1.28 is (2, 1), (1, 2), (0, 2), (2, 0).



(2, 1), (1, 2), (0, 2), (2, 0)

Figure 1.28 *A degree sequence of a digraph*

A sequence $(s_1, t_1), (s_2, t_2), \ldots, (s_p, t_p)$ of ordered pairs of nonnegative integers is called a *digraphical sequence* if it is a degree sequence of some digraph. Clearly, if the sequence $(s_1, t_1), (s_2, t_2), \ldots, (s_p, t_p)$ is digraphical, then $\sum_{i=1}^{p} s_i = \sum_{i=1}^{p} t_i$, and we have $s_i \leq p - 1$ and $t_i \leq p - 1$ for all i. That these conditions are not sufficient for a sequence to be digraphical is illustrated by the sequence (1, 1), (0, 0). However, necessary and sufficient conditions for a sequence to be digraphical were discovered independently by Fulkerson [F7] and Ryser [R11].

Theorem 1.6 (Fulkerson-Ryser) *A sequence* $(s_1, t_1), (s_2, t_2), \ldots, (s_p, t_p)$ *of ordered pairs of nonnegative integers with* $s_1 \geq s_2 \geq \ldots \geq s_p$ *is digraphical if and only if*

(a) $s_i \leq p-1$ *and* $t_i \leq p-1$ *for* $1 \leq i \leq p$,

(b) $\sum_{i=1}^{p} s_i = \sum_{i=1}^{p} t_i$, *and*

(c) $\sum_{i=1}^{n} s_i \leq \sum_{i=1}^{n} \min\{n-1, t_i\} + \sum_{i=n+1}^{p} \min\{n, t_i\}$ *for* $1 \leq n < p$.

Exercises 1.3

1.28 Determine whether the following sequences are graphical. If so, construct a graph with the appropriate degree sequence.

(a) 4, 4, 3, 2, 1, 0

(b) 3, 3, 2, 2, 2, 2, 1, 1, 0

(c) 7, 4, 3, 3, 2, 2, 2, 1, 1, 1, 0

1.29 Show that no nontrivial sequence with distinct terms is graphical.

1.30 Show that the sequence $d_1, d_2, \ldots, d_p$ is graphical if and only if the sequence $p-d_1-1, p-d_2-1, \ldots, p-d_p-1$ is graphical.

1.31 (a) Using Theorem 1.4, show that s: 7, 6, 5, 4, 4, 3, 2, 1 is graphical.

(b) Prove that there exists exactly one graph with degree sequence s.

1.32 Show that the condition given in Theorem 1.4 is necessary for a sequence to be graphical.

1.33 Find a graph G of order 8 having $\mathcal{D}_G = \{3, 4, 5, 7\}$.

1.34 Show that conditions (a), (b), and (c) in Theorem 1.6 are necessary for a sequence to be digraphical.

Chapter Two

Connected Graphs and Digraphs

The most basic property that a graph or digraph may possess is that of being connected; that is, where one may proceed from one vertex to another by means of a sequence of edges. This concept is now investigated.

2.1 Paths and Cycles

Let u and v be (not necessarily distinct) vertices of a graph G. A *u-v walk* of G is a finite, alternating sequence

$$u = u_0,\ e_1,\ u_1,\ e_2,\ \ldots,\ u_{n-1},\ e_n,\ u_n = v$$

of vertices and edges, beginning with vertex u and ending with vertex v, such that $e_i = u_{i-1}u_i$ for $i = 1, 2, \ldots, n$. The number n (the number of occurrences of edges) is called the *length* of the walk. A *trivial walk* contains no edges; that is, $n = 0$. We note that there may be repetition of vertices and edges in a walk. Often only the vertices of a walk are indicated since the edges present are then evident. Two u-v walks $u = u_0, u_1, \ldots, u_n = v$ and $u = v_0, v_1, \ldots, v_m = v$ are considered to be *equal* if and only if $n = m$ and $u_i = v_i$ for $0 \leq i \leq n$; otherwise, they are *different*. Observe that the edges of two different $u - v$ walks of G may very well induce the same subgraph of G.

A u-v walk is *closed* or *open* depending on whether $u = v$ or $u \neq v$. A *u-v trail* is a u-v walk in which no edge is repeated, while a *u-v path* is a u-v walk in which no vertex is repeated. A vertex u forms the *trivial u-u path*. Every path is therefore a trail. In the graph G of Figure 2.1, W_1: $v_1, v_2, v_3, v_2,$

v_5, v_3, v_4 is a v_1-v_4 walk that is not a trail, W_2: v_1, v_2, v_5, v_1, v_3, v_4 is a v_1-v_4 trail that is not a path, and W_3: v_1, v_3, v_4 is a v_1-v_4 path.

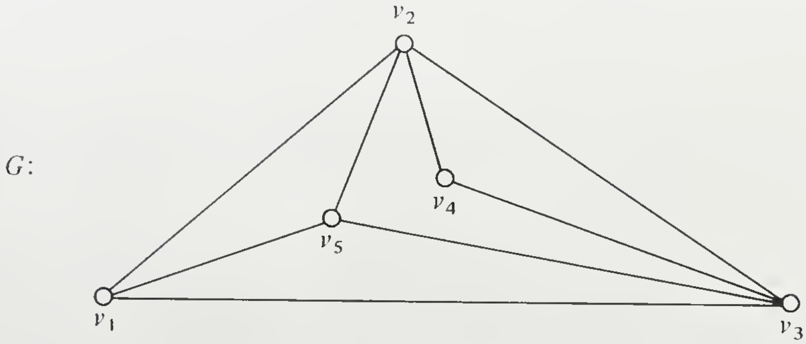


Figure 2.1 *Walks, trails, and paths*

By definition, every path is a walk. Although the converse of this statement is not true in general, we do have the following theorem. A walk W is said to *contain* a walk W' if W' is a subsequence of W.

Theorem 2.1 *Every u-v walk in a graph contains a u-v path.*

Proof Let W be a u-v walk in a graph G. If W is closed, the result is trivial. Let W: $u = u_0, u_1, u_2, \ldots, u_n = v$ be an open u-v walk of a graph G. (A vertex may have received more than one label.) If no vertex of G occurs in W more than once, then W is a u-v path. Otherwise, there are vertices of G that occur in W twice or more. Let i and j be distinct positive integers, with $i < j$ say, such that $u_i = u_j$. If the terms $u_i, u_{i+1}, \ldots, u_{j-1}$ are deleted from W, a u-v walk W_1 is obtained having fewer terms than that of W. If there is no repetition of vertices in W_1, then W_1 is a u-v path. If this is not the case, we continue the above procedure until finally arriving at a u-v walk that is a u-v path. ■

As the next theorem indicates, the nth power of the adjacency matrix of a graph can be used to compute the number of walks of various lengths in the graph.

Theorem 2.2 *If A is the adjacency matrix of a graph G with $V(G) = \{v_1, v_2, \ldots, v_p\}$, then the (i, j) entry of A^n, $n \geq 1$, is the number of different v_i-v_j walks of length n in G.*

Proof The proof is by induction on n. The result is obvious for $n = 1$ since there exists a v_i-v_j walk of length 1 if and only if $v_iv_j \in E(G)$. Let $A^{n-1} = [a_{ij}^{(n-1)}]$ and assume $a_{ij}^{(n-1)}$ is the number of different v_i-v_j walks of length $n-1$ in G; furthermore, let $A^n = [a_{ij}^{(n)}]$. Since $A^n = A^{n-1} \cdot A$, we have

$$a_{ij}^{(n)} = \sum_{k=1}^{p} a_{ik}^{(n-1)} a_{kj}. \tag{2.1}$$

Every v_i-v_j walk of length n in G consists of a v_i-v_k walk of length $n-1$, where v_k is adjacent to v_j, followed by the edge $v_k v_j$ and the vertex v_j. Thus by the inductive hypothesis and equation (2.1), we have the desired result. ■

A nontrivial closed trail of a graph G is referred to as a *circuit* of G, and a circuit $v_1, v_2, \ldots, v_n, v_1 (n \geq 3)$ whose n vertices v_i are distinct is called a *cycle*. An *acyclic graph* has no cycles. The subgraph of a graph G induced by the edges of a trail, path, circuit, or cycle is also referred to as a *trail*, *path*, *circuit*, or *cycle* of G. A cycle is *even* if its length is even; otherwise it is *odd*. A cycle of length n is an *n-cycle*; a 3-cycle is also called a *triangle*. A graph of order n that is a path or a cycle is denoted by P_n or C_n, respectively.

The concepts discussed in this section for graphs have analogues for digraphs. The important difference is that the directions of the arcs must be followed in a walk (and hence a trail, path, circuit, or cycle) in a digraph. Specifically, let u and v be vertices of a digraph D. By a *u-v walk* of D is meant a finite, alternating sequence

$$u = u_0, a_1, u_1, a_2, \ldots, u_{n-1}, a_n, u_n = v$$

of vertices and arcs, beginning with vertex u and ending with vertex v, such that $a_i = (u_{i-1}, u_i)$ for $i = 1, 2, \ldots, n$. As with walks in graphs, only the vertices need be listed since the arcs are then discernible. The number n is called the *length* of the walk. Trivial walks, equal walks, open and closed walks, trails, paths, circuits, and cycles are defined in the obvious manner for digraphs.

The following theorem is analogous to Theorem 2.1 (both in statement and proof).

Theorem 2.3 *Every u-v walk in a digraph contains a u-v path.*

We now consider a very basic concept in graph theory, namely connected and disconnected graphs. A vertex u is said to be *connected* to a vertex v in a graph G if there exists a u-v path in G. A graph G is *connected* if every two of its vertices are connected. A graph that is not connected is *disconnected*. The relation "is connected to" is an equivalence relation on the vertex set of every graph G. Each subgraph induced by the vertices in a resulting equivalence class is called a *connected component* or simply a *component* of G. Equivalently, a component of a graph G is a connected subgraph of G not properly contained in any other connected subgraph of G; that is, a component of G is a subgraph that is maximal with respect to the property of being connected. Hence, a connected subgraph F of a graph G is a component of G if for each connected graph H with $F \subset H \subset G$ where $V(F) \subseteq V(H)$ and $E(F) \subseteq E(H)$, it follows that $F = H$. The number of components of G is denoted by $k(G)$; of course, $k(G) = 1$ if and only if G is connected. For the graph G of Figure 2.2, $k(G) = 6$.

G:

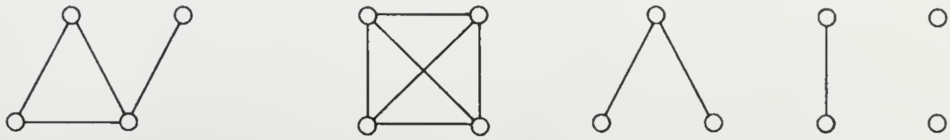


Figure 2.2 *A graph with six components*

For a connected graph G, we define the *distance* $d(u, v)$ between two vertices u and v as the minimum of the lengths of the u-v paths of G. Under this distance function, the set $V(G)$ is a metric space.

The *eccentricity* $e(v)$ of a vertex v of a connected graph G is the number $\max_{u \in V(G)} d(u, v)$. The *radius* rad G is defined as $\min_{v \in V(G)} e(v)$ while the *diameter* diam G is $\max_{v \in V(G)} e(v)$. It therefore follows that diam $G = \max_{u, v \in V(G)} d(u, v)$. A vertex v is a *central vertex* if $e(v) = \text{rad } G$ and the *center* $Z(G)$ of G consists of its central vertices.

For the graph G of Figure 2.3, rad $G = 3$ and diam $G = 5$. Here, $Z(G) = \{u, v, w\}$. The vertices x and y have maximum eccentricity 5.

The radius and diameter are related by the following inequalities.

G:

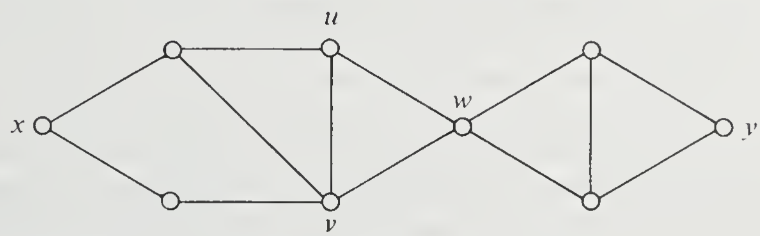


Figure 2.3 *A graph with radius 3 and diameter 5*

Theorem 2.4 *For every connected graph G,*

$$\text{rad } G \leq \text{diam } G \leq 2 \text{ rad } G.$$

Proof The inequality rad $G \leq$ diam G is a direct consequence of the definitions. In order to verify the second inequality, select vertices u and v in G such that $d(u, v) = \text{diam } G$. Furthermore, let w be a central vertex of G. Since d is a metric on $V(G)$,

$$d(u, v) \leq d(u, w) + d(w, v) \leq 2 \text{ rad } G. \quad \blacksquare$$

Theorem 2.4 gives a lower bound (namely, rad G) for the diameter of a connected graph G as well as an upper bound (namely, 2 rad G). This is the first of many results we shall encounter for which a question of "sharpness" is involved. In other words, just how good is this result? Ordinarily, there are many interpretations of such a question. We shall consider some possible interpretations in the case of the upper bound.

Certainly, the upper bound in Theorem 2.4 would not be considered sharp if diam $G < 2$ rad G for every graph G; however, it would be considered sharp indeed if diam $G = 2$ rad G for every graph G. In the latter case, we would have a formula, not just a bound. Actually, there are graphs G for which diam $G < 2$ rad G and graphs H for which diam $H = 2$ rad H. This alone may be a satisfactory definition of "sharpness". A more likely interpretation is the existence of an infinite class $\mathcal{H}$ of graphs H such that diam $H = 2$ rad H for each $H \in \mathcal{H}$. Such a class exists; for example, let $\mathcal{H}$ consist of the graphs of the type $K_n + \overline{K}_2$. One disadvantage of this example is that for each $H \in \mathcal{H}$, diam $H = 2$ and rad $H = 1$. Perhaps a more satisfactory class (which fills a more satisfactory requirement for sharpness) is the class of paths P_{2n+1}, $n \geq 1$. In this case, diam $P_{2n+1} = 2n$ and rad $P_{2n+1} = n$; that is, for each positive integer n, there exists a graph G such that diam $G = 2$ rad $G = 2n$. (See also Exercise 2.7.)

The concept of distance in graphs can be generalized in a most natural manner. By a *weighted graph*, we mean a graph in which each edge e is assigned a positive real number, called the *weight* of e, and denoted by $w(e)$. The *length of a path* P in a weighted graph G is the sum of the weights of the edges of P. For connected vertices u and v of G, the *distance* $d(u, v)$ between u and v is the minimum of the lengths of the u-v paths of G. If each edge of G has weight 1, then G can be regarded as a graph, where the definitions of lengths of paths and distance in G coincide with those given earlier for graphs.

We next introduce a good algorithm, due to Dijkstra [D2], which determines, for a fixed vertex u_0 in a connected weighted graph G, the distance $d(u_0, v)$ from u_0 to each vertex v of G, as well as a shortest u_0-v path in G. This algorithm will thus provide a means of solving the airline problem described in Section 1.1, where Figure 1.2(b) displays a weighted graph with the weight of an edge being the cost of a direct flight. Finding the distance between two vertices in this weighted graph corresponds to finding the minimum cost of flying from one city to another, while a shortest path is a flying route between two cities with the lowest cost.

Before stating Dijkstra's algorithm formally, we present some preliminary facts that help explain it. Let u_0 be a fixed vertex in a connected weighted graph G. Further, let $S \subseteq V(G)$ such that $u_0 \in S$ and define $\overline{S} = V(G) - S$. The distance $d(u_0, \overline{S})$ from u_0 to $\overline{S}$ is defined by

$$d(u_0, \overline{S}) = \min_{x \in \overline{S}} \{d(u_0, x)\}.$$

Necessarily, there exists at least one vertex $v \in \overline{S}$ such that $d(u_0, v) = d(u_0, \overline{S})$.

Further, if

$$P: u_0, u_1, u_2, \ldots, u_n, v$$

is a shortest u_0-v path in G, then

(a) $u_i \in S$ for $i = 0, 1, \ldots, n$, and

(b) $u_0, u_1, \ldots, u_n$ is a shortest u_0-u_n path.

Moreover,

$$d(u_0, \bar{S}) = \min \{d(u_0, u) + w(uv)\},$$

where the minimum is taken over all $u \in S$ and $v \in \bar{S}$ (such that uv is an edge of G). Finally, if this minimum is attained when $u = x$ and $v = y$, then

$$d(u_0, y) = d(u_0, x) + w(xy), \tag{2.2}$$

which provides an expression for the distance between u_0 and y.

The algorithm begins by defining $S_0 = \{u_0\}$. Then

$$d(u_0, \bar{S}_0) = \min_{u \in S_0,\ v \in \bar{S}_0} \{d(u_0, u) + w(uv)\},$$

so that

$$d(u_0, \bar{S}_0) = \min_{v \in \bar{S}_0} \{w(u_0 v)\}.$$

If this minimum occurs at $v = v_1$, then we have determined $d(u_0, v_1)$. We then define $S_1 = \{u_0, v_1\}$ and proceed, as above, until arriving at $\bar{S}_{p-1} = \varnothing$, where G has order p.

Throughout the algorithm, various labelings of the vertices of G are produced where, at its conclusion, a vertex $v (\neq u_0)$ will be labeled by the ordered pair $(L(v), u)$ such that $L(v) = d(u_0, v)$ and u is the predecessor of v on a shortest u_0-v path. Initially, u_0 is labeled $L(u_0) = 0$ and all other vertices are labeled ∞. For $v \neq u_0$, the label $L(v)$ changes (perhaps several times) from ∞ to $d(u_0, v)$ as this distance is determined.

Algorithm 2A (Dijkstra) *Given a connected weighted graph G of order p and a vertex u_0 of G:*

1. *Set $i = 0$, $S_0 = \{u_0\}$, $L(u_0) = 0$, and $L(v) = \infty$ for $v \neq u_0$. If $p = 1$, then stop; otherwise, go to Step 2.*
2. *For each $v \in \bar{S}_i$, replace $L(v)$ by*

$$\min \{L(v), L(u_i) + w(u_i v)\}.$$

If this produces a new value of $L(v)$, then label v by the ordered pair $(L(v), u_i)$.

3. *Determine* $\min_{v \in \bar{S}_i} \{L(v)\}$ *and let* u_{i+1} *be a vertex where the minimum is attained.*
4. *Let* $S_{i+1} = S_i \cup \{u_{i+1}\}$.
5. *Replace* i *by* $i+1$. *If* $i = p-1$, *then stop; otherwise, return to Step 2.*

Theorem 2A *Algorithm 2A determines the distance from a fixed vertex* u_0 *of a connected weighted graph* G *of order* p *to every vertex of* G*; namely, at the termination of the algorithm,*

$$L(v) = d(u_0, v) \qquad \text{for all } v \in V(G). \tag{2.3}$$

Further, for $v \neq u_0$,

$$u_0 = w_0, w_1, w_2, \ldots, w_k = v \tag{2.4}$$

is a shortest u_0-v path, where w_i is labeled $(L(w_i), w_{i-1})$ for $i = 1, 2, \ldots, k$.

Proof First we verify (2.3). To do this, we proceed by induction and show after $S_i (0 \leq i \leq p-1)$ is constructed that

$$L(v) = d(u_0, v) \qquad \text{for all } v \in S_i. \tag{2.5}$$

This is certainly true for $i = 0$. Assume that (2.5) holds for a given i $(0 \leq i < p-1)$; we show that (2.5) holds for $i+1$. It suffices to prove that $L(u_{i+1}) = d(u_0, u_{i+1})$. By Algorithm 2A, u_{i+1} is a vertex where $\min_{v \in \bar{S}_i} \{L(v)\}$ is attained. However,

$$\begin{aligned} L(u_{i+1}) &= \min_{v \in \bar{S}_i} \{L(v)\} \\ &= \min_{u \in S_i,\, v \in \bar{S}_i} \{L(u) + w(uv)\} \\ &= \min_{u \in S_i,\, v \in \bar{S}_i} \{d(u_0, u) + w(uv)\}, \end{aligned} \tag{2.6}$$

where the last equality follows by the inductive hypothesis. The minimum in the expression (2.6) occurs for $v = u_{i+1}$ so by (2.2),

$$L(u_{i+1}) = d(u_0, u_{i+1}).$$

In order to verify (2.4), let $v \in V(G)$, with $v \neq u_0$, and suppose at the completion of the algorithm that v has been labeled $(L(v), v_1)$. We know, however, that

$$L(v) = L(v_1) + w(v_1 v)$$

or, equivalently, that

$$d(u_0, v) = d(u_0, v_1) + w(v_1 v).$$

This implies that v_1 is the next to last vertex on some shortest u_0-v path. Continuing in this manner, we produce a shortest u_0-v path,

$$P: u_0 = v_n, v_{n-1}, \ldots, v_1, v,$$

where v_i is labeled $(L(v_i), v_{i+1})$ for $i = 1, 2, \ldots, n-1$, verifying (2.4). ■

Consider the weighted graph G of Figure 2.4. We compute $d(u_0, v)$ for each $v \in V(G)$ and determine a shortest u_0-v_3 path.

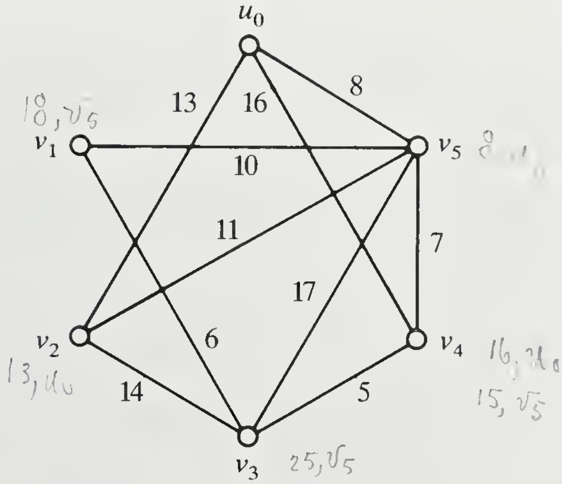


Figure 2.4 *A weighted graph*

Step 1. Set $i = 0$, $S_0 = \{u_0\}$, $L(u_0) = 0$ and $L(v) = \infty$ for $v \neq u_0$.

Step 2. Set $L(v_2) = 13$. Label v_2 by $(L(v_2), u_0)$.
Set $L(v_4) = 16$. Label v_4 by $(L(v_4), u_0)$.
Set $L(v_5) = 8$. Label v_5 by $(L(v_5), u_0)$.

Step 3. Among the vertices in $\overline{S}_0$, the minimum label is 8, which occurs at v_5.

Step 4. Let $S_1 = \{u_0, v_5\}$.

Step 5. Set $i = 1$. Since $i < 5$, return to Step 2.

Step 2. Set $L(v_1) = 18$. Label v_1 by $(L(v_1), v_5)$.
Set $L(v_3) = 25$. Label v_3 by $(L(v_3), v_5)$.
Set $L(v_4) = 15$. Label v_4 by $(L(v_4), v_5)$.

Step 3. Among the vertices in $\overline{S}_1$, the minimum label is 13, which occurs at v_2.

Step 4. Let $S_2 = \{u_0, v_2, v_5\}$.

Step 5. Set $i = 2$. Since $i < 5$, return to Step 2.

Step 2. No change in labels.

Step 3. Among the vertices in $\overline{S}_2$, the minimum label is 15, which occurs at v_4.

Step 4. Let $S_3 = \{u_0, v_2, v_4, v_5\}$.

Step 5. Set $i = 3$. Since $i < 5$, return to Step 2.

Step 2. Set $L(v_3) = 20$. Label v_3 by $(L(v_3), v_4)$.

Step 3. Among the vertices in $\overline{S}_0$, the minimum label is 18, which occurs at v_1.

Step 4. Let $S_4 = \{u_0, v_1, v_2, v_4, v_5\}$.

Step 5. Set $i = 4$. Since $i < 5$, return to Step 2.

Step 2. No change in labels.

Step 3. Among the vertices in $\overline{S}_4$, the minimum label is 20, which occurs at v_3.

Step 4. Let $S_5 = \{u_0, v_1, v_2, v_3, v_4, v_5\}$.

Step 5. Set $i = 5$. Since $i = 5$, the algorithm stops.

We have thus computed the following distances:

$$\begin{aligned} d(u_0, u_0) &= L(u_0) = 0 & \qquad d(u_0, v_3) &= L(v_3) = 20 \\ d(u_0, v_1) &= L(v_1) = 18 & d(u_0, v_4) &= L(v_4) = 15 \\ d(u_0, v_2) &= L(v_2) = 13 & d(u_0, v_5) &= L(v_5) = 8 \end{aligned}$$

To determine a shortest u_0-v_3 path, we note that the second label of v_3 is v_4, that of v_4 is v_5, and that of v_5 is u_0. Hence, this path is u_0, v_5, v_4, v_3.

We now present a useful characterization of bipartite graphs.

Theorem 2.5 *A nontrivial graph is bipartite if and only if it contains no odd cycles.*

Proof Let G be a bipartite graph with partite sets V_1 and V_2. Suppose C: $v_1, v_2, \ldots, v_k, v_1$ is a cycle of G. Without loss of generality, we may assume $v_1 \in V_1$. However, then $v_2 \in V_2$, $v_3 \in V_1$, $v_4 \in V_2$, and so on. This implies $k = 2s$ for some positive integer s; hence, C has even length.

For the converse, it suffices to prove that every nontrivial connected graph G without odd cycles is bipartite, since a nontrivial graph is bipartite if and only if each of its nontrivial components is bipartite. Let $v \in V(G)$ and denote by V_1 the subset of $V(G)$ consisting of v and all vertices u of G with the property that any shortest u-v path of G has even length. Let $V_2 = V(G) - V_1$. We now prove that the partition $V_1 \cup V_2$ of $V(G)$ has the appropriate properties to show that G is bipartite.

Let u and w be elements of V_1, and suppose $uw \in E(G)$. Necessarily, then, neither u nor w is the vertex v. Let $v = u_1, u_2, \ldots, u_{2n+1} = u$, $n \geq 1$, and $v = w_1, w_2, \ldots, w_{2m+1} = w$, $m \geq 1$, be a shortest v-u path and a shortest v-w path of G, respectively, and suppose w' is a vertex that the two paths have in common such that the w'-u subpath and w'-w subpath have only w' in common. (Note that w' may be v.) The two v-w' subpaths so determined are then shortest v-w' paths. Thus, there exists an i such that $w' = u_i = w_i$. However, $u_i, u_{i+1}, \ldots, u_{2n+1}, w_{2m+1}, w_{2m}, \ldots, w_i = u_i$ is an odd cycle of G, which is a contradiction to our hypothesis. Similarly, no two vertices of V_2 are adjacent. ■

In Theorem 2.5, the graphs do not contain odd cycles. We now consider graphs that do not possess "small" cycles. The length of the shortest cycle in a graph G that contains cycles is called the *girth* of G and is denoted by $g(G)$. For the graph G of Figure 2.3, we have $g(G) = 3$. Moreover, $g(K_p) = 3$ for $p \geq 3$, $g(K(m, n)) = 4$ for $m, n \geq 2$, and $g(C_n) = n$ for $n \geq 3$. As might be expected, no formula exists for the girth of a graph in general. This, however, has not been the problem most often considered; instead, it has been the following problem: For positive integers $r \geq 2$ and $n \geq 3$, determine the smallest positive integer $f(r, n)$ for which there exists an r-regular graph with girth n having order $f(r, n)$. The r-regular graphs of order $f(r, n)$ with girth n have been the object of many investigations; such graphs are called *(r, n)-cages*. The $(3, n)$-cages are commonly referred to simply as *n-cages*. We introduce the notation *$[r, n]$-graph* to indicate an r-regular graph having girth n. Thus, an (r, n)-cage is an $[r, n]$-graph; indeed, it is one of minimum order.

It is clear that $f(r, n) \geq \max\{r+1, n\}$. Thus, $f(2, n) = n$ since C_n is a 2-regular graph with girth n. Likewise, $f(r, 3) = r + 1$ since K_{r+1} is an r-regular graph having girth 3. In fact, the complete graph K_4 is the unique 3-cage. That $f(r, n)$ always exists has been shown by Erdös and Sachs [ES1].

Theorem 2.6 *For every pair of positive integers $r, n \geq 3$, the number $f(r, n)$ exists and in fact*

$$f(r, n) \leq \left(\frac{r-1}{r-2}\right)[(r-1)^{n-1} + (r-1)^{n-2} + (r-4)].$$

Proof Since

$$\sum_{i=1}^{n-1} (r-1)^i = \left(\frac{r-1}{r-2}\right)[(r-1)^{n-1} - 1]$$

and

$$\sum_{i=1}^{n-2} (r-1)^i = \left(\frac{r-1}{r-2}\right)[(r-1)^{n-2} - 1],$$

it follows that

$$\left(\frac{r-1}{r-2}\right)[(r-1)^{n-1}+(r-1)^{n-2}+(r-4)]$$

is an integer. Denote this integer by p, and let S be the set of all graphs H of order p such that $g(H)=n$ and $\Delta(H)\leqslant r$. Note that $p\geqslant n$. The set S is nonempty since the graph consisting of an n-cycle and $p-n$ isolated vertices belongs to S. For each $H\in S$, define

$$M(H)=\{v\in V(H)|\deg v<r\}.$$

If for some $H\in S$, $M(H)=\varnothing$, then we have the desired result; thus we assume for all $H\in S$, $M(H)\neq\varnothing$. For $H\in S$, we define $m(H)$ to be the maximum distance between two vertices of $M(H)$. (We define $d(u_1, u_2)=+\infty$ if u_1 and u_2 are not connected.)

Let S_1 be those graphs of S containing the maximum number of edges, and denote by S_2 the set of all those graphs H of S_1 for which $|M(H)|$ is maximum. Now among the graphs of S_2 let G be chosen so that $m(G)$ is maximum.

Let $u, v\in M(G)$ such that $d(u, v)=m(G)$. Suppose $m(G)\geqslant n-1\geqslant 2$. By adding the edge uv to G, we obtain a graph G' of order p having $g(G')=\text{n}$ and $\Delta(G')\leqslant r$. Hence $G'\in S$; however, G' has more edges than G, and this produces a contradiction. Therefore, $m(G)\leqslant n-2$ and $d(u, v)\leqslant n-2$. (The vertices u and v may not be distinct.)

Denote by W the set of all those vertices w of G such that $d(u, w)\leqslant n-2$ or $d(v, w)\leqslant n-1$. From our earlier remark, it follows that u, $v\in W$. The number of vertices different from u and at a distance at most $n-2$ from u cannot exceed

$$\sum_{i=1}^{n-2}(r-1)^i=\left(\frac{r-1}{r-2}\right)[(r-1)^{n-2}-1],$$

while the number of vertices different from v and at a distance at most $n-1$ from v cannot exceed

$$\sum_{i=1}^{n-1}(r-1)^i=\left(\frac{r-1}{r-2}\right)[(r-1)^{n-1}-1].$$

Hence the number of elements in W is at most

$$\left(\frac{r-1}{r-2}\right)[(r-1)^{n-2}-1]+\left(\frac{r-1}{r-2}\right)[(r-1)^{n-1}-1];$$

however, $[(r-1)/(r-2)][(r-1)^{n-1}+(r-1)^{n-2}-2]=p-r+1<p$. Therefore, there is a vertex $w_1\in V(G)-W$, so $d(u, w_1)\geqslant n-1$ and $d(v, w_1)\geqslant n$.

Since $d(u, w_1)>m(G)$ and $u\in M(G)$, it follows that $w_1\notin M(G)$ and $\deg w_1=r\geqslant 3$. Therefore, there exists an edge e incident with w_1 whose removal from G results in a graph having girth n. Suppose $e=w_1w_2$. Clearly, $d(v, w_2)\geqslant n-1$ so that $w_2\notin M(G)$ and $\deg w_2=r$.

We now add the edge uw_1 to G and delete the edge w_1w_2, producing the graph G_1. The graph G_1 also belongs to S and, in fact, belongs to S_1. The set $M(G_1)$ contains all the members of $M(G)$ except possibly u and, in addition, contains w_2. From the manner in which G was chosen, $|M(G_1)| \leq |M(G)|$, so that $u \notin M(G_1)$ and $|M(G_1)| = |M(G)|$. Therefore, $\deg u = r$ in G_1, implying that, in G, $\deg u = r - 1$. Furthermore, G_1 belongs to S_2.

We now show that u is not the only vertex of $M(G)$, for suppose it is. Since there is an even number of odd vertices, we must have r and p odd; however, this cannot occur since p is even when r is odd. We conclude that u and v are distinct vertices of $M(G)$.

The vertices v and w_2 are distinct vertices of $M(G_1)$. If there exists no v-w_2 path in G_1, then $m(G_1) = +\infty$, and this is contrary to the fact that $m(G_1) \leq m(G)$. Thus v and w_2 are connected in G_1. Let P be a shortest v-w_2 path in G_1. If P is also in G, then P has length at least $d_G(v, w_2)$ in G, but

$$d_G(v, w_2) \geq n - 1 > m(G),$$

which is impossible. If P is not in G, then P contains the edge uw_1 and a u-v path of length $d_G(u, v)$ as a subpath. Hence P has length exceeding $d_G(u, v) = m(G)$, again a contradiction.

It follows for some H in S that $M(H) = \emptyset$, that is, H is an r-regular graph of order p having girth n. ■

We now determine the value of the number $f(r, 4)$.

Theorem 2.7 *For $r \geq 2$, $f(r, 4) = 2r$. Furthermore, there is only one $(r, 4)$-cage; namely, $K(r, r)$.*

Proof Suppose G is an $[r, 4]$-graph, and let $u_1 \in V(G)$. Denote by $v_1, v_2, \ldots, v_r$ the vertices of G adjacent with u_1. Since $g(G) = 4$, v_1 is adjacent to none of the vertices v_i, $2 \leq i \leq r$; hence G contains at least $r - 1$ additional vertices $u_2, u_3, \ldots, u_r$. Therefore, $f(r, 4) \geq 2r$. Obviously, the graph $K(r, r)$ is r-regular, has girth 4, and has order $2r$, thus implying $f(r, 4) = 2r$.

To show that $K(r, r)$ is the only $(r, 4)$-cage, let G be a $[r, 4]$-graph of order $2r$ whose vertices are labeled as just stated. Since every vertex has degree r and G contains no triangle, each u_i is adjacent to every v_j; therefore, $G \cong K(r, r)$. ■

By Theorem 2.7, there is a unique 4-cage: the graph $K(3, 3)$. There is no other value of $n > 4$ for which $f(r, n)$ is known for all values of r, nor is there a value of $r > 2$ for which $f(r, n)$ is known for all values of n. In these cases, only bounds have been determined. We illustrate this type of result by establishing a lower bound for $f(r, 5)$. It was shown in [B7] that $f(r, 5) \geq r^2 + 1$ for all $r \geq 2$. Hoffman and Singleton [HS1] proved that equality holds for $r = 2$, 3, and 7, and perhaps 57, since it is not known whether there is a [57, 5]-graph of order $57^2 + 1$.

Theorem 2.8 *For $r \geq 2$, $f(r, 5) \geq r^2 + 1$. Furthermore, for $r \neq 57$, equality holds if and only if $r = 2$, 3, or 7.*

Proof Let G be an $[r, 5]$-graph, and let $v_1 \in V(G)$. Denote by $v_2, v_3, \ldots, v_{r+1}$ the vertices of G adjacent with v_1. Since $g(G) = 5$, no two vertices v_i and v_j, $1 < i < j \leq r + 1$, are adjacent with each other or with a vertex different from v_1. Thus each vertex v_i, $2 \leq i \leq r + 1$, is adjacent with $r - 1$ new vertices. Hence G has at least $r(r-1) + (r+1) = r^2 + 1$ vertices, so that $f(r, 5) \geq r^2 + 1$.

Suppose now that G has order $r^2 + 1$ and that the remaining vertices are labeled so that $V(G) = \{v_i \mid i = 1, 2, \ldots, r^2 + 1\}$. Let A be the adjacency matrix of G.

First, we show that

$$A^2 + A = J + (r-1)I, \tag{2.7}$$

where J is the $(r^2+1) \times (r^2+1)$ matrix, all of whose entries are 1, and I is the $(r^2+1) \times (r^2+1)$ identity matrix. Since every vertex of G has degree r, each diagonal entry of A^2 is r, and therefore each diagonal entry of $A^2 + A$ is r. It is easy to verify that each diagonal entry of $J + (r-1)I$ is r also. Because the (i, j)-entry, $i \neq j$, of $J + (r-1)I$ is 1, it remains to show that each such entry of $A^2 + A$ is 1 also. Denote the (i, j)-entry of A by a_{ij}, and that of A^2 by $a_{ij}^{(2)}$. By definition, $a_{ij} = 1$ if and only if $v_i v_j \in E(G)$, and $a_{ij} = 0$ otherwise. By Theorem 2.2, $a_{ij}^{(2)}$ represents the number of paths of length 2 between v_i and v_j. Since $g(G) = 5$, $a_{ij}^{(2)} = 0$ or 1. If $a_{ij} = 1$, then $a_{ij}^{(2)} \neq 1$, for otherwise, G contains a triangle. Hence in this case $a_{ij}^{(2)} = 0$. Suppose next that $a_{ij} = 0$. Because G is an $[r, 5]$-graph having order $r^2 + 1$, no two vertices can have a distance exceeding 2. Thus since $d(v_i, v_j) \neq 1$, we have $d(v_i, v_j) = 2$, thereby proving equation (2.7).

Next we show that $r^2 + r$ is an eigenvalue of $A^2 + A$ of multiplicity 1 and that $r - 1$ is an eigenvalue of $A^2 + A$ of multiplicity r^2. Since $A^2 + A = J + (r-1)I$, the eigenvalues of $A^2 + A$ are the roots of the equation

$$|A^2 + A - \lambda I| = \begin{vmatrix} r-\lambda & 1 & \cdots & 1 \\ 1 & r-\lambda & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & r-\lambda \end{vmatrix} = 0.$$

If we add to the first row all other rows and factor out the common term $r^2 + r - \lambda$, we obtain

$$|A^2+A-\lambda I| = [r^2+r-\lambda] \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & r-\lambda & 1 & \cdots & 1 \\ 1 & 1 & r-\lambda & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdots & r-\lambda \end{vmatrix}.$$

Subtracting the first row from each of the other rows, we obtain $|A^2+A-\lambda I| = (r^2+r-\lambda)(r-1-\lambda)^{r^2}$, which gives us the desired result.

Let λ_i, $i=1, 2, \ldots, r^2+1$, denote the eigenvalues of A. Therefore, the eigenvalues of $\phi(A)=A^2+A$ are $\phi(\lambda_i)$, $i=1, 2, \ldots, r^2+1$. Let $\phi(\lambda_1)=r^2+r$ and $\phi(\lambda_i)=r-1$ for $2\leqslant i\leqslant r^2+1$.

Necessarily, $\lambda_1=r$, for if $\alpha=(1, 1, \ldots, 1)$ is a vector all of whose r^2+1 entries are 1, then $A\alpha'=r\alpha'$, where α' is the transpose of α. This shows that r is an eigenvalue of A.

The remaining r^2 eigenvalues are roots of the equation

$$\lambda^2+\lambda=r-1.$$

Hence, each λ_i, $2\leqslant i\leqslant r^2+1$, has either the value

$$\frac{(-1+\sqrt{4r-3})}{2} \quad \text{or} \quad \frac{(-1-\sqrt{4r-3})}{2}.$$

Assume then that k of the eigenvalues, $0\leqslant k\leqslant r^2$, are $(-1+\sqrt{4r-3})/2$, while the remaining r^2-k eigenvalues are $(-1-\sqrt{4r-3})/2$.

Since the sum of the eigenvalues is zero,

$$r+\frac{k(-1+\sqrt{4r-3})}{2}+\frac{(r^2-k)(-1-\sqrt{4r-3})}{2}=0. \tag{2.8}$$

Solving for $2k$ in equation (2.8), we obtain

$$2k=\frac{(r^2-2r)}{\sqrt{4r-3}}+r^2.$$

Since k is a nonnegative integer and $r\geqslant 2$, either $r=2$ or $4r-3$ is the square of an odd positive integer, say $4r-3=(2m+1)^2$, where m is a positive integer. In the latter case, $r=m^2+m+1$, which implies that

$$2k=2m-1+\frac{m^2}{4}\left(2m+3-\frac{15}{2m+1}\right)+(m^2+m+1)^2.$$

The integers m^2 and $2m+1$ are relatively prime so $2m+1$ divides 15; hence, $m=1, 2,$ or 7. Then $r=m^2+m+1$ implies that $r=3, 7,$ or 57 so that r has one of the values 2, 3, 7, or 57.

Conversely, if $r=2, 3,$ or 7, there is known to exist an $[r, 5]$-graph of order r^2+1. As mentioned earlier, it is not known whether a [57, 5]-graph of order $(57)^2+1$ exists. ■

When $r = p^m + 1$ for some prime p and positive integer m, then

$$f(r, 6) = \frac{2(r-1)^3 - 2}{r-2}$$

and the $(r, 6)$-cage is unique (see [B10], Chap. 23). For example, $f(4, 6) = 26$ and the (4, 6)-cage is shown in Figure 2.5.

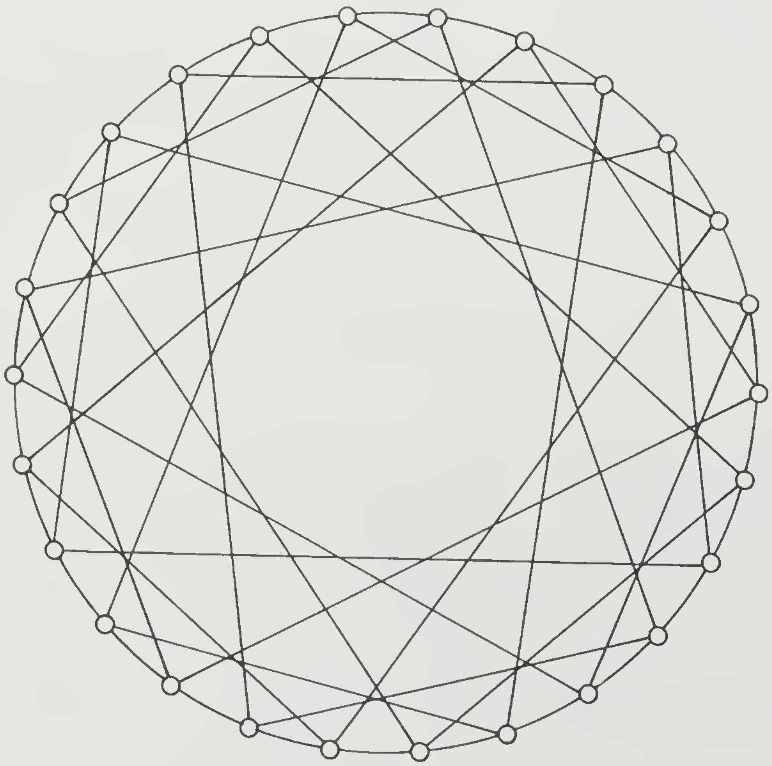


Figure 2.5 *The* (4, 6)*-cage*

Two other "small" cages, namely the (4, 5)-cage and (5, 5)-cage, are shown in Figure 2.6. The (6, 5)-cage has order 40 while the (7, 5)-cage, as mentioned earlier, is known to have order 50.

Returning to Theorem 2.8, we note that $f(3, 5) = 10$. It is not difficult to verify that the graph of Figure 2.7 is a 5-cage; in fact it is the *only* 5-cage. This graph is called the *Petersen graph* and it is quite possibly the most famous graph in all of graph theory. Its fame stems from the unusual list of properties it possesses and fails to possess. (For more information about this well-known graph, see [CW2].) We now verify that it is the only 5-cage.

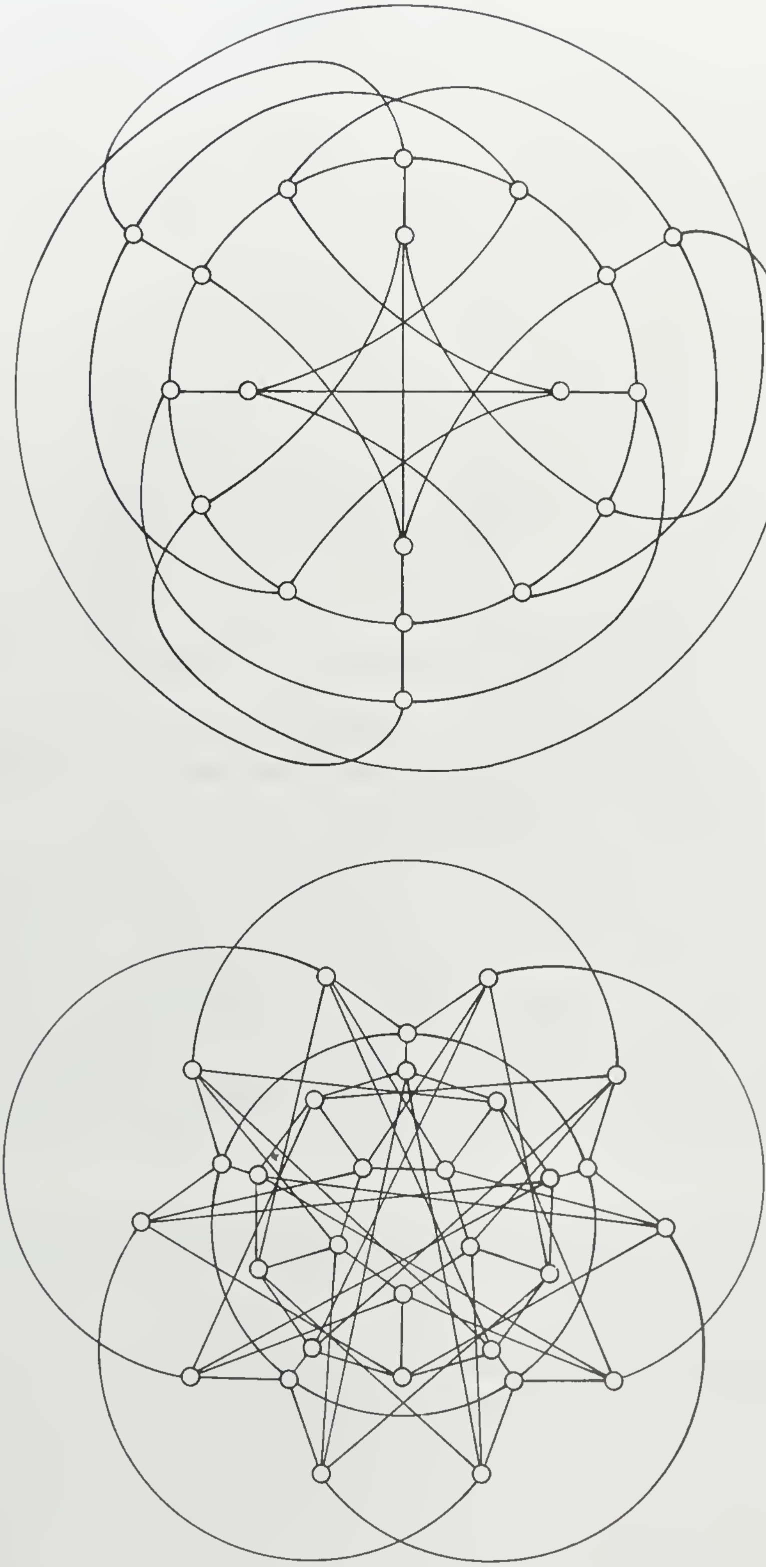


Figure 2.6 *The* (4, 5)-*cage and* (5, 5)-*cage*

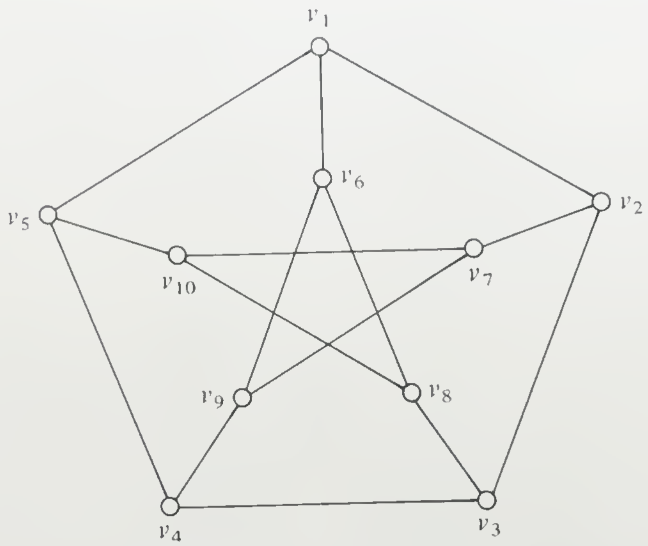


Figure 2.7 *The Petersen graph*

Theorem 2.9 *The Petersen graph is the unique 5-cage.*

Proof As mentioned earlier, it is not difficult to show that the Petersen graph is a 5-cage. In order to see that it is unique, assume G is a [3, 5]-graph of order 10. We show that G is isomorphic to the Petersen graph.

Let $v_1 \in V(G)$, and suppose v_2, v_3, and v_4 are the vertices adjacent to v_1. Since $g(G) = 5$, each v_i, $i = 2, 3, 4$, is adjacent to two new vertices of G. Let v_5 and v_6 be adjacent with v_2, v_7 and v_8 with v_3, and v_9 and v_{10} with v_4. Hence $V(G) = \{v_i | i = 1, 2, \ldots, 10\}$. The fact that the girth of G is 5 and that every vertex of G has degree 3 implies that v_5 is adjacent with one of v_7 and v_8 and one of v_9 and v_{10}. Without loss of generality, we assume v_5 to be adjacent to v_7 and v_9. We must now have v_6 adjacent to v_8 and v_{10}. Therefore, edges v_7v_{10} and v_8v_9 are also present and no others. Thus, G is isomorphic to the Petersen graph. ■

There is only one 6-cage, referred to as the *Heawood graph*, and this is shown in Figure 2.8.

There are only three other known n-cages, $n \geq 7$. The 7-cage (known as the *McGee graph*) and the 8-cage (the so-called *Tutte-Coxeter graph*) are shown in Figure 2.9. The 12-cage has order 126.

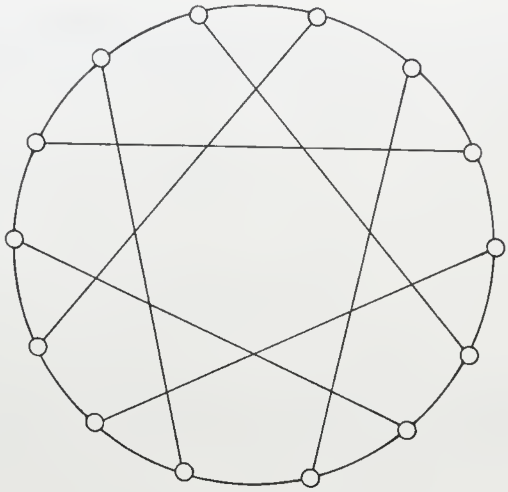


Figure 2.8 *The Heawood graph: the unique 6-cage*

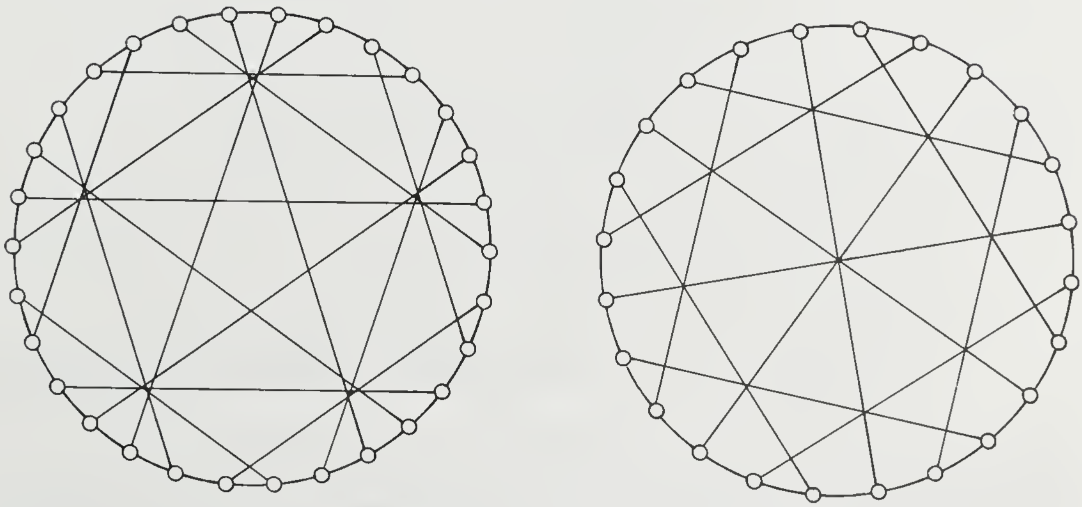


Figure 2.9 *The 7-cage and 8-cage*

Exercises 2.1

2.1 Let u and v be arbitrary vertices of a connected graph G. Show that there exists a u-v walk containing all vertices of G.

2.2 Prove that "is connected to" is an equivalence relation on the vertex set of a graph.

2.3 (a) Let G be a graph of order p such that $\deg v \geq (p-1)/2$ for every $v \in V(G)$. Prove that G is connected.

(b) Examine the sharpness of the result in (a).

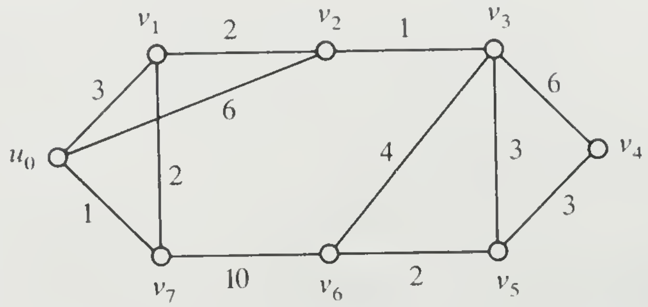
2.4 (a) Let G be a (p, q) graph. Determine a sharp bound $f(p)$ such that if $q > f(p)$, then G is connected.

(b) Show that if G is a (p, q) graph for which $q < p - 1$, then G is disconnected.

2.5 Show that if G is a graph of order p and size $p^2/4$, then either G contains odd cycles or $G \cong K(p/2, p/2)$.

2.6 (a) Let G be the weighted graph shown below. Use Dijkstra's algorithm to compute $d(u_0, v)$ for each $v \in V(G)$ and to determine a shortest u_0-v_6 path.

(b) Let H be the graph obtained by deleting the weights from the edges of the weighted graph in (a). Use Dijkstra's algorithm to compute $d(u_0, v)$ for each $v \in V(H)$ and to determine a shortest u_0-v_6 path.



2.7 Let n and m be positive integers such that $n \leq m \leq 2n$. Prove that there exists a graph G such that rad $G = n$ and diam $G = m$.

2.8 (a) Prove that every circuit of a graph G contains a cycle of G.

(b) Prove that if a vertex is repeated in a trail of G, then the trail contains a cycle of G.

2.9 Give another proof of Theorem 2.5 using the distance concept.

2.10 Prove Theorem 2.3.

2.11 Prove that if a digraph D contains a u-u circuit, it also contains a u-u cycle.

2.12 Prove that every walk in an acyclic digraph is a path.

2.13 Prove that each of the following conditions is sufficient for a digraph D to contain cycles.

(a) Every vertex of D has positive outdegree.

(b) Every vertex of D has positive indegree.

2.14 Prove that if A is the adjacency matrix of a digraph D with $V(D) = \{v_1, v_2, \ldots, v_p\}$, then the (i, j) entry $a_{ij}^{(n)}$ of A^n, $n \geq 1$, is the number of different v_i-v_j walks of length n in D. (See Exercise 1.27.)

2.15 Prove that a graph G is connected if and only if for any partition $V(G) = V_1 \cup V_2 (V_1 \neq \emptyset, V_2 \neq \emptyset)$, there exists an edge of G joining a vertex of V_1 and a vertex of V_2.

2.16 Define a connected graph G to be *degree linear* if G contains a path P with the property that for each $d \in \mathcal{D}_G$ (the degree set of G), there exists a vertex of degree d on P.

(a) Let G be a connected graph with $\mathcal{D}_G = \{d_1, d_2\}$, $d_1 < d_2$. Prove that G is degree linear by proving that G contains a path of length 1 containing vertices of degrees d_1 and d_2.

(b) Determine the maximum value of k such that *every* connected graph having a k-element degree set is degree linear.

2.17 Characterize those graphs G having the property that every induced subgraph of G is a connected subgraph of G.

2.18 Prove that if G is a disconnected graph, then $\bar{G}$ is connected.

2.19 Prove that there exist exactly two 4-regular graphs G of order 7. (Hint: Consider $\bar{G}$.)

2.20 Let G be a connected graph with cycles. Show that $g(G) \leq 2 \operatorname{diam}(G) + 1$.

2.21 (a) Prove that $f(3, 6) = 14$.
(b) Prove that the Heawood graph is the only 6-cage.

2.22 Let G be an $[r, n]$-graph ($r \geq 2$, $n \geq 3$) of order $f(r, n)$; that is, let G be an (r, n)-cage. Prove that if $H = G \times K_2$ is an $[s, n]$-graph, then H cannot be an (s, n)-cage.

2.2 Cut-Vertices, Bridges, and Blocks

Some graphs are connected so slightly that they can be disconnected by removing a single vertex or single edge. Such vertices and edges play a special role in graph theory, and we discuss these next.

A vertex v of a graph G is called a *cut-vertex* of G if $k(G - v) > k(G)$. Thus, a vertex of a connected graph is a cut-vertex if its removal produces a disconnected graph. In general, a vertex v of a graph G is a cut-vertex of G if its removal disconnects a component of G. The following theorem characterizes cut-vertices.

Theorem 2.10 *A vertex v of a connected graph G is a cut-vertex of G if and only if there exist vertices u and w ($u, w \neq v$) such that v is on every u-w path of G.*

Proof Let v be a cut-vertex of G so that the graph $G - v$ is disconnected. If u and w are vertices in different components of $G - v$, then there are no u-w paths in

$G - v$; however, since G is connected, there are u-w paths in G. Therefore, every u-w path of G contains v.

Conversely, assume that there exist vertices $u, w \in V(G)$ such that the vertex v lies on every u-w path of G. Then there are no u-w paths in $G - v$, implying that $G - v$ is disconnected and that v is a cut-vertex of G. ■

The complete graphs have no cut-vertices while, at the other extreme, each nontrivial path contains only two vertices that are not cut-vertices. In order to see that this is the other extreme, we prove the following theorem.

Theorem 2.11 *Every nontrivial graph contains at least two vertices that are not cut-vertices.*

Proof Assume the theorem is false. Then there exists a nontrivial connected graph G containing at most one vertex that is not a cut-vertex; that is, every vertex of G with at most one exception is a cut-vertex. Let u and v be vertices of G such that $d(u, v) = \text{diam } G$.

At least one of u and v is a cut-vertex, say v. Let w be a vertex belonging to a component of $G - v$ not containing u. Since every u-w path in G contains v, we conclude that

$$d(u, w) > d(u, v) = \text{diam } G,$$

which is impossible. The desired result now follows. ■

Analogous to the cut-vertex is the concept of a bridge. A *bridge* of a graph G is an edge e such that $k(G - e) > k(G)$. If e is a bridge of G, then it is immediately evident that $k(G - e) = k(G) + 1$. Furthermore, if $e = uv$, then u is a cut-vertex of G if and only if $\deg u > 1$. Indeed, the complete graph K_2 is the only connected graph containing a bridge but no cut-vertices. Bridges are characterized in a manner similar to that of cut-vertices; the proof too is similar to that of Theorem 2.10 and is omitted.

Theorem 2.12 *An edge e of a connected graph G is a bridge of G if and only if there exist vertices u and w such that e is on every u-w path of G.*

For bridges, there is another useful characterization.

Theorem 2.13 *An edge e of a graph G is a bridge of G if and only if e is on no cycle of G.*

Proof We assume G to be connected. Let $e = uv$ be an edge of G, and suppose e lies on a cycle C of G. Further, let w_1 and w_2 be arbitrary distinct vertices of G. If e does not lie on a w_1-w_2 path P of G, then P is also a w_1-w_2 path of $G - e$. If, however, e lies on a w_1-w_2 path Q of G, then replacing e by the u-v path (or v-u path) on C not containing e produces a w_1-w_2 walk in $G - e$. By Theorem 2.1, there is a w_1-w_2 path in $G - e$. Hence e is not a bridge.

Conversely, suppose $e = uv$ is an edge of G that is on no cycle of G, and assume e is not a bridge. Thus $G - e$ is connected. Hence there exists a u-v path P in $G - e$; however, P together with e produce a cycle in G containing e, which is a contradiction. ■

A *cycle edge* is an edge that lies on a cycle. From Theorem 2.13, a cycle edge of a graph G is an edge that is not a bridge of G. A bridge incident with an end-vertex is called a *terminal edge.*

Many of the graphs we encounter do not contain cut-vertices; we discuss these next. A nontrivial connected graph with no cut-vertices is called a *block.* Nontrivial connected graphs that are not blocks contain special subgraphs in which we are also interested. A *block of a graph G* is a subgraph of G, which is itself a block and which is maximal with respect to that property. A block is necessarily an induced subgraph, and, moreover, the blocks of a graph partition its edge set. Every two blocks have at most one vertex in common, namely a cut-vertex. The graph of Figure 2.10 has five blocks B_i, $1 \leq i \leq 5$, as indicated. The vertices v_3, v_5, and v_8 are cut-vertices, while v_3v_5 and v_4v_5 are bridges; moreover, v_4v_5 is a terminal edge.

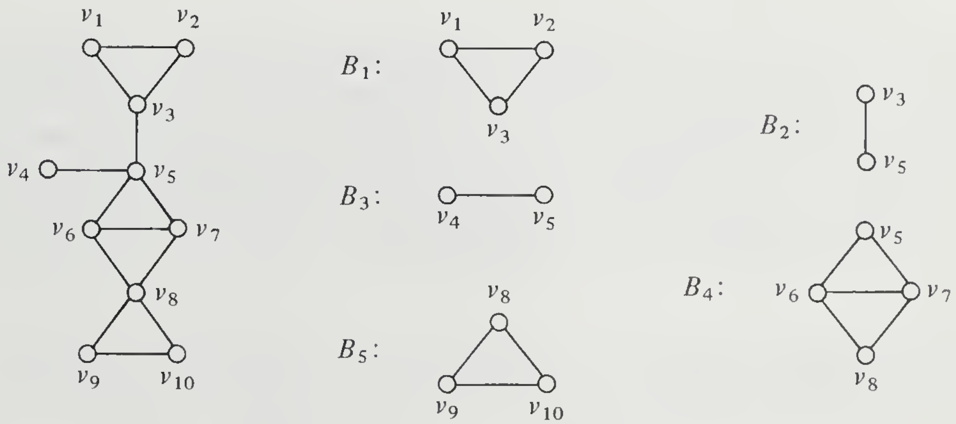


Figure 2.10 *A graph and its five blocks*

Two useful criteria for a graph to be a block are now presented.

Theorem 2.14 *A graph G of order $p \geq 3$ is a block if and only if every two vertices of G lie on a common cycle of G.*

Proof Let G be a graph such that each two of its vertices lie on a cycle. Thus G is connected. Suppose G is not a block; hence G contains a cut-vertex v. By Theorem 2.10, there exist vertices u and w such that v is on every u-w path. Let C be a cycle of G containing u and w. The cycle C determines two distinct u-w paths, one of which cannot contain v, contradicting the fact that every u-w path contains v. Therefore, G is a block.

Conversely, let G be a block with $p \geq 3$ vertices. We show that every two vertices of G lie on a common cycle of G. Let u be an arbitrary vertex of G, and denote by U the set of all vertices that lie on a cycle containing u. We now prove $U = V = V(G)$. Assume $U \neq V$ so that there exists a vertex $v \in V - U$. Since G is a block, it contains no cut-vertices, and furthermore, since $p \geq 3$, the graph G contains no bridge. By Theorem 2.13, every edge of G lies on a cycle of G; hence, every vertex adjacent with u is an element of U. Since G is connected, there exists a u-v path W: $u = u_0, u_1, u_2, \ldots, u_n = v$ in G. Let i be the smallest integer, $2 \leq i \leq n$, such that $u_i \notin U$; thus $u_{i-1} \in U$. Let C be a cycle containing u and u_{i-1}. Because u_{i-1} is not a cut-vertex of G, there exists a u_i-u path P: $u_i = v_0, v_1, v_2, \ldots, v_m = u$ not containing u_{i-1}. If the only vertex common to P and C is u, then a cycle containing u and u_i exists, which produces a contradiction. Hence P and C have a vertex in common different from u. Let j be the smallest integer, $1 \leq j < m$, such that v_j belongs to both P and C. A cycle containing u and u_i can now be constructed by beginning with the u_i-v_j subpath of P, proceeding along C from v_j to u and then to u_{i-1}, and finally taking the edge $u_{i-1}u_i$ back to u_i. Thus, a contradiction arises again, implying that the vertex v does not exist and that every two vertices lie on a cycle. ■

An *internal vertex* of a u-v path P is any vertex of P different from u or v. A collection $\{P_1, P_2, \ldots, P_n\}$ of paths is called *internally disjoint* if each internal vertex of $P_i (i = 1, 2, \ldots, n)$ lies on no $P_j (j \neq i)$. In particular, two u-v paths are internally disjoint if they have no vertices in common, other than u and v. *Edge-disjoint* u-v paths have no edges in common. A second characterization of blocks is now apparent.

Corollary 2.14 *A graph G of order $p \geq 3$ is a block if and only if there exist two internally disjoint u-v paths for every two distinct vertices u and v of G.*

Theorem 2.14 suggests the following definitions: A block of order $p \geq 3$ is called a *cyclic block* while the block K_2 is called the *acyclic block*.

We now state a theorem of which Theorem 2.11 is a corollary.

Theorem 2.15 *Let G be a connected graph with one or more cut-vertices. Then among the blocks of G there are at least two, each of which contains exactly one cut-vertex of G.*

In view of Theorem 2.15, we define an *end-block* of a graph G as a block containing exactly one cut-vertex of G. Hence every connected graph with at least one cut-vertex contains at least two end-blocks. In this context, another result that is often useful is presented. Its proof is left to the reader.

Theorem 2.16 *Let G be a graph with at least one cut-vertex. Then G contains a cut-vertex v with the property that, with at most one exception, all blocks of G containing v are end-blocks.*

Another interesting property of blocks of graphs was pointed out by Harary and Norman [HN2].

Theorem 2.17 *The center of every connected graph G lies in a single block of G.*

Proof Suppose G is a connected graph whose center $Z(G)$ does not lie within a single block of G. Then G has a cut-vertex v such that $G - v$ contains components G_1 and G_2, each of which contains elements of $Z(G)$. Let u be a vertex such that $d(u, v) = e(v)$, and let P_1 be a v-u path of G having length $e(v)$. At least one of G_1 and G_2, say G_2, contains no vertices of P_1. Let w be an element of $Z(G)$ belonging to G_2, and let P_2 be a w-v path of minimum length. The paths P_1 and P_2 together form a u-w path P_3, which is necessarily a u-w path of length $d(u, w)$. However, then $e(w) > e(v)$, which contradicts the fact that $w \in Z(G)$. Thus $Z(G)$ lies in a single block of G. ■

A graph G is a *critical block* if G is a block and for every vertex v, the graph $G - v$ is not a block. Hence a block G is noncritical if and only if there exists a vertex v of G such that $G - v$ is also a block. There is an analogous concept concerning edges. A graph G is a *minimal block* if G is a block and for every edge e, the graph $G - e$ is not a block.

The block G_1 of Figure 2.11 is minimal and noncritical, while the block G_2 is critical but nonminimal.

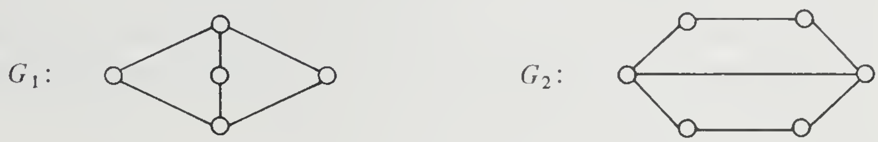


Figure 2.11 *Minimal and critical blocks*

In each of the graphs of Figure 2.11, there are vertices of degree 2. All minimal and critical blocks have this property, as we shall see.

Theorem 2.18 *Every critical block of order at least 4 contains a vertex of degree 2.*

Proof Let G be a critical block of order at least 4. Thus, for each vertex x in G, there exists a vertex y in $G-x$ such that $G-x-y$ is disconnected. Among all pairs x, y of distinct vertices of G, let u, v be a pair for which $G-u-v$ contains a component G_1 of minimum order n, and let G_2 be the union of the remaining components of $G-u-v$.

We prove that $n=1$, which implies that the only vertex of G_1 has degree 2. Assume, to the contrary, that $n \geq 2$. Let

$$H = \langle V(G_1) \cup \{u, v\} \rangle,$$

which, then, is connected. Let $w_1 \in V(G_1)$, and let w_2 be a vertex of $G-w_1$ such that $G-w_1-w_2$ is disconnected. We consider two cases.

Case 1: Suppose that $w_2 \in V(H)$. Since both $\langle V(G_2) \cup \{u\} \rangle$ and $\langle V(G_2) \cup \{v\} \rangle$ are connected, some component of $G-w_1-w_2$ has order less than n, producing a contradiction.

Case 2: Suppose that $w_2 \in V(G_2)$. Since $G-w_1-w_2$ is disconnected, $H-w_1$ must contain exactly two components, namely a component H_u containing u and a component H_v containing v. If either H_u or H_v is trivial, then G has a vertex (namely u or v) of degree 2; so we may assume that H_u and H_v are nontrivial. However, then, there is a component H_1 of $G-w_1-u$ that is a subgraph of H_u-u, so that H_1 has order less than n, yielding a contradiction. ■

Corollary 2.18 *If G is a minimal block of order at least 4, then G contains a vertex of degree 2.*

Proof Suppose that G is a minimal block of order at least 4, but that G contains no vertices of degree 2. By Theorem 2.18, G is not a critical block. Thus, G contains a vertex w such that $G-w$ is a block. Let e be an edge of G incident with w. Since G is a minimal block, $G-e$ is not a block, and therefore $G-e$ contains a cut-vertex $u(\neq w)$. Hence $G-e-u(=G-u-e)$ is disconnected so that e is a bridge of $G-u$. On the other hand, since $G-u-w(=G-w-u)$ is connected, e is a terminal edge of $G-u$ and w is an end-vertex of $G-u$. Therefore, w has degree 1 in $G-u$ and degree 2 in G. This is a contradiction. ■

Exercises 2.2

2.23 Prove Theorem 2.12.

2.24 Show that every connected $(p, p-1)$ graph, $p \geq 3$, contains a cut-vertex.

2.25 Prove that every connected (p, q) graph, $3 \leq p \leq q$, contains a cycle edge.

2.26 Prove that every graph containing only even vertices is bridgeless.

2.27 Prove that if v is a cut-vertex of a connected graph G, then v is *not* a cut-vertex of $\bar{G}$.

2.28 Write out the details of the proof of Corollary 2.14.

2.29 Let G be a block of order $p \geq 3$, and let u and v be distinct vertices of G. If P is a given u-v path of G, does there always exist a u-v path Q such that P and Q are internally disjoint u-v paths?

2.30 For a nontrivial connected graph G, define the *block-cut-vertex graph* $bc(G)$ of G as that graph whose vertices are the blocks and cut-vertices of G and where two vertices of $bc(G)$ are adjacent if and only if one is a cut-vertex of G and the other is a block of G containing the cut-vertex.

(a) Prove that $bc(G)$ is connected and acyclic.

(b) Prove Theorem 2.15.

(c) Prove Theorem 2.16.

2.31 Assuming Theorem 2.15, prove Theorem 2.11.

2.32 Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$, $p \geq 3$. Let H be a graph with $V(H) = \{u_1, u_2, \ldots, u_p\}$.

(a) Vertices u_i and u_j are adjacent in H if and only if v_i and v_j belong to a common cycle in G. Characterize those graphs G for which H is complete.

(b) Vertices u_i and u_j are adjacent in H if and only if $\deg_G v_i + \deg_G v_j$ is odd. Prove that H is bipartite.

2.33 An *element* of a graph G is a vertex or edge of G. Prove that a graph G of order $p \geq 3$ is a block if and only if each pair of elements of G lie on a common cycle of G.

2.34 The *block index* $b(v)$ of a vertex v of a graph G is the number of blocks of G to which v belongs. Let $b(G)$ denote the number of blocks of G. Prove that

$$b(G) = k(G) + \sum_{v \in V(G)} [b(v) - 1].$$

2.35 Let G be a graph having four blocks with $V(G) = \{v_1, v_2, \ldots, v_8\}$. Suppose each v_i, $1 \leq i \leq 6$, lies in exactly one block while each of v_7 and v_8 belongs to exactly two blocks. Prove that G is disconnected.

2.36 Let G be a cyclic block, and let $v \in V(G)$. Prove that there exists a vertex u of G such that $uv \in E(G)$ and $G - v - u$ is connected.

2.37 Let v be a vertex of a cyclic block G. Suppose that $G - v$ contains a cut-vertex u. Let F be a component of $G - v - u$. Prove that $\langle V(F) \cup \{u, v\} \rangle$ is a connected subgraph of G.

2.38 Does there exist a noncritical block G containing an edge $e = uv$ such that $G - e$ is a block, but neither $G - u$ nor $G - v$ is a block?

2.39 Does there exist a graph other than K_2 and the n-cycles, $n \geq 4$, that is a critical block as well as a minimal block?

2.3 Eulerian Graphs and Digraphs

In this section we discuss those trails and circuits in graphs and digraphs which are historically the most famous.

It is difficult to say just when and where graphs originated, but there is justification to the belief that graphs and graph theory may have begun in Switzerland in the early 18th century. In any case, it is evident that the great Swiss mathematician Leonhard Euler [E7] was thinking in graphical terms when he considered the problem of the seven Königsberg bridges.

Figure 2.12 shows a map of Königsberg as it appeared in the 18th century. The river Pregel was crossed by seven bridges, which connected two

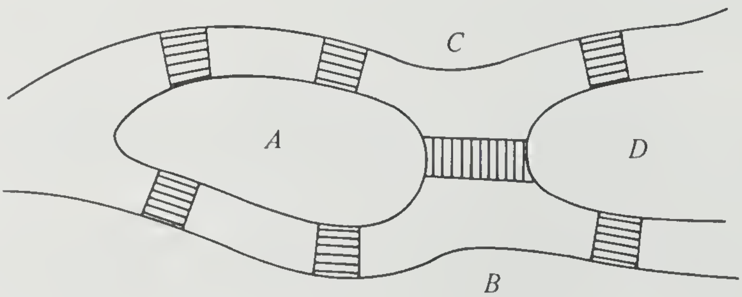


Figure 2.12 *The bridges of Königsberg*

islands in the river with each other and with the opposite banks. We denote the land regions by the letters A, B, C and D (as Euler himself did). It is said that the townsfolk of Königsberg amused themselves by trying to devise a route that crossed each bridge just once. (For a more detailed account of the Königsberg Bridge Problem, see Biggs, Lloyd, and Wilson [BLW1, p. 1].)

Euler proved that such a route over the bridges of Königsberg is impossible—a fact of which many of the people of Königsberg had already convinced themselves. However, it is probable that Euler's approach to the problem was a bit more sophisticated.

Euler observed that if such a route were possible it could be represented by a sequence of eight letters, each chosen from A, B, C, and D. A term of the sequence would indicate the particular land area to which the route had progressed while two consecutive terms would denote a bridge traversed while proceeding from one land area to another. Since each bridge was to be crossed only once, the letters A and B would necessarily appear in the sequence as consecutive terms twice, as would A and C. Also, since five bridges lead to region A, Euler saw that the letter A must appear in the sequence a total of three times—twice to indicate an entrance to and exit from land area A, and once to denote either an entrance to A or exit from A. Similarly, each of the letters B, C, and D must appear in the sequence twice. However, this implies nine terms are needed in the sequence, an impossibility; hence the desired route around Königsberg is also impossible.

The Königsberg Bridge Problem has graphical overtones in many ways; indeed, even Euler's representation of a route around Königsberg is essentially that of a walk in a graph. If each land region of Königsberg is represented by a vertex and two vertices are joined by a number of edges equal to the number of bridges joining corresponding land areas, then the resulting structure (see Figure 2.13) is referred to as a multigraph. In general, if one allows more than one edge (but a finite number) to join pairs of vertices, the result is called a *multiple graph* or *multigraph.*

The Königsberg Bridge Problem is then equivalent to the problem of determining whether the multigraph of Figure 2.13 has a trail containing all its edges.

The Königsberg Bridge Problem suggests the following two concepts. An *eulerian trail* of a connected graph (multigraph) G is an open trail of G containing all the edges of G, while an *eulerian circuit* of G is a circuit containing all the edges of G. A graph (multigraph) possessing an eulerian circuit is called an *eulerian graph* (*multigraph*). The graph G_1 of Figure 2.14 contains an eulerian trail while G_2 is an eulerian graph.

Simple but useful characterizations of both eulerian multigraphs and

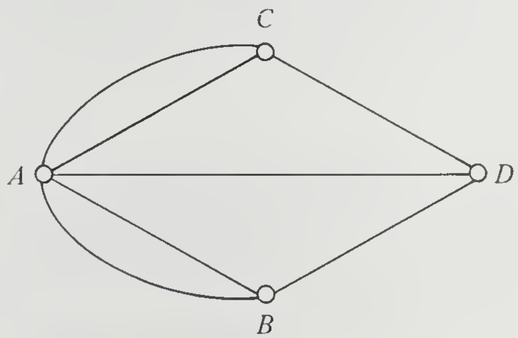


Figure 2.13 *The multigraph of Königsberg*

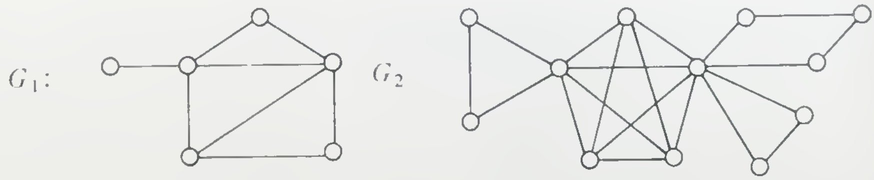


Figure 2.14 *Graphs with eulerian trails and eulerian circuits*

multigraphs with eulerian trails exist; in fact, in each case the characterization was known to Euler [E7]. Complete proofs of these results were not given until 1873, however, in a paper by Hierholzer [H12]. The proof of the next result is based on one due to Fowler [F5].

Theorem 2.19 (Euler) *Let G be a nontrivial connected multigraph. Then G is eulerian if and only if every vertex of G is even.*

Proof Let G be an eulerian multigraph with eulerian circuit C, and let v be an arbitrary vertex of G. If v is not the initial vertex of C (and therefore not the final vertex either), then each time v is encountered on C, it is entered and left by means of distinct edges. Thus each occurrence of v in C represents a contribution of 2 to the degree of v so that v has even degree. If v is the initial vertex of C, then C begins and ends with v, each term representing a contribution of 1 to its degree while every other occurrence of v indicates an addition of 2 to its degree. This gives v an even degree. In either case, v is even.

For the converse, we proceed by induction on the size q of a nontrivial connected multigraph in which every vertex is even. If $q = 2$, then the multigraph consists of two vertices joined by two edges, which has an eulerian circuit. Assume that every nontrivial connected multigraph of size less than $q \geq 3$ and having only even vertices contains an eulerian circuit, and let G be such a multigraph of size q.

If G has order 2, then the two vertices are joined by an even number (at least 4) of edges and G is eulerian. Otherwise, G contains a vertex v adjacent with distinct vertices u and w. Let G' be the multigraph obtained by deleting one of each of the edges uv and vw and adding an edge uw. If G' is connected, then G' contains an eulerian circuit C' by the inductive hypothesis. Replacing an edge uw on C' by deleted edges uv and vw produces an eulerian circuit of G.

Assume now that G' is disconnected. Then G' contains two components, namely a component G_1' containing u and w and a (possibly trivial) component G_2' containing v. By the inductive hypothesis, G_1' contains an eulerian circuit C_1' and, if G_2' is nontrivial, it contains an eulerian circuit C_2'. An eulerian circuit C of G can then be produced by replacing an edge uw on C_1' by deleted edges uv and vw, where at v we insert C_2' if this circuit exists. ■

A characterization of graphs containing eulerian trails can now be presented. Obvious analogues of this and the next two results exist for multigraphs. It will therefore follow from Theorem 2.19 and the multigraph analogue of Theorem 2.20 that the multigraph of Figure 2.13 contains neither an eulerian trail nor an eulerian circuit.

Theorem 2.20 (Euler) *Let G be a nontrivial connected graph. Then G contains an eulerian trail if and only if G has exactly two odd vertices. Furthermore, the trail begins at one of these odd vertices and terminates at the other.*

Proof If G contains an eulerian u-v trail, then, as in the proof of Theorem 2.19, every vertex of G different from u and v is even. It is likewise immediate that each of u and v is odd.

Conversely, let G be a connected graph having exactly two odd vertices u and v. If G does not contain the edge $e = uv$, the graph $G + e$ is eulerian. If the edge e is deleted from an eulerian circuit of $G + e$, then an eulerian trail of G results. In any case, however, a new vertex w can be added to G together with the edges uw and vw, obtaining a graph H in which every vertex is even. Therefore, H is eulerian and contains an eulerian circuit C. The circuit C necessarily contains uw and vw as consecutive edges so that their deletion from C yields an eulerian trail of G. Moreover, this trail begins at u or v and terminates at the other. ∎

Naturally the eulerian trail of Theorem 2.20 has even or odd length, according to whether the graph G has even or odd size. Theorem 2.20 was extended in [CPS1].

Theorem 2.21 *Let G be a connected graph with $2n$ odd vertices, $n \geq 1$. Then $E(G)$ can be partitioned into subsets $E_1, E_2, \ldots, E_n$ so that for each i, $\langle E_i \rangle$ is a trail connecting odd vertices and such that at most one of these trails has odd length.*

Proof Let H denote a graph obtained by adding to G a total of n new vertices $x_i (i = 1, 2, \ldots, n)$ of degree 2 such that each odd vertex G is adjacent to exactly one x_i. The graph H is eulerian and therefore contains an eulerian circuit C. If the x_i are deleted from C, we obtain trails $T_i (i = 1, 2, \ldots, n)$ connecting odd vertices u_i and v_i of G such that every edge of G lies on precisely one such trail.

If at most one trail T_i has odd length, then, of course, there is nothing further to prove. Thus, suppose there are at least two trails T_i of odd length.

If two trails T_j and T_k of odd length have a common vertex w, then there exist trails T_j^* and T_k^* of even length such that

$$E(T_j^*) \cup E(T_k^*) = E(T_j) \cup E(T_k).$$

In order to see this, let T_j' denote a u_j-w subtrail of T_j, and let T_j'' denote the remaining w-v_j subtrail of T_j. Similarly, let T_k' denote a u_k-w subtrail of T_k, and let T_k'' denote the remaining w-v_k subtrail of T_k. The lengths of T_j' and T_j'' are of opposite parity, as are the lengths of T_k' and T_k''. Thus, T_j' may be paired with either T_k' or T_k'' to produce a trail T_j^* of even length, while T_j'' and that subtrail T_k' or T_k'' not on T_j^* form a trail T_k^* also of even length.

We may continue this process until no two trails T_i of odd length remain having a vertex in common. If at most one trail T_i of odd length remains, then the proof is complete. Assume then that there exist two trails, T_r and T_s of odd length that have no vertex in common.

We define the distance $d(T_r, T_s)$ between T_r and T_s as the minimum of the distances between a vertex of T_r and a vertex of T_s. Thus, $d(T_r, T_s) \geq 1$. Let w_r and w_s be vertices of T_r and T_s, respectively, such that $d(w_r, w_s) = d(T_r, T_s)$, and let P be a w_r-w_s path length $d(w_r, w_s)$.

Let $w_r w_0$ be the edge of P incident with w_r. In the present decomposition of G into n trails, assume $w_r w_0$ belongs to trail T. Necessarily T has even length. Suppose that T is a u-v trail. Let T' be a u-w_r subtrail of T, and let T'' be the remaining w_r-v subtrail. Without loss of generality, we assume that T'' contains the edge $w_r w_0$. Also, let T_r' be a u_r-w_r subtrail of T_r, and let T_r'' be the remaining w_r-v_r subtrail of T_r. Either the lengths of T_r' and T'' are of opposite parity or the lengths of T_r'' and T'' are of opposite parity; without loss of generality, assume the former. Define $T_r^{(1)}$ to be the trail composed of T_r' and T'': the trail $T_r^{(1)}$ has odd length and $d(T_r^{(1)}, T_s) < d(T_r, T_s)$. Note that the trail composed of T_r'' and T' has even length.

If $T_r^{(1)}$ and T_s have a vertex in common, then, as we have seen, these trails may be replaced by two trails of even length. Otherwise, we may repeat the above process with T_r and T_s replaced by $T_r^{(1)}$ and T_s, obtaining trails $T_r^{(2)}$ and T_s of odd length for which $d(T_r^{(2)}, T_s) < d(T_r^{(1)}, T_s)$. This process may be continued until trails T_r^* and T_s of odd length that have a common vertex are obtained. In this process no other trail of odd length is altered. The trails T_r^* and T_s may then be replaced by two trails of even length.

This argument may then be repeated as many times as there are pairs of trails having odd length, arriving at a collection of n trails, at most one of which has odd length. ■

We next present a characterization of eulerian graphs of a completely different nature. The necessity is due to Toida [T7] and the sufficiency to McKee [M4].

Theorem 2.22 *A nontrivial connected graph G is eulerian if and only if every edge of G lies on an odd number of cycles.*

Proof First, let G be an eulerian graph and let $e = uv$ be an edge of G. Then $G - e$ is connected. Consider the set of all u-v trails in $G - e$ for which v appears only once, namely as the terminal vertex. There is an odd number of edges possible for the initial edge of such a trail. Once the initial edge has been chosen and the trail has then proceeded to the next vertex, say w, then again there is an odd number of choices for edges incident with w and different from uw. We continue this process until we arrive at vertex v. At each vertex different from v in such a trail, there is an odd number of edges available for a continuation of the trail. Hence there is an odd number of these trails.

Suppose that T_1 is a u-v trail containing v only once, which is not a u-v path. Then some vertex $v_1(\neq v)$ occurs at least twice on T_1, implying that T_1 contains a v_1-v_1 circuit, say C: $v_1, v_2, \ldots, v_n, v_1$. Hence, there exists a u-v trail T_2 identical to T_1 except that C is replaced by the "reverse" circuit C': $v_1, v_n, v_{n-1}, \ldots, v_2, v_1$. This implies that the u-v trails that are not u-v paths occur in pairs. Therefore, there is an even number of such u-v trails that are not u-v paths and, consequently, there is an odd number of u-v paths in $G-e$. This, in turn, implies that there is an odd number of cycles containing e.

For the converse, suppose that G is a nontrivial connected graph that is not eulerian. Then G contains a vertex v of odd degree. For each edge e incident with v, denote by $k(e)$ the number of cycles of G containing e. Since each such cycle contains two edges incident with v, it follows that $\Sigma\, k(e)$ equals twice the number of cycles containing v. Because there is an odd number of terms in the sum $\Sigma\, k(e)$, some $k(e)$ is even. ■

The preceding characterizations presented of eulerian graphs and graphs possessing an eulerian trail are existential in nature. We now introduce a good algorithm, due to Fleury (see [BM2]), which allows us to construct an eulerian circuit in an eulerian graph.

Algorithm 2B (Fleury) *Given an eulerian graph G:*

1. *Select an arbitrary vertex v_0 of G and define T_0:v_0.*
2. *Given that the trail T_i:$v_0, e_1, v_1, e_2, \ldots, e_i, v_i$ has been constructed, select an edge e_{i+1} from*

$$E(G)-\{e_1, e_2, \ldots, e_i\}$$

   subject to the conditions:
   (a) e_{i+1} is incident with v_i;
   (b) unless there is no other choice, e_{i+1} is not a bridge of the graph

$$G_i = G - \{e_1, e_2, \ldots, e_i\}.$$

   If no such edge e_{i+1} exists, then stop.
3. *Define T_{i+1}: $v_0, e_1, v_1, e_2, \ldots, e_{i+1}, v_{i+1}$, where $e_{i+1}=v_iv_{i+1}$.*
4. *Replace i by $i+1$ and return to Step 2.*

Theorem 2B *Any trail constructed by Fleury's algorithm in an eulerian graph is an eulerian circuit.*

Proof Let v_0 be an arbitrary vertex in an eulerian graph G. Suppose that the trail

$$T_n\colon v_0, e_1, v_1, e_2, \ldots, e_n, v_n$$

is constructed by Fleury's algorithm. We show that T_n is an eulerian circuit. For $i = 1, 2, \ldots, n$, let

$$G_i = G - \{e_1, e_2, \ldots, e_i\}.$$

First we show that $v_0 = v_n$. Certainly $\deg_{G_n} v_n = 0$ since Fleury's algorithm terminates at v_n. Every vertex different from v_0 and v_n is incident with an even number of edges in T_n and so has even degree in G_n. If $v_0 \neq v_n$, then every occurrence of v_0 in T_n after its first must be an interior vertex so that $\deg_{G_n} v_0$ is odd. However, then v_0 is the only odd vertex of G_n, which is impossible by Corollary 1.1. Thus, $v_0 = v_n$ and every vertex of G_n has even degree.

It now remains to show that T_n contains every edge of G. Suppose that this is not the case. Since G is connected, there are vertices on T_n that have positive degree in G_n. Let S be the set of all vertices of G having positive degree in G_n and let $\overline{S} = V(G) - S$. Let v_k be the last vertex of T_n that belongs to S. Since $v_n \in \overline{S}$, it follows that $k < n$.

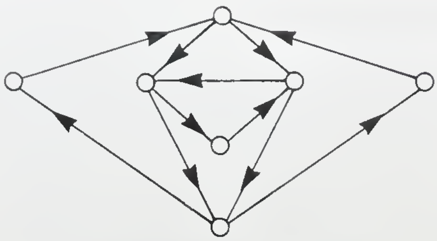
From the definition of $\overline{S}$, each edge of G_k that joins S and $\overline{S}$ is on T_n. We conclude then, from the choice of v_k, that e_{k+1} is the only edge of G_k that joins S and $\overline{S}$. Hence, e_{k+1} is a bridge of G_k.

Let e' be any other edge of G_k that is incident with v_k. By Step 2, since e_{k+1} is a bridge of G_k, so too is e'. Let $H_k = \langle S \rangle_{G_k}$ and $H_n = \langle S \rangle_{G_n}$. Since e' is a bridge of H_k and $H_n \subset H_k$, it follows that e' is a bridge of H_n. Moreover, since e_{k+1} is a bridge of G_k and v_k is the last vertex of T_n that belongs to S, we see that $H_k = H_n$, and $\deg_{H_n} v = \deg_{G_n} v$ for every vertex v of H_n. Thus, every vertex of H_n has even degree, implying that H_n is bridgeless (see Exercise 2.26). This produces a contradiction. ■

We now turn our attention to the directed analogue of eulerian graphs. Two definitions will be helpful. First, the *underlying graph* of a digraph D is that graph G obtained from D by deleting all directions from the arcs of D (equivalently, replacing each arc (u, v) by the edge uv) and deleting an edge from a pair of multiple edges if multiple edges should be produced. A digraph D is then said to be *connected* if its underlying graph G is connected.

The concepts of eulerian trail, eulerian circuit, and eulerian digraph are very much patterned after their graphical counterparts. An *eulerian trail* of a connected digraph D is an open trail of D containing all the arcs of D; an *eulerian circuit* of D is a circuit containing every arc of D. A digraph that contains an eulerian circuit is called an *eulerian digraph*. The digraph D_1 of Figure 2.15 is eulerian while D_2 contains an eulerian trail.

D_1:



D_2:

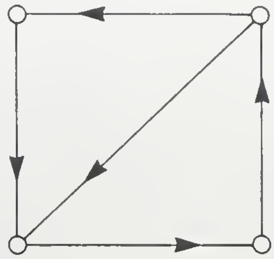


Figure 2.15 *Digraphs with eulerian circuits and eulerian trails*

We now present a characterization of eulerian digraphs whose statement and proof is very similar to Theorem 2.19.

Theorem 2.23 *Let D be a nontrivial connected digraph. Then D is eulerian if and only if* $\text{od}\, v = \text{id}\, v$ *for every vertex v of D.*

With the aid of Theorem 2.23, it is easy to give a characterization of digraphs containing eulerian trails.

Theorem 2.24 *Let D be a nontrivial connected digraph. Then D has an eulerian trail if and only if D contains vertices u and v such that*

$$\text{od}\, u = \text{id}\, u + 1 \quad \textit{and} \quad \text{id}\, v = \text{od}\, v + 1$$

and $\text{od}\, w = \text{id}\, w$ *for all other vertices w of D. Furthermore, the trail begins at u and ends at v.*

Theorem 2.24 can be extended as follows.

Theorem 2.25 *Let D be a connected digraph such that*

$$\sum_{v \in V(D)} |\text{od}\, v - \text{id}\, v| = 2\ell,$$

where $\ell \geq 1$. Then $E(D)$ can be partitioned into subsets $E_1, E_2, \ldots, E_\ell$ so that $\langle E_i \rangle$ is a trail for each $i = 1, 2, \ldots, \ell$.

A related result is also of interest.

Theorem 2.26 *Let D be a connected digraph with vertices u and v such that*

$$\text{od}\, u = \text{id}\, u + \ell \quad \textit{and} \quad \text{id}\, v = \text{od}\, v + \ell,$$

where ℓ is a positive integer and such that $\text{od}\, w = \text{id}\, w$ *for all other vertices w of*

D. Then D contains ℓ arc-disjoint u-v paths.

The concepts of eulerian graphs and digraphs are applicable to many problems such as the routing of street-cleaning and snow-removal vehicles or the inspection of highways in a county system. If a single vehicle or inspector is to be used, then in both cases one would like to find a route that includes each section of street or highway exactly once, i.e., an eulerian circuit or trail in the associated graph. For example, the graph G associated with a county highway system has vertices corresponding to the cities in the county, and two vertices are adjacent if and only if the corresponding cities are connected directly by a highway. Theorems 2.19 and 2.20 give the exact conditions under which the highways can be optimally inspected—if the number of vertices of odd degree in G is 0 or 2. If G has more than two vertices of odd degree, then an alternative is to use an inspection crew, where each member of the crew inspects highways corresponding to an open trail in G. Then Theorem 2.21 indicates that the smallest such crew would have n members, where $2n$ is the number of odd vertices in G. If, however, we are restricted to a single vehicle, then some highway must be traveled more than once. The problem of minimizing repeated highways is equivalent to the well-known Chinese Postman Problem, first formulated by Kwan [K13].

A mail carrier picks up mail at the post office, delivers the mail to each block in the territory, and then returns to the post office. The carrier wishes to choose a route that minimizes the distance traveled. The vertices and edges of the graph G modeling this situation correspond to the street corners and connecting blocks of the mail carrier's territory. The solution, then, to the Chinese Postman Problem is a closed walk of minimum length in G that uses every edge at least once. If G has two or more odd vertices, Goodman and Hedetniemi [GH1] observed that the problem of finding a solution is equivalent to finding the minimum number of edges of G that need to be doubled in order to obtain an eulerian multigraph. By considering all possible pairings of the vertices of odd degree and taking the minimum of the sum of the distances between paired vertices, a solution is obtained. An observation of Edmonds [E2] provides an efficient means of implementing such an approach.

Exercises 2.3

2.40 In present-day Königsberg (Kaliningrad), there are two additional bridges, one between regions B and C, and one between regions B and D. Is it now possible to devise a route over all bridges of Königsberg without recrossing any of them?

2.41 Prove that a nontrivial connected graph G is eulerian if and only if $E(G)$ can be partitioned into subsets E_i, $1 \leq i \leq n$, where $\langle E_i \rangle$ is a cycle of G for each i.

2.42 Let G be a connected graph with $2n$ odd vertices, $n \geq 1$. Show that if $m < n$, then

$E(G)$ cannot be partitioned into subsets $E_1, E_2, \ldots, E_m$ so that $\langle E_i \rangle$ is an open trail for each i.

2.43 Determine an algorithm to construct an eulerian trail in a graph containing an eulerian trail.

2.44 A graph G is *randomly eulerian from a vertex* v if every trail of G with initial vertex v can be extended to an eulerian v-v circuit of G. Give examples of eulerian graphs that are randomly eulerian from none, one, two, or all of their vertices.

2.45 Prove that an eulerian graph G is randomly eulerian from a vertex v if and only if every cycle of G contains v.

2.46 Prove that if a graph G is randomly eulerian from v, then $\Delta(G) = \deg v$, where $\Delta(G)$ is the maximum degree among the vertices of G.

2.47 Let G be an eulerian graph of order $p \geq 3$. Prove that G is randomly eulerian from exactly none, one, two, or all of its vertices.

2.48 Let G be a graph that is randomly eulerian from a vertex v. If $\deg u = \Delta(G)$, where $u \neq v$, then prove G is randomly eulerian from u.

2.49 Using Exercise 2.46 and 2.48, determine a necessary condition for an eulerian graph to be randomly eulerian from one or more vertices.

2.50 Prove Theorem 2.23.

2.51 Prove that a nontrivial connected digraph D is eulerian if and only if $E(D)$ can be partitioned into subsets $E_1, E_2, \ldots, E_\ell$ such that $\langle E_i \rangle$ is a cycle for each $i (1 \leq i \leq \ell)$.

2.52 Prove Theorem 2.24.

2.53 Prove Theorem 2.25.

2.54 Prove Theorem 2.26.

2.55 A digraph D is *randomly eulerian from a vertex* v if every trail of D with initial vertex v can be extended to an eulerian v-v circuit of D. Give examples of eulerian digraphs that are randomly eulerian from none, one, two, or all of their vertices.

2.56 Prove that an eulerian digraph D is randomly eulerian from a vertex v of D if and only if every cycle of D contains v.

2.57 Prove that if an eulerian digraph D is randomly eulerian from a vertex v, then the maximum outdegree among the vertices of D equals $\text{od}\, v$.

2.58 Let D be a digraph that is randomly eulerian from a vertex v and let $u \in V(D)$ such that $\text{od}\, u = \text{od}\, v$. Prove that D is randomly eulerian from u.

2.59 Use the result of Exercises 2.57 and 2.58 to establish a property of randomly eulerian digraphs.

2.60 Prove that if an eulerian digraph D of order $p (\geq 2)$ is randomly eulerian from exactly n vertices, then either $0 \leq n \leq p/2$ or $n = p$.

2.4 An Unsolved Problem in Graph Theory: The Reconstruction Problem

Probably the foremost unsolved problem in graph theory is the Reconstruction Problem. This problem is due to Paul J. Kelly and S. M. Ulam and its origin dates back to 1941. We discuss it briefly in this section.

A graph G with $V(G) = \{v_1, v_2, \ldots, v_p\}$ is said to be *reconstructible* if for every graph H having $V(H) = \{u_1, u_2, \ldots, u_p\}$, $G - v_i \cong H - u_i$ for $i = 1, 2, \ldots, p$ implies that $G \cong H$. Hence, if G is a reconstructible graph, then the subgraphs $G - v$, $v \in V(G)$ determine G uniquely.

We now state a conjecture of Kelly and Ulam, the following formulation of which is due to Frank Harary.

The Reconstruction Conjecture *Every graph of order at least 3 is reconstructible.*

The *Reconstruction Problem* is thus to determine the truth or falsity of the Reconstruction Conjecture.

The condition on the order in the Reconstruction Conjecture is needed since if $G_1 \cong K_2$, then G_1 is *not* reconstructible. If $G_2 \cong 2\,K_1$, then the subgraphs $G_1 - v$, where $v \in V(G_1)$, and the subgraphs $G_2 - v$, for $v \in V(G_2)$, are precisely the same. Thus G_1 is not uniquely determined by its subgraphs $G_1 - v$, $v \in V(G_1)$. By the same reasoning, $G_2 \cong 2K_1$ is also nonreconstructible. The Reconstruction Conjecture claims that K_2 and $2K_1$ are the only nonreconstructible graphs.

Before proceeding further, we note that there is a related problem we shall not consider. Given graphs $G_1, G_2, \ldots, G_p$, does there exist a graph G with $V(G) = \{v_1, v_2, \ldots, v_p\}$ such that $G_i \cong G - v_i$ for $i = 1, 2, \ldots, p$? The answer to this question is not known in general. Although there is a similarity between this question and the Reconstruction Problem, the question is quite distinct from the problem we are interested in.

If there is a counterexample to the Reconstruction Conjecture, then it must have order at least 10, for, with the aid of computers, McKay [M3] and Nijenhuis [N3] have shown that all graphs of order less than 10 (and greater than 2) are reconstructible.

There are several properties of a graph G that can be found by considering the subgraphs $G - v$, $v \in V(G)$. We begin with the most elementary properties.

Theorem 2.27 *If G is a (p, q) graph with $p \geq 3$, then p and q as well as the degrees of the vertices of G are determined from the p subgraphs $G - v$, $v \in V(G)$.*

Proof It is trivial to determine the number p, which is necessarily one greater than the order of any subgraph $G - v$. Also, p is equal to the number of subgraphs $G - v$.

To determine q, label these subgraphs by G_i, $i=1, 2, \ldots, p$, and suppose $G_i \cong G - v_i$, where $v_i \in V(G)$. Let q_i denote the size of G_i. Consider an arbitrary edge e of G, say $e = v_j v_k$. Then e belongs to $p-2$ of the subgraphs G_i, namely all except G_j and G_k. Hence, $\Sigma_{i=1}^{p} q_i$ counts each edge $p-2$ times; that is, $\Sigma_{i=1}^{p} q_i = (p-2)q$. Therefore.

$$q = \frac{\sum_{i=1}^{p} q_i}{p-2}$$

The degrees of the vertices of G can be determined by simply noting that $\deg v_i = q - q_i$, $i = 1, 2, \ldots, p$. ■

We illustrate Theorem 2.27 with the six subgraphs $G - v$ shown in Figure 2.16 of some unspecified graph G. From these subgraphs we determine p, q, and $\deg v_i$ for $i = 1, 2, \ldots, 6$. Clearly, $p = 6$. By calculating the q_i, $i = 1, 2, \ldots, 6$, we find that $q = 9$. Thus, $\deg v_1 = \deg v_2 = 2$, $\deg v_3 = \deg v_4 = 3$, and $\deg v_5 = \deg v_6 = 4$.

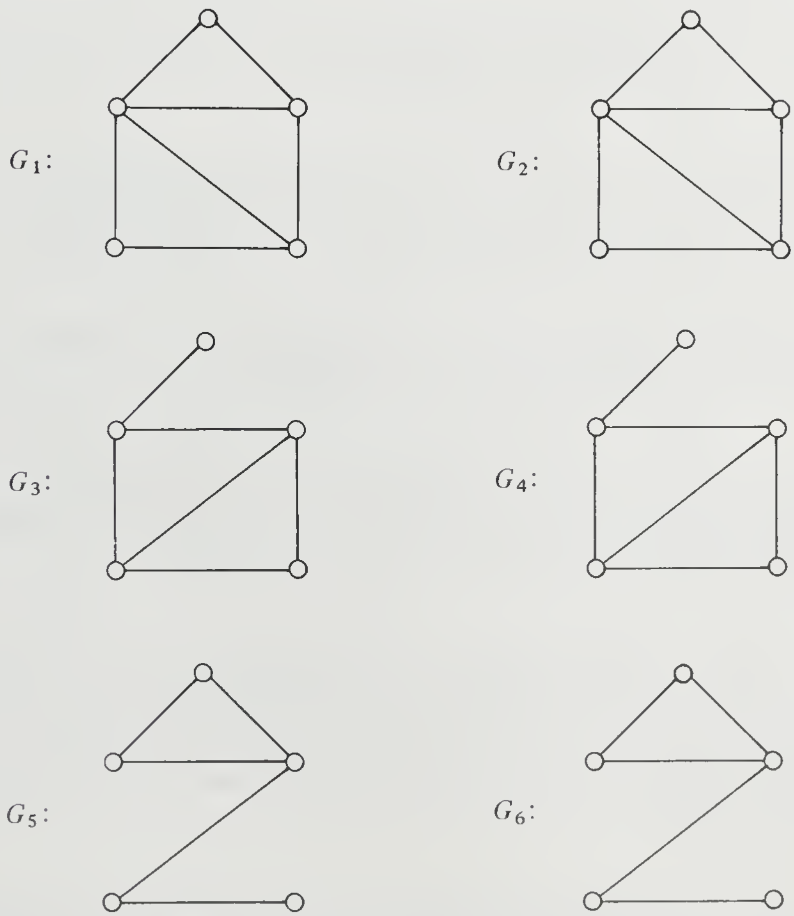


Figure 2.16 *The subgraphs $G - v$ of a graph G*

We say that a graphical parameter or graphical property is *recognizable* if, for each graph G of order at least 3, it is possible to determine the value of the parameter for G or whether G has the property from the subgraphs $G - v$, $v \in V(G)$. Theorem 2.27 thus states that for a graph of order at least 3, the order, the size, and the degrees of its vertices are recognizable parameters. From Theorem 2.27, it also follows that the property of graph regularity is recognizable; indeed, the degree of regularity is a recognizable parameter. For regular graphs, much more can be said.

Theorem 2.28 *Every regular graph of order at least 3 is reconstructible.*

Proof As we have already mentioned, regularity and the degree of regularity are recognizable. Thus, without loss of generality, we may assume that G is an r-regular graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$, $p \geq 3$. It remains to show that G is uniquely determined by its subgraphs $G - v_i$, $i = 1, 2, \ldots, p$. Consider $G - v_1$, say. Add vertex v_1 to $G - v_1$ together with all those edges $v_1 v$ where $\deg_{G-v_1} v = r - 1$. This produces a graph isomorphic to G. ■

If G has order $p \geq 3$, then it is discernible whether G is connected from the p subgraphs $G - v$, $v \in V(G)$.

Theorem 2.29 *For graphs of order at least 3, connectedness is a recognizable property. In particular, if G is a graph with $V(G) = \{v_1, v_2 \ldots, v_p\}$, $p \geq 3$, then G is connected if and only if at least two of the subgraphs $G - v_i$ are connected.*

Proof Let G be a connected graph. By Theorem 2.11, G contains at least two vertices that are not cut-vertices, implying the result.

Conversely, assume there exist vertices $v_1, v_2 \in V(G)$ such that both $G - v_1$ and $G - v_2$ are connected. Thus, in $G - v_1$ and also in G, vertex v_2 is connected to v_i, $i \geq 3$. Moreover, in $G - v_2$ (and thus in G), v_1 is connected to each v_i, $i \geq 3$. Hence every pair of vertices of G are connected and so G is connected. ■

Since connectedness is a recognizable property, it is possible to determine from the subgraphs $G - v$, $v \in V(G)$, whether a graph G of order at least 3 is disconnected. We now show that disconnected graphs are reconstructible. There have been several proofs of this fact. The proof given here is from Manvel [M1].

Theorem 2.30 *Disconnected graphs of order at least 3 are reconstructible.*

Proof We have already noted that disconnectedness in graphs of order at least 3 is a recognizable property. Thus, we assume without loss of generality that G is a disconnected graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$, $p \geq 3$. Further, let $G_i = G - v_i$ for $i = 1, 2, \ldots p$. From Theorem 2.27, the degrees of the v_i, $i = 1, 2, \ldots, p$, can be determined from the $G - v_i$. Hence, if G contains an isolated vertex, then G is reconstructible. Assume then that G has no isolated vertices.

Since every component of G is nontrivial, it follows that $k(G_i) \geq k(G)$ for $i = 1, 2, \ldots, p$ and that $k(G_j) = k(G)$ for some j satisfying $1 \leq j \leq p$. Hence the number of components of G is $\min\{k(G_i) | i = 1, 2, \ldots, p\}$. Suppose F is a component of G of maximum order. Necessarily, F is a component of maximum order among the components of the graphs G_i; that is, F is recognizable. Delete a vertex that is not a cut-vertex from F, obtaining F'.

Assume there are $n(\geq 1)$ components of G isomorphic to F. The number n is recognizable, as we shall see. Let

$$S = \{G_i | k(G_i) = k(G)\},$$

and let S' be the subset of S consisting of all those graphs G_i having a minimum number m of components isomorphic to F. (Observe that if $n = 1$, there exist graphs G_i in S containing no components isomorphic to F; that is, $m = 0$.) In general, then, $n = m + 1$. Next let S'' denote the set of those graphs G_i in S' having a maximum number of components isomorphic to F'.

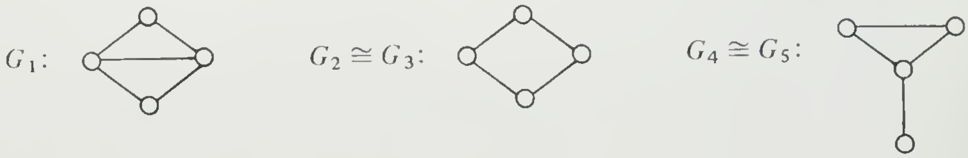
Assume $G_1, G_2, \ldots, G_t$ $(t \geq 1)$ are the elements of S''. *Each G_i in S'' has $k(G)$ components.* Since each $G_i (1 \leq i \leq t)$ has a minimum number of components isomorphic to F, each vertex $v_i (1 \leq i \leq t)$ belongs to a component F_i of G isomorphic to F, where the components F_i of $G (1 \leq i \leq t)$ are not necessarily distinct. Further, since each $G_i (1 \leq i \leq t)$ has a maximum number of components isomorphic to F', it follows that $F_i - v_i \cong F'$ for each $i = 1, 2, \ldots, t$. Hence, every two of the graphs $G_1, G_2, \ldots, G_t$ are isomorphic, and G can be produced from G_1, say, by replacing a component of G_1 isomorphic to F' by a component isomorphic to F. ■

It can be shown that (connected) graphs of order at least 3 whose complements are disconnected are reconstructible (see Exercise 2.64). However, it remains to be shown that *all* connected graphs of order at least 3 are reconstructible.

Exercises 2.4

2.61 Reconstruct the graph G whose subgraphs $G - v$, $v \in V(G)$, are given in Figure 2.16.

2.62 Reconstruct the graph G whose subgraphs $G - v$, $v \in V(G)$, are given in the accompanying figure.



2.63 Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_7\}$ such that $G - v_i \cong K(2, 4)$ for $i = 1, 2, 3$, and $G - v_i \cong K(3, 3)$ for $i = 4, 5, 6, 7$. Show that G is reconstructible.

2.64 (a) Prove that if G is reconstructible, then $\bar{G}$ is reconstructible.

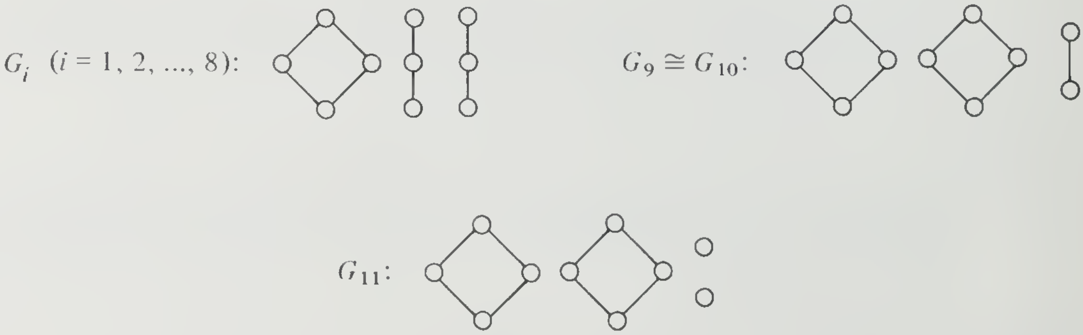
(b) Prove that every graph of order $p(\geq 3)$ whose complement is disconnected is reconstructible.

(a) Prove that the property of a graph being eulerian is recognizable.

(b) Prove that eulerian graphs are reconstructible.

2.65 Prove that bipartiteness is a recognizable property.

2.66 Reconstruct the graph G whose subgraphs $G - v$, $v \in V(G)$, are given in the accompanying figure.



Chapter Three

Trees

Among the connected graphs, the simplest yet most important are the trees. Moreover, "spanning" trees appear in all connected graphs. Trees are rich in applications, particularly in computer science.

3.1 Elementary Properties of Trees

A *tree* is an acyclic connected graph. There are several observations that can be made regarding trees. First, by Theorem 2.13, it follows that every edge of a tree G is a bridge; that is, every block of G is acyclic. Conversely, if every edge of a connected graph G is a bridge, then G is a tree.

There is one tree of each of the orders 1, 2, and 3; while there are two trees of order 4, three trees of order 5, and six trees of order 6. Figure 3.1 shows all trees of order 6.

If u and v are any two nonadjacent vertices of a tree G, then $G + uv$ contains precisely one cycle C. If, in turn, e is any edge of C in $G + uv$, then the graph $G + uv - e$ is once again a tree.

In a nontrivial tree G, it is immediate that the number of blocks to which a vertex v of G belongs equals $\deg v$. Thus, every vertex of G that is not an end-vertex belongs to at least two blocks and is necessarily a cut-vertex. By Theorem 2.11, we have the following.

Theorem 3.1 *Every nontrivial tree has at least two end-vertices.*

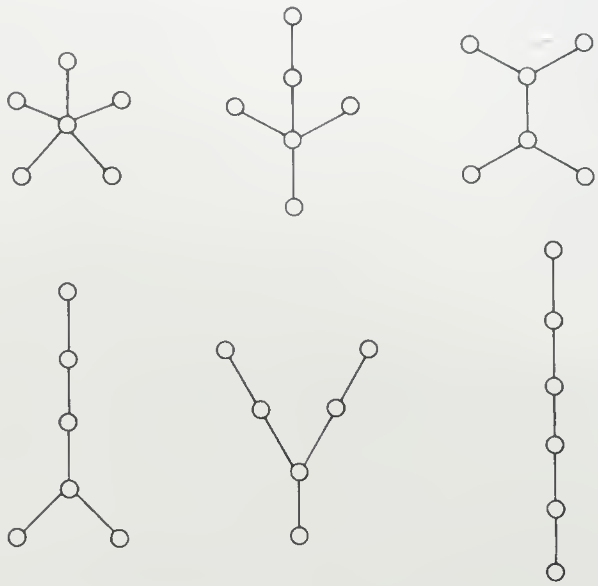


Figure 3.1 *The trees of order 6*

There are a number of ways to characterize trees (for example, see [B7, p. 152] and [H7, p. 32]); three of these are particularly useful.

Theorem 3.2 *A (p, q) graph G is a tree if and only if it is acyclic and $p = q + 1$.*

Proof If G is a tree, then it is acyclic by definition. To verify the equality $p = q + 1$, we employ induction on p. For $p = 1$, the result (and graph) is trivial. Assume, then, that the equality $p = q + 1$ holds for all (p, q) trees with $p \geq 1$ vertices, and let G_1 be a tree with $p + 1$ vertices. Let v be an end-vertex of G_1. The graph $G_2 = G_1 - v$ is a tree of order p, and so $p = q(G_2) + 1$. Since G_1 has one more vertex and one more edge than does G_2, $p(G_1) = p + 1 = (q(G_2) + 1) + 1 = q(G_1) + 1$.

Conversely, let G be an acyclic (p, q) graph with $p = q + 1$. To show G is a tree, we need only verify that G is connected. Denote by $G_1, G_2, \ldots, G_k$ the components of G, where $k \geq 1$. Furthermore, let G_i be a (p_i, q_i) graph. Since each G_i is a tree, $p_i = q_i + 1$. Hence,

$$p - 1 = q = \sum_{i=1}^{k} q_i = \sum_{i=1}^{k} (p_i - 1) = p - k$$

so that $k = 1$ and G is connected. ■

A *forest* is an acyclic graph. Thus each component of a forest is a tree. The proof of Theorem 3.2 provides us with the following result.

Corollary 3.2 *A forest G of order p has $p-k(G)$ edges.*

Another characterization of trees is presented next.

Theorem 3.3 *A (p, q) graph G is a tree if and only if G is connected and $p=q+1$.*

Proof Let G be a (p, q) tree. By definition, G is connected and by Theorem 3.2, $p=q+1$. For the converse, we assume G is a connected (p, q) graph with $p=q+1$. It suffices to show that G is acyclic. If G contains a cycle C and e is an edge of C, then $G-e$ is a connected graph of order p having $p-2$ edges. This is impossible by Exercise 2.4(b); therefore, G is acyclic and is a tree. ■

Hence, any two of the properties (1) connected, (2) acyclic, and (3) $p=q+1$ characterize a tree. There is yet another interesting characterization of trees that deserves mention.

Theorem 3.4 *A graph G is a tree if and only if every two distinct vertices of G are joined by a unique path of G.*

Proof If G is a tree, then certainly every two vertices u and v are joined by at least one path. If u and v are joined by two different paths, then a cycle of G is determined, producing a contradiction.

On the other hand, suppose G is a graph for which every two distinct vertices are joined by a unique path. This implies that G is connected. If G has a cycle C containing vertices u and v, then u and v are joined by at least two paths. This contradicts our hypothesis. Thus, G is acyclic so that G is a tree. ■

Every connected graph G contains a spanning tree. If G itself is a tree, then this is a trivial observation; if G is not a tree, then a spanning tree of G may be obtained by removing cycle edges, one at a time, until finally only bridges remain. If G has q edges, then, of course, it is necessary to delete $q-p+1$ edges in order to obtain a spanning tree of G. A much stronger statement than this can be made, however. A spanning subgraph H of a connected graph G is said to be *distance-preserving from a vertex v in G* if $d_H(v, u)=d_G(v, u)$ for every vertex u. (The following result can be found in [O2, p. 102].)

Theorem 3.5 *For every vertex v of a connected graph G, there exists a spanning tree H that is distance-preserving from v.*

Proof For $i=0, 1, 2, \ldots,$ let

$$A_i(v) = \{u \in V(G) \mid d(u, v) = i\}.$$

Since G is connected, for $u \neq v$ it follows that $u \in A_i(v)$ for some $i \neq 0$. Furthermore, such a vertex u is adjacent with at least one vertex of $A_{i-1}(v)$ and possibly with vertices in $A_i(v)$ and $A_{i+1}(v)$ as well. Delete all but one edge of the type uw, where $w \in A_{i-1}(v)$. Also, remove every edge of the type uw, where $w \in A_i(v)$. Repeat this process for each $u \neq v$: the resulting graph is denoted by H.

From the manner in which H was constructed, it is clear that H is connected since a u-v path exists in H for each $u \neq v$. It is likewise obvious that H is distance-preserving from v. To verify that H is a tree, it remains only to show that H is acyclic. Suppose H contains a cycle C. Let w be a vertex of C whose distance from v is maximum, and let w_1 and w_2 be the vertices adjacent with w on C. Suppose $w \in A_k(v)$; hence $w_i \in A_k(v)$ or $w_i \in A_{k-1}(v)$ for $i = 1, 2$. If either $w_1 \in A_k(v)$ or $w_2 \in A_k(v)$, then we have a contradiction due to the manner in which H was constructed. Thus, $w_1 \in A_{k-1}(v)$ and $w_2 \in A_{k-1}(v)$, which again gives a contradiction. Therefore H is acyclic and hence is a tree. ■

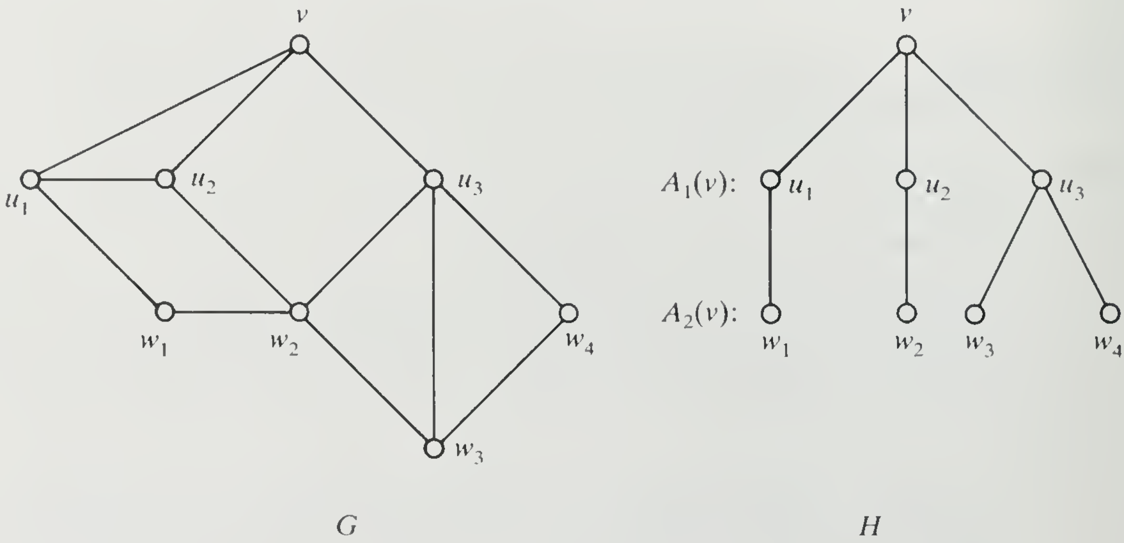


Figure 3.2 *A connected graph G (with vertex v) and a spanning tree H that is distance-preserving from v*

Figure 3.2 shows a connected graph G and a spanning tree that is distance-preserving from a vertex of G.

There are a number of applications involving spanning trees in connected graphs. The *weight of a spanning tree T* in a connected weighted graph G is the sum of the weights of the edges of T. A *minimum spanning tree* is a spanning tree of G of minimum weight.

Suppose we wish to construct a railroad system connecting certain cities and we know the cost to build tracks between each pair of these cities. This situation can be modeled naturally by a connected weighted graph G. Finding a least expensive railroad system connecting all cities is equivalent to determining a minimum spanning tree of G.

We now describe a good algorithm, due to Kruskal [K11], that allows us to construct a minimum spanning tree of a connected weighted graph.

Algorithm 3A (Kruskal) *Given a nontrivial connected weighted graph G:*

1. *Set $i = 1$ and let $E_0 = \emptyset$.*
2. *Select an edge e_i of minimum value not in E_{i-1} such that $T_i = \langle E_{i-1} \cup \{e_i\} \rangle$ is acyclic and define $E_i = E_{i-1} \cup \{e_i\}$. If no such edge exists, let $T = \langle E_{i-1} \rangle$ and stop.*
3. *Replace i by $i + 1$. Return to Step 2.*

Theorem 3A *Algorithm 3A produces a minimum spanning tree in a nontrivial connected weighted graph.*

Proof Let G be a nontrivial connected weighted graph of order p. Certainly, the subgraph T produced by this algorithm is a spanning tree. By Theorem 3.2, T has size $p - 1$. Thus,

$$E(T) = \{e_1, e_2, \ldots, e_{p-1}\},$$

and the weight $w(T)$ of T is given by

$$w(T) = \sum_{i=1}^{p-1} w(e_i).$$

Suppose that T is not a minimum spanning tree. Among the minimum spanning trees of G, let H be one having a maximum number of edges in common with T. Since H and T are not identical, T has one or more edges that are not in H. Let e_i, $1 \le i \le p - 1$, be the first edge of T that is not in H. We add the edge e_i to H, producing a graph G_0 that contains a cycle. Since T is acyclic, there exists a cycle edge e_0 in G_0 that is not in T. The graph $T_0 = G_0 - e_0$ is also a spanning tree of G, and

$$w(T_0) = w(H) + w(e_i) - w(e_0).$$

Since $w(H) \le w(T_0)$, it follows that $w(e_0) \le w(e_i)$. However, by the algorithm, e_i is an edge of minimum weight such that $\langle \{e_1, e_2, \ldots, e_i\} \rangle$ is acyclic. Since $\langle \{e_1, e_2, \ldots, e_{i-1}, e_0\} \rangle$ is acyclic, $w(e_i) = w(e_0)$ so that $w(T_0) = w(H)$. This says that T_0 is also a minimum spanning tree of G, but T_0 has more edges in common with T than does H, contrary to assumption. ■

Knowledge of properties of trees is often useful when attempting to prove certain results about graphs in general. Because of the simplicity of the structure of trees, every graph ordinarily contains a number of trees as subgraphs. Of course, every tree of order p or less is a subgraph of K_p. A more general result is given next.

The *minimum degree* of the vertices of a graph G is denoted by $\delta(G)$.

Theorem 3.6 *Let T be any tree of order m, and let G be a graph with $\delta(G) \geq m-1$. Then T is a subgraph of G.*

Proof The proof is by induction on m. The result is obvious for $m=1$ since K_1 is a subgraph of every graph and for $m=2$ since K_2 is a subgraph of every nonempty graph.

Assume for any tree T' of order $m-1$, $m \geq 3$, and any graph H with $\delta(H) \geq m-2$ that T' is a subgraph of H. Let T be a tree of order m and let G be a graph with $\delta(G) \geq m-1$. We show that $T \subset G$.

Let v be an end-vertex of T and let u be the vertex of T adjacent with v. The graph $T-v$ is necessarily a tree of order $m-1$. The graph G has $\delta(G) \geq m-1 > m-2$; thus by the inductive hypothesis, $T-v$ is a subgraph of G. Let u' denote the vertex of G that corresponds to u. Since $\deg_G u' \geq m-1$ and $T-v$ has order $m-1$, the vertex u' is adjacent to a vertex w that corresponds to no vertex of $T-v$. Therefore, $T \subset G$. ∎

Although no convenient closed formula is known for the number of nonisomorphic trees of order p, such a formula does exist when one considers nonidentical trees. This result is due to Cayley [C3], but since the original proof, the result has been established by a variety of mathematicians using a variety of methods [M9]. The proof given here is due to Clarke [C6].

Theorem 3.7 (Cayley's Tree Formula) *The number of nonidentical trees of order p is p^{p-2}.*

Proof Let N be the number of trees, no two of which are identical, on the p vertices labeled $v_1, v_2, \ldots, v_p$. For $d = 1, 2, \ldots, p-1$, denote by N_d the number of such trees with $\deg v_p = d$. We refer to these N_d graphs as *trees of type d*. We note further that

$$N = \sum_{d=1}^{p-1} N_d.$$

For $d = 2, 3, \ldots, p-1$, let G' be a tree of type $d-1$ and let $v_i(\neq v_p)$ be one of the $p-d$ vertices of G' not adjacent with v_p. Suppose that v_j is the vertex on the unique v_i-v_p path that is adjacent with v_i. Define a *linkage* as an ordered pair (G, G') of trees for which $G = G' + v_i v_p - v_i v_j$. Since each tree of type $d-1$ is linked to $p-d$ trees of type d and no two of these trees of type d are identical, the total number of linkages is $(p-d)N_{d-1}$.

We now derive another expression for the number of linkages. Let G be a tree of type d, and assume that the vertices adjacent with v_p are $v_1, v_2, \ldots, v_d$. Denote by G_i the component of $G - v_p$ containing v_i, $i = 1, 2, \ldots, d$, and let G_i have order p_i. Any tree of type $d-1$ linked to G may be obtained by adding to $G - v_i v_p$ (for some $i = 1, 2, \ldots, d$) an edge joining v_i and a vertex v_j not in G_i. The number of different such edges is $p - 1 - p_i$; hence the number of trees linked to G is $\Sigma_{i=1}^{d}(p - 1 - p_i)$, and no two of these trees are identical. Since

$$\sum_{i=1}^{d} (p - 1 - p_i) = (d-1)(p-1),$$

it follows that the total number of linkages is $(d-1)(p-1)N_d$.

We therefore arrive at the recursive relation $(p-d)N_{d-1} = (d-1)(p-1)N_d$ for $d = 2, 3, \ldots, p-1$.

Using the fact that $N_{p-1} = 1$, we see that $N_{p-2} = (p-2)(p-1)$, and calculating N_{d-1} from N_d, we arrive at

$$N_d = \binom{p-2}{d-1}(p-1)^{p-d-1}.$$

Thus,

$$N = \sum_{d=1}^{p-1} \binom{p-2}{d-1}(p-1)^{p-d-1} = \sum_{d=0}^{p-2} \binom{p-2}{d}(p-1)^{p-d-2},$$

the latter expression being equal to the binomial expansion

$$\sum_{d=0}^{p-2} \binom{p-2}{d}(p-1)^{p-2-d} \cdot 1^d = [(p-1)+1]^{p-2} = p^{p-2},$$

which completes the proof. ■

Theorem 3.7 might be considered as a formula for determining the number of nonidentical spanning trees in the labeled graph K_p. We now consider the same question for graphs in general.

The next result, namely Theorem 3.8, is due to Kirchhoff [K3] and is often referred to as the Matrix-Tree Theorem. The proof given here is based on that given in [H6].

This proof will employ a useful result of matrix theory. Let M and M' be $m \times n$ and $n \times m$ matrices, respectively, with $m \leq n$. An $m \times m$ submatrix M_i of M is said to correspond to the $m \times m$ submatrix M_i' of M' if the column numbers of M determining M_i are the same as the row numbers of M'

determining M_i'. Then

$$\det(M \cdot M') = \sum (\det M_i)(\det M_i'),$$

where the sum is taken over all $m \times m$ submatrices M_i of M, and where M_i' is the $m \times m$ submatrix of M' corresponding to M_i. The numbers det M_i and det M_i' are called the *major determinants* of M and M', respectively.

As an illustration, we have

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 4 & 6 \end{bmatrix},$$

which has a determinant of -36. Writing $|A| = \det A$, we see that

$$\begin{vmatrix} 1 & -2 \\ 2 & 0 \end{vmatrix} \cdot \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} \cdot \begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} -2 & 3 \\ 0 & 4 \end{vmatrix} \cdot \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} = -36.$$

For a graph G with $V(G) = \{v_1, v_2, \ldots, v_p\}$, the *degree matrix* $C(G) = C = [c_{ij}]$ is the $p \times p$ matrix with $c_{ii} = \deg v_i$ and $c_{ij} = 0$ for $i \neq j$.

Theorem 3.8 (The Matrix-Tree Theorem) *If G is a nontrivial labeled graph with adjacency matrix A and degree matrix C, then the number of nonidentical spanning trees of G is the value of any cofactor of the matrix $C - A$.*

Proof We note first that the sum of the entries of row i (column i) of A is deg v_i so that every row (column) sum of $C - A$ is zero. It is a result of matrix theory that all cofactors of $C - A$ have the same value.

Assume first that G is a disconnected graph of order p, and that G_1 is a component of G with $V(G_1) = \{v_1, v_2, \ldots, v_n\}$. Let C' be the $(p-1) \times (p-1)$ submatrix obtained by deleting from $C - A$ the last row and last column. Since the sum of the first n rows of C' is the zero vector with $p - 1$ entries, the rows of C' are linearly dependent, implying that det $C' = 0$. Hence one cofactor of $C - A$ has value zero. This is, of course, the number of spanning trees of G.

We henceforth assume G to be a connected (p, q) graph, where $q \geq p - 1$. Let B denote the incidence matrix of G and in each column of B, replace one of the two nonzero entries by -1. Denote the resulting matrix by $M = [m_{ij}]$. We now show that the product of M and its transpose M^t is $C - A$. The (i, j) entry of MM^t is

$$\sum_{k=1}^{q} m_{ik} m_{jk},$$

which has the value $\deg v_i$ if $i = j$, the value -1 if $v_i v_j \in E(G)$, and 0 otherwise. Therefore, $MM^t = C - A$.

Consider a spanning subgraph H of G containing $p-1$ edges. Let M' be the $(p-1)\times(p-1)$ submatrix of M determined by the columns associated with the edges of H and by all rows of M with one exception, say row k.

We now determine $|\det M'|$. If H is not connected, then H has a component H_1 not containing v_k. The sum of the row vectors of M' corresponding to the vertices of H_1 is the zero vector with $p-1$ entries; hence $\det M' = 0$.

Assume now that H is connected so that H is (by Theorem 3.3) a spanning tree of G. Let $u_1(\neq v_k)$ be an end-vertex of H, and e_1 the edge incident with it. Next, let $u_2(\neq v_k)$ be an end-vertex of the tree $H-u_1$ and e_2 the edge of $H-u_1$ incident with u_2. We continue this procedure until finally only v_k remains. A matrix $M''=[m''_{ij}]$ can now be obtained by a permutation of the rows and columns of M' such that $|m''_{ij}|=1$ if and only if u_i and e_j are incident. From the manner in which M'' was defined, any vertex u_i is incident only with edges e_j, where $j\leq i$. This, however, implies that M'' is lower triangular, and since $|m''_{ii}|=1$ for all i, we conclude that $|\det M''|=1$. However, the permutation of rows and columns of a matrix affects only the sign of its determinant, implying that $|\det \mathrm{M}'|=|\det M''|=1$.

Since every cofactor of $C-A$ has the same value, we evaluate only the ith principal cofactor; that is, the determinant of the matrix obtained by deleting from $C-A$ both row i and column i. Denote by M_i the matrix obtained from M by removing row i, so that the aforementioned cofactor equals det $(M_iM_i^t)$, which, by the remark preceding the statement of this theorem, implies that this number is the sum of the products of the corresponding major determinants of M_i and M_i^t. However, corresponding major determinants have the same value and their product is 1 if the defining columns correspond to a spanning tree of G and is 0 otherwise. This completes the proof. ■

Exercises 3.1

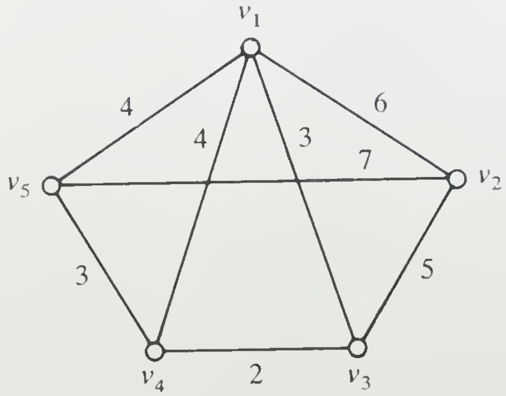
3.1 Draw all forests of order 6.

3.2 Prove that the number of end-vertices in a nontrivial tree T with $V(T)=\{v_1, v_2, \ldots, v_p\}$ equals

$$2+\sum_{\deg v_i\geq 3}(\deg v_i-2).$$

3.3 Let v be a vertex of degree 2 in the complete bipartite graph $K(2, 4)$. Show that this graph contains nonisomorphic spanning trees that are distance-preserving from v.

3.4 Use Kruskal's algorithm to find a minimum spanning tree of the weighted graph in the accompanying figure.



3.5 Show that every tree of order at least 3 contains a cut-vertex v such that every vertex adjacent to v, with at most one exception, is an end-vertex.

3.6 Prove Theorem 3.7 as a corollary to Theorem 3.8.

3.7 (a) Prove that a graph G is a forest if and only if every induced subgraph of G contains a vertex of degree at most 1.

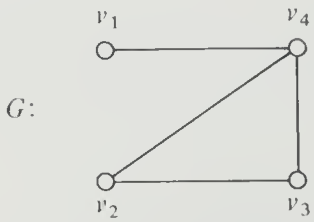
(b) Characterize those graphs with the property that every connected subgraph is an induced subgraph.

3.8 A tree is called *central* or *bicentral* depending on whether its center consists of one vertex or two adjacent vertices.

(a) Prove that every tree is central or bicentral.

(b) Let T be a tree with diameter d and radius r. Prove that T is central or bicentral according to whether $d = 2r$ or $d = 2r - 1$.

3.9 Let G be the labeled graph below.



(a) Use the Matrix-Tree Theorem to compute the number of nonidentical spanning trees of G.

(b) Draw all nonidentical spanning trees of G.

3.10 Let T be a nontrivial tree of order p such that $T \not\cong K(1, p-1)$. Prove that $T \subset \bar{T}$.

3.11 A (p, q) graph G is called *graceful* if it is possible to label the vertices of G with distinct positive integers in such a way that the induced edge labeling, which prescribes the integer $|i-j|$ to the edge joining the vertices labeled i and j, assigns the labels 1, 2, ..., q to the q edges of G. It has been conjectured that every tree is graceful.

(a) Prove that every path is graceful.

(b) Prove that every star is graceful.

(c) Show that every tree of order 6 is graceful.

3.12 A *unicyclic graph* is a connected graph that contains exactly one cycle. Prove that a (p, q) graph G is unicyclic if and only if G has any two of the properties: (a) G is connected, (b) G has exactly one cycle, (c) $p = q$.

3.13 A graph G is *geodetic* if, between every two vertices u and v of G, there is a unique u-v path of length $d(u, v)$. A *cactus* is a connected graph, every cyclic block of which is a cycle. Thus every tree is both geodetic and a cactus. Prove the following.

(a) If G is a geodetic graph, then any cycle of G having smallest length is odd.

(b) If every cycle of a connected graph G is odd, then G is both geodetic and a cactus.

3.2 n-Ary Trees

A *directed tree* is an asymmetric digraph whose underlying graph is a tree. A *rooted tree* is a directed tree T with some vertex r (called the *root*) such that T contains an r-v path for every vertex v of T. Thus a rooted tree with root r contains no v-r path for each vertex $v \neq r$. Furthermore, id $r = 0$ and id $v = 1$ for all $v \neq r$. Figure 3.3 shows a directed tree T_1 that is not rooted and a rooted tree T_2 (with root r).

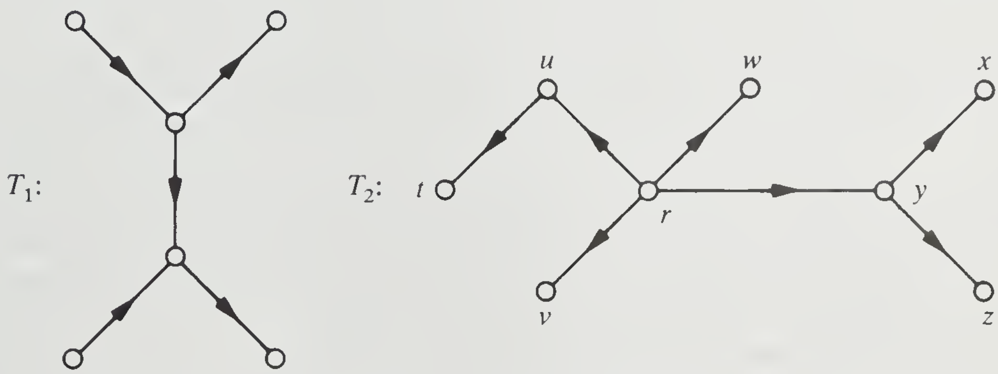


Figure 3.3 *A rooted tree T_2 and a directed tree T_1 that is not rooted*

The customary way to draw a rooted tree T is to place the root at the top. The vertices of T adjacent from the root are placed one level below the root;

the vertices of T adjacent from these vertices are placed another level below, and so on. Thus, the rooted tree T_2 of Figure 3.3 can be redrawn as in Figure 3.4(a). In fact, since all arcs are directed downward, we may remove all arrows and draw T_2 as in Figure 3.4(b). Again we emphasize that the drawing of Figure 3.4(b) represents a rooted tree (a type of digraph), not a tree.

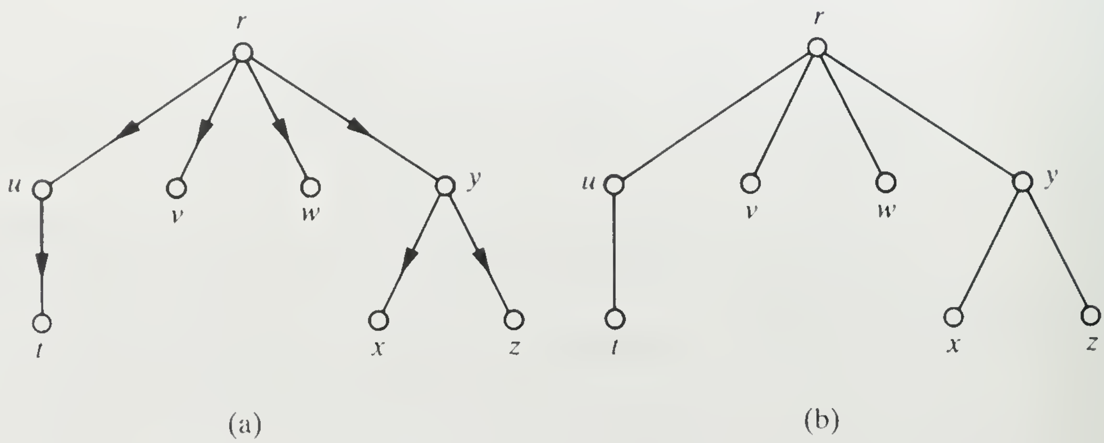


Figure 3.4 *A rooted tree*

If T is a rooted tree with root r and v is a vertex of T, then the *level number* of v is the length of the unique r-v path in T. The maximum of the level numbers of the vertices of T is called the *height* of T and is denoted by $h(T)$. For the rooted tree T_2 of Figure 3.4, the root r is at level 0, the vertices u, v, w, and y are at level 1, and the vertices t, x, and z are at level 2. Also $h(T_2) = 2$.

In working with rooted trees, some rather descriptive terminology is often used. Let T be a rooted tree with root r. For any vertex $v \neq r$, the *father* of v is that unique vertex u that is adjacent to v. Conversely, v is the *son* of u. Two vertices having the same father are *brothers*. In the rooted tree of Figure 3.4, y is the father of x, while x is the son of y; the vertices x and z are brothers.

Vertices of a rooted tree having no sons (having outdegree 0) are called *leaves*. All other vertices (those with sons) are called *internal vertices*. The rooted tree of Figure 3.4 has five leaves and three internal vertices.

In most applications of rooted trees, there is a limit as to how many sons a vertex can have. If every vertex of a rooted tree T has n or fewer sons, then T is called an *n-ary tree*. If every vertex of T has either n or no sons, then T is a *complete n-ary tree*. A rooted tree T is *ordered* if the sons of each vertex of T are ordered (as first son, second son, and so on). In a drawing of an ordered tree, the sons are ordered from left to right.

In Figure 3.5, the rooted tree T_1 is a complete 3-ary tree while T_2 is a 3-ary tree that is not a complete 3-ary tree. The rooted trees T_3 and T_4 are complete 2-ary trees. As ordered trees they are not equal since w is the first son of t in T_3 while w is the second son of t in T_4. As unordered trees, they are considered equal.

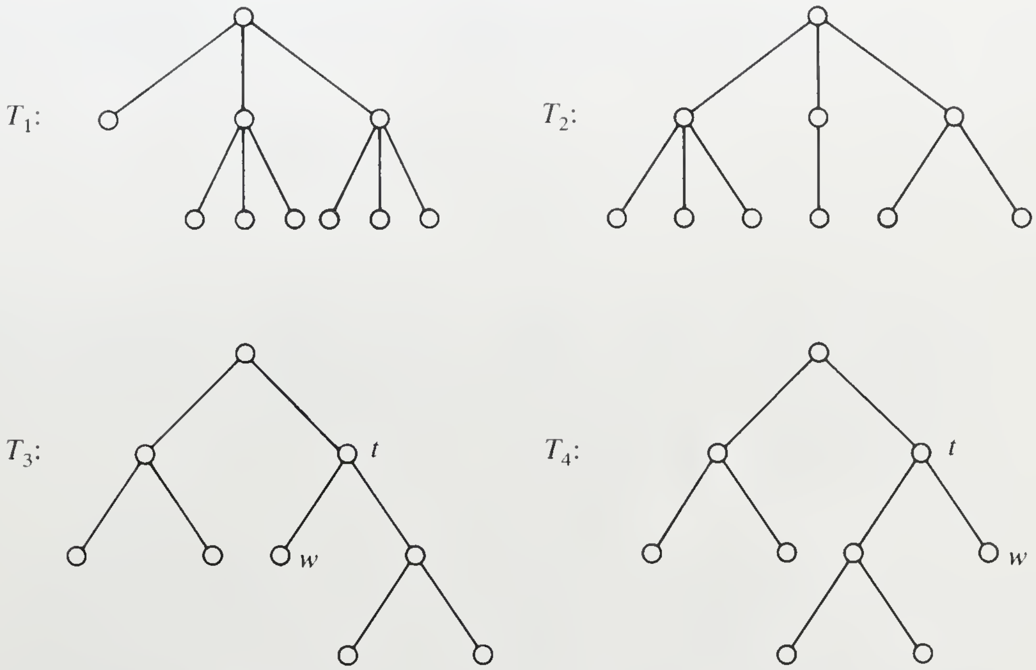


Figure 3.5 *Ordered and unordered n-ary trees*

The following result provides us with an elementary formula involving the order and the number of internal vertices in a complete n-ary tree.

Theorem 3.9 *A complete n-ary tree T with i internal vertices has order* $p = ni + 1$.

Proof Since every internal vertex has n sons and every son has only one father, there are exactly ni sons in T. Because only the root is not a son, it follows that $p = ni + 1$. ∎

Corollary 3.9 *A complete 2-ary tree T with i internal vertices has* $2i + 1$ *vertices,* $i + 1$ *of which are leaves.*

Proof By Theorem 3.9, T has order $2i + 1$. Since every vertex of T is an internal vertex or a leaf, T has $i + 1$ leaves. ∎

We now present a relationship between the order of a 3-ary tree and its height.

Theorem 3.10 *If T is a 2-ary tree of height h and order p, then*

$$h + 1 \leq p \leq 2^{h+1} - 1.$$

Proof Let p_k $(0 \leq k \leq h)$ be the number of vertices at level k; hence, $\sum_{k=0}^{h} p_k = p$. Since $1 \leq p_k \leq 2^k$, it follows that

$$h + 1 = \sum_{k=0}^{h} 1 \leq \sum_{k=0}^{h} p_k \leq \sum_{k=0}^{h} 2^k = 2^{h+1} - 1,$$

producing the desired result. ■

A rooted tree T of height h is *balanced* if every leaf is at level h or $h-1$. Figure 3.6 shows two complete 2-ary trees; the first is balanced while the second is not. An immediate corollary of Theorem 3.10 can now be given. For any real number x, we denote by $\lceil x \rceil$ the *ceiling function* of x—that is, the smallest integer not less than x.

Figure 3.6 *Balanced and unbalanced complete 2-ary trees*

Corollary 3.10 *If T is a 2-ary tree of height h and order p, then $h \geq \log_2((p+1)/2)$. If T is a balanced complete 2-ary tree, then $h = \lceil \log_2((p+1)/2) \rceil$.*

Proof Let T be a 2-ary tree of height h and order p. By Theorem 3.10, $p \leq 2^{h+1} - 1$ or $2^h \geq (p+1)/2$. Consequently, $h \geq \log_2((p+1)/2)$. If T is a balanced complete 2-ary tree, then it follows that $p > 2^h - 1$ so that $2^h - 1 < p \leq 2^{h+1} - 1$ and

$$2^{h-1} < \frac{p+1}{2} \leq 2^h.$$

Hence, $h - 1 < \log_2((p+1)/2) \leq h$, implying that $h = \lceil \log_2((p+1)/2) \rceil$. ■

One use of 2-ary trees is in scheduling playoffs among a certain number of individuals or teams. For example, suppose there are thirteen contestants for a tennis championship. Then a playoff scheme can be developed by means of any complete 2-ary tree with thirteen leaves. By Corollary 3.9, such a complete 2-ary tree has twelve internal vertices and order 25. Two such complete 2-ary trees T_1 and T_2 are shown in Figure 3.7. The root c in each case denotes the champion.

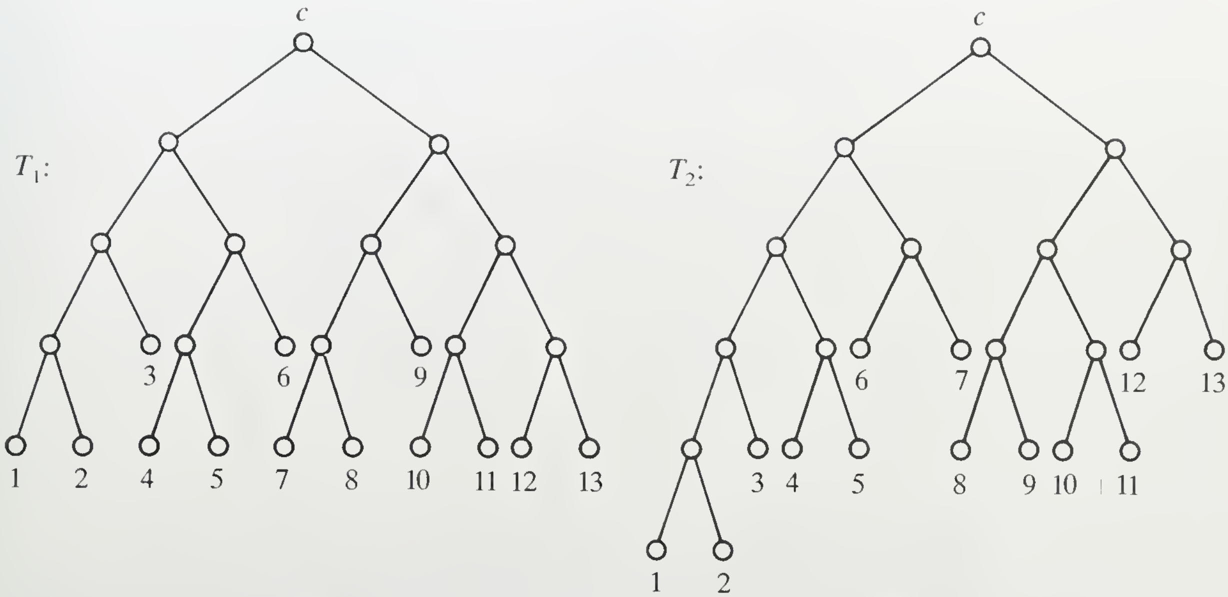


Figure 3.7 *Two complete 2-ary trees with thirteen leaves*

When using a complete 2-ary tree to represent a playoff scheme, we normally draw the tree with the root at the bottom. For example if we were to draw the complete 2-ary tree T_1 of Figure 3.7 in this manner, the result is the diagram of Figure 3.8. The labeling indicates that contestant 2 defeated contestant 1, contestant 4 defeated contestant 5, and so on. The champion is the winner of the match between contestants 6 and 9.

In each complete 2-ary tree of Figure 3.7, $p=25$ so, by Corollary 3.10, the height h is at least $\log_2 13$; so $h \geqslant 4$. The complete 2-ary tree T_1 is balanced so that $h=4$. This implies that every contestant must win either three or four matches to win the championship. On the other hand, T_2 is not balanced; the height of T_2 is 5. In this case, contestant 1 must win five matches to become champion while contestant 12, for example, need win only three matches. Such a disparity always occurs when one is dealing with an unbalanced 2-ary tree. An unbalanced 2-ary tree may be appropriate if certain contestants are required to win additional matches to qualify for a main tournament and/or past champions are required to win fewer matches.

In many applications of 2-ary trees, every vertex other than the root is specified as the left son or right son of its father. Such 2-ary trees are referred to as *binary trees.* As we shall see, in binary trees the left son is often "less than" its father and the right son is "greater than" its father. Since a binary tree is a 2-ary tree, Corollary 3.9, Theorem 3.10, and Corollary 3.10 provide results on binary trees.

Binary trees are commonly used in a variety of searches. For example, they may be used in a search for a word W against words in some set containing W. The maximum number of tests needed to recognize W is the

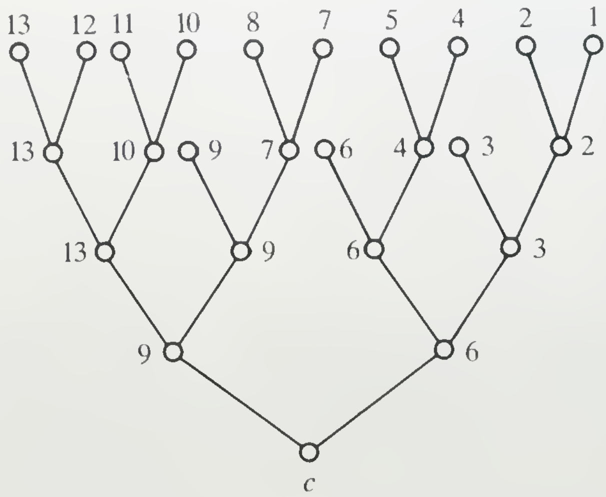


Figure 3.8 *Representing a playoff scheme by a complete 2-ary tree*

height of an appropriate balanced binary tree. Suppose we are searching for a letter of the alphabet. A search process can be illustrated by means of the binary tree of Figure 3.9. We choose the "middle" letter M as the root. This indicates that we first test to see if M is the letter we are searching for. If not, we test to determine whether the letter precedes M or follows M. If, for example, we learn that our letter follows M, we move to the right son of M and test to see if it is the letter T. If it is not T, we apply a test to determine whether it precedes T (and follows M) or follows T, and so on. The height of this tree is 4, implying that at most four tests are required to locate the letter.

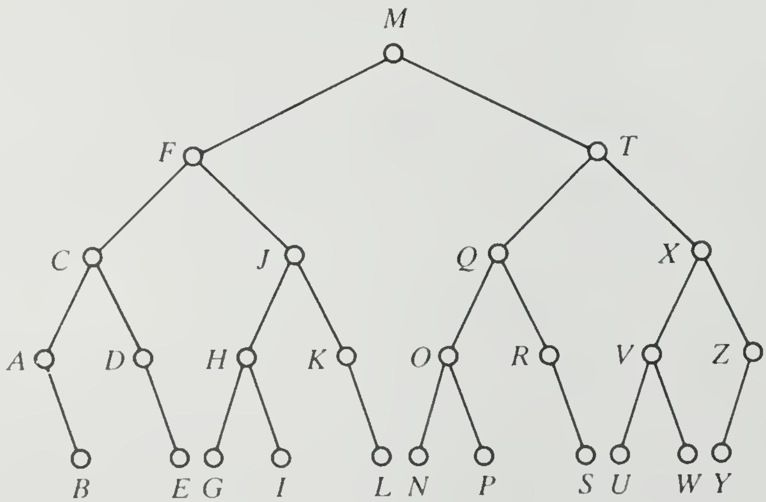


Figure 3.9 *A binary search tree*

Exercises 3.2

3.14 Let T be a directed tree with the property that id $r = 0$ for some vertex r of T but id $v = 1$ for all $v \neq r$. Prove that T is a rooted tree with root r.

3.15 Prove that the bounds given in Theorem 3.10 are sharp.

3.16 Determine and prove a result that generalizes Theorem 3.10 to n-ary trees.

3.3 Decomposition of Graphs into Acyclic Subgraphs

We now return to our main topic in this chapter—(undirected) acyclic graphs in general, and trees in particular.

One of the most common problems in graph theory deals with the decomposition of a graph into various subgraphs possessing some prescribed property. There are ordinarily two problems of this type, one dealing with a decomposition of the vertex set and the other with a decomposition of the edge set. One such property that has been the subject of investigation is that of being acyclic, which we now consider.

For any graph G, it is possible to partition $V(G)$ into subsets V_i, $1 \leq i \leq n$, such that each induced subgraph $\langle V_i \rangle$ is acyclic; that is, is a forest. This can always be done by selecting each V_i so that $|V_i| \leq 2$; however, the major problem is to partition $V(G)$ so that as few subsets as possible are involved. This suggests our next concept. The *vertex-arboricity* $a(G)$ of a graph G is the minimum number of subsets into which $V(G)$ can be partitioned so that each subset induces an acyclic subgraph. It is obvious that $a(G) = 1$ if and only if G is acyclic. For a few classes of graphs, the vertex-arboricity is easily determined. For example, $a(C_p) = 2$. If p is even, $a(K_p) = p/2$, while if p is odd, $a(K_p) = (p+1)/2$. Also, $a(K(m, n)) = 1$ if $m = 1$ or $n = 1$, and $a(K(m, n)) = 2$ otherwise. No formula is known in general, however, for the vertex-arboricity of a graph although some bounds for this number exist. First, it is clear that for any graph G of order p,

$$a(G) \leq \left\lceil \frac{p}{2} \right\rceil. \tag{3.1}$$

The bound (3.1) is not particularly good. In order to present a sharper bound, a new concept is introduced at this point.

A graph G is called *critical with respect to vertex-arboricity* if $a(G-v) < a(G)$ for all vertices v of G. This is the first of several occasions when a graph will be defined as critical with respect to a certain parameter. In order to avoid cumbersome phrases, we will simply use the term "critical" when the parameter involved is clear by context. In particular, a graph G that is critical with respect to vertex-arboricity will be referred to in this section as a critical graph and, further, as an *n-critical* graph if $a(G)=n$. The complete graph K_{2n-1} is n-critical while each cycle is 2-critical. It is not difficult to locate critical graphs; indeed, every graph G with $a(G)=n\geq 2$ contains an induced n-critical subgraph. In fact, any induced subgraph G' of G with $a(G')=n$ and having minimum order is n-critical.

Before presenting the aforementioned bound for $a(G)$, we give another result.

Theorem 3.11 *If G is a graph having $a(G)=n\geq 2$ that is critical with respect to vertex-arboricity, then $\delta(G)\geq 2(n-1)$.*

Proof Let G be an n-critical graph, $n\geq 2$, and suppose G contains a vertex v of degree $2n-3$ or less. Since G is n-critical, $a(G-v)=n-1$ and there is a partition $V_1, V_2, \ldots, V_{n-1}$ of the vertex set of $G-v$ such that each subgraph $\langle V_i\rangle$ is acyclic. Because $\deg v \leq 2n-3$, at least one of these subsets, say V_j, contains at most one vertex adjacent with v in G. The subgraph $\langle V_j \cup \{v\}\rangle$ is necessarily acyclic. Hence $V_1, V_2, \ldots, V_j\cup\{v\}, \ldots, V_{n-1}$ is a partition of the vertex set of G into $n-1$ subsets, each of which induces an acyclic subgraph. This contradicts the fact that $a(G)=n$. ■

We are now in a position to present the desired upper bound [CK1]. (Recall that the notation $H < G$ indicates that H is an induced subgraph of G.) The symbol $\lfloor x\rfloor$, for a real number x, is called the *floor function* of x and represents the greatest integer not exceeding x.

Theorem 3.12 *For each graph G,*

$$a(G)\leq 1+\left\lfloor \frac{\max \delta(G')}{2}\right\rfloor,$$

where the maximum is taken over all induced subgraphs G' of G.

Proof The result is obvious for acyclic graphs; thus, let G be a graph with $a(G)=n\geq 2$. Furthermore, let H be an induced n-critical subgraph of G. Since H itself is an induced subgraph of G,

$$\delta(H)\leq \max_{G'<G} \delta(G'). \tag{3.2}$$

By Theorem 3.11, $\delta(H) \geqslant 2n-2$, so by (3.2),

$$\max_{G' < G} \delta(G') \geqslant 2n - 2 = 2a(G) - 2.$$

This inequality now produces the desired result. ■

Let $\Delta(G)$ denote the *maximum degree* among the vertices of a graph G. Since $\delta(G') \leqslant \Delta(G)$ for $G' < G$, we note the following consequence of the preceding result.

Corollary 3.12 *For any graph G,*

$$a(G) \leqslant 1 + \left\lfloor \frac{\Delta(G)}{2} \right\rfloor.$$

We now turn to the second decomposition problem. The *edge-arboricity*, or simply the *arboricity*, $a_1(G)$ of a nonempty graph G is the minimum number of subsets into which $E(G)$ can be partitioned so that each subset induces an acyclic subgraph. As with vertex-arboricity, a nonempty graph has arboricity 1 if and only if it is a forest. Unlike vertex-arboricity, however, there is a formula for the arboricity of any graph [N1].

Theorem 3.13 (Nash-Williams) *For any nonempty graph G,*

$$a_1(G) = \max_{H < G} \left\lceil \frac{q(H)}{p(H)-1} \right\rceil,$$

where the maximum is taken over all nontrivial induced subgraphs H of G.

As a consequence of Theorem 3.13, it follows that

$$a_1(K_p) = \left\lceil \frac{p}{2} \right\rceil \quad \text{and} \quad a_1(K(m, n)) = \left\lceil \frac{mn}{m+n-1} \right\rceil.$$

It is interesting to note that when p is even, K_p can be expressed as the edge sum of $p/2$ spanning paths, as shown by Beineke [B2], and when p is odd, K_p can be expressed as the edge sum of $(p+1)/2$ subgraphs, $(p-1)/2$ of which are isomorphic to $P_{p-1} \cup K_1$ and the other isomorphic to $K(1, p-1)$. Decomposing a graph into pairwise edge-disjoint acyclic subgraphs is a special case of the more general subject of "factorization", which will be considered in Chapter 8.

Exercises 3.3

3.17 What upper bounds for $a(K(1, n))$ are given by Theorem 3.12 and Corollary 3.12?

3.18 Let G be an n-critical graph with respect to vertex-arboricity ($n \geq 3$). Prove that for each vertex v of G, the graph $G - v$ is *not* $(n-1)$-critical with respect to vertex-arboricity.

3.19 Give an example of a graph G that has a nonempty induced subgraph H such that

$$\left\lceil \frac{q(G)}{p(G)-1} \right\rceil < \left\lceil \frac{q(H)}{p(H)-1} \right\rceil,$$

thereby proving that, in general, $a_1(G) \neq \lceil q(G)/(p(G)-1) \rceil$. Determine $a_1(G)$ for this graph.

Chapter Four

Graph Embeddings

We now consider graphs that can be drawn in the plane without their edges crossing. A formula developed by Euler plays a central role in the study of these "planar" graphs. We then extend this concept to graphs that can only be drawn on more complex surfaces.

4.1 Euler's Formula

A (p, q) graph G is said to be *realizable* or *embeddable* on a surface S if it is possible to distinguish a collection of p distinct points of S that correspond to the vertices of G and a collection of q curves, pairwise disjoint except possibly for endpoints, on S that correspond to the edges of G such that if a curve A corresponds to the edge $e = uv$, then only the endpoints of A correspond to vertices of G, namely u and v. Intuitively, G is embeddable on S if G can be drawn on S so that edges (more precisely, the curves corresponding to edges) intersect only at a vertex (that is, a point corresponding to a vertex) mutually incident with them. In this section we are concerned exclusively with the case in which S is a plane or sphere.

A graph is *planar* if it can be embedded in the plane. Embedding a graph in the plane is equivalent to embedding it on the sphere. In order to see this, we perform a *stereographic projection*. Let S be a sphere tangent to a plane π, where A is the point of S diametrically opposite to the point of tangency. If a graph G is embedded on S in such a way that no vertex of G is A and no edge of G passes through A, then G may be projected onto π to produce an embedding of G on π. The inverse of this projection shows that any graph that can be embedded in the plane can also be embedded on the sphere.

If a planar graph is embedded in the plane, then it is called a *plane* graph. The graph $G_1 \cong K(2, 3)$ of Figure 4.1 is planar, although, as drawn, it is not plane; however, $G_2 \cong K(2, 3)$ is both planar and plane. The graph $G_3 \cong K(3, 3)$ is nonplanar. This last statement will be proved presently.

Probably the most practical use of planar graphs is in the design of electrical circuits. Indeed, printed circuit boards and integrated semiconductor chips are essentially planar graphs. If the graph corresponding to a printed circuit is planar, then a single circuit board suffices. Otherwise, modifications are necessary to avoid short circuits; for example, drilling holes in the board and using both sides, or using more than one board and connecting them by jumpers.

Given a plane graph G, a *region* of G is a maximal portion of the plane for which any two points may be joined by a curve A such that each point of A neither corresponds to a vertex of G nor lies on any curve corresponding to an edge of G. Intuitively, the regions of G are the connected portions of the

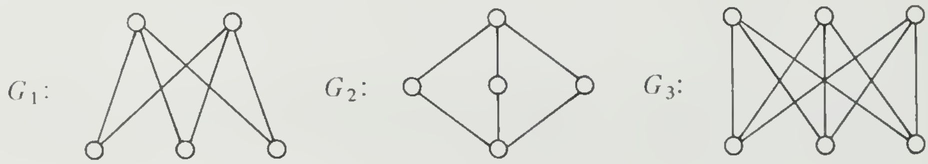


Figure 4.1 *Planar, plane, and nonplanar graphs*

plane remaining after all curves and points corresponding, respectively, to edges and vertices of G have been deleted. For a plane graph G, the *boundary* of a region R consists of all those points x corresponding to vertices and edges of G having the property that x can be joined to a point of R by a curve, all of whose points different from x belong to R. Every plane graph G contains an unbounded region called the *exterior region* of G. If G is embedded on the sphere, then no region of G can be regarded as being exterior. On the other hand, it is equally clear that a plane graph G can always be embedded in the plane so that a given region of G becomes the exterior region. Hence a plane graph G can always be realized in the plane so that any vertex or edge lies on the boundary of its exterior region. The plane graph G_2 of Figure 4.1 has three regions, and the boundary of each is a 4-cycle.

The order, size, and number of regions of any connected plane graph are related by a well-known formula discovered by Euler [E8].

Theorem 4.1 (Euler's Formula) *If G is a connected plane graph with p vertices, q edges, and r regions, then*

$$p - q + r = 2.$$

Proof We employ induction on q, the result being obvious for $q=0$ since in this case $p=1$ and $r=1$. Assume the result is true for all connected plane graphs with fewer than q edges, where $q \geq 1$, and suppose G has q edges. If G is a tree, then $p=q+1$ and $r=1$ so that the desired formula follows. On the other hand, if G is not a tree, let e be a cycle edge of G and consider $G-e$. The connected plane graph $G-e$ has p vertices, $q-1$ edges, and $r-1$ regions so that by the inductive hypothesis, $p-(q-1)+(r-1)=2$, which implies that $p-q+r=2$. ■

From the preceding theorem, it follows that any two embeddings of a connected planar graph in the plane result in plane graphs having the same number of regions; thus one can speak of the number of regions of a connected planar graph. For planar graphs in general, we have the following result.

Corollary 4.1 *If G is a plane graph with p vertices, q edges, and r regions, then* $p-q+r=1+k(G)$.

A planar graph G is called *maximal planar* if, for every pair of non-adjacent vertices u and v of G, the graph $G+uv$ is nonplanar. Thus in any embedding of a maximal planar graph G having order $p \geq 3$, the boundary of every region of G is a triangle. For this reason, maximal planar graphs are also referred to as *triangulated planar graphs*; triangulated plane graphs are often called simply *triangulations*.

On a given number p of vertices, a planar graph is quite limited as to how large its size q can be. A bound on q follows from our next result.

Theorem 4.2 *If G is a maximal planar (p, q) graph with $p \geq 3$, then*

$$q=3p-6.$$

Proof Denote by r the number of regions of G. In G the boundary of every region is a triangle, and each edge is on the boundary of two regions. Therefore, if the number of edges on the boundary of a region is summed over all regions, the result is $3r$. On the other hand, such a sum counts each edge twice so that $3r=2q$. Applying Theorem 4.1, we obtain $q=3p-6$. ■

Corollary 4.2a *If G is a planar (p, q) graph with $p \geq 3$, then*

$$q \leq 3p-6.$$

Proof Add to G sufficiently many edges so that the resulting (p', q') graph G' is maximal planar. Clearly, $p=p'$ and $q \leq q'$. By Theorem 4.2, $q'=3p-6$ and so $q \leq 3p-6$. ■

An immediate but important consequence of Corollary 4.2a is given next.

Corollary 4.2b *Every planar graph contains a vertex of degree at most 5.*

Proof Let G be a planar (p, q) graph with $V(G)=\{v_1, v_2, \ldots, v_p\}$. If $p \leq 6$, then the result is obvious. Otherwise, $q \leq 3p-6$ implies that

$$\sum_{i=1}^{p} \deg v_i = 2q \leq 6p-12.$$

Not all p vertices of G have degree 6 or more, for then $2q \geq 6p$. Thus G contains a vertex of degree 5 or less. ■

We next consider another corollary involving degrees. In it we make use of the fact that the minimum degree is at least 3 in a maximal planar graph of order at least 4.

Corollary 4.2c *Let G be a maximal planar graph of order $p \geq 4$, and let p_i denote the number of i-vertices of G, where $i=3, 4, \ldots, n=\Delta(G)$. Then*

$$3p_3+2p_4+p_5=p_7+2p_8+\ldots+(n-6)p_n+12.$$

Proof Let G have size q. Then, by Theorem 4.2, $q=3p-6$. Since

$$p=\sum_{i=3}^{n} p_i \quad \text{and} \quad 2q=\sum_{i=3}^{n} ip_i,$$

it follows that

$$\sum_{i=3}^{n} ip_i = 6\sum_{i=3}^{n} p_i - 12$$

and, consequently,

$$3p_3+2p_4+p_5=p_7+2p_8+\ldots+(n-6)p_n+12. \quad ■$$

An interesting feature of planar graphs is that they can be embedded in the plane so that every edge is a straight line segment. This result was proved independently by Fáry [F1] and Wagner [W1].

The theory of planar graphs is very closely allied with the study of polyhedra; in fact, with every polyhedron P is associated a connected planar graph $G(P)$ whose vertices and edges are the vertices and edges of P. Necessarily, then, every vertex of $G(P)$ has degree at least 3. Moreover, if $G(P)$ is a plane graph, then the faces of P are the regions of $G(P)$ and every edge of $G(P)$ is on the boundary of two regions. A polyhedron and its associated plane graph are shown in Figure 4.2.

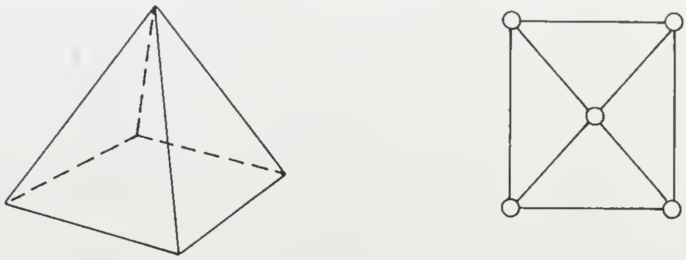


Figure 4.2 *A polyhedron and its associated graph*

It is customary to denote the number of vertices, edges, and faces of a polyhedron P by V, E, and F, respectively. However, these are the number of vertices, number of edges, and number of regions of a connected planar graph, namely $G(P)$. According to Theorem 4.1, V, E, and F are related. In this form, the statement of this result is known as the Euler Polyhedron Formula.

Theorem 4.3 (Euler Polyhedron Formula) *If V, E, and F are the number of vertices, edges, and faces of a polyhedron, then*

$$V - E + F = 2.$$

When dealing with a polyhedron P (as well as the graph $G(P)$), it is customary to represent the number of vertices of degree n by V_n and the number of faces (regions) bounded by an n-cycle by F_n. It follows then that

$$2E = \sum_{n \geq 3} nV_n = \sum_{n \geq 3} nF_n. \tag{4.1}$$

By Corollary 4.2b, every polyhedron has at least one vertex of degree 3, 4, or 5. As an analogue to this result, we have the following.

Theorem 4.4 *At least one face of every polyhedron is bounded by an n-cycle for some $n = 3, 4, 5$.*

Proof Assume that $F_3 = F_4 = F_5 = 0$ so that by equation (4.1),

$$2E = \sum_{n \geq 6} nF_n \geq \sum_{n \geq 6} 6F_n = 6 \sum_{n \geq 6} F_n = 6F.$$

Hence $E \geq 3F$. Also,

$$2E = \sum_{n \geq 3} nV_n \geq \sum_{n \geq 3} 3V_n = 3 \sum_{n \geq 3} V_n = 3V.$$

By Theorem 4.3, $V-E+F=2$; therefore, $E \leqslant \frac{2}{3}E+\frac{1}{3}E-2=E-2$. This is a contradiction. ■

A *regular polyhedron* is a polyhedron whose faces are bounded by congruent regular polygons and whose polyhedral angles are congruent. In particular, for a regular polyhedron, $V=V_k$ for some k, and $F=F_h$ for some h. For example, a cube is a regular polyhedron with $V=V_3$ and $F=F_4$. There are only four other regular polyhedra. These five regular polyhedra are also called platonic solids. The Greeks were aware, over two thousand years ago, that there are only five such polyhedra.

Theorem 4.5 *There are exactly five regular polyhedra.*

Proof Let P be a regular polyhedron and let $G(P)$ be an associated planar graph. Then $V-E+F=2$, where V, E, and F denote the number of vertices, edges, and faces of P and $G(P)$. Therefore,

$$\begin{aligned}-8 &= 4E-4V-4F\\ &= 2E+2E-4V-4F\\ &= \sum_{n\geqslant 3} nF_n+\sum_{n\geqslant 3} nV_n-4\sum_{n\geqslant 3} V_n-4\sum_{n\geqslant 3} F_n\\ &= \sum_{n\geqslant 3}(n-4)F_n+\sum_{n\geqslant 3}(n-4)V_n.\end{aligned}$$

Since P is regular, there exist integers $h(\geqslant 3)$ and $k(\geqslant 3)$ such that $F=F_h$ and $V=V_k$. Hence $-8=(h-4)F_h+(k-4)V_k$. Moreover, we note that $3\leqslant h\leqslant 5$, $3\leqslant k\leqslant 5$, and $hF_h=2E=kV_k$. This gives us nine cases to consider.

Case 1: ($h=3$, $k=3$) Here we have

$$-8=-F_3-V_3 \quad\text{and}\quad 3F_3=3V_3,$$

so that $F_3=V_3=4$. Thus P is the *tetrahedron*. (That the tetrahedron is the only regular polyhedron with $V_3=F_3=4$ follows from geometric considerations.)

Case 2: ($h=3$, $k=4$) Therefore

$$-8=-F_3 \quad\text{and}\quad 3F_3=4V_4.$$

Hence $F_3=8$ and $V_4=6$, implying that P is the *octahedron*.

Case 3: ($h=3$, $k=5$) In this case,

$$-8=-F_3+V_5 \quad\text{and}\quad 3F_3=5V_5,$$

so that $F_3 = 20$, $V_5 = 12$, and P is the *icosahedron.*

Case 4: $(h = 4,\ k = 3)$ We find here that

$$-8 = -V_3 \quad \text{and} \quad 4F_4 = 3V_3.$$

Thus $V_3 = 8$, $F_4 = 6$, and P is the *cube.*

Case 5: $(h = 4,\ k = 4)$ This is impossible since $-8 \neq 0$.

Case 6: $(h = 4,\ k = 5)$ This case, too, cannot occur, for otherwise $-8 = V_5$.

Case 7: $(h = 5,\ k = 3)$ For these values,

$$-8 = F_5 - V_3 \quad \text{and} \quad 5F_5 = 3V_3.$$

Solving for F_5 and V_3, we find that $F_5 = 12$ and $V_3 = 20$ so that P is the *dodecahedron.*

Case 8: $(h = 5,\ k = 4)$ Here $-8 = F_5$, which is impossible.

Case 9: $(h = 5,\ k = 5)$ This, too, is impossible since $-8 \neq F_5 + V_5$. This completes the proof. ■

The graphs of the five regular polyhedra are shown in Figure 4.3.

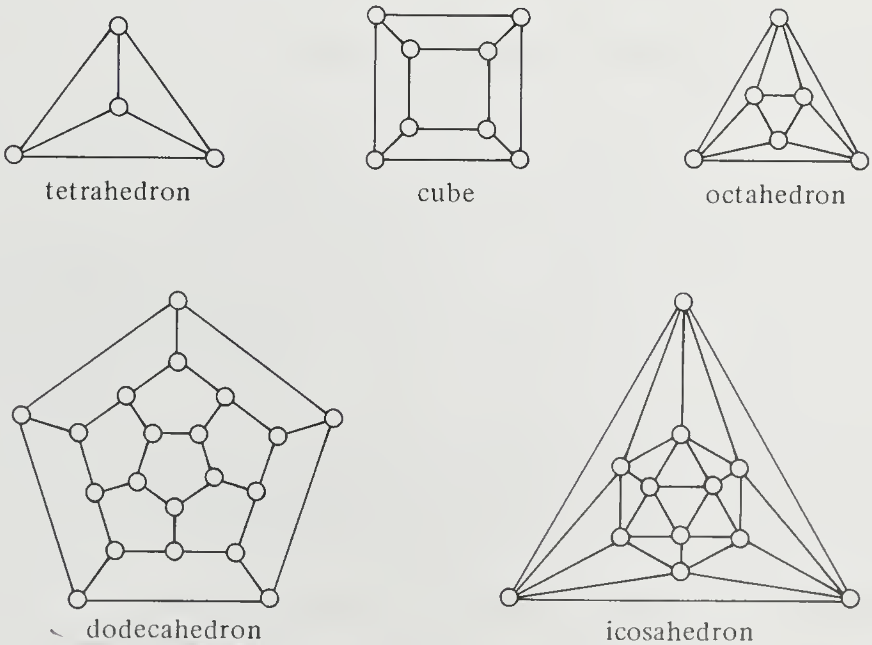


Figure 4.3 *The graphs of the regular polyhedra*

Exercises 4.1

4.1 Give an example of a planar graph that contains no vertex of degree less than 5.

4.2 Show that every planar graph of order $p \geq 4$ has at least four vertices of degree less than or equal to 5.

4.3 Prove Corollary 4.1.

4.4 (a) Prove that a planar (p, q) graph with $p \geq 3$ is maximal planar if and only if $q = 3p - 6$.

(b) Prove that there exists only one 4-regular maximal planar graph.

4.5 Let $n \geq 3$ be an integer, and let G be a (p, q) plane graph where $p \geq n$.

(a) If $g(G) \geq n$, then determine an upper bound B for q in terms of p and n.

(b) Show that your bound B in (a) is sharp by determining, for arbitrary $n \geq 3$, a (p, q) plane graph G with $g(G) \geq n$ such that $q = B$.

4.6 Let T be a nontrivial tree with p_i vertices of degree i ($i = 1, 2, \ldots, n = \Delta(T)$). Prove that

$$p_1 = p_3 + 2p_4 + 3p_5 + \cdots + (n-2)p_n + 2.$$

4.2 Characterizations of Planar Graphs

There are two graphs, namely K_5 and $K(3, 3)$ (shown in Figure 4.4), that play an important role in the study of planar graphs.

Figure 4.4 *The nonplanar graphs K_5 and $K(3, 3)$*

Theorem 4.6 *The graphs K_5 and $K(3, 3)$ are nonplanar.*

Proof Suppose, to the contrary, that K_5 is a planar graph. Since K_5 has $p = 5$ vertices and $q = 10$ edges,

$$10 = q > 3p - 6 = 9,$$

which contradicts Corollary 4.2a. Thus K_5 is nonplanar.

Suppose next that $K(3, 3)$ is a planar graph, and consider any plane embedding of it. Since $K(3, 3)$ is bipartite, it has no triangles; thus each of its regions is bounded by at least four edges. Let the number of edges bounding a region be summed over all r regions of $K(3, 3)$, denoting the result by N. Thus, $N \geq 4r$. Since the sum N counts each edge twice and $K(3, 3)$ contains $q = 9$ edges, $N = 18$ so that $r \leq 9/2$. However, by Theorem 4.1, $r = 5$, and this is a contradiction. Hence $K(3, 3)$ is nonplanar. ■

For the purpose of presenting two useful, interesting criteria for graphs to be planar, we describe two relations on graphs in this section.

An *elementary subdivision* of a nonempty graph G is a graph obtained from G by removing some edge $e = uv$ and adding a new vertex w and edges uw and vw. A *subdivision* of G is a graph obtained from G by a succession of elementary subdivisions. A graph H is defined to be *homeomorphic from* G if either $H \cong G$ or H is isomorphic to a subdivision of G. A graph G_1 is *homeomorphic with* a graph G_2 if there exists a graph G_3 such that each of G_1 and G_2 is homeomorphic from G_3.

In Figure 4.5 the graphs G_1 and G_2 are homeomorphic with each other since each is homeomorphic from G_3. However, neither G_1 nor G_2 is homeomorphic from the other.

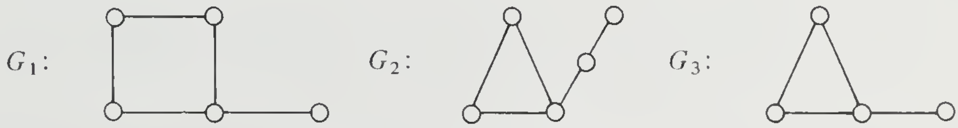


Figure 4.5 *Homeomorphism*

The relation "is homeomorphic with" is an equivalence relation on graphs. We thus refer to two graphs as being *homeomorphic* if either is homeomorphic with the other. Hence the set of graphs may be partitioned into equivalence classes, two graphs belonging to the same class if and only if they are homeomorphic.

It should be clear that any graph homeomorphic from a graph G is planar or nonplanar according to whether G is planar or nonplanar. Also it is an elementary observation that if a graph G contains a nonplanar subgraph, then G is nonplanar. Combining these facts with our preceding results, we obtain the following.

Theorem 4.7 *If a graph G contains a subgraph homeomorphic with either K_5 or $K(3, 3)$, then G is nonplanar.*

The remarkable property of Theorem 4.7 is that its converse is also true. These two results provide a characterization of planar graphs that is undoubtedly one of the best known theorems in the theory of graphs. Before presenting a proof of this result, first discovered by Kuratowski [K12], we need one additional fact about planar graphs.

Theorem 4.8 *A graph is planar if and only if each of its blocks is planar.*

Proof Certainly, a graph G is planar if and only if each of its components is planar, so we may assume G to be connected. It is equally clear that if G is planar, then each block of G is planar. For the converse, we employ induction on the number of blocks of G. If G has only one block and this block is planar, then, of course, G is planar. Assume every graph with fewer than $n \geq 2$ blocks, each of which is planar, is a planar graph, and suppose G has n blocks, all of which are planar. Let B be an end-block of G, and denote by v the cut-vertex of G common to B. Delete from G all vertices of B different from v, calling the resulting graph G'. By the inductive hypothesis, G' is a planar graph. Since the block B is planar, it may be embedded in the plane so that v lies on the exterior region. In any region of a plane embedding of G' containing v, the plane block B may now be suitably placed so that the two vertices of G' and B labeled v are "identified". The result is a plane graph of G; hence G is a planar. ■

We can now give a characterization of planar graphs. The proof of the following result is based on a proof by Dirac and Schuster [DS1].

Theorem 4.9 (Kuratowski) *A graph is planar if and only if it contains no subgraph homeomorphic with K_5 or $K(3, 3)$.*

Proof The necessity is precisely the statement of Theorem 4.7; thus we need only consider the sufficiency. In view of Theorem 4.8, it is sufficient to show that if a block contains no subgraph homeomorphic with K_5 or $K(3, 3)$, then it is planar. Assume, to the contrary, that such is not the case. Hence among all nonplanar blocks not containing subgraphs homeomorphic with either K_5 or $K(3, 3)$, let G be one of minimum size.

First we verify that $\delta(G) \geq 3$. Since G is a block, it contains no end-vertices. Assume, then, that G contains a vertex v with $\deg v = 2$, such that v is adjacent with u and w. We consider two possibilities. Suppose $uw \in E(G)$. Then $G - v$ is also a block. Since $G - v$ is a subgraph of G, it follows that $G - v$ also contains no subgraph homeomorphic with K_5 or $K(3, 3)$; however, G is a nonplanar block of minimum size having this property so that $G - v$ is planar. However, in any plane graph of $G - v$, the vertex v and edges uv and

vw may be inserted so that the resulting graph G is plane, which contradicts the fact that G is nonplanar. Next, suppose that $uw \notin E(G)$. The graph $G' = G - v + uw$ is a block having smaller size than G. Furthermore, G' contains no subgraph homeomorphic with either K_5 or $K(3, 3)$; for suppose it contained such a subgraph F. If F failed to contain the edge uw, then G would also contain F, which is impossible; thus F contains uw. If to $F - uw$ we add the vertex v and edges uv and wv, the resulting graph F' is homeomorphic from F. However, F' is a subgraph of G, which is impossible. Thus G' is a block of size less than G that contains no subgraph homeomorphic with either K_5 or $K(3, 3)$, so that G' is planar. However, since G is homeomorphic from G', this implies that G too is planar, which is a contradiction. Thus, G cannot contain a vertex of degree 2, so that $\delta(G) \geq 3$, as claimed.

By Corollary 2.18, G is not a minimal block so that there exists an edge $e = uv$, such that $H = G - e$ is also a block. Since H has no subgraph homeomorphic with either K_5 or $K(3, 3)$ and H has fewer edges than does G, the graph H is planar. Since H is a cyclic block, it follows by Theorem 2.14 that H possesses cycles containing both u and v. We henceforth assume H to be a plane graph having a cycle, say C, containing u and v such that the number of regions interior to C is maximum. Assume C to be given by

$$u = v_0, v_1, \ldots, v_i = v, \ldots, v_n = u,$$

where $1 < i < n - 1$.

Several observations regarding the plane graph H can now be made. In order to do this, it is convenient to define two special subgraphs of H. By the *exterior subgraph* (*interior subgraph*) of H, we mean the subgraph of G induced by those edges lying exterior (interior) to the cycle C. First, since the graph G is nonplanar, both the exterior and interior subgraphs exist, for otherwise, the edge e could be added to H (either exterior to C or interior to C) so that the resulting graph, namely G, is planar.

We note further that no two distinct vertices of the set $\{v_0, v_1, \ldots, v_i\}$ are connected by a path in the exterior subgraph of H, for this would contradict the choice of C as being that cycle containing u and v having the maximum number of regions interior to it. A similar statement can be made regarding the set $\{v_i, v_{i+1}, \ldots, v_n\}$. These remarks in connection with the fact that $H + e$ is nonplanar imply the existence of a $v_j - v_k$ path P, $0 < j < i < k < n$, in the exterior subgraph of H such that no vertex of P different from v_j and v_k belongs to C. This structure is illustrated in Figure 4.6. We further note that no vertex of P different from v_j and v_k is adjacent to a vertex of C other than v_j or v_k, and, moreover, any path joining a vertex of P with a vertex of C must contain at least one of v_j and v_k.

Let H_1 be the component of $H - \{v_m | 0 \leq m < n, m \neq j, k\}$ containing P. By the choice of C, the subgraph H_1 cannot be inserted in the interior of C in a plane manner. This, together with the assumption that G is nonplanar, implies that the interior subgraph of H must contain one of the following:

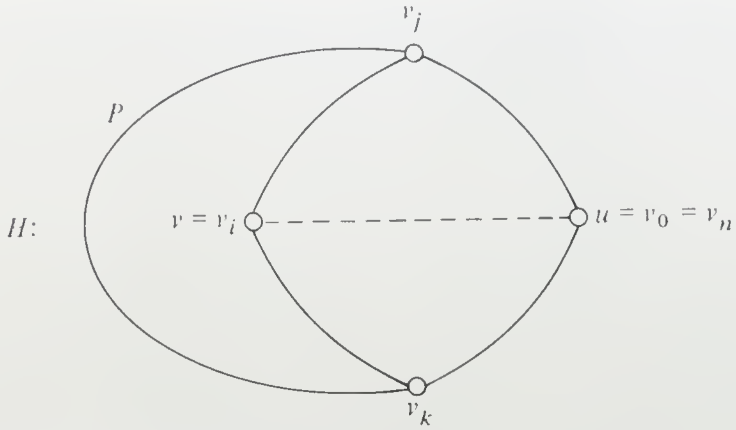


Figure 4.6 *Structure of the graph H of Theorem 4.9*

(a) A v_r-v_s path Q, $0<r<j$, $i<s<k$, (or, equivalently, $j<r<i$ and $k<s<n$) none of whose vertices different from v_r and v_s belongs to C.

(b) A vertex w not on C that is connected to C by three internally disjoint paths such that the end-vertex of one such path P is one of v_0, v_j, v_i, and v_k. If P ends at v_0, the end-vertices of the other paths are v_r and v_s, where $j \leq r < i$ and $i < s \leq k$ but not both $r = j$ and $s = k$ hold. If P ends at any of v_j, v_i or v_k, there are three analogous cases.

(c) A vertex w not on C that is connected to C by three internally disjoint paths P_1, P_2, P_3 such that the end-vertices of the paths (different from w) are three of the four vertices v_0, v_j, v_i, v_k, say v_0, v_i, v_j, respectively, together with a $v_t - v_k$ path $P_4(v_t \neq v_0, v_i, w)$ where v_t is on P_1 or P_2, and P_4 is disjoint from P_1, P_2, and C except for v_t and v_k. The remaining choices for P_1, P_2 and P_3 produce three analogous cases.

(d) A vertex w not on C that is connected to the vertices v_0, v_j, v_i, v_k by four internally disjoint paths.

These four cases exhaust the possibilities. (This is a fact of which one must convince oneself.) In each of the first three cases, the graph G has a subgraph homeomorphic with $K(3, 3)$ while in the fourth case, G has a subgraph homeomorphic with K_5. However, in any case, this is contrary to assumption. Thus no such graph G exists, and the proof is complete. ■

Thus the Petersen graph (see Figure 4.7(a)) is nonplanar since it contains the subgraph of Figure 4.7(b) that is homeomorphic with $K(3, 3)$. Despite its resemblance to the complete graph K_5, the Petersen graph does *not* contain a subgraph homeomorphic with K_5.

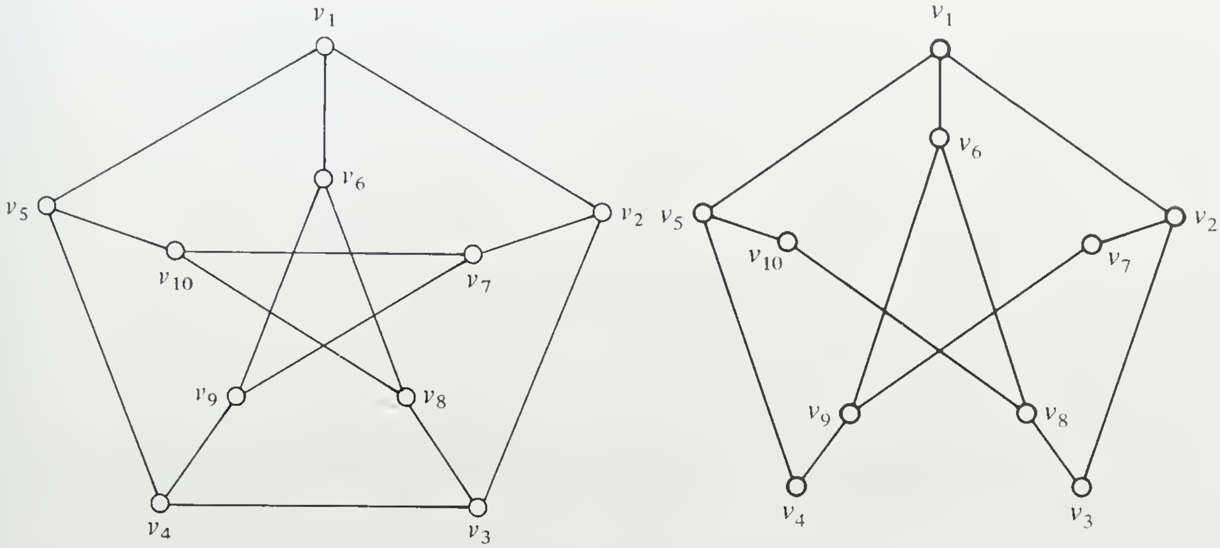


Figure 4.7 *The Petersen graph and a subgraph homeomorphic with K(3, 3)*

For graphs G_1 and G_2, a mapping ϕ from $V(G_1)$ onto $V(G_2)$ is called an *elementary contraction* if there exist adjacent vertices u and v of G_1 such that

(a) $\phi u = \phi v$, and $\{u_1, v_1\} \neq \{u, v\}$ implies $\phi u_1 \neq \phi v_1$,

(b) $\{u_1, v_1\} \cap \{u, v\} = \varnothing$ implies $u_1 v_1 \in E(G_1)$ if and only if $\phi u_1 \phi v_1 \in E(G_2)$, and

(c) for $w \in V(G_1)$, $w \neq u, v$, then $uw \in E(G_1)$ or $vw \in E(G_1)$ if and only if $\phi u \phi w \in E(G_2)$.

We say here that G_2 is obtained from G_1 by the *identification of the adjacent vertices u and v*. A *contraction* is then a mapping from $V(G_1)$ onto $V(G_2)$ that is either an isomorphism or a composition of finitely many elementary contractions.

If there exists a contraction from $V(G_1)$ onto $V(G_2)$, then G_2 is a *contraction of* G_1, and G_1 *contracts to* or is *contractible to* G_2. A *subcontraction* of a graph G is a contraction of a subgraph of G.

There is an alternative and more intuitive manner in which to define "contraction". A graph G_2 may be defined as a contraction of a graph G_1 if there exists a one-to-one correspondence between $V(G_2)$ and the elements of a partition of $V(G_1)$ such that each element of the partition induces a connected subgraph of G_1, and two vertices of G_2 are adjacent if and only if the subgraph induced by the union of the corresponding subsets is connected.

In Figure 4.8, the graph G is a contraction of H, obtained by the identification v_2 and v_5. It might also be considered as the contraction resulting from the partition

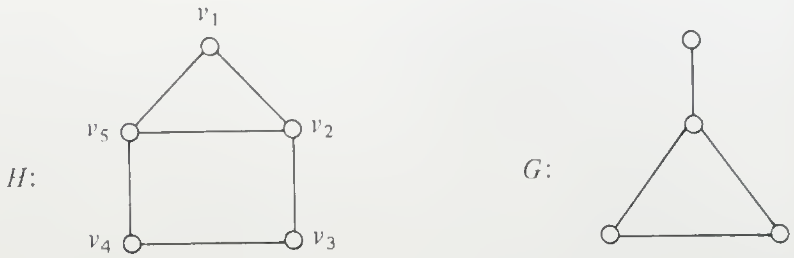


Figure 4.8 *Contraction*

$$V(H) = \{v_1\} \cup \{v_2, v_5\} \cup \{v_3\} \cup \{v_4\}.$$

A relationship between contraction and homeomorphism is given in the following theorem.

Theorem 4.10 *If a graph H is homeomorphic from a graph G, then G is a contraction of H.*

Proof If $G \cong H$, then clearly G is a contraction of H. Hence we may assume H is obtained from G by a sequence of elementary subdivisions. Suppose G' is an elementary subdivision of G; then G' is obtained from G by removing some edge uv and adding a vertex w together with the edges uw and vw. However, then G' is contractible to G by an elementary contraction ϕ, which fixes every element of $V(G)$ and $\phi w = \phi u$. Hence G can be obtained from H by a mapping that is a composition of finitely many elementary contractions so that G is a contraction of H. ■

Corollary 4.10 will actually prove to be of more use than the theorem itself.

Corollary 4.10 *If a graph H contains a subgraph homeomorphic from a connected nontrivial graph G, then G is a subcontraction of H.*

We can now present our second characterization [H3, HT2, W2] of planar graphs.

Theorem 4.11 *A graph G is planar if and only if neither K_5 nor $K(3, 3)$ is a subcontraction of G.*

Proof Let G be a nonplanar graph. By Theorem 4.9, G contains a subgraph homeomorphic with (or equivalently here, homeomorphic from) K_5 or $K(3, 3)$. Thus by Corollary 4.10, K_5 or $K(3, 3)$ is a subcontraction of G.

In order to verify the converse, we first suppose that G is a graph such that $H \cong K(3, 3)$ is a subcontraction of G. We show, in this case, that G contains a subgraph homeomorphic with $K(3, 3)$, implying that G is nonplanar. Denote the vertices of H by u_i and u'_i, $1 \leq i \leq 3$, such that every edge of H is of the type $u_i u'_j$. Taking the alternate definition of contraction, we let G_i, $1 \leq i \leq 3$, be the connected subgraph of G corresponding to u_i and let G'_i correspond to u'_i. Since $u_i u'_j \in E(H)$ for $1 \leq i \leq 3$, $1 \leq j \leq 3$, in the graph G there exists a vertex v_{ij} of G_i adjacent with a vertex v'_{ij} of G'_j. Among the vertices v_{i1}, v_{i2}, v_{i3} of G_i, two or possibly all three may actually represent the same vertex. If $v_{i1} = v_{i2} = v_{i3}$, we set each $v_{ij} = v_i$; otherwise, we define v_i to be a vertex of G_i connected to the distinct elements of $\{v_{i1}, v_{i2}, v_{i3}\}$ with internally disjoint paths in G_i. (It is possible that $v_i = v_{ij}$ for some j.) We now proceed as above with the subgraphs G'_i, thereby obtaining vertices v'_i. The subgraph of G induced by the nine edges $v_{ij}v'_{ij}$ together with the edge sets of any necessary aforementioned paths from a vertex v_i or v'_i is homeomorphic with $K(3, 3)$.

Assume now that $H \cong K_5$ is a subcontraction of G. Let $V(H) = \{u_i | \leq i \leq 5\}$, and suppose G_i is the connected subgraph of G that corresponds to u_i. As before, there exists vertex v_{ij} of G_i adjacent with vertex v_{ji} of G_j, $i \neq j$, $1 \leq i$, $j \leq 5$. For a fixed i, $1 \leq i \leq 5$, we consider the vertices v_{ij}, $j \neq i$. If the vertices v_{ij} represent the same vertex, we denote this vertex by v_i. If the vertices v_{ij} are distinct and there exists a vertex (possibly some v_{ij}) from which there are internally disjoint paths (one of which may be trivial) to the v_{ij}, then denote this vertex by v_i. If three of the vertices v_{ij} are the same vertex, call this vertex v_i. If two vertices v_{ij} are the same while the other two are distinct, then denote the two coinciding vertices by v_i if there exist internally disjoint paths to the other two vertices. Hence in several instances we have defined a vertex v_i, for $1 \leq i \leq 5$. Should v_i exist for each $i = 1, 2, \ldots, 5$, then G contains a subgraph homeomorphic with K_5.

Otherwise, for some i, there exist distinct vertices w_i and w'_i of G_i, each of which is connected to two of the v_{ij} by internally disjoint (possibly trivial) paths of G_i while w_i and w'_i are connected by a path of G_i, none of whose internal vertices are the vertices v_{ij}. If two vertices v_{ij} coincide, then this vertex is w_i. If the other two vertices v_{ij} should also coincide, then this vertex is w'_i. Without loss of generality, we assume $i = 1$ and that w_1 is connected to v_{12} and v_{13} while w'_1 is connected to v_{14} and v_{15} as described above.

Denote the edge set of these five paths of G_1 by E_1. We now turn to G_2. If $v_{21} = v_{24} = v_{25}$, we set $E_2 = \emptyset$; otherwise, there is a vertex w_2 of G_2 (which may coincide with v_{21}, v_{24}, or v_{25}) joined by pairwise internally disjoint (possibly trivial) paths in G_2 to the distinct elements of $\{v_{21}, v_{24}, v_{25}\}$. We then let E_2 denote the edge sets of these paths. In an analogous manner, we define accordingly the sets E_3, E_4, and E_5 with the aid of the sets $\{v_{31}, v_{34}, v_{35}\}$, $\{v_{41}, v_{42}, v_{43}\}$, and $\{v_{51}, v_{52}, v_{53}\}$, respectively. The subgraph induced by the union of the sets E_i and the edges $v_{ij}v_{ji}$ contains a subgraph F homeomorphic with $K(3, 3)$ such that the vertices of degree 3 of F are w_1, w'_1 and the vertices w_i, $i = 2, 3, 4, 5$. In either case, G is nonplanar. ■

As an application of this theorem, we again note the nonplanarity of the Petersen graph of Figure 4.7(a). The Petersen graph contains K_5 as a subcontraction that follows by considering the partition $V_1, V_2, V_3, V_4; V_5$ of its vertex set, where $V_i = \{v_i, v_{i+5}\}$.

Next we describe a good algorithm, for deciding whether a given graph is planar, from Demoucron, Malgrange, and Pertuiset [DMP1]. In order to do this, we introduce some additional terminology.

Let G be a graph and H a subgraph of G. Define a relation $\sim$ on $E(G) - E(H)$ by $e \sim f$ if there exists a walk W in $G - E(H)$ whose first and last edges are e and f and no internal vertex of W belongs to H. Then $\sim$ is an equivalence relation on $E(G) - E(H)$. The subgraphs of $G - E(H)$ induced by the resulting equivalence classes are called the *fragments* of H in G. The subgraph H of the graph G of Figure 4.9 has the three fragments F_1, F_2, and F_3.

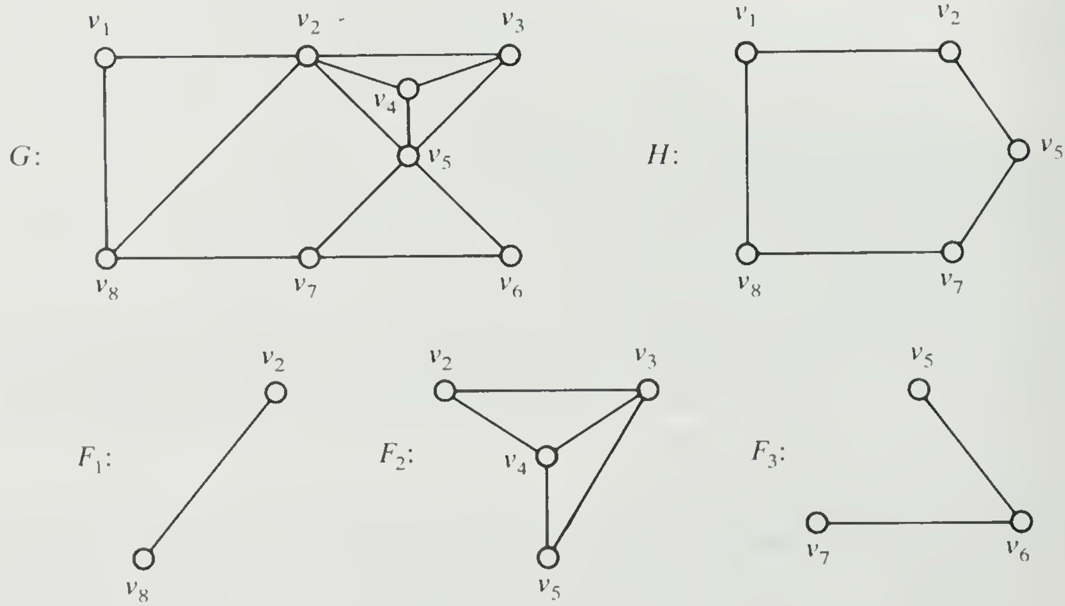


Figure 4.9

Let H be a plane subgraph of a graph G. We say that H is *G-extendable* if G is planar and the plane embedding of H can be extended to a plane embedding of G. Let G be the graph of Figure 4.10. For the plane subgraphs H_1 and H_2 of Figure 4.10, where $H_1 \cong H_2$, we note that H_1 is G-extendable but H_2 is not G-extendable.

Let H be a plane subgraph of a graph G and R a region of H. A fragment F of H in G is an *R-fragment* if all vertices of F belonging to H lie in the boundary of R. The set of regions R for which F is an R-fragment is denoted by $\mathfrak{R}(F, H)$. Let H be the plane subgraph of G shown in Figure 4.11. Then F (also shown in Figure 4.11) is a fragment of H in G, and $\mathfrak{R}(F, H) = \{R_1\}$.

It is now apparent that if a plane subgraph H of a planar graph G is G-

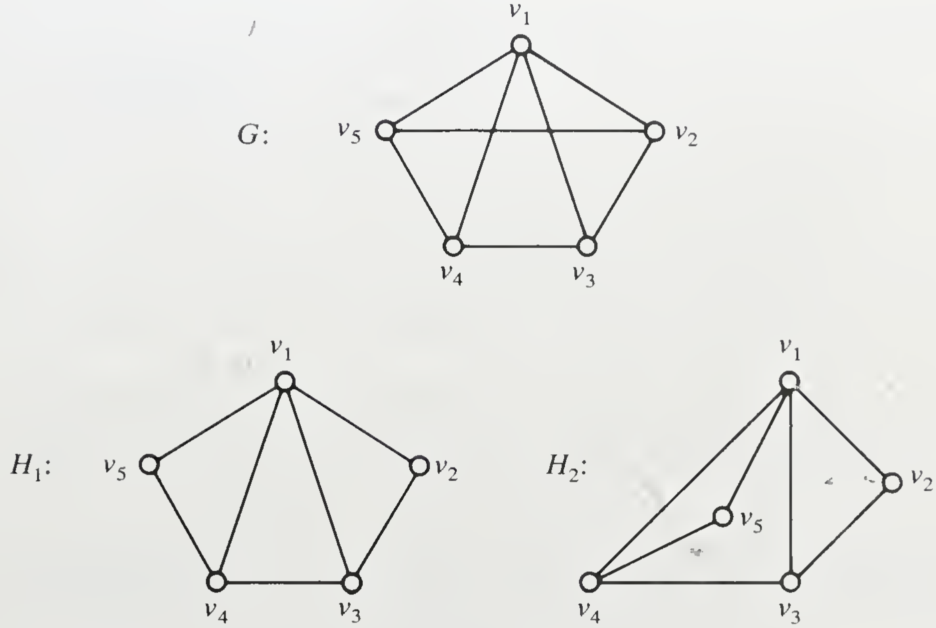


Figure 4.10

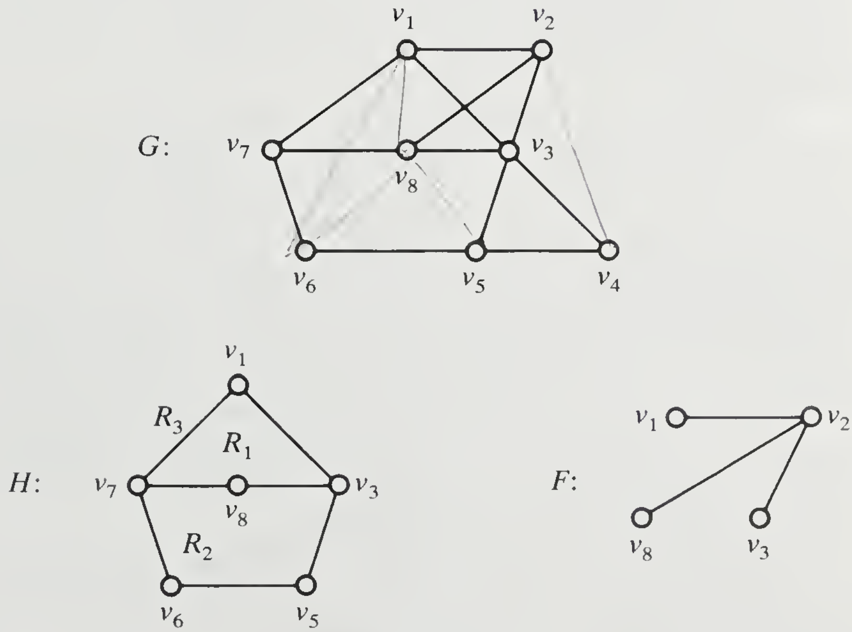


Figure 4.11

extendable, then for every fragment F of H in G, $\mathcal{R}(F, H) \neq \emptyset$. We are now prepared to present the algorithm. Given a graph G, the algorithm describes a sequence $G_1, G_2, \ldots, G_n$ of plane subgraphs of G such that $G_i \subset G_{i+1}$ for $i = 1, 2, \ldots, n-1$. Then G is planar if and only if $G_n \cong G$.

Algorithm 4A *Given a connected graph G:*

1. *If G is a tree, then G is planar and stop. Otherwise, go to Step 2.*
2. *Let G_1 be a cycle of G such that G_1 is embedded in the plane. Set $i = 1$.*
3. *If $E(G) - E(G_i) = \emptyset$, then G is planar and stop. Otherwise, determine all fragments of the plane subgraph G_i in G, and for each such fragment F, determine the set $\mathfrak{R}(F, G_i)$.*
4. *If there exists a fragment F of G_i in G for which $\mathfrak{R}(F, G_i) = \emptyset$, then G is nonplanar and stop. If there exists a fragment F such that $|\mathfrak{R}(F, G_i)| = 1$, then let $\mathfrak{R}(F, G_i) = \{R\}$. Otherwise, let F be an R-fragment of G_i in G, where R is a region of G_i.*
5. *Select a path P in F connecting two vertices of G_i, and let G_{i+1} denote that plane subgraph obtained by drawing P in R. Replace i by $i + 1$ and go to Step 3.*

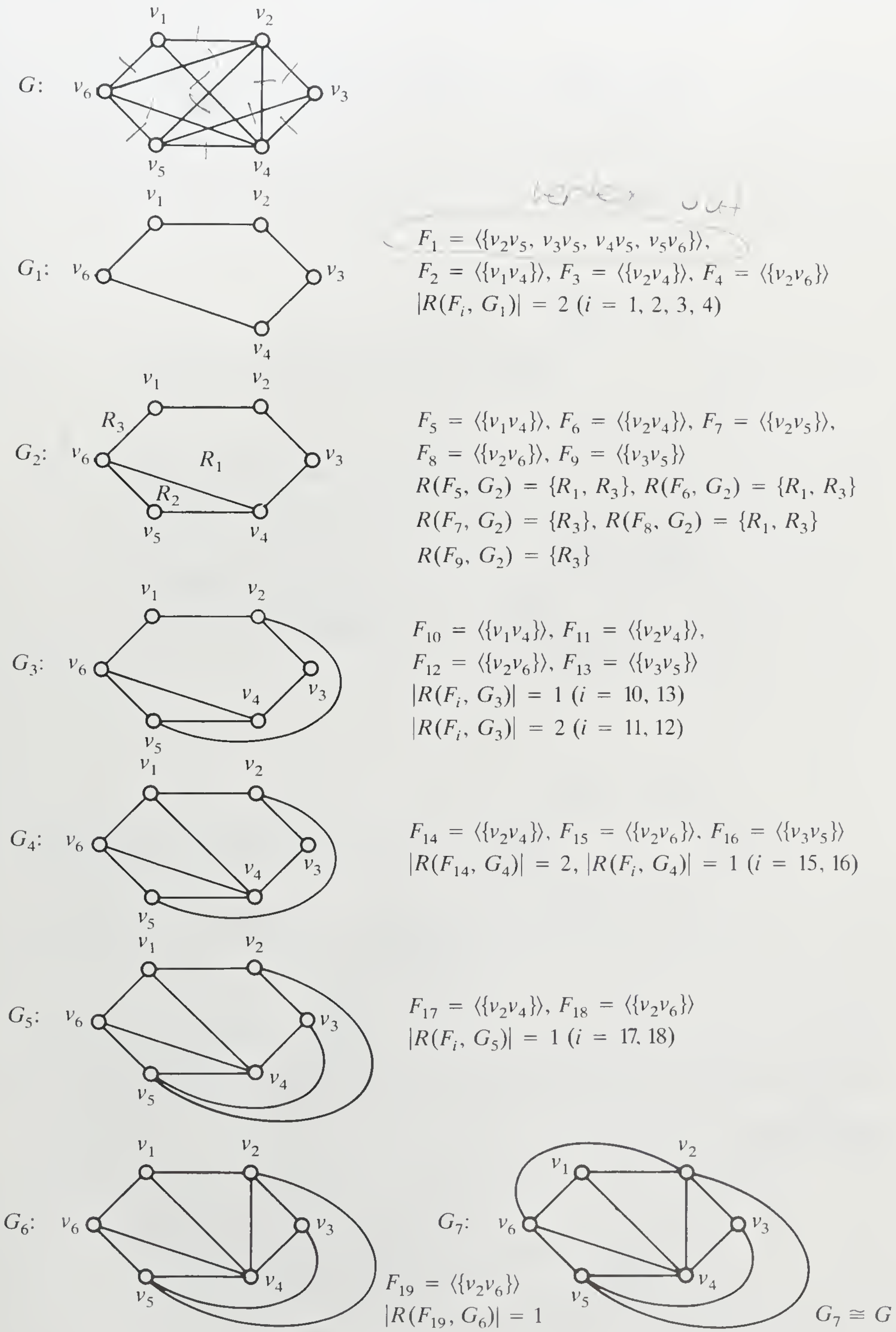
Theorem 4A *Algorithm 4A determines whether a given connected graph is planar.*

Although the proof of Theorem 4A is too involved to present here, we describe the essential steps. It suffices to show that if G is planar, then each term of the sequence $G_1, G_2, \ldots, G_n$ of plane subgraphs is G-extendable. This is verified by induction. Certainly a plane embedding of G_1 is G-extendable.

It is assumed that G_i is G-extendable ($1 \leq i < n$) and shown that G_{i+1} is G-extendable. Since G_i is G-extendable, we can extend the embedding of G_i to a plane embedding of G. Let F and R be selected as in Step 4. If $|\mathfrak{R}(F, G_i)| = 1$, then F is drawn in R (in the extension of G_i to G) and G_{i+1} is G-extendable. Otherwise, $|\mathfrak{R}(F, G_i)| \geq 2$ and there is no fragment of G_i in G that is an S-fragment for only one region S of G_i. If F is drawn in R, then, once again, G_{i+1} is G-extendable. Suppose, to the contrary, that F is drawn in a region R' of G that is different from R. Further, assume that F' is a fragment of G_i in G that is drawn in R whose vertices of G_i belong to the common boundary of R and R'. Then F and F' may be interchanged across this common boundary to produce a new embedding of G in the plane in which F is drawn in R. Consequently, G_{i+1} is G-extendable.

We now present two examples to illustrate Algorithm 4A. We determine whether the graph G of Figure 4.12 is planar, where G_1 is the indicated 5-cycle. The plane subgraph G_1 has four fragments F_1, F_2, F_3, F_4. A sequence $G_1, G_2, \ldots, G_7$ (which is not unique) is shown together with the sets $\mathfrak{R}(F, G_2)$ for each fragment F of G_2 in G. Since $G \cong G_7$, G is a planar graph. For graph G of Figure 4.13, a sequence G_1, G_2, G_3 is shown. Since $\mathfrak{R}(F_6, G_3) = \emptyset$, graph G is nonplanar (actually $G \cong K(3, 3)$).

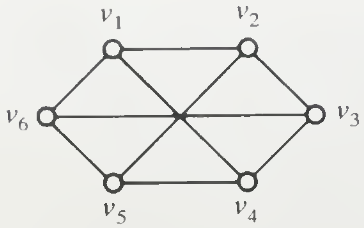
More information on this algorithm may also be found in [BM2].



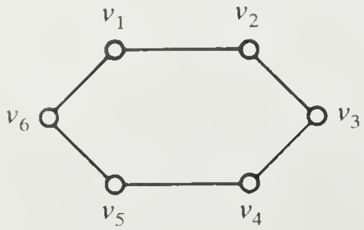
G: v_1 v_2 v_3 v_4 v_5 v_6
G_1: v_1 v_2 v_3 v_4 v_6
$F_1 = \langle\{v_2v_5, v_3v_5, v_4v_5, v_5v_6\}\rangle$,
$F_2 = \langle\{v_1v_4\}\rangle$, $F_3 = \langle\{v_2v_4\}\rangle$, $F_4 = \langle\{v_2v_6\}\rangle$
$|R(F_i, G_1)| = 2$ $(i = 1, 2, 3, 4)$
G_2: v_1 v_2 v_3 v_4 v_5 v_6 R_1 R_2 R_3
$F_5 = \langle\{v_1v_4\}\rangle$, $F_6 = \langle\{v_2v_4\}\rangle$, $F_7 = \langle\{v_2v_5\}\rangle$,
$F_8 = \langle\{v_2v_6\}\rangle$, $F_9 = \langle\{v_3v_5\}\rangle$
$R(F_5, G_2) = \{R_1, R_3\}$, $R(F_6, G_2) = \{R_1, R_3\}$
$R(F_7, G_2) = \{R_3\}$, $R(F_8, G_2) = \{R_1, R_3\}$
$R(F_9, G_2) = \{R_3\}$
G_3: v_1 v_2 v_3 v_4 v_5 v_6
$F_{10} = \langle\{v_1v_4\}\rangle$, $F_{11} = \langle\{v_2v_4\}\rangle$,
$F_{12} = \langle\{v_2v_6\}\rangle$, $F_{13} = \langle\{v_3v_5\}\rangle$
$|R(F_i, G_3)| = 1$ $(i = 10, 13)$
$|R(F_i, G_3)| = 2$ $(i = 11, 12)$
G_4: v_1 v_2 v_3 v_4 v_5 v_6
$F_{14} = \langle\{v_2v_4\}\rangle$, $F_{15} = \langle\{v_2v_6\}\rangle$, $F_{16} = \langle\{v_3v_5\}\rangle$
$|R(F_{14}, G_4)| = 2$, $|R(F_i, G_4)| = 1$ $(i = 15, 16)$
G_5: v_1 v_2 v_3 v_4 v_5 v_6
$F_{17} = \langle\{v_2v_4\}\rangle$, $F_{18} = \langle\{v_2v_6\}\rangle$
$|R(F_i, G_5)| = 1$ $(i = 17, 18)$
G_6: v_1 v_2 v_3 v_4 v_5 v_6
$F_{19} = \langle\{v_2v_6\}\rangle$
$|R(F_{19}, G_6)| = 1$
G_7: v_1 v_2 v_3 v_4 v_5 v_6
$G_7 \cong G$

Figure 4.12

G:

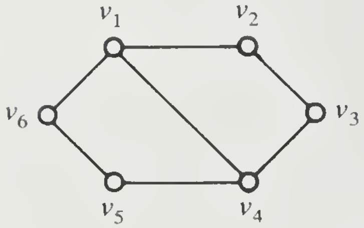


G_1:



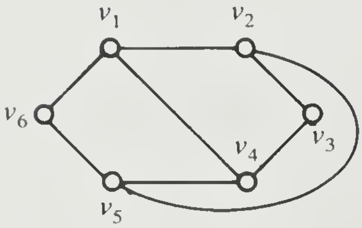
$F_1 = \langle\{v_1v_4\}\rangle$, $F_2 = \langle\{v_2v_5\}\rangle$, $F_3 = \langle\{v_3v_6\}\rangle$
$|R(F_i, G_1)| = 2 \ (i = 1, 2, 3)$

G_2:



$F_4 = \langle\{v_2v_5\}\rangle$, $F_5 = \langle\{v_3v_6\}\rangle$
$|R(F_i, G_2)| = 1 \ (i = 4, 5)$

G_3:



$F_6 = \langle\{v_3v_6\}\rangle$
$|R(F_6, G_3)| = 0$

Figure 4.13

Exercises 4.2

4.7 Show that the converse of Theorem 4.10 is not, in general, true.

4.8 Show that the Petersen graph (of Figure 4.7(a)) is nonplanar by
(a) showing that it has $K(3, 3)$ as a subcontraction, and
(b) using Exercise 4.5(a).

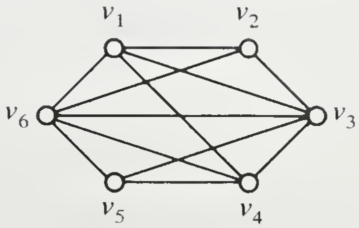
4.9 Let T be a tree of order at least 4, and let $e_1, e_2, e_3 \in E(\bar{T})$. Prove that $T+e_1+e_2+e_3$ is planar.

4.10 Use Algorithm 4A to test

(a) K_5 for planarity by taking G_1 to be a 4-cycle;

(b) graph G below for planarity by taking G_1 to be the cycle $v_1, v_2, v_3, v_4, v_6, v_1$.

G:



4.11 A graph G is *outerplanar* if it can be embedded in the plane so that every vertex of G lies on the boundary of the exterior region. Prove the following:

(a) A graph G is outerplanar if and only if $G+K_1$ is planar.

(b) A graph is outerplanar if and only if it contains no subgraph homeomorphic from K_4 or $K(2, 3)$.

(c) If G is a (p, q) outerplanar graph with $p \geq 2$, then $q \leq 2p-3$.

4.3 Nonplanar Graphs

There are a variety of ways of measuring how nonplanar a graph is. In the remainder of this chapter, we discuss several of these "measures".

Nonplanar graphs cannot, of course, be embedded in the plane. Hence, whenever a nonplanar graph is "drawn" in the plane, some of its edges must cross. This rather simple observation suggests our next concept.

The *crossing number* $\nu(G)$ of a graph G is the minimum number of crossings (of its edges) among the drawings of G in the plane. Before proceeding further, we comment on the assumptions we are making regarding the idea of "drawings". In all drawings under consideration, we assume that

(a) adjacent edges never cross,

(b) two nonadjacent edges cross at most once,

(c) no edge crosses itself,

(d) no more than two edges cross at a point of the plane, and

(e) the (open) arc in the plane corresponding to an edge of the graph contains no vertex of the graph.

A few observations will prove useful. Clearly a graph G is planar if and only if $\nu(G)=0$. Further, if $G \subset H$, then $\nu(G) \leqslant \nu(H)$, while if H is obtained from G by inserting vertices of degree 2 into the edges of G, then $\nu(G)=\nu(H)$. For very few classes of graphs is the crossing number known. It has been shown by Blažek and Koman [BK1] and Guy [G8], among others, that for complete graphs,

$$\nu(K_p) \leqslant \frac{1}{4}\left\lfloor\frac{p}{2}\right\rfloor\left\lfloor\frac{p-1}{2}\right\rfloor\left\lfloor\frac{p-2}{2}\right\rfloor\left\lfloor\frac{p-3}{2}\right\rfloor, \tag{4.2}$$

and Guy has conjectured that equality holds in (4.2) for all p. As far as exact results are concerned, the best obtained is the following (see Guy [G9]).

Theorem 4.12 *For* $1 \leqslant p \leqslant 10$,

$$\nu(K_p) = \frac{1}{4}\left\lfloor\frac{p}{2}\right\rfloor\left\lfloor\frac{p-1}{2}\right\rfloor\left\lfloor\frac{p-2}{2}\right\rfloor\left\lfloor\frac{p-3}{2}\right\rfloor. \tag{4.3}$$

Since K_p is planar for $1 \leqslant p \leqslant 4$, Theorem 4.12 is obvious for $1 \leqslant p \leqslant 4$. Further, K_5 is nonplanar; thus, $\nu(K_5) \geqslant 1$. On the other hand, there exists a drawing (see Figure 4.14) of K_5 in the plane with one crossing so that $\nu(K_5)=1$.

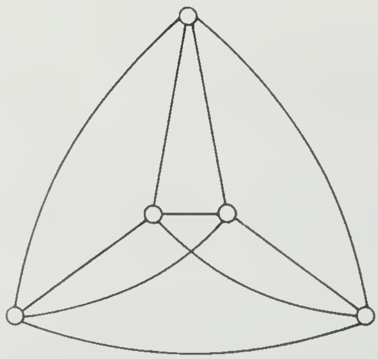


Figure 4.14 *A drawing of* K_5 *with one crossing*

The inequality $\nu(K_6) \leqslant 3$ follows from Figure 4.15, where a drawing of K_6 with three crossings is shown. We now verify that $\nu(K_6) \geqslant 3$, completing the proof that $\nu(K_6)=3$. Let there be given a drawing of K_6 in the plane with $c=\nu(K_6)$ crossings, where, of course, $c \geqslant 1$. At each crossing we introduce a new vertex, producing a connected plane graph G of order $6+c$ and size $15+2c$. By Corollary 4.2a,

$$15+2c \leqslant 3(6+c)-6$$

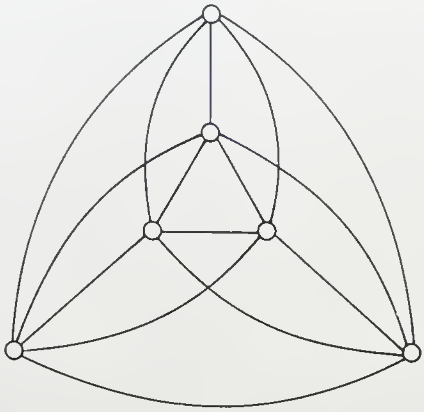


Figure 4.15 *A drawing of K_6 with three crossings*

so that $c \geqslant 3$ and, consequently, $\nu(K_6) \geqslant 3$.

Considerably more specialized techniques are required to verify Theorem 4.12 for $7 \leqslant p \leqslant 10$.

It was mentioned in Section 4.1 that every planar graph can be embedded in the plane so that each edge is a straight line segment. Thus, if a graph G has crossing number 0, this fact can be realized by considering only drawings in the plane in which the edges are straight line segments. One may very well ask if, in general, it is sufficient to consider only drawings of graphs in which edges are straight line segments in determining crossing numbers. With this question in mind, we introduce a variation of the crossing number.

The *rectilinear crossing number* $\bar{\nu}(G)$ of a graph G is the minimum number of crossings among all those drawings of G in the plane in which each edge is a straight line segment. Since the crossing number $\nu(G)$ considers *all* drawings of G in the plane (not just those for which edges are straight line segments), we have the obvious inequality

$$\nu(G) \leqslant \bar{\nu}(G). \tag{4.4}$$

As previously stated, $\nu(G) = \bar{\nu}(G)$ for every planar graph G. It has also been verified that $\nu(K_p) = \bar{\nu}(K_p)$ for $1 \leqslant p \leqslant 7$ and $p = 9$; *however*,

$$\nu(K_8) = 18 \quad \text{and} \quad \bar{\nu}(K_8) = 19$$

(see [G9]) so that strict inequality in (4.4) is indeed a possibility.

We return to our chief interest, namely the crossing number, and consider the complete bipartite graphs. The problem of determining $\nu(K(m, n))$ has a rather curious history. It is sometimes referred to as Turán's Brick-Factory Problem (named for Paul Turán). We quote from Turán [T11]:

> We worked near Budapest, in a brick factory. There were some kilns where the bricks were made and some open storage yards where the bricks were stored. All the kilns were connected by rail with all the storage yards. The

bricks were carried on small wheeled trucks to the storage yards. All we had to do was to put the bricks on the trucks at the kilns, push the trucks to the storage yards, and unload them there. We had a reasonable piece rate for the trucks, and the work itself was not difficult; the trouble was only at the crossings. The trucks generally jumped the rails there, and the bricks fell out of them; in short this caused a lot of trouble and loss of time which was precious to all of us. We were all sweating and cursing at such occasions, I too; but *nolens volens* the idea occurred to me that this loss of time could have been minimized if the number of crossings of the rails had been minimized. But what is the minimum number of crossings? I realized after several days that the actual situation could have been improved, but the exact solution of the general problem with m kilns and n storage yards seemed to be very difficult . . . the problem occurred to me again . . . at my first visit to Poland where I met Zarankiewicz. I mentioned to him my "brick-factory"-problem . . . and Zarankiewicz thought to have solved (it). But Ringel found a gap in his published proof, which nobody has been able to fill so far—in spite of much effort. This problem has also become a notoriously difficult unsolved problem . . . .

Zarankiewicz [Z2], thus, thought that he had proved

$$\nu(K(m, n)) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \tag{4.5}$$

but, in actuality, he only verified that the right hand expression of (4.5) is an upper bound for $\nu(K(m, n))$. As it turned out, both P.C. Kainen and G. Ringel found flaws in Zarankiewicz's proof. Hence, (4.5) remains only a conjecture. It is further conjectured that $\nu(K(m, n)) = \bar{\nu}(K(m, n))$. The best general result on crossing numbers of complete bipartite graphs is the following due to Kleitman [K6].

Theorem 4.13 *For* $1 \leq min\{m, n\} \leq 6$,

$$\nu(K(m, n)) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

It follows, therefore, from Theorem 4.13 that

$$\nu(K(3, n)) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \nu(K(4, n)) = 2 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor,$$

$$\nu(K(5, n)) = 4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{and} \quad \nu(K(6, n)) = 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

for all n. For example, $\nu(K(3, 3)) = 1$, $\nu(K(4, 4)) = 4$, $\nu(K(5, 5)) = 16$, and $\nu(K(6, 6)) = 36$. A drawing of $K(4, 4)$ with four crossings is shown in Figure 4.16. Thus, the simplest complete bipartite graph whose crossing number is

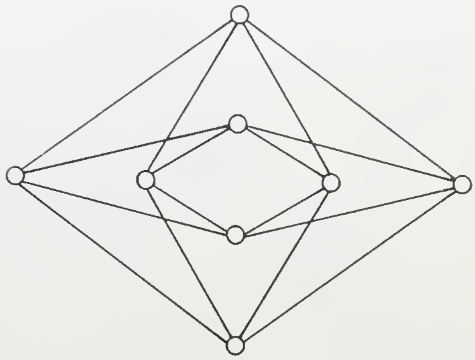


Figure 4.16 *A drawing of* $K(4, 4)$ *with four crossings*

unknown is $K(7, 7)$. Kleitman [K6], however, has verified that $\nu(K(7, 7))$ has one of the values 77, 79, or 81.

As would be expected, the situation regarding crossing numbers of complete n-partite graphs, $n \geq 3$, is even more complex. For the most part, only bounds and highly specific results have been obtained in these cases. On the other hand, some of the proof techniques employed have been enlightening. As an example, we establish the crossing number of $K(2, 2, 3)$ (see White [W4, p. 77]).

Theorem 4.14 *The crossing number of* $K(2, 2, 3)$ *is* $\nu(K(2, 2, 3)) = 2$.

Proof Let $\nu(K(2, 2, 3)) = c$. Since $K(3, 3)$ is nonplanar and $K(3, 3) \subset K(2, 2, 3)$, it follows that $K(2, 2, 3)$ is nonplanar so that $c \geq 1$. Let there be given a drawing of $K(2, 2, 3)$ in the plane with c crossings. At each crossing we introduce a new vertex, producing a connected plane graph G of order $p = 7 + c$ and size $q = 16 + 2c$. By Corollary 4.2a, $q \leq 3p - 6$.

Let u_1u_2 and v_1v_2 be two (nonadjacent) edges of $K(2, 2, 3)$ that cross in the given drawing, giving rise to a new vertex. If G is a triangulation, then C: u_1, v_1, u_2, v_2, u_1, is a cycle of G, implying that the induced subgraph $\langle \{u_1, u_2, v_1, v_2\} \rangle$ in $K(2, 2, 3)$ is isomorphic to K_4. However, $K(2, 2, 3)$ contains no such subgraph; thus, G is not a triangulation so that $q < 3p - 6$. We have

$$16 + 2c < 3(7 + c) - 6,$$

from which it follows that $c \geq 2$. The inequality $c \leq 2$ follows from the fact that there exists a drawing of $K(2, 2, 3)$ with two crossings (see Figure 4.17). ■

Other graphs whose crossing numbers have been investigated with little success are the n-cubes Q_n. Since Q_n is planar for $n = 1, 2, 3$, of course,

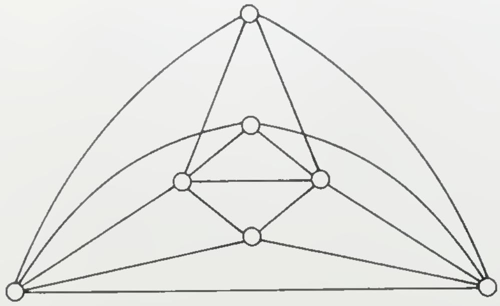


Figure 4.17 *A drawing of $K(2, 2, 3)$ with two crossings*

$\nu(Q_n) = 0$ for such n. Eggleton and Guy [EG1] have shown that $\nu(Q_4) = 8$ but $\nu(Q_n)$ is unknown for $n \geq 5$. One might observe that

$$Q_4 \cong K_2 \times K_2 \times K_2 \times K_2 \cong C_4 \times C_4$$

so that $\nu(C_4 \times C_4) = 8$. This raises the problem of determining $\nu(C_m \times C_n)$ for $m, n \geq 3$. For the case $m = n = 3$, Harary, Kainen, and Schwenk [HKS1] showed that $\nu(C_3 \times C_3) = 3$. Their proof consisted of the following three steps:

1. exhibiting a drawing of $C_3 \times C_3$ with three crossings so that $\nu(C_3 \times C_3) \leq 3$;
2. showing that $C_3 \times C_3 - e$ is nonplanar for every edge e of $C_3 \times C_3$ so that $\nu(C_3 \times C_3) \geq 2$; and
3. showing, by case exhaustion, that it is impossible to have a drawing of $C_3 \times C_3$ with exactly two crossings so that $\nu(C_3 \times C_3) \geq 3$ (see Exercise 4.17).

Ringeisen and Beineke [RB1] then extended this result significantly by determining $\nu(C_3 \times C_n)$ for all integers $n \geq 3$.

Theorem 4.15 *For all $n \geq 3$,*

$$\nu(C_3 \times C_n) = n.$$

Proof We label the vertices of $C_3 \times C_n$ by the $3n$ ordered pairs $(0, j)$, $(1, j)$, and $(2, j)$, where $j = 0, 1, \ldots, n-1$, and, for convenience, we let

$$u_j = (0, j), \quad v_j = (1, j), \quad \text{and} \quad w_j = (2, j).$$

First, we note that $\nu(C_3 \times C_n) \leq n$. This observation follows from the fact that there exists a drawing of $C_3 \times C_n$ with n crossings. A drawing of $C_3 \times C_4$ with four crossings is shown in Figure 4.18. Drawings of $C_3 \times C_n$ with n crossings for other values of n can be given similarly.

To complete the proof, we show that $\nu(C_3 \times C_n) \geq n$. We verify this by induction on $n \geq 3$. For $n = 3$, we recall the previously mentioned result $\nu(C_3 \times C_3) = 3$.

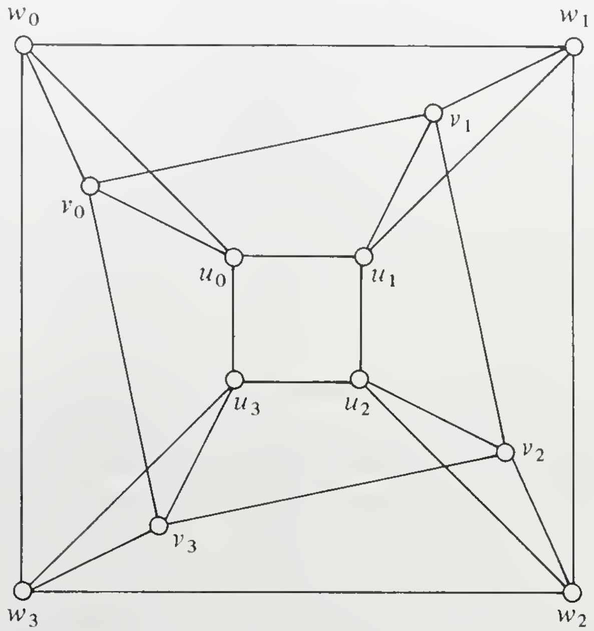


Figure 4.18 *A drawing of $C_3 \times C_4$ with four crossings*

Assume that $\nu(C_3 \times C_k) \geq k$, where $k \geq 3$, and consider the graph $C_3 \times C_{k+1}$. We show that $\nu(C_3 \times C_{k+1}) \geq k+1$. Let there be given a drawing of $C_3 \times C_{k+1}$ with $\nu(C_3 \times C_{k+1})$ crossings. We consider two cases.

Case 1: Suppose that no edge of any triangle $T_j = \langle \{u_j, v_j, w_j\} \rangle$, $j = 0, 1, \ldots, k$, is crossed. For $j = 0, 1, \ldots, k$, define

$$H_j = \langle \{u_j, v_j, w_j, u_{j+1}, v_{j+1}, w_{j+1}\} \rangle,$$

where the subscripts are expressed modulo $k+1$. We show that for each $j = 0, 1, \ldots, k$, the number of times edges of H_j are crossed totals at least two. Since, by assumption, no triangle T_j has an edge crossed and since every edge not in any T_j belongs to exactly one subgraph H_j, it will follow that there are at least $k+1$ crossings in the drawing because then every crossing of an edge in H_j involves either two edges of H_j or an edge of H_j and an edge of H_i for some $i \neq j$.

If two of the edges u_ju_{j+1}, v_jv_{j+1}, and w_jw_{j+1} cross each other, then two edges of H_j are crossed. Assume then that no two edges of H_j cross each other. Thus, H_j is a plane subgraph in the drawing of $C_3 \times C_{k+1}$ (see Figure 4.19). The triangle T_{j+2} must lie within some region of H_j. If T_{j+2} lies in a region of H_j bounded by a triangle, say T_j, then at least one edge of the cycle $u_0, u_1, \ldots, u_k, u_0$, for example, must cross an edge of T_j, contradicting our assumption. Thus, T_{j+2} must lie in a region of H_j bounded by a 4-cycle, say $u_j, u_{j+1}, w_{j+1}, w_j, u_j$, without loss of generality. However then edges of the cycle $v_0, v_1, \ldots, v_k, v_0$ must cross edges of the cycle $u_j, u_{j+1}, w_{j+1}, w_j, u_j$ at least twice and hence edges of H_j at least twice, as asserted.

H_j:

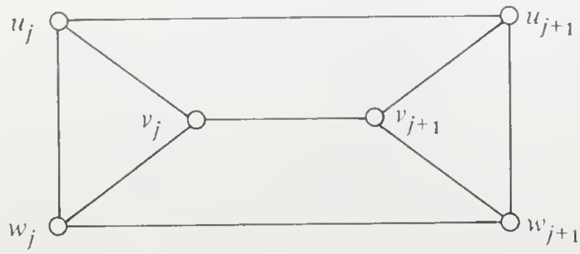


Figure 4.19 *The subgraph H_j in the proof of Theorem 4.15*

Case 2: Assume that some triangle, say T_0, has at least one of its edges crossed. Suppose that $\nu(C_3 \times C_{k+1}) < k+1$. Then the graph $C_3 \times C_{k+1} - E(T_0)$, which can be obtained by inserting vertices of degree 2 into the edges of $C_3 \times C_k$, is drawn with fewer than k crossings, contradicting the inductive hypothesis. ■

The only other result giving the crossing number of graphs $C_m \times C_n$ is the following formula by Beineke and Ringeisen [BR1].

Theorem 4.16 *For all $n \geq 4$,*

$$\nu(C_4 \times C_n) = 2n.$$

Beineke and Ringeisen [BR1] have also found a formula for $\nu(K_4 \times C_n)$.

Theorem 4.17 *For all $n \geq 3$,*

$$\nu(K_4 \times C_n) = 3n.$$

In addition to the crossing number, other topological parameters have proved to be interesting when applied to nonplanar graphs. Prior to defining these, however, we introduce a class of parameters whose definitions bear a striking similarity.

Let P be any property possessed by the trivial graph K_1 (such as being acyclic or planar). By the *vertex covering number* of a graph G *with respect to* P is meant the minimum number of elements V_i in a partition of $V(G)$ such that each induced subgraph $\langle V_i \rangle$ has property P. The *vertex packing number of G with respect to P* is the maximum number of mutually disjoint nonempty subsets V_i of $V(G)$ such that no subgraph $\langle V_i \rangle$ has property P. In a completely analogous manner, one can define the *edge covering number* and *edge packing number* of a nonempty graph G *with respect to any property possessed by the graph* K_2.

We have already seen examples of these types of parameters. The arboricity $a_1(G)$ of a nonempty graph G (defined in Chapter 3) is the edge covering number of G with respect to the property of being acyclic while the vertex-arboricity $a(G)$ of G is the corresponding vertex covering number of G. The number that might be considered "dual" to $a_1(G)$ is the edge packing number of G with respect to the property of being acyclic. It is not difficult to see that this is the maximum number of edge-disjoint cycles contained in G. For this reason, this number is referred to as the *cycle multiplicity* of G; we denote it by $\bar{a}_1(G)$. As expected, no formula exists for the cycle multiplicity of an arbitrary graph G; however, formulas have been found [CGH1] for $\bar{a}_1(G)$ when $G \cong K_p$ and $G \cong K(m, n)$.

Formulas for the vertex analogue of cycle multiplicity are very easy to derive in the case of complete graphs and complete bipartite graphs; in fact, a formula exists [CKW1] for any complete n-partite graph.

Another property that has given rise to parameters of the above type is planarity. The *edge-thickness* or simply the *thickness* $\theta_1(G)$ of a nonempty graph G is the edge covering number of G with respect to planarity; that is, $\theta_1(G)$ is the minimum number of pairwise edge-disjoint planar spanning subgraphs of G whose edge sum is G. This provides another measure of the nonplanarity of a graph. Once again, it is the complete graphs, complete bipartite graphs, and n-cubes that have received the most attention.

A formula for the thickness of the complete graphs was established primarily due to the efforts of Beineke [B3], Beineke and Harary [BH2], Vasak [V1], and Alekseev and Gonchakov [AG1].

Theorem 4.18 *The thickness of K_p is given by*

$$\theta_1(K_p) = \begin{cases} \left\lfloor \dfrac{p+7}{6} \right\rfloor & p \neq 9,\ 10 \\ 3 & p = 9,\ 10. \end{cases}$$

Although only partial results exist for the thickness of complete bipartite graphs (see [BHM1]), a formula is known for the thickness of the n-cubes, due to Kleinert [K5].

Theorem 4.19 *The thickness of Q_n is given by*

$$\theta_1(Q_n) = \left\lceil \frac{n+1}{4} \right\rceil.$$

The vertex analogue of thickness is the *vertex-thickness* $\theta(G)$; very little is known about this parameter. The edge packing number of a graph G with respect to planarity is denoted by $\overline{\theta}_1(G)$ and is referred to as the *coarseness* of G. There has been little success in the study of coarseness, though some progress has been made by Guy and Beineke [GB1].

Other parameters of the type discussed in this section will be encountered in Chapter 10.

Exercises 4.3

4.12 Draw K_7 in the plane with nine crossings.

4.13 Determine $\nu(K(3, 3))$ without using Theorem 4.13.

4.14 Show that $\nu(K(7, 7)) \leq 81$.

4.15 Determine $\nu(K(2, 2, 2))$.

4.16 Determine $\nu(K(1, 2, 3))$.

4.17 Show that $2 \leq \nu(C_3 \times C_3) \leq 3$.

4.18 Prove that $\bar{\nu}(C_3 \times C_n) = n$ for $n \geq 3$.

(a) It is known that $\nu(W_4 \times K_2) = 2$, where W_4 is the wheel $C_4 + K_1$ of order 5. Draw $W_4 \times K_2$ in the plane with two crossings.

(b) Prove or disprove: If G is a nonplanar graph containing an edge e such that $G - e$ is planar, then $\nu(G) = 1$.

4.19 Define the vertex analogue $\bar{a}(G)$ of the parameter cycle multiplicity. Derive formulas for $\bar{a}(K_p)$ and $\bar{a}(K(m, n))$.

4.20 Develop a formula for $\bar{a}(Q_n)$.

4.21 Develop a formula for $\bar{a}_1(Q_n)$.

4.22 Prove that $\theta_1(K_p) \geq \lfloor (p + 7)/6 \rfloor$ for all positive integers p.

4.23 Verify that $\theta_1(K_p) = \lfloor (p + 7)/6 \rfloor$ for $p = 4, 5, 6, 7, 8$.

4.24 Give a definition for $\theta(G)$. Develop a formula for $\theta(K_p)$.

4.25 Define the parameter $\bar{\theta}(G)$. Develop a formula for $\bar{\theta}(K_p)$.

4.26 Define the parameter $\bar{\theta}_1(G)$. Determine an upper bound for $\bar{\theta}_1(K_p)$.

4.27 Define the vertex covering number of a graph G with respect to the property of being disconnected or trivial. Determine the value of this parameter for all complete n-partite graphs.

4.28 Let P denote the property that a graph G has no induced subgraph isomorphic to the path P_n for any $n \geq 3$. Find the vertex covering number $\ell(G)$, the vertex packing number $\bar{\ell}(G)$, the edge covering number $\ell_1(G)$ and edge packing number $\bar{\ell}_1(G)$ of G with respect to P for $G \cong C_n (n \geq 3)$.

4.4 The Genus of a Graph

We now introduce the best known parameter involving nonplanar graphs. A compact orientable 2-manifold is a surface that may be thought of as a sphere on which has been placed a number of "handles" or, equivalently, a sphere in which has been inserted a number of "holes". The number of handles (or holes) is referred to as the *genus of the surface*. By the *genus* $\gamma(G)$ *of a graph* G is meant the smallest genus of all surfaces (compact orientable 2-manifolds) on which G can be embedded. Every graph has a genus; in fact, it is a relatively simple observation that a graph of size q can be embedded on a surface of genus q.

Since the embedding of graphs on spheres and planes is equivalent, the graphs of genus 0 are precisely the planar graphs. The graphs with genus 1 are therefore the nonplanar graphs that are embeddable on the torus. The (nonplanar) graphs K_5 and $K(3, 3)$ have genus 1. Embeddings of $K(3, 3)$ on the torus and on the surface of genus 2 are shown in Figure 4.20(a) and (b).

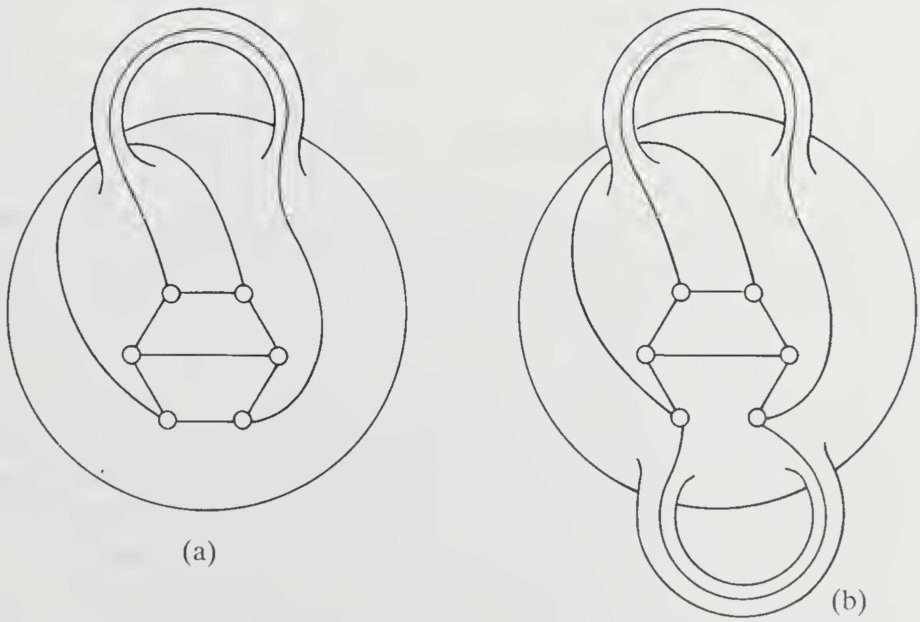


Figure 4.20 *Embeddings of* $K(3, 3)$ *on surfaces of genus* 1 *and* 2

Not only is K_5 embeddable on the torus, but so are K_6 and K_7. (The graph K_8 is not embeddable on the torus.) Figure 4.21 gives an embedding of K_7 on the torus. The torus is obtained by identifying opposite sides of the rectangle. The vertices of K_7 are labeled $v_0, v_1, \ldots, v_6$. Thus note that the "vertices" located at the corners of the rectangle actually represent the same vertex of K_7, namely the one labeled v_5.

For graphs embedded on surfaces of positive genus, the regions and the boundaries of the regions are defined in entirely the same manner as for

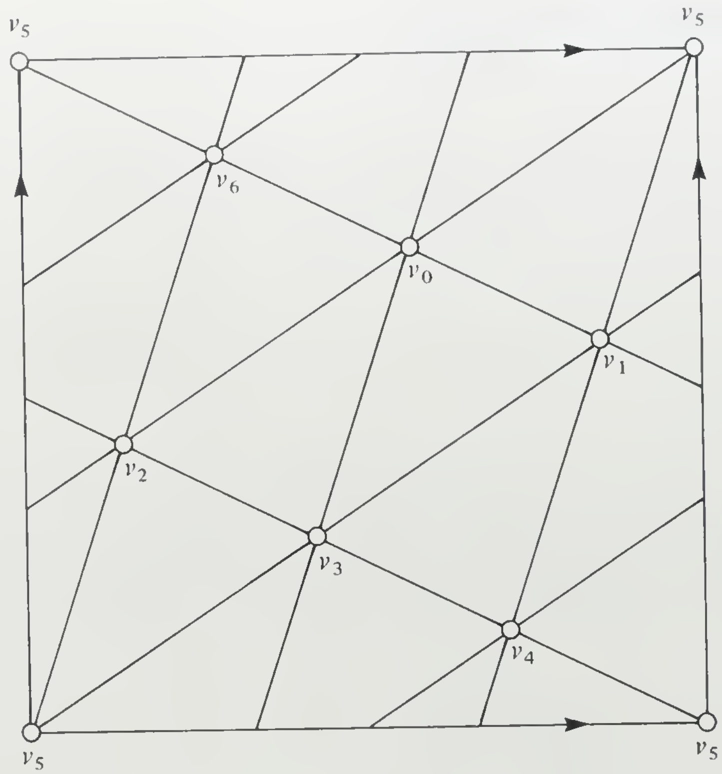


Figure 4.21 *An embedding of K_7 on the torus*

embeddings in the plane. Thus, if G is embedded on a surface S, then the components of $S - G$ are the regions of the embedding. In Figure 4.20(a) there are three regions, in Figure 4.20(b) there are two regions, and in Figure 4.21 there are 14 regions.

A region is called a *2-cell* if any simple closed curve in that region can be continuously deformed or contracted in that region to a single point. Equivalently, a region is a 2-cell if it is topologically homeomorphic to 2-dimensional Euclidean space. Although every region of a connected graph embedded on the sphere is necessarily a 2-cell, this need not be the case for connected graphs embedded on surfaces of positive genus. Of the two regions determined by the embedding of $K(3, 3)$ on the "double torus" in Figure 4.20(b), one is a 2-cell and the other is not. The boundary of the 2-cell is a 4-cycle while the boundary of the other region consists of all vertices and edges of $K(3, 3)$.

An embedding of a graph G on a surface S is called a *2-cell embedding* of G on S if all the regions so determined are 2-cells. The embeddings in Figure 4.20(a) and Figure 4.21 are both 2-cell embeddings.

In order to present an extension of Theorem 4.1 to surfaces of positive genus, we introduce some new terms. A *loop-graph* is a finite nonempty set V of vertices together with a (possibly empty) set E (of edges) consisting of one- or two-element subsets of V, each one-element subset being referred to as a *loop*. A loop-graph that admits multiple edges (including multiple loops) is called a *pseudograph*.

Theorem 4.20 *Let G be a connected (p, q) pseudograph with a 2-cell embedding on the surface of genus n and having r regions. Then*

$$p - q + r = 2 - 2n. \tag{4.6}$$

Proof The proof is by induction on n. For $n = 0$, the formula holds for connected graphs by Theorem 4.1. If G is a connected (p, q) pseudograph (which is not a graph) embedded in the plane and having r regions, then a plane graph H is obtained by deleting from G all loops and all but one edge in any set of multiple edges joining the same two vertices. If H has order p_1, size q_1, and r_1 regions, then $p_1 - q_1 + r_1 = 2$ by Theorem 4.1. We now add back the deleted edges to form the originally embedded pseudograph G. Note that the addition of each such edge increases the number of regions by one. If G has k more edges than does H, then $p = p_1$, $q = q_1 + k$, and $r = r_1 + k$ so that

$$p - q + r = p_1 - (q_1 + k) + (r_1 + k) = p_1 - q_1 + r_1 = 2,$$

producing the desired result for $n = 0$.

Assume the theorem to be true for all connected pseudographs that are 2-cell embedded on the surface of genus $n - 1$, where $n > 0$, and let G be a connected (p, q) pseudograph that is 2-cell embedded on the surface S of genus n and having r regions. We verify that (4.6) holds.

Since the surface S has genus n and $n > 0$, S has handles. Draw a curve C around a handle of S such that C contains no vertices of G. Necessarily, C will cross edges of G; for otherwise C lies in a region of G and cannot be contracted in that region to a single point, contradicting the fact that the embedding on S is a 2-cell embedding. By re-embedding G on S, if necessary, we may assume that the total number of intersections of C with edges of G is finite, say k, where $k > 0$. If $e_1, e_2, \ldots, e_m$ are the edges of G that are crossed by C, then $1 \leq m \leq k$ (see Figure 4.22). Moreover, if edge e_i, $1 \leq i \leq m$, is crossed by C a total of ℓ_i times, then $\Sigma_{i=1}^{m} \ell_i = k$.

At each of the k intersections of C with the edges of G we add a new vertex; further, each subset of C lying between consecutive new vertices is identified as a new edge. Moreover, each edge of G that is crossed by C, say a total of ℓ times, is subdivided into $\ell + 1$ new edges.

Let the new pseudograph so formed be denoted by G'; further, suppose G' has order p', size q', and r' regions. Since k new vertices have been introduced in forming G', it follows that $p' = p + k$. The curve C has resulted in an increase of k in the number of edges. Also, each edge e_i, $1 \leq i \leq m$, has given rise to an increase of ℓ_i edges and since $\Sigma_{i=1}^{m} \ell_i = k$, the total increase in size from G to G' is $2k$; that is, $q' = q + 2k$.

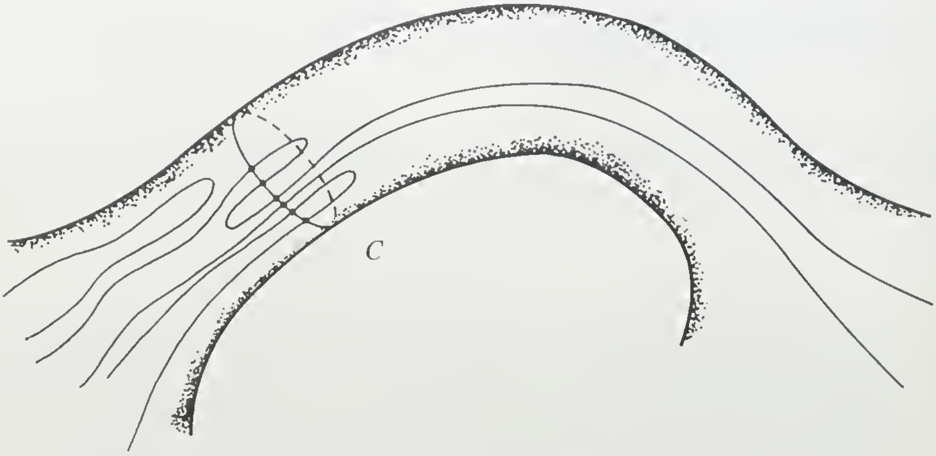


Figure 4.22 *A curve C drawn on a handle of the surface S*

Each portion of C that became an edge of G' is in a region of G. Thus, the addition of such an edge divides that region into two regions. Since there exist k such edges, $r' = r + k$. Because every region of G is a 2-cell, it follows that every region of G' is a 2-cell.

We now make a "cut" in the handle along C, separating the handle into two pieces (as shown in Figure 4.23). The two resulting holes are "patched" or "capped", producing a new (2-cell) region in each case. (This is called a "*capping*" *operation.*)

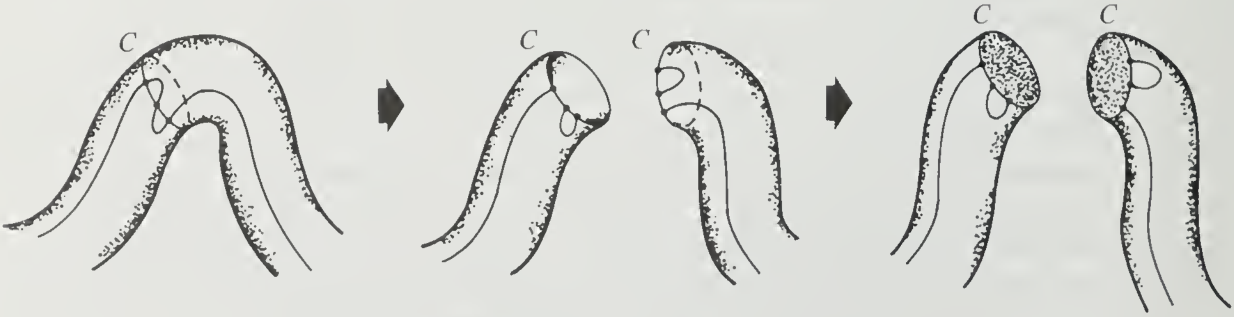


Figure 4.23 *Capping a cut handle*

In the process of performing this capping operation, several changes have occurred. First, the surface S has been transformed into a new surface S''. The two capped pieces of the handle of S are now part of the sphere of S''. Hence S'' has one less handle than S so that S'' has genus $n - 1$. Furthermore, the pseudograph G' itself has been altered. The vertices and edges resulting from the curve C have been divided into two copies, one copy on each of the two pieces of the capped handle. If G'' denotes this new pseudograph, then G'' has order $p'' = p' + k = p + 2k$ and size $q'' = q' + k = q + 3k$. Also, the number r'' of regions satisfies $r'' = r' + 2 = r + k + 2$. Since each of these r'' regions in the connected pseudograph G'' is a 2-cell, the inductive hypothesis applies so that $p'' - q'' + r'' = 2 - 2(n-1)$ or

$$(p + 2k) - (q + 3k) + (r + k + 2) = 2 - 2(n - 1);$$

thus,

$$p - q + r = 2 - 2n,$$

giving the desired result. ■

Restating Theorem 4.20 for graphs, we have the following.

Corollary 4.20 *Let G be a connected (p, q) graph with a 2-cell embedding on the surface of genus n and having r regions. Then*

$$p - q + r = 2 - 2n.$$

In connection with Corollary 4.20 is the following result. Proofs of this theorem (see Youngs [Y1], for example) are strictly topological in nature; we present no proof.

Theorem 4.21 *If G is a connected graph embedded on the surface of genus $\gamma(G)$, then every region of G is a 2-cell.*

Corollary 4.20 and Theorem 4.21 now immediately imply the following.

Theorem 4.22 *If G is a connected (p, q) graph embedded on the surface of genus $\gamma(G)$ and having r regions, then*

$$p - q + r = 2 - 2\gamma(G).$$

An important conclusion, which can be reached with the aid of Theorem 4.22, is that every two embeddings of a connected graph G on the surface of genus $\gamma(G)$ result in the same number of regions. With the theorems obtained thus far, we can now provide a lower bound for the genus of a connected graph in terms of its order and size.

Theorem 4.23 *If G is a connected (p, q) graph $(p \geq 3)$, then*

$$\gamma(G) \geq \frac{q}{6} - \frac{p}{2} + 1.$$

Proof The result is immediate for $p = 3$, so we assume that $p \geq 4$. Let G be embedded on the surface of genus $\gamma(G)$. By Theorem 4.22, $p - q + r = 2 - 2\gamma(G)$, where r is the number of regions of G. (Necessarily, each of these regions is a 2-cell by Theorem 4.21.) Since the boundary of every region contains at least three edges and every edge is on the boundary of at most two regions, $3r \leq 2q$. Thus,

$$2 - 2\gamma(G) = p - q + r \leq p - q + \frac{2q}{3},$$

and the desired result follows. ■

The lower bound for $\gamma(G)$ presented in Theorem 4.23 can be improved when more information on cycle lengths in G is available. The proof of the next theorem is entirely analogous to that of the preceding.

Theorem 4.24 *If G is a connected (p, q) graph with girth n, then*

$$\gamma(G) \geq q\left(1 - \frac{2}{n}\right)/2 - \frac{p}{2} + 1.$$

A special case of Theorem 4.24 that includes bipartite graphs is of special interest. A graph is often called *triangle-free* if it contains no triangles.

Corollary 4.24 *If G is a connected, triangle-free (p, q) graph $(p \geq 3)$, then*

$$\gamma(G) \geq \frac{q}{4} - \frac{p}{2} + 1.$$

As one might have deduced by now, no general formula for the genus of an arbitrary graph is known. Indeed, it is unlikely that such a formula will ever be developed in terms of quantities that are easily calculable. On the other hand, the following result by Battle, Harary, Kodama, and Youngs [BHKY1] implies that, as far as genus formulas are concerned, one need only investigate blocks. We omit the proof.

Theorem 4.25 (Battle, Harary, Kodama, and Youngs) *If G is a graph having blocks $B_1, B_2, \ldots, B_n$, then*

$$\gamma(G) = \sum_{i=1}^{n} \gamma(B_i).$$

The following corollary is a consequence of the preceding result.

Corollary 4.25 *If G is a graph with components $G_1, G_2, \ldots, G_k$, then*

$$\gamma(G) = \sum_{i=1}^{k} \gamma(G_i).$$

As is often the case, when no general formula exists for the value of a parameter for an arbitrary graph, formulas (or partial formulas) are established for certain families of graphs. Ordinarily the first classes to be considered are the complete graphs, the complete bipartite graphs, and the n-cubes. The genus offers no exception to this rule.

In 1968, Ringel and Youngs [RY1] completed a proof of a result that has a remarkable history. They solved a problem that became known as the *Heawood Map Coloring Problem*; this problem will be discussed in Chapter 10. The solution involved the verification of a conjectured formula for the genus of a complete graph; the proof can be found in (and, in fact, *is*) the book by Ringel [R6].

Theorem 4.26 (Ringel and Youngs) *The genus of the complete graph is given by*

$$\gamma(K_p) = \left\lceil \frac{(p-3)(p-4)}{12} \right\rceil, \qquad p \geqslant 3.$$

A formula for the genus of the complete bipartite graph was discovered by Ringel [R5]. We shall also omit the proof of this result.

Theorem 4.27 (Ringel) *The genus of the complete bipartite graph is given by*

$$\gamma(K(m, n)) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil, \qquad m, n \geqslant 2.$$

A formula for the genus of the n-cube was found by Ringel [R4] and, independently, by Beineke and Harary [BH1]. We prove this result to illustrate some of the techniques involved. We omit the obvious equality $\gamma(Q_1) = 0$.

Theorem 4.28 (Ringel; Beineke and Harary) *For $n \geqslant 2$, the genus of the n-cube is given by*

$$\gamma(Q_n) = (n-4) \cdot 2^{n-3} + 1.$$

Proof The n-cube is a triangle-free $(2^n, n \cdot 2^{n-1})$ graph; thus, by Corollary 4.24,

$$\gamma(Q_n) \geqslant (n-4) \cdot 2^{n-3} + 1.$$

To verify the inequality in the other direction, we employ induction on n.

For $n \geq 2$, define the statement $A(n)$ as follows: The graph Q_n can be embedded on the surface of genus $(n-4) \cdot 2^{n-3} + 1$ such that the boundary of every region is a 4-cycle and such that there exist 2^{n-2} regions with pairwise disjoint boundaries. That the statements $A(2)$ and $A(3)$ are true is trivial. Assume $A(k-1)$ to be true, $k \geq 4$, and, accordingly, let S be the surface of genus $(k-5) \cdot 2^{k-4} + 1$ on which Q_{k-1} is embedded such that the boundary of each region is a 4-cycle and such that there exist 2^{k-3} regions with pairwise disjoint boundaries. We note that since Q_{k-1} has order 2^{k-1}, each vertex of Q_{k-1} belongs to the boundary of precisely one of the aforementioned 2^{k-3} regions. Now let Q_{k-1} be embedded on another copy S' of the surface of genus $(k-5) \cdot 2^{k-4} + 1$ such that the embedding of Q_{k-1} on S' is a "mirror image" of the embedding of Q_{k-1} on S (that is, if v_1, v_2, v_3, v_4 are the vertices of a region of Q_{k-1} on S, where the vertices are listed clockwise about the 4-cycle, then there is a region on S', with the vertices v_1, v_2, v_3, v_4 on its boundary listed counterclockwise). We now consider the 2^{k-3} distinguished regions of S together with the corresponding regions of S', and join each pair of associated regions by a handle. The addition of the first handle produces the surface of genus $2[(k-5) \cdot 2^{k-4} + 1]$ while the addition of each of the other $2^{k-3} - 1$ handles results in an increase of one to the genus. Thus, the surface just constructed has genus $(k-4) \cdot 2^{k-3} + 1$. Now each set of four vertices on the boundary of a distinguished region can be joined to the corresponding four vertices on the boundary of the associated region so that the four edges are embedded on the handle joining the regions. It is now immediate that the resulting graph is isomorphic to Q_k and that every region is bounded by a 4-cycle. Furthermore, each added handle gives rise to four regions, "opposite" ones of which have disjoint boundary, so that there exist 2^{k-2} regions of Q_k that are pairwise disjoint.

Thus, $A(n)$ is true for all $n \geq 2$, proving the result. ■

Exercises 4.4

4.29 Determine $n = \gamma(K(4, 4))$ without using Theorem 4.27 and label the regions in a 2-cell embedding of $K(4, 4)$ on the surface of genus n.

4.30 (a) Show that $\gamma(G) \leq \nu(G)$ for every graph G.

(b) Prove that for every positive integer n, there exists a graph G such that $\gamma(G) = 1$ and $\nu(G) = n$.

4.31 Prove Theorem 4.24.

4.32 Use Theorem 4.25 to prove Corollary 4.25.

4.33 Show that $\gamma(K_p) \geq \left\lceil \frac{(p-3)(p-4)}{12} \right\rceil$, $p \geq 3$.

4.34 Show that $\gamma(K(m, n)) \geq \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$, $m, n \geq 2$.

4.35 (a) Find a lower bound for $\gamma(K(3, 3) + \bar{K}_n)$.
(b) Determine $\gamma(K(3, 3) + \bar{K}_n)$ exactly for $n = 1, 2,$ and 3.

4.36 Determine $\gamma(K_2 \times C_4 \times C_6)$.

4.37 Prove, for every positive integer n, that there exists a connected graph G of genus n.

4.38 Prove, for each positive integer n, that there exists a planar graph G such that $\gamma(G \times K_2) \geq n$.

4.5 2-Cell Embeddings of Graphs

In the preceding section we saw that every graph G has a genus; that is, there exists a surface (a compact orientable 2-manifold) of minimum genus on which G can be embedded. Indeed, by Theorem 4.21 if G is a connected graph that is embedded on the surface of genus $\gamma(G)$, then the embedding is necessarily a 2-cell embedding. On the other hand, if G is disconnected, then no embedding of G is a 2-cell embedding.

Our primary interest lies with embeddings of (connected) graphs that are 2-cell embeddings. In this section, we investigate graphs and the surfaces on which they can be 2-cell embedded. It is convenient to denote the surface of genus n by S_n. Thus, S_0 represents the sphere (or plane), S_1 represents the torus, and S_2 represents the double torus (or sphere with two handles).

We have already mentioned that the torus can be represented as a rectangle with opposite sides identified. More generally, the surface $S_n (n > 0)$ can be represented as a regular $4n$-gon whose $4n$ sides can be listed in clockwise order as

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \ldots a_n b_n a_n^{-1} b_n^{-1}, \tag{4.7}$$

where, for example, a_1 is a side directed clockwise and a_1^{-1} is a side also labeled a_1 *but* directed counterclockwise. These two sides are then identified in a manner consistent with their directions. Thus, the double torus can be represented as shown in Figure 4.24. The "two" points labeled X are actually the same point on S_2 while the "eight" points labeled Y are, in fact, a single point.

Although it is probably obvious that there exist a variety of graphs that can be embedded on the surface S_n for a given nonnegative integer n, it may not be entirely obvious that there always exist graphs for which a 2-cell embedding on S_n exists.

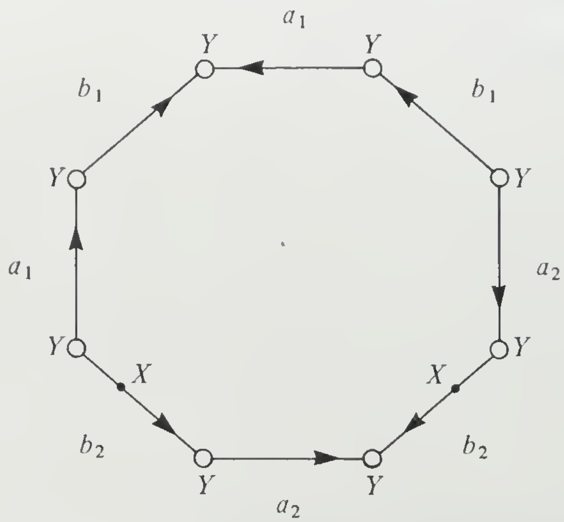


Figure 4.24 *A representation of the double torus*

Theorem 4.29 *For every nonnegative integer n, there exists a connected graph that has a 2-cell embedding on S_n.*

Proof For $n = 0$, every connected planar graph has the desired property; thus, we assume that $n > 0$.

We represent S_n as a regular $4n$-gon whose $4n$ sides are described and identified as in (4.7). First, we define a pseudograph H as follows. At each vertex of the $4n$-gon, let there be a vertex of H. Actually, the identification process associated with the $4n$-gon implies that this is only one vertex of H. Let each side of the $4n$-gon represent an edge of H. The identification produces $2n$ distinct edges, each of which is a loop. This completes the construction of H. Hence, the pseudograph H has order 1 and size $2n$. Furthermore, there is only one region, namely the interior of the polygon; this region is clearly a 2-cell. Therefore, there exists a 2-cell embedding of H on S_n.

To convert the pseudograph H into a graph, we subdivide each loop twice, producing a graph G having order $4n + 1$, size $6n$, and again a single 2-cell region. ∎

Figure 4.25 illustrates the construction given in the proof of Theorem 4.29 in the case of the torus S_1. The graph G so constructed is shown in Figure

4.25(a). In Figures 4.25(b) through (e) we see a variety of ways of visualizing the embedding. In Figure 4.25(b), a 3-dimensional embedding is described. In Figures 4.25(c) and (d), the torus is represented as a rectangle with opposite sides identified. (Figure 4.25(c) is the actual drawing described in the proof of the theorem.) In Figure 4.25(e), a portion of G is drawn in the plane, then two circular holes are made in the plane and a tube (or handle) is placed over the plane joining the two holes. The edge uv is then drawn over the handle, completing the 2-cell embedding.

The graphs G constructed in the proof of Theorem 4.29 are planar. Hence, for every nonnegative integer n, there exist planar graphs that can be 2-

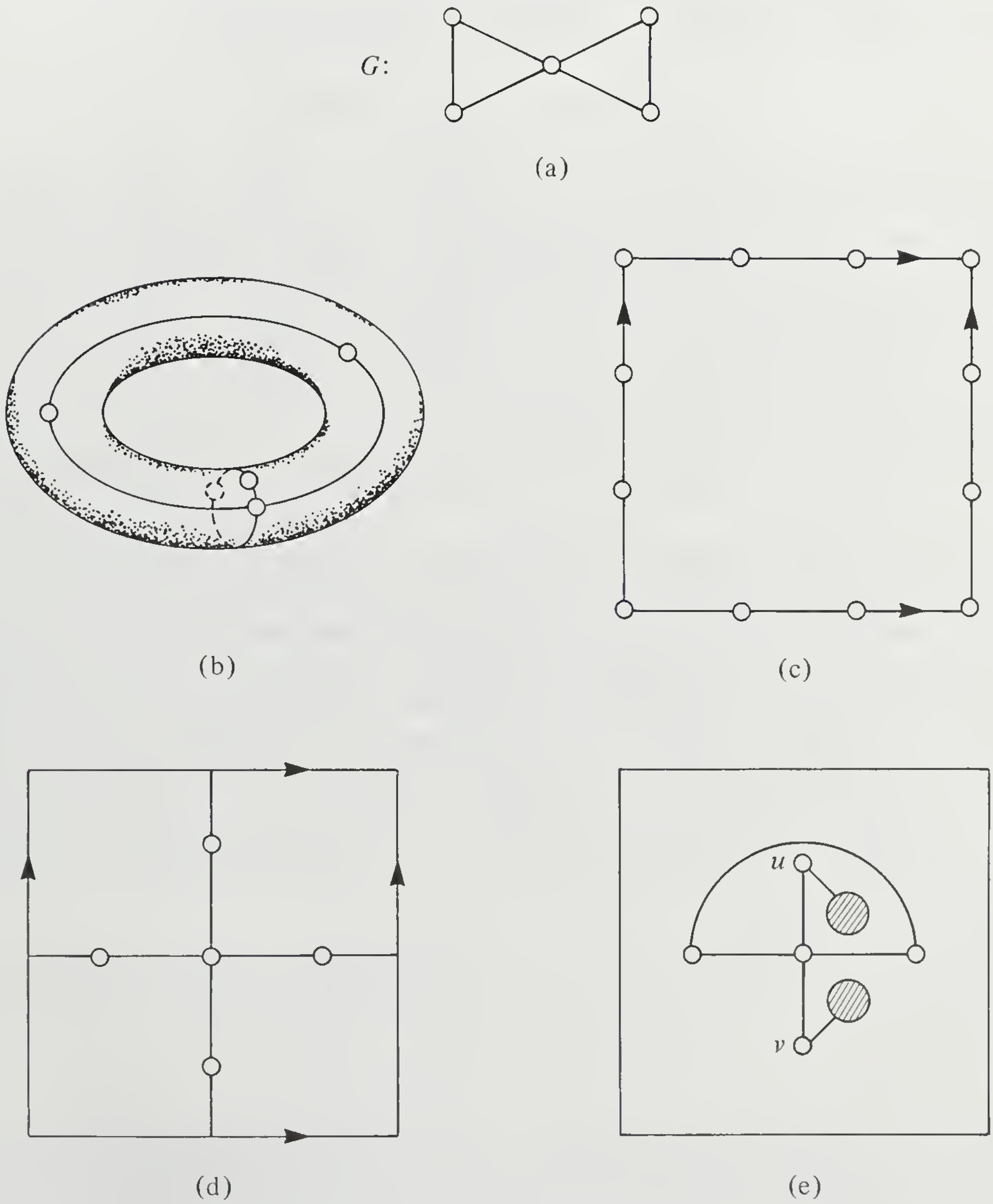


Figure 4.25 *A graph 2-cell embedded on the torus*

cell embedded on S_n. It is also true that for every planar graph G and *positive* integer n, there exists an embedding of G on S_n that is *not* a 2-cell embedding. In general, for a given graph G and positive integer n with $n > \gamma(G)$, there always exists an embedding of G on S_n that is not a 2-cell embedding, which can be obtained from an embedding of G on $S_{\gamma(G)}$ by adding $n - \gamma(G)$ handles to the interior of some region of G. If $n = \gamma(G)$ and G is connected, then by Theorem 4.21 every embedding of G on S_n is a 2-cell embedding while, of course, if $n < \gamma(G)$, there is no embedding whatsoever of G on S_n.

Thus far, whenever we have described a 2-cell embedding (or, in fact, any embedding) of a graph G on a surface S_n, we have resorted to a geometric description, such as one shown in Figure 4.25. There is a far more useful method, algebraic in nature, that we shall now discuss.

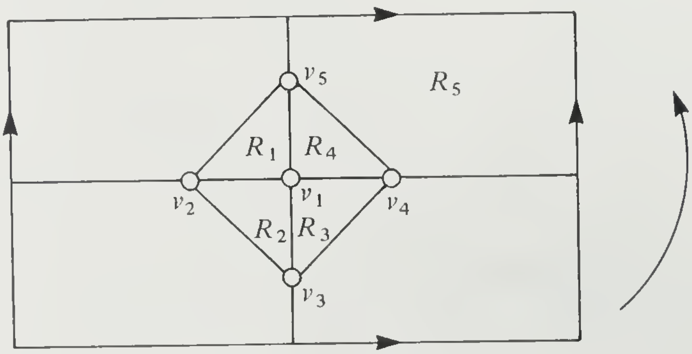


Figure 4.26 *A 2-cell embedding of K_5 on the torus*

Consider the 2-cell embedding of K_5 on S_1 shown in Figure 4.26, with the vertices of K_5 labeled as indicated. Observe that in this embedding the edges incident with v_1 are arranged cyclically counterclockwise about v_1 in the order v_1v_2, v_1v_3, v_1v_4, v_1v_5 (or, equivalently, v_1v_3, v_1v_4, v_1v_5, v_1v_2, and so on). This induces a cyclic permutation π_1 of the subscripts of the vertices adjacent with v_1, namely $\pi_1 = (2\ 3\ 4\ 5)$, expressed as a permutation cycle. Similarly, this embedding induces a cyclic permutation π_2 of the subscripts of the vertices adjacent with v_2; in particular, $\pi_2 = (1\ 5\ 4\ 3)$. In fact, for each vertex $v_i (1 \le i \le 5)$, one can associate a cyclic permutation π_i with v_i. In this case, we have

$$\begin{aligned} \pi_1 &= (2\ 3\ 4\ 5), \\ \pi_2 &= (1\ 5\ 4\ 3), \\ \pi_3 &= (1\ 2\ 5\ 4), \\ \pi_4 &= (1\ 3\ 2\ 5), \\ \pi_5 &= (1\ 4\ 3\ 2). \end{aligned}$$

In the 2-cell embedding of K_5 on S_1 shown in Figure 4.26, there are five regions, labeled R_1, R_2, ..., R_5. Each region $R_i (1 \le i \le 5)$ is, of course, a 2-cell. The boundary of the region R_1 consists of the vertices v_1, v_2, and v_5 and the edges v_1v_2, v_2v_5, and v_5v_1. If we trace out the edges of the boundary in a clockwise direction, that is, keeping the boundary at our left and the region to our right (see Figure 4.27), beginning with the edge v_1v_2, we have v_1v_2,

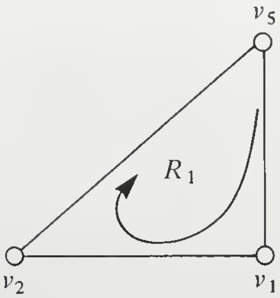


Figure 4.27 *Tracing out a region*

followed by v_2v_5, and finally v_5v_1. This information can also be obtained from the cyclic permutations $\pi_1, \pi_2, \ldots, \pi_5$; indeed, the edge following $v_1v_2 = v_2v_1$ as we trace the boundary edges of R_1 in a clockwise direction is precisely the edge incident with v_2 that follows v_2v_1 if one proceeds counterclockwise about v_2; that is, the edge following v_1v_2 in the boundary of R_1 is $v_2v_{\pi_2(1)} = v_2v_5$. Similarly, the edge following $v_2v_5 = v_5v_2$ as we trace out the edges of the boundary of R_1 in a clockwise direction is $v_5v_{\pi_5(2)} = v_5v_1$. Hence with the aid of the cyclic permutations $\pi_1, \pi_2, \ldots, \pi_5$, we can trace out the edges of the boundary of R_1. In a like manner, the boundary of every region of the embedding can be so described.

Since the direction (namely, clockwise) in which the edges of the boundary of a region are traced in the above description is of utmost importance, it is convenient to regard each edge of K_5 as a symmetric pair of arcs and, thus, to interpret K_5 itself as a digraph D. With this interpretation, the boundary of the region R_1 and thus R_1 itself can be described, starting at v_1, as

$$(v_1, v_2), (v_2, v_{\pi_2(1)}), (v_5, v_{\pi_5(2)})$$

or

$$(v_1, v_2), (v_2, v_5), (v_5, v_1). \tag{4.8}$$

We now define a mapping $\pi: E(D) \to E(D)$ as follows. Let $a \in E(D)$, where $a = (v_i, v_j)$. Then

$$\pi(a) = \pi((v_i, v_j)) = \pi(v_i, v_j) = (v_j, v_{\pi_j(i)}).$$

The mapping π is one-to-one and so is a permutation of $E(D)$. Thus, π can be expressed as a product of disjoint permutation cycles. In this context, each permutation cycle of π is referred to as an "orbit" of π. Hence (4.8) corresponds to an orbit of π and is often denoted more compactly as $v_1 - v_2 - v_5 -$

v_1. (Although this orbit corresponds to a cycle in the graph, this is not always the case for an arbitrary orbit in a graph that is 2-cell embedded.) For the embedding of K_5 on S_1 shown in Figure 4.26, the list of all five orbits (one for each region) is given below:

$$\begin{aligned}
R_1&: v_1 - v_2 - v_5 - v_1,\\
R_2&: v_1 - v_3 - v_2 - v_1,\\
R_3&: v_1 - v_4 - v_3 - v_1,\\
R_4&: v_1 - v_5 - v_4 - v_1,\\
R_5&: v_2 - v_3 - v_5 - v_2 - v_4 - v_5 - v_3 - v_4 - v_2.
\end{aligned}$$

The orbits of π form a partition of $E(D)$ and, as such, each arc of D appears in exactly one orbit of π. Since D is the digraph obtained by replacing each edge of K_5 by a symmetric pair of arcs, each edge of K_5 appears twice among the orbits of π, once for each of the two possible directions that are assigned to the edge.

The 2-cell embedding of K_5 on S_1 shown in Figure 4.26 uniquely determines the collection $\{\pi_1, \pi_2, \ldots, \pi_5\}$ of permutations of the subscripts of the vertices adjacent to the vertices of K_5. This set of permutations, in turn, completely describes the embedding of K_5 on S_1 shown in Figure 4.26.

This method of describing an embedding is referred to as the *Rotational Embedding Scheme.* Such a scheme was observed and used by Dyck [D9] in 1888 and by Heffter [H11] in 1891. It was formalized by Edmonds [E1] in 1960 and discussed in more detail by Youngs [Y1] in 1963.

We now describe the Rotational Embedding Scheme in a more general setting. Let G be a nontrivial connected graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$. Let $N(v_i)$ denote the *neighborhood* of the vertex v_i; that is,

$$N(v_i) = \{v_j \in V(G) \mid v_i v_j \in E(G)\}$$

and let

$$V(i) = \{j \mid v_j \in N(v_i)\}.$$

For each $i (1 \leq i \leq p)$, let $\pi_i\colon V(i) \to V(i)$ be a cyclic permutation (or rotation) of $V(i)$. Thus, each permutation π_i can be represented by a (permutation) cycle of length $|V(i)| = |N(v_i)| = \deg v_i$. The Rotational Embedding Scheme states that there is a one-to-one correspondence between the 2-cell embeddings of G (on all possible surfaces) and the p-tuples $(\pi_1, \pi_2, \ldots, \pi_p)$ of cyclic permutations.

Theorem 4.30 (The Rotational Embedding Scheme) *Let G be a nontrivial connected graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$. For each 2-cell embedding of G on a surface, there exists a unique p-tuple $(\pi_1, \pi_2, \ldots, \pi_p)$, where for $i = 1, 2, \ldots, p$, $\pi_i\colon V(i) \to V(i)$ is a cyclic permutation that describes the subscripts of the* *vertices adjacent to v_i in counterclockwise order about v_i. Conversely, for each such p-tuple $(\pi_1, \pi_2, \ldots, \pi_p)$, there exists a 2-cell embedding of G on some surface such that for $i = 1, 2, \ldots, p$, the subscripts of the vertices adjacent to v_i and in counterclockwise order about v_i are given by π_i.*

Proof Let there be given a 2-cell embedding of G on some surface. For each vertex v_i of G, define $\pi_i\colon V(i) \to V(i)$ as follows: If $v_iv_j \in E(G)$ and v_iv_k (possibly $k = j$) is the next edge encountered after v_iv_j as we proceed counterclockwise about v_i, then we define $\pi_i(j) = k$. Each π_i so defined is a cyclic permutation.

Conversely, assume that we are given a p-tuple $(\pi_1, \pi_2, \ldots, \pi_p)$ such that for each $i (1 \leq i \leq p)$, $\pi_i\colon V(i) \to V(i)$ is a cyclic permutation. We show that this determines a 2-cell embedding of G on some surface. (By necessity, this proof requires use of properties of compact orientable 2-manifolds.)

Let D denote the digraph obtained from G by replacing each edge of G by a symmetric pair of arcs. We define a mapping $\pi\colon E(D) \to E(D)$ by

$$\pi((v_i, v_j)) = \pi(v_i, v_j) = (v_j, v_{\pi_j(i)}).$$

The mapping π is one-to-one and, thus, is a permutation of $E(D)$. Hence, π can be expressed as a product of disjoint permutation cycles. Each of these permutation cycles is called an *orbit* of π. Thus, the orbits partition the set $E(D)$. Assume that

$$R\colon ((v_i, v_j)(v_j, v_k) \cdots (v_\ell, v_i))$$

is an orbit of π, which we also write as

$$R\colon v_i - v_j - v_k - \cdots - v_\ell - v_i.$$

Hence, this implies that in the desired embedding if we begin at v_i and proceed along (v_i, v_j) to v_j, the next arc we must encounter after (v_i, v_j) in a counterclockwise direction about v_j is $(v_j, v_{\pi_j(i)}) = (v_j, v_k)$. Continuing in this manner, we must finally arrive at the arc (v_ℓ, v_i) and return to v_i, in the process describing the boundary of a (2-cell) region (considered as a subset of the plane) corresponding to the orbit R. Therefore, each orbit of π gives rise to a 2-cell region in the desired embedding.

To obtain the surface S on which G is 2-cell embedded, pairs of regions, with their boundaries, are "pasted" along certain arcs; in particular, if (v_i, v_j) is an arc on the boundary of R_1 and (v_j, v_i) is an arc on the boundary of R_2, then (v_i, v_j) is identified with (v_j, v_i) as shown in Figure 4.28. The properties of compact orientable 2-manifolds imply that S is indeed an appropriate surface.

In order to determine the genus of S, one needs only to observe that the number r of regions equals the number of orbits. Thus, if G has order p and size q, then by Corollary 4.20, $S = S_n$ where n is the nonnegative integer satisfying the equation $p - q + \mathrm{r} = 2 - 2n$. ■

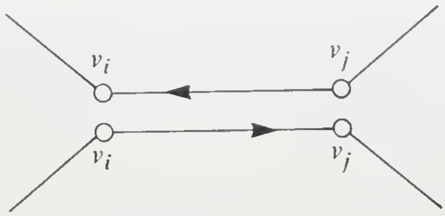


Figure 4.28 *A step in the proof of Theorem 4.30*

As an illustration of the Rotational Embedding Scheme, we once again consider the complete graph K_5, with $V(K_5) = \{v_1, v_2, v_3, v_4, v_5\}$. Let there be given the 5-tuple $(\pi_1, \pi_2, \pi_3, \pi_4, \pi_5)$, where

$$\pi_1 = (2\ 3\ 4\ 5),$$
$$\pi_2 = (1\ 3\ 4\ 5),$$
$$\pi_3 = (1\ 2\ 4\ 5),$$
$$\pi_4 = (1\ 2\ 3\ 5),$$
$$\pi_5 = (1\ 2\ 3\ 4).$$

Thus, by Theorem 4.30, this 5-tuple describes a 2-cell embedding of K_5 on some surface S_n. To evaluate n, we consider the digraph D obtained by replacing each edge of K_5 by a symmetric pair of arcs and determine the orbits of the permutation π: $E(D) \to E(D)$ defined in the proof of Theorem 4.30. The orbits are

$$R_1: v_1 - v_2 - v_3 - v_4 - v_5 - v_1,$$
$$R_2: v_1 - v_3 - v_2 - v_4 - v_3 - v_5 - v_4 - v_1 - v_5 - v_2 - v_1,$$
$$R_3: v_1 - v_4 - v_2 - v_5 - v_3 - v_1;$$

and each orbit corresponds to a 2-cell region. Thus, the number of regions in the embedding is $r = 3$. Since K_5 has order $p = 5$ and size $q = 10$, and since $p - q + r = -2 = 2 - 2n$, it follows that $n = 2$, so that the given 5-tuple describes an embedding of K_5 on S_2.

Given a p-tuple of cyclic permutations as we have described, it is not necessarily an easy problem to present a geometric description of the embedding, particularly on surfaces of high genus. For the example just presented, however, we give two geometric descriptions in Figure 4.29. In Figure 4.29(a), a portion of K_5 is drawn in the plane. Two handles are then inserted over the plane, as indicated, and the remainder of K_5 is drawn along these handles. The edge $e_1 = v_2v_5$ is drawn along the handle H_1, the edge $e_2 = v_3v_5$ is drawn along H_2 while $e_3 = v_1v_3$ is drawn along both H_1 and H_2. The three 2-cell regions produced are denoted by R_1, R_2, and R_3.

In Figure 4.29(b), this 2-cell embedding of K_5 on S_2 is shown on the regular octagon. The labeling of the eight sides (as in (4.7)) indicates the identification used in producing S_2.

As a more general illustration of Theorem 4.30 we determine the genus of the complete bipartite graph $K(2m, 2n)$. According to Theorem 4.27, $\gamma(K(2m, 2n)) = (m-1)(n-1)$. That $(m-1)(n-1)$ is a lower bound for $\gamma(K(2m, 2n))$ follows from Exercise 4.36. We use Theorem 4.30 to show $K(2m, 2n)$ is 2-cell embeddable on $S_{(m-1)(n-1)}$, thereby proving that $\gamma(K(2m, 2n)) \leq (m-1)(n-1)$ and completing the argument.

Denote the partite sets of $K(2m, 2n)$ by U and W, where $|U| = 2m$ and $|W| = 2n$. Further, label the vertices so that

$$U = \{v_1, v_3, v_5, \ldots, v_{4m-1}\} \quad \text{and} \quad W = \{v_2, v_4, v_6, \ldots, v_{4n}\}.$$

Let there be given the $(2m+2n)$-tuple (assuming that $m \leq n$)

$$(\pi_1, \pi_2, \ldots, \pi_{4m-1}, \pi_{4m}, \pi_{4m+2}, \pi_{4m+4}, \ldots, \pi_{4n}),$$

where

$$\begin{aligned}
\pi_1 = \pi_5 = \cdots = \pi_{4m-3} &= (2\ 4\ 6 \cdots 4n), \\
\pi_3 = \pi_7 = \cdots = \pi_{4m-1} &= (4n \cdots 6\ 4\ 2), \\
\pi_2 = \pi_6 = \cdots = \pi_{4n-2} &= (1\ 3\ 5 \cdots 4m-1), \\
\pi_4 = \pi_8 = \cdots = \pi_{4n} &= (4m-1 \cdots 5\ 3\ 1).
\end{aligned}$$

By Theorem 4.30, then, this $(2m+2n)$-tuple describes a 2-cell embedding of $K(2m, 2n)$ on some surface S_h. In order to evaluate h, we let D denote the digraph obtained by replacing each edge of $K(2m, 2n)$ by a symmetric pair of arcs and determine the orbits of the permutation $\pi\colon E(D) \to E(D)$ defined in the proof of Theorem 4.30.

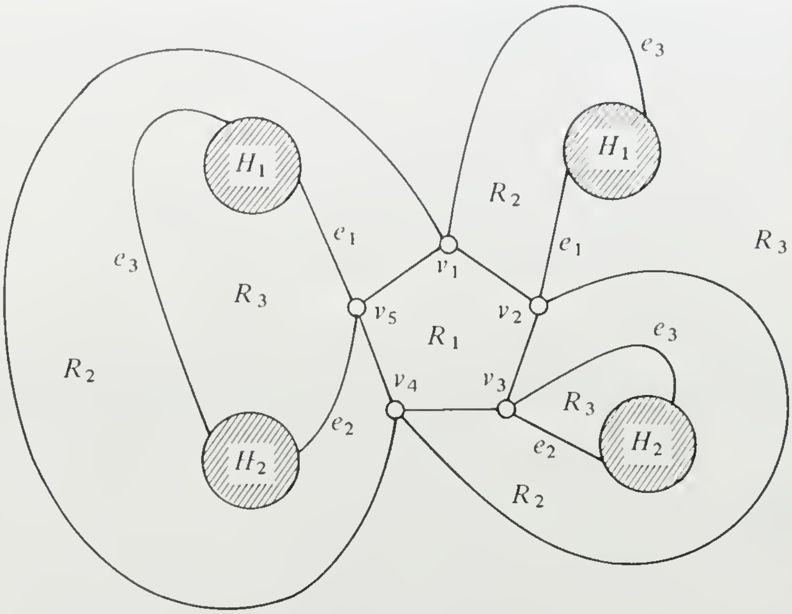
Every orbit of π contains an arc of the type (v_a, v_b), where $v_a \in U$ and $v_b \in W$. If $a \equiv 1 \pmod 4$ and $b \equiv 2 \pmod 4$, then the resulting orbit R containing (v_a, v_b) is

$$R\colon v_a - v_b - v_{a+2} - v_{b-2} - v_a,$$

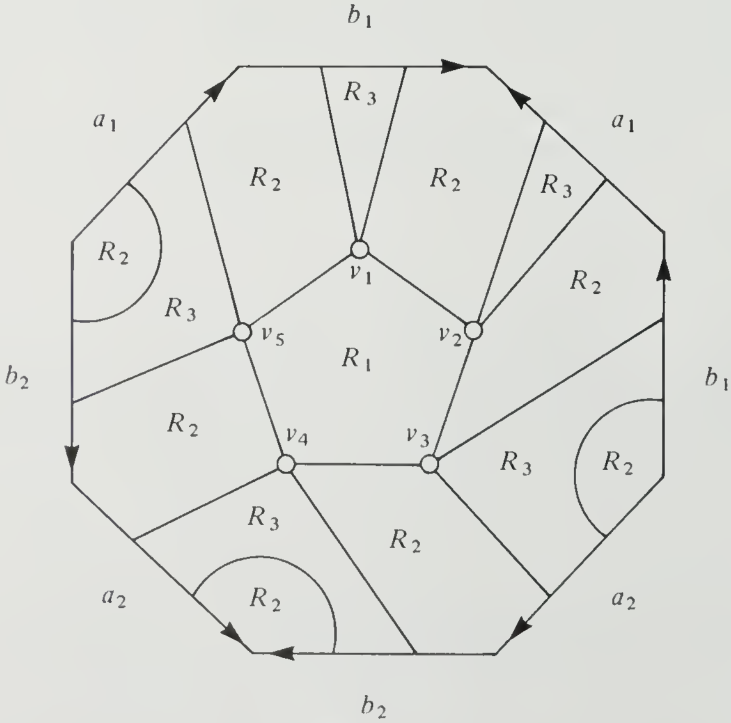
with $a+2$ expressed modulo $4m$ and $b-2$ expressed modulo $4n$. Note that R also contains the arc (v_{a+2}, v_{b-2}), where, then, $a+2 \equiv 3 \pmod 4$ and $b-2 \equiv 0 \pmod 4$. If $a \equiv 1 \pmod 4$ and $b \equiv 0 \pmod 4$, then the orbit R' containing (v_a, v_b) is

$$R'\colon v_a - v_b - v_{a-2} - v_{b-2} - v_a,$$

where, again, $a-2$ is expressed modulo $4m$ and $b-2$ expressed modulo $4n$. The orbit R' also contains the arc (v_{a-2}, v_{b-2}), where $a-2 \equiv 3 \pmod 4$ and $b-2 \equiv 2 \pmod 4$. Thus every orbit of π is either of the type R (where $a \equiv 1 \pmod 4$ and $b \equiv 2 \pmod 4$) or the type R' (where $a \equiv 1 \pmod 4$ and $b \equiv 0 \pmod 4$). Since there are m choices for a and n choices for b in each case, the total number of orbits is $2mn$; therefore, the number of regions in this embedding is $r = 2mn$.



(a)



(b)

Figure 4.29 *A 2-cell embedding of K_5 on the double torus*

Since $K(2m, 2n)$ has order $p = 2m + 2n$ and size $q = 4mn$ and because $p - q + r = 2 - 2h$, we have

$$(2m + 2n) - 4mn + 2mn = 2 - 2h$$

so that $h = (m-1)(n-1)$. Hence there is a 2-cell embedding of $K(2m, 2n)$ on $S_{(m-1)(n-1)}$, as we wished to show.

As a theoretical application of Theorem 4.30, we present a result that is referred to as the Ringeisen-White Edge-Adding Lemma (see [R3]).

Theorem 4.31 (Ringeisen-White Edge-Adding Lemma) *Let G be a connected graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$ such that v_i and v_j are distinct nonadjacent vertices. Suppose there exists a 2-cell embedding of G on some surface S_h with r regions such that v_i is on the boundary of region R_i and v_j is on the boundary of region R_j. Let $H = G + v_iv_j$. Then*

(a) if $R_i \neq R_j$, then there exists a 2-cell embedding of H on S_{h+1} with $r-1$ regions in which v_i and v_j are on the boundary of the same region, while

(b) if $R_i = R_j$, then there exists a 2-cell embedding of H on S_h with $r+1$ regions in which each of v_i and v_j belongs to the boundaries of (the same) two distinct regions.

Proof By hypothesis, there exists a 2-cell embedding of the (p, q) graph G on S_h with r regions such that v_i is on the boundary of region R_i and v_j is on the boundary of region R_j. By Theorem 4.30, a p-tuple $(\pi_1, \pi_2, \ldots, \pi_p)$ of cyclic permutations corresponds to this embedding, namely for $t = 1, 2, \ldots, p$, $\pi_t\colon V(t) \to V(t)$ is a cyclic permutation of the subscripts of the vertices of $N(v_t)$ in counterclockwise order about v_t.

Let D denote the symmetric digraph obtained from G by replacing each edge by a symmetric pair of arcs and let $\pi\colon E(D) \to E(D)$ be defined by $\pi(v_m, v_n) = (v_n, v_{\pi_n(m)})$. Since the given embedding has r regions, π has r orbits. Denote each region and its corresponding orbit by the same symbol; in particular, R_i and R_j are orbits of π.

Suppose that $R_i \neq R_j$. We can therefore represent orbits R_i and R_j as

$$R_i\colon v_i - v_k - \cdots - v_{k'} - v_i$$

and

$$R_j\colon v_j - v_\ell - \cdots - v_{\ell'} - v_j.$$

It therefore follows that

$$\pi_i(k') = k \quad \text{and} \quad \pi_j(\ell') = \ell.$$

We now consider the graph $H = G + v_i v_j$ and define

$$V'(t) = \{r \mid v_r v_t \in E(H)\}$$

for $t = 1, 2, \ldots, p$. Thus $V'(t) = V(t)$ for $t \neq i, j$, and $V'(i) = V(i) \cup \{j\}$ while $V'(j) = V(j) \cup \{i\}$. For the graph H, we define a p-tuple $(\pi'_1, \pi'_2, \ldots, \pi'_p)$ of cyclic permutations, where π'_t: $V'(t) \to V'(t)$ for $t = 1, 2, \ldots, p$ such that $\pi'_t = \pi_t$ for $t \neq i, j$. Furthermore,

$$\pi'_i(a) = \begin{cases} \pi_i(a) & \text{if} \quad a \neq k' \\ j & \text{if} \quad a = k' \\ k & \text{if} \quad a = j \end{cases}$$

and

$$\pi'_j(a) = \begin{cases} \pi_j(a) & \text{if} \quad a \neq \ell' \\ i & \text{if} \quad a = \ell' \\ \ell & \text{if} \quad a = i. \end{cases}$$

Let D' be the digraph obtained from H by replacing each edge of H by a symmetric pair of arcs. Define the permutation π': $E(D') \to E(D')$ by $\pi'(v_m, v_n) = (v_n, v_{\pi'_n(m)})$. The orbits of π' then consist of all orbits of π different from R_i and R_j together with the orbit

$$R\colon v_i - v_j - v_\ell - \cdots - v_{\ell'} - v_j - v_i - v_k - \cdots - v_{k'} - v_i.$$

Thus, π' has $r - 1$ orbits and the corresponding 2-cell embedding of H has $r - 1$ regions. Moreover, v_i and v_j lie on the boundary of R. Since $p - q + r = 2 - 2h$, it follows that $p - (q + 1) + (r - 1) = 2 - 2(h + 1)$ and H is 2-cell embedded on S_{h+1}. This completes the proof of (a).

Suppose that $R_i = R_j$. We can represent the orbit $R_i (= R_j)$ as

$$R_i\colon v_i - v_k - \cdots - v_{\ell'} - v_j - v_\ell - \cdots - v_{k'} - v_i.$$

(Note that v_i and v_j cannot be consecutive in R_i since $v_i v_j \notin E(G)$.) It follows that

$$\pi_i(k') = k \quad \text{and} \quad \pi_j(\ell') = \ell.$$

We again consider the graph $H = G + v_i v_j$ and once more define

$$V'(t) = \{r \mid v_r v_t \in E(H)\}$$

for $t = 1, 2, \ldots, p$. We define a p-tuple $(\pi'_1, \pi'_2, \ldots, \pi'_p)$ of cyclic permutations, where π'_t: $V'(t) \to V'(t)$ for $t = 1, 2, \ldots, p$ such that $\pi'_t = \pi_t$ for $t \neq i, j$. Also,

$$\pi_i'(a) = \begin{cases} \pi_i(a) & \text{if} \quad a \neq k' \\ j & \text{if} \quad a = k' \\ k & \text{if} \quad a = j \end{cases}$$

and

$$\pi_j'(a) = \begin{cases} \pi_j(a) & \text{if} \quad a \neq \ell' \\ i & \text{if} \quad a = \ell' \\ \ell & \text{if} \quad a = i. \end{cases}$$

Again we denote by D' the digraph obtained from H by replacing each edge of H by a symmetric pair of arcs and define the permutation π': $E(D') \to E(D')$ by $\pi'(v_m, v_n) = (v_n, v_{\pi_n'(m)})$. The orbits of π' consist of all orbits of π different from R_i together with the orbits

$$R'\colon v_i - v_j - v_\ell - \cdots - v_{k'} - v_i$$

and

$$R''\colon v_j - v_i - v_k - \cdots - v_{\ell'} - v_j.$$

Therefore, π' has $r+1$ orbits and the resulting 2-cell embedding of H has $r+1$ regions. Furthermore, each of v_i and v_j belongs to the boundaries of both R' and R''. Here $p - q + r = 2 - 2h$ implies that $p - (q+1) + (r+1) = 2 - 2h$, and H is 2-cell embedded on S_h, which verifies (b). ■

A consequence of Theorem 4.31 will prove to be useful.

Corollary 4.31 *Let e and f be adjacent edges of a connected graph G. If there exists a 2-cell embedding of $G' = G - e - f$ with one region, then there exists a 2-cell embedding of G with one region.*

Proof Let $e = uv$ and $f = vw$, where then $u \neq w$. Let there be given a 2-cell embedding of G' with one region R. Thus all vertices of G' belong to the boundary of R, including u and v. By Theorem 4.31(b), there exists a 2-cell embedding of $G' + e$ with two regions where u and v lie on the boundary of both regions. Therefore, v is on the boundary of one region and w is on the boundary of the other region in the 2-cell embedding of $G' + e$. Applying Theorem 4.31(a), we conclude that there exists a 2-cell embedding of $G' + e + f = G$ with one region. ■

We now turn our attention for the remainder of the section to the following question: Given a (connected) graph G, on which surfaces S_n do there exist 2-cell embeddings of G? As a major step towards answering this question, we present the following "interpolation theorem" of Duke [D8].

Theorem 4.32 (Duke) *If there exist 2-cell embeddings of a connected graph G on the surfaces S_m and S_n, where $m \leq n$, and k is any integer such that $m \leq k \leq n$, then there exists a 2-cell embedding of G on the surface S_k.*

Proof Observe that there exist 2-cell embeddings of K_1 only on the sphere; thus, we assume that G is nontrivial.

Assume that there exists a 2-cell embedding of G on some surface S_ℓ. Let $V(G) = \{v_1, v_2, \ldots, v_p\}$, $p \geq 2$. By Theorem 4.30, there exists a p-tuple $(\pi_1, \pi_2, \ldots, \pi_p)$ of cyclic permutations associated with this embedding such that for $i = 1, 2, \ldots, p$, $\pi_i: V(i) \to V(i)$ is a cyclic permutation of the subscripts of the vertices of $N(v_i)$ in counterclockwise order about v_i.

Let D be the symmetric digraph obtained from G by replacing each edge by a symmetric pair of arcs. Let π: $E(D) \to E(D)$ be the permutation defined by $\pi(v_i, v_j) = (v_j, v_{\pi_j(i)})$. Denote the number of orbits in π by r; that is, assume there are r 2-cell regions in the given embedding of G on S_ℓ.

Assume there exists some vertex of G, say v_1, such that $\deg v_1 \geq 3$. Then $\pi_1 = (a\ b\ c\ \cdots)$, where a, b, and c are distinct. Let v_x be any vertex adjacent with v_1 other than v_a and v_b, and suppose that $\pi_1(x) = y$. Thus

$$\pi_1 = (a\ b\ c\ \cdots\ x\ y\ \cdots)$$

where, possibly, $x = c$ or $y = a$. Let E_1 be the subset of $E(D)$ consisting of the three pairs of arcs

$$(v_a, v_1), (v_1, v_b); \qquad (v_b, v_1), (v_1, v_c); \qquad (v_x, v_1), (v_1, v_y). \tag{4.9}$$

Note that the six arcs listed in (4.9) are all distinct. By the definition of the permutation π, we have

$$\pi(v_a, v_1) = (v_1, v_b), \qquad \pi(v_b, v_1) = (v_1, v_c), \qquad \text{and} \qquad \pi(v_x, v_1) = (v_1, v_y).$$

This implies that the arc (v_a, v_1) is followed by the arc (v_1, v_b) in some orbit of π, and that the edge $v_a v_1$ of G is followed by the edge $v_1 v_b$ as we proceed clockwise around the boundary of the corresponding region in the given embedding of G in S_ℓ. Also, (v_b, v_1) is followed by (v_1, v_c) in some orbit of π and (v_x, v_1) is followed by (v_1, v_y) in some orbit.

We now define a new permutation π': $E(D) \to E(D)$ with the aid of the p-tuple $(\pi'_1, \pi'_2, \ldots, \pi'_p)$, where for $i = 1, 2, \ldots, p$, π'_i: $V(i) \to V(i)$ is a cyclic permutation defined by

$$\pi'_i = \begin{cases} (a\ c\ \cdots\ x\ b\ y\ \cdots) & \text{if} \quad i = 1 \\ \pi_i & \text{if} \quad 2 \leq i \leq p. \end{cases}$$

We then define $\pi'(v_i, v_j) = (v_j, v_{\pi'_j(i)})$. By Theorem 4.30, the p-tuple $(\pi'_1, \pi'_2,$

$\ldots, \pi'_p)$ determines a 2-cell embedding of G on some surface, where for $i = 1, 2, \ldots, p$, π'_i is a cyclic permutation of the subscripts of the vertices adjacent to v_i in counterclockwise order about v_i.

Three cases are now considered, depending on the possible distribution of the pairs (4.9) of arcs in E_1 among the orbits of π.

Case 1: Assume that all arcs of E_1 belong to a single orbit R of π. Suppose, first, that the orbit R has the form

$$R\colon v_1 - v_y - \cdots - v_b - v_1 - v_c - \cdots - v_a - v_1 - v_b - \cdots - v_x - v_1.$$

Here the orbits of π' are the orbits of π except that the orbit R is replaced by the three orbits

$$\begin{aligned} R'_1&\colon v_1 - v_y - \cdots - v_b - v_1, \\ R'_2&\colon v_1 - v_c - \cdots - v_a - v_1, \\ R'_3&\colon v_1 - v_b - \cdots - v_x - v_1. \end{aligned}$$

Hence, π' describes a 2-cell embedding of G with $r + 2$ regions on a surface S'. Necessarily, then, $S' = S_{\ell-1}$.

The other possible form that the oribit R may take is

$$R\colon v_1 - v_y - \cdots - v_a - v_1 - v_b - \cdots - v_b - v_1 - v_c - \cdots - v_x - v_1.$$

In this situation, the orbits of π' are the orbits of π, except for R, which is replaced by the orbit

$$R'\colon v_1 - v_y - \cdots - v_a - v_1 - v_c - \cdots - v_x - v_1 - v_b - \cdots - v_b - v_1.$$

Here π' has r orbits.

Case 2: Assume that π has two orbits, say R_1 and R_2, with R_1 containing two of the pairs of arcs in E_1 and R_2 containing the remaining pair of arcs. In this case, the orbits of π' are those of π, except for R_1 and R_2, which are replaced by two orbits R'_1 and R'_2, where one of R'_1 and R'_2 contains two arcs of E_1 and the other contains the remaining four arcs of E_1. In this case, π' has r orbits.

Case 3: Assume that π has three orbits R_1, R_2, and R_3 such that (v_a, v_1) is followed by (v_1, v_b) in R_1, *(v_b, v_1) is followed by (v_1, v_c) in R_2, and (v_x, v_1) is followed by (v_1, v_y) in R_3.* In this case, the orbits of π' are the orbits of π, except for R_1, R_2, and R_3, which are replaced by a single orbit R' of the form

$$R'\colon v_1 - v_y - \cdots - v_x - v_1 - v_b - \cdots - v_a - v_1 - v_c - \cdots - v_b - v_1.$$

In this case, π' has $r - 2$ orbits so that π' describes a 2-cell embedding of G on $S_{\ell+1}$.

Thus, we can now conclude that the shifting of a single term in π_1 (producing π'_1) changes the genus of the resulting surface on which G is 2-cell embedded by at most one. Having made this observation, we can now complete the proof.

Let $(\mu_1, \mu_2, \ldots, \mu_p)$ be the p-tuple of cyclic permutations associated with a 2-cell embedding of G on S_m and let $(\nu_1, \nu_2, \ldots, \nu_p)$ be the p-tuple of cyclic permutations associated with a 2-cell embedding of G on S_n. If $\deg v_i = 1$ or 2 for each i, $1 \leq i \leq p$, then $\mu_i = \nu_i$ so that $m = n$ and the desired result follows. Hence, we may assume that for some i, $1 \leq i \leq p$, $\deg v_i \geq 3$. For each such i, μ_i can be transformed into ν_i by a finite number of single term shifts, as described above. Each such single term shift describes an embedding of G on a surface whose genus differs by at most one from the genus of the surface on which G is embedded prior to the shift. Therefore, by performing sequences of single term shifts on those μ_i for which $\deg v_i \geq 3$, the p-tuple $(\mu_1, \mu_2, \ldots, \mu_p)$ can be transformed into $(\nu_1, \nu_2, \ldots, \nu_p)$. Since $m \leq k \leq n$, there must be at least one term $(\pi_1, \pi_2, \ldots, \pi_p)$ in the aforementioned sequence beginning with $(\mu_1, \mu_2, \ldots, \mu_p)$ and ending with $(\nu_1, \nu_2, \ldots, \nu_p)$ that describes a 2-cell embedding of G on S_k. ■

Exercises 4.5

4.39 (a) For the 2-cell embedding of $K(3, 3)$ shown in Figure 4.20(a), determine the 6-tuple of cyclic permutations π_i associated with this embedding. Determine the orbits of the resulting permutation π.

(b) For the 2-cell embedding of K_7 on S_1 shown in Figure 4.21, determine the 7-tuple of cyclic permutations π_i associated with this embedding. Determine the orbits of the resulting permutation π.

4.40 Let $G \cong K_4 \times K_2$.

(a) Show that G is nonplanar.

(b) Show, in fact, that $\gamma(G) = 1$ by finding an 8-tuple of cyclic permutations that describes a 2-cell embedding of G on S_1. Determine the orbits of the resulting permutation π.

4.41 Let G be a graph with $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and let $(\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6)$ describe a 2-cell embedding of G on the surface S_k, where

$$\pi_1 = (2\ 5\ 6\ 3), \quad \pi_2 = (3\ 6\ 1\ 4), \quad \pi_3 = (4\ 1\ 2\ 5),$$
$$\pi_4 = (5\ 2\ 3\ 6), \quad \pi_5 = (6\ 3\ 4\ 1), \quad \pi_6 = (1\ 4\ 5\ 2).$$

(a) What familiar graph is isomorphic to G?

(b) What is k?

(c) Is $k = \gamma(G)$?

4.42 How many of the 2-cell embeddings of K_4 are embeddings in the plane? On the torus? On the double torus?

4.43 (a) Describe an embedding of $K(3, 3)$ on S_2 by means of a 6-tuple of cyclic permutations.

(b) Show that there exists no 2-cell embedding of $K(3, 3)$ on S_3.

4.6 The Maximum Genus of a Graph

If G is a connected graph with $\gamma(G) = m$, and n is the largest positive integer such that G is 2-cell embeddable on S_n, then it follows from Theorem 4.32 that G can be 2-cell embedded on S_k if and only if $m \leq k \leq n$. This suggests the following concept.

Let G be a connected graph. The *maximum genus* $\gamma_M(G)$ of G is the maximum among the genera of all surfaces on which G can be 2-cell embedded. At the outset, it may not even be clear that every graph has a *maximum* genus since, perhaps, some graphs may be 2-cell embeddable on infinitely many surfaces. However, there are no graphs that can be 2-cell embedded on infinitely many surfaces, for suppose G is a nontrivial connected graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$. By Theorem 4.30, there exists a one-to-one correspondence between the set of all 2-cell embeddings of G and the p-tuples $(\pi_1, \pi_2, \ldots, \pi_p)$, where for $i = 1, 2, \ldots, p$, $\pi_i\colon V(i) \to V(i)$ is a cyclic permutation. Since the number of such p-tuples is finite, in fact, is equal to

$$\prod_{i=1}^{p} (\deg v_i - 1)!,$$

it follows that there are only finitely many 2-cell embeddings of G and therefore that there exists a surface of maximum genus on which G can be 2-cell embedded. We can now state an immediate consequence of Duke's Theorem.

Corollary 4.32 *A connected graph G has a 2-cell embedding on the surface S_k if and only if*

$$\gamma(G) \leq k \leq \gamma_M(G).$$

We now present an upper bound for the maximum genus of any connected graph. This bound employs a new but very useful concept.

The *Betti number* $\mathcal{B}(G)$ of a (p, q) graph G having k components is defined as

$$\mathcal{B}(G) = q - p + k.$$

Thus, if G is connected, then

$$\mathcal{B}(G) = q - p + 1.$$

The following result is due to Nordhaus, Stewart and White [NSW1].

Theorem 4.33 *If G is a connected graph, then*

$$\gamma_M(G) \leqslant \left\lfloor \frac{\mathcal{B}(G)}{2} \right\rfloor.$$

Furthermore, equality holds if and only if there exists a 2-cell embedding of G on the surface of genus $\gamma_M(G)$ with exactly one or two regions according to whether $\mathcal{B}(G)$ is even or odd, respectively.

Proof Let G be a connected (p, q) graph that is 2-cell embedded on the surface of genus $\gamma_M(G)$, producing r (2-cell) regions. By Theorem 4.20,

$$p - q + r = 2 - 2\gamma_M(G).$$

Thus,

$$\mathcal{B}(G) = q - p + 1 = 2\gamma_M(G) + r - 1$$

so that

$$\gamma_M(G) = \frac{\mathcal{B}(G) + 1 - r}{2} \leqslant \frac{\mathcal{B}(G)}{2},$$

producing the desired bound.

Moreover, we have

$$\gamma_M(G) = \frac{\mathcal{B}(G) + 1 - r}{2} = \left\lfloor \frac{\mathcal{B}(G)}{2} \right\rfloor$$

if and only if $r = 1$ (which can only occur when $\mathcal{B}(G)$ is even) or $r = 2$ (which is only possible when $\mathcal{B}(G)$ is odd). ■

A (connected) graph G is called *upper embeddable* if the maximum genus of G attains the upper bound given in Theorem 4.33; that is, if $\gamma_M(G) = \lfloor \mathcal{B}(G)/2 \rfloor$. The graph G is said to be *upper embeddable on a surface S* if $S = S_{\gamma_M(G)}$. We can now state an immediate consequence of Theorem 4.33.

Corollary 4.33 *Let G be a graph with even (odd) Betti number. Then G is upper embeddable on a surface S if and only if there exists a 2-cell embedding of G on S with one (two) region(s).*

A characterization of upper embeddable graphs was independently discovered by Jungerman [J2] and Xuong [X1]. In order to present this result it will be necessary to introduce a new concept.

A spanning tree T of a connected graph G is a *splitting tree* of G if at most one component of $G - E(T)$ has odd size. It follows therefore that if $G - E(T)$ is connected, then T is a splitting tree. For the graph G of Figure 4.30, the tree

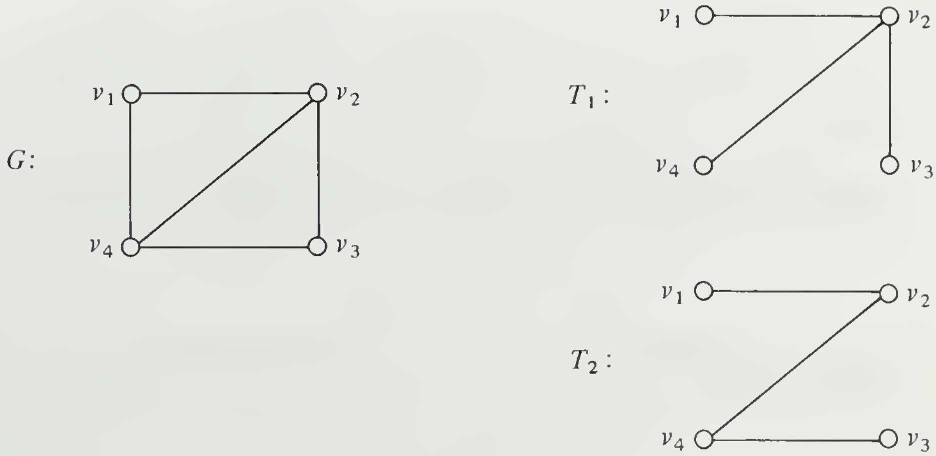


Figure 4.30 *Splitting trees of graphs*

T_1 is a splitting tree. On the other hand, T_2 is not a splitting tree of G.

The following observation that relates splitting trees and Betti numbers is elementary, but useful.

Theorem 4.34 *Let T be a splitting tree of a (p, q) graph G. Then every component of $G - E(T)$ has even size if and only if $\mathcal{B}(G)$ is even.*

Proof Suppose that every component of $G - E(T)$ has even size. Then $G - E(T)$ has even size. Since every tree of order p has size $p - 1$, the graph $G - E(T)$ has size $q - (p - 1) = q - p + 1$. Therefore, $\mathcal{B}(G) = q - p + 1$ is even.

Conversely, suppose that $\mathcal{B}(G)$ is even. The graph $G - E(T)$ has size $q - p + 1 = \mathcal{B}(G)$. Since T is a splitting tree of G, at most one component of $G - E(T)$ has odd size. Since the sum of the sizes of the components of $G - E(T)$ is even, it is impossible for exactly one such component to have odd size, producing the desired result. ■

We now give a characterization of upper embeddable graphs.

Theorem 4.35 (Jungerman-Xuong) *A graph G is upper embeddable if and only if G has a splitting tree.*

Proof Suppose that G is a (p, q) graph having a splitting tree T. We show that G is upper embeddable by considering two cases, depending on the parity of the Betti number of G.

Case 1: Assume that $\mathcal{B}(G) = q - p + 1$ is even. By Theorem 4.34, each component of $G - E(T)$ has even size. By Theorem 2.21, every component C of $G - E(T)$ is trivial or eulerian, or the edge set of C can be partitioned into subsets each of which induces an open trail of even length. Thus, the edge set of each nontrivial component of $G - E(T)$ can be partitioned into adjacent pairs of edges, implying that the graph of $G - E(T)$ itself can be expressed as the edge sum of $(q - p+1)/2 = \mathcal{B}(G)/2$ subgraphs of F_i, $1 \leq i \leq \mathcal{B}(G)/2$, each isomorphic to $P_3 \cup K_{p-3}$.

The tree T is 2-cell embeddable on the sphere with a single region. By repeated application of Corollary 4.31, it follows that the graph

$$T + \bigcup_{i=1}^{k} E(F_i)$$

is 2-cell embeddable for $k = 1, 2, \ldots, \mathcal{B}(G)/2$ on some surface with a single region. Since

$$G \cong T + \bigcup_{i=1}^{\mathcal{B}(G)/2} E(F_i),$$

we conclude that G is 2-cell embeddable on a surface with one region, which, by Corollary 4.33, implies that G is upper embeddable.

Case 2: Assume that $\mathcal{B}(G)$ is odd. Necessarily, $G - E(T)$ has exactly one component of odd size; denote this component by H. If H is a tree, let e be a terminal edge of H (an edge incident with an end-vertex); otherwise, let e be a cycle edge of H. Since any bridge of G must belong to T, it follows that $G - e$ is connected. Further, since $H - e$ has only components of even size, every component of $G - E(T) - e = G - e - E(T)$ has even size and T is a splitting tree of $G - e$. Since $\mathcal{B}(G - e)$ is even, we conclude from Case 1 that $G - e$ is upper embeddable and, so, is 2-cell embeddable on a surface with one region. Applying Theorem 4.31(b), we see that there is a 2-cell embedding of $(G - e) + e = G$ on a surface with two regions which, by Corollary 4.33, implies that G is upper embeddable.

For the converse, we suppose that G is a (p, q) upper embeddable graph (and consequently a connected graph) with $V(G) = \{v_1, v_2, \ldots, v_p\}$. To show that G has a splitting tree, we again consider two cases according to the parity of the Betti number of G.

Case 1: Assume that $\mathcal{B}(G) = q - p + 1$ is even. We proceed by induction on the size of G. If $q = 0$, then $G \cong K_1$ and G is a splitting tree of itself. For the inductive hypothesis we assume that all upper embeddable graphs with even Betti numbers having sizes less than q contain splitting trees.

Since G is upper embeddable, it follows from Corollary 4.33 that G is 2-cell embeddable on the surface of genus $\gamma_M(G)$ with one region, say R. By Theorem 4.30, corresponding to this 2-cell embedding, there exists a p-tuple $(\pi_1, \pi_2, \ldots, \pi_p)$, where for $i=1, 2, \ldots, p$, $\pi_i\colon V(i) \to V(i)$ is a cyclic permutation that describes the subscripts of the vertices adjacent to v_i in counterclockwise order about v_i. Denote by D the digraph obtained from G by replacing each edge of G by a symmetric pair of arcs. Since the embedding has one region, the mapping $\pi\colon E(D) \to E(D)$ defined by

$$\pi(v_i, v_j) = (v_j, v_{\pi_j(i)})$$

has one orbit, also denoted by R. Thus in R all arcs of D appear in a fixed (cyclic) order. Among the edges of G, let $e = v_\ell v_m$ be one with the property that in the orbit R, the resulting arcs $a_1 = (v_\ell, v_m)$ and $a_2 = (v_m, v_\ell)$ have the minimum number k of arcs between them. Without loss of generality, we assume that in R the number of arcs following a_1 but preceding a_2 is k. Thus, the number of arcs following a_2 and preceding a_1 is $2q-2-k$, where then $k \leq 2q-2-k$.

If $k=0$, than a_1 and a_2 appear consecutively in the orbit R, in the order a_1, a_2. This further implies that $\deg_G v_m = 1$. The graph $G - v_m$ is clearly 2-cell embeddable on the same surface on which G is 2-cell embedded and also with a single region. Thus, $G - v_m$ is upper embeddable. Since $G - v_m$ has size $q-1$ and since $\mathcal{B}(G - v_m)$ is even, we may apply the inductive hypothesis and conclude that $G - v_m$ has a splitting tree T_1. Thus, the tree T produced by adding the vertex v_m and the edge e to T_1 is a splitting tree of G since the components of $G - E(T)$ are the components of $G - v_m - E(T_1)$ together with an additional trivial component, which, of course, has even size.

Assume, then, that $k > 0$. Thus, by assumption, in the orbit R there are k arcs following a_1 but preceding a_2. Suppose that b_1 is the arc following a_1 in R, so that $b_1 = (v_m, v_n)$, where $n \neq \ell$. Further, let $f = v_m v_n$ denote the corresponding edge of G and let $b_2 = (v_n, v_m)$ denote the other related arc of D. Necessarily the orbit R is of the form

$$R\colon (a_1\ b_1\ \cdots\ a_2\ \cdots\ b_2\ \cdots);$$

that is, b_2 must follow a_2 and precede a_1. Otherwise, if b_2 follows b_1 but precedes a_2, then the number of arcs in R between b_1 and b_2 is less than k, contradicting the defining property of the edge e. The single orbit R of π can also be described as follows:

$$R\colon v_\ell - v_m - v_n - v_r - \cdots - v_s - v_m - v_\ell - v_t - \cdots - v_x - v_n - v_m - v_y - \cdots - v_z - v_\ell.$$

This implies that $G - e - f$ contains the closed spanning walk

$$v_\ell,\ v_t,\ \ldots,\ v_x,\ v_n,\ v_r,\ \ldots,\ v_s,\ v_m,\ v_y,\ \ldots,\ v_z,\ v_\ell;$$

thus, $G-e-f$ is connected. The description of the orbit R also implies, for example, that

$$\pi_\ell(m)=t, \qquad \pi_m(n)=y, \qquad \pi_n(m)=r.$$

We next show that $G-e-f$ is 2-cell embeddable on a surface with a single region. Define

$$V'(i)=\{j \mid v_iv_j \in E(G-e-f)\}$$

for i = 1, 2, ..., p. Therefore, $V(i)=V'(i)$ for $i \neq \ell$, m, n, while $V(\ell)=V'(\ell)\cup\{m\}$, $V(m)=V'(m)\cup\{\ell, n\}$ and $V(n)=V'(n)\cup\{m\}$. For the graph $G-e-f$, we define a p-tuple $(\pi_1', \pi_2', \ldots, \pi_p')$ of cyclic permutations, where $\pi_i'\colon V'(i)\to V'(i)$ for $i=1, 2, \ldots, p$ such that $\pi_i'=\pi_i$ for $i \neq \ell$, m, n. Moreover, $\pi_\ell'(j)=\pi_\ell(j)$ for $j \neq z$ while $\pi_\ell'(z)=\pi_\ell(m)=t$; also $\pi_m'(j)=\pi_m(j)$ for $j \neq s$ and $\pi_m'(s)=\pi_m(n)=y$; finally, $\pi_n'(j)=\pi_n(j)$ for $j \neq x$ while $\pi_n'(x)=\pi_n(m)=r$.

Let D' be the digraph obtained from $G-e-f$ by replacing each edge of $G-e-f$ by a symmetric pair of arcs. Define the permutation $\pi'\colon E(D')\to E(D')$ by $\pi'(v_i, v_j)=(v_j, v_{\pi_j'(i)})$. There is a single orbit of π', namely

$$R'\colon v_\ell - v_t - \cdots - v_x - v_n - v_r - \cdots - v_s - v_m - v_y - \cdots - v_z - v_\ell.$$

Hence, corresponding to the p-tuple $(\pi_1', \pi_2', \ldots, \pi_p')$ is a 2-cell embedding of $G-e-f$ with one region. Therefore, $G-e-f$ is upper embeddable. Since $\mathcal{B}(G-e-f)$ is even, $G-e-f$ has a splitting tree T by the inductive hypothesis.

By Theorem 4.34, every component of $G-e-f-E(T)$ has even size. Returning the two adjacent edges e and f to $G-e-f-E(T)$ produces either the same number of components (if e and f are incident only to vertices in the same component of $G-e-f-E(T)$) or fewer components (if e and f are incident with vertices in two or three components of $G-e-f-E(T)$). In either case, every component of $G-E(T)$ has even size and T is a splitting tree of G.

Case 2: Assume that $\mathcal{B}(G)$ is odd. Since G is upper embeddable, it follows by Theorem 4.33 that there exists a 2-cell embedding of G with two regions. Let e be an edge of G that is on the boundary of both regions. (Necessarily, e is not a bridge of G.) Deleting e produces a 2-cell embedding of $G-e$ with one region. Therefore, $G-e$ is upper embeddable and since $\mathcal{B}(G-e)$ is even, it follows by the preceding case that $G-e$ has a splitting tree T. Furthermore, by Theorem 4.34 every component of $G-e-E(T)$ has even size. Returning the edge e to $G-e-E(T)$ produces a graph, namely $G-E(T)$, having exactly one component of odd size. Therefore, T is also a splitting tree of G. ■

Returning to the graph G of Figure 4.30, we now see that G is upper embeddable since G contains T_1 as a splitting tree. On the other hand, neither the graph G_1 nor the graph G_2 of Figure 4.31 has a single splitting tree, so, by Theorem 4.35, neither of these graphs is upper embeddable.

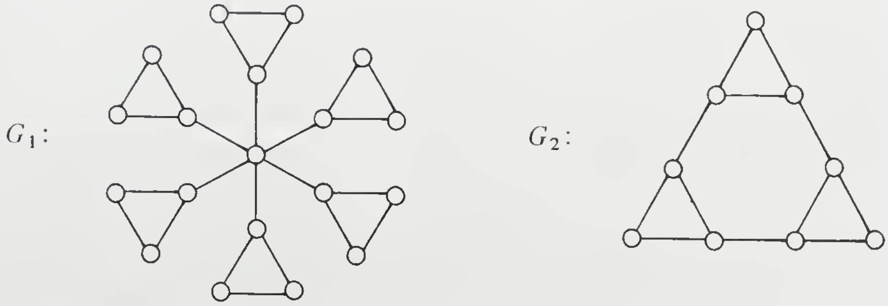


Figure 4.31 *Graphs that are not upper embeddable*

We mentioned earlier that no formula is known for the genus of an arbitrary graph. However such is not the case with maximum genus. With the aid of Theorem 4.35, Xuong [X1] developed a formula for the maximum genus of any connected graph.

For a graph H we denote by $\xi_0(H)$ the number of components of H of odd size. For a connected graph G, we define the number $\xi(G)$ as follows:

$$\xi(G) = \min \xi_0(G - E(T)),$$

where the minimum is taken over all spanning trees T of G.

Theorem 4.36 (Xuong) *The maximum genus of a connected graph G is given by*

$$\gamma_M(G) = \frac{1}{2}(\mathcal{B}(G) - \xi(G)).$$

Proof Assume that G has order p and size q. Let G be 2-cell embedded on the surface of genus $\gamma_M(G)$ such that r regions are produced. First we show that $r = 1 + \xi(G)$. Note that if $\xi(G) = 0$, then G contains a splitting tree and $\mathcal{B}(G)$ is even. Thus, G is upper embeddable on $S_{\gamma_M(G)}$ with one region; that is, $r = 1 + \xi(G)$ if $\xi(G) = 0$. Similarly, if $r = 1$, then $\mathcal{B}(G)$ is even and G is upper embeddable. Thus G has a splitting tree and $\xi(G) = 0$, so that $r = 1 + \xi(G)$ if $r = 1$. Therefore, we assume that $\xi(G) > 0$ and $r \geq 2$.

Let T_1 be a spanning tree of G such that

$$\xi_0(G - E(T_1)) = \xi(G).$$

Let G_i, $i = 1, 2, \ldots, \xi(G)$, be the components of odd size in $G - E(T_1)$. For $i = 1, 2, \ldots, \xi(G)$, let e_i be a terminal edge of G_i if G_i is a tree and let e_i be a cycle edge of G_i if G_i is not a tree. Define $H = G - \{e_1, e_2, \ldots, e_{\xi(G)}\}$. Since T_1 is a spanning tree of H, the graph H is connected. Since every component of $H - E(T_1)$ has even size, T_1 is a splitting tree of H. Therefore, by Theorem 4.35, H is upper embeddable. Also, by Theorem 4.34, $\mathcal{B}(H)$ is even. Hence, by Corollary 4.33, H can be 2-cell embedded on $S_{\gamma_M(H)}$ with one region. Adding the edges $e_1, e_2, \ldots, e_{\xi(G)}$ to H produces the graph G. By the Ringeisen-White edge-adding lemma (Theorem 4.31), there exists a 2-cell embedding of $H + e_1$ on some surface (namely on $S_{\gamma_M(H)}$ in this case) with two regions. By $\xi(G)$ applications of Theorem 4.31, it follows that there exists a 2-cell embedding of $G = H + e_1 + e_2 + \cdots + e_{\xi(G)}$ on some surface S with at most $1 + \xi(G)$ regions. Therefore, if G is 2-cell embedded on S with s regions, then necessarily $s \leq 1 + \xi(G)$. Since the minimum number of regions of any 2-cell embedding of G occurs when G is 2-cell embedded on $S_{\gamma_M(G)}$ and since such an embedding produces r regions, by assumption, we conclude that $r \leq s$ so that $r \leq 1 + \xi(G)$.

To verify that $r \geq 1 + \xi(G)$, we again assume that G is 2-cell embedded on the surface of genus $\gamma_M(G)$ with $r(\geq 2)$ regions. Let f_1 be an edge belonging to the boundary of two regions of G. (Necessarily, f_1 is not a bridge of G.) Then $G - f_1$ is 2-cell embeddable on the surface of genus $\gamma_M(G)$ with $r - 1$ regions. Furthermore, if $r > 2$, then for $k = 2, 3, \ldots, r - 1$, let f_k be an edge belonging to the boundary of two regions of $G - \{f_1, f_2, \ldots, f_{k-1}\}$. Then for $k = 1, 2, \ldots, r - 1$, the graph $G - \{f_1, f_2, \ldots, f_k\}$ is 2-cell embeddable on the surface of genus $\gamma_M(G)$ with $r - k$ regions; in particular, the graph $G' = G - \{f_1, f_2, \ldots, f_{r-1}\}$ is 2-cell embeddable on the surface of genus $\gamma_M(G)$ with one region. Therefore, $\mathcal{B}(G')$ is even and, by Corollary 4.33, the graph G' is upper embeddable on the surface of genus $\gamma_M(G)$. By Theorem 4.35, G' contains a splitting tree T', and all components of $G' - E(T')$ have even size. Thus, $\xi_0(G - E(T')) \leq r - 1$. Consequently, $\xi(G) \leq \xi_0(G - E(T')) \leq r - 1$ so that $r \geq 1 + \xi(G)$. Therefore, $r = 1 + \xi(G)$.

By Theorem 4.20,

$$p - q + r = 2 - 2\gamma_M(G).$$

Since $r = 1 + \xi(G)$, it follows that

$$2\gamma_M(G) = q - p + 1 - \xi(G)$$

or

$$2\gamma_M(G) = \frac{1}{2}(\mathcal{B}(G) - \xi(G)). \quad \blacksquare$$

Returning to the graph G_1 of Figure 4.31, we see that $\mathcal{B}(G_1) = 6$ and that $\xi_0(G_1 - E(T)) = 6$ for every spanning tree T. Therefore, $\xi(G_1) = 6$ so that

$$\gamma_M(G_1) = \frac{1}{2}(\mathcal{B}(G_1) - \xi(G_1)) = 0$$

and G_1 is 2-cell embeddable only on the sphere.

With the aid of Theorem 4.36 (or Theorem 4.35), it is possible to show that a wide variety of graphs are upper embeddable. The following result is due to Kronk, Ringeisen and White [KRW1].

Corollary 4.36a *Every complete n-partite graph, $n \geq 2$, is upper embeddable.*

From Corollary 4.36a, it of course follows at once that every complete graph is upper embeddable, a result due to Nordhaus, Stewart and White [NSW1]. We present a proof using Theorem 4.35.

Corollary 4.36b *The maximum genus of K_p is given by*

$$\gamma_M(K_p) = \left\lfloor \frac{(p-1)(p-2)}{4} \right\rfloor .$$

Proof If T is a spanning path of K_p, then $K_p - E(T)$ contains at most one nontrivial component. Therefore, T is a splitting tree of K_p and, by Theorem 4.35, K_p is upper embeddable. Since $\mathcal{B}(K_p) = (p-1)(p-2)/2$, the result follows. ■

A formula for the maximum genus of complete bipartite graphs was discovered by Ringeisen [R3].

Corollary 4.36c *The maximum genus of $K(m, n)$ is given by*

$$\gamma_M(K(m, n)) = \left\lfloor \frac{(m-1)(n-1)}{2} \right\rfloor .$$

Zaks [Z1] discovered a formula for the maximum genus of the n-cube.

Corollary 4.36d *The maximum genus of Q_n, $n \geq 2$, is given by*

$$\gamma_M(Q_n) = (n-2)2^{n-2}.$$

We close this section with a few remarks about other schemes that have been employed for providing 2-cell embeddings of connected graphs as well as a few comments on embedding graphs on other "surfaces".

The reader who wishes to pursue the topic of graph embeddings in more depth would do well to familiarize himself with the dual topics of "current graphs" and "voltage graphs". The theory of current graphs (see Gross and Alpert [GA1], [GA2], [GA3], Gustin [G6], and Youngs [Y2, Chap. 12]) was originally developed with the express purpose of generating rotational embedding schemes for embedding complete graphs. It was later extended to 2-cell embeddings of arbitrary graphs. Gross [G5] dualized the ideas involved to obtain the theory of voltage graphs, which has proved to be a powerful tool for graph embeddings.

Finally, we also note that it is possible to speak of embedding graphs on nonorientable surfaces such as the Möbius strip, projective plane, and Klein bottle. As might be expected, every planar graph (as well as some nonplanar graphs) can be embedded on such surfaces. Figure 4.32 shows K_5 embedded on the Möbius strip. These topics shall not be the subject of further discussion, however.

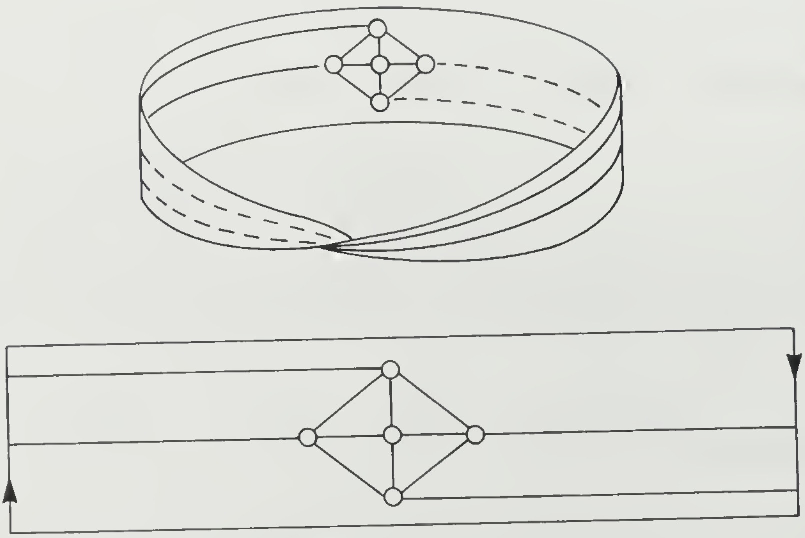


Figure 4.32 *An embedding of K_5 on the Möbius strip*

Exercises 4.6

4.44 Describe an embedding of K_5 on $S_{\gamma_M(K_5)}$ by means of a 5-tuple of cyclic permutations.

4.45 Determine the maximum genus of the graph G_2 of Figure 4.31.

4.46 Determine the maximum genus of the Petersen graph.

4.47 (a) Let G be a connected graph with blocks $B_1, B_2, \ldots, B_n$. Prove that

$$\gamma_M(G) \geq \sum_{i=1}^{n} \gamma_M(B_i).$$

(b) Show that the inequality in (a) may be strict.

4.48 Prove Theorem 4.35 as a corollary to Theorem 4.36.

4.49 Prove Corollary 4.36a.

4.50 Prove Corollary 4.36c.

4.51 Prove Corollary 4.36d.

4.52 Prove or disprove: For every positive integer n, there exists a connected graph G_n such that $\lfloor \mathcal{B}(G_n)/2 \rfloor - \gamma_M(G_n) = n$.

4.53 Prove or disprove: If H is a connected spanning subgraph of an upper embeddable graph G, then H is upper embeddable.

4.54 Determine the maximum genus of the Heawood graph.

4.55 For $G \cong C_m \times C_n (m, n \geq 3)$, determine $\gamma(G)$ and $\gamma_M(G)$.

4.56 Prove that if each vertex of a connected graph G lies on at most one cycle, then G is only 2-cell embeddable on the sphere.

4.57 Prove, for positive integers m and n with $m \leq n$, that there exists a graph G of genus m that can be 2-cell embedded on S_n.

4.58 Prove that if G is upper embeddable, then $G \times K_2$ is upper embeddable.

Chapter Five

Connectivity and Networks

The concepts of a cut-vertex, bridge, and block (introduced in Chapter 2) are generalized here to better describe the "connectivity" of a graph. An important and related result in digraphs, the max-flow min-cut theorem, is discussed, along with some of its many applications.

5.1 n-Connected and n-Edge-Connected Graphs

The *vertex-connectivity* or simply *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal from G results in a disconnected or trivial graph. The complete graph K_p cannot be disconnected by the removal of vertices, but the deletion of any $p-1$ vertices results in K_1; thus $\kappa(K_p)=p-1$. It is an immediate consequence of the definition that a nontrivial graph G has connectivity zero if and only if G is disconnected. Furthermore, a graph G has connectivity 1 if and only if $G \cong K_2$ or G is a connected graph with cut-vertices; $\kappa(G) \geqslant 2$ if and only if G is a cyclic block.

Connectivity has an edge analogue. The *edge-connectivity* $\kappa_1(G)$ of a graph G is the minimum number of edges whose removal from G results in a disconnected or trivial graph. Connectivity and edge-connectivity are related as we now see [W6].

Theorem 5.1 *For any graph G,*

$$\kappa(G) \leqslant \kappa_1(G) \leqslant \delta(G).$$

Proof Let $v \in V(G)$ such that $\deg v = \delta(G)$. The removal of the $\delta(G)$ edges of G incident with v results in a graph G' in which v is isolated so that G' is either disconnected or trivial; therefore, $\kappa_1(G) \leq \delta(G)$.

We now verify the other inequality. If $\kappa_1(G) = 0$, then G is disconnected or trivial so that $\kappa(G) = 0$. If $\kappa_1(G) = 1$, then G is connected and contains a bridge so that either $G \cong K_2$ or G is connected and contains cut-vertices; thus $\kappa(G) = 1$. In each of these cases, $\kappa(G) = \kappa_1(G)$. We henceforth assume $\kappa_1(G) \geq 2$.

There exists a set of $\kappa_1(G)$ edges in G whose removal disconnects G. The removal of $\kappa_1(G) - 1$ of these edges results in a connected graph with a bridge $e = uv$. For each of the $\kappa_1(G) - 1$ edges, select an incident vertex different from u and v. If the removal of these vertices results in a graph H that is disconnected, then $\kappa(G) < \kappa_1(G)$. If, on the other hand, H is connected, then either $H \cong K_2$ or H has a cut-vertex. In either case, there exists a vertex of H whose removal results in a disconnected or trivial graph. Therefore, $\kappa(G) \leq \kappa_1(G)$. ■

Figure 5.1 shows a graph G for which $\kappa(G) = 2$, $\kappa_1(G) = 3$, and $\delta(G) = 4$. It can be shown (see Exercise 5.8) that if a, b, and c are positive integers with $a \leq b \leq c$, then there is a graph G with $\kappa(G) = a$, $\kappa_1(G) = b$, and $\delta(G) = c$.

G:

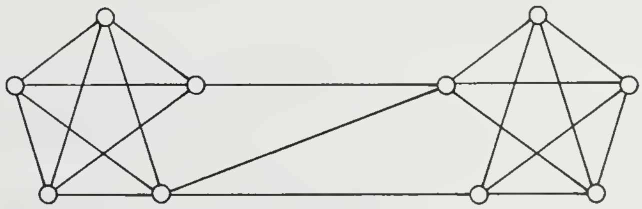


Figure 5.1 *A graph G for which $\kappa(G) = 2$, $\kappa_1(G) = 3$, and $\delta(G) = 4$*

A graph G is said to be *n-connected*, $n \geq 1$, if $\kappa(G) \geq n$. Thus, G is 1-connected if and only if G is nontrivial and connected, and G is 2-connected if and only if G is a cyclic block. It might be further noted that a graph G is n-connected if and only if the removal of fewer than n vertices results in neither a disconnected graph nor the trivial graph.

It is often the case that the knowledge that a graph is n-connected for some specified n is as valuable as knowing the connectivity itself. The following theorem gives a condition under which a graph is n-connected. The result is due to Bondy [B13]; however, it is stated in the form given by Boesch [B11].

Theorem 5.2 *Let G be a graph of order $p \geq 2$, the degrees d_i of whose vertices satisfy $d_1 \leq d_2 \leq \cdots \leq d_p$, and let n be an integer such that $1 \leq n \leq p - 1$.*
If

$$d_k \leqslant k+n-2 \Rightarrow d_{p-n+1} \geqslant p-k$$

for each k such that $1 \leqslant k \leqslant \lfloor (p-n+1)/2 \rfloor$, then G is n-connected.

Proof Suppose $\kappa(G)<n$. Then G is not complete, and there exists a set S of vertices of G such that $G-S$ is disconnected, where $\kappa(G) \leqslant |S| = n-1$. Let H be a component of $G-S$ of minimum order k. Then

$$k \leqslant \left\lfloor \frac{p-n+1}{2} \right\rfloor.$$

Clearly, each vertex of H has degree at most $k+n-2$ in G, where $k+n-2<p-k$. Since H has order k, this implies that $d_k \leqslant k+n-2$. By hypothesis, $d_{p-n+1} \geqslant p-k$. For $u \in V(G)-V(H)-S$, we have $\deg u \leqslant p-k-1$. Hence, only the vertices of S have degree at least $p-k$. Since

$$d_{p-n+1} \geqslant p-k \quad \text{and} \quad d_p \geqslant d_{p-1} \geqslant \cdots \geqslant d_{p-n+1},$$

it follows that S contains at least n vertices, which is a contradiction. ■

In a certain sense, which we now describe, the result given in Theorem 5.2 is sharp. For integers p, n, and m satisfying $2 \leqslant n \leqslant p-1$ and $1 \leqslant m \leqslant \lfloor (p-n+1)/2 \rfloor$, let

$$G \cong K_{n-1} + [K_m \cup K_{p-n-m+1}].$$

If we denote the degrees of the vertices of G by $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_p$, then

$$d_k = \begin{cases} m+n-2 & \text{for} \quad 1 \leqslant k \leqslant m \\ p-m-1 & \text{for} \quad m+1 \leqslant k \leqslant p-n+1. \\ p-1 & \text{for} \quad p-n+2 \leqslant k \leqslant p \end{cases}$$

For each k such that $1 \leqslant k \leqslant \lfloor (p-(n-1)+1)/2 \rfloor$, it can be verified that

$$d_k \leqslant k+(n-1)-2 \Rightarrow d_{p-(n-1)+1} \geqslant p-k.$$

Therefore, by Theorem 5.2, G is $(n-1)$-connected. Now suppose that $1 \leqslant k \leqslant \lfloor (p-n+1)/2 \rfloor$ and $k \neq m$. If $1 \leqslant k \leqslant m-1$, then $d_k = m+n-2 > k+n-2$. If $m+1 \leqslant k \leqslant \lfloor (p-n+1)/2 \rfloor$, then $d_{p-n+1} = p-(m+1) \geqslant p-k$. Thus, $d_k \leqslant k+n-2 \Rightarrow d_{p-n+1} \geqslant p-k$. However, for $k=m$ we have $d_k = k+n-2$ and $d_{p-n+1} = p-k-1$. Hence the hypothesis of Theorem 5.2 is "not quite" satisfied and, obviously, G is not n-connected.

Although Theorem 5.2 is often difficult to apply, it has a corollary that is more easily stated.

Corollary 5.2 *Let G be a graph of order $p \geq 2$, and let n be an integer such that $1 \leq n \leq p-1$. If*

$$\deg v \geq \left\lceil \frac{p+n-2}{2} \right\rceil$$

for every vertex v of G, then G is n-connected.

Proof Let k be a positive integer such that $k \leq \lfloor (p-n+1)/2 \rfloor$. Then $k+n-2 \leq \lfloor (p+n-3)/2 \rfloor$. Since

$$\deg v \geq \left\lceil \frac{p+n-2}{2} \right\rceil > k+n-2$$

for every vertex v of G, it follows that the hypothesis of Theorem 5.2 is satisfied vacuously. Thus, G is n-connected. ■

A graph G is *n-edge-connected*, $n \geq 1$, if $\kappa_1(G) \geq n$. Equivalently, G is n-edge-connected if the removal of fewer than n edges from G results in neither a disconnected graph nor a trivial graph. The class of n-edge-connected graphs is characterized in the following simple but useful theorem.

Theorem 5.3 *A nontrivial graph G is n-edge-connected if and only if there exists no nonempty proper subset W of V(G) such that the number of edges joining W and V(G) − W is less than n.*

Proof First, assume that there exists no nonempty proper subset W of $V(G)$ for which the number of edges joining W and $V(G) - W$ is less than n but that G is not n-edge-connected. Since G is nontrivial, this implies that there exist k edges, $0 \leq k < n$, such that their deletion from G results in a disconnected graph H. Let H_1 be a component of H. Since the number of edges joining $V(H_1)$ and $V(G) - V(H_1)$ is at most k, where $k < n$, this is a contradiction.

Conversely, suppose G is an n-edge-connected graph. If there should exist a subset W of $V(G)$ such that j edges, $j < n$, join W and $V(G) - W$, then the deletion of these j edges produces a disconnected graph—again a contradiction. The characterization now follows. ■

According to Theorem 5.1, $\kappa_1(G) \leq \delta(G)$ for every graph G. The following theorem of Plesník [P2] gives a sufficient condition for equality to hold.

Theorem 5.4 *If G is a graph of diameter 2, then $\kappa_1(G) = \delta(G)$.*

Proof Let S be a set of $\kappa_1(G)$ edges of G whose removal disconnects G, and let H_1 and H_2 denote the components of $G-S$. Without loss of generality, assume that $p(H_1) \leq p(H_2)$.

Suppose that some vertex u of H_1 is adjacent in G to no vertex of H_2. Then $d_G(u, v) = 2$ for each vertex v of H_2, and each vertex v of H_2 is adjacent to some vertex of H_1. Thus, either each vertex of H_1 is adjacent to some vertex of H_2 or each vertex of H_2 is adjacent to some vertex of H_1. In either case,

$$\kappa_1(G) = |S| \geq \min\{p(H_1), p(H_2)\} = p(H_1). \tag{5.1}$$

For each vertex $u \in V(H_1)$, let $d_i(u)$ denote the number of vertices of $H_i (i = 1, 2)$ adjacent to u in G. Then

$$\delta(G) \leq \deg u = d_1(u) + d_2(u) \leq p(H_1) - 1 + d_2(u). \tag{5.2}$$

Since $\delta(G) \geq \kappa_1(G)$, it follows from (5.1) and (5.2) that $d_2(u) \geq 1$ for each vertex u of H_1. Let $V(H_1) = \{u_1, u_2, \ldots, u_n\}$, where $n = p(H_1)$. Then

$$\begin{aligned}\kappa_1(G) = |S| = \sum_{i=1}^{n} d_2(u_i) &= \sum_{i=1}^{n-1} d_2(u_i) + d_2(u_n) \geq (n-1) + d_2(u_n)\\ &= p(H_1) - 1 + d_2(u_n).\end{aligned} \tag{5.3}$$

Again since $\delta(G) \geq \kappa_1(G)$, it follows from (5.2) and (5.3) that

$$p(H_1) - 1 + d_2(u_n) \geq \delta(G) \geq \kappa_1(G) \geq p(H_1) - 1 + d_2(u_n).$$

Thus, $\kappa_1(G) = \delta(G)$. ■

Corollary 5.4 *If G is a graph of order $p \geq 2$ such that for all distinct nonadjacent vertices u and v,*

$$\deg u + \deg v \geq p - 1,$$

then $\kappa_1(G) = \delta(G)$.

Exercises 5.1

5.1 Determine the connectivity and edge-connectivity of each complete n-partite graph.

5.2 Let G be an n-connected graph and let $v_1, v_2, \ldots, v_n$ be n distinct vertices. Let

H be the graph formed from G by adding a new vertex of degree n that is adjacent to each of $v_1, v_2, \ldots, v_n$. Show that $\kappa(H) = n$.

5.3 Let $H = G + K_1$, where G is n-connected. Prove that H is $(n+1)$-connected.

5.4 A unicyclic graph is a connected graph with exactly one cycle.
(a) If G is a unicyclic graph, show that $\kappa(G) \leq 2$ and $\kappa_1(G) \leq 2$.
(b) For which unicyclic graphs G does $\kappa(G) = \kappa_1(G) = 2$?
(c) Is there a unicyclic graph G with $\kappa(G) = 1$ and $\kappa_1(G) = 2$?

5.5 For a graph G of order $p \geq 2$. define the k-connectivity $\kappa_k(G)$ of G, $2 \leq k \leq p$, as the least number of vertices whose removal from G results in a graph with at least k components or a graph of order less than k. (Note that $\kappa_2(G) = \kappa(G)$.) A graph G is defined to be (n, k)-connected if $\kappa_k(G) \geq n$. Let G be a graph of order p containing a set of at least k nonadjacent vertices. Show that if

$$\deg_G v \geq \left\lceil \frac{p + (k-1)(n-2)}{k} \right\rceil$$

for every $v \in V(G)$, then G is (n, k)-connected.

5.6 Prove Corollary 5.4.

5.7 Let G be a graph of diameter 2. Show that if S is a set of $\kappa_1(G)$ edges whose removal disconnects G, then at least one of the components of $G - S$ is isomorphic to K_1 or $K_{\delta(G)}$.

5.8 Let a, b, and c be positive integers with $a \leq b \leq c$. Prove that there exists a graph G with $\kappa(G) = a$, $\kappa_1(G) = b$, and $\delta(G) = c$.

5.9 (a) Verify that Corollary 5.2 is best possible by showing that for each positive integer n, there exists a graph G of order $p (\geq n+1)$ such that $\delta(G) = \lceil (p+n-3)/2 \rceil$ and $\kappa(G) < n$.
(b) Verify that Theorem 5.4 is best possible by finding an infinite class of graphs G of diameter 3 for which $\kappa_1(G) \neq \delta(G)$.

5.2 Menger's Theorem

A nontrivial graph G is connected (or, equivalently, 1-connected) if between every two distinct vertices of G there exists at least one path. This fact can be generalized in many ways, most of which involve, either directly or indirectly, a theorem due to Menger [M5]. In this section, we discuss the major ones of these, beginning with Dirac's proof [D7] of Menger's Theorem itself.

A set S of vertices (or edges) of a graph G is said to *separate* two vertices u and v of G if the removal of the elements of S from G produces a disconnected graph in which u and v lie in different components.

In the graph G of Figure 5.2, there is a set $S = \{w_1, w_2, w_3\}$ of vertices of G that separate the vertices u and v. No set with fewer than three vertices separates u and v. As is guaranteed by the next theorem, there are three internally disjoint u-v paths in G.

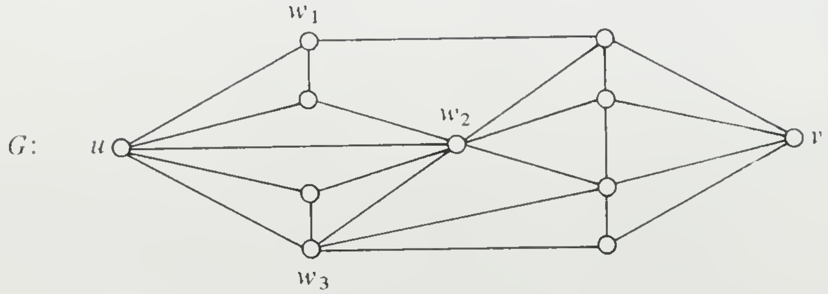


Figure 5.2 *A graph illustrating Menger's Theorem*

Theorem 5.5 (Menger) *Let u and v be nonadjacent vertices in a graph G. Then the minimum number of vertices that separate u and v is equal to the maximum number of internally disjoint u-v paths in G.*

Proof First, if u and v lie in different components of G, then the result is true; so we may assume that the graphs under consideration are connected. If the minimum number of vertices that separate u and v is $n(\geq 1)$, then the maximum number of internally disjoint u-v paths in G is at most n. Thus, if $n = 1$, the result is true (since we are assuming that G is connected). Denote by $S_n(u, v)$ the statement that the minimum number of vertices that separate u and v is n.

Suppose that the theorem is false. Then there exists a smallest positive integer $m(\geq 2)$ such that $S_m(u, v)$ is true in some graph G but there are fewer than m internally disjoint u-v paths. Among all such graphs G of smallest order, let H be one of minimum size.

We now establish three properties of the graph H.

(1) *For adjacent vertices v_1, v_2 of H, where neither v_1 nor v_2 is u or v, there exists a set U of $m-1$ vertices of H such that $U \cup \{v_i\}$, $i = 1, 2$, separates u and v.* To see this, let $e = v_1v_2$ and note that $S_m(u, v)$ is false for $H - e$ by the minimality of H. However, $S_{m-1}(u, v)$ is true for $H - e$; for suppose there exists a set U of vertices that separates u and v in $H - e$, where $|U| \leq m - 2$. Then $U \cup \{v_i\}$, $i = 1, 2$, separates u and v in H, contradicting the fact that the minimum number of vertices that separate u and v in H is m. Therefore, $S_{m-1}(u, v)$ is true in $H - e$ and so there exists a set U that separates u and v in $H - e$, where $|U| = m - 1$. However, then $U \cup \{v_i\}$, $i = 1$, 2, separates u and v in H.

(2) *For any vertex w $(\neq u, v)$ in H, not both uw and vw are edges of H.* If this were not the case, then $S_{m-1}(u, v)$ is true for $H - w$. The

minimality of m, however, then implies that $H - w$ contains $m-1$ internally disjoint u-v paths so that H contains m internally disjoint u-v paths, which is a contradiction.

(3) *If $W = \{w_1, w_2, \ldots, w_m\}$ is a set of vertices that separates u and v in H, then either $uw_i \in E(H)$ for all $i \in \{1, 2, \ldots, m\}$ or $vw_i \in E(H)$ for all $i \in \{1, 2, \ldots, m\}$.* To see this, define H_u as the subgraph of H induced by the edges on all u-w_i paths in H that contain only one vertex of W, and define H_v similarly. Observe that $V(H_u) \cap V(H_v) = W$. Suppose that it is not the case that $uw_i \in E(H)$ for all $i \in \{1, 2, \ldots, m\}$ or that $vw_i \in E(H)$ for all $i \in \{1, 2, \ldots, m\}$. Then $p(H_u) \geq m+2$ and $p(H_v) \geq m+2$. Define a new graph H_u^* to consist of H_u, a new vertex v^* together with all edges v^*w_i, and define H_v^* similarly. Observe that H_u^* and H_v^* have smaller order than H. Further, $S_m(u, v^*)$ is true in H_u^* and $S_m(u^*, v)$ is true in H_v^*. Therefore, there exist m internally disjoint u-v^* paths in H_u^* and m internally disjoint u^*-v paths in H_v^*. These $2m$ paths produce m internally disjoint u-v paths in H, a contradiction.

Let P be a shortest u-v path in H. By property (2) the length of P is at least 3. Thus we may denote P by $u, u_1, u_2, \ldots, v$, where $u_1, u_2 \neq v$. By property (1) there exists a set U of $m-1$ vertices such that both $U \cup \{u_1\}$ and $U \cup \{u_2\}$ separate u and v. In particular, $U \cup \{u_1\}$ separates u and v. Since $uu_1 \in E(H)$, it follows from properties (2) and (3) that every vertex of U is adjacent to u. Consider now $U \cup \{u_2\}$. No vertex of U is adjacent to v and hence, by property (3), u_2 is adjacent to u, contradicting our choice of P. ∎

With the aid of Menger's Theorem, it is now possible to present Whitney's characterization [W6] of n-connected graphs.

Theorem 5.6 (Whitney) *A nontrivial graph G is n-connected if and only if for each pair u, v of distinct vertices there are at least n internally disjoint u-v paths in G.*

Proof Assume G is an n-connected graph and that the maximum number of internally disjoint u-v paths in G is m, where $m < n$. If $uv \notin E(G)$, then by Theorem 5.5 $\kappa(G) \leq m < n$, which is contrary to hypothesis. If $uv \in E(G)$, then the maximum number of internally disjoint u-v paths in $G - uv$ is $m-1 < n-1$; hence $\kappa(G - uv) < n-1$. Therefore, there exists a set U of fewer than $n-1$ vertices such that $G - uv - U$ is a disconnected graph. Therefore, at least one of $G - (U \cup \{u\})$ and $G - (U \cup \{v\})$ is disconnected, implying that $\kappa(G) < n$. This also produces a contradiction.

Conversely, suppose that G is a nontrivial graph that is not n-connected but in which every pair of distinct vertices are connected by at least n internally disjoint paths. Certainly, G is not complete.

Since G is not n-connected, $\kappa(G) < n$. Let W be a set of $\kappa(G)$ vertices of G such that $G - W$ is disconnected, and let u and v be in different components of $G - W$. The vertices u and v are necessarily nonadjacent; however, by hypothesis, there are at least n internally disjoint u-v paths. By Theorem 5.5, u and v cannot be separated by fewer than n vertices, so a contradiction arises. ■

With the aid of Whitney's Theorem, the following result can now be established rather easily.

Theorem 5.7 *If G is an n-connected graph and $v, v_1, v_2, \ldots, v_n$ are $n+1$ distinct vertices of G, then for $i = 1, 2, \ldots, n$, there exist internally disjoint v-v_i paths.*

Proof Construct a new graph H from G by adding a new vertex u to G together with the edges uv_i, $i = 1, 2, \ldots, n$. Since G is n-connected, H is n-connected. (See Exercise 5.2.) By Theorem 5.6, there exist n internally disjoint u-v paths in H. The restriction of these paths to G yields the desired internally disjoint v-v_i paths. ■

One of the interesting properties of 2-connected graphs is that every two vertices of such graphs lie on a common cycle. (This is a direct consequence of Theorem 2.14.) There is a generalization of this fact to n-connected graphs by Dirac [D5].

Theorem 5.8 *Let G be an n-connected graph, $n \geq 2$. Then every n vertices of G lie on a common cycle of G.*

Proof For $n = 2$, the result follows from Theorem 2.14; hence, we assume $n \geq 3$. Let W be a set of n vertices of G. Among all cycles of G, let C be a cycle containing a maximum number, say m, of vertices of W. We observe that $m \geq 2$. We wish to show that $m = n$. Assume, to the contrary, that $m < n$. Let w be a vertex of W such that w does not lie on C.

Necessarily, C contains at least $m+1$ vertices; for if this were not the case, then the vertices of C could be labeled so that C: $w_1, w_2, \ldots, w_m, w_1$, where $w_i \in W$ for $1 \leq i \leq m$. By Theorem 5.7, there exist internally disjoint w-w_i paths Q_i, $1 \leq i \leq m$. Replacing the edge w_1w_2 on C by the w_1-w_2 path determined by Q_1 and Q_2, we obtain a cycle containing at least $m+1$ vertices of W, which is a contradiction. Therefore, C contains at least $m+1$ vertices.

Thus we may assume that C contains vertices $w_1, w_2, \ldots, w_m, w_{m+1}$, where $w_i \in W$ for $1 \leq i \leq m$ and $w_{m+1} \notin W$. Since $n \geq m+1$, we may apply Theorem 5.7 again to conclude that there exist $m+1$ internally disjoint w-w_i paths $P_i (1 \leq i \leq m+1)$. For each $i = 1, 2, \ldots, m+1$, let v_i be the first vertex on P_i that belongs to C (possibly $v_i = w_i$) and let P_i' denote the w-v_i subpath of P_i. Since C contains exactly m vertices of W, there are distinct integers j and k, $1 \leq j, k \leq m+1$, such that one of the two v_j-v_k paths, say P, determined by C contains no interior vertex belonging to W. Replacing P by the v_j-v_k path determined by P_j' and P_k', we obtain a cycle of G containing at least $m+1$ vertices of W. This contradiction gives the desired result that $m = n$. ■

It follows from Corollary 5.2 and Theorem 5.6 that if G is a graph of order $p \geq 2$ and n is an integer with $1 \leq n \leq p-1$ such that $\deg v \geq \lceil (p+n-2)/2 \rceil$ for every vertex v of G, then for each pair u, w of distinct vertices of G, there exist n internally disjoint u-w paths in G. Hedman [H10] has shown that a considerably stronger conclusion can be made with only a slightly stronger hypothesis.

Theorem 5.9 *Let G be a graph of order $p \geq 2$, and let n be an integer such that $1 \leq n \leq p-1$. If*

$$\deg v \geq \left\lceil \frac{p+n-1}{2} \right\rceil$$

for every vertex v of G, then for every two distinct vertices u and w of G there exist n internally disjoint u-w paths, each of length at most 2.

Both Theorems 5.5 and 5.6 have "edge" analogues; the analogue to Menger's Theorem was proved in [EFS1], [FF1]. It is not surprising that the edge analogue of Menger's Theorem can be proved in a manner that bears a striking similarity to the proof of Menger's Theorem.

Theorem 5.10 *If u and v are distinct vertices of a graph G, then the maximum number of edge-disjoint u-v paths in G equals the minimum number of edges of G that separate u and v.*

Proof We actually prove a stronger result here by allowing G to be a multigraph.

If u and v are vertices in different components of a multigraph G, then the theorem is true. Thus, without loss of generality, we may assume that the multigraphs under consideration are connected. If the minimum number of edges that separate u and v is n, where $n \geq 1$, then the maximum number of edge-disjoint u-v paths is at most n. Thus, the result is true if $n = 1$.

For vertices u and v of a multigraph G, let $S_n(u, v)$ denote the statement that the minimum number of edges that separate u and v is n.

If the theorem is not true, then there exists a positive integer $m(\geq 2)$ for which there are multigraphs G containing vertices u and v such that $S_m(u, v)$ is true, but there is no set of m edge-disjoint u-v paths. Among all such multigraphs G, let F denote one of minimum size.

If every u-v path of F has length 1 or 2, then since the minimum number of edges of F that separate u and v is m, it follows that there are m edge-disjoint u-v paths in F, a contradiction. Thus F contains at least one u-v path P of length 3 or more. Let e_1 be an edge of P incident with neither u nor v. Then for $F - e_1$, the statement $S_m(u, v)$ is false but $S_{m-1}(u, v)$ is true. This implies that e_1 belongs to a set of m edges of F that separate u and v, say $\{e_1, e_2, \ldots, e_m\}$. We now subdivide each of the edges e_i, $1 \leq i \leq m$; that is, let $e_i = u_i v_i$, replace each e_i by a new vertex w_i, and add the $2m$ edges $u_i w_i$ and $w_i v_i$. The vertices w_i are now identified, producing a new vertex w and a new multigraph H. The vertex w in H is a cut-vertex, and every u-v path of H contains w.

Denote by H_u the submultigraph of H determined by all u-w paths of H; the submultigraph H_v is defined similarly. Each of the multigraphs H_u and H_v has fewer edges than does F (since e_1 was chosen to be an edge of a u-v path in F incident with neither u nor v). Also, the minimum number of edges separating u and w in H_u is m, and the minimum number of edges separating v and w in H_v is m. Thus, the multigraph H_u satisfies $S_m(u, w)$, and the multigraph H_v satisfies $S_m(w, v)$. This implies that H_u contains a set of m edge-disjoint u-w paths and H_v contains a set of m edge-disjoint w-v paths. For each $i = 1, 2, \ldots, m$, a u-w path and w-v path can be paired off to produce a u-v path in H containing the two edges $u_i w$ and $w v_i$. These m u-v paths of H are edge-disjoint. The process of subdividing the edges $e_i = u_i v_i$ of F and identifying the vertices w_i to obtain w can now be reversed to produce m edge-disjoint u-v paths in F. This, however, produces a contradiction.

Since the theorem has been proved for multigraphs G, its validity follows in the case where G is a graph. ■

With the aid of Theorem 5.10, it is now possible to present an edge analogue of Theorem 5.6.

Theorem 5.11 *A nontrivial graph G is n-edge-connected if and only if for every two distinct vertices u and v of G, there exist at least n edge-disjoint u-v paths in G.*

Exercises 5.2

5.10 Let G be a graph with $\kappa(G)=3$ such that for some pair u, v of distinct nonadjacent vertices, the minimum number of vertices that separate u and v is 4.

(a) What is the maximum number of internally disjoint u-v paths in G?

(b) Give an example of a graph G satisfying the above properties.

5.11 Prove that a graph G of order $p \geq n+1 \geq 3$ is n-connected if and only if for each set S of n distinct vertices of G and for each two-vertex subset T of S, there is a cycle of G that contains the vertices of T and avoids the vertices of $S-T$.

5.12 Prove that a graph G of order $p \geq 2n$ is n-connected if and only if for every two disjoint sets V_1 and V_2 of n vertices each, there exist n disjoint paths connecting V_1 and V_2.

5.13 Let G be an n-connected graph and let v be a vertex of G. For each positive integer k, define G_k to be the graph obtained from G by adding k new vertices $u_1, u_2, \ldots, u_k$ and all edges of the form $u_i w$, where $1 \leq i \leq k$ and $vw \in E(G)$. Show that G_k is n-connected.

5.14 Show that if G is an n-connected graph with nonempty disjoint subsets S_1 and S_2 of $V(G)$, then there exist n internally disjoint paths $P_1, P_2, \ldots, P_n$ such that P_i is a u_i-v_i path, where $u_i \in S_1$ and $v_i \in S_2$, for $i=1, 2, \ldots, n$, and $|S_1 \cap V(P_i)| = |S_2 \cap V(P_i)| = 1$.

5.15 Prove Theorem 5.9.

5.16 Prove Theorem 5.11.

5.17 Prove or disprove: If G is an n-edge-connected graph and $v, v_1, v_2, \ldots, v_n$ are $n+1$ distinct vertices of G, then for $i=1, 2, \ldots, n$, there exist v-v_i paths P_i such that each P_i contains exactly one vertex of $\{v_1, v_2, \ldots, v_n\}$, namely v_i, and for $i \neq j$, P_i and P_j are edge-disjoint.

5.18 Prove or disprove: If G is an n-edge-connected graph with nonempty disjoint subsets S_1 and S_2 of $V(G)$, then there exist n edge-disjoint paths $P_1, P_2, \ldots, P_n$ such that P_i is a u_i-v_i path, where $u_i \in S_1$ and $v_i \in S_2$, for $i=1, 2, \ldots, n$, and $|S_1 \cap V(P_i)| = |S_2 \cap V(P_i)| = 1$.

5.19 Prove that $\kappa(Q_n)=\kappa_1(Q_n)=n$ for all positive integers n.

5.20 Assume that G is a *graph* in the proof of Theorem 5.10. Does the proof go through? If not, where does it fail?

5.21 Let G be a graph with $\kappa(G) \geq 1$. Prove that

$$p(G) \geq \kappa(G)[\operatorname{diam} G - 1] + 2.$$

5.3 Networks

In Sections 5.1 and 5.2 we investigated connectivity properties in graphs. In the next two sections we turn our attention to related questions in directed graphs called networks.

A *network* N is a digraph D with two distinguished vertices u and v, called the *source* and *sink* of N, respectively, and a nonnegative integer-valued function c on $E(D)$. The digraph D is called the *underlying digraph* of N. The function c is the *capacity function* of N and its value $c(a) = c(x, y)$ on an arc $a = (x, y)$ is referred to as the *capacity* of a.

Intuitively, the capacity of an arc (x, y) may be thought of as the maximum amount of some material that can be transported from x to y per unit time. For example, the capacity of the arc (x, y) may represent the number of seats available on a direct flight from city x to city y in some airline system. On the other hand, this capacity might be the capacity of a pipeline from city x to city y in an oil network, or perhaps the maximum weight of items that can be transported by truck along a highway from city x to city y. The problem in general, then, is to maximize the "flow" from the source u to the sink v without exceeding the capacities of the arcs.

A network may be represented by drawing its underlying digraph D and labeling each arc of D with its capacity. Note, for example, that $c(x, y) = 4$ in the network of Figure 5.3.

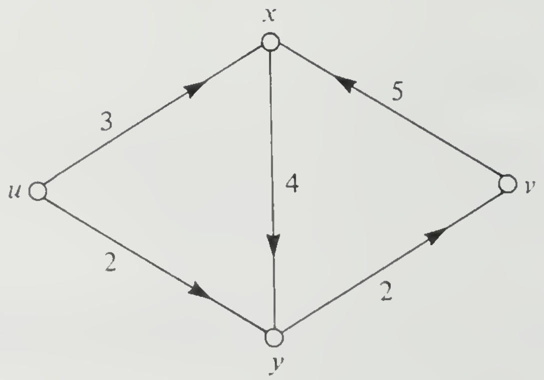


Figure 5.3 *A network*

If x is a vertex of a digraph D, let $N^+(x)$ (respectively, $N^-(x)$) denote the set of vertices of D adjacent from x (respectively, adjacent to x). Thus, $N^+(x) = \{y \in V(D) | (x, y) \in E(D)\}$ and $N^-(x) = \{y \in V(D) | (y, x) \in E(D)\}$.

A *flow* in a network N (with underlying digraph D, source u, sink v, and capacity function c) is an integer-valued function f on $E(D)$ such that

$$0 \leq f(a) \leq c(a) \quad \text{for every } a \in E(D), \tag{5.4}$$

and

$$\sum_{y \in N^+(x)} f(x, y) = \sum_{y \in N^-(x)} f(y, x) \quad \text{for every } x \in V(D) - \{u, v\}. \tag{5.5}$$

The *net flow out of a vertex* x is defined as $\Sigma_{y \in N^+(x)} f(x, y) - \Sigma_{y \in N^-(x)} f(y, x)$, while the *net flow into* x is, of course, $\Sigma_{y \in N^-(x)} f(y, x) - \Sigma_{y \in N^+(x)} f(x, y)$. Thus by condition (5.5), if x is an *intermediate vertex*, that is, if x is neither the source nor the sink, then the net flow out of x equals the net flow into x, and this common value is zero.

The value $f(a) = f(x, y)$ on an arc $a = (x, y)$, called the *flow in arc a*, can be interpreted as the amount of material that is transported under the flow f along the arc (x, y). Condition (5.4) requires that this amount cannot exceed the capacity of (x, y). Condition (5.5), referred to as a *conservation equation*, requires that for an intermediate vertex x, the amount of material transported into x per unit time equals the amount of material transported out of x.

An example of a flow is given in Figure 5.4. The first number associated with an arc is the capacity of the arc, and the second number associated with an arc is the flow in the arc. The arc (x, t) is said to be *saturated* (with respect to the given flow) since the flow in (x, t) equals the capacity of (x, t). On the other hand, the arc (u, x) is *unsaturated*. We note that in this example, the net flow out of the source u is equal to the net flow into the sink v. That this is always true will be shown in Theorem 5.12.

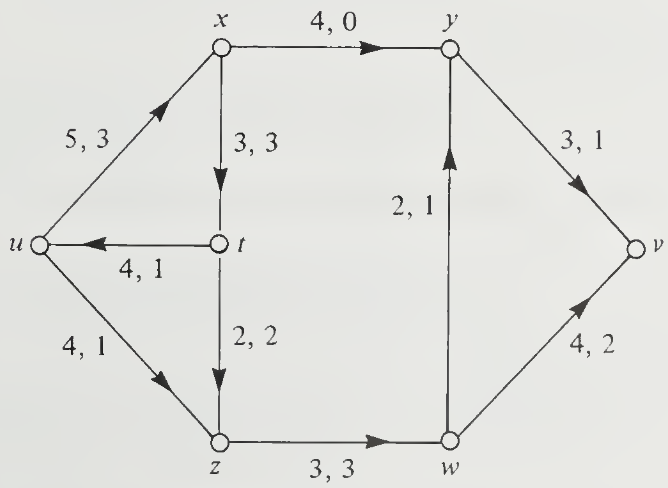


Figure 5.4 *A network flow*

Before presenting our first result on networks, it is helpful to introduce some notation. Let D be a digraph and let X and Y be nonempty subsets of $V(D)$. The symbol (X, Y) denotes the following subset of $E(D)$:

$$(X, Y) = \{(x, y) \in E(D) | x \in X, y \in Y\}.$$

If $X = \{x\}$, we often write (x, Y) rather than $(\{x\}, Y)$. Similarly, (X, y) denotes $(X, \{y\})$. If g is a real-valued function defined on $E(D)$, then $g(X, Y)$ is defined by

$$g(X, Y) = \sum_{(x, y) \in (X, Y)} g(x, y),$$

where $g(X, Y) = 0$ if $(X, Y) = \emptyset$. Observe that if X, Y, Z are subsets of $V(D)$, then

$$g(X \cup Z, Y) = g(X, Y) + g(Z, Y) - g(X \cap Z, Y).$$

Furthermore, if $x \in V(D)$, then

$$g(x, V(D)) = \sum_{y \in N^+(x)} g(x, y) \quad \text{and} \quad g(V(D), x) = \sum_{y \in N^-(x)} g(y, x).$$

Finally,

$$g(X, V(D)) = \sum_{x \in X} g(x, V(D)) \quad \text{and} \quad g(V(D), X) = \sum_{x \in X} g(V(D), x).$$

Theorem 5.12 *Let u and v be the source and sink, respectively, of a network N with underlying digraph D, and let f be a flow in N. Then*

$$f(u, V(D)) - f(V(D), u) = f(V(D), v) - f(v, V(D)).$$

Proof We first observe that

$$\sum_{x \in V(D)} f(x, V(D)) = \sum_{x \in V(D)} f(V(D), x) \tag{5.6}$$

since each side of the equation in (5.6) equals $f(V(D), V(D))$. However, by (5.5),

$$f(x, V(D)) = f(V(D), x) \quad \text{if } x \neq u, v.$$

Thus, (5.6) becomes

$$f(u, V(D)) + f(v, V(D)) = f(V(D), u) + f(V(D), v). \blacksquare$$

The *value* of a flow f in a network N, denoted val f, is defined as the net flow out of the source of the network; equivalently (by Theorem 5.12), val f equals the net flow into the sink of N. A flow f in N is called a *maximum flow* if val $f \geq$ val f' for every flow f' in N. In Section 5.4 we will determine the value of a maximum flow in a given network and present an algorithm for constructing such a flow.

Let N be a network with underlying digraph D, source u, and sink v. A *cut* in N is a set of arcs of the form $(X, V(D) - X)$ such that $u \in X$ and $v \in V(D) - X$. We often write $\overline{X}$ in lieu of $V(D) - X$. If $K = (X, \overline{X})$ is a cut in N, then the *capacity* of K, denoted cap K, is given by cap $K = c(X, \overline{X})$, where

c is the capacity function of N. In the network illustrated in Figure 5.4, the set $K = \{(x, y), (x, t), (u, z)\}$ is a cut since $K = (\{u, x\}, \overline{\{u, x\}})$, with $u \in \{u, x\}$ and $v \in \overline{\{u, x\}}$. The capacity of K is cap $K = c(x, y) + c(x, t) + c(u, z) = 4 + 3 + 4 = 11$.

If K is a cut in a network N, then any directed path from the source u to the sink v must contain at least one arc of K. Thus, if all the arcs of K were deleted from the underlying digraph D, there would be no path from u to v. In a certain sense, then, K "separates" u and v (this observation will be pursued further in the next section) and, intuitively, it appears that the value of a flow in N cannot exceed the capacity of K. That this is indeed the case is verified in Theorem 5.13.

Theorem 5.13 *Let f be a flow in a network N and let $K = (X, \overline{X})$ be a cut in N. Then*

$$\text{val}\, f = f(X, \overline{X}) - f(\overline{X}, X) \leq \text{cap}\, K.$$

Proof Let D be the underlying digraph of N and let u and v be the source and sink, respectively. Since f is a flow in N, f satisfies the equations

$$f(x, V(D)) - f(V(D), x) = 0, \qquad x \in X - \{u\}$$
$$f(u, V(D)) - f(V(D), u) = \text{val}\ f.$$

It follows that $\Sigma_{x \in X}\{f(x, V(D)) - f(V(D), x)\} = \text{val}\ f$ since $u \in X$. However,

$$\sum_{x \in X} \{f(x, V(D)) - f(V(D), x)\} = \sum_{x \in X} f(x, V(D)) - \sum_{x \in X} f(V(D), x)$$
$$= f(X, V(D)) - f(V(D), X).$$

Furthermore, $f(X, V(D)) = f(X, X \cup \overline{X}) = f(X, X) + f(X, \overline{X}) - f(X, X \cap \overline{X})$, and $f(V(D), X) = f(X \cup \overline{X}, X) = f(X, X) + f(\overline{X}, X) - f(X \cap \overline{X}, X)$. Since $f(X, X \cap \overline{X}) = f(X \cap \overline{X}, X) = 0$, we conclude that $\Sigma_{x \in X}\{f(x, V(D)) - f(V(D), x)\} = f(X, \overline{X}) - f(\overline{X}, X)$ and that $\text{val} f = f(X, \overline{X}) - f(\overline{X}, X)$.

The inequality $f(X, \overline{X}) - f(\overline{X}, X) \leq \text{cap}\ K$ follows from the facts that $f(X, \overline{X}) \leq \text{cap}\ K$ and $f(\overline{X}, X) \geq 0$. ■

A cut K in a network N is called a *minimum cut* if cap $K \leq$ cap K' for every cut K' in N.

Corollary 5.13a *Let f be a flow and K be a cut in a network N. If* val $f = $ cap K, *then f is a maximum flow and K is a minimum cut.*

Proof By Theorem 5.13, if f^* is a maximum flow in N and K^* is a minimum cut, then val $f^* \leq$ cap K^*. However, val $f \leq$ val f^* and cap $K^* \leq$ cap K, so that

$$\text{val } f \leqslant \text{val } f^* \leqslant \text{cap } K^* \leqslant cap\ K.$$

Since val $f = \text{cap } K$, we have val $f = \text{val } f^*$ and cap $K^* = \text{cap } K$. Thus f is a maximum flow and K is a minimum cut. ■

Corollary 5.13b *If f is a flow in a network N with capacity function c, and $(X, \bar{X})$ is a cut in N such that*

$$f(a) = c(a) \quad \textit{for every } a \in (X, \bar{X})$$

and

$$f(a) = 0 \quad \textit{for every } a \in (\bar{X}, X),$$

then f is a maximum flow in N and $(X, \bar{X})$ is a minimum cut.

Consider the network N with source u and sink v illustrated in Figure 5.5 where, as before, the first number associated with an arc a is its capacity $c(a)$, and the second number, $f(a)$, is the flow in a. We see that val $f = 4$. If X is the

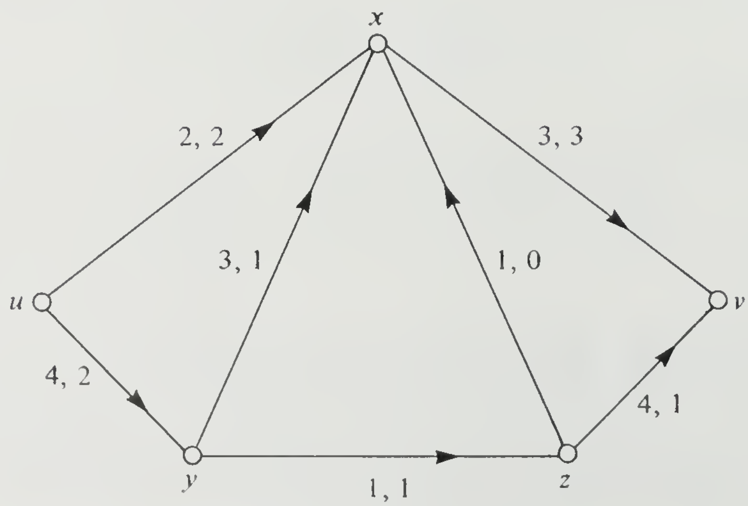


Figure 5.5 *A maximum flow*

set $\{u, x, y\}$, then $(X, \bar{X})$ is a cut in N and $c(X, \bar{X}) = 4$. By Corollary 5.13a, then, f is a maximum flow and (X, X) is a minimum cut.

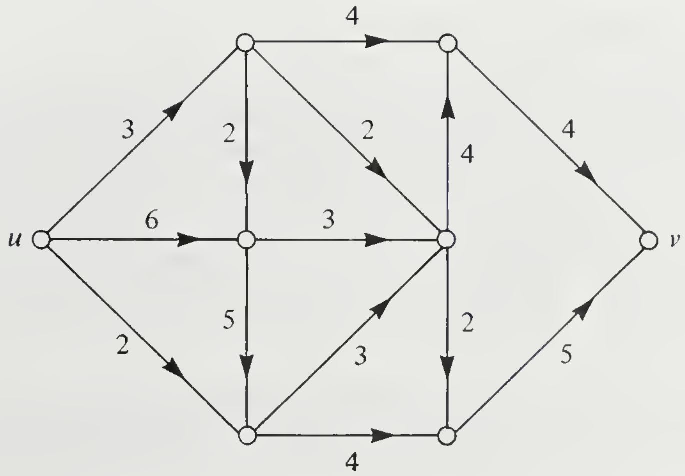
Exercises 5.3

5.22 Show that every network has at least one flow and at least one maximum flow.

5.23 Let N be the network with source u and sink v illustrated, where the label on each arc is its capacity.

(a) Give an example of a flow f in N such that val $f > 5$.

(b) Determine val f.



5.24 Let u and v be two vertices of a digraph D and let A be a set of arcs of D such that every u-v path in D contains at least one arc of A.

(a) Show that there exists a set of arcs of the form $(X, \overline{X})$, where $X \subseteq V(D)$, $u \in X$, and $v \in \overline{X}$, such that $(X, \overline{X}) \subseteq A$.

(b) Show that $(X, \overline{X})$ may be a proper subset of A.

5.25 Prove Corollary 5.13b.

5.26 Let N be a network with underlying digraph D, source u, and sink v. Show that if D contains no u-v path, then the value of a maximum flow in N and the value of a minimum cut in N are both zero.

5.4 The Max-Flow Min-Cut Theorem

It was shown in Corollary 5.13a that if val f = cap K for a flow f and a cut K in a network N, then f is a maximum flow in N and K is a minimum cut. In this section we prove the converse of this result; that is, in any network N, the value of a maximum flow is equal to the capacity of a minimum cut. Before presenting a proof of this result, we need some additional definitions.

A *u-v semipath* in a digraph D is a finite, alternating sequence

$$P\colon u = u_0, a_1, u_1, a_2, \ldots, u_{n-1}, a_n, u_n = v$$

of vertices and arcs, beginning with vertex u and ending with vertex v, such that no vertex of P is repeated, and either $a_i = (u_i, u_{i-1})$ or $a_i = (u_{i-1}, u_i)$ for $i = 1, 2, \ldots, n$. Let f be a flow in a network N with underlying digraph D and capacity function c. A semipath $w_0, a_1, w_1, a_2, w_2, \ldots, w_{n-1}, a_n, w_n$ in D is said to be *f-unsaturated* if, for $1 \leq i \leq n$,

(a) $f(a_i) < c(a_i)$ whenever $a_i = (w_{i-1}, w_i)$, and
(b) $f(a_i) > 0$ whenever $a_i = (w_i, w_{i-1})$.

If P is an f-unsaturated u-v semipath, where u and v are the source and sink of N, respectively, then P is called an *f-augmenting* semipath. For example, if f is the flow given in Figure 5.4, then u, (t, u), t, (x, t), x, (x, y), y, (y, v), v is an f-augmenting semipath.

The relationship between augmenting semipaths and maximum flows is given in the following theorem of Ford and Fulkerson [FF1].

Theorem 5.14 *Let N be a network with underlying digraph D. A flow f in N is a maximum flow if and only if there is no f-augmenting semipath in D.*

Proof Let u and v be the source and sink of N, respectively, and let c be the capacity function of N. Assume, first, that D contains an f-augmenting semipath. Thus there is a u-v semipath P

$$u = w_0, a_1, w_1, a_2, w_2, \ldots, w_{n-1}, a_n, w_n = v$$

such that for $1 \leq i \leq n$,

(a) $f(a_i) < c(a_i)$ whenever $a_i = (w_{i-1}, w_i)$, and
(b) $f(a_i) > 0$ whenever $a_i = (w_i, w_{i-1})$.

Let $\varepsilon_1 = \min(c(a_i) - f(a_i))$, where the minimum is taken over all i such that $a_i = (w_{i-1}, w_i)$; if no such i exists, set $\varepsilon_1 = 1$. Similarly, let $\varepsilon_2 = \min f(a_i)$, where the minimum is taken over all i such that $a_i = (w_i, w_{i-1})$; again, if no such i exists, set $\varepsilon_2 = 1$. Finally, let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$.

Define a function f^* on $E(D)$ as follows:

$$f^*(a) = \begin{cases} f(a) + \varepsilon & \text{if } a = (w_{i-1}, w_i) \quad \text{for some } i,\ 1 \leq i \leq n, \\ f(a) - \varepsilon & \text{if } a = (w_i, w_{i-1}) \quad \text{for some } i,\ 1 \leq i \leq n, \\ f(a) & \text{if } a \notin E(P). \end{cases}$$

Then clearly, f^* is an integer-valued function on $E(D)$ such that $0 \leqslant f^*(a) \leqslant c(a)$ for every $a \in E(D)$. Thus f^* satisfies condition (5.4). We show that f^* also satisfies (5.5). Let $x \in V(D) - \{u, v\}$. If x does not lie on P, then $f^*(x, V(D)) = f(x, V(D)) = f(V(D), x) = f^*(V(D), x)$. Therefore, f^* satisfies condition (5.5) for all vertices x not on P. If, on the other hand, x lies on P, then we consider three possible cases.

Case 1: Suppose the vertex x is incident to exactly two arcs of P. Then for some i, where $1 \leqslant i \leqslant n-1$, we have $x = w_i$, $a_i = (w_i, w_{i-1})$ and $a_{i+1} = (w_i, w_{i+1})$. Therefore,

$$f^*(x, V(D)) = f^*(a_i) + f^*(a_{i+1}) + \sum f(a)$$

where the sum is taken over all arcs $a \neq a_i, a_{i+1}$ that are incident from x. Furthermore, $f^*(a_i) = f(a_i) - \varepsilon$ and $f^*(a_{i+1}) = f(a_{i+1}) + \varepsilon$, so that $f^*(x, V(D)) = f(x, V(D))$. Since $f^*(V(D), x) = f(V(D), x)$ and $f(V(D), x) = f(x, V(D))$, it follows that condition (5.5) holds in this case.

In a similar manner it can be shown that $f^*(x, V(D)) = f^*(V(D), x)$ in each of the remaining two cases.

Case 2: Suppose the vertex x is incident from exactly two arcs of P.

Case 3: Suppose the vertex x is incident from exactly one arc of P and incident to exactly one arc of P.

Thus we conclude that f^* is a flow in N, and that

$$\text{val } f^* = f^*(u, V(D)) - f^*(V(D), u).$$

If $a_1 = (w_0, w_1)$, then $f^*(u, V(D)) = f(u, V(D)) + \varepsilon$ and $f^*(V(D), u) = f(V(D), u)$. If $a_1 = (w_1, w_0)$, then $f^*(u, V(D)) = f(u, V(D))$ and $f^*(V(D), u) = f(V(D), u) - \varepsilon$. In either case, $f^*(u, V(D)) - f^*(V(D), u) = f(u, V(D)) - f(V(D), u) + \varepsilon$. Thus, $\text{val } f^* = \text{val } f + \varepsilon > \text{val } f$, the last inequality holding since $\varepsilon \geqslant 1$, so that f is not a maximum flow.

Conversely, assume that there is no f-augmenting semipath in D. According to Corollary 5.13b, the proof will be complete if we can exhibit a cut $(X, \bar{X})$ in N such that $f(a) = c(a)$ for every $a \in (X, \bar{X})$, and $f(a) = 0$ for every $a \in (\bar{X}, X)$. (Such a cut will, in fact, be a minimum cut in N.)

Let X denote the set of all vertices x in D for which there exists an f-unsaturated u-x semipath. Then $u \in X$ and, by assumption, $v \notin X$. Thus $(X, \bar{X})$ is a cut in N. Suppose $(y, w) \in (X, X)$. Since $y \in X$, there exists an f-saturated u-y semipath P in D. Therefore $f(y, w) = c(y, w)$; otherwise, the semipath Q defined as P, (y, w), w would be an f-unsaturated u-w semipath, contradicting the fact that $w \notin X$. Similarly, if $(y, w) \in (\bar{X}, X)$, then $f(y, w) = 0$. This completes the proof. ■

The following theorem, due to Ford and Fulkerson [FF1] and often referred to as the *max-flow min-cut theorem*, verifies the converse of Corollary 5.13a.

Theorem 5.15 (Ford and Fulkerson) *In any network, the value of a maximum flow equals the capacity of a minimum cut.*

Proof Let f be a maximum flow in a network N (see Exercise 5.22). By Theorem 5.14, there is no f-augmenting semipath in the underlying digraph D of N. However, it was shown in the proof of Theorem 5.14 that if D contains no f-augmenting semipath, then there is a (minimum) cut $K=(X, \bar{X})$ such that each arc in $(X, \bar{X})$ is saturated, while the flow in each arc in $(\bar{X}, X)$ is zero. Thus $f(X, \bar{X})=\text{cap } K$ and $f(\bar{X}, X)=0$. An application of Theorem 5.13 yields $\text{val } f=f(X, \bar{X})-f(\bar{X}, X)=\text{cap } K$. ■

The proof of Theorem 5.14 provides the basis of an algorithm, also due to Ford and Fulkerson [FF2], for finding a maximum flow and minimum cut in a network. A slight refinement of this algorithm, suggested by Edmonds and Karp [EK1], results in a good graph-theoretic algorithm. Both of these algorithms provide a systematic method for finding an f-augmenting semipath in a network with a given flow f. The Edmonds-Karp algorithm, which we describe next, searches for a shortest such semipath, i.e., one with the fewest number of arcs.

Let u and v be the source and sink, respectively, of a network N with underlying digraph D. We begin the procedure with a given flow f in N (perhaps the zero flow). The computation progresses by a sequence of labelings of the vertices of D. A vertex w receives a label only if there is an f-unsaturated u-w semipath P in D. The label assigned to w is an ordered pair. If x is the vertex preceding w on P, then the first component of the label is $x+$ or $x-$, depending on whether the arc preceding w is (x, w) or (w, x). The second component of the label is a positive integer reflecting the potential change in f along P. If the sink v is labeled, then a new flow of greater value is obtained and the process is repeated. If the sink is not labeled, then f is a maximum flow in N, and the labels can be used to find a minimum cut.

Throughout the algorithm, a vertex is considered to be in one of three states: unlabeled, labeled and unscanned, or labeled and scanned. Initially, all vertices are unlabeled. When a vertex receives a label, it is added to the bottom of the "labeled but unscanned" list L. These vertices are scanned on a "first-labeled first-scanned" basis, which insures that a shortest f-augmenting semipath is selected.

Algorithm 5A (Edmonds and Karp) *Given a network N with underlying digraph D, source u, sink v, and capacity function c:*

1. *Assign values of an initial flow f to the arcs of D.*
2. *Label u with $(-, \infty)$ and add u to L, the list of labeled and unscanned vertices.*
3. *Select and remove the first element of L, say x, with label $(z+, \varepsilon(x))$ or $(z-, \varepsilon(x))$. If L is empty, then stop.*
   (*a*) *To all vertices y that are unlabeled and such that*

$$(x, y) \in E(D) \quad \text{and} \quad f(x, y) < c(x, y),$$

   assign the label $(x+, \varepsilon(y))$, where

$$\varepsilon(y) = \min\{\varepsilon(x), c(x, y) - f(x, y)\}$$

   and add y to the end of L.

   (*b*) *To all vertices y that are unlabeled and such that $(y, x) \varepsilon E(D)$ and $f(y, x) > 0$, assign the label $(x-, \varepsilon(y))$, where*

$$\varepsilon(y) = \min\{\varepsilon(x), f(y, x)\}$$

   and add y to the end of L.
4. *If v has been labeled, go to Step 5; otherwise, go to Step 3.*
5. *The labels describe an f-augmenting semipath*

$$u = w_0, a_1, w_1, a_2, w_2, \ldots, w_{n-1}, a_n, w_n = v$$

   where, for $1 \leqslant i \leqslant n$, w_i is labeled $(w_{i-1}+, \varepsilon(w_i))$ if $a_i = (w_{i-1}, w_i)$ and w_i is labeled $(w_{i-1}-, \varepsilon(w_i))$ if $a_i = (w_i, w_{i-1})$. In the first case, replace $f(w_{i-1}, w_i)$ by $f(w_{i-1}, w_i) + \varepsilon(v)$; in the second case, replace $f(w_i, w_{i-1})$ by $f(w_i, w_{i-1}) - \varepsilon(v)$.
6. *Discard all labels, remove all vertices from L, and go to Step 2.*

Theorem 5A *Algorithm 5A terminates with a maximum flow f in N. Furthermore, if X is the set of labeled vertices upon termination, then $(X, \bar{X})$ is a minimum cut.*

Proof It follows from Theorem 5.14 that each time Step 5 is completed, a new flow in N with larger value has been constructed. Furthermore if, in Step 3, the list L is empty, then there is no f-augmenting semipath and so, by Theorem 5.14, f is a maximum flow and $(X, \bar{X})$ is a minimum cut. It remains to see that the process must, in fact, terminate, since for every flow $\tilde{f}$ in N,

$$\text{val } \tilde{f} \leqslant c(u, V(D)),$$

and thus Step 5 can be repeated at most $c(u, V(D))$ times. ■

As an illustration of Algorithm 5A, let N be the network with source u and sink v given in Figure 5.6(a). The labels on each arc a are the capacity $c(a)$ of a and the initial flow $f(a)$ in a, respectively. Initially, u is labeled $(-, \infty)$, and L consists only of u. As the algorithm proceeds through Step 3 the first time, the set of labeled vertices and the value of L changes as follows:

Labeled Vertices	L
u: $(-, \infty)$	u
u: $(-, \infty)$, r: $(u+, 3)$	r
u: $(-, \infty)$, r: $(u+, 3)$, s: $(u-, 2)$	r, s

Continuing through Step 3 again, we have

u: $(-, \infty)$, r: $(u+, 3)$, s: $(u-, 2)$, y: $(r+, 2)$	s, y
u: $(-, \infty)$, r: $(u+, 3)$, s: $(u-, 2)$, y: $(r+, 2)$, w: $(s-, 2)$	y, w
u: $(-, \infty)$, r: $(u+, 3)$, s: $(u-, 2)$, y: $(r+, 2)$, w: $(s-, 2)$, t: $(s+, 2)$,	y, w, t

Finally, we obtain

u: $(-, \infty)$, r: $(u+, 3)$, s: $(u-, 2)$, y: $(r+, 1)$, w: $(s-, 2)$, t: $(s+, 2)$, v: $(y+, 2)$	w, t

and we reach Step 4.

Thus, from Step 5 we obtain the f-augmenting semipath

$$u, (u, r), r, (r, y), y, (y, v), v$$

and the flow in each of the arcs in this semipath is increased by $\varepsilon(v) = 2$. The resulting flow f is shown in Figure 5.6(b).

Proceeding through Step 6, Step 2, and then repeating Step 3 until v is again labeled, we have the following set of labeled vertices:

u: $(-, \infty)$, r: $(u+, 1)$, s: $(u-, 2)$, w: $(s-, 2)$, t: $(s+, 2)$, v: $(w+, 2)$.

Using the resulting f-augmenting semipath

$$u, (s, u), s, (w, s), w, (w, v), v$$

with $\varepsilon(v) = 2$, we obtain the flow f indicated in Figure 5.6(c).

Beginning again with Step 2, we label the vertices

u: $(-, \infty)$, r: $(u+, 1)$.

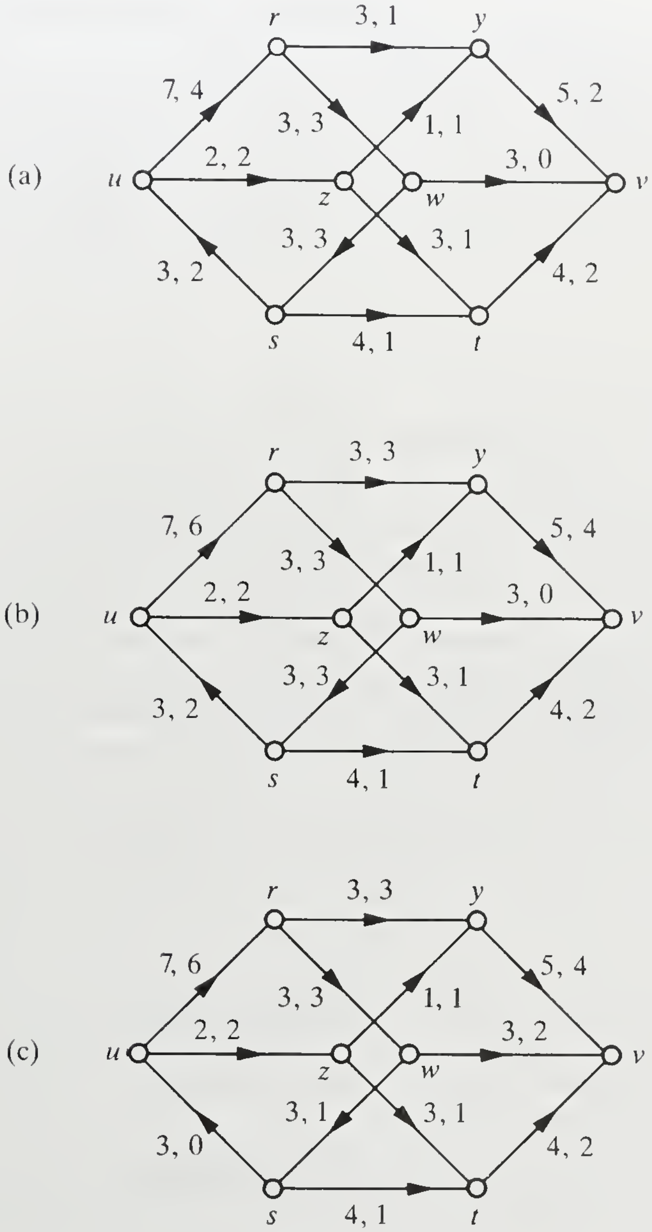


Figure 5.6

However, once we "scan r", i.e., remove r from L, the list L is empty and v is unlabeled. Thus we have a maximum flow, and the corresponding minimum cut is $(X, \bar{X})$, where $X = \{u, r\}$.

The definitions of network and flow in a network may be altered slightly to allow nonnegative real-valued capacity functions and flows. The max-flow min-cut theorem remains valid in this situation, although the aforementioned Ford and Fulkerson algorithm may not produce a maximum flow in such a case. However, the algorithm of Edmonds and Karp may still be used here.

Although the networks considered so far have (by definition) a single source and a single sink, a generalized structure with multiple sources and sinks is frequently more practical.

We define a (*generalized*) *network* N as a digraph D with two distinguished sets of vertices S and T, the *sources* and *sinks* of N, respectively, and a nonnegative integer-valued function c on $E(D)$. The sets S and T are assumed to be nonempty and disjoint; the function c is the *capacity function* of N. A *flow* in N is an integer-valued function f on $E(D)$ such that

$$0 \leq f(a) \leq c(a) \quad \text{for every } a \in E(D),$$

and

$$f(x, V(D)) = f(V(D), x) \quad \text{for every } x \in V(D) - (S \cup T).$$

The *value* of the flow f, denoted val f, is given by

$$\text{val } f = f(S, V(D)) - f(V(D), S).$$

The problem of determining a maximum flow in a generalized network can be reduced to the case of a network with a single source and single sink, and thus easily handled by the theory developed in this section.

For other examples of extensions of networks, see [FF3].

Exercises 5.4

5.27 Verify Cases 2 and 3 in the proof of Theorem 5.14.

5.28 Show that the converse of Corollary 15.3b is true. That is, let N be a network with capacity function c, and show that if f is a maximum flow in N and $(X, \overline{X})$ is a minimum cut, then

$$f(a) = c(a) \quad \text{for every } a \in (X, \overline{X}),$$

and

$$f(a) = 0 \quad \text{for every } a \in (\overline{X}, X).$$

5.29 Let N be a network with capacity function c and suppose that $(X, \overline{X})$ is a minimum cut in N.

(a) Prove or disprove: If f_1 and f_2 are flows in N that agree on $(X, \overline{X})$ and $(\overline{X}, X)$, then f_1 and f_2 are maximum flows in N.

(b) Prove or disprove: If f_1 and f_2 are maximum flows in N, then f_1 and f_2 agree on $(X, \overline{X})$ and $(\overline{X}, X)$.

5.30 Begin with the flow in Figure 5.4 and use Algorithm 5A to find a maximum flow and the corresponding minimum cut in the given network.

5.5 Applications of the Max-Flow Min-Cut Theorem

As observed in Section 5.3, deleting the arcs of a cut "separates" the source and the sink of a network. Since Menger's Theorem and its edge analogue each deal with sets that separate two vertices in a graph, it is, perhaps, not surprising that these two theorems are closely related to the max-flow min-cut theorem. It is this relationship that we now consider.

Menger's Theorem states that if u and v are distinct nonadjacent vertices of a graph G, then the maximum number of internally disjoint u-v paths in G equals the minimum number of vertices of G that separate u and v. There are other theorems that are often referred to as "forms of Menger's Theorem". We have seen one such theorem, Theorem 5.10, which is the edge form of Menger's Theorem. The vertex and edge forms of Menger's Theorem have natural analogues in the directed case. All four of the aforementioned results can be proved either directly or indirectly using the max-flow min-cut theorem. In each case, the key is to construct an appropriate network from the given graph or digraph. Furthermore, Algorithm 5A can be used to determine, for example, the minimum number of vertices of a given graph G that separate nonadjacent vertices u and v of G as well as such a set of vertices.

We now introduce some additional terms involving separation. A set S of vertices of a digraph D (graph G) is said to be a *u-v separating set of vertices*, where $u, v \in V(D) - S$ (respectively, $u, v \in V(G) - S$) if every u-v path in D (in G) contains at least one vertex of S. Similarly, a set S of arcs (edges) of a digraph D (graph G) is said to be a *u-v separating set of arcs* (*u-v separating set of edges*) if every u-v path in D (in G) contains at least one element of S. Observe that if S is a set of vertices (edges) of a graph G and $u, v \in V(G)$, then S is a u-v separating set of vertices (edges) in G if and only if S separates u and v.

Theorem 5.16 (The Arc Form of Menger's Theorem) *If u and v are distinct vertices of a digraph D, then the maximum number of arc-disjoint u-v paths in D equals the minimum number of arcs in a u-v separating set of arcs in D.*

Proof If m denotes the maximum number of arc-disjoint u-v paths in D, and n equals the minimum number of arcs in a u-v separating set of arcs in D, then clearly $m \leq n$. Thus we need only show that $m \geq n$.

Let N be the network with underlying digraph D, source u, and sink v, whose capacity function c satisfies $c(a) = 1$ for every $a \in E(D)$. By Theorem 5.15, the value of a maximum flow f in N equals the capacity of a minimum cut K. The proof will be complete if we show that $n \leq \text{cap } K$ and val $f \leq m$.

Since K is a cut in N, the set K is a u-v separating set of arcs in D. Therefore $n \leq |K| = \text{cap } K$.

Since f is a nonnegative integer-valued function, $f(a) = 0$ or 1 for every

$a \in E(D)$. Let D_1 be the digraph obtained from D by deleting all arcs a such that $f(a) = 0$. Thus $f(a) = 1$ for each $a \in E(D_1)$. Since f is a flow in D, we know that $f(x, V(D)) = f(V(D), x)$ for each $x \in V(D) - \{u, v\}$, and that $f(u, V(D)) - f(V(D), u) = \text{val } f = f(V(D), v) - f(v, V(D))$. However, for each $w \in V(D)$, $f(w, V(D)) = \text{od}_{D_1} w$ and $f(V(D), w) = \text{id}_{D_1} w$. Therefore,

$$\text{id}_{D_1} x = \text{od}_{D_1} x \quad \text{if} \quad x \in V(D) - \{u, v\}$$

and

$$\text{od}_{D_1} u - \text{id}_{D_1} u = \text{val } f = \text{id}_{D_1} v - \text{od}_{D_1} v.$$

By Theorem 2.26, the digraph D_1, and hence D, contains val f arc-disjoint u-v paths. Therefore $m \geqslant val\ f$, and so

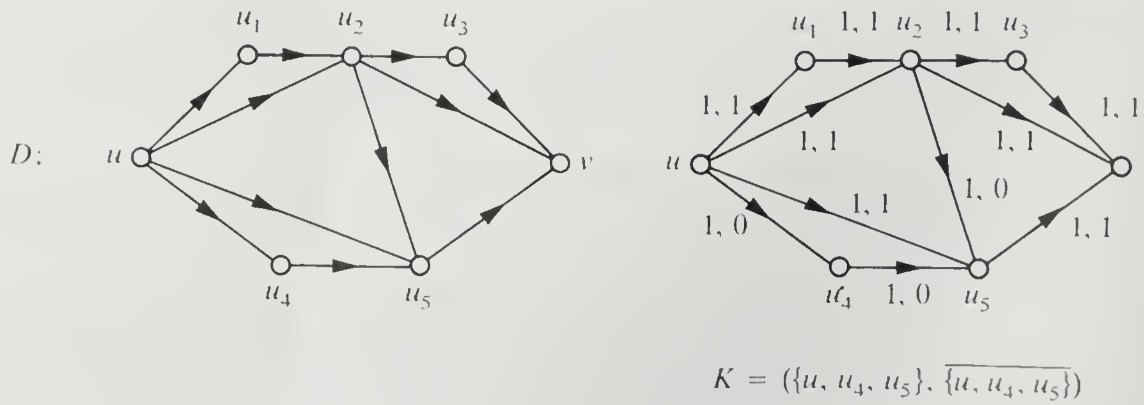
$$m = \text{val } f = \text{cap } K = n. \blacksquare$$

To determine, for example, the maximum number m of arc-disjoint u-v paths in the digraph D of Figure 5.7(a), we apply Algorithm 5A to the network N with underlying digraph D, source u, sink v, and whose capacity function satisfies $c(a) = 1$ for every $a \in E(D)$. We obtain the maximum flow f and minimum cut K indicated in Figure 5.7(b), so that $m = \text{val } f = \text{cap } K = 3$. In fact, f and K indicate a maximum set of arc-disjoint u-v paths in D and a minimum u-v separating set of arcs, namely

$$\{P_1: u, u_1, u_2, u_3, v,\ P_2: u, u_2, v,\ P_3: u, u_5, v\}$$

and $\{(u, u_1), (u, u_2), (u_5, v)\}$, respectively.

An alternate proof of the edge form of Menger's Theorem (see Theorem 5.10) can be obtained by applying Theorem 5.16 to the symmetric digraph D



(a) (b)

Figure 5.7

corresponding naturally to a given graph G, and then verifying that if u, $v \in V(G)$, there is a one-to-one correspondence between the u-v paths in G and the (directed) u-v paths in D.

Theorem 5.17 (The Edge Form of Menger's Theorem) *If u and v are distinct vertices of a graph G, then the maximum number of edge-disjoint u-v paths in G equals the minimum number of edges in a u-v separating set of edges in G.*

We now turn to vertex forms of Menger's Theorem. Two u-v paths P and Q in a digraph are called *internally disjoint* if $V(P) \cap V(Q) = \{u, v\}$.

Theorem 5.18 (The Directed Vertex Form of Menger's Theroem) *If u and v are distinct vertices of a digraph D such that u is not adjacent to v, then the maximum number of internally disjoint u-v paths in D equals the minimum number of vertices in a u-v separating set of vertices in D.*

Proof If m is the maximum number of internally disjoint u-v paths in D and n is the minimum number of vertices in a u-v separating set of vertices in D, then obviously $m \leq n$. Thus we must show that $m \geq n$.

Let $\tilde{D}$ be the digraph obtained from D in the following manner. Replace each $t \in V(D) - \{u, v\}$ with a pair of vertices, t' and t'', together with the arc (t', t''). If $(x, y) \in E(D)$ and $\{x, y\} \cap \{u, v\} = \varnothing$, then replace (x, y) with (x'', y'). Replace (u, x) with (u, x') if $(u, x) \in E(D)$ and replace (x, u) with (x'', u) if $(x, u) \in E(D)$ with $x \neq v$. Similarly, replace (x, v) with (x'', v) if $(x, v) \in E(D)$ and replace (v, x) with (v, x') if $(v, x) \in E(D)$ and $x \neq u$. By Theorem 5.16, if $\tilde{m}$ is the maximum number of arc-disjoint u-v paths in $\tilde{D}$ and $\tilde{n}$ is the minimum number of arcs in a u-v separating set of arcs in $\tilde{D}$, then $\tilde{n} = \tilde{m}$. Thus it suffices to show that $n \leq \tilde{n}$ and $\tilde{m} \leq m$.

Let A be a u-v separating set of arcs in $\tilde{D}$ with $|A| = \tilde{n}$. Observe that A contains no arcs incident to u or incident from v. For each $a \in A$, let w_a be defined as follows. Since $a \in A$, either $a = (u, x')$, $a = (x', x'')$, $a = (y'', x')$, or $a = (x'', v)$, where $x, y \in V(D) - \{u, v\}$. In any case, let $w_a = x$, and let $W = \{w_a | a \in A|$. Then $W \subseteq (D) - \{u, v\}$, and $|W| \leq |A| = \tilde{n}$. Furthermore, that W is a u-v separating set of vertices is easily verified. Therefore, $n \leq |W| \leq \tilde{n}$.

Let $\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_{\tilde{m}}$ be a collection of $\tilde{m}$ arc-disjoint u-v paths in $\tilde{D}$. Each $\tilde{P}_i$ is of the form

$$\tilde{P}_i\colon u, x_1', x_1'', x_2', x_2'', \ldots, x_\ell', x_\ell'', v,$$

and gives rise to a path P_i in D, where P_i: $u, x_1, x_2, \ldots, x_\ell, v$. By the way in which $\tilde{D}$ was constructed, if $i \neq j$, then P_i and P_j are (distinct) internally disjoint u-v paths in D. Thus, $m \geq \tilde{m}$, and so

$$m = \tilde{m} = \tilde{n} = n. \quad \blacksquare$$

If D is the digraph of Figure 5.8(a), then in order to determine the maximum number m of internally disjoint u-v paths, we first construct the digraph $\tilde{D}$ of Figure 5.8(b).

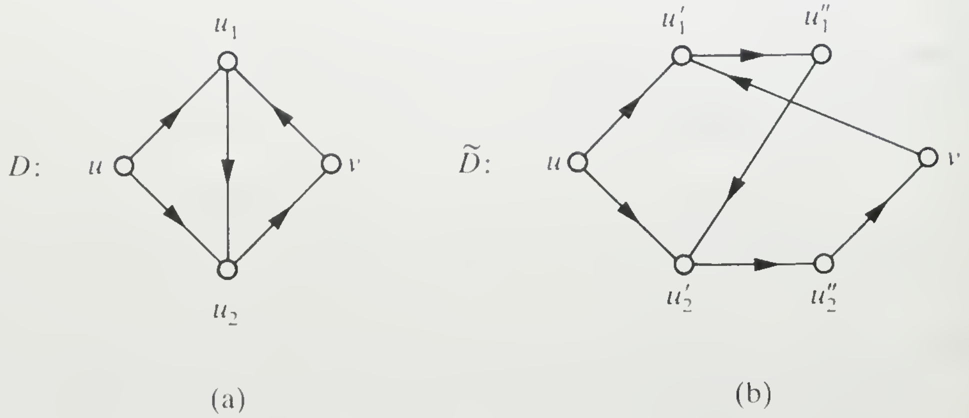


Figure 5.8

We then proceed as in the example following Theorem 5.16 (applying Algorithm 5A) to find that the maximum number of arc-disjoint u-v paths in $\tilde{D}$ is 1, and so $m = 1$. We can, in fact, use Algorithm 5A to find a maximum set of arc-disjoint u-v paths in $\tilde{D}$; namely, $\{P_1\colon u, u_2', u_2'', v\}$, and a minimum u-v separating set of arcs $\{(u_2', u_2'')\}$. This results in the u-v path u, u_2, v and the u-v separating set $\{u_2\}$ in D.

The undirected vertex form of Menger's Theorem can be obtained by applying Theorem 5.18 to the symmetric digraph that corresponds to a given graph.

Theorem 5.19 (The Undirected Vertex Form of Menger's Theorem) *If u and v are distinct nonadjacent vertices of a graph G, then the maximum number of internally disjoint u-v paths in G equals the minimum number of vertices in a u-v separating set of vertices in G.*

It is interesting to note that the directed vertex form of Menger's Theorem was the basic tool used by Robacker [R7] in a proof of the max-flow min-cut theorem.

Exercises 5.5

5.31 Use Theorem 5.16 to prove Theorem 5.17.

5.32 Use Theorem 5.18 to prove Theorem 5.19.

5.33 Describe a method to determine the connectivity (edge-connectivity) of a nonempty graph using Algorithm 5A.

Chapter Six

Hamiltonian Graphs and Digraphs

We now consider graphs and digraphs that contain spanning cycles, and possess even more stringent properties.

6.1 Hamiltonian Graphs

A graph G is defined to be *hamiltonian* if it has a cycle containing all the vertices of G. The name "hamiltonian" is derived from Sir William Rowan Hamilton, the well-known Irish mathematician. Surprisingly, though, Hamilton's relationship with the graphs bearing his name is not strictly mathematical (see [BLW1, p. 31]). In 1857, Hamilton introduced a game consisting of a solid regular dodecahedron made of wood, twenty pegs (one inserted at each corner of the dodecahedron), and a supply of string. Each corner was marked with an important city of the time. The aim of the game was to find a route along the edges of the dodecahedron that passed through each city exactly once and that ended at the city where the route began. In order for the player to recall which cities in a route he had already visited, the string was used to connect the appropriate pegs in the appropriate order. The dodecahedron proved to be rather awkward to manage so that Hamilton also produced a "planar graph" version of the game (see Figure 6.1). There is no indication that either version of the game proved successful.

The object of Hamilton's game may be described in graphical terms, namely, to determine whether the graph of the dodecahedron has a cycle containing each of its vertices. It is from this that we get the term "hamiltonian".

It is interesting to note that in 1855, two years before Hamilton introduced his game, the English mathematician Thomas P. Kirkman posed the following question in a paper submitted to the Royal Society. Given the graph of a polyhedron, can one always find a circuit that passes through each vertex once and only once? Thus, Kirkman apparently introduced the general study of "hamiltonian graphs" although Hamilton's game generated interest in the problem.

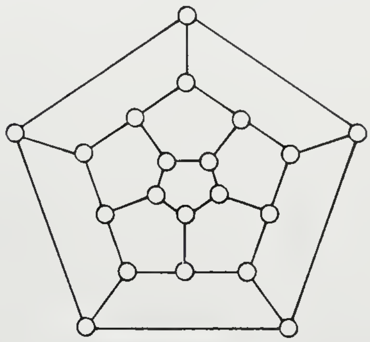


Figure 6.1 *The graph of the dodecahedron*

A cycle of a graph G containing every vertex of G is called a *hamiltonian cycle* of G; thus, a hamiltonian graph is one that possesses a hamiltonian cycle. Because of the similarity in the definitions of eulerian graphs and hamiltonian graphs, and because a particularly useful characterization of eulerian graphs exists, one might well expect an analogous criterion for hamiltonian graphs. However, such is not the case; indeed it must be considered one of the major unsolved problems of graph theory to develop an applicable characterization of hamiltonian graphs.

There have been several sufficient conditions established for a graph to be hamiltonian. We consider some of these in this section. The following result is due to Ore [O1].

Theorem 6.1 *If G is a graph of order $p \geq 3$ such that for all distinct nonadjacent vertices u and v,*

$$\deg u + \deg v \geq p,$$

then G is hamiltonian.

Proof Assume the theorem is false. Hence there exists a maximal nonhamiltonian graph G of order $p \geq 3$ that satisfies the hypothesis of the theorem; that is, G is nonhamiltonian and for every two nonadjacent vertices w_1 and w_2 of G, the graph $G + w_1w_2$ is hamiltonian. Since $p \geq 3$, G is not complete.

Let u and v be two nonadjacent vertices of G. Thus, $G + uv$ is hamiltonian, and, furthermore, every hamiltonian cycle of $G + uv$ contains the edge uv. Thus there is a u-v path P: $u=u_1, u_2, \ldots, u_p=v$ in G containing every vertex of G.

If $u_1u_i \in E(G)$, $2 \leq i \leq p$, then $u_{i-1}u_p \notin (G)$; for otherwise,

$$u_1, u_i, u_{i+1}, \ldots, u_p, u_{i-1}, u_{i-2}, \ldots, u_1$$

is a hamiltonian cycle of G. Hence for each vertex of $\{u_2, u_3, \ldots, u_p\}$ adjacent to u_1 there is a vertex of $\{u_1, u_2, \ldots, u_{p-1}\}$ not adjacent with u_p. Thus, $\deg u_p \leq (p-1) - \deg u_1$ so that

$$\deg u + \deg v \leq p-1.$$

This presents a contradiction, so G is hamiltonian. ■

If a graph G is hamiltonian, then certainly so is the graph $G+uv$, where u and v are distinct nonadjacent vertices of G. Conversely, suppose that G is a graph of order p with nonadjacent vertices u and v such that $G+uv$ is hamiltonian; furthermore, suppose that $\deg_G u + \deg_G v \geq p$. If G is not hamiltonian, then, as in the proof of Theorem 6.1, we arrive at the contradiction that $\deg_G u + \deg_G v \leq p-1$. Hence we have the following result, which was first observed by Bondy and Chvátal [BC1].

Theorem 6.2 *Let u and v be distinct nonadjacent vertices of a graph G of order p such that $\deg u + \deg v \geq p$. Then $G+uv$ is hamiltonian if and only if G is hamiltonian.*

Theorem 6.2 motivates our next definition. The *closure* of a graph G of order p, denoted by $C(G)$, is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least p (in the resulting graph at each stage) until no such pair remains. Figure 6.2 illustrates the closure function. That $C(G)$ is well-defined is established next.

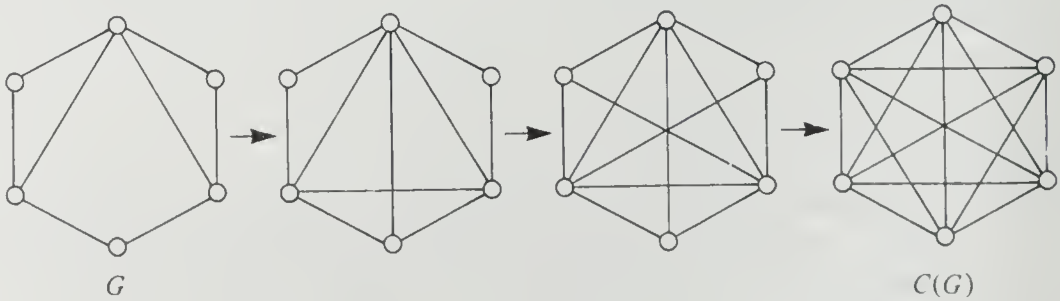


Figure 6.2 *The closure function*

Theorem 6.3 *If G_1 and G_2 are two graphs obtained from a graph G of order p by recursively joining pairs of nonadjacent vertices whose degree sum is at least p, then $G_1 = G_2$.*

Proof Let $e_1, e_2, \ldots, e_j$ and $f_1, f_2, \ldots, f_k$ be the sequences of edges added to G to obtain G_1 and G_2, respectively. It suffices to show that each e_i is an edge of $G_2 (1 \leq i \leq j)$ and that each f_i is an edge of $G_1 (1 \leq i \leq k)$. Assume, to the contrary, that this is not the case. Thus we may assume, without loss of generality, that for some m satisfying $0 \leq m \leq j-1$, the edge $e_{m+1} = uv$ does not belong to G_2; furthermore, $e_i \in E(G_2)$ for $i \leq m$. Let G_3 be the graph obtained from G by adding the edges $e_1, e_2, \ldots, e_m$. It follows from the definition of G_1 that $\deg_{G_3} u + \deg_{G_3} v \geq p$. Since $G_3 \subset G_2$, we have that $\deg_{G_2} u + \deg_{G_2} v \geq p$. This is a contradiction, however, since u and v are nonadjacent vertices of G_2. Thus each e_i is an edge of G_2 and each f_i is an edge of G_1; that is, $G_1 = G_2$. ■

Our next theorem is a simple consequence of the definition of closure and Theorem 6.2.

Theorem 6.4 *A graph is hamiltonian if and only if its closure is hamiltonian.*

Since each complete graph with at least three vertices is hamiltonian, we obtain a sufficient condition for a graph to be hamiltonian.

Theorem 6.5 (Bondy and Chvátal) *Let G be a graph with at least three vertices. If $C(G)$ is complete, then G is hamiltonian.*

According to Theorem 6.4, if the closure $C(G)$ of a graph G is hamiltonian, so too is G. We next present a good algorithm, also due to Bondy and Chvátal, that describes how to determine a hamiltonian cycle of G given a hamiltonian cycle of $C(G)$. [If, for example, $C(G)$ is complete, then it is trivial to find a hamiltonian cycle of $C(G)$.] The key step of this algorithm is the following modification process.

Suppose C is a hamiltonian cycle of some graph H of order $p \geq 4$ and that e is an edge of C. Label the vertices of H so that C: $v_1, v_2, \ldots, v_p, v_1$ and $e = v_1 v_p$. If there is an integer $i (3 \leq i \leq p-1)$ such that $v_1 v_i$ and $v_{i-1} v_p$ are edges of H, then C': $v_1, v_2, \ldots, v_{i-1}, v_p, v_{p-1}, \ldots, v_i, v_1$ is a hamiltonian cycle of H and $E(C') = E(C) - \{v_1 v_p, v_{i-1} v_i\} \cup \{v_1 v_i, v_{i-1} v_p\}$. We say that $v_1 v_i$ and $v_{i-1} v_p$ are *modifying edges with respect to* (C, e) and that C' is obtained by *modifying C via* $(v_1 v_p, v_1 v_i, v_{i-1} v_p)$.

In the following algorithm, we assume that in constructing $C(G)$ each edge e of G is labeled $\ell(e) = 1$ and that if f is the kth edge added to G, then f is labeled $\ell(f) = k+1$. (This labeling, of course, need not be unique.)

Algorithm 6A (Bondy and Chvátal) *Given the (edge-labeled) closure $C(G)$ of a graph G and a hamiltonian cycle C of $C(G)$:*

1. *Set* $m = \max_{e \in E(C)} \{\ell(e)\}$. *If $m = 1$, then stop; otherwise, let e be the unique edge of C labeled m and go to Step 2.*
2. *Select edges e_1 and e_2* of $C(G)$ *satisfying:*
   (a) $\ell(e_1), \ell(e_2) < m$;
   (b) e_1 and e_2 are modifying edges with respect to (C, e).
3. *Modify C via (e, e_1, e_2).*
4. *Go to Step 1.*

Theorem 6A *Algorithm 6A terminates with a hamiltonian cycle C of G.*

Proof We first show that Step 2 can always be completed. Let $p = p(G)$ and label the vertices of $C(G)$ so that C: $v_1, v_2, \ldots, v_p, v_1$ and $e = v_1v_p$. Let G' denote the spanning subgraph of $C(G)$ with edge set $E(G) \cup \{f \in E(C(G)) \mid \ell(f) < m\}$. Then $C - e$ is a subgraph of G' and, by the way in which $C(G)$ was constructed,

$$\deg_{G'} v_1 + \deg_{G'} v_p \geq p.$$

If follows (from an argument similar to the proof of Theorem 6.1) that there is an integer $i (3 \leq i \leq p-1)$ such that v_1v_i and $v_{i-1}v_p$ are edges of G'; i.e., e_1 and e_2 are modifying edges with respect to (C, e) with $\ell(e_1), \ell(e_2) < m$.

In order to complete the proof, we observe that since the value of m decreases by at least 1 each time Step 1 is repeated, the algorithm does indeed terminate with $m = 1$; i.e., with C a hamiltonian cycle of G. ∎

If a graph G satisfies the conditions of Theorem 6.1, then $C(G)$ is complete and so, by Theorem 6.5, G is hamiltonian. Thus, Ore's Theorem is an immediate corollary of Theorem 6.5 (although, chronologically, it preceded the theorem of Bondy and Chvátal by several years). Prior to the Bondy-Chvátal result, a number of sufficient conditions for a graph to be hamiltonian appeared in the literature. All of these can be deduced from (and, of course, are corollaries of) Theorem 6.5. We present one of the best known of these, due to Pósa [P5]. The proof we present, however, is independent of Theorem 6.5 in an attempt to give the reader a better idea of the kind of proof technique that is more typical in this area of graph theory.

Theorem 6.6 (Pósa) *If G is a graph of order $p \geq 3$ such that for every integer j with $1 \leq j < p/2$, the number of vertices of degree not exceeding j is less than j, then G is hamiltonian.*

Proof Assume the theorem is false. Then there exists a nonhamiltonian graph G of minimum size satisfying the hypothesis of the theorem. Since $p \geq 3$, G is not complete.

Among all pairs of nonadjacent vertices of G, let v_1 and v_p be two nonadjacent vertices such that $\deg v_1 + \deg v_p$ is a maximum. Suppose $\deg v_1 \leq \deg v_p$. Because of the way G is defined, the graph $G + v_1v_p$ is hamiltonian; indeed, every hamiltonian cycle of $G + v_1v_p$ contains the edge v_1v_p. This implies that v_1 and v_p are the end-vertices of a path P: $v_1, v_2, \ldots, v_p$ in G containing every vertex of G. Now if a vertex v_i, $2 \leq i \leq p$, is adjacent to v_1, then v_{i-1} is not adjacent to v_p; for otherwise,

$$v_1, v_i, v_{i+1}, \ldots, v_p, v_{i-1}, v_{i-2}, \ldots, v_1$$

would be a hamiltonian cycle of G. Therefore for every vertex v_i, $2 \leq i \leq p$, adjacent to v_1, there is a vertex v_{i-1} not adjacent to v_p, so that there are at least $\deg v_1$ vertices in G that are not adjacent to v_p. Hence there are at most $p - 1 - \deg v_1$ vertices adjacent to v_p; consequently,

$$\deg v_1 \leq \deg v_p \leq p - 1 - \deg v_1$$

so that $\deg v_1 \leq (p-1)/2$. From the manner in which v_1 and v_p were chosen, it follows that $\deg v_{i-1} \leq \deg v_1$ for all vertices v_{i-1} not adjacent to v_p. Thus there are at least $\deg v_1$ vertices having degree not exceeding $\deg v_1$. However, $1 \leq \deg v_1 < p/2$, so by hypothesis, there are fewer than $\deg v_1$ vertices having degree not exceeding $\deg v_1$—a contradiction. ■

Perhaps the simplest sufficient condition is due to Dirac [D4]. This result is a corollary of each of Theorems 6.1, 6.5, and 6.6.

Corollary 6.6 (Dirac) *If G is a graph of order $p \geq 3$ such that $\deg v \geq p/2$ for every vertex v of G, then G is hamiltonian.*

According to Dirac's result, if a graph G of order $p \geq 3$ satisfies $\delta(G) \geq p/2$, then G is hamiltonian, i.e., G contains a cycle of length $p = 2(p/2)$. Dirac [D4] has also shown that if G is a 2-connected graph of order $p \geq 2d$ and $\delta(G) \geq d > 1$, then G contains a cycle of length at least $2d$. This result can be used to obtain the result of Nash-Williams [N2] that every r-regular graph ($r \geq 2$) of order $2r+1$ is hamiltonian. We note that such a graph G satisfies none of the sufficient conditions presented thus far since all of these conditions require the existence of vertices of degree at least $p/2$. In the same vein, Jackson [J1] has shown that every 2-connected r-regular graph of order at most $3r$ is hamiltonian.

The sufficient conditions for a graph to be hamiltonian that we have presented involve the degrees of the vertices of the graph. In order to present a result of a different nature, we define the independence number of a graph. An *independent set* of vertices of a graph G is a set of vertices of G whose elements are pairwise nonadjacent. The *independence number* $\beta(G)$ of G is the maximum cardinality among all independent sets of vertices of G. Our next result, due to Chvátal and Erdös [CE1], involves the independence number $\beta(G)$ and the connectivity $\kappa(G)$ of a graph G.

Theorem 6.7 *Let G be a graph with at least three vertices. If $\kappa(G) \geq \beta(G)$, then G is hamiltonian.*

Proof If $\beta(G)=1$, the result follows since G is complete. Hence we assume that $\beta(G) \geq 2$. Let $\kappa(G)=n$. Since $n \geq 2$, G contains at least one cycle. Among all cycles of G, let C be one of maximum length. By Theorem 5.8, there are at least n vertices on C. We wish to show that C is a hamiltonian cycle of G. Assume, to the contrary, that there is a vertex w of G that does not lie on C. Since $|V(C)| \geq n$, we may apply Theorem 5.7 to conclude that there are n paths $P_1, P_2, \ldots, P_n$ having initial vertex w that are pairwise disjoint, apart from w, and that share with C only their terminal vertices $v_1, v_2, \ldots, v_n$, respectively. For each $i=1, 2, \ldots, n$, let u_i be the vertex following v_i in some fixed cyclic ordering of C. No vertex u_i is adjacent to w in G; for otherwise we could replace the edge v_iu_i in C by the v_i-u_i path determined by the path P_i and the edge u_iw to obtain a cycle having length at least $|V(C)|+1$, which is impossible. Let $S=\{w, u_1, u_2, \ldots, u_n\}$. Since $|S|=n+1>\beta(G)$ and $wu_i \notin E(G)$ for $i=1, 2, \ldots, n$, there are integers j and k such that $u_ju_k \in E(G)$. Thus by deleting the edges v_ju_j and v_ku_k from C and adding the edge u_ju_k together with the paths P_j and P_k, we obtain a cycle of G that is longer than C. This produces a contradiction, so that C is a hamiltonian cycle of G. ■

A well-known problem of practical importance is the determination of a minimum hamiltonian cycle in a weighted complete graph G of order $p(\geq 3)$. This is referred to as the *Traveling Salesman Problem*. For example, if $V(G)=\{v_1, v_2, \ldots, v_p\}$ represents p cities and the weight $w(v_iv_j)$ of the edge v_iv_j indicates the cost of a nonstop flight between v_i and v_j, then the Traveling Salesman Problem asks for the minimum cost of a round trip passing through all the cities $v_i (1 \leq i \leq p)$. Such a round trip can be determined by inspecting all $(p-1)!$ hamiltonian cycles. Any algorithm that does this, however, is clearly not efficient; indeed, there is no known good algorithm to solve the Traveling Salesman Problem. Instead, we describe a good algorithm that produces a hamiltonian cycle in G which, though not necessarily a minimum one, is very close to the actual minimum. In order to apply this algorithm, the weighted graph G must satisfy the triangle inequality $w(v_iv_k) \leq w(v_iv_j) + w(v_jv_k)$, which is not normally an unusual assumption.

For the purpose of describing the next algorithm, we refer to a single vertex as a 1-cycle and a closed walk of length 2 as a 2-cycle.

Algorithm 6B *Given a complete graph G of order $p(\geq 3)$:*

1. *Select any vertex of G to form a 1-cycle C_1 in G.*
2. *If the n-cycle C_n of G is given and $1 \leq n < p$, then find a vertex v_n not on C_n that is closest to a vertex u_n on C_n, and go to Step 3. If $n = p$, then C is the desired hamiltonian cycle, and stop.*
3. *Let C_{n+1} be the $(n+1)$-cycle obtained by inserting v_n immediately before u_n in C_n, and return to Step 2.*

As an example, let G be a weighted complete graph with $V(G) = \{v_1, v_2, \ldots, v_6\}$ where the weights of its edges are given by the matrix $M = [m_{ij}]$ of Figure 6.3, with $m_{ij} = w(v_i v_j)$.

$$M = \begin{bmatrix} 0 & 3 & 3 & 2 & 7 & 3 \\ 3 & 0 & 3 & 4 & 5 & 5 \\ 3 & 3 & 0 & 1 & 4 & 4 \\ 2 & 4 & 1 & 0 & 5 & 5 \\ 7 & 5 & 4 & 5 & 0 & 4 \\ 3 & 5 & 4 & 5 & 4 & 0 \end{bmatrix}$$

Figure 6.3

We apply Algorithm 6B by defining $C_1 : v_3$. Since v_4 is the vertex not on C_1 closest to v_3, we have C_2: v_3, v_4, v_3. Now v_1 is the vertex not on C_2 closest to a vertex of C_2 (namely v_4) so that C_3: v_3, v_1, v_4, v_3. This procedure produces the following "cycles" of G:

$C_1 : v_3$
$C_2 : v_3, v_4, v_3$
$C_3 : v_3, v_1, v_4, v_3$
$C_4 : v_3, v_2, v_1, v_4, v_3$
$C_5 : v_3, v_2, v_6, v_1, v_4, v_3$
$C_6 : v_3, v_2, v_6, v_1, v_4, v_5, v_3.$

The weight of the hamiltonian cycle C_6 is 22. This compares with 18, which is the weight of a minimum hamiltonian cycle in G. Indeed, it can be shown that, in general, any hamiltonian cycle produced by Algorithm 6B is always less than twice the weight of a minimum hamiltonian cycle. (More information on this and other algorithms can be found in Tucker [T9].)

As we have already indicated, obtaining an applicable characterization of hamiltonian graphs remains an unsolved problem in graph theory. In view of the lack of success in developing such a characterization, it is not surprising that special subclasses of hamiltonian graphs have been singled out for investigation as well as certain classes of nonhamiltonian graphs. We now discuss several types of "highly hamiltonian" graphs and then briefly consider graphs that are "nearly hamiltonian."

A path in a graph G containing every vertex of G is called a *hamiltonian* path. A graph G is *hamiltonian-connected* if for every pair u, v of distinct vertices of G, there exists a hamiltonian u-v path. It is immediate that a hamiltonian-connected graph with at least three vertices is hamiltonian. We define the *$(p+1)$-closure* $C_{p+1}(G)$ of a graph G of order p to be the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least $p+1$ until no such pair remains. We then have the following analogue to Theorem 6.5, also due to Bondy and Chvátal [BC1].

Theorem 6.8 *Let G be a graph of order p. If $C_{p+1}(G)$ is complete, then G is hamiltonian-connected.*

Proof If $p=1$, the result is obvious; so we assume that $p \geq 2$. Let G be a graph of order p whose $(p+1)$-closure is complete, and let u and v be any two vertices of G. We show that G contains a hamiltonian u-v path, which will then give us the desired result.

Define the graph H to consist of G together with a new vertex w and the edges uw and vw. Note that H has order $p'=p+1$. We determine $C(H)$. Since

$$\deg_H x + \deg_H y \geq p'$$

for all nonadjacent vertices x and y of G, it follows that $\langle V(G)\rangle_{C(H)} \cong K_p$. Hence if $x \in V(G)$ and $xw \notin E(H)$, then $\deg_{C(H)} x + \deg_{C(H)} w \geq p'$; so $C(H) \cong K_{p'}$. By Theorem 6.5, H is hamiltonian. Any hamiltonian cycle C of H necessarily contains the edges uw and vw, implying that G has a hamiltonian u-v path. ■

Two immediate corollaries now follow, the first of which is due to Ore [O3].

Corollary 6.8a *If G is a graph of order p such that for all distinct nonadjacent vertices u and v,*

$$\deg u + \deg v \geq p+1,$$

then G is hamiltonian-connected.

Corollary 6.8b *If G is a graph of order p such that* $\deg v \geq (p+1)/2$ *for every vertex v of G, then G is hamiltonian-connected.*

A number of other sufficient conditions for a graph to be hamiltonian-connected can be deduced from Theorem 6.8. One of these is the analogue to Theorem 6.6 (see [CKK1]).

Corollary 6.8c *If G is a graph of order $p \geq 3$ such that for every integer j with $2 \leq j \leq p/2$, the number of vertices of degree not exceeding j is less than $j-1$, then G is hamiltonian-connected.*

A connected graph G of order p is said to be *panconnected* if for each pair u, v of distinct vertices of G, there exists a u-v path of length ℓ for each ℓ satisfying $d(u, v) \leq \ell \leq p-1$. If a graph is panconnected, then it is hamiltonian-connected; the next example indicates that these concepts are not equivalent.

For $k \geq 3$, let G_k be that graph such that $V(G_k) = \{v_1, v_2, \ldots, v_{2k}\}$ and $E(G) = \{v_i v_{i+1} | i = 1, 2, \ldots, 2k\} \cup \{v_i v_{i+3} | i = 2, 4, \ldots, 2k-4\} \cup \{v_1 v_3, v_{2k-2} v_{2k}\}$, where all subscripts are to be taken modulo $2k$. Although for each pair u, v of distinct vertices and for each ℓ satisfying $k \leq \ell \leq 2k-1$, the graph G_k contains a u-v path of length ℓ, there is no v_1-v_{2k} path of length ℓ if $1 < \ell < k$. Since $d(v_1, v_{2k}) = 1$, it follows that G_k is not panconnected.

A sufficient condition [W7] for a graph G to be panconnected can be given in terms of the minimum degree of G.

Theorem 6.9 *If G is a graph of order $p \geq 4$ such that* $\deg v \geq (p+2)/2$ *for every vertex v of G, then G is panconnected.*

Proof If $p = 4$, then $G \cong K_4$ and the statement is true.

Suppose that the theorem is false. Thus there exists a graph G of order $p \geq 5$ with $\delta(G) \geq (p+2)/2$ that is not panconnected; that is, there are vertices u and v of G that are joined by no path of length ℓ for some ℓ satisfying $d(u, v) \leq \ell \leq p-1$. Let $G^* = G - \{u, v\}$. Then G^* has order $p^* = p - 2 \geq 3$ and $\delta(G^*) \geq (p+2)/2 - 2 = p^*/2$. Therefore by Corollary 6.6, the graph G^* contains a hamiltonian cycle C: $v_1, v_2, \ldots, v_{p^*}, v_1$.

If $uv_i \in E(G)$, $1 \leq i \leq p^*$, then $vv_{i+\ell-2} \notin E(G)$, where the subscripts are to be taken modulo p^*; for otherwise,

$$u, v_i, v_{i+1}, \ldots, v_{i+\ell-2}, v$$

is a u-v path of length ℓ in G. Thus for each vertex of C that is adjacent with u in G, there is a vertex of C that is not adjacent with v in G. Since $\deg_G u \geq (p+2)/2$, we conclude that u is adjacent with at least $p/2$ vertices of C, so that

$$\deg_G v \leq 1 + p^* - \frac{p}{2} = \frac{p}{2} - 1.$$

This, however, produces a contradiction. ■

A graph G of order p is called *pancyclic* if G contains a cycle of length ℓ for each ℓ satisfying $3 \leq \ell \leq p$. We say that G is *vertex-pancyclic* if for each vertex v of G and for every integer ℓ satisfying $3 \leq \ell \leq p$, there is a cycle of G of length ℓ that contains v. Certainly every pancyclic graph is hamiltonian, as is every vertex-pancyclic graph, although the converse is not true. The next theorem, due to Bondy [B14], gives a sufficient condition for a hamiltonian graph to be pancyclic. In order to present a proof due to C. Thomassen, a preliminary definition will be useful.

Let G be a hamiltonian graph and C: $v_1, v_2, \ldots, v_p, v_1$ a hamiltonian cycle of G. With respect to this cycle, every edge of G either lies on C or joins two nonconsecutive vertices of C and is referred to as a *diagonal*. Any cycle of G containing precisely one diagonal is an *outer cycle* of G (with respect to the fixed hamiltonian cycle C).

Theorem 6.10 *If G is a hamiltonian (p, q) graph, where $q \geq p^2/4$, then either G is pancyclic or p is even and $G \cong K(p/2, p/2)$.*

Proof We first show that if G is a hamiltonian (p, q) graph, where $p \geq 4$ and $q \geq p^2/4$, and G contains no $(p-1)$-cycle, then p is even and $G \cong K(p/2, p/2)$.

Let C: $v_1, v_2, \ldots, v_p, v_1$ be a hamiltonian cycle of G and let v_j and v_{j+1} be any two consecutive vertices of C (where all subscripts are taken modulo p). If $1 \leq k \leq p$ but $k \neq j-1$ and $k \neq j$, then at most one of v_jv_k and $v_{j+1}v_{k+2}$ is an edge of G; otherwise,

$$v_{j+1}, v_{j+2}, \ldots, v_k, v_j, v_{j-1}, v_{j-2}, \ldots, v_{k+2}, v_{j+1}$$

is a $(p-1)$-cycle of G. Thus for each of the $\deg v_j - 1$ vertices in $V(G) - \{v_{j-1}, v_j\}$ that is adjacent to v_j, there is a vertex in $V(G) - \{v_{j+1}, v_{j+2}\}$ that is not adjacent to v_{j+1}. Thus $\deg v_{j+1} \leq (p-2) - (\deg v_j - 1) + 1$, so that

$$\deg v_j + \deg v_{j+1} \leq p. \tag{6.1}$$

Suppose p is odd. Then by (6.1) there is some vertex, say v_p, such that $\deg v_p \leq (p-1)/2$. But then

$$\begin{aligned} 2q &= \sum_{i=1}^{p-1} \deg v_i + \deg v_p \\ &\leq \frac{p(p-1)}{2} + \frac{(p-1)}{2} < \frac{p^2}{2}, \end{aligned}$$

which contradicts the fact that $q \geq p^2/4$. Thus p is even and $2q = \Sigma_{i=1}^{p} \deg v_i \leq p^2/2$, so that $q \leq p^2/4$. Since $q \geq p^2/4$, we have that $q = p^2/4$. This implies that equality is attained in (6.1) for each j. Therefore,

$$v_jv_k \in E(G) \quad \text{if and only if} \quad v_{j+1}v_{k+2} \notin E(G) \qquad k \neq j-1, j. \tag{6.2}$$

Suppose that $G \not\cong K(p/2, p/2)$. Since $q = p^2/4$, by Exercise 2.5, G has an odd cycle. This implies that G contains an outer cycle of odd length. Let $v_j, v_{j+1}, \ldots, v_{j+m}, v_j$ be a shortest outer cycle of G of odd length $m+1$ where, then, m is even and $4 \leq m \leq p-4$ since G contains no $(p-1)$-cycle. Since

$v_j v_{j+m} \in E(G)$, by (6.2), $v_{j-1}v_{j+m-2} \notin E(G)$. Then, again by (6.2), $v_{j-2}v_{j+m-4} \in E(G)$. Therefore $v_{j-2}, v_{j-1}, \ldots, v_{j+m-4}, v_{j-2}$ is an outer cycle of (odd) length $m-1$, a contradiction. Thus $G \cong K(p/2, p/2)$.

We now show by induction on p that if G is a hamiltonian (p, q) graph, where $q \geq p^2/4$, then either G is pancyclic or p is even and $G \cong K(p/2, p/2)$. If $p=3$, then $G \cong C_3$ and G is pancyclic. Assume for all hamiltonian graphs H of order $p-1$ (≥ 3) with at least $(p-1)^2/4$ edges that either H is pancyclic or $p-1$ is even and $H \cong K((p-1)/2, (p-1)/2)$. Let G be a hamiltonian (p, q) graph with $q \geq p^2/4$. Assume that either (a) p is even and $G \not\cong K(p/2, p/2)$ or (b) p is odd. We show that G is pancyclic. Under these assumptions, it follows from the first part of the proof that G contains a $(p-1)$-cycle C^*: $w_1, w_2, \ldots, w_{p-1}, w_1$. Let w be the single vertex of G not on C^*.

If $\deg w \geq p/2$, then for each integer m satisfying $3 \leq m \leq p$, the vertex w lies on an m-cycle of G; otherwise, whenever $ww_i \in E(G)$, $1 \leq i \leq p-1$, then $ww_\ell \notin E(G)$, where $\ell \equiv i+m-2 \pmod{p-1}$. This, however, implies that $\deg w \leq (p-1)/2$, a contradiction. Thus G is pancyclic if $\deg w \geq p/2$.

If $\deg w < p/2$, then $G-w$ is a hamiltonian graph of order $p-1$ with at least $p^2/4-(p-1)/2$ edges. Since $p^2/4-(p-1)/2 > (p-1)^2/4$, it follows that $G-w \not\cong K((p-1)/2, (p-1)/2)$. Applying the inductive hypothesis, we conclude that $G-w$ is pancyclic. Thus G is pancyclic and the proof is complete. ■

If the sum of the degrees of each pair of nonadjacent vertices of a graph G is at least p, where $p = |V(G)| \geq 3$, then by Theorem 6.1, G is hamiltonian. Our next result shows that the condition of Theorem 6.1 actually implies much more about the cycle structure of G.

Corollary 6.10 *Let G be a graph of order $p \geq 3$ such that for all distinct nonadjacent vertices u and v,*

$$\deg u + \deg v \geq p.$$

Then either G is pancyclic or p is even and $G \cong K(p/2, p/2)$.

Proof We need only show that $|E(G)| \geq p^2/4$, since G is hamiltonian by Theorem 6.1. Let k be the minimum degree among the vertices of G. If $k \geq p/2$, then clearly $|E(G)| \geq p^2/4$. Thus we may assume that $k < p/2$.

Let n denote the number of vertices of G of degree k. These n vertices induce a subgraph H that is complete; for if any two vertices of H were not adjacent, then there would exist two nonadjacent vertices the sum of whose degrees would be less than p. This implies that $n \leq k+1$. However, $n \neq k+1$; for otherwise, each vertex of H is adjacent only to vertices of H, which is impossible since G is connected.

Let u be a vertex of degree k. Since $n \leq k$, one of the k vertices adjacent to u has degree at least $k+1$, while each of the other $k-1$ vertices adjacent to u has degree at least k. If $w \neq u$ is one of the $p-k-1$ vertices of G that is not adjacent to u, then $\deg w + \deg u \geq p$, so that $\deg w \geq p-k$. Hence,

$$|E(G)| = \frac{1}{2} \sum_{v \in V(G)} \deg v \geq \frac{1}{2}[(p-k-1)(p-k) + k^2 + k + 1]$$
$$= \frac{1}{2}[2k^2 + (2-2p)k + (p^2 - p + 1)] \geq \frac{p^2+1}{4},$$

the last inequality holding since for $k \leq (p-1)/2$, the expression $\frac{1}{2}[2k^2 + (2-2p)k + (p^2-p+1)]$ takes on its minimum value when $k = (p-1)/2$. ■

The results presented in the preceding theorem and corollary have been extended in [HFS1]. It is interesting to note that many other conditions that imply that a graph is hamiltonian have been shown to imply that either the graph is pancyclic or else belongs to a simple family of exceptional graphs.

We close this section with a brief discussion of nonhamiltonian graphs that are, in certain senses, "nearly hamiltonian". Of course, if G is hamiltonian, then G has a hamiltonian path. Sufficient conditions for a graph to possess a hamiltonian path can be obtained from the sufficient conditions for a graph to be hamiltonian. For example, suppose that G is a graph of order $p \geq 2$ such that for all distinct nonadjacent vertices u and v, we have $\deg u + \deg v \geq p-1$. Then the graph $G + K_1$ satisfies the hypothesis of Theorem 6.1 and so is hamiltonian. This, of course, implies that G contains a hamiltonian path.

If a graph G of order p has a hamiltonian path, then the length of a longest path in G is $p-1$. The next result involves longest paths in graphs that are not hamiltonian.

Theorem 6.11 *Let G be a connected graph of order 3 or more that is not hamiltonian. If for all distinct nonadjacent vertices u and v,*

$$\deg u + \deg v \geq m,$$

where m is a positive integer, then G contains a path of length m.

Proof Let $P: u_0, u_1, \ldots, u_k$ be a longest path in G. Since P is a longest path, each of u_0 and u_k is adjacent only to vertices of P.

If $u_0 u_i \in E(G)$, $1 \leq i \leq k$, then $u_{i-1}u_k \notin E(G)$; for otherwise the cycle

$$C: u_0, u_1, \ldots, u_{i-1}, u_k, u_{k-1}, \ldots, u_i, u_0$$

of length $k+1$ is present in G. The cycle C cannot contain all vertices of G since G is not hamiltonian. Therefore there exists a vertex w not on C adjacent with a vertex of C; however, this implies that G contains a path of length $k+1$, which is impossible. Hence u_0 and u_k are nonadjacent vertices of G. Furthermore, for each vertex of $\{u_1, u_2, \ldots, u_k\}$ adjacent to u_0 there is a vertex of $\{u_0, u_1, \ldots, u_{k-1}\}$ not adjacent with u_k. Thus $\deg u_k \leq k - \deg u_0$ so that

$$k \geq \deg u_0 + \deg u_k \geq m. \blacksquare$$

A graph G is *hypohamiltonian* if it is not hamiltonian but $G - v$ is hamiltonian for every vertex v of G. Herz, Gaudin, and Rossi [HGR1] showed that hypohamiltonian graphs exist; in fact, they showed that the Petersen graph (see Figure 2.7) is the hypohamiltonian graph of smallest order. It was further shown by Herz, Duby, and Vigué [HDV1] that there is no hypohamiltonian graph of order 11 or 12 and that there is a nonregular hypohamiltonian graph of order 13. That there are infinitely many hypohamiltonian graphs is verified by the class of graphs independently discovered by Lindgren [L2] and Sousselier (see [HDV1]). Results by Chvátal [C4] and Thomassen [T2] have shown hypohamiltonian graphs of every order exist with only a small number of exceptions.

Exercises 6.1

6.1 Show that if a graph G is hamiltonian, then for every proper subset S of $V(G)$,

$$k(G - S) \leq |S|.$$

6.2 (a) Prove that $K(n, 2n, 3n)$ is hamiltonian for every positive integer n.

(b) Prove that $K(n, 2n, 3n + 1)$ is hamiltonian for no positive integer n.

6.3 (a) Prove that if G and H are hamiltonian graphs, then $G \times H$ is hamiltonian.

(b) Prove that the n-cube Q_n, $n \geq 2$, is hamiltonian.

6.4 Let G be a graph with $\delta(G) \geq 2$. Prove that G contains a cycle of length at least $1 + \delta(G)$.

6.5 Let G be a (p, q) graph, where $p \geq 3$ and $q \geq (p^2 - 3p + 6)/2$. Prove that G is hamiltonian.

6.6 Let G be a bipartite graph with partite sets U and W such that $|U| = |W| = n \geq 2$. Prove that if $\deg v > n/2$ for every vertex v of G, then G is hamiltonian.

6.7 Let G be a graph of order $p \geq 3$ that contains n vertices of degree $p - 1$. Prove that if $n \geq \beta(G)$, then G is hamiltonian.

6.8 In the weighted graph defined by the matrix of Figure 6.3, apply Algorithm 6.3 by defining C_1: v_i for each $i \neq 3$.

6.9 The nth *power* G^n of a connected graph G, where $n \geq 1$, is that graph with $V(G^n) = V(G)$ for which $uv \in E(G^n)$ if and only if $1 \leq d_G(u, v) \leq n$. The graphs G^2 and G^3 are also referred to as the *square* and *cube*, respectively, of G.

Although it is not true that the square of each connected graph of order 3 or more is hamiltonian, it was conjectured independently by C. Nash-Williams and M. D. Plummer that for 2-connected graphs, this is the case. In 1974, Fleishner [F3] proved the conjecture to be correct.

(a) Give an example of a connected graph G of order 3 or more such that G^2 is not hamiltonian.

(b) Let T be a tree of order 3 or more. Prove that T^3 is hamiltonian.

(c) Let G be a graph of order 3 or more. Prove that G^3 is hamiltonian.

6.10 Give a direct proof of Corollary 6.6 (without using Theorem 6.1 or 6.6).

6.11 Show that if G is a (p, q) graph, where $p \geq 4$ and $q \geq \binom{p-1}{2} + 3$, then G is hamiltonian-connected.

6.12 Let G be a hamiltonian-connected graph of order 4 or more. Prove that G is 3-connected.

6.13 Give an example of a graph G of order 8 that is pancyclic but not panconnected.

6.14 Prove or disprove: If G is any graph of order $p \geq 4$ such that for all distinct nonadjacent vertices u and v,

$$\deg u + \deg v \geq p + 2,$$

then G is panconnected.

6.2 Hamiltonian Planar Graphs

We have encountered many sufficient conditions for a graph to be hamiltonian. None of these, however, is also a necessary condition. For example, the cycle C_n, the simplest hamiltonian graph of all, does not satisfy any of these conditions when n is large. In this section we are able to reverse our point of view in the case of planar graphs and consider a necessary condition for a graph to be hamiltonian.

Let G be a hamiltonian plane graph of order p and let C be a fixed hamiltonian cycle in G. With respect to this cycle, a diagonal is, as before, an edge of G that does not lie on C. Let $r_i (i = 3, 4, \ldots, p)$ denote the number of regions of G in the interior of C whose boundary contains exactly i edges; similarly, let r_i' denote the number of regions of G in the exterior of C whose boundary contains i edges. To illustrate these definitions, let G be the plane graph of Figure 6.4 with hamiltonian cycle C: $v_1, v_6, v_7, v_8, v_9, v_{10}, v_5, v_4, v_3, v_2, v_1$. Then $r_i = 0$ if $i \neq 4$ and $r_4 = 4$. Also, $r_i' = 0$ if $i \neq 4, 5$ while $r_4' = 1$ and $r_5' = 2$.

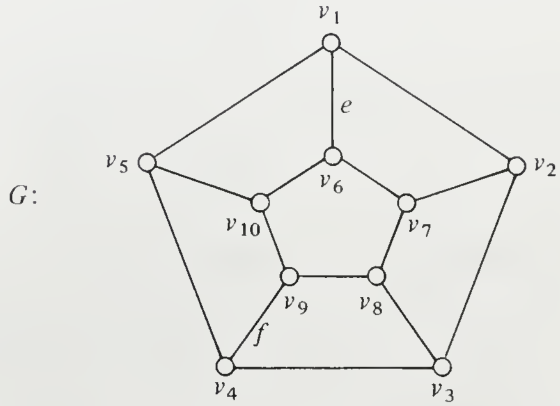


Figure 6.4 *A hamiltonian plane graph*

Using the notation of the previous paragraph, we have the following necessary condition, due to Grinberg [G4], for a plane graph to be hamiltonian.

Theorem 6.12 (Grinberg) *Let G be a plane graph of order p with hamiltonian cycle C. Then with respect to this cycle C,*

$$\sum_{i=3}^{p} (i-2)(r_i - r_i') = 0.$$

Proof We first consider the interior of C. If d denotes the number of diagonals of G in the interior of C, then exactly $d+1$ regions of G lie inside C. Therefore,

$$\sum_{i=3}^{p} r_i = d+1,$$

implying that

$$d = \left(\sum_{i=3}^{p} r_i\right) - 1. \tag{6.3}$$

Let the number of edges bounding a region interior to C be summed over all $d+1$ such regions, denoting the result by N. Hence $N = \Sigma_{i=3}^{p} i r_i$. However, N counts each interior diagonal twice and each edge of C once, so that $N = 2d + p$. Thus,

$$\sum_{i=3}^{p} i r_i = 2d + \text{p}. \tag{6.4}$$

Substituting (6.3) into (6.4) we obtain

$$\sum_{i=3}^{p} ir_i = 2\sum_{i=3}^{p} r_i - 2 + p,$$

so that

$$\sum_{i=3}^{p} (i-2)r_i = p - 2. \tag{6.5}$$

By considering the exterior of C, we conclude in a similar fashion that

$$\sum_{i=3}^{p} (i-2)r_i' = p - 2. \tag{6.6}$$

It follows from (6.5) and (6.6) that

$$\sum_{i=3}^{p} (i-2)(r_i - r_i') = 0. \blacksquare$$

The following observations often prove quite useful in applying Theorem 6.12. Let G be a plane graph with hamiltonian cycle C. Furthermore, suppose the edge e of G is on the boundary of two regions R_1 and R_2 of G. If e is an edge of C, then one of R_1 and R_2 is in the interior of C and the other is in the exterior of C. If, on the other hand, e is not an edge of C, then R_1 and R_2 are either both in the interior of C or both in the exterior of C.

In 1880, the English mathematician P.G. Tait conjectured that every 3-connected cubic planar graph is hamiltonian. This conjecture was disproved in 1946 by Tutte [T12], who produced the graph G in Figure 6.5 as a counterexample. In addition to disproving Tait's conjecture, Tutte [T14] proved that every 4-connected planar graph is hamiltonian. This result was later extended by Thomassen [T6].

Theorem 6.13 *Every 4-connected planar graph is hamiltonian-connected.*

As an illustration of Grinberg's theorem, we now verify that Tutte's graph is not hamiltonian.

Assume that the graph G of Figure 6.5, which has order 46, contains a hamiltonian cycle C. Observe that C must contain exactly two of the edges e, f_1, and f_2. Let G_1 denote the component of $G - \{e, f_1, f_2\}$ containing w.

Consider the regions R_1, R_2, and R_3 of G. Suppose that two of them, say R_1 and R_2, lie in the exterior of C. Then the edges f_1 and f_2 do not belong to C since the unbounded region of G also lies in the exterior of C. This, however, is impossible; thus at most one of the regions R_1, R_2, and R_3 lies in the exterior of C. We conclude that at least two of these regions, say R_1 and R_2, lie in the interior of C. This, of course, implies that their common boundary edge e does not belong to C. Therefore, f_1 and f_2 are edges of C. Furthermore, C contains a v_1-v_2 subpath P that is a hamiltonian path of G_1. Consider the graph

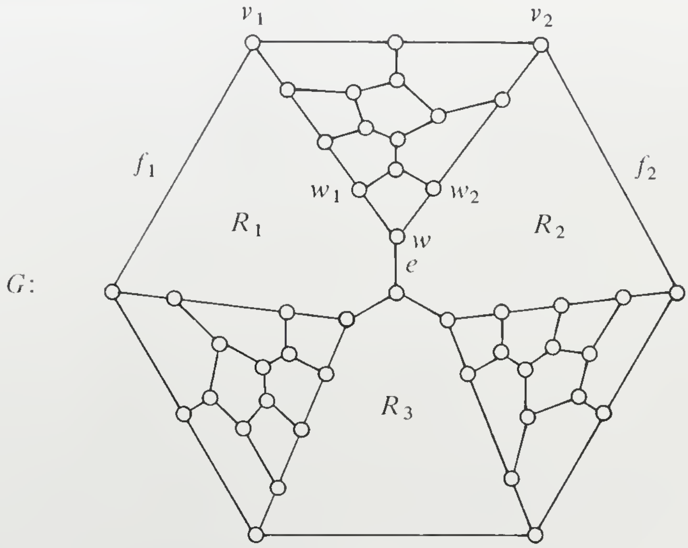


Figure 6.5 *The Tutte graph*

$G_2 = G_1 + v_1v_2$. Then G_2 has a hamiltonian cycle C_2 consisting of P together with the edge v_1v_2.

An application of Theorem 6.12 to G_2 and C_2 yields that

$$1(r_3 - r_3') + 2(r_4 - r_4') + 3(r_5 - r_5') + 6(r_8 - r_8') = 0. \tag{6.7}$$

Since v_1v_2 is an edge of C_2 and since the unbounded region of G_2 lies in the exterior of C_2, we have that

$$r_3 - r_3' = 1 - 0 = 1 \quad \text{and} \quad r_8 - r_8' = 0 - 1 = -1.$$

Therefore, from (6.7) we obtain

$$2(r_4 - r_4') + 3(r_5 - r_5') = 5.$$

Since $\deg_{G_2} w = 2$, both ww_1 and ww_2 are edges of C_2. This implies that $r_4 \geq 1$ so that

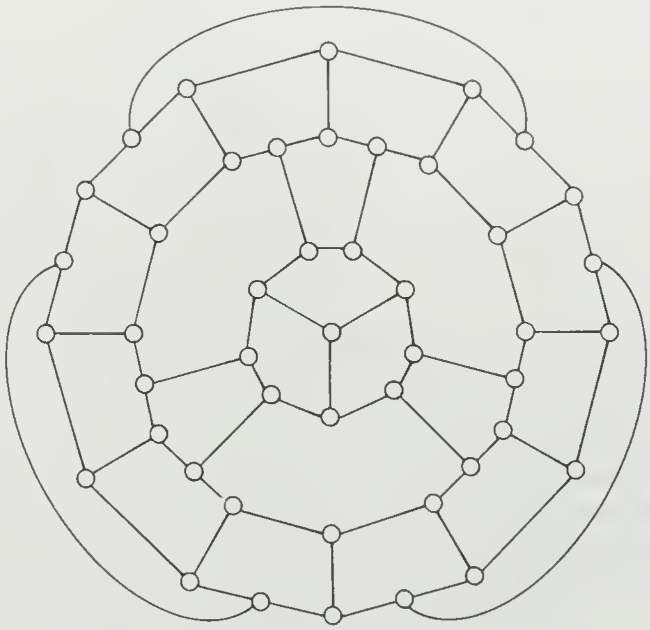
$$r_4 - r_4' = 1 - 1 = 0 \quad \text{or} \quad r_4 - r_4' = 2 - 0 = 2.$$

If $r_4 - r_4' = 0$, then $3(r_5 - r_5') = 5$, which is impossible. If, on the other hand, $r_4 - r_4' = 2$, then $3(r_5 - r_5') = 1$, again impossible. We conclude that Tutte's graph is not hamiltonian.

For many years Tutte's graph was the only known example of a 3-connected cubic planar graph that was not hamiltonian. More recently, however, other such graphs have been found; for example, Grinberg himself provided the graph in Exercise 6.15 as another counterexample to Tait's conjecture.

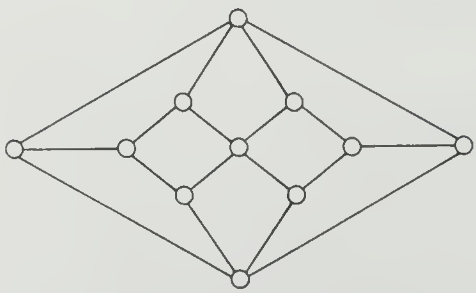
Exercises 6.2

6.15 Show, by applying Theorem 6.12, that the Grinberg graph (below) is non-hamiltonian.



The Grinberg graph

6.16 Show, by applying Theorem 6.12, that the Herschel graph (below) is non-hamiltonian.



The Herschel graph

6.17 Show, by applying Theorem 6.12, that no hamiltonian cycle in the graph of Figure 6.4 contains both the edges e and f.

6.3 Hamiltonian Digraphs

The concept of hamiltonian graphs has a most natural counterpart in directed graphs. A digraph D is called *hamiltonian* if it contains a spanning cycle; such a cycle is called a *hamiltonian cycle.* As with hamiltonian graphs, no characterization of hamiltonian digraphs exists. Indeed, if anything, the situation for hamiltonian digraphs is even more complex than it is for hamiltonian graphs. While there are sufficient conditions for a digraph to be hamiltonian, they are analogues of the simpler sufficient conditions for hamiltonian graphs.

We say that a vertex v is *reachable* from a vertex u in a digraph D if D contains a u-v path. A digraph D is called *strongly connected*, or more simply *strong*, if for every two distinct vertices of D, each vertex is reachable from the other. Clearly, every hamiltonian digraph is strong (though not conversely).

We state without proof the following theorem of Meyniel [M6] that gives a sufficient condition for a digraph to be hamiltonian. It should remind the reader of Ore's Theorem (Theorem 6.1).

Theorem 6.14 (Meyniel) *Let D be a strong nontrivial digraph of order p such that for every pair u, v of distinct nonadjacent vertices,*

$$\deg u + \deg v \geq 2p - 1.$$

Then D is hamiltonian.

The bound presented in Theorem 6.14 is sharp in the following sense. Let n and p be positive integers such that $2 \leq n \leq p-1$. Let D be the digraph obtained by identifying a vertex of (the complete symmetric digraph) K_n^* and a vertex of K_{p-n+1}^*. (See Figure 6.6 for the case where $n=3$ and $p=4$.) The

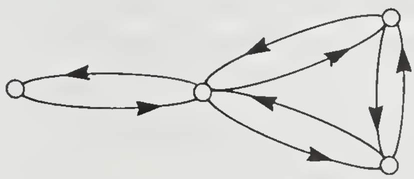


Figure 6.6 *A nonhamiltonian digraph*

strong digraph D of order p is nonhamiltonian and if u and v are nonadjacent vertices of D, then $\deg u + \deg v = 2p - 2$.

Theorem 6.14 has a large number of consequences. We consider these now, beginning with a result originally discovered by Woodall [W9].

Corollary 6.14a (Woodall) *Let D be a nontrivial digraph of order p such that whenever u and v are distinct vertices and $(u, v) \notin E(D)$, then*

$$\text{od}\, u + \text{id}\, v \geq p. \tag{6.8}$$

Then D is hamiltonian.

Proof First we show that condition (6.8) implies that D is strong. Let u and v be any two vertices of D. We show that v is reachable from u. If $(u, v) \in E(D)$, then this is obvious. If $(u, v) \notin E(D)$, then by (6.8), there must exist a vertex w in D, with $w \neq u, v$, such that $(u, w), (w, v) \in E(D)$. However, then u, w, v is a path in D and v is reachable from u. Therefore, D is strong.

To complete the proof we apply Meyniel's Theorem (Theorem 6.14). Let u and v be any two nonadjacent vertices of D. Then by (6.8), $\text{od}\, u + \text{id}\, v \geq p$ and $\text{od}\, v + \text{id}\, u \geq p$ so that $\deg u + \deg v \geq 2p$. Thus, by Theorem 6.14, D is hamiltonian. ■

The following well-known theorem is due to Ghouila-Houri [G3]. The proof is an immediate consequence of Theorem 6.14.

Corollary 6.14b (Ghouila-Houri) *Let D be a strong digraph such that* $\deg v \geq p$ *for every vertex v of D. Then D is hamiltonian.*

This result also has a rather immediate corollary.

Corollary 6.14c *Let D be a digraph such that*

$$\text{od}\, v \geq p/2 \quad \textit{and} \quad \text{id}\, v \geq p/2$$

for every vertex v of D. Then D is hamiltonian.

A spanning path in a digraph D is called a *hamiltonian path* of D. With the aid of Theorem 6.14, we can present some sufficient conditions for a digraph to possess a hamiltonian path.

Corollary 6.14d *Let D be a digraph of order p such that for every pair u, v of distinct nonadjacent vertices,*

$$\deg u + \deg v \geq 2p - 3. \tag{6.9}$$

Then D contains a hamiltonian path.

Proof We construct a new digraph D' from D by adding a new vertex w and joining w in both directions to every vertex of D. The digraph D' is necessarily strong. Let u and v be nonadjacent vertices of D'. Then u and v are nonadjacent vertices of D and by (6.9),

$$\deg_{D'} u + \deg_{D'} v \geq (2p - 3) + 4 = 2p + 1 = 2(p + 1) - 1.$$

Since D' has order $p+1$, it follows from Meyniel's Theorem that D' contains a hamiltonian cycle C. Deleting w and its incident arcs on C produces a hamiltonian path in D. ∎

Analogues to Corollaries 6.14a, 6.14b, and 6.14c for digraphs poss hamiltonian paths are now easily obtained and are stated below.

Corollary 6.14e *Let D be a digraph of order p such that whenever u and v are distinct vertices and $(u, v) \notin E(D)$, then*

$$\text{od}\, u + \text{id}\, v \geq p - 1.$$

Then D contains a hamiltonian path.

Corollary 6.14f *If D is a digraph of order p such that $\deg v \geq p-1$ for each vertex v of D, then D contains a hamiltonian path.*

Corollary 6.14g *If D is a digraph of order p such that $\text{od}\, v \geq (p-1)/2$ and $\text{id}\, v \geq (p-1)/2$ for every vertex v of D, then D contains a hamiltonian path.*

If, in the previous results, we restrict our attention to symmetric digraphs, then we have sufficient conditions for a *graph* to be hamiltonian and for a graph to possess hamiltonian paths. The theorems we obtain are those due to Ore (Theorem 6.1) and Dirac (Corollary 6.6f) as well as their analogues for graphs containing hamiltonian paths.

We now consider some special classes of hamiltonian digraphs. A digraph D is *hamiltonian-connected* if D contains a hamiltonian u-v path for every two distinct vertices u, v of D. Clearly, every nontrivial hamiltonian-connected digraph is hamiltonian. The following sufficient condition for a digraph to be hamiltonian-connected is analogous to Woodall's Theorem (Corollary 6.14a) on hamiltonian digraphs; the result is due to Overbeck-Larisch [O4].

Theorem 6.15 *Let D be a nontrivial digraph of order p such that for every two distinct vertices u, v of D with $(u, v) \notin E(D)$,*

$$\text{od}\, u + \text{id}\, v \geq p + 1. \tag{6.10}$$

Then D is hamiltonian-connected.

Proof If $p = 2$, then $D \cong K_2$ and the result follows, so we assume that $p \geq 3$. Let u and v be any two distinct vertices of D. We show that D contains a hamiltonian u-v path.

Let $D' = D - u - v$. We construct a digraph D'' from D' by adding a new vertex w to D', an arc (w, z), $z \in V(D-u-v)$, if (u, z) is an arc in D and an arc (z, w), $z \in V(D-u-v)$, if (z, v) is an arc in D.

We next verify that the digraph D'' is strong. Let x and y be distinct vertices of D''. We show that y is reachable from x. Assume first that $w \neq x, y$. If $(x, y) \in E(D)$, the result is obvious, so suppose that $(x, y) \notin E(D)$. By hypothesis,

$$\text{od}_D x + \text{id}_D y \geqslant p + 1;$$

thus, since there are $p-2$ vertices of D different from x or y, there must be at least three vertices of D that are both adjacent from x and adjacent to y. In particular, there must be a vertex $z \neq u, v$ such that $(x, z), (z, y) \in E(D)$ so that x, z, y is a path in D'' and y is reachable from x. Suppose, next, that $x = w$. If $(w, y) \in E(D'')$, then y is reachable from w in D''. If $(w, y) \notin E(D'')$, then $(u, y) \notin E(D)$. But, then, as we have seen before, this implies that there exists $z \in V(D-u-v)$ such that $(u, z), (z, y) \in E(D)$. However, if $(u, z) \in E(D)$, then $(w, z) \in E(D'')$, implying that w, z, y is a path in D'' and y is reachable from w. Finally, assume that $y = w$. We show that w is reachable from x. This is immediate if $(x, w) \in E(D'')$, so we assume that $(x, w) \notin E(D'')$. Thus, $(x, v) \notin E(D)$. As before, there exists $z \in V(D-u-v)$ such that (x, z), $(z, v) \in E(D)$; hence, $(x, z), (z, w) \in E(D'')$ so that x, z, w is a path in D'' and w is reachable from x. Therefore, D'' is strong and has order $p-1$.

Let x and y be any two nonadjacent vertices of D'', where $w \neq x, y$. Then x and y are nonadjacent in D and by (6.10),

$$\deg_{D''} x + \deg_{D''} y \geqslant 2(p+1) - 4 = 2(p-1).$$

If x and w are nonadjacent in D'', so that $(u, x), (x, v) \notin E(D)$, then

$$\begin{aligned}\deg_{D''} x + \deg_{D''} w &\geqslant (\deg_D x - 2) + (\text{od}_D u + \text{id}_D v - 2) \\ &= (\text{od}_D u + \text{id}_D x) + (\text{od}_D x + \text{id}_D v) - 4 \\ &\geqslant 2(p+1) - 4 = 2(p-1).\end{aligned}$$

Hence, by Meyniel's Theorem, D'' is hamiltonian and, thus, contains a hamiltonian cycle C. Deleting w from C produces a hamiltonian path P in D'; say P is a w_1-w_2 path. Since $(w, w_1), (w_2, w) \in E(D'')$, it follows that (u, w_1), $(w_2, v) \in E(D)$. Thus, D contains a hamiltonian u-v path. ■

An immediate corollary is given below.

Corollary 6.15 *Let D be a digraph of order p such that $\text{od}\, v \geqslant (p+1)/2$ and $\text{id}\, v \geqslant (p+1)/2$ for every vertex v of D. Then D is hamiltonian-connected.*

One might be tempted to conjecture that if D is a strong digraph of order p such that $\deg v \geqslant p+1$ for every vertex v of D, then D is hamiltonian-connected. However, such is not the case. Indeed, for nearly every positive integer p, Thomassen [T4] has constructed a strong digraph D of order p such that all vertices have degree at least $p+1$ and D is not hamiltonian-connected. One such digraph is shown in Figure 6.7; this digraph has no hamiltonian u-v path.

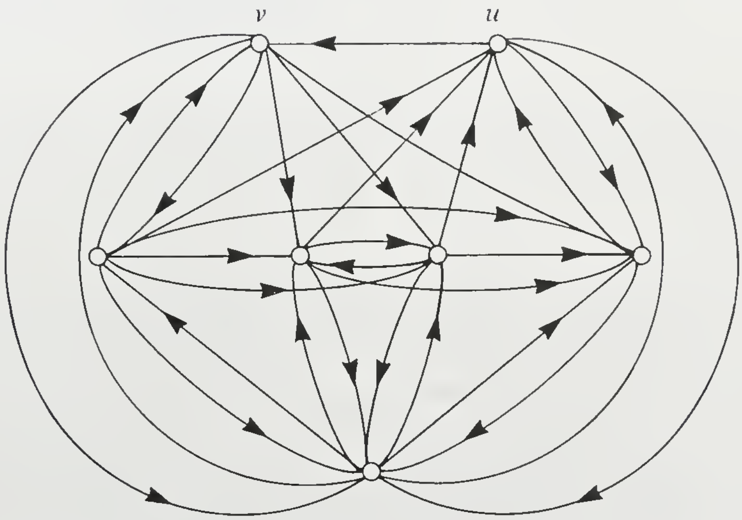


Figure 6.7 *A strong digraph that is not hamiltonian-connected*

A digraph D of order $p \geq 3$ is *pancyclic* if D contains cycles of every length ℓ for $3 \leq \ell \leq p$. Of course, every pancyclic digraph is hamiltonian. If a pancyclic digraph D contains a symmetric pair of arcs, then D contains cycles of every length ℓ for $2 \leq \ell \leq p$.

In Corollary 6.10 we stated the result by Bondy [B14] that if G is a graph of order $p \geq 3$ such that for all distinct nonadjacent vertices u and v, $\deg u + \deg v \geq p$, then either G is pancyclic, or p is even and $G \cong K(p/2, p/2)$. This result has been extended to digraphs by Thomassen [T3]. We state this theorem without proof. By the digraph $K^*(m, n)$, $m, n \geq 1$, we mean the symmetric digraph whose underlying graph is isomorphic to the graph $K(m, n)$.

Theorem 6.16 *If D is a strong digraph of order $p \geq 3$ such that $\deg u + \deg v \geq 2p$ for all distinct nonadjacent vertices u and v of D, then either D is pancyclic or p is even and $D \cong K^*(p/2, p/2)$.*

As a direct consequence of Theorem 6.16, we have the following result of Overbeck-Larisch [O5].

Corollary 6.16a *If D is a strong digraph of order $p \geq 3$ such that $\deg u + \deg v \geq 2p+1$ for all distinct nonadjacent vertices u and v of D, then D is pancyclic.*

Another corollary was initially discovered by Häggkvist and Thomassen [HT1].

Corollary 6.16b *If D is a strong digraph of order $p \geq 3$ such that $\deg v \geq p$ for every vertex v of D, then D is pancyclic, or p is even and $D \cong K^*(p/2, p/2)$.*

Exercises 6.3

6.18 Show that the condition that the digraph D in the statement of Theorem 6.14 be strong cannot be removed from the hypothesis.

6.19 (a) Show that Meyniel's Theorem (Theorem 6.14) is stronger than Corollary 6.14a by giving an example of a (hamiltonian) digraph that satisfies the hypothesis of Theorem 6.14 but not the hypothesis of Corollary 6.14a.

(b) Repeat (a) with Corollary 6.14a replaced in each instance by Corollary 6.14b.

6.20 Show that the condition that the digraph D in the statement of Corollary 6.14b be strong cannot be removed from the hypothesis.

6.21 Prove Ghouila-Houri's Theorem (Corollary 6.14b).

6.22 Show that Meyniel's Theorem is stronger than Ghouila-Houri's Theorem.

6.23 Show that neither Corollary 6.14a nor Corollary 6.14b is a corollary of the other.

6.24 Prove that the bound presented in Corollary 6.14d is best possible.

6.25 Prove Corollaries 6.14e and 6.14f.

6.26 Prove that the bound presented in Theorem 6.15 is best possible.

6.27 Give an example of a (hamiltonian-connected) digraph that satisfies the hypothesis of Theorem 6.15 but does not satisfy the hypothesis of Corollary 6.15.

6.28 Let D be a digraph of order $p \geq 3$ such that $\text{od}\, v \geq (p+1)/2$ and $\text{id}\, v \geq (p+1)/2$ for every vertex v of D. Prove that D is pancyclic.

6.29 Use Theorem 6.16 to prove Corollaries 6.14a and 6.14b.

6.30 Show that if a digraph D satisfies the hypothesis of Theorem 6.16 but contains no symmetric pair of arcs, then D is a pancyclic tournament.

6.31 Show that the bound presented in Theorem 6.16 is sharp by verifying that the number $2p$ cannot be reduced to $2p - 1$.

Chapter Seven

Oriented Graphs

We now turn our attention to the topic of asymmetric digraphs. Among such digraphs, the tournaments are probably the most studied and most applicable, and it is these digraphs that we emphasize here.

7.1 Robbins' Theorem

Recall that an oriented graph D can be obtained from a graph G by assigning a direction to (or by "orienting") each edge of G; the digraph D is also called an orientation of G.

One application of oriented digraphs is in traffic flow problems, for example when two-way streets are changed to one-way streets during periods of heaviest traffic. Certainly, we want the digraph modeling the one-way system to be strongly connected so that it is possible to travel from any location to any other. If G is the graph of the two-way system and G has a bridge, then it is impossible to assign a direction to each edge of G so that the resulting digraph is strong. However, if G is a bridgeless connected graph, then G always has a *strong orientation*; that is, the edges of G can be directed in such a way that the resulting digraph is strong. This observation was first made by Robbins [R8].

Theorem 7.1 (Robbins) *Every 2-edge-connected graph has a strong orientation.*

Proof Suppose the theorem is false. Then there exists a 2-edge connected graph G that has no strong orientation. Among the subgraphs of G, let H be one of maximum order that has a strong orientation; such a subgraph exists since for each $v \in V(G)$, the subgraph $\langle \{v\} \rangle$ trivially has a strong orientation. Thus, among the orientations of G, the maximum number of vertices that are pairwise mutually reachable is $|V(H)|$. Since H has a strong orientation, so too does $\langle V(H) \rangle_G$. Thus $|V(H)| < |V(G)|$ since, by assumption, G has no strong orientation.

Assign directions to the edges of H so that the resulting digraph D is strong, but assign no directions to the edges of $G - E(H)$. Let $u \in V(H)$ and let $v \in V(G) - V(H)$. Since G is 2-edge-connected, there exist two edge-disjoint (graphical) u-v paths in G. Let P be one of these u-v paths and let Q be the v-u path that results from the other u-v path. Further, let u_1 be the last vertex of P that belongs to H, and let v_1 be the first vertex of Q belonging to H. Next, let P_1 be the u_1-v subpath of P and let Q_1 be the v-v_1 subpath of Q. Direct the edges of P_1 from u_1 toward v, producing the directed path P_1', and direct the edges of Q_1 from v toward v_1, producing the directed path Q_1'.

Define the digraph D' by $V(D') = V(D) \cup V(P_1') \cup V(Q_1')$ and $E(D') = E(D) \cup E(P_1') \cup E(Q_1')$. Let $w \in V(D)$. Since D is strong, u_1 is reachable from w. Also, v is reachable from u_1 in D'. Therefore, v is reachable from w and, similarly, w is reachable from v in D'. It follows that there exists an orientation of G that results in at least $|V(H)| + 1$ pairwise mutually reachable vertices, which contradicts the choice of H. ■

Exercises 7.1

7.1 A digraph D is called *unilateral* if for every two distinct vertices u and v of D there is either a u-v path or a v-u path in D.

(a) Prove or disprove: Every connected graph has a unilateral orientation.

(b) Prove or disprove: Every orientation of a 2-edge-connected graph is a unilateral orientation.

7.2 Tournaments

The class of oriented graphs that has received the greatest attention is the class of tournaments; that is, those digraphs obtained by orienting the edges of complete graphs.

The number of nonisomorphic tournaments increases sharply with order. For example, there is only one tournament of order 1 and one of order 2. There are two tournaments of order 3, namely the tournaments T_1 and T_2 shown in Figure 7.1. There are four tournaments of order 4, twelve of order 5, and over nine million of order 10.

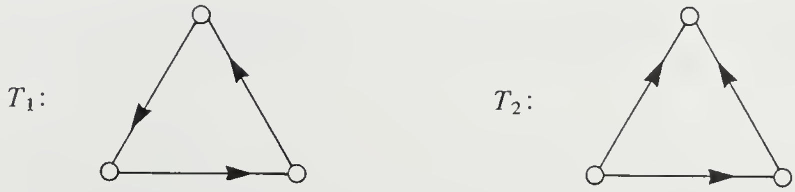


Figure 7.1 *The tournaments of order* 3

If T is a tournament of order p, then it follows since T is complete that its size is $\binom{p}{2}$ and that, consequently,

$$\sum_{v \in V(T)} \text{od } v = \sum_{v \in V(T)} \text{id } v = \binom{p}{2}.$$

A tournament T is *transitive* if, whenever (u, v) and (v, w) are arcs of T, then (u, w) is also an arc of T. The tournament T_2 of Figure 7.1 is transitive while T_1 is not. The following result gives an elementary property of transitive tournaments.

Theorem 7.2 *A tournament is transitive if and only if it is acyclic.*

Proof Let T be an acyclic tournament, and suppose (u, v) and (v, w) are arcs of T. Since T is acyclic, $(w, u) \notin E(T)$; therefore, $(u, w) \in E(T)$ and T is transitive.

Conversely, suppose T is a transitive tournament, and assume that T contains a cycle, say C: $v_1, v_2, \ldots, v_n, v_1$ (where $n \geq 3$ since T is asymmetric). Since (v_1, v_2) and (v_2, v_3) are arcs of the transitive tournament T, (v_1, v_3) is an arc of T. Similarly, $(v_1, v_4), (v_1, v_5), \ldots, (v_1, v_n)$ are arcs of T. However, this contradicts the fact that (v_n, v_1) is an arc of T. Thus, T is acyclic. ■

Every tournament of order p can be thought of as representing or modeling a round robin tournament involving competition among p teams. In a round robin tournament, each team plays every other team exactly once and ties are not permitted. Let $v_1, v_2, \ldots, v_p$ represent the teams as well as the vertices of the corresponding tournament T. If, in the competition between v_i and v_j, $i \neq j$, team v_i defeats team v_j, then (v_i, v_j) is an arc of T. The number of victories by team v_i is the number $\operatorname{od} v_i$; for this reason, the outdegree of the vertex v_i in a tournament is also referred to as the *score* of v_i.

A sequence $s_1, s_2, \ldots, s_p$ of nonnegative integers is called a *score sequence* (*of a tournament*) if there exists a tournament T of order p whose vertices can be labeled as $v_1, v_2, \ldots, v_p$ such that $\operatorname{od} v_i = s_i$ for $i = 1, 2, \ldots, p$. The following result indicates exactly which sequences are score sequences of transitive tournaments.

Theorem 7.3 *A nondecreasing sequence $\mathcal{S}$ of $p(\geq 1)$ nonnegative integers is a score sequence of a transitive tournament of order p if and only if $\mathcal{S}$ is the sequence $0, 1, \ldots, p-1$.*

Proof First we show that $\mathcal{S}$: $0, 1, \ldots, p-1$ is a score sequence of a transitive tournament. Let T be a tournament defined by $V(T) = \{v_1, v_2, \ldots, v_p\}$ and $E(T) = \{(v_i, v_j) \mid 1 \leq j < i \leq p\}$. Then $\operatorname{od} v_i = i - 1$ for $i = 1, 2, \ldots, p$; so $\mathcal{S}$ is a score sequence of the transitive tournament T.

Conversely, assume that T is a transitive tournament of order p. We show that $\mathcal{S}$: $0, 1, \ldots, p-1$ is a score sequence of T. It suffices to show that no two vertices of T have the same score (outdegree). Let $u, v \in V(T)$ and assume, without loss of generality, that $(u, v) \in E(T)$. If W denotes the set of vertices of T adjacent from v, then $\operatorname{od} v = |W|$. Since $(v, w) \in E(T)$ for each $w \in W$ and $(u, v) \in E(T)$, it follows that $(u, w) \in E(T)$ for each $w \in W$, since T is transitive. Thus, $\operatorname{od} u \geq 1 + |W| = 1 + \operatorname{od} v$. ■

Corollary 7.3a. *A nondecreasing sequence $\mathcal{S}$ of $p(\geq 1)$ nonnegative integers is a sequence of indegrees of the vertices of a transitive tournament if and only if $\mathcal{S}$ is the sequence $0, 1, \ldots, p-1$.*

Another result that follows readily from Theorem 7.3 is given next.

Corollary 7.3b *For every positive integer p, there is exactly one transitive tournament of order p.*

Combining this corollary with Theorem 7.2, we arrive at yet another corollary.

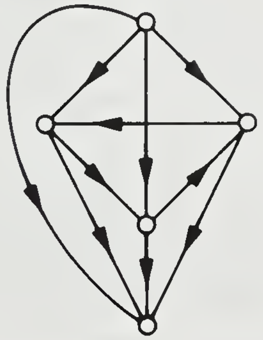
Corollary 7.3c *For every positive integer p, there is exactly one acyclic tournament of order p.*

We shall discuss score sequences of arbitrary tournaments in Section 7.4.

Although there is only one transitive tournament of each order p, in a certain sense that we now explore, every tournament has the structure of a transitive tournament. The relationship "mutually reachable" is an equivalence relation on the vertex set of a tournament T and, as such, this relation partitions $V(T)$ into equivalence classes $V_1, V_2, \ldots, V_n (n \geq 1)$. Let $S_i = \langle V_i \rangle$ for $i = 1, 2, \ldots, n$. Then it is easy to see that each S_i is a strong subdigraph and, indeed, is maximal with respect to the property of being strong. The subdigraphs $S_1, S_2, \ldots, S_n$ are called the *strong components* of T.

Let T be a tournament with strong components $S_1, S_2, \ldots, S_n$, and let $\tilde{T}$ denote that digraph whose vertices $u_1, u_2, \ldots, u_n$ can be put in one-to-one correspondence with the strong components (u_i corresponds to S_i, $i = 1, 2, \ldots, n$) such that (u_i, u_j) is an arc of $\tilde{T}$, $i \neq j$, if and only if some vertex of S_i is adjacent to at least one vertex of S_j. A tournament T and associated digraph $\tilde{T}$ are shown in Figure 7.2.

T:



$\tilde{T}$:

Figure 7.2 *A tournament T and $\tilde{T}$*

Observe that for the tournament T of Figure 7.2, $\tilde{T}$ is also a tournament and, in fact, a transitive tournament. That this always occurs follows from Theorem 7.4 (see Exercise 7.6).

Theorem 7.4 *If T is a tournament with (exactly) n strong components, then $\tilde{T}$ is the transitive tournament of order n.*

Since for every tournament T we have that $\tilde{T}$ is a transitive tournament, it is easy to show that if T is a tournament that is not strong, then $V(T)$ can be partitioned as $V_1 \cup V_2 \cup \ldots \cup V_n (n \geq 2)$ such that $\langle V_i \rangle$ is a strong tournament for each i, and if $v_i \in V_i$ and $v_j \in V_j$, where $i > j$, then $(v_i, v_j) \in E(T)$. This decomposition proves useful in studying the properties of tournaments that are not strong.

If u and v are vertices of a digraph D, and D contains at least one u-v path, then the length of a shortest u-v path is called the *distance* from u to v and is denoted by $d_D(u, v)$ or simply $d(u, v)$. Our next three results involve the concept of distance in a tournament.

Theorem 7.5 *Let v be a vertex of maximum score in a nontrivial tournament T. If u is a vertex of T different from v, then $d(v, u) \leq 2$.*

Proof Assume $\text{od}\, v = n$. Necessarily, $n \geq 1$. Let $v_1, v_2, \ldots, v_n$ denote the vertices of T adjacent from v. Then $d(v, v_i) = 1$ for $i = 1, 2, \ldots, n$. If $V(T) = \{v, v_1, v_2, \ldots, v_n\}$, the proof is complete.

Assume, then, that $V(T) - \{v, v_1, v_2, \ldots, v_n\}$ is nonempty, and let $u \in V(T) - \{v, v_1, v_2, \ldots, v_n\}$. If the vertex u is adjacent from some v_i, $1 \leq i \leq n$, then $d(v, u) = 2$, producing the desired result. Thus, assume this is not the case. Then u is adjacent to all of the vertices $v_1, v_2, \ldots, v_n$, as well as to v, so that $\text{od}\, u \geq 1 + n = 1 + \text{od}\, v$. However, this contradicts the fact that v is a vertex of maximum score. ■

Theorem 7.5 was first discovered by the sociologist Landau [L1] during a study of pecking orders and domination among chickens. In the case of chickens, the theorem says that if chicken c pecks the largest number of other chickens, then for every other chicken d, either c pecks d, or c pecks some chicken that pecks d. Thus c dominates every other chicken either directly or indirectly in two steps.

Let D be a strong digraph. The *eccentricity* $e(v)$ of a vertex v of D is defined as $e(v) = \max_{w \in V(D)} d(v, w)$. The *radius* of D is $\text{rad}\, D = \min_{v \in V(D)} e(v)$ and the *center* $Z(D)$ of D is defined by $Z(D) = \{v \mid e(v) = \text{rad}\, D\}$. Theorem 7.5 provides an immediate result dealing with the radius of a strong tournament.

Corollary 7.5 *Every nontrivial strong tournament has radius 2.*

We conclude this section with a result on the center of a strong tournament.

Theorem 7.6 *The center of every nontrivial strong tournament contains at least three vertices.*

Proof Let T be a nontrivial strong tournament. By Corollary 7.5, $\text{rad}\, T = 2$. Let w be a vertex having eccentricity 2. Since T is strong, there are vertices adjacent to w; let v be one of these having maximum score. Among the vertices adjacent to v, let u be one of maximum score. We show that each of the vertices u and v has eccentricity 2, which will complete the proof.

Assume, to the contrary, that one of the vertices u and v does *not* have eccentricity 2. Suppose, then, that $x \in \{u, v\}$ and $e(x) \geq 3$. Hence, there exists a vertex y in T such that $d(x, y) \geq 3$. Thus, y is adjacent to x. Moreover, y is adjacent to every vertex adjacent from x. These observations imply that $\text{od}\, y > \text{od}\, x$.

Suppose that $x = v$. Since x is adjacent to w, it follows that y is adjacent to w. However, $\text{od}\, y > \text{od}\, v$, which contradicts the defining property of v. Therefore, $x = u$. Here x is adjacent to v so that y is adjacent to v, but $\text{od}\, y > \text{od}\, u$. Hence, $x \neq u$ and the proof is complete. ■

Exercises 7.2

7.2 Draw all four (nonisomorphic) tournaments of order 4.

7.3 (a) Show that every tournament has at most one vertex with score zero.
(b) Show that every tournament has at most one vertex with indegree zero.

7.4 (a) Prove Corollary 7.3a.
(b) Prove Corollary 7.3b.

7.5 Give an example of two nonisomorphic regular tournaments of the same order.

7.6 Prove Theorem 7.4.

7.7 Determine those positive integers p for which there exist regular tournaments of order p.

7.8 Prove that every regular tournament is strong.

7.9 Prove that every two vertices in a nontrivial regular tournament lie on a 3-cycle.

7.10 Prove that if T is a nontrivial regular tournament, then $\text{diam}\, T = 2$.

7.11 Prove that every vertex of a nontrivial strong tournament lies on a 3-cycle.

7.12 Prove Corollary 7.5.

7.13 (a) A vertex v of a tournament T is called a *winner* if $d(v, u) \leq 2$ for every $u \in V(T)$. Show that no tournament has exactly two winners.
(b) Show that if p is a positive integer, $p \neq 2, 4$, then there is a tournament of order p in which every vertex is a winner.

7.3 Hamiltonian Tournaments

The large number of arcs a tournament possesses produces a variety of paths and cycles. In this section we investigate these types of subdigraphs in tournaments. We begin with perhaps the most basic result of this type, a property of tournaments first observed by Rédei [R2].

Theorem 7.7 (Rédei) *Every tournament contains a hamiltonian path.*

Proof Let T be a tournament of order p, and let P: $v_1, v_2, \ldots, v_k$ be a longest path in T. If P is not a hamiltonian path of T, then $1 \leq k < p$ and there is a vertex v of T not on P. Since P is a longest path, $(v, v_1), (v_k, v) \notin E(T)$, and so (v_1, v), $(v, v_k) \in E(T)$. This implies that there is an integer i, $1 \leq i \leq p-1$, such that $(v_i, v) \in E(T)$ and $(v, v_{i+1}) \in E(T)$. But then

$$v_1, v_2, \ldots, v_i, v, v_{i+1}, \ldots, v_k$$

is a path whose length exceeds that of P, producing a contradiction. ■

A simple but useful consequence of Theorem 7.7 concerns transitive tournaments.

Corollary 7.7 *Every transitive tournament contains exactly one hamiltonian path.*

The preceding corollary is a special case of a result by Szele [S3], who showed that every tournament contains an odd number of hamiltonian paths.

While not every tournament is hamiltonian, such is the case for strong tournaments, a fact first observed by Camion [C1]. It is perhaps surprising that if a tournament is hamiltonian, then it must possess a significantly stronger property. A digraph D of order $p \geq 3$ is *vertex-pancyclic* if each vertex of D lies on a cycle of length ℓ for each $\ell = 3, 4, \ldots, p$. The following result was discovered by Moon [M8]; the proof here is due to C. Thomassen.

Theorem 7.8 (Moon) *Every nontrivial strong tournament is vertex-pancyclic.*

Proof Let T be a strong tournament of order $p \geq 3$, and let v_1 be a vertex of T. We show that v_1 lies on an ℓ-cycle for each $\ell = 3, 4, \ldots, p$. We proceed by induction on ℓ.

Since T is strong, it follows from Exercise 7.11 that v_1 lies on a 3-cycle. Assume v_1 lies on an ℓ-cycle $v_1, v_2, \ldots, v_\ell, v_1$, where $3 \leq \ell \leq p-1$. We prove that v_1 lies on an $(\ell+1)$-cycle.

Case 1: Suppose there is a vertex v not on C that is adjacent to at least one vertex of C and is adjacent from at least one vertex of C. This implies that for some i, $1 \leq i \leq \ell$, both (v_i, v) and (v, v_{i+1}) are arcs of T (where all subscripts are expressed modulo ℓ). Thus, v_1 lies on the $(\ell+1)$-cycle

$$v_1, v_2, \ldots, v_i, v, v_{i+1}, \ldots, v_\ell, v_1.$$

Case 2: No vertex v exists as in Case 1. Let A denote the set of all vertices in $V(T) - V(C)$ that are adjacent to every vertex of C, and let B be the set of all vertices in $V(T) - V(C)$ that are adjacent from every vertex of C. Then $A \cup B = V(T) - V(C)$. Since T is strong, neither A nor B is empty. Furthermore, there is a vertex b in B and a vertex a in A such that $(b, a) \in E(T)$. Thus, v_1 lies on the $(\ell + 1)$-cycle

$$a, v_1, v_2, \ldots, v_{\ell-1}, b, a. \blacksquare$$

Corollary 7.8 (Camion) *A tournament of order $p \geq 3$ is hamiltonian if and only if it is strong.*

Exercises 7.3

7.14 Prove that if T is not a transitive tournament, then T has at least three hamiltonian paths.

7.15 Use Corollary 7.7 to give an alternative proof of Theorem 7.3.

7.16 Prove Corollary 7.8.

7.17 Prove or disprove: Every arc of a nontrivial strong tournament T lies on a hamiltonian cycle of T.

7.18 Prove or disprove: Every vertex-pancyclic tournament is hamiltonian-connected.

7.19 Show that if a tournament T has an ℓ-cycle, then T has an s-cycle for $s = 3, 4, \ldots, \ell$.

7.4 Score Sequences of Tournaments

Recall that a sequence $s_1, s_2, \ldots, s_p$ of nonnegative integers is a score sequence (of a tournament) if there exists a tournament of order p whose vertices can be labeled as $v_1, v_2, \ldots, v_p$ such that $\text{od}\, v_i = s_i$ for $i = 1, 2, \ldots, p$. From Theorem 7.3 and Corollary 7.3b, it follows that a nondecreasing sequence $\mathcal{S}$: $s_1, s_2, \ldots, s_p$ of nonnegative integers is a score sequence of the transitive tournament if and only if $s_i = i - 1$ for all i. In this section we investigate score sequences in more generality. We begin with a theorem similar to the Havel-Hakimi theorem on graphical sequences (Theorem 1.3).

Theorem 7.9 *A nondecreasing sequence* $\mathcal{S}$: $s_1, s_2, \ldots, s_p$ $(p \geq 2)$ *of nonnegative integers is a score sequence if and only if the sequence* $\mathcal{S}_1$: $s_1, s_2, \ldots, s_{s_p}, s_{s_p+1} - 1, \ldots, s_{p-1} - 1$ *is a score sequence.*

Proof Assume that $\mathcal{S}_1$ is a score sequence. Then there exists a tournament T_1 of order $p-1$ such that $\mathcal{S}_1$ is a score sequence of T_1. Hence the vertices of T_1 can be labeled as $v_1, v_2, \ldots, v_{p-1}$ so that

$$\operatorname{od} v_i = \begin{cases} s_i & \text{for } 1 \leq i \leq s_p. \\ s_i - 1 & \text{for } i > s_p. \end{cases}$$

We construct a tournament T by adding a vertex v_p to T_1. Furthermore, for $1 \leq i \leq p$, v_p is adjacent to v_i if $1 \leq i \leq s_p$, and v_p is adjacent from v_i otherwise. The tournament T then has $\mathcal{S}$ as a score sequence.

For the converse, we assume that $\mathcal{S}$ is a score sequence. Hence there exist tournaments of order p whose score sequence is $\mathcal{S}$. Among all such tournaments, let T be one such that $V(T) = \{v_1, v_2, \ldots, v_p\}$, $\operatorname{od} v_i = s_i$ for $i = 1, 2, \ldots, p$, and the sum of the scores of the vertices adjacent from v_p is minimum. We consider two cases.

Case 1: *Suppose that* T *contains a vertex* u *with score* s_p *such that* u *is adjacent to vertices having scores* $s_1, s_2, \ldots, s_{s_p}$. Then $T - u$ is a tournament having score sequence $\mathcal{S}_1$.

Case 2: *Suppose no vertex* u *exists as in Case 1.* Thus the vertex v_p is not adjacent to vertices having scores $s_1, s_2, \ldots, s_{s_p}$. Necessarily, then, there exist vertices v_j and v_k, with $s_j < s_k$, such that v_p is adjacent to v_k and v_p is adjacent from v_j. Since the score of v_k exceeds the score of v_j, there must exist a vertex v_n distinct from v_p, v_j, and v_k such that v_k is adjacent to v_n and v_n is adjacent to v_j. (See Figure 7.3(a).) Thus, a 4-cycle C: v_p, v_k, v_n, v_j, v_p is produced. If we reverse the directions of the arcs of C, a tournament T' is obtained also having $\mathcal{S}$ as a score sequence. (See Figure 7.3(b).) However, in T', the vertex v_p is adjacent to v_j rather than v_k. Hence the sum of the scores of the vertices adjacent from v_p is smaller in T' than in T, which is impossible. Thus, Case 2 cannot occur, returning us to Case 1 and the fact that $\mathcal{S}_1$ is a score sequence. ■

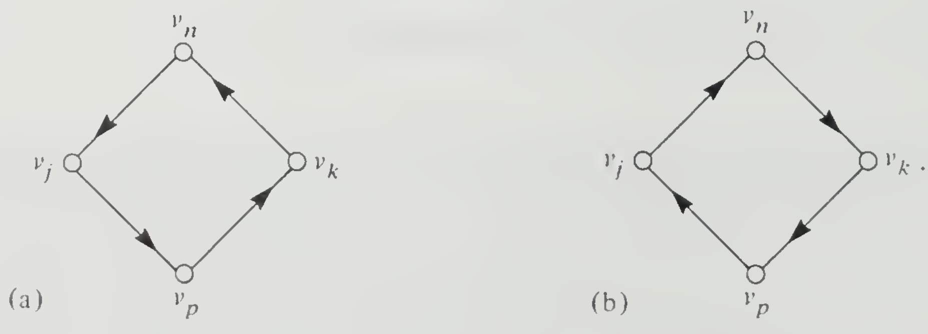


Figure 7.3 *A step in the proof of Theorem 7.8*

As an illustration of Theorem 7.9, we consider the sequence

$$\mathcal{S}: 1, 2, 2, 3, 3, 4.$$

In this case, s_p (actually s_6) has the value 4; thus, we delete the last term, repeat the first $s_p = 4$ terms, and subtract one from the remaining terms, obtaining

$$\mathcal{S}_1': 1, 2, 2, 3, 2.$$

Rearranging, we have

$$\mathcal{S}_1: 1, 2, 2, 2, 3.$$

Repeating this process twice more, we have

$$\mathcal{S}_2': 1, 2, 2, 1$$
$$\mathcal{S}_2: 1, 1, 2, 2$$
$$\mathcal{S}_3: 1, 1, 1.$$

The sequence $\mathcal{S}_3$ is clearly a score sequence. We can use this information to construct a tournament with score sequence $\mathcal{S}$. The sequence $\mathcal{S}_3$ is the score sequence of the tournament T_3 of Figure 7.4. Proceeding from $\mathcal{S}_3$ to $\mathcal{S}_2$, we add a new vertex to T_3 and join it *to* two vertices of T_3 and *from* the other, producing a tournament T_2 with score sequence $\mathcal{S}_2$. To proceed from $\mathcal{S}_2$ to

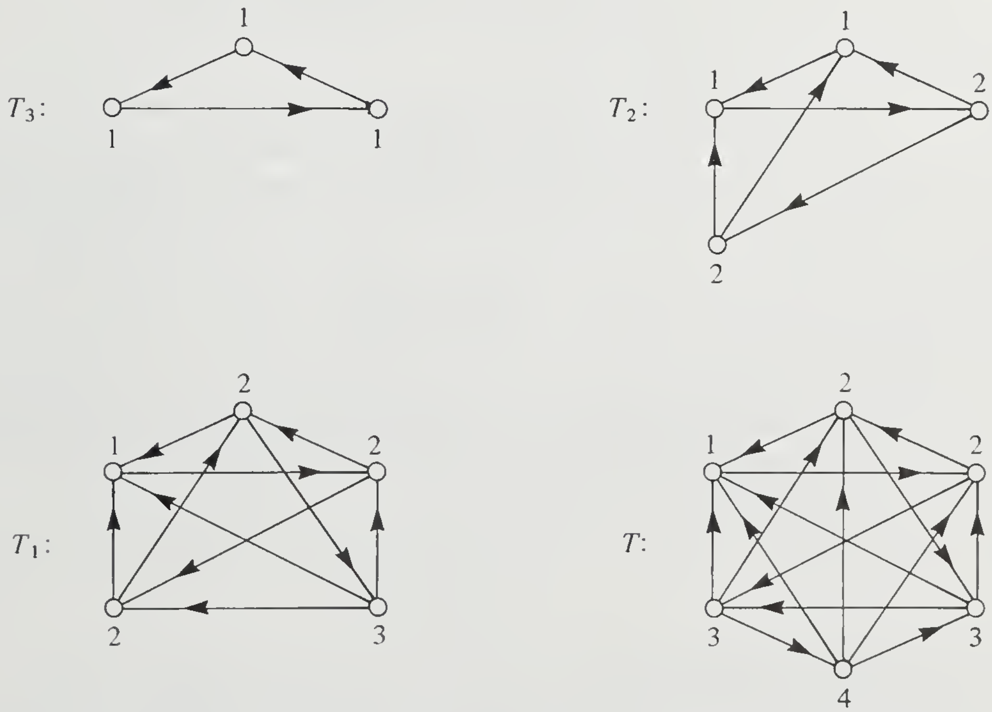


Figure 7.4 *Construction of a tournament with a given score sequence*

$\mathcal{S}_1$, we add a new vertex to T_2 and join it *to* vertices having scores 1, 2, and 2 and *from* the remaining vertex of T_2, producing a tournament T_1 with score sequence $\mathcal{S}_1$. Continuing in the same fashion, we finally produce the desired tournament T with score sequence $\mathcal{S}$ by adding a new vertex to T_1 and joining it *to* vertices having scores 1, 2, 2, and 3 and joining it *from* the other vertex.

The following theorem by Landau [L1] gives a nonconstructive criterion for a sequence to be a score sequence. There are many proofs of this result; the one we give is due to Thomassen [T5].

Theorem 7.10 *A nondecreasing sequence $\mathcal{S}$: $s_1, s_2, \ldots, s_p$ of nonnegative integers is a score sequence if and only if for each $k (1 \leqslant k \leqslant p)$,*

$$\sum_{i=1}^{k} s_i \geqslant \binom{k}{2}, \tag{7.1}$$

with equality holding when $k = p$.

Proof Assume that $\mathcal{S}$: $s_1, s_2, \ldots, s_p$ is a score sequence. Then there exists a tournament T of order p with $V(T) = \{v_1, v_2, \ldots, v_p\}$ such that $\mathrm{od}_T v_i = s_i$ for $i = 1, 2, \ldots, p$. Let k be an integer with $1 \leqslant k \leqslant p$. Then $T_1 = \langle\{v_1, v_2, \ldots, v_k\}\rangle$ is a tournament of order k and size $\binom{k}{2}$. Since $\mathrm{od}_T v_i \geqslant \mathrm{od}_{T_1} v_i$ for $1 \leqslant i \leqslant k$, it follows that

$$\sum_{i=1}^{k} s_i = \sum_{i=1}^{k} \mathrm{od}_T v_i \geqslant \sum_{i=1}^{k} \mathrm{od}_{T_1} v_i = \binom{k}{2},$$

with equality holding when $k = p$.

We prove the converse by contradiction. Assume that $\mathcal{S}$: $s_1, s_2, \ldots, s_p$ is a counterexample to the theorem, chosen so that p is as small as possible and so that s_1 is as small as possible among all counterexamples of length p.

Suppose first that there exists an integer k with $1 \leqslant k \leqslant p - 1$ such that

$$\sum_{i=1}^{k} s_i = \binom{k}{2}. \tag{7.2}$$

Thus the sequence $\mathcal{S}_1$: $s_1, s_2, \ldots, s_k$ satisfies (7.1) and so, by the minimality of p, there exists a tournament T_1 of order k having score sequence $\mathcal{S}_1$.

Consider the sequence $\mathcal{T}$: $t_1, t_2, \ldots, t_{p-k}$, where $t_i = s_{k+i} - k$ for $i = 1, 2, \ldots, p - k$. Since

$$\sum_{i=1}^{k+1} s_i \geqslant \binom{k+1}{2},$$

it follows from (7.2) that

$$s_{k+1} = \sum_{i=1}^{k+1} s_i - \sum_{i=1}^{k} s_i \geqslant \binom{k+1}{2} - \binom{k}{2} = k.$$

Thus, since $\mathcal{S}$ is a nondecreasing sequence,

$$t_i = s_{k+i} - k \geqslant s_{k+1} - k \geqslant 0$$

for $i = 1, 2, \ldots, p-k$, and so $\mathcal{T}$ is a nondecreasing sequence of nonnegative integers. We show that $\mathcal{T}$ satisfies (7.1).

For each m satisfying $1 \leqslant m \leqslant p-k$, we have

$$\sum_{i=1}^{m} t_i = \sum_{i=1}^{m} (s_{k+i} - k) = \sum_{i=1}^{m} s_{k+i} - mk = \sum_{i=1}^{m+k} s_i - \sum_{i=1}^{k} s_i - mk.$$

Since $\sum_{i=1}^{m+k} s_i \geqslant \binom{m+k}{2}$ and $\sum_{i=1}^{k} s_i = \binom{k}{2}$, it follows that

$$\sum_{i=1}^{m} t_i \geqslant \binom{m+k}{2} - \binom{k}{2} - mk = \binom{m}{2},$$

with equality holding for $m = p-k$. Thus, $\mathcal{T}$ satisfies (7.1) and so, by the minimality of p, there exists a tournament T_2 of order $p-k$ having score sequence $\mathcal{T}$.

We construct a tournament T as follows: $V(T) = V(T_1) \cup V(T_2)$ and

$$E(T) = E(T_1) \cup E(T_2) \cup \{(u, v) \mid u \in V(T_2),\ v \in V(T_1)\}.$$

Then $\mathcal{S}$ is a score sequence for T, contrary to assumption. Thus for $k = 1, 2, \ldots, p-1$,

$$\sum_{i=1}^{k} s_i > \binom{k}{2}.$$

In particular, $s_1 > 0$.

Consider the sequence $\mathcal{S}'$: $s_1 - 1, s_2, s_3, \ldots, s_{p-1}, s_p + 1$. Clearly $\mathcal{S}'$ is a nondecreasing sequence of nonnegative integers that satisfies (7.1). By the minimality of s_1, then, there exists a tournament T' of order p having score sequence $\mathcal{S}'$. Let x and y be vertices of T' such that $\operatorname{od}_{T'} x = s_p + 1$ and $\operatorname{od}_{T'} y = s_1 - 1$. Since $\operatorname{od}_{T'} x \geqslant \operatorname{od}_{T'} y + 2$, there is a vertex $w \neq x, y$ such that $(x, w) \in E(T')$ and $(y, w) \notin E(T')$. Thus, x, w, y is a path in T'.

Let T be the tournament obtained from T' by reversing the directions of the arcs of P. Then $\mathcal{S}$ is a score sequence for T, again producing a contradiction and completing the proof. ■

With a slight alteration in the hypothesis of the preceding theorem, we obtain a necessary and sufficient condition for a score sequence of a strong tournament. This result is due to L. Moser and may be found in [HM1].

Theorem 7.11 *A nondecreasing sequence $\mathcal{S}$: $s_1, s_2, \ldots, s_p$ of nonnegative integers is a score sequence of a strong tournament if and only if*

$$\sum_{i=1}^{k} s_i > \binom{k}{2}.$$

for $1 \le k \le p-1$ *and*

$$\sum_{i=1}^{p} s_i = \binom{p}{2}.$$

Furthermore, if $\mathcal{S}$ is a score sequence of a strong tournament, then every tournament with $\mathcal{S}$ as a score sequence is strong.

Proof Let T be a strong tournament and suppose that $\mathcal{S}$: $s_1, s_2, \ldots, s_p$ is a score sequence of T, where $s_1 \le s_2 \le \cdots \le s_p$. Since T is a tournament of order p,

$$\sum_{i=1}^{p} s_i = \binom{p}{2}.$$

Let $1 \le k \le p-1$ and define $T_1 = \langle \{v_1, v_2, \ldots, v_k\} \rangle$. Since T_1 is a tournament of order k,

$$\sum_{i=1}^{k} \mathrm{od}_{T_1} v_i = \binom{k}{2}.$$

Since T is a strong tournament, some vertex v_j in $T_1 (1 \le j \le k)$ must be adjacent in T to a vertex not in T_1 so that $\mathrm{od}_T v_j > \mathrm{od}_{T_1} v_j$. Since $\mathrm{od}_T v_i \ge \mathrm{od}_{T_1} v_i$ for all i, $1 \le i \le k$, we obtain

$$\sum_{i=1}^{k} s_i = \sum_{i=1}^{k} \mathrm{od}_T v_i > \sum_{i=1}^{k} \mathrm{od}_{T_1} v_i = \binom{k}{2}.$$

For the converse, we assume that $\mathcal{S}$: $s_1, s_2, \ldots, s_p$ is a nondecreasing sequence of nonnegative integers such that

$$\sum_{i=1}^{k} s_i > \binom{k}{2}$$

for $1 \le k \le p-1$ and

$$\sum_{i=1}^{p} s_i = \binom{p}{2}.$$

By Theorem 7.10, $\mathcal{S}$ is the score sequence of a tournament. Let T be a tournament with score sequence $\mathcal{S}$; we show that T is strong.

If T is not strong, it follows from Theorem 7.4 (and the discussion preceding it) that $V(T)$ can be partitioned as $U \cup W$ such that $(u, w) \in E(T)$ for every $u \in U$ and $w \in W$. Thus, if $T_1 = \langle W \rangle$, then $\mathrm{od}_T w = \mathrm{od}_{T_1} w$ for every

$w \in W$. Let $k = |W|$, where, then, $1 \leqslant k \leqslant p-1$. Since $\mathscr{S}$ is a nondecreasing sequence and T_1 is a tournament of order k, we have

$$\sum_{i=1}^{k} s_i \leqslant \sum_{w \in W} \mathrm{od}_T w = \sum_{w \in W} \mathrm{od}_{T_1} w = \binom{k}{2},$$

contradicting the hypothesis. ■

In a similar fashion, the following result is obtained.

Theorem 7.12 *Let T be a tournament with score sequence $\mathscr{S}$: $s_1, s_2, \ldots, s_p$, where $s_1 \leqslant s_2 < \cdots \leqslant s_p$. Suppose that*

$$\sum_{i=1}^{k} s_i = \binom{k}{2}, \quad \sum_{i=1}^{\ell} s_i = \binom{\ell}{2},$$

and

$$\sum_{i=1}^{n} s_i > \binom{n}{2} \qquad \text{for } n = k+1,\ k+2,\ \ldots,\ \ell-1,$$

where $0 \leqslant k < \ell \leqslant p$.
Then the subtournament induced by $\{v_{k+1}, v_{k+2}, \ldots, v_\ell\}$ is a strong component of T with score sequence $s_{k+1}-k,\ s_{k+2}-k,\ \ldots,\ s_\ell - k$, where $\mathrm{od}\, v_j = s_j$, $j = k+1,\ k+2,\ \ldots,\ \ell$.

We close this section with a brief discussion of one other family of oriented graphs. A *bipartite tournament* is an orientation of a complete bipartite graph. If T is a bipartite tournament whose underlying graph G has partite sets U and W, then U and W are also referred to as the partite sets of T. Furthermore, if $|U| = m$ and $|W| = n$, then we say that T is an $m \times n$ bipartite tournament. Figure 7.5 shows the four (nonisomorphic) 2×2 bipartite tournaments.

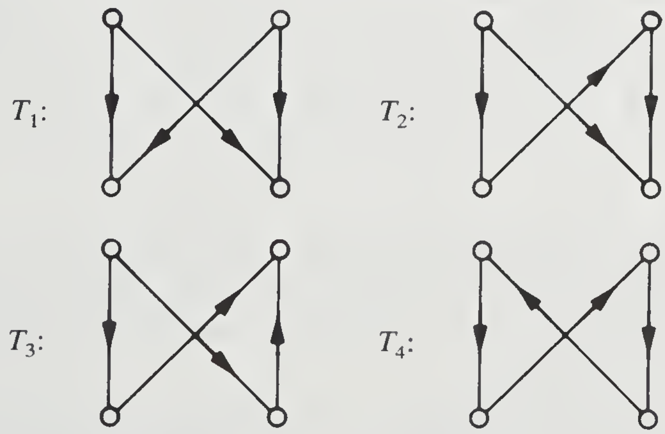


Figure 7.5 *The four 2×2 bipartite tournaments*

It will be convenient to have another method of illustrating a bipartite tournament. If T is a bipartite tournament with partite sets U and W, then we indicate those arcs from U to W by an edge. The remaining arcs are then obvious. Using this convention, we again show the four 2×2 bipartite tournaments in Figure 7.6.

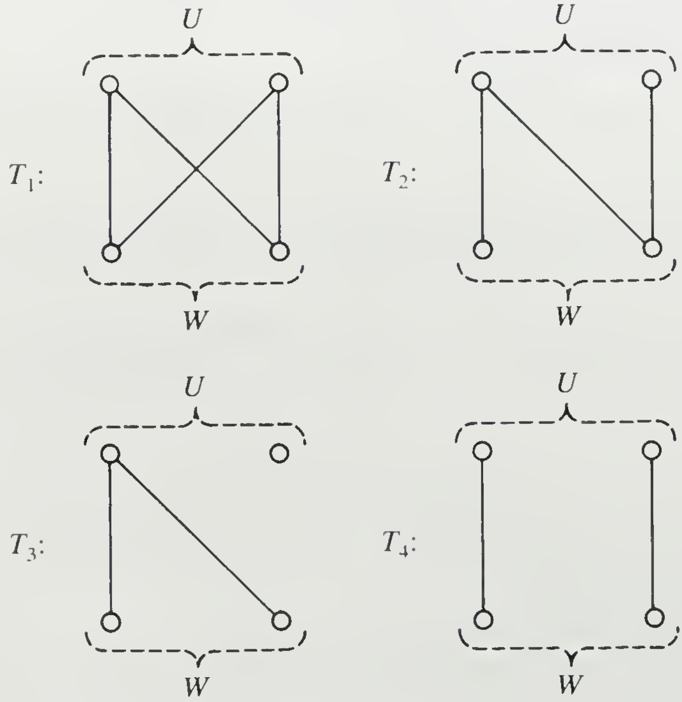


Figure 7.6 *The four 2×2 bipartite tournaments*

Just as tournaments may be used to represent competition, so may bipartite tournaments. In the bipartite case there are two teams, and two individuals compete if and only if they are on opposing teams.

There are many results for bipartite tournaments that are similar to tournament results (see, for example, [BM1]). Two such results involve score sequences.

Two sequences $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$ of nonnegative integers are called *score sequences of a bipartite tournament* if there exists a bipartite tournament T with partite sets $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ such that $\text{od}\, u_i = a_i$, $1 \leq i \leq m$, and $\text{od}\, w_i = b_i$, $1 \leq i \leq n$. Similar to Theorem 7.9 (both in statement and in proof) is the following.

Theorem 7.13 *Let $\mathcal{A}$: $a_1, a_2, \ldots, a_m$ and $\mathcal{B}$: $b_1, b_2, \ldots, b_n$ be nondecreasing sequences of nonnegative integers. Then $\mathcal{A}$ and $\mathcal{B}$ are score sequences of a bipartite tournament if and only if the sequences $\mathcal{A}_1$:$a_1, a_2, \ldots, a_{m-1}$ and $\mathcal{B}_1$: $b_1, \ldots, b_{a_m}, b_{a_m+1} - 1, \ldots, b_n - 1$ are score sequences of a bipartite tournament.*

To illustrate Theorem 7.13, we consider the sequences

$$\mathcal{A}: \quad 1, 1, 3, 5, 5 \qquad \mathcal{B}: \quad 1, 1, 2, 3, 4, 4.$$

We obtain the sequence $\mathcal{A}_1$ by deleting the largest term a_m from $\mathcal{A}$; the sequence $\mathcal{B}_1'$ is obtained by repeating the first $a_m = 5$ terms of $\mathcal{B}$ and subtracting 1 from the remaining terms. Thus, we have

$$\mathcal{A}_1: \quad 1, 1, 3, 5 \qquad \mathcal{B}_1': \quad 1, 1, 2, 3, 4, 3.$$

Rearranging $\mathcal{B}_1'$ produces

$$\mathcal{A}_1: \quad 1, 1, 3, 5 \qquad \mathcal{B}_1: \quad 1, 1, 2, 3, 3, 4.$$

Repeating this process twice more, we have

$$\begin{array}{llll} \mathcal{A}_2: & 1, 1, 3 & \mathcal{B}_2 = \mathcal{B}_2': & 1, 1, 2, 3, 3, 3 \\ \mathcal{A}_3: & 1, 1 & \mathcal{B}_3 = \mathcal{B}_3': & 1, 1, 2, 2, 2, 2. \end{array}$$

Sequences $\mathcal{A}_3$ and $\mathcal{B}_3$ are score sequences of the bipartite tournament T_3 of Figure 7.7, with partite sets U_3 and W. Proceeding from $\mathcal{A}_3$ and $\mathcal{B}_3$ to $\mathcal{A}_2$ and $\mathcal{B}_2$, we add a new vertex to U_3 and join it *to* three vertices of W having scores 1, 1, and 2 and *from* the other vertices of W, producing a bipartite tournament T_2 with partite sets U_2 and W whose score sequences are $\mathcal{A}_2$ and $\mathcal{B}_2$. Continuing in this fashion, we add a new vertex to U_2 and join it *to* vertices of W having scores 1, 1, 2, 3, and 3 and *from* the remaining vertex of W, producing a bipartite tournament T_1 with partite sets U_1 and W and score sequences $\mathcal{A}_1$ and $\mathcal{B}_1$. We complete the construction by adding one vertex to U_1 joined *to* vertices of W having scores 1, 1, 2, 3, 4 and *from* the remaining vertex of W. This produces the desired bipartite tournament T with partite sets U and W and score sequences $\mathcal{A}$ and $\mathcal{B}$.

The final result of this section, which is stated without proof, is the analogue of Theorems 7.9 and 7.10 for bipartite tournaments.

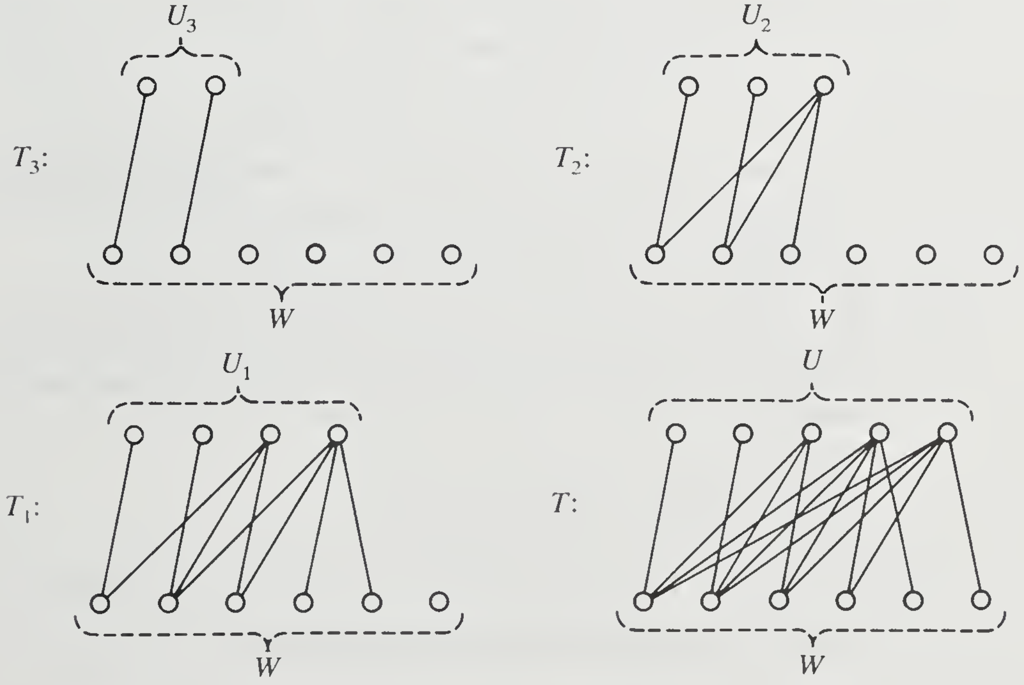


Figure 7.7 *Construction of a bipartite tournament with given score sequences*

Theorem 7.14 *Let $\mathcal{A}$: $a_1, a_2, \ldots, a_m$ and $\mathcal{B}$: $b_1, b_2, \ldots, b_n$ be nondecreasing sequences of nonnegative integers. Then $\mathcal{A}$ and $\mathcal{B}$ are score sequences of a bipartite tournament if and only if*

$$\sum_{i=1}^{k} a_i + \sum_{j=1}^{\ell} b_j \geq k\ell$$

for $1 \leq k \leq m$ and $1 \leq \ell \leq n$, with equality when $k = m$ and $\ell = n$.

Furthermore, the bipartite tournament is strong if and only if the inequality is strict except when $k = m$ and $\ell = n$.

Exercises 7.4

7.20 Which of the following sequences are score sequences? Which are score sequences of strong tournaments? For each sequence that is a score sequence, construct a tournament having the given sequence as a score sequence.

(a) 0, 1, 1, 4, 4

(b) 1, 1, 1, 4, 4, 4

(c) 1, 3, 3, 3, 3, 3, 5

(d) 2, 3, 3, 4, 4, 4, 4, 4

7.21 What can be said about a tournament T with score sequences $s_1, s_2, \ldots, s_p$ such that equality holds in (7.1) for every k, $1 \leq k \leq p$?

7.22 Show that if $\mathcal{S}$: $s_1, s_2, \ldots, s_p$ is a score sequence, then $\mathcal{S}_1$: $p-1-s_1, p-1-s_2, \ldots, p-1-s_p$ is a score sequence.

7.23 (a) Use Theorem 7.11 to determine the score sequences of the strong components of a tournament T with score sequence

$$\mathcal{S}: \quad 1, 1, 2, 3, 4, 4, 6, 8, 8, 8, 10.$$

(b) A score sequence $\mathcal{S}$ of a tournament is called *simple* if whenever T_1 and T_2 are tournaments, each having score sequence $\mathcal{S}$, then $T_1 \cong T_2$. Show that the sequence $\mathcal{S}$ of Exercise 7.23(a) is not simple.

7.24 Use Theorem 7.11 to prove that every regular tournament is strong.

7.25 Which of the following pairs of sequences are score sequences of a bipartite tournament? For each pair of sequences that are score sequences of a bipartite tournament, construct a bipartite tournament with the given pair as score sequences

(a) $\mathcal{A}$: 2, 2, 2, 3 $\quad$ $\mathcal{B}$: 1, 1, 2, 3, 4

(b) $\mathcal{A}$: 1, 2, 2, 3 $\quad$ $\mathcal{B}$: 0, 1, 1, 5, 5

7.26 Use Theorem 7.14 to prove that every regular bipartite tournament is strong.

Chapter Eight

Factors and Factorizations

We now consider special subgraphs that a graph may contain or into which a graph may be decomposed. In particular, we emphasize isomorphic decompositions.

8.1 Matchings

Two distinct vertices or edges in a graph G are *independent* if they are not adjacent in G. A set of pairwise independent edges of G is called a *matching* in G, while a matching of maximum cardinality is a *maximum matching* in G. In the graph G of Figure 8.1, the set $M_1 = \{e_1, e_4\}$ is a matching that is not a maximum matching, while $M_2 = \{e_1, e_3, e_5\}$ and $M_3 = \{e_1, e_3, e_6\}$ are maximum matchings in G.

If M is a matching in a graph G with the property that every vertex of G is incident with an edge of M, then M is a *perfect matching* in G. Clearly, if G has a perfect matching M, then G has even order and $\langle M \rangle$ is a 1-regular spanning subgraph of G. Thus, the graph G of Figure 8.1 cannot have a perfect matching.

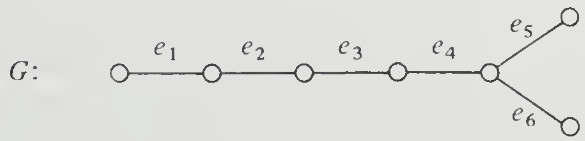


Figure 8.1 *Matchings and maximum matchings*

In order to present a characterization of maximum matchings, we introduce a few new terms. Let M be a specified matching in a graph G. An edge e of G that is not in M is called a *weak edge* (with respect to M). A *weak vertex* (with respect to M) is a vertex of G incident only with weak edges. An *alternating path* of G is a path whose edges are alternately in M and not in M. The following theorem will prove to be useful.

Theorem 8.1 *Let M_1 and M_2 be matchings in a graph G. Then each component of the spanning subgraph H of G with $E(H) = (M_1 - M_2) \cup (M_2 - M_1)$ is one of the following types:*
(a) an isolated vertex,
(b) an even cycle whose edges are alternately in M_1 and in M_2,
(c) a nontrivial path whose edges are alternately in M_1 and in M_2 and such that each end-vertex of the path is weak with respect to exactly one of M_1 and M_2.

Proof First we note that $\Delta(H) \leq 2$, for if H contains a vertex v such that $\deg_H v \geq 3$, then v is incident with at least two edges in the same matching. Since $\Delta(H) \leq 2$, every component of H is a path (possibly trivial) or a cycle. Since no two edges in a matching are adjacent, the edges of each cycle and path in H are alternately in M_1 and in M_2. Thus each cycle in H is even.

Suppose $e = uv$ is an edge of H and u is the end-vertex of a path P that is a component of H. The proof will be complete once we have shown that u is weak with respect to exactly one of M_1 and M_2. Since $e \in E(H)$, $e \in M_1 - M_2$ or $e \in M_2 - M_1$. If $e \in M_1 - M_2$, then u is not weak with respect to M_1. We show that u is weak with respect to M_2. If this is not the case, then there is an edge f in M_2 (thus $f \neq e$) such that f is incident to u. Since e and f are adjacent, $f \notin M_1$. Thus, $f \in M_2 - M_1 \subseteq E(H)$. This, however, is impossible since u is the end-vertex of P. Therefore, u is weak with respect to M_2; similarly, if $e \in M_2 - M_1$, then u is weak with respect to M_1. ■

The following characterization of maximum matchings is due to Berge [B5].

Theorem 8.2 *A matching M in a graph G is a maximum matching if and only if there exists no alternating path between any two distinct weak vertices of G.*

Proof Assume M is a maximum matching in G and that there exists an alternating path P between two distinct weak vertices of G. Necessarily, P has odd length. Let M' denote the edges of P belonging to M, and let $M'' = E(P) - M'$. Since $|M''| = |M'| + 1$, the set $(M - M') \cup M''$ is a matching having cardinality exceeding that of M, producing a contradiction.

Conversely, let M_1 be a matching in a graph G, and suppose there exists no alternating path between any two distinct weak vertices of G. We verify that M_1 is a maximum matching. Let M_2 be a maximum matching in G. By the first part of the proof, there exists, with respect to M_2, no alternating path between any two distinct weak vertices of G. Let H be the spanning subgraph of G with $E(H) = (M_1 - M_2) \cup (M_2 - M_1)$. Suppose H_1 is a component of H that is neither an isolated vertex nor an even cycle. Then it follows from Theorem 8.1 that H_1 is a path of even length whose edges are alternately in M_1 and in M_2, for otherwise, there would exist an alternating path between two vertices that are both weak with respect to M_1 or both weak with respect to M_2. This is impossible, however. It now follows by Theorem 8.1 that $|M_1 - M_2| = |M_2 - M_1|$, which, in turn, implies that $|M_1| = |M_2|$. Hence, M_1 is a maximum matching. ■

According to Theorem 8.2, if a matching M is given, it is possible to decide whether M is a maximum matching by searching for all alternating u-v paths, where u and v are distinct weak vertices of G.

In applications, maximum matchings in bipartite graphs have proved to be most useful. The next result, namely Theorem 8.3, attributed to König [K8] and Hall [H5], is of interest in its own right.

In a graph G, a nonempty subset U_1 of $V(G)$ is said to be *matched* to a subset U_2 of $V(G)$ disjoint from U_1 if there exists a matching M in G such that each edge of M is incident with a vertex of U_1 and a vertex of U_2 and every vertex of U_1 is incident with an edge of M, as is every vertex of U_2. If $M \subseteq M^*$, where M^* is also a matching in G, we also say that U_1 is *matched under* M^* to U_2.

Let U be a nonempty set of vertices of a graph G and let its neighborhood $N(U)$ denote the set of all vertices of G adjacent with at least one element of U. Then the set U is said to be *nondeficient* if $|N(S)| \geq |S|$ for every nonempty subset S of U.

Theorem 8.3 *Let G be a bipartite graph with partite sets V_1 and V_2. The set V_1 can be matched to a subset of V_2 if and only if V_1 is nondeficient.*

Proof Suppose that V_1 can be matched to a subset of V_2 under a matching M^*. Then every nonempty subset S of V_1 can be matched under M^* to some subset of V_2, implying that $|N(S)| \geq |S|$ so that V_1 is nondeficient.

To verify the converse, let G be a bipartite graph for which V_1 is nondeficient and suppose that V_1 cannot be matched to a subset of V_2. Let M be a maximum matching in G. By assumption, there is a vertex v in V_1 that is weak with respect to M. Let S be the set of all vertices of G that are connected to v by an alternating path. Since M is a maximum matching, an application of Theorem 8.2 yields that v is the only weak vertex in S.

Let $W_1 = S \cap V_1$ and let $W_2 = S \cap V_2$. Using the definition of the set S, together with the fact that no vertex of $S - \{v\}$ is weak, we conclude that $W_1 - \{v\}$ is matched under M to W_2. Therefore, $|W_2| = |W_1| - 1$ and $W_2 \subseteq N(W_1)$. Furthermore, for every $w \in N(W_1)$, the graph G contains an alternating v-w path so that $N(W_1) \subseteq W_2$. Thus, $N(W_1) = W_2$, and

$$|N(W_1)| = |W_2| = |W_1| - 1 < |W_1|.$$

This, however, contradicts the fact that V_1 is nondeficient. ■

We are now in a position to present a well-known theorem due to Hall [H5]. A collection $S_1, S_2, \ldots, S_n$, $n \geq 1$, of finite nonempty sets is said to have a *system of distinct representatives* or a *transversal* if there exists a set $\{s_1, s_2, \ldots, s_n\}$ of distinct elements such that $s_i \in S_i$ for $1 \leq i \leq n$. (For a thorough treatment of transversals, see [M7].)

Theorem 8.4 (Hall) *A collection $S_1, S_2, \ldots, S_n$, $n \geq 1$, of finite nonempty sets has a system of distinct representatives if and only if the union of any k of these sets contains at least k elements, for each k such that $1 \leq k \leq n$.*

Proof From the collection $S_1, S_2, \ldots, S_n$, $n \geq 1$, of finite, nonempty sets we construct a bipartite graph G with partite sets V_1 and V_2 in the following manner. Let V_1 be the set $\{v_1, v_2, \ldots, v_n\}$ of distinct vertices, where v_i corresponds to the set S_i, and let V_2 be a set of vertices disjoint from V_1 such that $|V_2| = |\bigcup_{i=1}^{n} S_i|$, where there is a one-to-one correspondence between the elements of V_2 and those of $\bigcup_{i=1}^{n} S_i$. The construction of G is completed by joining a vertex v of V_1 with a vertex w of V_2 if and only if v corresponds to a set S_i and w corresponds to an element of S_i. From the manner in which G is defined, it follows that V_1 is nondeficient if and only if the union of any k of the sets S_i contains at least k elements. Now obviously, the sets S_i have a system of distinct representatives if and only if V_1 can be matched to a subset of V_2. Theorem 8.3 now produces the desired result. ■

The preceding discussion is directly related to a well-known combinatorial problem called the *Marriage Problem*: Given a set of boys and a set of girls where each girl knows some of the boys, under what conditions can all girls get married, each to a boy she knows? In this context, Theorem 8.4 may be reformulated to produce what is often referred to as *Hall's Marriage Theorem*: If there are n girls, then the Marriage Problem has a solution if and only if every subset of k girls ($1 \leq k \leq n$) collectively know at least k boys.

Exercises 8.1

8.1 Show that a tree has at most one perfect matching.

8.2 Determine the maximum size of a graph of order p having a maximum matching of k edges, where (a) $p = 2k$ and (b) $p = 2k + 2$.

8.3 Use Menger's Theorem to prove Theorem 8.3.

8.2 Factorizations

A *factor* of a graph G is a (possibly empty) spanning subgraph of G. If G_1, G_2, ..., $G_n (n \geq 2)$ are edge-disjoint factors of a graph G such that $\bigcup_{i=1}^{n} E(G_i) = E(G)$, then we write $G = G_1 \oplus G_2 \oplus \cdots \oplus G_n$ and say G is the *edge sum* of the factors $G_1, G_2, \ldots, G_n$. This edge sum is called a *factorization* of G into the factors $G_1, G_2, \ldots, G_n$. We have actually already considered factorizations when we discussed the arboricity of a graph; namely, a nonempty graph G can be factored into $a_1(G)$ acyclic factors.

An r-regular factor of a graph G is referred to as an *r-factor* of G. Hence, a graph has a 1-factor if and only if it contains a perfect matching. If there exists a factorization of a graph G such that each factor is an r-factor (for a fixed r), then G is *r-factorable.* If G is an r-factorable graph, then necessarily G is k-regular for some k that is a multiple of r.

More generally, a spanning subgraph H of a graph G is called an *isofactor* of G if G contains a factorization, each factor of which is isomorphic to H. If H is an isofactor of G, then we also say that G is *H-factorable* and that G has an *isomorphic factorization* into the factor H. (Clearly, if a graph G is H-factorable, then $q(H)|q(G)$.) Consequently, a graph G of order $p = 2n$ (≥ 2) is 1-factorable if and only if nK_2 is an isofactor of G (or, equivalently, G is nK_2-factorable).

The problems involving these concepts that have received the most attention deal with whether a given graph contains a 1-factor and whether a given regular graph is 1-factorable. Graphs that contain 1-factors have been characterized by Tutte [T13].

The following proof of Tutte's Theorem is due to Anderson [A1]. An *odd component* of a graph is a component of odd order.

Theorem 8.5 (Tutte) *A nontrivial graph G has a 1-factor if and only if for every proper subset S of $V(G)$, the number of odd components of $G-S$ does not exceed $|S|$.*

Proof Let F be a 1-factor of G. Assume, to the contrary, that there exists a proper subset W of $V(G)$ such that the number of odd components of $G-W$ exceeds $|W|$. For each odd component H of $G-W$, there is necessarily an edge of F joining a vertex of H with a vertex of W. This implies, however, that at least one vertex of W is incident with at least two edges of F, which is impossible. This establishes the necessity.

Next we consider the sufficiency. For a subset S of $V(G)$, denote the number of odd components of $G-S$ by $k_0(G-S)$. Hence, the hypothesis for G may now be restated as $k_0(G-S) \leq |S|$ for every proper subset S of $V(G)$. In particular, $k_0(G-\varnothing) \leq |\varnothing| = 0$, implying that G has only even components and therefore has even order p. Furthermore, we note that for each proper subset S of $V(G)$, the numbers $k_0(G-S)$ and $|S|$ are of the same parity, since p is even.

We proceed by induction on even positive integers p. If G is a graph of order $p=2$ such that $k_0(G-S) \leq |S|$ for every proper subset S of $V(G)$, then $G \cong K_2$ and G has a 1-factor.

Assume for all graphs H of even order less than p (where $p \geq 4$ is an even integer) that if $k_0(H-W) \leq |W|$ for every proper subset W of $V(H)$, then H has a 1-factor. Let G be a graph of order p and assume that $k_0(G-S) \leq |S|$ for each proper subset S of $V(G)$. We consider two cases.

Case 1: Suppose that $k_0(G-S) < |S|$ for all subsets S of $V(G)$ with $2 \leq |S| < p$. Since $k_0(G-S)$ and $|S|$ are of the same parity, $k_0(G-S) \leq |S|-2$ for all subsets S of $V(G)$ with $2 \leq |S| < p$. Let $e = uv$ be an edge of G and consider $G-u-v$. Let T be a proper subset of $V(G-u-v)$. It follows that $k_0(G-u-v-T) \leq |T|$, for suppose, to the contrary, that $k_0(G-u-v-T) > |T|$. Then

$$k_0(G-u-v-T) > |T| = |T \cup \{u, v\}| - 2,$$

so that $k_0(G-(T \cup \{u, v\})) \geq |T \cup \{u, v\}|$, contradicting our supposition. Thus, by the inductive hypothesis, $G-u-v$ has a 1-factor and, hence, so does G.

Case 2: Suppose there exists a subset R of $V(G)$ such that $k_0(G-R) = |R|$, where $2 \leq |R| < p$. Among all such sets R, let S be one of maximum cardinality, where $k_0(G-S) = |S| = n$. Further, let $G_1, G_2, \ldots, G_n$ denote the odd components of $G-S$. These are the only components of $G-S$, for if G_0 were an even component of $G-S$ and $u_0 \in V(G_0)$, then $k_0(G-(S \cup \{u_0\})) \geq n+1 = |S \cup \{u_0\}|$, implying necessarily that $k_0(G-(S \cup \{u_0\})) = |S \cup \{u_0\}|$, which contradicts the maximum property of S.

For $i = 1, 2, \ldots, n$, let S_i denote the set of those vertices of S adjacent to one or more vertices of G_i. Each set S_i is nonempty; otherwise some G_i would be an odd component of G. The union of any k of the sets $S_1, S_2, \ldots, S_n$ contains at least k vertices for each k with $1 \leqslant k \leqslant n$; for otherwise, there exists k $(1 \leqslant k \leqslant n)$ such that the union T of some k sets contains less than k vertices. This would imply, however, that $k_0(G-T) > |T|$, which is impossible. Thus, we may employ Theorem 8.4 to produce a system of distinct representatives for $S_1, S_2, \ldots, S_n$. This implies that S contains vertices $v_1, v_2, \ldots, v_n$, and each G_i contains a vertex u_i $(1 \leqslant i \leqslant n)$ such that $u_i v_i \in E(G)$ for $i = 1, 2, \ldots, n$.

Let W be a proper subset of $V(G_i - u_i)$, $1 \leqslant i \leqslant n$. We show that $k_0(G_i - u_i - W) \leqslant |W|$, for suppose that $k_0(G_i - u_i - W) > |W|$. Since $G_i - u_i$ has even order, $k_0(G_i - u_i - W)$ and $|W|$ are of the same parity and so $k_0(G_i - u_i - W) \geqslant |W| + 2$. Thus,

$$\begin{aligned} k_0(G - (S \cup W \cup \{u_i\})) &= k_0(G_i - u_i - W) + k_0(G - S) - 1 \\ &\geqslant |S| + |W| + 1 \\ &= |S \cup W \cup \{u_i\}|. \end{aligned}$$

This, however, contradicts the maximum property of S. Therefore, $k_0(G_i - u_i - W) \leqslant |W|$ as claimed, implying by the inductive hypothesis that, for $i = 1, 2, \ldots, n$, the subgraph $G_i - u_i$ has a 1-factor. This fact, together with the existence of the edges $u_i v_i$ $(1 \leqslant i \leqslant n)$, produces a 1-factor in G. ■

By definition, every 1-regular graph contains a 1-factor and, trivially, is 1-factorable. A 2-regular graph G contains a 1-factor if and only if every component of G is an even cycle; such graphs are, of course, also 1-factorable. This brings us to the 3-regular or cubic graphs. First, not all cubic graphs contain 1-factors, as is shown by the graph of Figure 8.2.

Petersen [P1], however, proved that every cubic graph that fails to contain a 1-factor possesses bridges.

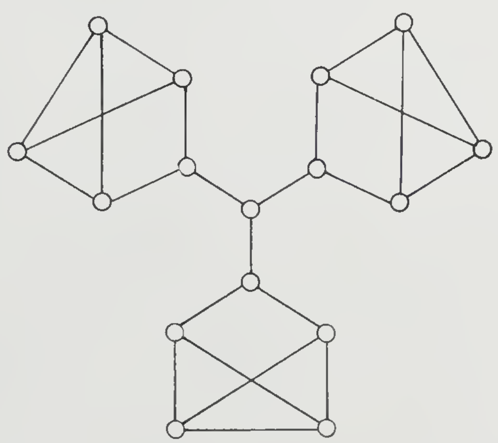


Figure 8.2 *A cubic graph containing no 1-factors*

Theorem 8.6 (Petersen) *Every bridgeless cubic graph can be expressed as the edge sum of a 1-factor and a 2-factor.*

Proof It is sufficient to show that every bridgeless cubic graph G has a 1-factor. Assume, to the contrary, that G has no 1-factor. Then by Theorem 8.5, $V(G)$ has a proper subset S such that the number of odd components of $G-S$ exceeds $|S|$. Let $k=|S|$ and let $G_1, G_2, \ldots, G_n$ $(n>k)$ be the odd components of $G-S$. There must be at least one edge joining a vertex of G_i to a vertex of S, for each $i=1, 2, \ldots, n$; for otherwise, G_i is a cubic graph of odd order. On the other hand, since G contains no bridges, there cannot be exactly one such edge; that is, there are at least two edges joining G_i and S, for each $i=1, 2, \ldots, n$.

Suppose that for some $i=1, 2, \ldots, n$, there are exactly two edges joining G_i and S. Then there are an odd number of odd vertices in the component G_i of $G-S$, which cannot happen. Hence, for each $i=1, 2, \ldots, n$, there are at least three edges joining G_i and S. Therefore, the total number of edges joining $\bigcup_{i=1}^{n} V(G_i)$ and S is at least $3n$. However, since each of the k vertices of S has degree 3, the number of edges joining $\bigcup_{i=1}^{n} V(G_i)$ and S is at most $3k$. Therefore, $3k \geq 3n$, which is a contradiction since $3n>3k$. Hence, no such set S exists. By Theorem 8.5, then, we conclude that G has a 1-factor. ■

Theorem 8.6 states that every bridgeless cubic graph G can be factored into a 1-factor and a 2-factor. If the 2-factor can be factored into two 1-factors, then G is 1-factorable. However, not every bridgeless cubic graph is 1-factorable, as the Petersen graph (see Figure 8.3) illustrates and as is verified in the following theorem.

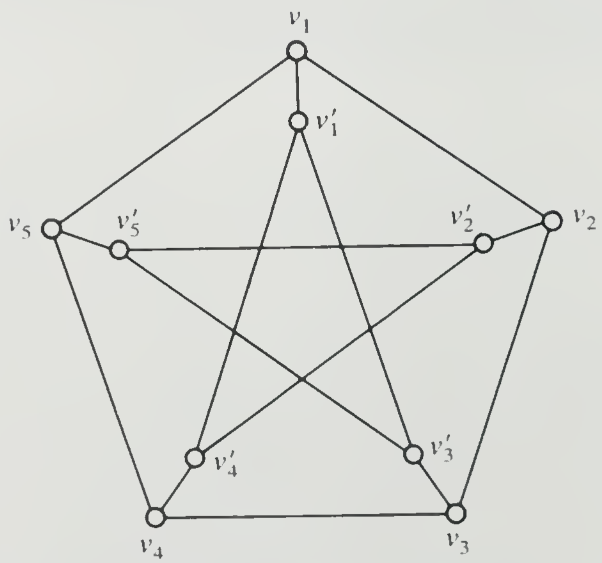


Figure 8.3 *The Petersen graph*

Theorem 8.7 *The Petersen graph is not 1-factorable.*

Proof Suppose, to the contrary, that the Petersen graph is 1-factorable. Then it can be factored into three 1-factors, one of which, say F, contains at least two edges of the set $S = \{v_i v_i' \mid i = 1, 2, 3, 4, 5\}$ (see Figure 8.3). Deleting any two edges of S and their incident vertices from the Petersen graph gives the graph shown in Figure 8.4, where the edges belonging to S are so labeled. The only 1-factor of this subgraph consists of the three edges in S, and so F must be the 1-factor induced by the elements of S. But then this implies that the graph $2C_5$ that results by deleting the edges of S from the Petersen graph is 1-factorable, which is impossible. ■

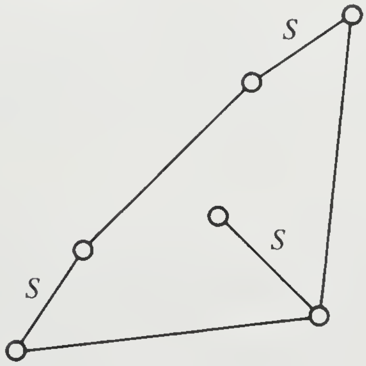


Figure 8.4 *A subgraph of the Petersen graph*

The preceding result shows that the graph $5K_2$ is not an isofactor of the Petersen graph; however, S. Ruiz verified that the Petersen graph has exactly ten isofactors of size 5. They are shown in Figure 8.5.

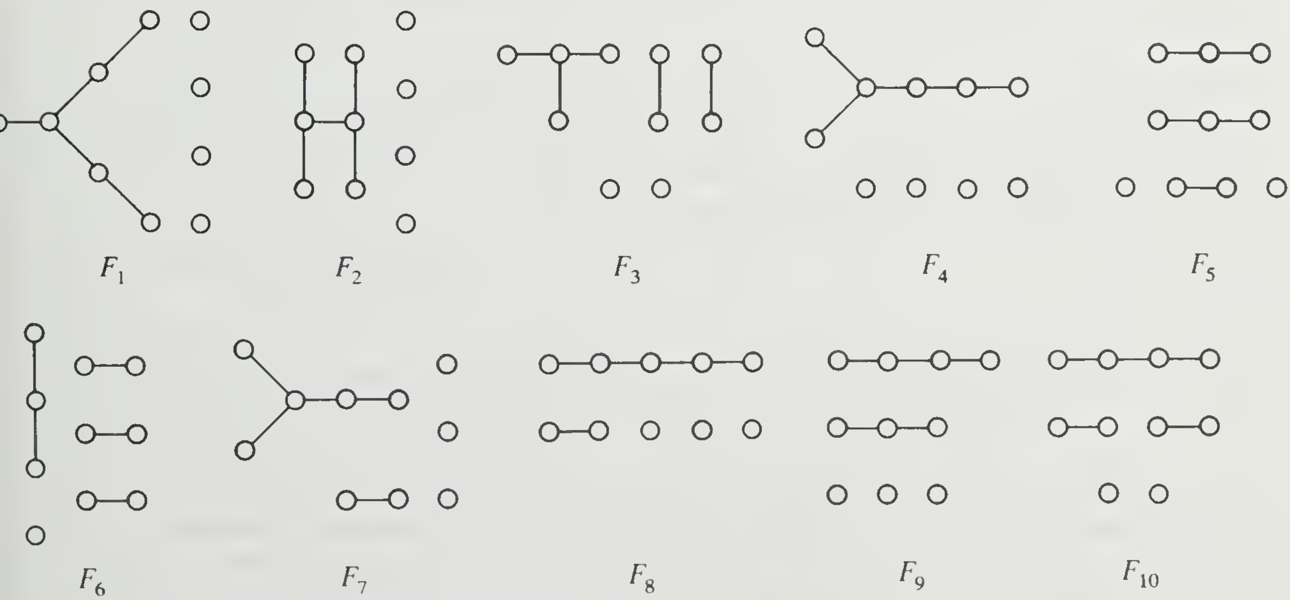


Figure 8.5 *The isofactors of size 5 of the Petersen graph*

A useful device for showing that the graphs of Figure 8.5 are in fact isofactors of the Petersen graph involves the alternative drawing of the Petersen graph shown in Figure 8.6(a). We illustrate the method for F_1 (drawn in Figure 8.6(b)). Rotating the edges of F_1 about vertex v in angles of 120° and 240°, respectively, produces an F_1-factorization of the Petersen graph.

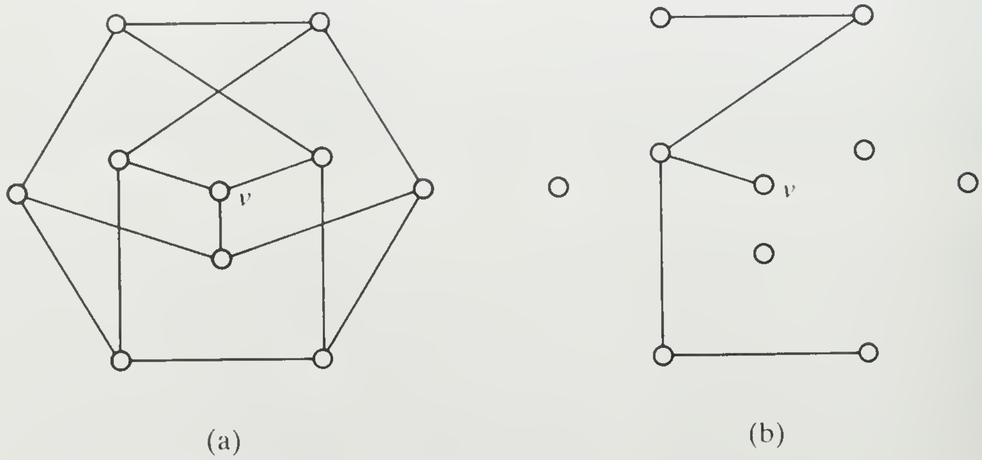


Figure 8.6 *An isofactor of the Petersen graph*

We next consider a somewhat reverse question: Is the Petersen graph an isofactor of some graph? Of course, the Petersen graph is an isofactor of itself. Also, it is not difficult to show that the Petersen graph is an isofactor of a 6-regular graph. What is considerably less clear is whether the Petersen graph is an isofactor of K_{10}. This question is answered in the negative in the following result, the proof of which is due to A. J. Schwenk.

Theorem 8.8 *The Petersen graph is not an isofactor of* K_{10}.

Proof Assume, to the contrary, that $K_{10} = P_1 \oplus P_2 \oplus P_3$, where P_i, $i = 1, 2, 3$, is isomorphic to the Petersen graph. Thus,

$$A(P_1) + A(P_2) + A(P_3) = J - I, \tag{8.1}$$

where J is the 10×10 matrix each of whose entries is 1, I is the 10×10 identity matrix, and $A(P_i)$, $i = 1, 2, 3$, is the adjacency matrix of P_i.

The Petersen graph is known to have eigenvalues 3, 1 (of multiplicity 5) and -2 (of multiplicity 4). Since the Petersen graph is cubic, the column vector $\vec{v} = (1, 1, \ldots, 1)'$ is an eigenvector corresponding to the eigenvalue 3 for $A(P_i)$, $i = 1, 2, 3$. Therefore, the eigenspaces corresponding to the eigenvalues 1 and -2 lie in the 9-dimensional orthogonal complement S of the eigenspace spanned by $\vec{v}$. For the eigenvalue 1, the eigenspaces of $A(P_1)$ and $A(P_2)$ are both 5-dimensional subspaces of S so that there exists a nonzero vector $\vec{w}$ in the intersection of these subspaces. Therefore, $A(P_i)\vec{w} = 1\vec{w} = \vec{w}$ for $i = 1, 2$. Equation (8.1) implies that

$$A(P_1)\vec{w} + A(P_2)\vec{w} + A(P_3)\vec{w} = J\vec{w} - I\vec{w}. \tag{8.2}$$

Since $\vec{v}$ and $\vec{w}$ are orthogonal, it follows that $J\vec{w} = 0$ and, consequently, (8.2) yields

$$\vec{w} + \vec{w} + A(P_3)\vec{w} = \vec{0} - \vec{w}$$

or, equivalently, $A(P_3)\vec{w} = -3\vec{w}$. This implies that -3 is an eigenvalue of $A(P_3)$, which produces a contradiction. ■

We now present some results concerning r-factorability for $r = 1, 2$. Although Theorem 8.7 shows that a regular graph of even order need not be 1-factorable, König [K7] has shown that every such bipartite graph is 1-factorable.

Theorem 8.9 *Every regular bipartite graph of degree $r \geq 1$ is 1-factorable.*

Proof We proceed by induction on r, the result being obvious for $r = 1$. Assume, then, that every regular bipartite graph of degree $r-1$, $r \geq 2$, is 1-factorable, and let G be a regular bipartite graph of degree r, where V_1 and V_2 are the partite sets of G.

We now show that V_1 is nondeficient. Let S be a nonempty subset of V_1. The number of edges of G incident with the vertices of S is $r|S|$. These edges are, of course, also incident with the vertices of $N(S)$. Since G is r-regular, the number of edges joining S and $N(S)$ cannot exceed $r|N(S)|$. Hence, $r|N(S)| \geq r|S|$ so that $|N(S)| \geq |S|$. Therefore, V_1 is nondeficient, implying by Theorem 8.3 that V_1 can be matched to a subset of V_2. Since G is regular of positive degree, $|V_1| = |V_2|$; thus, G has a 1-factor F. The removal of the edges of F from G results in a bipartite graph G' that is regular of degree $r-1$. By the inductive hypothesis, G' is 1-factorable, implying that G also is 1-factorable. ■

The 2-factorable graphs have been characterized by Petersen [P1]. We present this result next.

Theorem 8.10 *A nonempty graph G is 2-factorable if and only if G is $2n$-regular for some $n \geq 1$.*

Proof Certainly if G is 2-factorable, then G is regular of even positive degree. Conversely, suppose G is $2n$-regular for some $n \geq 1$. Without loss of generality, we assume that G is connected. Hence, G is eulerian and contains an eulerian circuit C.

Let $V(G) = \{v_1, v_2, \ldots, v_p\}$. We define a bipartite graph H with partite sets $U = \{u_1, u_2, \ldots, u_p\}$ and $W = \{w_1, w_2, \ldots, w_p\}$, where

$$E(H) = \{u_i w_j | v_j \text{ immediately follows } v_i \text{ on } C\}.$$

The graph H is n-regular and so, by Theorem 8.9, is 1-factorable. Hence, $H = F_1 \oplus F_2 \oplus \cdots \oplus F_n$, where $F_k (1 \leq k \leq n)$ is a 1-factor.

Corresponding to each 1-factor F_k of H is a permutation α_k on the set $\{1, 2, \ldots, p\}$, defined by $\alpha_k(i) = j$ if $u_i w_j \in E(F_k)$. Let α_k be expressed as a product of disjoint permutation cycles. There is no permutation cycle of length 1 in this product; for if (i) were a permutation cycle, then this would imply that $\alpha_k(i) = i$. However, this further implies that $u_i w_i \in E(F_k)$ and that $v_i v_i \in E(C)$, which is impossible. Also there is no permutation cycle of length 2 in this product; for if (ij) were a permutation cycle, then $\alpha_k(i) = j$ and $\alpha_k(j) = i$. This would indicate that $u_i w_j$, $u_j w_i \in E(F_k)$ and that v_j both immediately follows and precedes v_i on C, contradicting the fact that no edge is repeated on a circuit. Thus, every permutation cycle in α_k has length at least 3.

Each permutation cycle in α_k therefore gives rise to a cycle in G, and the product of disjoint permutation cycles in α_k produces a collection of mutually disjoint cycles in G containing all vertices of G; that is, a 2-factor in G. Since the 1-factors F_k in H are mutually edge-disjoint, the resulting 2-factors in G are mutually edge-disjoint. Hence, G is 2-factorable. ■

We now consider 1-factorization and 2-factorization of a very special class of graphs, namely the complete graphs. If K_p is 1-factorable, then certainly p must be even; if K_p is 2-factorable, then p is odd (and at least three) since K_p is $(p-1)$-regular. As the next two theorems indicate, it is, in fact, the case that K_{2n} is 1-factorable and that $K_{2n+1} (n \geq 1)$ can be factored into connected 2-factors.

Theorem 8.11 *For every positive integer n, the graph K_{2n} is 1-factorable.*

Proof The result is obvious for $n = 1$. Thus we assume $n \geq 2$. Let $V(K_{2n}) = \{v_0, v_1, \ldots, v_{2n-1}\}$. Arrange the vertices $v_1, v_2, \ldots, v_{2n-1}$ in a regular $(2n-1)$-gon, and place v_0 in the center. Join every two vertices by a straight line segment. For $i = 1, 2, \ldots, 2n-1$, define the edge set of the factor F_i to be the edge $v_0 v_i$ together with all those edges perpendicular to $v_0 v_i$. Then $K_{2n} = F_1 \oplus F_2 \oplus \cdots \oplus F_{2n-1}$, where F_i is a 1-factor of K_{2n}, $i = 1, 2, \ldots, 2n-1$. ■

A generalization of this theorem will be presented in the next section. Theorem 8.11 and its proof are illustrated in Figure 8.7 for K_6.

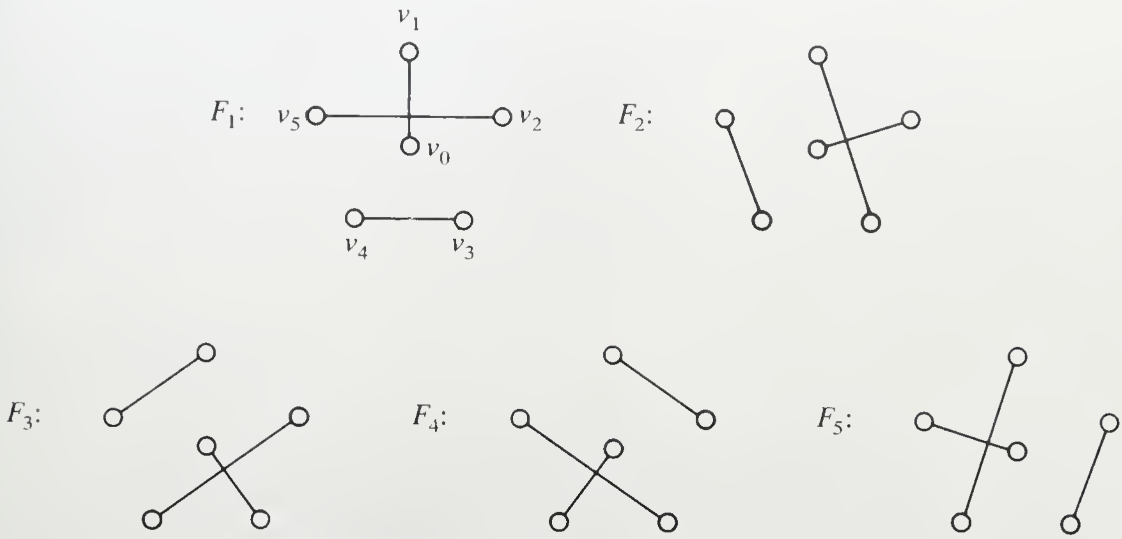


Figure 8.7 *A 1-factorization of K_6*

Theorem 8.12 *For every positive integer n, the graph K_{2n+1} can be factored into n hamiltonian cycles.*

Proof Since the result is clear for $n = 1$, we may assume that $n \geq 2$. Let $V(K_{2n+1}) = \{v_0, v_1, \ldots, v_{2n}\}$. Arrange the vertices $v_1, v_2, \ldots, v_{2n}$ in a regular $2n$-gon and place v_0 in some convenient position. Join every two vertices by a straight line segment, thereby producing K_{2n+1}. We define the edge set of F_1 to consist of v_0v_1, v_0v_{n+1}, all edges parallel to v_1v_2 and all edges parallel to $v_{2n}v_2$. (See F_1 in Figure 8.8 for the case $n = 3$.) In general, for $i = 1, 2, \ldots, n$, we define the edge set of the factor F_i to consist of v_0v_i, v_0v_{n+i}, all edges parallel to v_iv_{i+1} and all edges parallel to $v_{i-1}v_{i+1}$, where the subscripts are expressed modulo $2n$. Then $K_{2n+1} = F_1 \oplus F_2 \oplus \cdots \oplus F_n$, where F_i is the hamiltonian cycle

$$v_0, v_i, v_{i+1}, v_{i-1}, v_{i+2}, v_{i-2}, \ldots, v_{n+i-1}, v_{n+i+1}, v_{n+i}, v_0. \blacksquare$$

This result is illustrated in Figure 8.8 for K_7.

Corollary 8.12 *For every positive integer n, the graph K_{2n} is the edge sum of n hamiltonian paths.*

Using the construction employed in the proof of Theorem 8.11, we can obtain the following related result.

Theorem 8.13 *For every positive integer n, the graph K_{2n} is the edge sum of $n - 1$ hamiltonian cycles and a 1-factor.*

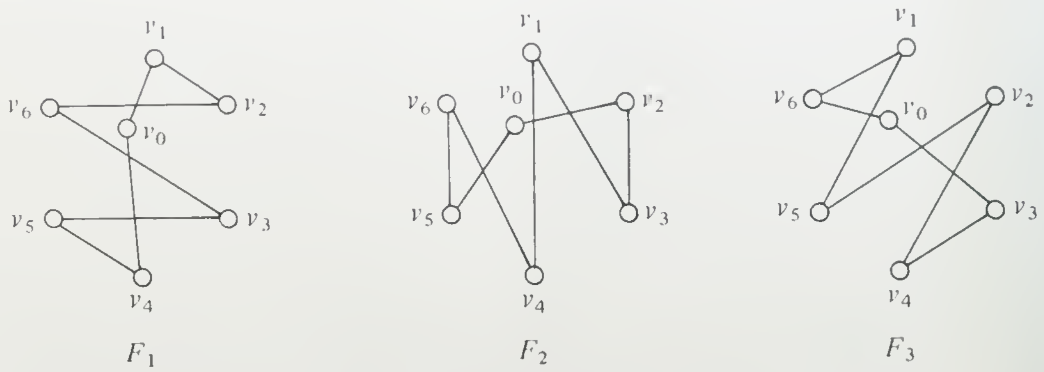


Figure 8.8 *A factorization of K_7 into hamiltonian cycles*

Exercises 8.2

8.4 Show that each of the graphs of Figure 8.5 is an isofactor of the Petersen graph.

8.5 Show that $P_6 \cup 4K_1$ is not an isofactor of the Petersen graph.

8.6 Determine all isofactors of size 3 of the Petersen graph.

8.7 An isofactor F of a graph G is called *proper* if $1 < q(F) < q(G)$. A graph G is *prime* if it has no proper isofactors.

(a) Show that every graph of prime size is prime.

(b) Give an example of a prime graph of composite size.

8.8 Give an example of a connected graph G of composite size having the property that whenever F is a factor of G and $q(F)|q(G)$, then F is an isofactor of G.

8.9 (a) Prove that Q_n is 1-factorable for all $n \geq 1$.
(b) Prove that Q_n is m-factorable if and only if $m|n$.

8.10 Prove that every cubic graph with at most two bridges contains a 1-factor by using a technique similar to that used in the proof of Theorem 8.6.

8.11 (a) Let G be a graph, all of whose vertices are odd, and let $V_1 \cup V_2$ be a partition of $V(G)$, where E' is the set of edges joining V_1 and V_2. Prove that $|V_1|$ and $|E'|$ are of the same parity.

(b) Prove that every $(2m+1)$-regular, $2m$-edge-connected graph, $m \geq 1$, can be factored into a 1-factor and m 2-factors.

8.12 Let n be a nonnegative even integer and $p \geq 5$ an odd integer with $n \leq p-3$. Prove that there exists a graph G of order p having degree set $\mathcal{D}_G = \{n, n+2\}$.

8.13 Using the proof of Theorem 8.10, give a 2-factorization of the graph of the octahedron (the graph $K_{3(2)}$).

8.14 Prove Corollary 8.12.

8.15 Prove that $K_{2n+1} (n \geq 1)$ cannot be factored into hamiltonian paths.

8.16 Give a constructive proof of Theorem 8.13.

8.3 Decompositions

Very similar to the concept of a factorization of a graph G is a decomposition of G. A *decomposition* of G is a collection $\{H_i\}$ of subgraphs of G such that $H_i = \langle E_i \rangle$ for some subset E_i of $E(G)$ and where $\{E_i\}$ is a partition of $E(G)$. If $\{H_i\}$ is a decomposition of G, then we write G as the edge sum $H_1 \oplus H_2 \oplus \cdots \oplus H_n$, where $n = |\{H_i\}|$, as we do with factorizations. Indeed, if $G = H_1 \oplus H_2 \oplus \cdots \oplus H_n$ is a decomposition of a graph G of order p and we define $F_i = H_i \cup [p - p(H_i)] K_1$, then $F_1 \oplus F_2 \oplus \cdots \oplus F_n$ is a factorization of G. If $\{H_i\}$ is a decomposition of a graph G and $H_i \cong H$ for each i, then G is *H-decomposable.*

A K_3-decomposable complete graph is called a *Steiner triple system.* Kirkman [K4] has characterized Steiner triple systems.

Theorem 8.14 *The complete graph K_p is K_3-decomposable if and only if p (≥ 3) is odd and $3 | \binom{p}{2}$.*

For K_p to be K_{n+1}-decomposable, the conditions $n | (p-1)$ and $\binom{n+1}{2} | \binom{p}{2}$ are necessary. These conditions are not sufficient in general, however. For $p = n^2 + n + 1$, Ryser [R11] showed that K_p is K_{n+1}-decomposable if and only if there exists a projective plane of order n; and in order for a projective plane of order n to exist, n must satisfy the Bruck-Ryser conditions [BR2] that $n \equiv 0$ or 1 (mod 4) and $n = x^2 + y^2$ for integers x and y. The smallest value of n for which the existence of a projective plane of order n is unknown is $n = 10$.

Whenever K_p is decomposed into graphs K_{n+1} where $n + 1 < p$, then we have an example of the combinatorial structure referred to as a *balanced incomplete block design.* Thus the concept of graph decompositions may be thought of as a generalized block design.

The vast majority of factorization and decomposition results deal with factoring or decomposing complete graphs into a specific graph or graphs. R. M. Wilson [W8] proved that for every nonempty graph H, there exist infinitely many positive integers p such that K_p is H-decomposable. As a consequence, there exists a regular H-decomposable graph for each nonempty graph H. This result also appears in [F2]. The proof we give is due to Fink and

Ruiz [FR1] and was inspired by a proof technique of Rosa [R9].

Theorem 8.15 *Let H be a graph without isolated vertices. Then there exists a regular H-decomposable graph.*

Proof Label the vertices of H with distinct positive integers so that the induced edge-labeling is one-to-one; i.e., let $f\colon V(H) \rightarrow \mathbb{N}$ (where $\mathbb{N}$ is the set of natural numbers) be a one-to-one function such that $uv, xy \in E(H)$ and $uv \neq xy$ imply that the label $|f(u)-f(v)|$ of uv is different from the label $|f(x)-f(y)|$ of xy. (Although there are many ways to produce such a labeling, one method is to label the ith vertex of H by 2^{i-1}.)

We now construct a regular, H-decomposable graph G of order p, where

$$p = 1 + 2\max\{|f(u)-f(v)| \mid uv \in E(H)\}.$$

Let $V(G) = \{v_1, v_2, \ldots, v_p\}$ and arrange the vertices cyclically in counter-clockwise order about a regular p-gon. Next we define a graph H_1 by

$$V(H_1) = \{v_{f(x)} \mid x \in V(H)\}$$

and

$$E(H_1) = \{v_{f(x)}v_{f(y)} \mid xy \in E(H)\}.$$

For $i = 2, 3, \ldots, p$, define H_i by cyclically rotating H_1 about the p-gon through a counterclockwise angle of $2\pi(i-1)/p$ radians; in particular

$$V(H_i) = \{v_{f(x)+i-1} \mid x \in V(H)\}$$

and

$$E(H_i) = \{v_{f(x)+i-1}v_{f(y)+i-1} \mid xy \in E(H)\}$$

for $i = 1, 2, \ldots, p$. The definition of G is completed by defining $E(G) = \bigcup_{i=1}^{p} E(H_i)$. (See Figure 8.9 for a given graph H, a possible labeling of the vertices of H, the induced edge labels of H, the vertices of G, and the graphs H_1 and H_2, where the edges of H_2 are drawn with dashed lines.)

The graph G is therefore decomposable into the graphs $H_1, H_2, \ldots, H_p$, each of which is isomorphic to H, and G is $2q$-regular, where q is the size of H. This completes the proof. ■

Suppose, in the preceding proof, that H is graceful (see Exercise 3.11); i.e., it is possible to label the vertices of H so that the induced edge labels are $1, 2, \ldots, q$. Then $G \cong K_{2q+1}$ and, consequently, K_{2q+1} is H-decomposable.

Certainly, every result dealing with factorizations can be reformulated in terms of decompositions. For example, Theorem 8.11 states that K_{2n} is nK_2-

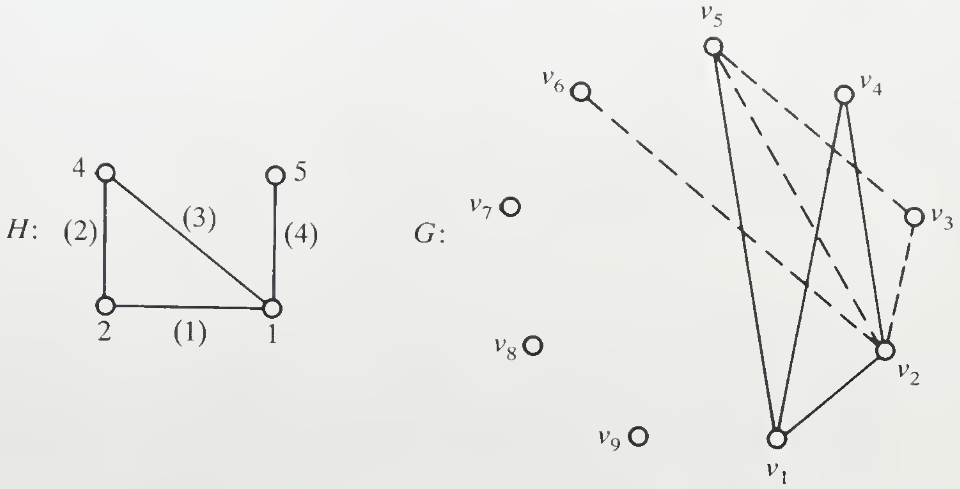


Figure 8.9 *Construction of a regular H-decomposable graph*

decomposable for every integer $n \geq 1$. This theorem has been generalized by Ruiz [R10].

A *linear forest* is a forest each component of which is a path. Thus every 1-factor is a linear forest.

Theorem 8.16 *If F is a linear forest of size n having no isolated vertices, then K_{2n} is F-decomposable.*

Proof Since the result is obvious for $n = 1$, we assume that $n \geq 2$. Let the vertex set of K_{2n} be denoted by $\{v_0, v_1, v_2, \ldots, v_{2n-1}\}$. Arrange the vertices $v_1, v_2, \ldots, v_{2n-1}$ cyclically in counterclockwise order about a regular $(2n-1)$-gon, calling the resulting cycle C, and place v_0 in the center of the $(2n-1)$-gon. Join every two vertices by a straight line segment to obtain the edges of K_{2n}. For the purpose of this proof, we refer to any edge joining v_0 to a vertex of C as a 0-edge (there are obviously $2n-1$ such edges). Every other edge of K_{2n} joins two vertices of C. If uv is an edge joining two vertices of C, then we call uv an i-edge if $d_C(u, v) = i$. Note that $1 \leq i \leq n-1$ and that for each $i = 1, 2, \ldots, n-1$, the graph K_{2n} contains $2n-1$ i-edges.

We now describe two paths P and Q of length n in K_{2n}. If n is even, then

$$P: v_0, v_1, v_{2n-1}, v_2, v_{2n-2}, v_3, \ldots, v_{n/2}, v_{3n/2}$$

and

$$Q: v_0, v_n, v_{n+1}, v_{n-1}, v_{n+2}, v_{n-2}, \ldots, v_{(n+2)/2}, v_{3n/2};$$

while if n is odd, then

$$P: v_0, v_1, v_{2n-1}, v_2, v_{2n-2}, v_3, \ldots, v_{(3n+1)/2}, v_{(n+1)/2}$$

and

$$Q: v_0, v_n, v_{n+1}, v_{n-1}, v_{n+2}, v_{n-2}, \ldots, v_{(3n-1)/2}, v_{(n+1)/2}.$$

Observe that, in either case, the ith edge of P and the ith edge of Q are $(i-1)$-edges for $i = 1, 2, \ldots, n$.

Assume that the linear forest

$$F \cong P_{n_1+1} \cup P_{n_2+1} \cup \cdots \cup P_{n_k+1},$$

where then $\sum_{i=1}^{k} n_i = n$. We define a subgraph H of K_{2n} as follows. The edge set E of H consists of the first n_1 edges of P, edges n_1+1 through n_1+n_2 of Q, edges n_1+n_2+1 through $n_1+n_2+n_3$ of P, and so on until finally the last n_k edges of Q if k is even or the last n_k edges of P if k is odd. Define $H = \langle E \rangle$. Note that $H \cong F$ and that H contains exactly one i-edge for each $i = 0, 1, \ldots, n-1$.

Now for $j = 1, 2, \ldots, 2n-1$, define H_j to be the subgraph of K_{2n} obtained by revolving H about the $(2n-1)$-gon in a counterclockwise angle of $2\pi(j-1)/(2n-1)$ radians. Observe that for each $i = 0, 1, \ldots, n-1$ and each $j = 1, 2, \ldots, 2n-1$, the subgraph H_j contains exactly one i-edge. Since $H_j \cong H$ for each $j = 1, 2, \ldots, 2n-1$ and K_{2n} is decomposed into the subgraphs $H_1, H_2, \ldots, H_{2n-1}$, it follows that K_{2n} is H-decomposable. ■

The preceding theorem and its proof are illustrated in Figure 8.10 for $2n = 12$ and $F \cong P_2 \cup P_3 \cup P_4$. The labeling of the vertices of K_{12} is shown along with the subgraph H (or H_1).

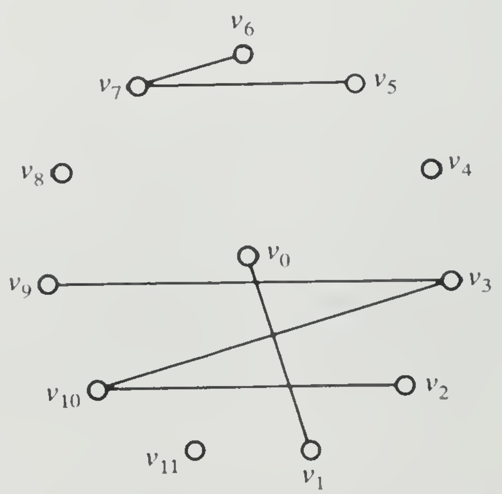


Figure 8.10 *A step in the construction of an F-decomposition of K_{12} for $F = P_2 \cup P_3 \cup P_4$*

Exercises 8.3

8.17 (a) Find a K_3-decomposition of K_7 by using the fact that K_3 is graceful.

(b) Find a noncomplete, regular graph that is K_3-decomposable.

8.18 Find an F-decomposition of K_{12} where $F = 2P_2 \cup 2P_3$.

8.19 Find a P_6-decomposition of K_{10}.

8.20 For each integer $n \geq 1$, prove that

(a) K_{2n+1} is $K(1, n)$-decomposable.

(b) K_{2n} is $K(1, n)$-decomposable.

8.21 Let G be a connected graph.

(a) Prove that G is P_3-decomposable if and only if G has even size.

(b) Show that the result in (a) is false if P_3 is replaced by $2K_2$.

8.4 Coverings

Recall that an independent set of vertices in a graph G is one whose elements are pairwise independent (nonadjacent) and that the vertex independence number $\beta(G)$ of G is the maximum cardinality among the independent sets of vertices in G. One can similarly define an *independent set of edges* in G as a set of edges, each two of which are independent (nonadjacent). The *edge independence number* $\beta_1(G)$ of G is the maximum cardinality among the independent sets of edges in G. For example, if $m \leq n$, then $\beta(K(m, n)) = n$ and $\beta_1(K(m, n)) = m$.

A vertex and an edge are said to *cover* each other in a graph G if they are incident in G. A *vertex cover* in G is a set of vertices that covers all edges of G. An *edge cover* in a graph G without isolated vertices is a set of edges that covers all vertices of G.

The minimum cardinality of a vertex cover in a graph G is called the *vertex covering number* of G and is denoted by $\alpha(G)$. Suppose we represent a street system in a town by a graph G, in which the streets themselves (actually street segments) are the edges of G and the intersections are the vertices of G. Suppose further that we wish to station one law officer at each of various intersections so that some law officer can get to any location in the town quickly, i.e., we wish to have a law officer stationed at one or both intersections at the end of each street segment. Then the fewest such law officers needed is $\alpha(G)$.

As expected, the *edge covering number* $\alpha_1(G)$ of a graph G (without isolated vertices) is the minimum cardinality of an edge cover in G. For $m \leqslant n$, we have $\alpha(K(m, n)) = m$ and $\alpha_1(K(m, n)) = n$.

As an added illustration of the four parameters just presented, we note that for $p \geqslant 2$, $\beta(K_p) = 1$, $\beta_1(K_p) = \lfloor p/2 \rfloor$, $\alpha(K_p) = p - 1$, and $\alpha_1(K_p) = \lceil p/2 \rceil$. Observe that for the two graphs G of order p considered above, namely $K(m, n)$, with $p = m + n$, and K_p, we have

$$\alpha(G) + \beta(G) = \alpha_1(G) + \beta_1(G) = p.$$

These two examples serve to illustrate the next theorem, due to Gallai [G1].

Theorem 8.17 (Gallai) *If G is a graph of order p having no isolated vertices, then*

$$\alpha(G) + \beta(G) = p \tag{8.3}$$

and

$$\alpha_1(G) + \beta_1(G) = p. \tag{8.4}$$

Proof We begin with (8.3). Let U be an independent set of vertices of G with $|U| = \beta(G)$. Clearly, the set $V(G) - U$ is a vertex cover in G. Therefore, $\alpha(G) \leqslant p - \beta(G)$. If, however, W is a set of $\alpha(G)$ vertices that covers all edges of G, then $V(G) - W$ is independent; thus $\beta(G) \geqslant p - \alpha(G)$. This proves (8.3).

To verify (8.4), let E_1 be an independent set of edges of G with $|E_1| = \beta_1(G)$. Obviously, E_1 covers $2\beta_1(G)$ vertices of G. For each vertex of G not covered by E_1, select an incident edge and define E_2 to be the union of this set of edges and E_1. Necessarily, E_2 is an edge cover in G so that $|E_2| \geqslant \alpha_1(G)$. Also we note that $|E_1| + |E_2| = p$; hence $\alpha_1(G) + \beta_1(G) \leqslant p$. Now suppose E' is an edge cover in G with $|E'| = \alpha_1(G)$. The minimality of E' implies that each component of $\langle E' \rangle$ is a tree. Select from each component of $\langle E' \rangle$ one edge, denoting the resulting set of edges by E''. We observe that $|E''| \leqslant \beta_1(G)$ and that $|E'| + |E''| = p$. These two facts imply that $\alpha_1(G) + \beta_1(G) \geqslant p$, completing the proof of (8.4) and the theorem. ■

If C is a vertex cover in a graph G and E is an independent set of edges, then for each edge e of E there is a vertex v_e in C that is incident with e. Furthermore, if $e, f \in E$, then $v_e \neq v_f$. Thus for any independent set E of edges and any vertex cover C in G, we have $|C| \geqslant |E|$. This, of course, implies that $\alpha(G) \geqslant \beta_1(G)$. In general, equality does not hold here. If, however, G is bipartite, then we do have $\alpha(G) = \beta_1(G)$, as was shown by König [K8].

The proof of König's Theorem presented here employs the max-flow min-cut theorem and indicates a method for finding a maximum set of independent edges and a minimum vertex cover in a bipartite graph.

Theorem 8.18 (König) *If G is a bipartite graph, then*

$$\alpha(G) = \beta_1(G).$$

Proof Since $\alpha(G) \geq \beta_1(G)$, we need only show $\beta_1(G) \geq \alpha(G)$. Suppose G has partite sets V_1 and V_2. Let D be the digraph obtained from G by adding two new vertices u and v, all arcs from u to vertices in V_1, and all arcs from vertices in V_2 to v, and by replacing each edge v_1v_2, where $v_1 \in V_1$ and $v_2 \in V_2$, with the arc (v_1, v_2). Then N is taken to be the network with underlying digraph D, source u, sink v, and capacity function c on $E(D)$ defined by

$$c(a) = \begin{cases} 1, & \text{if } a = (u, x) \text{ or } a = (x, v) \text{ for some } x \in V_1 \cup V_2, \\ |V_1| + 1, & \text{otherwise.} \end{cases}$$

By Theorem 5.15, the value of a maximum flow f in N equals the capacity of a minimum cut $K = (X, \bar{X})$. The proof will be complete if we show that $\beta_1(G) \geq \operatorname{val} f$ and $\operatorname{cap} K \geq \alpha(G)$. Since the capacity of each arc a incident from u or to v is 1, it follows that $f(a) = 0$ or 1 for each such arc a. Furthermore, $f(a) = 0$ or 1 for every arc a of D since $f(x, V(D)) = f(V(D), x)$ whenever $x \in V(D) - \{u, v\}$. Suppose that $\operatorname{val} f = m$. Thus there are distinct vertices u_1, $u_2, \ldots, u_m$ of V_1 such that $f(u, u_i) = 1$ for $i = 1, 2, \ldots, m$. It follows that $f(u_i, V_2) = 1$, $1 \leq i \leq m$, and that there is exactly one vertex $w_i \in V_2$ such that $(u_i, w_i) \in E(D)$ and $f(u_i, w_i) = 1$. Since $c(w_i, v) = 1$, $1 \leq i \leq m$, we conclude that if $i \neq j$, then $w_i \neq w_j$. Thus, $\{u_iw_i \mid 1 \leq i \leq m\}$ is an independent set of edges of G, and so $\beta_1(G) \geq m = \operatorname{val} f$.

Let $A = V_1 \cap \bar{X}$ and $B = V_2 \cap X$. By the way in which D was defined, then,

$$K = (u, A) \cup (V_1 - A, V_2 - B) \cup (B, v).$$

However, since $\operatorname{cap} K \leq |V_1|$ and $c(a) = |V_1| + 1$ if $a \in (V_1 - A, V_2 - B)$, we have that $(V_1 - A, V_2 - B) = \emptyset$. Thus all arcs from V_1 to V_2 in D are incident from a vertex of A or to a vertex of B, and so $A \cup B$ is a vertex cover in G. Furthermore,

$$\operatorname{cap} K = c(u, A) + c(B, v) = |A \cup B|.$$

Hence, $\operatorname{cap} K = |A \cup B| \geq \alpha(G)$, and so

$$\alpha(G) = |A \cup B| = \operatorname{cap} K = \operatorname{val} f = \beta_1(G). \blacksquare$$

To determine, for example, a maximum independent set of edges and a minimum vertex cover in the bipartite graph G of Figure 8.11(a), we apply Algorithm 5A to the network N with underlying digraph D in Figure 8.11(b), source u, sink v, and whose capacity function c satisfies

$$c(a) = \begin{cases} 1, & \text{if } a = (u, x) \text{ or } a = (x, v) \text{ for some } x \in V_1 \cup V_2, \\ 6, & \text{otherwise.} \end{cases}$$

We obtain the maximum flow and minimum cut K indicated in Figure 8.11(c), so that $\alpha(G) = \beta_1(G) = 4$. Proceeding as in the proof of Theorem 8.18, we have

$$A = V_1 \cap \bar{X} = \{u_2, u_3, u_4, u_5\} \quad \text{and} \quad B = V_2 \cap X = \emptyset.$$

Thus, $\{u_2, u_3, u_4, u_5\}$ is a minimum vertex cover and $\{u_2w_2, u_3w_5, u_4w_3, u_5w_4\}$ is a maximum independent set of edges in G.

Next we present upper and lower bounds for the edge independence number, due to Weinstein [W3].

Theorem 8.19 *Let G be a graph of order p without isolated vertices. Then*

$$\left\lceil \frac{p}{1 + \Delta(G)} \right\rceil \leq \beta_1(G) \leq \left\lfloor \frac{p}{2} \right\rfloor.$$

Furthermore, these bounds are sharp.

Proof It suffices to prove the theorem for connected graphs. The upper bound for $\beta_1(G)$ is immediate and clearly sharp.

In order to verify the lower bound, we employ induction on the size q of a connected graph. If $q = 1$ or $q = 2$, then the lower bound follows. Assume that the lower bound holds for all connected graphs of positive size not exceeding k, where $k \geq 2$, and let G be a connected graph of order p having size $k + 1$. If G has a cycle edge e, then

$$\beta_1(G) \geq \beta_1(G - e) \geq \frac{p}{1 + \Delta(G - e)} \geq \frac{p}{1 + \Delta(G)}.$$

Otherwise, G is a tree. If $G \cong K(1, p-1)$, then $\beta_1(G) = p/(1 + \Delta(G)) = 1$ (which also shows the sharpness of the lower bound). If $G \not\cong K(1, p-1)$, then G contains an edge e such that $G - e$ has two nontrivial components G_1 and G_2. Let p_i denote the order of G_i, $i = 1, 2$. Applying the inductive hypothesis to G_1 and G_2, we obtain

$$\beta_1(G) \geq \beta_1(G_1) + \beta_1(G_2) \geq \frac{p_1}{1 + \Delta(G_1)} + \frac{p_2}{1 + \Delta(G_2)}$$

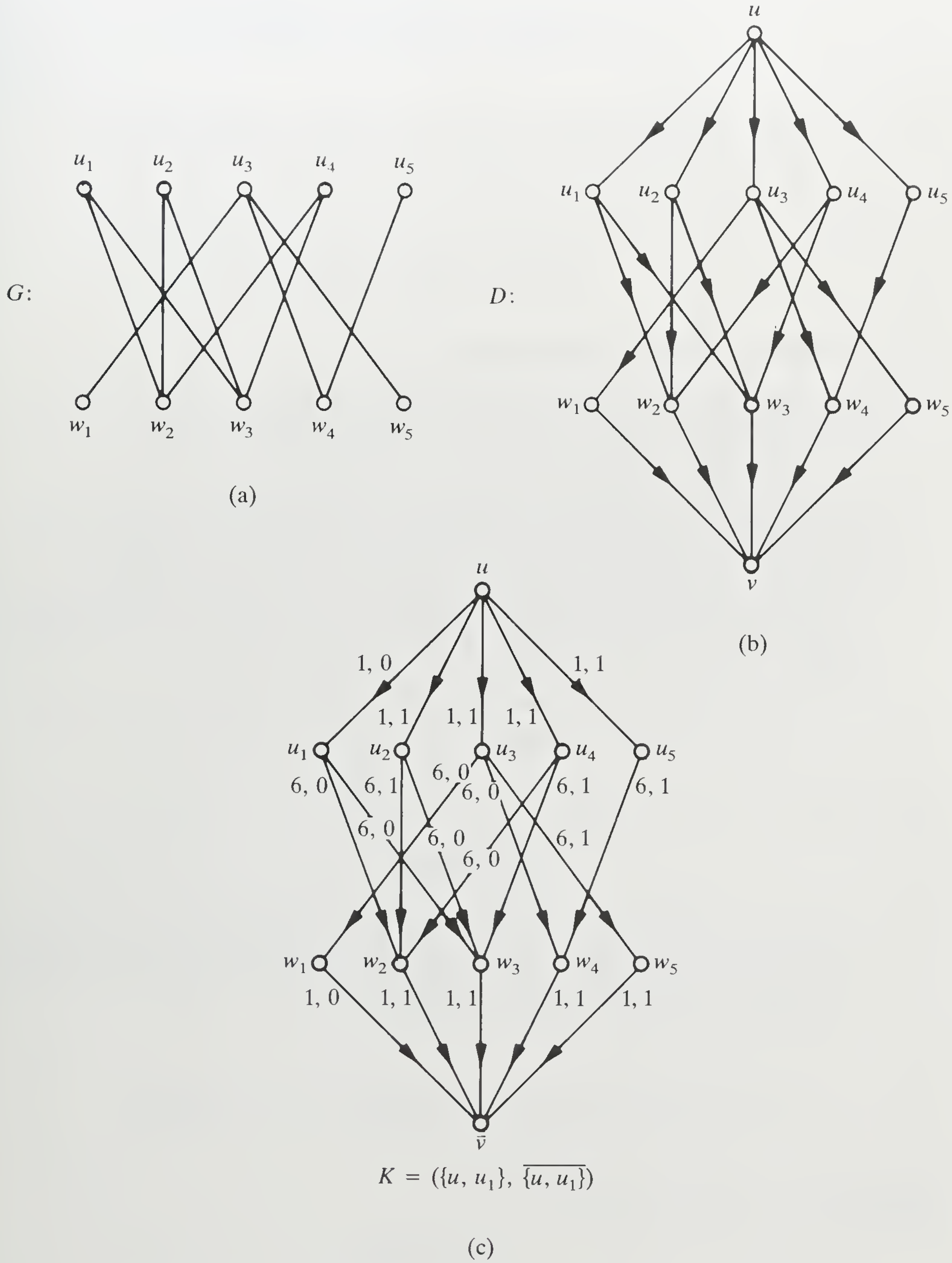


Figure 8.11 *Maximum independent sets of edges and minimum vertex covers*

$$\geq \frac{p_1}{1+\Delta(G)} + \frac{p_2}{1+\Delta(G)} = \frac{p}{1+\Delta(G)}. \blacksquare$$

Combining Theorems 8.17 and 8.19, we have our next result.

Corollary 8.19 *Let G be a graph of order p without isolated vertices. Then*

$$\left\lceil \frac{p}{2} \right\rceil \leq \alpha_1(G) \leq \left\lfloor \frac{p \cdot \Delta(G)}{1+\Delta(G)} \right\rfloor.$$

Furthermore, these bounds are sharp.

It is easy to see for a graph G of order p without isolated vertices that $1 \leq \beta(G) \leq p-1$ and that these bounds are sharp. This implies that $1 \leq \alpha(G) \leq p-1$ are sharp bounds for $\alpha(G)$.

We briefly consider two other kinds of "covers". A *vertex dominating set* for a graph G is a set S of vertices such that every vertex of G belongs to S or is adjacent to a vertex of S. *Edge dominating sets* are defined analogously. The minimum cardinality of a vertex dominating set in a graph G is called the *vertex dominating number* of G and is denoted by $\sigma(G)$. *The edge dominating number* $\sigma_1(G)$ is defined similarly. For the complete graph K_p, $\sigma(K_p)=1$ and $\sigma_1(K_p)=\lfloor p/2 \rfloor$.

We next establish an inequality involving the independence and vertex dominating numbers of a graph.

Theorem 8.20 *For every graph G,*

$$\sigma(G) \leq \beta(G).$$

Proof Let U be an independent set of vertices of G with $|U| = \beta(G)$. Every vertex $v \in V(G) - U$ is adjacent to some vertex of U; otherwise, $U \cup \{v\}$ is an independent set of vertices having cardinality $\beta(G)+1$. Thus U is a vertex dominating set so that $\sigma(G) \leq \beta(G)$. $\blacksquare$

In a similar fashion, the following result can be established.

Theorem 8.21 *For every graph G,*

$$\sigma_1(G) \leq \beta_1(G).$$

Exercises 8.4

8.22 Verify Gallai's Theorem for the regular complete n-partite graph of order kn, where n is even.

8.23 Prove or disprove: A graph G without isolated vertices has a 1-factor if and only if $\alpha_1(G) = \beta_1(G)$.

8.24 Show that a graph G is bipartite if and only if $\beta(H) \geq |V(H)|/2$ for every subgraph H of G.

8.25 Prove that if G is a bipartite graph without isolated vertices, then $\alpha_1(G) = \beta(G)$.

8.26 An independent set of vertices (edges) in a graph G is *maximal independent* if it is not properly contained in any other independent set of vertices (edges). Characterize those nonempty graphs with the property that every pair of maximal independent sets of vertices is disjoint.

8.27 Let G be a graph and let $U \subseteq V(G)$. Use Theorem 8.5 to prove that G has a set of independent edges covering U if and only if for every proper subset S of $V(G)$, the number of odd components of $G - S$ containing only vertices of U does not exceed $|S|$.

8.28 (a) Let $\beta_1^*(G)$ denote the minimum cardinality among the maximal independent sets of edges of a graph G. Prove that $\beta_1^*(G) = \sigma_1(G)$ for every graph G.

(b) Prove Theorem 8.21.

8.29 For each nonnegative integer n, determine a graph G_n for which $\beta(G_n) - \sigma(G_n) = n$ and a graph H_n for which $\beta_1(H_n) - \sigma_1(H_n) = n$.

8.30 Use Menger's Theorem to prove Theorem 8.18.

8.31 Use Theorem 8.2 to prove Theorem 8.18.

8.32 Use the max-flow min-cut theorem to prove Theorem 8.3.

Chapter Nine

Graphs and Groups

With every set on which a relation or operation is defined, there exists a group of permutations that preserve that relation or operation. Graphs are no exception. We describe here three groups and one graph defined in terms of certain adjacencies in a given graph. Conversely, some digraphs that arise very naturally from a given finite group are discussed.

9.1 The Group and Edge-Group of a Graph

An *automorphism* of a graph G is an isomorphism of G with itself, that is, a permutation on $V(G)$ that preserves adjacency. It is an immediate consequence of the definition that if ϕ is an automorphism of G and $v \in V(G)$, then $\deg \phi v = \deg v$.

It is straightforward to verify that (under the operation of composition) the set of all automorphisms of a graph G forms a group, denoted by $\mathscr{A}(G)$ and referred to as the *automorphism group*, the *vertex-group*, or simply *the group* of G. For example, $\mathscr{A}(K_p)$ is the symmetric group S_p of order $p!$ while $\mathscr{A}(C_p)$ is the dihedral group D_p of order $2p$.

If Γ' and Γ'' are isomorphic groups, then we write $\Gamma' \cong \Gamma''$. Our first theorem gives a simple, but useful, consequence of the definitions.

Theorem 9.1 *For any graph G and its complement $\overline{G}$, $\mathscr{A}(G) \cong \mathscr{A}(\overline{G})$.*

Proof Every element ϕ of $\mathscr{A}(G)$ is a permutation on $V(G)$ that preserves adjacency in G. However, ϕ preserves adjacency if and only if ϕ preserves nonadjacency. Thus a permutation on $V(G)$ is an automorphism of G if and only if it is an automorphism of $\bar{G}$, implying that $\mathscr{A}(G) \cong \mathscr{A}(\bar{G})$. ■

We have already mentioned that $\mathscr{A}(K_p) \cong S_p$. Certainly, if G is a graph of order p containing adjacent vertices as well as nonadjacent vertices, then $\mathscr{A}(G)$ is isomorphic to a proper subgroup of S_p. Combining this observation with Theorem 9.1 and Lagrange's Theorem on the order of subgroups of finite groups, we arrive at the following.

Corollary 9.1 *The order $|\mathscr{A}(G)|$ of the group of a graph G of order p is a divisor of $p!$ and equals $p!$ if and only if $G \cong K_p$ or $G \cong \bar{K}_p$.*

With the aid of the group of a graph G of order p, it is possible to determine the number of nonidentical graphs that are isomorphic to G and labeled from the same set of p labels.

Theorem 9.2 *Let G be a graph of order p. The number of labelings of G from a set of p labels such that no two resulting graphs are identical is $p!/|\mathscr{A}(G)|$.*

Proof Let $\{v_1, v_2, \ldots, v_p\}$ be a set of p labels. Certainly, there exist $p!$ labelings of G from this set of labels without regard to the number of resulting labeled graphs that may be identical. If G_1 and G_2 are two labeled graphs obtained from G, then the relation "G_1 is identical to G_2" is an equivalence relation on the set of labeled graphs obtained from G. For a given labeled graph G_1, each automorphism of G gives rise to a labeled graph that is identical to G_1, and conversely. Hence each equivalence class so determined contains $|\mathscr{A}(G)|$ elements, thus implying there are $p!/|\mathscr{A}(G)|$ equivalence classes in all. This proves the theorem. ■

We now turn our attention to a second group associated with a graph. Two nonempty graphs G and G' are called *edge-isomorphic* if there exists a one-to-one mapping ϕ from $E(G)$ onto $E(G')$ such that two edges e and f of G are adjacent if and only if the edges ϕe and ϕf of G' are adjacent. In this case, ϕ is called an *edge-isomorphism* from G to G'.

If G and G' are nonempty isomorphic graphs, then they are edge-isomorphic. In order to see this, let ϕ be an isomorphism from a nonempty graph G to a graph G'. Then $u_1u_2 \in E(G)$ if and only if $\phi u_1 \phi u_2 \in E(G')$. Moreover, the edges u_1u_2 and v_1v_2 of G are adjacent if and only if the edges $\phi u_1 \phi u_2$ and $\phi v_1 \phi v_2$ of G' are adjacent. Hence, each isomorphism from G to G' gives rise to an edge-isomorphism from G to G'. Whenever an edge-isomorphism can be obtained (in this sense) from an isomorphism, we refer to the edge-isomorphism as *induced*. Although every pair of nonempty isomorphic graphs are edge-isomorphic, the converse is not true in general; that is, if G and G' are edge-isomorphic, then G and G' need not be isomorphic. This fact is illustrated in Figure 9.1 with the graphs G_1 and G_2.

An *edge-automorphism* of a nonempty graph G is an edge-isomorphism of G with itself. The set of all edge-automorphisms of G (under composition) forms a group, called the *edge-group* of G and denoted by $\mathscr{A}_1(G)$. As examples, we note that $\mathscr{A}_1(K(1, n)) \cong S_n$ and $\mathscr{A}_1(C_p) \cong D_p$. We have already noted that each isomorphism from G to G' induces an edge-isomorphism from G to G'.

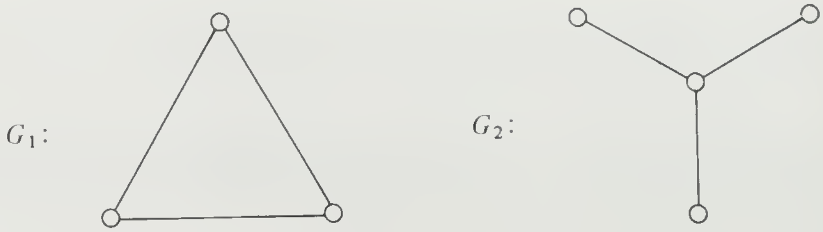


Figure 9.1 *Edge-isomorphic graphs that are not isomorphic*

In the case that G' also denotes the graph G, then we speak of an *induced edge-automorphism*. The set of all induced edge-automorphisms of G forms a group, called the *induced edge-group* of G and denoted by $\mathscr{A}^*(G)$. Obviously, $\mathscr{A}^*(G)$ is a subgroup of $\mathscr{A}_1(G)$. To see that $\mathscr{A}^*(G)$ may be a proper subgroup of $\mathscr{A}_1(G)$, consider the graphs G_3, G_4, and G_5 shown in Figure 9.2. Using the labelings as indicated, we observe that each mapping ϕ_i, $i = 3, 4, 5$, which follows, is an edge-automorphism of G_i that is not an induced edge-automorphism.

$$\phi_3 = \begin{pmatrix} v_1v_2 & v_2v_3 & v_2v_4 & v_3v_4 \\ v_3v_4 & v_2v_3 & v_2v_4 & v_1v_2 \end{pmatrix},$$

$$\phi_4 = \begin{pmatrix} v_1v_2 & v_2v_3 & v_2v_4 & v_3v_4 & v_1v_4 \\ v_2v_3 & v_3v_4 & v_2v_4 & v_1v_4 & v_1v_2 \end{pmatrix},$$

$$\phi_5 = \begin{pmatrix} v_1v_2 & v_2v_3 & v_2v_4 & v_3v_4 & v_1v_4 & v_1v_3 \\ v_3v_4 & v_2v_3 & v_2v_4 & v_1v_2 & v_1v_4 & v_1v_3 \end{pmatrix}.$$

As an additional observation regarding the graphs G_i of Figure 9.2, we remark that $|\mathscr{A}_1(G_i)| = 2|\mathscr{A}^*(G_i)|$. We now consider a relationship between the group and the induced edge-group of a graph.

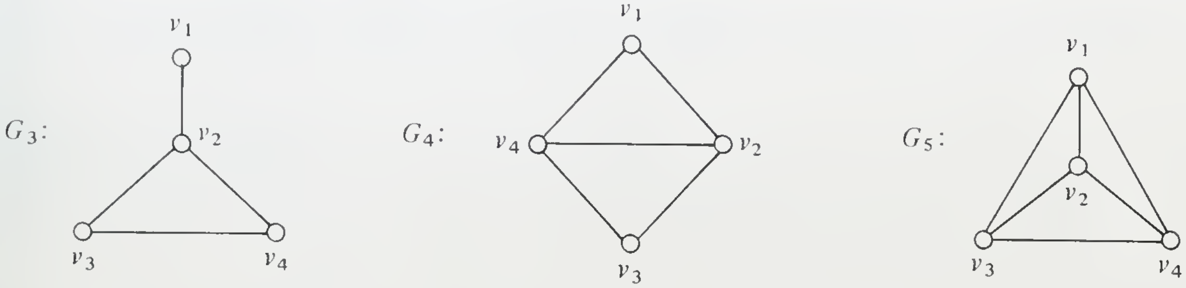


Figure 9.2 *Graphs having edge-automorphisms not induced by any automorphism*

Theorem 9.3 *Let G be a nontrivial connected graph. Then $\mathscr{A}(G) \cong \mathscr{A}^*(G)$ if and only if $G \not\cong K_2$.*

Proof Since $\mathscr{A}(K_2) \cong S_2$ while $\mathscr{A}^*(K_2) \cong S_1$, the necessity is clear. For the sufficiency, we assume G to be a connected graph of order at least 3 so that $|E(G)| \geq 2$.

Define a mapping ϕ: $\mathscr{A}(G) \to \mathscr{A}^*(G)$ such that $\phi\alpha = \alpha^*$ for $\alpha \in \mathscr{A}(G)$, where α^* is the edge-automorphism of G induced by α. We show that ϕ is a group isomorphism. By definition, the mapping ϕ is onto $\mathscr{A}^*(G)$.

We next verify that ϕ is one-to-one. Let $\alpha, \beta \in \mathscr{A}(G)$ such that $\alpha \neq \beta$. We must show that $\alpha^* \neq \beta^*$; that is, there exists an edge e of G for which $\alpha^* e \neq \beta^* e$. Let $v \in V(G)$ such that $\alpha v \neq \beta v$, and let u be a vertex of G adjacent with v. If either $\alpha u \neq \beta v$ or $\beta u \neq \alpha v$, then for the edge $e = uv$, we have $\alpha^* e \neq \beta^* e$. Thus, we assume that $\alpha u = \beta v$ and $\beta u = \alpha v$. There exists a vertex w in G adjacent with at least one of u and v, where $w \notin \{u, v\}$. If $e_1 = vw$ is an edge of G, then $\alpha^* e_1 \neq \beta^* e_1$. If $e_2 = uw$ is an edge of G, then $\alpha^* e_2 \neq \beta^* e_2$. Hence, in any case, ϕ is one-to-one.

It remains to show that ϕ is operation-preserving; that is, for any $e \in E(G)$, we have $\phi(\alpha\beta)(e) = (\phi\alpha)(\phi\beta)(e)$. Let $e = uv$ and

$$\beta u = u', \quad \beta v = v', \quad \alpha u' = u'', \quad \alpha v' = v''.$$

Then

$$\phi(\alpha\beta)(e) = \phi(\alpha\beta)(uv) = (\alpha\beta)u(\alpha\beta)v = \alpha u' \alpha v' = u''v''$$

and

$$(\phi\alpha)(\phi\beta)(e) = (\phi\alpha)(\phi\beta)(uv) = (\phi\alpha)(\beta u \beta v) = (\phi\alpha)(u'v') = \alpha u' \alpha v' = u''v''.$$

Hence ϕ is operation-preserving and is therefore an isomorphism between the groups $\mathscr{A}(G)$ and $\mathscr{A}^*(G)$. ■

Theorem 9.3 may now be generalized to arbitrary graphs.

Corollary 9.3 *Let G be a nonempty graph. Then $\mathscr{A}(G) \cong \mathscr{A}^*(G)$ if and only if G contains neither K_2 as a component nor two or more isolated vertices.*

Figure 9.2 shows three graphs G for which $\mathscr{A}_1(G)$ and $\mathscr{A}^*(G)$ are not isomorphic. We now begin an investigation to determine those graphs G such that $\mathscr{A}_1(G) \cong \mathscr{A}^*(G)$. For the purpose of doing this, we present a preliminary result that is due to Whitney [W6].

Theorem 9.4 *Let ϕ be an edge-isomorphism from a connected graph H_1 to a connected graph H_2, where H_1 is different from the graphs G_i ($i = 1, 2, 3, 4, 5$) of Figures 9.1 and 9.2. Then ϕ is induced by an isomorphism from H_1 to H_2.*

Proof We consider two cases.

Case 1: Assume H_1 has a vertex v_0 of degree $d \geq 4$. Denote by $v_1, v_2, \ldots, v_d$ the vertices of H_1 adjacent with v_0. Let $\phi(v_0v_i) = e_i$, for $i = 1, 2, \ldots, d$. Since the edges v_0v_i are mutually adjacent, the edges e_i of H_2 are also mutually adjacent. Since $d \geq 4$, there is a vertex u_0 incident with all edges e_i. Let $e_i = u_0u_i$, $i = 1, 2, \ldots, d$. If v_i and v_j are adjacent vertices in H_1, $i, j \neq 0$, then the edge $\phi(v_iv_j)$ of H_2 is adjacent with each of u_0u_i and u_0u_j but not with u_0u_k, $k \neq i, j$. This implies that $\phi(v_iv_j) = u_iu_j$. Let $A_1 = \langle\{v_0, v_1, \ldots, v_d\}\rangle$ and $B_1 = \langle\{u_0, u_1, \ldots, u_d\}\rangle$. Then the mapping ψ: $V(A_1) \to V(B_1)$ defined by $\psi v_i = u_i$, $i = 0, 1, \ldots, d$, is an isomorphism from A_1 to B_1. The mapping ψ induces an edge-isomorphism ϕ_1 from A_1 to B_1; namely, ϕ_1 is the restriction of ϕ to $E(A_1)$. If $A_1 = H_1$, then the proof is complete.

Assume $A_1 \neq H_1$. Since H_1 is connected, there exists a vertex v_{d+1} in $V(H_1) - V(A_1)$ such that v_{d+1} is adjacent with a vertex v_r of A_1. Let v_s be a vertex of A_1 adjacent with v_r. The edge $\phi(v_rv_{d+1})$ is not in B_1, but it is adjacent with the edge $\phi(v_rv_s) = u_ru_s$ in B_1. Thus there exists a vertex u_{d+1} in H_2 not belonging to B_1 such that $\phi(v_rv_{d+1})$ is either u_ru_{d+1} or u_su_{d+1}. Now the edge $\phi(v_rv_{d+1})$ is adjacent with u_ru_s and with every edge of B_1 that is incident with u_r. However, $\phi(v_rv_{d+1})$ is adjacent with no edge of B_1 that is incident with u_s, except u_ru_s. Since at least one of u_r and u_s has degree at least 2 in B_1, it follows that $\phi(v_rv_{d+1}) = u_ru_{d+1}$.

The preceding argument applies to every edge of H_1 joining v_{d+1} and a vertex of A_1. Hence, if v_jv_{d+1} is an edge of H_1, where $1 \leq j \leq d$, then $\phi(v_jv_{d+1}) = u_ju_{d+1}$. Let $A_2 = \langle V(A_1) \cup \{v_{d+1}\}\rangle$ and $B_2 = \langle V(B_1) \cup \{u_{d+1}\}\rangle$. If we extend the aforementioned mapping ψ by defining $\psi v_{d+1} = u_{d+1}$, we note that ψ is an isomorphism from the connected graph A_2 to the connected graph B_2. Furthermore, ψ induces an edge-isomorphism ϕ_2 from A_2 to B_2, and ϕ_2 is the restriction of ϕ to $E(A_2)$. If $A_2 = H_1$, then the desired result follows. Otherwise, we proceed inductively until arriving at graphs A_{p-d} and B_{p-d} (with p denoting the order of H_1), where $A_{p-d} = H_1$, $B_{p-d} = H_2$, and the mapping ψ has been extended to an isomorphism from H_1 to H_2 such that ϕ is induced by ψ.

Case 2: Assume that the degree of no vertex of H_1 *exceeds* 3. We may further assume that H_1 contains a vertex v_0 of degree 3 since the theorem is obvious for all paths and for all cycles other than C_3. Let v_1, v_2, and v_3 be the vertices of H_1 incident with v_0. The subgraph $A_1 = \langle\{v_0, v_1, v_2, v_3\}\rangle$ is either the graph G_2 of Figure 9.1 or one of the graphs G_i, $i=3, 4, 5$, of Figure 9.2. By hypothesis, A_1 is a proper subgraph of H_1. Therefore, since H_1 is connected, there is at least one other vertex v_4 in H_1 (but not in A_1) adjacent with some vertex of A_1 different from v_0, say v_1. The edge $\phi(v_1v_4)$ of H_2 is adjacent with $\phi(v_0v_1)$ but adjacent with neither $\phi(v_0v_2)$ nor $\phi(v_0v_3)$. This implies that the edges $\phi(v_0v_i)$, $i=1, 2, 3$, do not form a triangle in H_2; however, since these three edges are mutually adjacent, they are all incident with a vertex u_0 of H_2. Let $\phi(v_0v_i)=u_0u_i$, $i=1, 2, 3$. For each edge v_iv_j of $A_1(i, j\neq 0)$, the edge $\phi(v_iv_j)$ of H_2 is adjacent with both u_0u_i and u_0u_j but not with the other edge incident with u_0. Hence $\phi(v_iv_j)=u_iu_j$.

Let $B_1=\langle\{u_0, u_1, u_2, u_3\}\rangle$. Define the function ψ: $V(A_1)\to V(B_1)$ by $\psi v_i = u_i$, $i=0, 1, 2, 3$. The function ψ is an isomorphism from the connected graph A_1 to the connected graph B_1, and, moreover, induces an edge-isomorphism ϕ_1 from A_1 to B_1, where ϕ_1 is the restriction of ϕ to $E(A_1)$. By employing an argument identical to that used in Case 1, we obtain a proof of the theorem in this case also. ∎

We are now in a position to characterize those graphs G for which $\mathscr{A}_1(G)\cong\mathscr{A}^*(G)$.

Theorem 9.5 *Let G be a nonempty graph. Then* $\mathscr{A}_1(G)\cong\mathscr{A}^*(G)$ *if and only if*
(a) not both G_1 *and* G_2 *(of Figure 9.1) are components of G, and*
(b) none of the graphs G_i, $i=3, 4, 5$, *(of Figure 9.2) is a component of G.*

Proof If $\mathscr{A}_1(G)\cong\mathscr{A}^*(G)$, then conditions (a) and (b) must hold. We therefore consider the converse, and assume G to be a graph satisfying (a) and (b). Since $\mathscr{A}^*(G)$ is a subgroup of $\mathscr{A}_1(G)$, it remains only to show that every edge-automorphism of G is induced by an automorphism of G. If G is connected, then Theorem 9.4 immediately implies that $\mathscr{A}_1(G)\cong\mathscr{A}^*(G)$.

Suppose that G is disconnected. Let α be an edge-automorphism of G. For every nontrivial component H of G, the subgraph $\langle\alpha(E(H))\rangle$ is also a component of G. If $H\cong G_1$ or $H\cong G_2$, then since G satisfies (a), we have $H\cong\langle\alpha(E(H))\rangle$. Therefore, if α is restricted to H, then α is induced by an automorphism of H. If H is different from G_1 and G_2, then by (b), $H\not\cong G_i$, $i=1, 2, 3, 4, 5$. In this case, Theorem 9.4 implies that if α is restricted to H, then α is induced by an automorphism of H. Hence, by applying the above argument to every nontrivial component of G, we observe that α is induced by an automorphism of G, so that $\mathscr{A}_1(G)\cong\mathscr{A}^*(G)$. ∎

Corollary 9.5a *Let G be a nonempty connected graph. Then $\mathscr{A}_1(G) \cong \mathscr{A}^*(G)$ if and only if G is different from the graphs G_i, $i = 3, 4, 5$ (of Figure 9.2).*

Combining Theorem 9.3 and Corollary 9.5a, we obtain the following.

Corollary 9.5b *Let G be a connected graph of order $p \geq 3$. Then the groups $\mathscr{A}(G)$, $\mathscr{A}_1(G)$, and $\mathscr{A}^*(G)$ are isomorphic to one another if and only if G is different from the graphs G_i, $i = 3, 4, 5$ (of Figure 9.2).*

Exercises 9.1

9.1 (a) How many nonidentical labelings (from a fixed set of n labels) are there for (i) C_n $(n \geq 3)$; (ii) P_n $(n \geq 2)$; (iii) $K(1, n)$ $(n \geq 2)$?

(b) How many nonidentical labelings (from a fixed set of $2n$ labels) are there for $K(n, n)$?

9.2 Let v be a vertex of a graph G. Define a vertex u of G (not necessarily distinct from v) to be *similar* to v if there exists $\alpha \in \mathscr{A}(G)$ such that $\alpha v = u$. For which pairs n, p of positive integers, with $n \leq p$, does there exist a graph G of order p and a vertex v of G such that there are exactly n vertices of G similar to v?

9.3 Let G be a nonempty graph. Prove that $\mathscr{A}_1(G)$ and $\mathscr{A}^*(G)$ are indeed groups.

9.4 Determine $\mathscr{A}_1(K(2, 3))$.

9.5 Determine $\mathscr{A}(G_i)$ for the graphs G_i, $i = 1, 2, 3$, of Figures 9.1 and 9.2.

9.6 Let G be a nonempty graph. Determine necessary and sufficient conditions for G such that $\mathscr{A}(G) \cong \mathscr{A}_1(G) \cong \mathscr{A}^*(G)$.

9.2 Cayley Color Graphs

We have seen that we can associate a group (in fact, three groups) with every graph. We now consider the reverse question of associating a graph with a given group. We consider only finite groups in this context. A nontrivial group Γ is said to be *generated* by the nonidentity elements $h_1, h_2, \ldots, h_k$ (and these elements are called *generators*) if every element of Γ can be expressed as a (finite) product of generators. Every nontrivial finite group has a finite generating set (often several such sets) since the set of all nonidentity elements of the group is always a generating set.

Let Γ be a given nontrivial finite group with $\Delta = \{h_1, h_2, \ldots, h_k\}$ a generating set for Γ. We associate a digraph with Γ and Δ called the *Cayley color graph of* Γ *with respect to* Δ and denoted by $D_\Delta(\Gamma)$. The vertex set of $D_\Delta(\Gamma)$ is the set of group elements of Γ; therefore, $D_\Delta(\Gamma)$ has order $|\Gamma|$. Each generator h_i is now regarded as a color. For $g_1, g_2 \in \Gamma$, there exists an arc (g_1, g_2) colored h_i in $D_\Delta(\Gamma)$ if and only if $g_2 = g_1 h_i$. If h_i is a group element of order 2 (and therefore self-inverse) and $g_2 = g_1 h_i$, then necessarily $g_1 = g_2 h_i$. When a Cayley color graph $D_\Delta(\Gamma)$ contains each of the arcs (g_1, g_2) and (g_2, g_1), both colored h_i, then it is customary to represent this symmetric pair of arcs by the single edge $g_1 g_2$.

We now illustrate the concepts just introduced. Let Γ denote the symmetric group S_3 of all permutations on the set $\{1, 2, 3\}$, and let $\Delta = \{a, b\}$, where $a = (123)$ and $b = (12)$. The Cayley color graph $D_\Delta(\Gamma)$ in this case is shown in Figure 9.3.

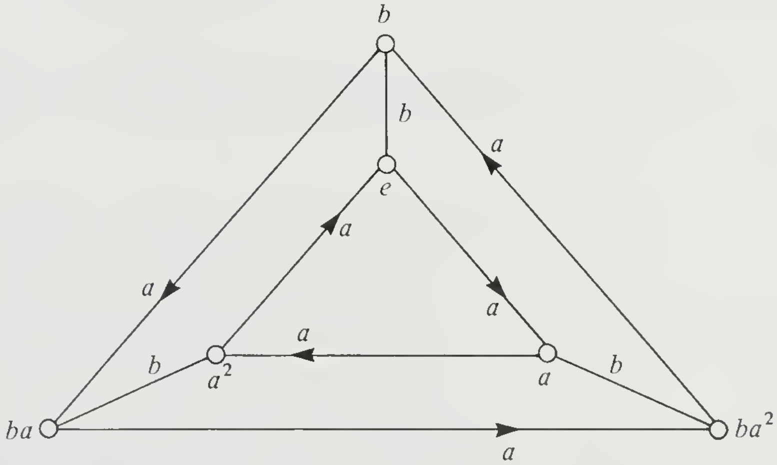


Figure 9.3 *A Cayley color graph*

If the generating set Δ of a given nontrivial finite group Γ is chosen to be the set of all nonidentity group elements, then for every two vertices g_1, g_2 of $D_\Delta(\Gamma)$, both (g_1, g_2) and (g_2, g_1) are arcs (although not necessarily of the same color) and $D_\Delta(\Gamma)$ is a complete symmetric digraph in this case.

An *automorphism of a digraph D* is a permutation α on $V(D)$ such that (u, v) is an arc of D if and only if $(\alpha u, \alpha v)$ is an arc of D. The set of all automorphisms of D forms a group under composition, and we denote this group by $\mathscr{A}(D)$. Let Γ be a nontrivial finite group with generating set Δ. An element $\alpha \in \mathscr{A}(D_\Delta(\Gamma))$ is said to be *color-preserving* if for every arc (g_1, g_2) of $D_\Delta(\Gamma)$, the arcs (g_1, g_2) and $(\alpha g_1, \alpha g_2)$ have the same color.

For a given nontrivial finite group Γ with generating set Δ, it is a routine exercise to prove that the set of all color-preserving automorphisms of $D_\Delta(\Gamma)$ forms a subgroup of $\mathscr{A}(D_\Delta(\Gamma))$. A useful characterization of color-preserving automorphisms is given in the next result.

Theorem 9.6 *Let Γ be a nontrivial finite group with generating set Δ and let α be a permutation on $V(D_\Delta(\Gamma))$. Then α is a color-preserving automorphism of $D_\Delta(\Gamma)$ if and only if*

$$\alpha(gh) = (\alpha g)h$$

for every $g \in \Gamma$ and $h \in \Delta$.

The major significance of the group of color-preserving automorphisms of a Cayley color graph is contained in the following theorem.

Theorem 9.7 *Let Γ be a nontrivial finite group with generating set Δ. Then the group of color-preserving automorphisms of $D_\Delta(\Gamma)$ is isomorphic to Γ.*

Proof Let $\Gamma = \{g_1, g_2, \ldots, g_p\}$. For $i = 1, 2, \ldots, p$, define α_i: $V(D_\Delta(\Gamma)) \to V(D_\Delta(\Gamma))$ by $\alpha_i g_m = g_i g_m$ for $1 \leq m \leq p$. Since Γ is a group, the mapping α_i is one-to-one and onto. Let $h \in \Delta$. Then for each i, $1 \leq i \leq p$, and for each m, $1 \leq m \leq p$,

$$\alpha_i(g_m h) = g_i(g_m h) = (g_i g_m)h = (\alpha_i g_m)h.$$

Hence, by Theorem 9.6, α_i is a color-preserving automorphism of $D_\Delta(\Gamma)$.

Next we verify that the mapping ϕ, defined by $\phi g_i = \alpha_i$, is an isomorphism from Γ to the group of color-preserving automorphisms of $D_\Delta(\Gamma)$. The mapping ϕ is clearly one-to-one since $\alpha_i \neq \alpha_j$ for $i \neq j$.

To show that ϕ preserves operations, let $g_i, g_j \in \Gamma$ be given, and suppose that $g_i g_j = g_k$. Then $\phi(g_i g_j) = \phi g_k = \alpha_k$ and $(\phi g_i)(\phi g_j) = \alpha_i \alpha_j$. Now, for each m, $1 \leq m \leq p$, $\alpha_k g_m = g_k g_m$. Furthermore, $g_k g_m = (g_i g_j) g_m = g_i(g_j g_m) = \alpha_i(g_j g_m) = \alpha_i(\alpha_j g_m) = (\alpha_i \alpha_j) g_m$. Hence, for each m, $1 \leq m \leq p$, $\alpha_k g_m = (\alpha_i \alpha_j) g_m$ so that $\alpha_k = \alpha_i \alpha_j$; that is, $\phi(g_i g_j) = (\phi g_i)(\phi g_j)$.

Finally, we show that the mapping ϕ is onto. Let α be a color-preserving automorphism of $D_\Delta(\Gamma)$. We show that $\alpha = \alpha_i$ for some i, $1 \leq i \leq p$. Suppose that $\alpha g_1 = g_r$, where g_1 is the identity of Γ. Let $g_m \in \Gamma$. Then g_m can be expressed as a product of generators, say

$$g_m = h_1 h_2 \cdots h_t,$$

where $h_j \in \Delta$, $1 \leq j \leq t$. Hence,

$$\alpha g_m = \alpha(g_1 g_m) = \alpha(g_1 h_1 h_2 \cdots h_t).$$

By successive applications of Theorem 9.6, it follows that

$$\alpha g_m = (\alpha g_1)h_1 h_2 \cdots h_t = (\alpha g_1)g_m = g_r g_m = \alpha_r g_m.$$

Therefore, $\alpha = \alpha_r$ and the proof is complete. ■

For more information on Cayley color graphs, see White [W4].

In 1936 the first book on graph theory was published. In this book the author König [K9, p. 5] proposed the problem of determining all finite groups Γ for which there exists a graph G such that $\mathcal{A}(G) \cong \Gamma$. The problem was solved in 1938 by Frucht [F6] who proved that every finite group has this property. We are now in a position to present a proof of this result.

If Γ is the trivial group, then $\mathcal{A}(G) \cong \Gamma$ for $G \cong K_1$. Therefore, let $\Gamma = \{g_1, g_2, \ldots, g_p\}$ be a given nontrivial finite group, and let $\Delta = \{h_1, h_2, \ldots, h_n\}$ be a generating set for Γ. We first construct the Cayley color graph $D_\Delta(\Gamma)$ of Γ with respect to Δ; the Cayley color graph is actually a digraph, of course. By Theorem 9.7, the group of color-preserving automorphisms of $D_\Delta(\Gamma)$ is isomorphic to Γ. We now transform the digraph $D_\Delta(\Gamma)$ into a graph G by the following technique. Let (g_i, g_j) be an arc of $D_\Delta(\Gamma)$ colored h_k. Delete this arc and replace it by the "graphical" path $g_i, u_{ij}, u'_{ij}, g_j$. At the vertex u_{ij} we construct a new path P_{ij} of length $2k-1$ and at the vertex u'_{ij} a path P'_{ij} of length $2k$. This construction is now performed with every arc of $D_\Delta(\Gamma)$, and is illustrated in Figure 9.4 for $k = 1$, 2, and 3.

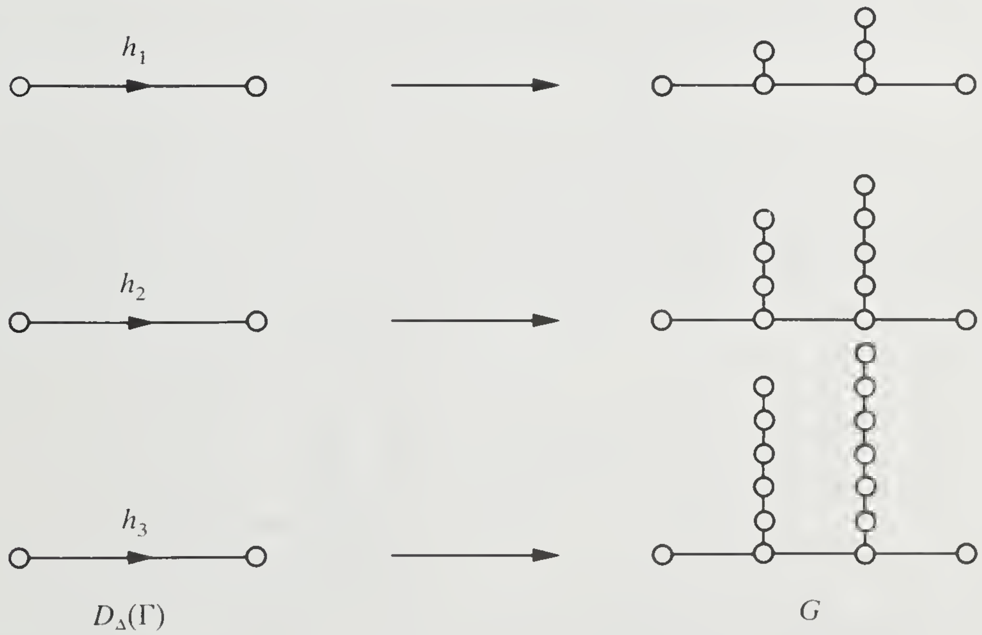


Figure 9.4 *Constructing a graph G from a given group* Γ

The addition of the paths P_{ij} and P'_{ij} in the formation of G is, in a sense, equivalent to the direction and the color of the arcs in the construction of $D_\Delta(\Gamma)$.

It now remains to observe that every color-preserving automorphism of $D_\Delta(\Gamma)$ induces an automorphism of G, and conversely. We state this below.

Theorem 9.8 (Frucht) *For every finite group Γ, there exists a graph G such that $\mathscr{A}(G) \cong \Gamma$.*

The condition of having a given group prescribed is not a particularly stringent one for graphs. For example, Babai [B1] showed that for every two finite groups Γ_1 and Γ_2, there exists a graph G and an edge e of G such that $\mathscr{A}(G) \cong \Gamma_1$ and $\mathscr{A}(G-e) \cong \Gamma_2$. Moreover, Izbicki [I2] has shown that it is possible to prescribe a finite group Γ, a connectivity $\kappa \geq 2$, a degree $r \geq 3$ of regularity, and some values of certain other parameters (as well as combinations of these) and then construct a graph possessing all these characteristics.

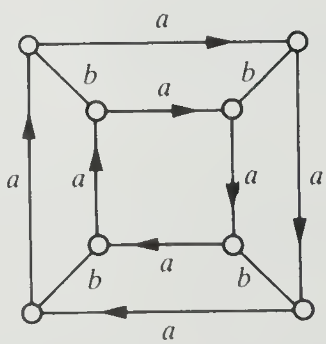
Exercises 9.2

9.7 Construct the Cayley color graph of the cyclic group of order 4 when the generating set Δ has
(a) one element;
(b) three elements.

9.8 Prove Theorem 9.6.

9.9 Determine the group of color-preserving automorphisms for the Cayley color graph $D_\Delta(\Gamma)$ below.

$D_\Delta(\Gamma)$:



9.10 Determine the smallest integer $p > 1$ such that there exists a connected graph G of order p such that $|\mathscr{A}(G)| = 1$.

9.11 Find a block G whose group is isomorphic to the cyclic group of order 3.

9.12 For a given finite group Γ, determine an infinite number of mutually non-isomorphic graphs whose groups are isomorphic to Γ.

9.13 For a given finite group Γ, find two nonhomeomorphic graphs whose groups are isomorphic to Γ.

9.14 For a finite group Γ generated by $\Delta(\subseteq \Gamma)$, let $D_\Delta(\Gamma)$ denote the Cayley color graph and $G_\Delta(\Gamma)$ the underlying graph.

(a) Show that the group of color-preserving automorphisms of $D_\Delta(\Gamma)$ is a subgroup of $\mathcal{A}(G_\Delta(\Gamma))$.

(b) If the group of color-preserving automorphisms of $D_\Delta(\Gamma)$ is isomorphic to $\mathcal{A}(G_\Delta(\Gamma))$ and $G_\Delta(\Gamma) \cong K_n$, then find Γ.

(c) Prove or disprove: If the group of color-preserving automorphisms of $D_{\Delta_1}(\Gamma_1)$ is isomorphic to the group of color-preserving automorphisms of $D_{\Delta_2}(\Gamma_2)$, then $\Gamma_1 \cong \Gamma_2$.

(d) Prove or disprove: If $\mathcal{A}(G_{\Delta_1}(\Gamma_1)) \cong \mathcal{A}(G_{\Delta_2}(\Gamma_2))$, then $\Gamma_1 \cong \Gamma_2$.

9.3 Line Graphs

We have seen that for each nonempty graph G, there is the associated group of edge symmetries of G, namely the group $\mathcal{A}_1(G)$ of edge automorphisms of G. In a very natural manner one can likewise associate with G a graph that describes the adjacencies among the edges of G.

Given a nonempty graph G, we define the *line graph* $L(G)$ of G as that graph whose vertices can be put in one-to-one correspondence with the edges of G in such a way that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent. (The line graph has also been referred to as the "interchange graph", "adjoint", "derived graph", and "derivative".) A graph and its line graph are shown in Figure 9.5.

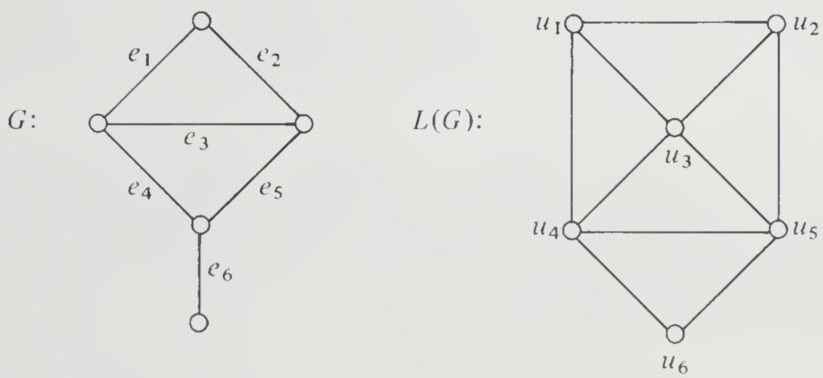


Figure 9.5 *A graph and its line graph*

It is relatively easy to determine the number of vertices and the number of edges of the line graph $L(G)$ of a graph G in terms of easily computed quantities in G. Indeed, if G is a (p, q) graph with degree sequence $d_1, d_2, \ldots, d_p$ and $L(G)$ is a (p', q') graph, then $p' = q$ and

$$q' = \sum_{i=1}^{p} \binom{d_i}{2},$$

since each edge of $L(G)$ corresponds to a pair of adjacent edges of G. Further, it is immediate that for each nonempty graph G, $\mathcal{A}(L(G)) \cong \mathcal{A}_1(G)$.

Obviously, two nonempty isomorphic graphs have isomorphic line graphs. The converse is not true, however, since the graphs K_3 and $K(1, 3)$ have K_3 as their line graph. Among connected graphs, though, these are the only two nonisomorphic graphs with this property. This fact was first discovered by Whitney [W6].

Theorem 9.9 (Whitney) *Let G_1 and G_2 be nontrivial connected graphs. Then $L(G_1) \cong L(G_2)$ if and only if $G_1 \cong G_2$, or one of G_1 and G_2 is the graph K_3 and the other is $K(1, 3)$.*

Proof We have already noted that $L(K_3) \cong L(K(1, 3))$ and that if $G_1 \cong G_2$, then $L(G_1) \cong L(G_2)$. Conversely, suppose that G_1 and G_2 are nontrivial connected graphs such that $L(G_1) \cong L(G_2)$. We assume that G_1 and G_2 are different from K_3 and $K(1, 3)$ and show that $G_1 \cong G_2$.

By the definition of the line graph function, for $i = 1, 2$, there exists a one-to-one mapping from $E(G_i)$ onto $V(L(G_i))$ such that two vertices of $L(G_i)$ are adjacent if and only if the corresponding edges of G_i are adjacent. Since $L(G_1) \cong L(G_2)$, this implies the existence of a one-to-one mapping α from $E(G_1)$ onto $E(G_2)$ with the property that two edges of G_1 are adjacent if and only if the corresponding edges of G_2 are adjacent, that is, α is an edge-isomorphism from G_1 to G_2. If G_1 is different from the graphs G_3, G_4, and G_5 of Figure 9.2, then it follows directly from Theorem 9.4 that α is induced by an isomorphism from G_1 to G_2. It is a straightforward exercise to verify for $i = 3$, 4, 5 that G_i is the only connected graph whose line graph is $L(G_i)$. ■

A graph H is called *a line graph* if there exists a graph G such that $H \cong L(G)$. A natural question to ask is whether a given graph H is a line graph. The following theorem of Krausz [K10] gives one answer to this question; further, if H is a line graph, this theorem gives a method of finding those graphs G with $H \cong L(G)$.

Theorem 9.10 (Krausz) *A nonempty graph H is a line graph if and only if $E(H)$ can be partitioned into subsets so that*

(a) *the subgraph induced by each member of the partition is complete, and*
(b) *no vertex of H lies in more than two of these induced subgraphs.*

Proof Since each trivial component of a graph is a line graph, it suffices to prove the result in the case that H has no isolated vertices.

Suppose first that H is a line graph. Then there is a graph G, each of whose components has order at least 3, such that $H \cong L(G)$. Each vertex v of G gives rise to a complete subgraph of H, which we denote by $K(v)$, having order $\deg_G v$. Let $v_1, v_2, \ldots, v_p$ denote the vertices of G where, then, $K(v_1)$, $K(v_2), \ldots, K(v_p)$ are the corresponding complete subgraphs of H.

Each vertex u of H corresponds to an edge ww' of G, so that $u \in V(K(v))$ if and only if $v = w$ or $v = w'$. Since no component of G has order 2, $K(w) \neq K(w')$. Therefore each vertex of H lies in exactly two of the p distinct subgraphs $K(v_i)$, $i = 1, 2, \ldots, p$. If e is an edge of H, then e corresponds to a pair of (adjacent) edges of G incident with a common vertex v_j, where $1 \leq j \leq p$. Thus e is an edge of $K(v_i)$ if and only if $i = j$.

By deleting any empty sets from the collection $E(K(v_1))$, $E(K(v_2)), \ldots$, $E(K(v_p))$, we obtain a partition of $E(H)$ with the desired properties.

Conversely, suppose there is a partition $F_1, F_2, \ldots, F_m$ of $E(H)$ such that each subgraph $\langle F_i \rangle_H$, $1 \leq i \leq m$, is complete and such that no vertex of H lies in more than two of these induced subgraphs. For $i = 1, 2, \ldots, m$, define $H_i = \langle F_i \rangle_H$. Each vertex u of H is incident with at least one edge, and so belongs to H_i for at least one value of i $(1 \leq i \leq m)$. By hypothesis, $u \in V(H_i)$ for at most two values of i $(1 \leq i \leq m)$. Let $u_1, u_2, \ldots, u_\ell$ be the vertices of H that belong to exactly one subgraph H_i (if no such vertices exist, set $\ell = 0$), and for $m + 1 \leq i \leq m + \ell$, define $H_i = \langle \{u_{i-m}\} \rangle_H$. Then each vertex of H belongs to exactly two of the (distinct) subgraphs $H_1, H_2, \ldots, H_{m+\ell}$, and each edge of H belongs to exactly one such subgraph. Furthermore, $0 \leq |V(H_i) \cap V(H_j)| \leq 1$ for $1 \leq i < j \leq m + \ell$.

Let G be the graph defined by $V(G) = \{v_1, v_2, \ldots, v_{m+\ell}\}$ and $E(G) = \{v_i v_j | V(H_i) \cap V(H_j) \neq \emptyset\}$. The proof will be complete once we have shown that $H \cong L(G)$. For $1 \leq i < j \leq m + \ell$, if $V(H_i) \cap V(H_j) \neq \emptyset$, let u_{ij} denote the unique vertex common to H_i and H_j. Define the mapping α: $E(G) \to V(H)$ by $\alpha(v_i v_j) = u_{ij}$. The map α is one-to-one since no vertex of H belongs to more than two of the subgraphs $H_i (1 \leq i \leq m + \ell)$; the map α is onto since every vertex of H belongs to at least two of the subgraphs H_i $(1 \leq i \leq m + \ell)$. Suppose e and f are adjacent edges of G, say $e = v_1 v_2$ and $f = v_1 v_3$. Then $V(H_1) \cap V(H_2) = \{u_{12}\}$ and $V(H_1) \cap V(H_3) = \{u_{13}\}$, where $u_{12} \neq u_{13}$. Since u_{12}, $u_{13} \in V(H_1)$, it follows that $u_{12} u_{13} \in E(H_1) \subseteq E(H)$; that is, $\alpha(e)\alpha(f) \in E(H)$. Suppose, on the other hand, that e and f are nonadjacent edges of G. Without loss of generality, assume $e = v_1 v_2$ and $f = v_3 v_4$, so that $\alpha(e) = u_{12}$ and $\alpha(f) = u_{34}$, where $V(H_1) \cap V(H_2) = \{u_{12}\}$ and $V(H_3) \cap V(H_4) = \{u_{34}\}$. Then $u_{12} u_{34} \notin E(H)$; otherwise, $u_{12} u_{34} \in E(H_j)$ for some $j \neq 1, 2, 3, 4$, which contradicts the fact that $u_{12} \notin V(H_i)$ if $i \neq 1, 2$. ■

A *Krausz decomposition* of a graph H is a collection of complete subgraphs of H (some possibly trivial) such that every edge of H lies in exactly one of these subgraphs and every vertex of H lies in exactly two subgraphs. It follows that if H_1 and H_2 are members of a Krausz decomposition of H, then $0 \leq |V(H_1) \cap V(H_2)| \leq 1$. Furthermore, a graph H has a Krausz decomposition if and only if H has no isolated vertices and there is a partition of $E(H)$ satisfying properties (a) and (b) of Theorem 9.10.

It was shown in the proof of Theorem 9.10 that if the vertices of a graph G can be put in one-to-one correspondence with the subsets in some Krausz decomposition of a graph H in such a way that two vertices of G are adjacent if and only if the corresponding subgraphs of H have a common vertex, then $H \cong L(G)$. Conversely, we saw that if $H \cong L(G)$, where G and H are graphs without isolated vertices and $V(G) = \{v_1, v_2, \ldots, v_p\}$, then the collection $\{K(v_1), K(v_2), \ldots, K(v_p)\}$ of p complete subgraphs of H is a Krausz decomposition of H. Moreover, it is clear that in such a situation, $v_iv_j \in E(G)$ if and only if $V(K(v_i)) \cap V(K(v_j)) \neq \emptyset$. Thus we have the following.

Corollary 9.10 *Let H be a graph without isolated vertices. Then H is a line graph if and only if H has a Krausz decomposition. Furthermore, $H \cong L(G)$ for some graph G if and only if the nonisolated vertices of G can be put in one-to-one correspondence with the subsets in some Krausz decomposition of H in such a way that two vertices of G are adjacent if and only if the corresponding subgraphs of H have a common vertex.*

If H is the graph of Figure 9.6(a), then $\{E(\langle\{v_1, v_2, v_3, v_4\}\rangle), E(\langle\{v_2, v_5\}\rangle), E(\langle\{v_3, v_6\}\rangle), E(\langle\{v_4, v_7\}\rangle), E(\langle\{v_5, v_6, v_7\}\rangle)\}$ is a partition of $E(H)$ that satisfies properties (a) and (b) of Theorem 9.10. The corresponding Krausz decomposition (as constructed in the proof of Theorem 9.10) is $\{\langle\{v_1\}\rangle, \langle\{v_1, v_2, v_3, v_4\}\rangle, \langle\{v_2, v_5\}\rangle, \langle\{v_3, v_6\}\rangle, \langle\{v_4, v_7\}\rangle, \langle\{v_5, v_6, v_7\}\rangle\}$. Finally, the graph G constructed in the proof of Theorem 9.10 so that $H \cong L(G)$ is shown in Figure 9.6(b). The vertices of G are labeled with the corresponding complete subgraphs of H.

If H is a graph without isolated vertices, then finding a Krausz decomposition of H (or, equivalently, finding a partition of the edge set of H satisfying properties (a) and (b) of Theorem 9.10) can be simplified somewhat by observing that if K is any Krausz decomposition of H, then every four mutually adjacent vertices of H lie (together) in some member of K. Suppose, to the contrary, that there are four mutually adjacent vertices of H that are common to no member of K. If some member of K contains three of these vertices, then the three edges joining them to the fourth vertex, say v, must be in distinct members of K, since each member of K is a complete subgraph of H and each edge of H lies in exactly one member of K. However, this contradicts the fact that v lies in exactly two members of K. Similarly, if no member of K contains more than two of these four vertices, a contradiction arises. Thus

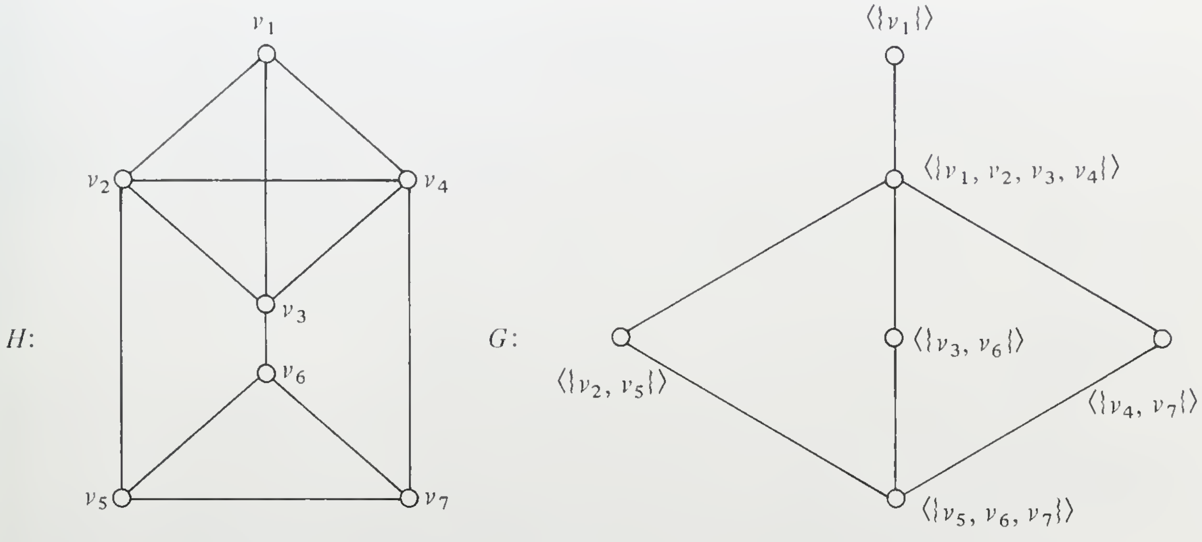


Figure 9.6 *An illustration of Theorem 9.10*

every four, or more, mutually adjacent vertices of H lie in a common member of K.

It follows from Theorem 9.10 and its corollary that given any graph H, one can find all graphs G (without isolated vertices) such that $H \cong L(G)$ by first determining all partitions of the edge set of H satisfying properties (a) and (b), and by then determining the corresponding graph constructed in the proof of the theorem. We know by Theorem 9.9 that if H is a connected line graph and $H \not\cong K_3$, then such a graph G is unique. However, it is not necessarily true that in this case there is a unique partition of $E(H)$ satisfying properties (a) and (b) of Theorem 9.10. For example, if H is the graph H_2 of Figure 9.7, then both $P_1 = \{\{e_1, e_2, e_3\}, \{e_4\}, \{e_5\}\}$ and $P_2 = \{\{e_1\}, \{e_2\}, \{e_3, e_4, e_5\}\}$ are such partitions of $E(H)$. However, it can be shown that if H is a connected line graph and H is not one of the graphs H_1, H_2, H_3, H_4 of Figure 9.7, then H does indeed have a unique edge-partition satisfying properties (a) and (b). If H is one of the aforementioned four graphs, then H has more than one such partition, but for $H \cong H_2$, H_3, or H_4, these partitions give rise to the same

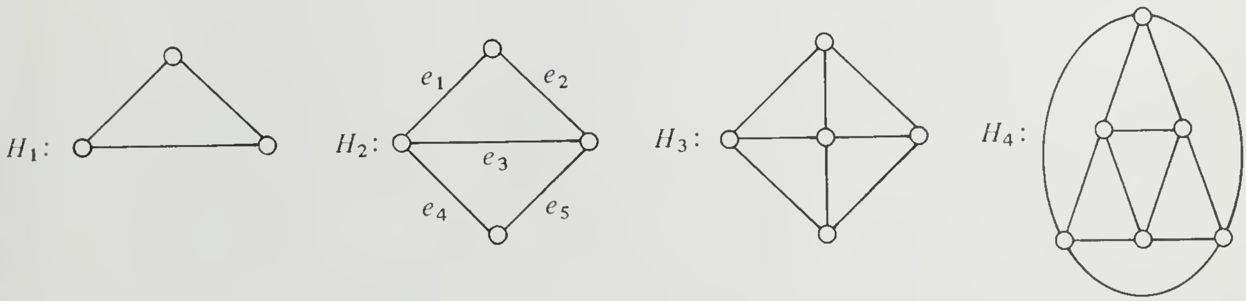


Figure 9.7 *Line graphs with two or more edge-partitions*

graph G such that $H \cong L(G)$. We note, for $i = 2, 3, 4$, that $H_i \cong L(G_{i+1})$, where the graphs G_3, G_4, G_5 are illustrated in Figure 9.2, and that $H_1 \cong L(K_3) \cong L(K(1, 3))$.

In order to state another characterization of line graphs, due to van Rooij and Wilf [VW1], we make the following definition. A triangle T in a graph H is *even* if every vertex of H is adjacent with an even number (that is, zero or two) of vertices of T; otherwise, T is *odd*.

Theorem 9.11 *A graph H is a line graph if and only if*
(a) $K(1, 3)$ *is not an induced subgraph of H, and*
(b) if $K(1, 1, 2)$ *is an induced subgraph of H, then at least one of its two triangles is even.*

With the aid of Theorem 9.11, it is possible to give a "forbidden subgraph" characterization of line graphs. These subgraphs were initially found by Beineke [B4].

Corollary 9.11 *A graph H is a line graph if and only if none of the graphs of Figure 9.8 is an induced subgraph of H.*

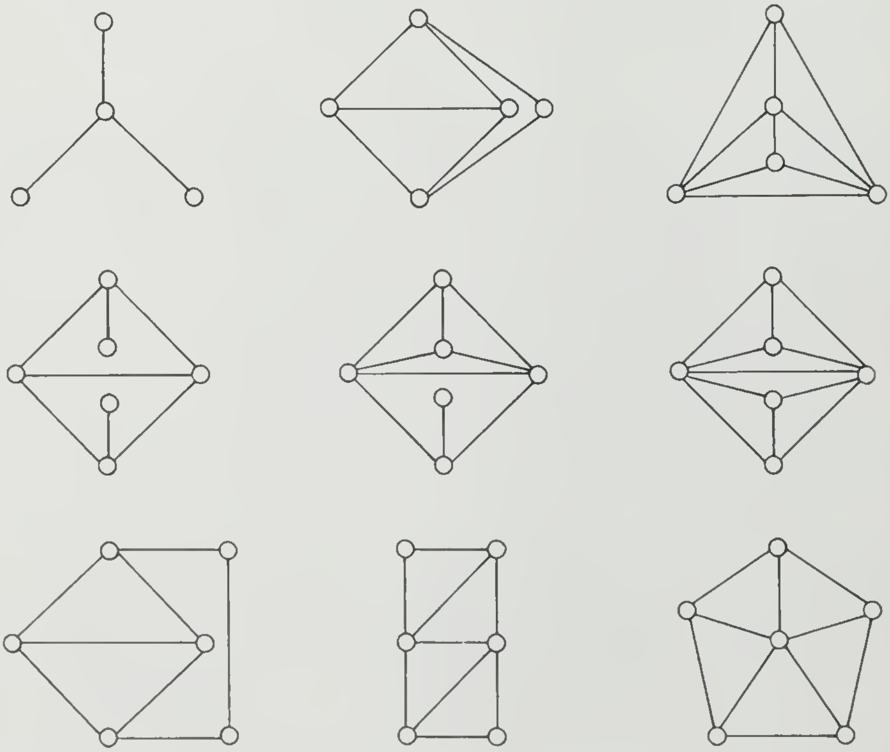


Figure 9.8 *The induced subgraphs not contained in any line graph*

We now change our direction and ask, for a given graphical property A, what property G must possess in order for $L(G)$ to have property A. We give two examples of this—the first of which deals with the property of being hamiltonian. Harary and Nash-Williams [HN1] characterized those graphs G for which $L(G)$ is hamiltonian.

Recall that a set E of edges of a graph G is a dominating set of edges if every edge of G either belongs to E or is adjacent to an edge of E. If $\langle E \rangle$ is a circuit C, then C is called a *dominating circuit* of G. Equivalently, a circuit C in a graph G is a dominating circuit if every edge of G is incident with a vertex of C.

Theorem 9.12 *Let G be a graph without isolated vertices. Then $L(G)$ is hamiltonian if and only if $G \cong K(1, n)$, for some $n \geq 3$, or G contains a dominating circuit.*

Proof If $G \cong K(1, n)$ for some $n \geq 3$, then $L(G)$ is hamiltonian since $L(G) \cong K_n$. Suppose, then, that G contains a dominating circuit

$$C: v_1, v_2, \ldots, v_t, v_1.$$

It suffices to show that there exists an ordering S: $e_1, e_2, \ldots, e_q$ of the q edges of G such that e_i and e_{i+1} are adjacent edges of G, for $1 \leq i \leq q-1$, as are e_1 and e_q, since such an ordering S corresponds to a hamiltonian cycle of $L(G)$. Begin the ordering S by selecting, in any order, all edges of G incident with v_1 that are not edges of C, followed by the edge v_1v_2. At each successive v_i, $2 \leq i \leq t-1$, select, in any order, all edges of G incident with v_i that are neither edges of C nor previously selected edges, followed by the edge v_iv_{i+1}. This process terminates with the edge $v_{t-1}v_t$. The ordering S is completed by adding all edges incident with v_t that are neither edges of C nor previously selected edges, followed by the edge v_tv_1. Since C is a dominating circuit of G, every edge of G appears exactly once in S. Furthermore, consecutive edges of S as well as the first and last edges of S are adjacent in G.

Conversely, suppose that G is not a star graph but that $L(G)$ is hamiltonian. We show that G contains a dominating circuit. Since $L(G)$ is hamiltonian, there is an ordering S: $e_1, e_2, \ldots, e_q$ of the q edges of G such that e_i and e_{i+1} are adjacent edges of G, for $1 \leq i \leq q-1$, as are e_1 and e_q. For $1 \leq i \leq q-1$, let v_i be the vertex of G incident with both e_i and e_{i+1}. (Note that $1 \leq k \neq m \leq q-1$ does not necessarily imply that $v_k \neq v_m$.) Since G is not a star graph, there is a smallest integer j_1 exceeding 1 such that $v_{j_1} \neq v_1$. The vertex v_{j_1-1} is incident with e_{j_1}, the vertex v_{j_1} is incident with e_{j_1}, and $v_{j_1-1} = v_1$. Thus, $e_{j_1} = v_1v_{j_1}$. Next, let j_2 (if it exists) be the smallest integer exceeding j_1 such that $v_{j_2} \neq v_{j_1}$. The vertex v_{j_2-1} is incident with e_{j_2}, the vertex v_{j_2} is incident with e_{j_2}, and $v_{j_2-1} = v_{j_1}$. Thus, $e_{j_2} = v_{j_1}v_{j_2}$. Continuing in this fashion, we finally arrive at a vertex v_{j_t} such that $e_{j_t} = v_{j_{(t-1)}}v_{j_t}$, where $v_{j_t} = v_{q-1}$. Since every edge of G appears exactly once in S and since $1 < j_1 < j_2 < \cdots < j_t \leq q-1$, this construction yields a trail

$$T\colon v_1, e_{j1}, v_{j1}, e_{j2}, v_{j2}, \ldots, v_{j(t-1)}, e_{j_t}, v_{j_t} = v_{q-1}$$

in G with the properties that

(a) every edge of G is incident with a vertex of T, and

(b) neither e_1 nor e_q is an edge of T.

Let w be the vertex of G incident with both e_1 and e_q. We consider four possible cases.

Case 1: Suppose $w = v_1 = v_{q-1}$. Then T itself is a dominating circuit of G.

Case 2: Suppose $w = v_1$ and $w \neq v_{q-1}$. Since e_q is incident with both w and v_{q-1}, it follows that $e_q = v_{q-1}w = v_{q-1}v_1$. Thus $C\colon T, e_q, v_1$ is a dominating circuit of G.

Case 3: Suppose $w = v_{q-1}$ *and* $w \neq v_1$. Since e_1 is incident with both w and v_1, we have that $e_1 = wv_1 = v_{q-1}v_1$. Thus $C\colon T, e_1, v_1$ is a dominating circuit of G.

Case 4: Suppose $w \neq v_{q-1}$ *and* $w \neq v_1$. Since e_q is incident with both w and v_{q-1}, it follows that $e_q = v_{q-1}w$. Since e_1 is incident with both w and v_1, we have that $e_1 = wv_1$. Thus $v_1 \neq v_{q-1}$, and $C\colon T, e_q, w, e_1, v_1$ is a dominating circuit of G. ■

For integers $n \geq 2$, the nth *iterated line graph* $L^n(G)$ of a graph G is defined to be $L(L^{n-1}(G))$, where $L^1(G)$ denotes $L(G)$ and $L^{n-1}(G)$ is assumed to be nonempty.

With the aid of Theorem 9.12, it is possible to present two results involving hamiltonian (iterated) line graphs (see [CW1]).

Theorem 9.13 *If G is a connected graph such that $\delta(G) \geq 3$, then $L(L(G))$ is hamiltonian.*

Proof For each vertex v of G, the corresponding complete subgraph $K(v)$ of $L(G)$ has order at least 3 since $\delta(G) \geq 3$, and so contains a hamiltonian cycle $C(v)$. Let H be the (spanning) subgraph of $L(G)$ defined by

$$V(H) = V(L(G)) \quad \text{and} \quad E(H) = \bigcup_{v \in V(G)} E(C(v)).$$

Each edge of $L(G)$ belongs to exactly one of the complete subgraphs $K(v)$ and each vertex belongs to exactly two; thus each edge of H belongs to exactly one of the cycles $C(v)$ and each vertex belongs to exactly two. It follows that H is a 4-regular graph.

Let u_1 and u_2 be vertices of H. Then for some vertices v_1 and v_2 of G (not necessarily distinct), u_1 is a vertex of $C(v_1)$ and u_2 is a vertex of $C(v_2)$. Since G is connected, there is a v_1-v_2 path P in G. Then the subgraph H' of H, defined by

$$V(H') = \bigcup_{v \in V(P)} V(C(v)) \quad \text{and} \quad E(H') = \bigcup_{v \in V(P)} E(C(v)),$$

is a connected subgraph of H containing a u_1-u_2 path. Thus, H is connected.

Since H is a 4-regular connected graph, by Theorem 2.19, the graph H is eulerian and contains an eulerian circuit C. Since H is a spanning subgraph of $L(G)$, it follows that C is a dominating circuit of $L(G)$. Thus, by Theorem 9.12, $L(L(G))$ is hamiltonian. ■

Corollary 9.13 *If G is a connected graph that is not a path, then some iterated line graph of G is hamiltonian.*

For a second example, we state without proof a result by Sedláček [S1] that characterizes those graphs having a planar line graph.

Theorem 9.14 *A nonempty graph G has a planar line graph if and only if*
(a) G is planar,
(b) $\Delta(G) \leq 4$, and
(c) if $\deg_G v = 4$, then v is a cut-vertex of G.

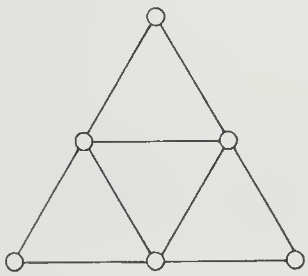
Exercises 9.3

9.15 Determine a formula for the number of triangles in the line graph $L(G)$ in terms of quantities in G.

9.16 Prove that $L(G)$ is connected if and only if G is a graph with exactly one nontrivial component.

9.17 Generalize Theorem 9.9 to arbitrary graphs.

9.18 Verify that the graph below is a line graph in two ways:
(a) by showing it satisfies the criteria of Theorem 9.10 for a graph to be a line graph, and
(b) by using the result of (a) to find a graph G such that $H \cong L(G)$.



9.19 Show that if P_1 and P_2 are partitions of the edge set of a connected graph H that both satisfy the hypothesis of Theorem 9.10, and if $P_1 \cap P_2 \neq \emptyset$ (that is, some set E of edges is a member of both P_1 and P_2), then $P_1 = P_2$.

9.20 Show that if there exist two distinct partitions of the edge set of a connected graph H that satisfy the hypothesis of Theorem 9.10, then H has order less than 7.

9.21 For each of the graphs H_1, H_3, H_4 of Figure 9.7, find two edge-partitions that satisfy properties (a) and (b) of Theorem 9.10.

9.22 Which of the complete n-partite graphs $K(p_1, p_2, \ldots, p_n)$ have planar line graphs?

9.23 (a) Find a necessary and sufficient condition for a graph G to have the property that $G \cong L(G)$.

(b) Find a necessary and sufficient condition for a graph G to have the property that $L(G) \cong L(L(G))$.

9.24 Prove that $L(G)$ is eulerian if G is eulerian.

9.25 Prove that if G is connected and $L(L(L(G)))$ is eulerian, than $L(L(G))$ is eulerian.

9.26 Show that each of the following conditions is sufficient for a graph G to have a hamiltonian line graph:
(a) G is eulerian; (b) G is hamiltonian.

9.27 Prove Corollary 9.13.

9.28 The *total graph* $T(G)$ *of a graph* G is that graph whose vertex set can be put in one-to-one correspondence with the set $V(G) \cup E(G)$ such that two vertices of $T(G)$ are adjacent if and only if the corresponding elements of G are adjacent or incident. Prove the following:

(a) Not all graphs are total graphs.

(b) The total graph of every nontrivial connected graph has a spanning eulerian subgraph.

(c) If G contains a spanning eulerian subgraph, then $T(G)$ is hamiltonian.

(d) If G is a nontrivial connected graph, then $T(T(G))$ is hamiltonian.

Chapter Ten

Graph Colorings

The graph-theoretic parameter that has probably received the most attention over the years is the chromatic number. Its prominence in graph theory is undoubtedly due to its involvement with the Four Color Problem, which is discussed in this chapter. The main object of this chapter, however, is to describe the variety of ways in which a graph can be colored.

10.1 Vertex Colorings

An assignment of colors (elements of some set) to the vertices of a graph G, one color to each vertex, so that adjacent vertices are assigned different colors is called a *coloring of G*; a coloring in which n colors are used is an *n-coloring.* A graph G is *n-colorable* if there exists an m-coloring of G for some $m \leq n$. It is obvious that if G has order p, then G can be p-colored, so that G is p-colorable.

The minimum n for which a graph G is n-colorable is called the *vertex chromatic number*, or simply the *chromatic number* of G, and is denoted by $\chi(G)$. If G is a graph for which $\chi(G) = n$, then G is *n-chromatic.* In a given coloring of a graph G, a set consisting of all those vertices assigned the same color is referred to as a *color class.*

The chromatic number of G may be defined alternatively as the minimum number of independent subsets into which $V(G)$ may be partitioned. Each such independent set is then a color class in the $\chi(G)$-coloring of G so defined.

Suppose that a company periodically sets aside a workday for several one-hour meetings for its employees. It is imperative, of course, that two meetings not be scheduled at the same time if some employee is to attend both meetings. Further, it is more efficient to minimize the number of one-hour periods used for meetings. This situation can be represented by a graph G. Let the meetings be represented by vertices and join two vertices by an edge if and only if there is at least one employee who is to attend both of the corresponding meetings. The minimum number of time periods required is then $\chi(G)$.

For several special classes of graphs, the chromatic number is quite easy to determine. For example,

$$\chi(C_{2n})=2, \qquad \chi(C_{2n+1})=3, \qquad \chi(K_p)=p$$

and, in general,

$$\chi(K(p_1, p_2, \ldots, p_n))=n.$$

Further, if G is n-partite, then $\chi(G)\leq n$. If G is a 2-chromatic graph, then necessarily G is bipartite; for in any 2-coloring of G, the color classes so determined are the partite sets of a bipartite graph. On the other hand, every nonempty bipartite graph is 2-chromatic. By Theorem 2.5, we therefore conclude that a nonempty graph G is 2-chromatic if and only if it contains no odd cycles. From this observation it follows that $\chi(T)=2$ for every nontrivial tree T. The graph G of Figure 10.1 is 3-chromatic; a 3-coloring of G is indicated, with the colors denoted by the integers 1, 2, 3. This graph is therefore n-colorable for every $n\geq 3$.

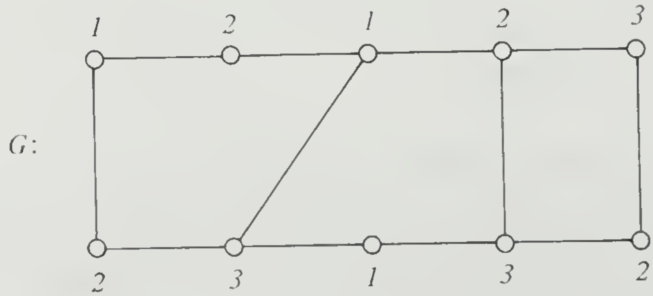


Figure 10.1 *A 3-chromatic graph*

We need only be concerned with determining the chromatic numbers of blocks, since the chromatic number of a disconnected graph is the maximum of the chromatic numbers of its components, and the chromatic number of a connected graph with cut-vertices is the maximum of the chromatic numbers of its blocks.

The chromatic number most properly belongs to the collection of graphical parameters discussed in Section 3.3 or the set of parameters introduced in Exercise 10.26. Although the quantity of literature dealing with the chromatic number far surpasses that of these other specialized parameters, no formula exists for the chromatic number of an arbitrary graph. Thus, for the most part, one must be content with supplying bounds for the chromatic number of graphs. In order to present such bounds, we now discuss graphs that are critical or minimal with respect to chromatic number.

For an integer $n \geq 2$, we say that a graph G is *critically n-chromatic* if $\chi(G) = n$ and $\chi(G - v) = n - 1$ for all $v \in V(G)$; G is *minimally n-chromatic* if $\chi(G) = n$ and $\chi(G - e) = n - 1$ for all $e \in E(G)$. There are several results dealing with critically n-chromatic graphs and minimally n-chromatic graphs, many of which are due to Dirac [D3]. We shall consider here only one of the more elementary of these.

Every critically n-chromatic graph is a block, while every minimally n-chromatic graph without isolated vertices is a block. Furthermore, every minimally n-chromatic graph (without isolated vertices) is critically n-chromatic. The converse is not true in general, however: for example, the graph of Figure 10.2 is critically 4-chromatic but not minimally 4-chromatic. For $n = 2$ and $n = 3$, the converse is true. In fact, K_2 is the only critically 2-chromatic graph as well as the only minimally 2-chromatic graph without isolated vertices, while the odd cycles are the only critically 3-chromatic graphs and the only minimally 3-chromatic graphs having no isolated vertices. For $n \geq 4$, neither the critically n-chromatic graphs nor the minimally n-chromatic graphs have been characterized. Although it is quite difficult, in general, to determine whether a given n-chromatic graph G is critical or minimal, G contains both critically n-chromatic subgraphs and minimally n-chromatic subgraphs. An n-chromatic subgraph of G of minimum order is critically n-chromatic while an n-chromatic subgraph of G of minimum size is minimally n-chromatic.

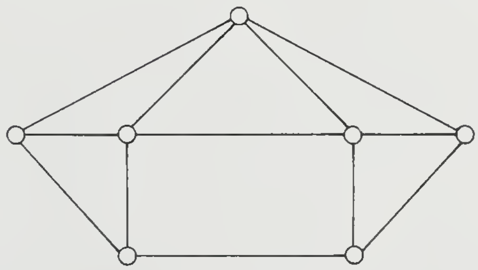


Figure 10.2 *A critically 4-chromatic graph that is not minimally 4-chromatic*

The first theorem of this chapter concerns the structure of critically (and minimally) n-chromatic graphs.

Theorem 10.1 *Every critically n-chromatic graph, $n \geq 2$, is $(n-1)$-edge-connected.*

Proof Let G be critically n-chromatic, $n \geq 2$. If $n = 2$ or $n = 3$, then $G \cong K_2$ or G is an odd cycle, respectively; therefore, G is 1-edge-connected or 2-edge-connected.

Assume $n \geq 4$ and that G is not $(n-1)$-edge-connected. Hence by

Theorem 5.3, there exists a partition of $V(G)$ into subsets V_1 and V_2 such that the set E' of edges joining V_1 and V_2 contains fewer than $n-1$ elements. Since G is critically n-chromatic, the subgraphs $G_1 = \langle V_1 \rangle$ and $G_2 = \langle V_2 \rangle$ are $(n-1)$-colorable. Let each of G_1 and G_2 be colored with at most $n-1$ colors, using the same set of $n-1$ colors. If each edge in E' is incident with vertices of different colors, then G is $(n-1)$-colorable. This contradicts the fact that $\chi(G) = n$. Hence we may assume that there are edges of E' incident with vertices assigned the same color. We show that the colors assigned to the elements of V_1 may be permuted so that each edge in E' joins vertices assigned different colors. Again this will imply that $\chi(G) \leq n-1$, produce a contradiction, and complete the proof.

In the coloring of G_1, let $U_1, U_2, \ldots, U_m$ be those color classes of G_1 such that for each i, $1 \leq i \leq m \leq n-2$, there is at least one edge joining U_i and G_2. For $i = 1, 2, \ldots, m$, assume there are n_i edges joining U_i and G_2. Hence, for each i, $1 \leq i \leq m$, it follows that $n_i > 0$ and $\Sigma_{i=1}^{m} n_i \leq n-2$.

If for each u_1 in U_1, the vertex u_1 is adjacent only with vertices assigned colors different from that assigned to u_1, then the assignment of colors to the vertices of G is not altered. On the other hand, if some vertex u_1 of U_1 is adjacent with a vertex of G_2 that is assigned the same color as that of u_1, then in G_1 we may permute the $n-1$ colors so that in the new assignment of colors to the vertices of G, no vertex of U_1 is adjacent to a vertex of G_2 having the same color. This is possible since the vertices of U_1 may be assigned any one of at least $n-1-n_1$ colors and $n-1-n_1 > 0$.

If, in this new assignment of colors to the vertices of G, each vertex u_2 of U_2 is adjacent only with vertices assigned colors different from that assigned to u_2, then no (additional) permutation of colors G_1 occurs. However, if some vertex u_2 of U_2 is adjacent with a vertex of G_2 that is assigned the same color as that of u_2, then in G_1 we may permute the $n-1$ colors, leaving the color assigned to U_1 fixed, so that no vertex of $U_1 \cup U_2$ is adjacent to a vertex of G_2 having the same color. This can be done since the vertices of U_2 can be assigned any of $(n-1)-(n_2+1)$ colors, and $(n-1)-(n_2+1) \geq (n-1)-(n_1+n_2) > 0$. Continuing this process, we arrive at an $(n-1)$-coloring of G, producing the desired contradiction. ■

Since every connected, minimally n-chromatic graph is critically n-chromatic, the preceding result has an immediate consequence.

Corollary 10.1a *If G is a connected, minimally n-chromatic graph, $n \geq 2$, then G is $(n-1)$-edge-connected.*

Theorem 10.1 and Corollary 10.1a imply that $\kappa_1(G) \geq n-1$ for every critically n-chromatic graph G or connected, minimally n-chromatic graph G. The next corollary now follows directly from Theorem 5.1.

Corollary 10.1b *If G is critically n-chromatic or connected and minimally n-chromatic, then $\delta(G) \geq n-1$.*

We are now prepared to present bounds for the chromatic number of a graph. We give here several upper bounds, beginning with the best known and most applicable. The theorem is due to Brooks [B15] but the proof here is due to Lovász [L6].

Theorem 10.2 (Brooks) *If G is a connected graph that is neither an odd cycle nor a complete graph, then*

$$\chi(G) \leq \Delta(G).$$

Proof Let G be a connected graph that is neither an odd cycle nor a complete graph, and suppose $\chi(G)=n$, where, necessarily, $n \geq 2$. Let H be a critically n-chromatic subgraph of G. Then H is a block and $\Delta(H) \leq \Delta(G)$.

Suppose $H \cong K_n$ or H is an odd cycle. Then $G \ncong H$; therefore, since G is connected, $\Delta(G) > \Delta(H)$. If $H \cong K_n$, then $\Delta(H) = n-1$ and $\Delta(G) \geq n$ so

$$\chi(G) = n \leq \Delta(G).$$

If H is an odd cycle, then

$$\Delta(G) \geq 3 = n = \chi(G).$$

Hence, we may assume that H is critically n-chromatic and is neither an odd cycle nor a complete graph, which implies that $n \geq 4$.

Let H have order p. Since $\chi(H) = n \geq 4$ and H is not complete, it follows that $p \geq 5$. We now consider two cases, depending on the connectivity of H.

Case 1: Suppose that H is 3-connected. Let x and y be vertices of H such that $d_H(x, y) = 2$, and suppose x, w, y is a path in H. We arrange the vertices of H in a sequence $v_1, v_2, \ldots, v_p$, where $v_1 = x$, $v_2 = y$, and the vertices v_i, $i \geq 3$, are arranged in nonincreasing order according to their distance from w. Thus, $v_p = w$ and for $1 \leq i < p$, v_i is adjacent to some v_j for $j > i$. That v_{p-1} is adjacent to $v_p = w$ is guaranteed by the fact that H is 3-connected and hence $\deg_H w \geq 3$. This implies that for $1 \leq i < p$, v_i is adjacent to at most $\Delta(H)-1$ vertices v_j with $j < i$. We now show that the vertices of H can be colored with the colors 1, 2, ..., $\Delta(H)$. Assign color 1 to vertices v_1 and v_2. We successively color each of the vertices $v_3, v_4, \ldots, v_{p-1}$ with one of the colors 1, 2, ..., $\Delta(H)$ that was not used in coloring adjacent vertices preceding it in the sequence. Such a color is available since, as we have seen, each v_i, $1 \leq i < p$, is adjacent to at most $\Delta(H)-1$ vertices preceding v_i in the sequence. The vertex $v_p = w$ is adjacent to $v_1 = x$ and $v_2 = y$, both colored 1, so a color is available for v_p. Therefore,

$$\chi(G) = \chi(H) \leq \Delta(H) \leq \Delta(G).$$

Case 2: Suppose that $\kappa(H)=2$. We begin with an observation in this case; namely, H does not contain only vertices of degree 2 and $p-1$. Since $\chi(H)\geq 4$, H cannot contain only vertices of degree 2. Since H is not complete, H cannot contain only vertices of degree $p-1$. If H contains vertices of both degrees (and no others), then H must contain two vertices of degree $p-1$ and $p-2$ vertices of degree 2; that is, $H \cong K(1, 1, p-2)$. However, then $\chi(H)=3$, which is impossible.

Let $u \in V(H)$ such that $2<\deg_H u<p-1$. If $\kappa(H-u)=2$, then let v be any vertex such that $d_H(u, v)=2$. (The vertex v exists since u is not adjacent to all vertices of $H-u$.) Let $x=u$ and $y=v$, and proceed as in Case 1.

If $\kappa(H-u)=1$, then we consider two end-blocks B_1 and B_2 containing cut-vertices w_1 and w_2, respectively, of $H-u$. Since H is 2-connected, there exist vertices u_1 in B_1-w_1 and u_2 in B_2-w_2 that are adjacent to u. Let $x=u_1$ and $y=u_2$, and proceed as in Case 1.

This completes the proof. ■

We may now state an upper bound on the chromatic number of an arbitrary graph.

Corollary 10.2 *For every graph G,*

$$\chi(G)\leq 1+\Delta(G).$$

The bound for the chromatic number given in Theorem 10.2 is not particularly good for certain classes of graphs. For example, the bound provided for star graphs $K(1, n)$ differs from its chromatic number by $n-2$. We shall see in the next chapter that 4 serves as an upper bound for the chromatic number of all planar graphs; however, Theorem 10.2 gives no bound for the entire class. Thus, there are several important classes of graphs for which the bound $\chi(G)\leq\Delta(G)$ is poor indeed. A better bound in many cases is given by an inequality observed by Halin [H4] and by Szekeres and Wilf [SW1]. The reader will notice the similarity of this result and Theorem 3.12.

Theorem 10.3 *For every graph G,*

$$\chi(G)\leq 1+\max\delta(G'),$$

where the maximum is taken over all induced subgraphs G′ of G.

Proof The result follows immediately for empty graphs, so we assume G is a graph with $\chi(G)=n\geq 2$. Let H be an induced n-critical subgraph of G. Since H is an induced subgraph of G,

$$\delta(H) \leqslant \max_{G' < G} \delta(G'). \tag{10.1}$$

By Corollary 10.1b, $\delta(H) \geqslant n-1$, so by (10.1),

$$\max_{G' < G} \delta(G') \geqslant n-1 = \chi(G) - 1,$$

giving the result. ■

Theorem 10.3 gives an upper bound of 2 for the chromatic numbers of the graphs $K(1, n)$, which is exact. Since every planar graph has minimum degree at most 5 (by Corollary 4.2b) and since every subgraph of a planar graph is planar, a bound of 6 is provided for the chromatic number of planar graphs by Theorem 10.3. In each of these two cases, a marked improvement is shown over the result offered by Theorem 10.2. If G is a regular graph of degree r, then both Theorems 10.2 and 10.3 give $r+1$ as an upper bound for $\chi(G)$; however, this bound is poor for many r-regular graphs, such as $K(r, r)$.

There are other upper bounds that have been obtained for chromatic numbers. We consider two of these. The first gives an upper bound in terms of the length of a longest path; this result is due to Gallai [G2].

Theorem 10.4 *For any graph G,*

$$\chi(G) \leqslant 1 + m(G),$$

where $m(G)$ denotes the length of a longest path in G.

Proof The result is obvious if G is empty, so we assume that $\chi(G) = n \geqslant 2$. Let H be a critically n-chromatic subgraph of G, so that by Corollary 10.1b, $\delta(H) \geqslant n-1$. By Theorem 3.6, H (and therefore G) contains a path of length $n-1$. Hence $m(G) \geqslant n-1 = \chi(G)-1$, producing the desired result. ■

There is no known efficient algorithm for determining the chromatic number of a graph. The following algorithm, called the sequential coloring algorithm, is good and produces a coloring of a labeled graph G. The number of colors required by this algorithm is a function of the labeling of G and may, in fact, differ markedly from $\chi(G)$ (see Exercise 10.13).

Algorithm 10A (Sequential Coloring Algorithm) *Given a graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$:*

1. Let $i = 1$.

2. *Let* $c = 1$.
3. *If no vertex adjacent with* v_i *is assigned color* c, *then assign color* c *to* v_i *and go to Step 5.*
4. *Replace* c *by* $c+1$ *and return to Step 3.*
5. *If* $i < p$, *then replace* i *with* $i+1$ *and return to Step 2; otherwise, stop.*

If we apply Algorithm 10A to the labeled (4-chromatic) graph G_1 of Figure 10.3(a), we produce the optimal 4-coloring of G_1 shown in Figure 10.3(b). Indeed, for any labeling of G_1, a 4-coloring of G_1 is produced by Algorithm 10A. Applying this algorithm to the labeled 3-chromatic graph G_2 of Figure 10.3(c), however, yields the 4-coloring shown in Figure 10.3(d).

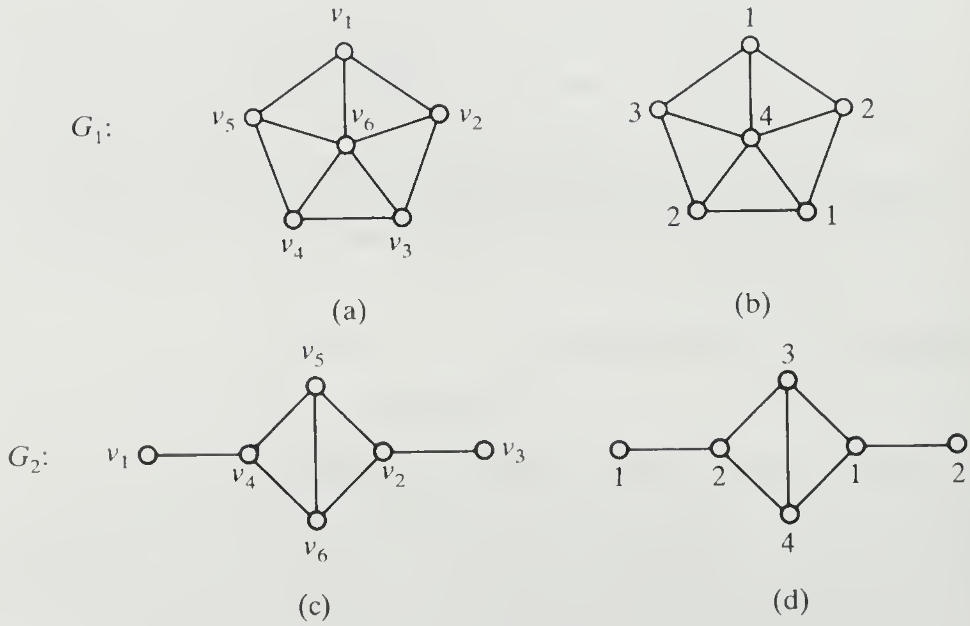


Figure 10.3 *Sequential colorings of graphs*

Algorithm 10A provides another upper bound for the chromatic number of a graph, a result first observed by Welsh and Powell [WP1].

Theorem 10.5 *Let* G *be a graph with* $V(G) = \{v_1, v_2, \ldots, v_p\}$. *Then*

$$\chi(G) \leq \max_{1 \leq i \leq p} \{\min \{i, \deg v_i + 1\}\}.$$

Proof We verify by induction that for each j, with $1 \leq j \leq p$, the number of colors produced by Algorithm 10A in coloring $\langle \{v_1, v_2, \ldots, v_j\} \rangle$ is at most

$$\max_{1 \leq i \leq j} \{\min \{i, \deg v_i + 1\}\}.$$

This is true for $j=1$. Assume that the number of colors assigned the first k vertices, where $1 \leq k < p$, is

$$n \leq \max_{1 \leq i \leq k} \{\min \{i, \deg v_i + 1\}\}.$$

In particular, $n \leq k$. If not all of the colors 1, 2, ..., n have been assigned to vertices adjacent to v_{k+1}, then one of these colors may be assigned to v_{k+1}, giving the desired result. Suppose then that all of the colors 1, 2, ..., n have been assigned to vertices adjacent to v_{k+1}. Then v_{k+1} is assigned the color $n+1$. It remains to show that

$$n+1 \leq \max_{1 \leq i \leq k+1} \{\min \{i, \deg v_i + 1\}\}. \tag{10.2}$$

Since $n \leq k$, it follows that $n+1 \leq k+1$. Further, $\deg v_{k+1} \geq n$ so that $\deg v_{k+1} + 1 \geq n+1$. Therefore,

$$n+1 \leq \min \{k+1, \deg v_{k+1} + 1\},$$

which verifies (10.2). ■

We now direct our attention briefly to lower bounds for the chromatic number. If $H \subset G$, then $\chi(H) \leq \chi(G)$. The *clique number* $\omega(G)$ of a graph G is the maximum order among the complete subgraphs of G. If $K_n \subset G$ for some n, then $\chi(G) \geq n$; so in general,

$$\chi(G) \geq \omega(G)$$

for every graph G. From what we have seen of upper bounds for $\chi(G)$, one might conjecture that this lower bound for $\chi(G)$ is not particularly good in general. There is, however, one (rather unusual) situation under which the lower bound is very good indeed [S2].

Theorem 10.6 *If a graph G does not contain P_4 as an induced subgraph, then $\chi(G) = \omega(G)$.*

As an immediate consequence of this result, we arrive at a characterization of n-colorable graphs, which, unfortunately, is quite difficult to apply.

Corollary 10.6a *A graph is n-colorable if it contains neither P_4 nor K_{n+1} as an induced subgraph.*

If a graph G satisfies the hypothesis of Theorem 10.6, then so too does each induced subgraph H of G, so that $\chi(H) = \omega(H)$. This suggests the following definition. A graph G is called *perfect* if $\chi(H) = \omega(H)$ for each induced subgraph H of G.

Corollary 10.6b *If a graph G does not contain P_4 as an induced subgraph, then G is perfect.*

Corollary 10.6b gives a sufficient condition for a graph to be perfect. This condition is not, in general, necessary. For example, the graph P_4 itself is perfect.

If H is a nonempty bipartite graph, then $\chi(H)=2=\omega(H)$, while if H is any empty graph, then $\chi(H)=1=\omega(H)$. It follows readily, then, that every bipartite graph is perfect. We now show that the complement $\bar{G}$ of a bipartite graph G is perfect. Each induced subgraph of $\bar{G}$ is of the form $\bar{H}$, where H is an induced subgraph of G. If H has no isolated vertices, then by Exercise 8.25, we have that $\alpha_1(H)=\beta(H)$. Since $\beta(H)=\omega(\bar{H})$, in this case we need only show that $\chi(\bar{H})=\alpha_1(H)$ in order to verify that $\chi(\bar{H})=\omega(\bar{H})$. Clearly, the chromatic number of $\bar{H}$ equals the minimum number of elements in a partition of $V(H)$ such that each element of the partition induces a complete subgraph in H. Since H contains no triangles, each such complete subgraph has order 1 or 2. It follows that such a partition contains $\alpha_1(H)$ elements and so $\chi(\bar{H})=\alpha_1(H)$. A similar argument can be applied if H has isolated vertices.

The concept of a perfect graph was introduced by Berge [B6], who conjectured that a graph G is perfect if and only if $\bar{G}$ is perfect. This conjecture (sometimes referred to as the *Perfect Graph Conjecture*) was proved by Lovász [L5].

Since $\chi(C_{2k+1})\neq\omega(C_{2k+1})$, $k=2, 3, \ldots$, if an induced subgraph of a graph G is an odd cycle of length at least 5, then G is not perfect. Now suppose that an induced subgraph of $\bar{G}$ is an odd cycle C_{2k+1}, where $k\geq 2$. Then $\bar{G}$ is not perfect and so, by the Perfect Graph Conjecture, G is not perfect. However, this conclusion can be reached independently of the conjecture, as we now see. Since C_{2k+1} is an induced subgraph of $\bar{G}$, it follows that G contains $\bar{C}_{2k+1}$ as an induced subgraph. However, $\omega(\bar{C}_{2k+1})=\beta(C_{2k+1})=k$, while $\chi(\bar{C}_{2k+1})=k+1$ since $\chi(\bar{C}_{2k+1})$ equals the minimum number of elements in a partition of $V(C_{2k+1})$ such that each element of the partition induces a complete subgraph in C_{2k+1}. Thus, G is not perfect if an induced subgraph of $\bar{G}$ is an odd cycle of length at least 5.

The preceding example is related to a second conjecture of Berge (see [B8, p. 361]) known as the *Strong Perfect Graph Conjecture*.

Strong Perfect Graph Conjecture *A graph G is perfect if and only if no induced subgraph of G or $\bar{G}$ is an odd cycle of length at least 5.*

This conjecture remains unproved, although it has been verified for several classes of graphs including planar graphs [T8] and graphs that do not contain $K(1, 3)$ as an induced subgraph [PR1].

We have mentioned for every graph G that $\chi(G)\geq\omega(G)$. Hence if G contains triangles, then $\chi(G)\geq 3$. However there exist graphs G that are triangle-free such that $\chi(G)\geq 3$. For example, the odd cycles C_{2n+1}, with

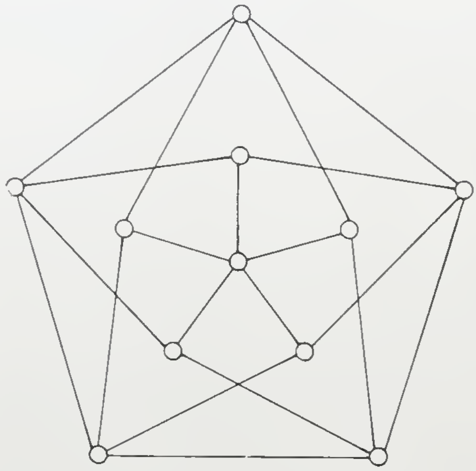


Figure 10.4 *The Grötzsch graph: a 4-chromatic triangle-free graph*

$n \geq 2$, have chromatic number 3 and are, of course, triangle-free. The graph of Figure 10.4, called the *Grötzsch graph*, is 4-chromatic and triangle-free and is, in fact, the smallest such graph (in terms of order).

It may be surprising that there exist triangle-free graphs with arbitrarily large chromatic number. This fact has been established by a number of mathematicians, including Descartes [D1], Kelly and Kelly [KK1], and Zykov [Z3]. The following construction is due to Mycielski [M10], however.

Theorem 10.7 *For every positive integer n, there exists an n-chromatic triangle-free graph.*

Proof The proof is by induction on n. For $n = 1$, 2, and 3, the graphs K_1, K_2, and C_5, respectively, have the required properties. Assume H is a triangle-free graph with $\chi(H) = k$, where $k \geq 3$. We show there exists a triangle-free graph with chromatic number $k + 1$. Let $V(H) = \{v_1, v_2, \ldots, v_p\}$. We construct a graph G from H by adding $p + 1$ new vertices $u, u_1, u_2, \ldots, u_p$. The vertex u is joined to each vertex u_i and, in addition, u_i is joined to each vertex to which v_i is adjacent.

To see that G is triangle-free, first observe that u belongs to no triangle. Since no two vertices u_i are adjacent, any triangle would consist of a vertex u_i and vertices v_j and v_k, $i \neq j, k$, but by the construction, this would imply that $\langle\{v_i, v_j, v_k\}\rangle$ is a triangle in H, which is impossible.

Let a k-coloring of H be given. Now assign to u_i the same color assigned to v_i and assign a $(k + 1)$st color to u. This produces a $(k + 1)$-coloring of G. Hence $\chi(G) \leq k + 1$. Suppose $\chi(G) \leq k$, and let there be given a k-coloring of G, with colors 1, 2, $\ldots$, k, say. Necessarily the vertex u is colored differently from each u_i. Suppose u is assigned the color k. Since $\chi(H) = k$, the color k is assigned to some vertices of H. Recolor each v_i assigned color k with the color assigned to vertex u_i. This produces a $(k-1)$-coloring of H and a contradiction. Thus, $\chi(G)=k+1$, and the proof is complete. ■

This result has been extended significantly by Erdös [E4] and Lovász [L4].

Theorem 10.8 (Erdös-Lovász) *For every two integers m, $n \geq 2$, there exists an n-chromatic graph whose girth exceeds m.*

We now turn our attention to a result on chromatic numbers and factorization [CP1].

Theorem 10.9 *If a graph G is factored as $G = G_1 \oplus G_2 \oplus \cdots \oplus G_k$, then*

$$\chi(G) \leq \prod_{i=1}^{k} \chi(G_i).$$

Proof For $i=1, 2, \ldots, k$, let a $\chi(G_i)$-coloring be given for G_i. Assign the color $(c_1, c_2, \ldots, c_k)$ to vertex v of G, where c_i is the color assigned to v in G_i. Vertices adjacent in G are adjacent in some G_i, $1 \leq i \leq k$, and are assigned different colors in that factor. Thus, this is a coloring of G using at most $\prod_{i=1}^{k} \chi(G_i)$ colors. Hence, $\chi(G) \leq \prod_{i=1}^{k} \chi(G_i)$. ■

In the case where $G \cong K_p$ and $k=2$, we obtain a result on the chromatic numbers of complementary graphs.

Corollary 10.9 *If G is a graph of order p, then*

$$\chi(G) \cdot \chi(\bar{G}) \geq p.$$

The best known result on chromatic numbers and complementary graphs, however, is the following theorem of Nordhaus and Gaddum [NG1]. The proof is based on one of H. V. Kronk.

Theorem 10.10 (Nordhaus and Gaddum) *If G is a graph of order p, then*
(a) $2\sqrt{p} \leq \chi(G) + \chi(\bar{G}) \leq p+1$, *and*
(b) $p \leq \chi(G) \cdot \chi(\bar{G}) \leq ((p+1)/2)^2$.

Proof The lower bound in (b) is a restatement of Corollary 10.9. Since the arithmetic mean of two positive numbers is always at least as large as their geometric mean, we have

$$\sqrt{p} \leqslant \sqrt{\chi(G)\cdot\chi(\bar{G})} \leqslant \frac{\chi(G)+\chi(\bar{G})}{2}.$$

This verifies the lower bound of (a).

To verify the upper bound in (a), we make use of Theorem 10.3. Let G be an arbitrary graph of order p and suppose that $k = \max \delta(G')$ where the maximum is taken over all induced subgraphs G' of G. Hence every induced subgraph of G has minimum degree at most k. Next we show that every induced subgraph of $\bar{G}$ has minimum degree at most $p-k-1$. Assume, to the contrary, that there exists an induced subgraph H' of $\bar{G}$ such that $\delta(H') \geqslant p-k$. Since $H' < \bar{G}$, we have that $H' \cong \bar{H}$ for some $H < G$. Denote the order of $\bar{H}$ (and, of course, of H) by h. Every vertex of H has degree at most $h-(p-k)-1 = h-p+k-1$ in H. Moreover, the vertices in H have degrees at most

$$(h-p+k-1)+(p-h) = k-1$$

in G. Since $\max_{G'<G}\delta(G') = k$, there exists an induced subgraph F of G such that $\delta(F) = k$. However, then, no vertex of H can be a vertex of F; that is, F is an induced subgraph of $G - V(H)$. The subgraph F has order at least $k+1$, implying that H has order at most $p-k-1$, contradicting the fact that $\delta(\bar{H}) \geqslant p-k$. We may therefore conclude that

$$\max_{G'<\bar{G}} \delta(G') \leqslant p-k-1.$$

Hence,

$$\chi(G) \leqslant 1 + \max_{G'<G} \delta(G') = 1+k$$

and

$$\chi(\bar{G}) \leqslant 1 + \max_{G'<\bar{G}} \delta(G') \leqslant 1 + (p-k-1) = p-k,$$

so that

$$\chi(G)+\chi(\bar{G}) \leqslant (1+k)+(p-k) = p+1,$$

completing the proof of the upper bound in (a).

Again, since the geometric mean of $\chi(G)$ and $\chi(\bar{G})$ never exceeds their arithmetic mean, we have

$$\sqrt{\chi(G)\cdot\chi(\bar{G})} \leqslant \frac{\chi(G)+\chi(\bar{G})}{2} \leqslant \frac{p+1}{2}.$$

This produces the upper bound in (b). ■

The proof of Theorem 10.10 implies, and rightly so, that the upper bound $\chi(G)+\chi(\bar{G})\leq p+1$ is the most complex of the four bounds. Plesník [P3] extended this result by showing that if $K_p = G_1 \oplus G_2 \oplus G_3$, then $\chi(G_1)+\chi(G_2)+\chi(G_3)\leq p+3$. Plesník further made the following conjecture.

Conjecture *If $K_p = G_1 \oplus G_2 \oplus \cdots \oplus G_n$, where $n \geq 1$, then*

$$\chi(G_1)+\chi(G_2)+\cdots+\chi(G_n)\leq p+\binom{n}{2}.$$

Exercises 10.1

10.1 Prove for every graph G of order p that $p/\beta(G)\leq\chi(G)\leq p+1-\beta(G)$.

10.2 Determine and prove a result analogous to Exercise 10.1 for vertex-arboricity.

10.3 Prove a result analogous to Theorem 10.2 for disconnected graphs.

10.4 What bound is given for $\chi(G)$ by Theorems 10.2 and 10.3 in the case that G is (a) a tree? (b) an outerplanar graph (see Exercise 4.11)?

10.5 Let G be a 4-regular graph of order 10. What bound for $\chi(G)$ is given by (a) Theorem 10.2? (b) Theorem 10.3?

10.6 Show that every n-chromatic graph is a subgraph of some complete n-partite graph.

10.7 Let G be an n-chromatic graph, where $n \geq 2$, and let r be a positive integer such that $r \geq \Delta(G)$. Prove that there exists an r-regular n-chromatic graph H such that G is an induced subgraph of H.

10.8 Determine (and prove) a necessary and sufficient condition for a graph to have a 2-colorable line graph.

10.9 Let G be a connected, cubic graph of order $p>4$ having girth 3. Determine $\chi(G)$.

10.10 Discuss the sharpness of Theorem 10.4.

10.11 Prove for every graph G that there exists a labeling of the vertices of G so that the number of colors assigned to G by Algorithm 10A is $\chi(G)$.

10.12 Determine the unique graph G of minimum order for which the number of colors assigned to G by Algorithm 10A differs from $\chi(G)$.

10.13 Prove that for every positive integer n there exists a (labeled) graph G such that the number of colors assigned to G by Algorithm 10A exceeds $\chi(G)$ by n.

10.14 (a) Prove that if the vertices $v_1, v_2, \ldots, v_p$ of a graph G are labeled so that $\deg v_1 \geq \deg v_2 \geq \cdots \geq \deg v_p$, then the upper bound for $\chi(G)$ given in Theorem 10.5 never exceeds the upper bound given by any other labeling of the vertices of G.

(b) Use Theorem 10.5 to prove Corollary 10.2.

10.15 Let $G_1, G_2, \ldots, G_n$ be pairwise disjoint graphs, and define $G = G_1 + G_2 + \cdots + G_n$. Prove that

$$\chi(G) = \sum_{i=1}^{n} \chi(G_i) \quad \text{and} \quad \omega(G) = \sum_{i=1}^{n} \omega(G_i).$$

10.16 Prove Corollary 10.6a.

10.17 The *co-chromatic number* $\tilde{\chi}(G)$ of a graph G is the minimum number of subsets into which $V(G)$ can be partitioned so that each subset is independent in G *or* in $\bar{G}$. Give an example of a graph H such that $\tilde{\chi}(H) = 3$.

10.18 For each integer $n \geq 7$, give an example of a graph G_n of order n such that no induced subgraph of G_n is an odd cycle of length at least 5 but G_n is *not* perfect.

10.19 Show that if H is a bipartite graph, then $\chi(\bar{H}) = \omega(\bar{H})$.

10.20 Determine G if, in the proof of Theorem 10.7,
(a) $H \cong K_2$; (b) $H \cong C_5$.

10.21 Use Theorem 10.5 to prove Theorem 10.10.

10.22 Prove that for every two integers $n \geq 3$ and $k \geq 3$, with $n \geq k$, there exists a graph G such that $\chi(G) = n$ and $\omega(G) = k$.

10.23 Show that if $K_p = G_1 \oplus G_2 \oplus \cdots \oplus G_n$, $n \geq 1$, then
$n\sqrt[n]{p} \leq \chi(G_1) + \chi(G_2) + \cdots + \chi(G_n)$.

10.24 Show that all the bounds given in Theorem 10.10 are sharp.

10.25 Determine and prove a theorem analogous to Theorem 10.10 for vertex-arboricity.

10.26 Define a graph G to be *k-degenerate*, $k \geq 0$, if for every induced subgraph H of G, $\delta(H) \leq k$. Then the 0-degenerate graphs are the empty graphs, and by Exercise 3.7(a), the 1-degenerate graphs are precisely the forests. By Corollary 4.2b, every planar graph is 5-degenerate. A k-degenerate graph is *maximal k-degenerate* if, for every two nonadjacent vertices u and v of G, the graph $G + uv$ is not k-degenerate.

For $k \geq 0$, the *vertex partition number* $\rho_k(G)$ of a graph G is defined as the minimum number of subsets into which $V(G)$ can be partitioned so that each subset induces a k-degenerate subgraph of G. Hence, $\rho_0(G) = \chi(G)$ and $\rho_1(G) = a(G)$. A graph G is said to be *n-critical with respect to* ρ_k, $n \geq 2$, if $\rho_k(G) = n$ and $\rho_k(G - v) = n - 1$ for every $v \in V(G)$.

(a) Prove that if G is a maximal k-degenerate graph of order p, where $p \geq k + 1$, then $\delta(G) = k$.

(b) Determine $\rho_k(K_p)$.

(c) Prove that if G is a graph that is n-critical with respect to ρ_k, then $\delta(G) \geq (k+1)(n-1)$.

10.2 Edge Colorings

An assignment of colors to the edges of a nonempty graph G so that adjacent edges are colored differently is an *edge coloring* of G (an *n-edge coloring* if n colors are used). The graph G is *n-edge colorable* if there exists an m-edge coloring of G for some $m \leq n$. The minimum n for which a graph G is n-edge colorable is its *edge chromatic number* (or *chromatic index*) and is denoted by $\chi_1(G)$.

The edge chromatic number of a graph has other interpretations. For example, the edge chromatic number of a nonempty graph G is the minimum number of 1-regular subgraphs of G into which G can be decomposed. Also, the determination of $\chi_1(G)$ can be transformed into a problem dealing with chromatic numbers; namely, from the definitions, it is immediate that

$$\chi_1(G) = \chi(L(G)),$$

where $L(G)$ is the line graph of G. This observation appears to be of little value in computing edge chromatic numbers, however, since chromatic numbers are extremely difficult to evaluate in general.

It is obvious that $\Delta(G)$ is a lower bound for $\chi_1(G)$. In what must be considered the fundamental result on edge colorings, Vizing [V2] proved that $\chi_1(G)$ equals $\Delta(G)$ or $1 + \Delta(G)$.

Theorem 10.11 (Vizing) *If G is a nonempty graph, then*

$$\chi_1(G) \leq 1 + \Delta(G).$$

Proof Suppose the theorem is not true. Then among the graphs for which the theorem is false, let G be one of minimum size. Hence G is not $(1+\Delta)$-edge colorable, where $\Delta = \Delta(G)$; however, if $e = uv$ is an edge of G, then $G - e$ is $(1 + \Delta(G - e))$-edge colorable. Since $\Delta(G-e) \leq \Delta(G)$, we have that $G - e$ is $(1+\Delta)$-edge colorable.

Let there be given a $(1+\Delta)$-edge coloring of $G - e$; that is, every edge of G except e is assigned one of $1 + \Delta$ colors so that adjacent edges are colored differently. For each edge $e' = uv'$ of G that is incident with u, we define its *dual color* as any one of the $1+\Delta$ colors that is not used to color edges incident with v'. Since no vertex of G has degree exceeding Δ, there is at least one color available for the dual color. It may occur that distinct edges have the same dual color.

Let $e=e_0$ have dual color α_1. (The color α_1 is not the color of any edge of G incident with v.) There must be some edge e_1 incident with u that has been assigned the color α_1; for if not, then the edge e could be colored α_1, thereby producing a $(1+\Delta)$-edge coloring of G. Let α_2 be the dual color of e_1. If there is an edge incident with u that has been assigned the color α_2, then we denote it by e_2 and call its dual color α_3. In this manner, we construct a sequence e_0, $e_1, \ldots, e_k$, $k \geq 1$, containing a maximum number of distinct edges. The final edge e_k of this sequence is therefore colored α_k and has dual color α_{k+1}.

If there is no edge of G incident with u that is assigned the color α_{k+1}, then we may assign each of the edges $e_0, e_1, \ldots, e_k$ with its dual color and obtain a $(1+\Delta)$-edge coloring of G. This, of course, is impossible. Hence we may assume that there exists an edge e_{k+1} of G incident with u that is colored α_{k+1}. Since $e_0, e_1, \ldots, e_k$ is maximum as to the number of distinct edges, we must have $e_{k+1}=e_j$ for some j, $1 \leq j \leq k$, or equivalently, $\alpha_{k+1}=\alpha_j$. Now certainly $\alpha_{k+1} \neq \alpha_k$ since the color assigned to e_k cannot be the same as its dual color; thus, $1 \leq j < k$. It is convenient to let $j=t+1$, where then $0 \leq t < k-1$. Hence $\alpha_{k+1}=\alpha_{t+1}$, so that e_k and e_t have the same dual color.

We now make some observations that will be important in the remainder of the proof. Since the edge e cannot be assigned any of the $1+\Delta$ colors without producing two adjacent edges having the same color, it follows that for each color α among the $1+\Delta$ colors, there is an edge of G adjacent with e that is colored α. This implies that there must be colors assigned to edges incident with v that are not assigned to any edge incident with u. Let β be one such color. Furthermore, let $e_i=uv_i$, $i=0, 1, \ldots, k$, where then $v_0=v$. The color β must be assigned to some edge incident with v_i for each $i=1, 2, \ldots, k$; for suppose there is a vertex v_m, $1 \leq m \leq k$, such that no edge incident with v_m is colored β. Then we may change the color of e_m to β and color each e_i, $0 \leq i < m$, with its dual color to obtain a $(1+\Delta)$-edge coloring of G.

We define two paths P and Q as follows. Let P be a path with initial vertex v_k of maximum length whose edges are alternately colored β and α_{k+1}, while Q is a path with initial vertex v_t having maximum length whose edges are alternately colored β and $\alpha_{t+1}=\alpha_{k+1}$. Suppose P terminates at w and Q at w'. We consider four cases according to certain possibilities for w and w'.

Case 1: $w=v_m$ *for some* m, $0 \leq m \leq k-1$. In this case, the initial and terminal edges of P are colored β. Also, no edge incident with v_m is assigned the color α_{k+1}. We note also that unless $v_m=v_t$, the vertex v_t is not on P. Interchange the colors β and α_{k+1} of the edges of P. Upon doing this, we have no edge incident with v_m that is assigned the color β; and, moreover, the dual color of each e_i, $i<m$, is not altered. If $m=0$, assign e the color β. If $m>0$, then, as described earlier, we may change the color of e_m to β and color each e_i,

$0 \leq i < m$, with its dual color. This implies that G is $(1 + \Delta)$-edge colorable, which is contradictory.

Case 2: $w' = v_m$ *for some* m, $0 \leq m \leq k$, $m \neq t$. Here also, the initial and terminal edges of Q are assigned the color β, and no edge incident with v_m is colored α_{k+1}. Also, Q does not contain v_k unless $v_m = v_k$. Interchange the colors β and α_{k+1} of the edges of Q. If $m < t$, then we proceed as in Case 1. If $m > t$, change the color of e to β if $t = 0$, while if $m > t > 0$, change the color of e_t to β and color each e_i, $0 \leq i < t$, with its dual color. Once again this implies that G is $(1 + \Delta)$-edge colorable, which is impossible.

Case 3: $w \neq v_m$, $0 \leq m \leq k - 1$, *and* $w \neq u$; *or* $w' \neq v_m$ *for any* $m \neq t$ *and* $w' \neq u$. We consider w only, the conclusion being identical for w'. Observe that by interchanging the colors β and α_{k+1} of P, the color β is assigned to no edge incident with v_k and the dual color of each e_i, $0 \leq i < k$, remains the same. This situation, as we have seen, yields a contradiction.

Thus, only one other case remains.

Case 4: $w = u$ *and* $w' = u$. Since u is incident with no edge colored β, the initial edge of both paths P and Q is colored β while each terminal edge is assigned α_{k+1}. If P and Q are edge-disjoint, then u is incident with two distinct edges colored α_{k+1}, which cannot occur. Thus P and Q have an edge in common. But then there is a vertex incident with three edges belonging to P or Q. At least two of these edges are colored either β or α_{k+1}. This is a contradiction. ∎

With the aid of Theorem 10.11 the set of all graphs can be divided very naturally into two classes. A graph G is said to be of *class one* if $\chi_1(G) = \Delta(G)$ and of *class two* if $\chi_1(G) = 1 + \Delta(G)$. The main problem, then, is to determine whether a given graph belongs to class one or class two.

The set of all edges of a graph G receiving the same color in an edge coloring of G is an *edge color class*. By Vizing's Theorem, the edge chromatic number of an r-regular graph $G (r \geq 1)$ is either r or $r + 1$. If $\chi_1(G) = r$ for such a graph G, then necessarily each edge color class in a $\chi_1(G)$-edge coloring of G induces a 1-factor of G; and it follows that an r-regular graph has edge chromatic number r if and only if it is 1-factorable. Hence, K_p is of class one if p is even and of class two if p is odd. It is also immediate that $C_n (n \geq 3)$ is of class one if n is even and of class two if n is odd. More generally, every regular graph of odd order is of class two. Also, since every bipartite graph G is a subgraph of a $\Delta(G)$-regular bipartite graph (Exercise 10.27), it follows by Theorem 8.9 that every bipartite graph is of class one.

Although it is probably not obvious, there are considerably more class one graphs than class two graphs, relatively speaking. Indeed, Erdös and Wilson [EW1] have proved that the probability that a graph of order p is of class one approaches 1 as p approaches infinity. However, the problem of determining which graphs belong to which class is unsolved.

Despite the fact that class two graphs are rather rare, graphs having relatively large sizes in relation to their orders are more likely to be of class two, as was discovered by Beineke and Wilson [BW1].

Theorem 10.12 *Let G be a graph of size q. If*

$$q > \Delta(G) \cdot \beta_1(G),$$

then G is of class two.

Proof If G is of class one, then $\chi_1(G) = \Delta(G)$. Let a $\chi_1(G)$-edge coloring of G be given. Each edge color class of G has at most $\beta_1(G)$ elements; therefore, $q \leqslant \Delta(G) \cdot \beta_1(G)$, which produces a contradiction and the desired result. ■

Since $\beta_1(G) \leqslant \lfloor p/2 \rfloor$ for every graph G of order p, we have an immediate consequence of the preceding result.

Corollary 10.2 *If G is a (p, q) graph for which*

$$q > \Delta(G) \cdot \lfloor p/2 \rfloor,$$

then G is of class two.

It should be emphasized that the results of Theorem 10.12 and its corollary are strictly sufficient conditions. There exist graphs with relatively few edges that are of class two. Of course, the odd cycles are of class two, but, then, they are regular of odd order. The Petersen graph is of class two; in fact, Isaacs [I1] has shown that there exist infinitely many cubic graphs of class two. For example, the graph of Figure 10.5, called the "double star", is a class two graph.

When discussing vertex colorings, we found it useful to consider graphs that are critical with respect to chromatic number. Now that we are investigating edge colorings, it proves valuable to consider certain minimal graphs.

A graph G with at least two edges is *minimal with respect to edge chromatic number* (or simply *minimal* if the parameter is clear from context) if $\chi_1(G-e) < \chi_1(G)$ for every edge e of G. Since isolated vertices have no effect on edge colorings, it is natural to rule out isolated vertices when considering such minimal graphs. Also, since the edge chromatic number of a disconnected graph G having only nontrivial components is the maximum of the edge chromatic numbers of the components of G, every minimal graph without isolated vertices is connected. Therefore, the added hypothesis that a minimal graph G is connected is equivalent to the assumption that G has no isolated vertices.

Two of the most useful results dealing with these minimal graphs are the

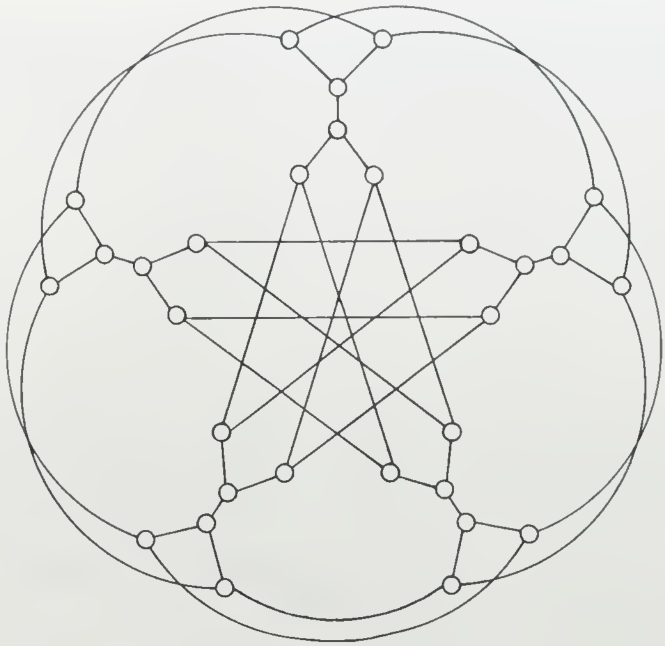


Figure 10.5 *The double star: a cubic, class two graph*

following so-called "adjacency lemmas" of Vizing [V3], which are presented without proof.

Theorem 10.13 (Vizing's Adjacency Lemma—First Form) *Let G be a connected graph of class two that is minimal with respect to edge chromatic number. Then every vertex of G is adjacent to at least two vertices of degree $\Delta(G)$. In particular, G contains at least three vertices of degree $\Delta(G)$.*

Theorem 10.14 (Vizing's Adjacency Lemma—Second Form) *Let G be a connected graph of class two that is minimal with respect to edge chromatic number. If $\Delta(G) = n$, and u and v are adjacent vertices with $\deg u = m$, then v is adjacent to at least $n - m + 1$ vertices of degree n.*

We next examine to which class a graph belongs if it is minimal with respect to edge chromatic number.

Theorem 10.15 *Let G be a connected graph with $\Delta(G) = n \geq 2$. Then G is minimal with respect to edge chromatic number if and only if either:*

(a) G is of class one and $G \cong K(1, n)$, or

(b) G is of class two and $G - e$ is of class one for every edge e of G.

Proof Assume first that $G \cong K(1, n)$. Then $\chi_1(G) = \Delta(G) = n \geq 2$ while $\chi_1(G-e) = n-1$ for every edge e of G. Next, suppose that G is of class two and $G-e$ is of class one for every edge e of G. Then, for an arbitrary edge e of G, we have

$$\chi_1(G-e) = \Delta(G-e) < 1 + \Delta(G) = \chi_1(G).$$

Conversely, assume that $\chi_1(G-e) < \chi_1(G)$ for every edge e of G. If G is of class one, then

$$\Delta(G) \leq \Delta(G-e) + 1 \leq \chi_1(G-e) + 1 = \chi_1(G) = \Delta(G).$$

Therefore, $\Delta(G-e) = \Delta(G) - 1$ for every edge e of G, which implies that $G \cong K(1, n)$.

If G is of class two, then

$$\chi_1(G-e) + 1 = \chi_1(G) = \Delta(G) + 1$$

so that $\chi_1(G-e) = \Delta(G)$ for every edge e of G. Suppose G contains an edge e_1 such that $G - e_1$ is of class two. Then $\chi_1(G-e_1) = \Delta(G-e_1) + 1$. Hence, $\Delta(G-e_1) < \Delta(G)$, implying that G has at most two vertices of degree $\Delta(G)$. This, however, contradicts Theorem 10.13 and completes the proof. ∎

A graph G with at least two edges is called *class minimal* if G is of class two and $G-e$ is of class one for every edge e of G. It follows that a class minimal graph without isolated vertices is necessarily connected. On the basis of Theorem 10.15, we conclude that except for star graphs, connected class minimal graphs are connected graphs that are minimal with respect to edge chromatic number, and conversely.

A lower bound on the size of class minimal graphs is given next in another result by Vizing [V3].

Theorem 10.16 *If G is a class minimal graph of size q with $\Delta(G) = n$, then*

$$q \geq \frac{1}{8}(3n^2 + 6n - 1).$$

Proof Without loss of generality, we assume that G is connected. Suppose that $\delta(G) = m$ and that $\deg u = m$. By Theorem 10.13, the vertex u is adjacent to at least two vertices of degree n; let v be such a vertex. By Theorem 10.14, v is adjacent to at least $n - m + 2$ vertices of degree n. Since the order of G is at least $n+1$, we arrive at the following lower bound on the sum of the degrees of G:

$$2q \geq [n(n-m+2) + m(m-1)] = [m^2 - (n+1)m + (n^2 + 2n)]. \qquad (10.3)$$

However, expression (10.3) is minimum when $m = (n+1)/2$ so that

$$2q \geq \left(\frac{n+1}{2}\right)^2 - \frac{(n+1)^2}{2} + n^2 + 2n$$

or

$$q \geq \frac{1}{8}(3n^2 + 6n - 1). \blacksquare$$

In the next section we shall be discussing various colorings of planar graphs, primarily vertex colorings and region colorings. We briefly consider edge colorings of planar graphs here. In this context, our chief problem remains to determine which planar graphs are of class one and which are of class two. It is easy to find planar graphs G of class one for which $\Delta(G) = n$ for each $n \geq 2$, since all star graphs are planar and of class one. There exist planar graphs G of class two with $\Delta(G) = n$ for $n = 2$, 3, 4, and 5. For $n = 2$, K_3 has the desired properties. For $n = 3$, 4, and 5, the graphs of Figure 10.6 satisfy the required conditions.

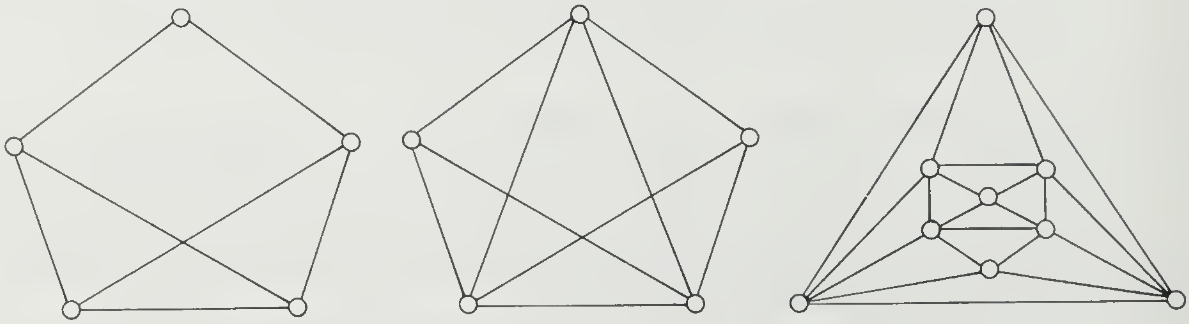


Figure 10.6 *Planar graphs of class two*

It is not known whether there exist planar graphs of class two having maximum degree 6 or 7; however, Vizing [V3] proved that if G is planar and $\Delta(G) \geq 8$, then G is of class one. We prove the following, somewhat weaker result.

Theorem 10.17 *If G is a planar graph with $\Delta(G) \geq 10$, then G is of class one.*

Proof Suppose the theorem is not true. Then among the graphs for which the theorem is false, let G be a connected graph of minimum size. Thus, G is planar, $\Delta(G) = n \geq 10$, and $\chi_1(G) = n + 1$. Furthermore, G is minimal with respect to edge chromatic number. By Corollary 4.2b, G contains vertices of degree 5 or less; let S denote the set of all such vertices. Define $H = G - S$. Since H is planar, H contains a vertex w such that $\deg_H w \leq 5$. Because $\deg_G w > 5$, the vertex w is adjacent to vertices of S. Let $v \in S$ such that

$wv \in E(G)$. Let $\deg_G v = m \leq 5$. Then, by Vizing's Adjacency Lemma (second form), w is adjacent to at least $n - m + 1$ vertices of degree n, but $n - m + 1 \geq 6$ so that w is adjacent to at least six vertices of degree n. Since $n \geq 10$, w is adjacent to at least six vertices of H, contradicting the fact that $\deg_H w \leq 5$. ■

More on edge colorings can be found in Fiorini and Wilson [FW1], which is devoted to that subject.

Exercises 10.2

10.27 Show that every bipartite graph G is a subgraph of a $\Delta(G)$-regular bipartite graph.

10.28 Show that every nonempty regular graph of odd order is of class two.

10.29 Let H be a nonempty regular graph of odd order, and let G be a graph obtained from H by deleting $(\Delta(H) - 1)/2$ or fewer edges. Show that G is of class two.

10.30 Prove or disprove: If G_1 and G_2 are class one graphs and H is a graph with $G_1 \subset H \subset G_2$, then H is of class one.

10.31 Show that the Petersen graph is of class two.

10.32 Prove that every hamiltonian cubic graph is of class one.

10.33 (a) Show that each graph in Figure 10.6 is of class two.

(b) Show that the two graphs of order 5 in Figure 10.6 are class minimal.

10.34 Determine the class of each of the five regular polyhedra.

10.35 Show that there are no connected class minimal graphs of order 4 or 6.

10.36 If G is a class minimal graph and u and v are adjacent vertices of G, then prove that $\deg u + \deg v \geq 2 + \Delta(G)$.

10.37 If E_1 is an independent set of edges in a class minimal graph, then prove that $\chi_1(G - E_1) = \chi_1(G) - 1$.

10.38 Let G be a graph of class two. Prove that G contains a class minimal subgraph H such that $\Delta(H) = \Delta(G)$.

10.39 A *total coloring* of a graph G is an assignment of colors to the elements (vertices and edges) of G so that adjacent elements and incident elements of G are colored differently. An *n-total coloring* is a total coloring that uses n colors. The minimum n for which a graph G admits an n-total coloring is called the *total chromatic number* of G and is denoted by $\chi_2(G)$. The following conjecture is known as the *Total Coloring Conjecture:* For every graph G,

$$\chi_2(G) \leq 2 + \Delta(G).$$

(a) Prove that $\chi_2(G) \geq 1 + \Delta(G)$ for every graph G.

(b) Verify the Total Coloring Conjecture for graphs G with $\Delta(G) \leq 2$.

(c) Determine $\chi(G)$, $\chi_1(G)$, and $\chi_2(G)$ for the (5, 7) graph G in Figure 10.6.

10.3 Map Colorings

It has been said that the mapmakers of many centuries past were aware of the "fact" that any map on the plane (or sphere) could be colored with four or fewer colors so that no two adjacent countries were colored alike. Two countries are considered to be *adjacent* if they share a common boundary line (not simply a single point). As was pointed out in [M2], however, there has been no indication in ancient atlases, books on cartography, or books on the history of mapmaking that people were familiar with this so-called fact. Indeed, it is probable that the Four Color Problem, that is, the problem of determining whether the countries of any map on the plane (or sphere) can be colored with four or fewer colors such that adjacent countries are colored differently, originated and grew in the minds of mathematicians.

What, then, is the origin of the Four Color Problem? The first written reference to the problem appears to be in a letter, dated October 23, 1852, by Augustus De Morgan, mathematics professor at University College, London, to Sir William Rowan Hamilton (after whom "hamiltonian graphs" are named) of Trinity College, Dublin. The letter by De Morgan reads in part:

> A student of mine asked me today to give him a reason for a fact which I did not know was a fact—and do not yet. He says that if a figure be anyhow divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured—four colours may be wanted, but no more .... Query cannot a necessity for five or more be invented. ... But it is tricky work ... what do you say? And has it, if true, been noticed? My pupil says he guessed it in colouring a map of England. The more I think of it, the more evident it seems. If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did....

The student referred to by De Morgan was Frederick Guthrie. By 1880 the problem had become quite well known. During that year, Frederick Guthrie published a note in which he stated that the originator of the question asked of

De Morgan was his brother, Francis Guthrie. We quote from Frederick Guthrie's note [G7]:

> Some thirty years ago, when I was attending Professor De Morgan's class, my brother, Francis Guthrie, who had recently ceased to attend them (and who is now professor of mathematics at the South African University, Cape Town), showed me the fact that the greatest necessary number of colours to be used in colouring a map so as to avoid identity of colour in lineally contiguous districts is four. I should not be justified, after this lapse of time, in trying to give his proof . . . .
>
> With my brother's permission I submitted the theorem to Professor De Morgan, who expressed himself very pleased with it; accepted it as new; and, as I am informed by those who subsequently attended his classes, was in the habit of acknowledging whence he got his information.
>
> If I remember rightly, the proof which my brother gave did not seem altogether satisfactory to himself; but I must refer to him those interested in the subject.

On the basis of this note, we seem to be justified in proclaiming that the Four Color Problem was the creation of one Francis Guthrie.

Returning to the letter of De Morgan to Hamilton, we note the very prompt reply of disinterest by Hamilton to De Morgan on October 26, 1852:

> I am not likely to attempt your "quaternion of colours" very soon.

Before proceeding further with this brief historical encounter with the Four Color Problem, we pause in order to give a more precise mathematical statement of the problem.

A plane graph G is said to be *n-region colorable* if the regions of G can be colored with n or fewer colors so that adjacent regions are colored differently. The *Four Color Problem* is thus the problem of settling the following conjecture.

The Four Color Conjecture *Every map (plane graph) is* 4*-region colorable.*

In dealing with the Four Color Conjecture, one need not consider *all* plane graphs, as we shall now see.

The *region chromatic number* $\chi^*(G)$ of a plane graph G is the minimum n for which G is n-region colorable. Since $\chi^*(G)$ is the maximum region chromatic number among its blocks, the Four Color Problem can be restated as determining whether every plane block is 4-region colorable.

In graph theory the Four Color Problem is more often stated in terms of coloring the vertices of a graph; that is, coloring the graph. In this form, the Four Color Conjecture is stated as follows.

The Four Color Conjecture *Every planar graph is 4-colorable.*

It is in terms of this second statement that the Four Color Problem will be primarily considered. We now verify that these two formulations of the Four Color Conjecture are indeed equivalent. Before doing this, however, we require one additional concept.

For a given connected plane graph G, we construct a pseudograph G^* as follows. A vertex is placed in each region of G, and these vertices constitute the vertex set of G^*. Two distinct vertices of G^* are then joined by an edge for each edge common to the boundaries of the two corresponding regions of G. In addition, a loop is added at a vertex v of G^* for each bridge of G that belongs to the boundary of the corresponding region. Each edge of G^* is drawn so that it crosses its associated edge of G but no other edge of G or G^* (which is always possible); hence, G^* is planar. The pseudograph G^* is referred to as the *dual* of G. In addition to being planar, G^* has the property that it has the same size as G and can be drawn so that each region of G^* contains a single vertex of G; indeed, $(G^*)^* \cong G$. If each set of multiple edges of G^* joining the same two vertices is replaced by a single edge and all loops are deleted, the result is a graph, referred to as the *underlying graph* $\widetilde{G}^*$ of G^*. These concepts are illustrated in Figure 10.7, with the vertices of G^* represented by solid circles.

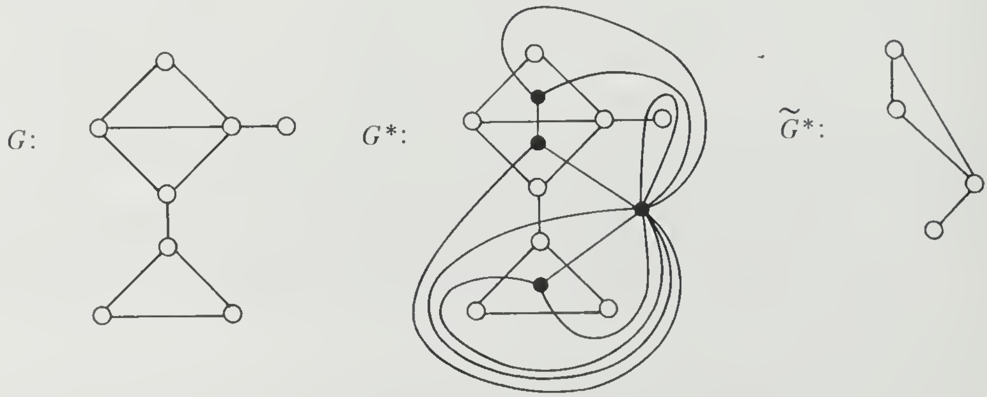


Figure 10.7 *The dual (and its underlying graph) of a plane graph*

Theorem 10.18 *Every planar graph is 4-colorable if and only if every plane graph is 4-region colorable.*

Proof Without loss of generality, we may assume that the graphs under consideration are connected.

Suppose every planar graph is 4-colorable. Let G be an arbitrary connected plane graph, and consider $\widetilde{G}^*$, *the underlying graph of its dual* G^*. Two regions of G are adjacent if and only if the corresponding vertices of $\widetilde{G}^*$

are adjacent. Since $\tilde{G}^*$ is planar, it follows, by hypothesis, that $\tilde{G}^*$ is 4-colorable; thus, G is 4-region colorable.

For the converse, assume that every plane graph is 4-region colorable, and let G be an arbitrary connected plane graph. As we have noted, the dual G^* of G can be embedded in the plane so that each region of G^* contains exactly one vertex of G. If G^* is not a graph, then it can be converted into a graph G' by inserting two vertices into each loop of G^* and by placing a vertex in all but one edge in each set of multiple edges joining the same two vertices. Two vertices of G are adjacent if and only if the corresponding regions of G' are adjacent. Since G' is 4-region colorable, G is 4-colorable. ■

With these concepts at hand, we now return to our historical account of the Four Color Problem. We indicated that this problem was evidently invented in 1852 by Francis Guthrie. The growing awareness of the problem was quite probably aided by De Morgan, who spoke often of it to other mathematicians. The first known published reference to the Four Color Problem is attributed to De Morgan in an anonymous article in the April 14, 1860 issue of the journal *Athenaeum*. By the 1860's the problem was becoming rather widely known. The Four Color Problem received added attention when on June 13, 1878, Arthur Cayley asked, during a meeting of the London Mathematical Society, whether the problem had been solved. Soon afterwards, Cayley [C2] published a paper in which he presented his views on why the problem appeared to be so difficult. From his discussion, one might very well infer the existence of planar graphs with an arbitrarily large chromatic number.

One of the most important events related to the Four Color Problem occurred on July 17, 1879, when the magazine *Nature* carried an announcement that the Four Color Conjecture had been verified by Alfred Bray Kempe. His proof of the conjecture appeared in a paper [K1] published in 1879 and was also described in a paper [K2] published in 1880. For approximately ten years, the Four Color Conjecture was considered to be settled. Then in 1890, Percy John Heawood [H9] discovered an error in Kempe's proof. However, using Kempe's technique, Heawood was able to prove that every planar graph is 5-colorable. This result was referred to, quite naturally, as the Five Color Theorem.

Theorem 10.19 (The Five Color Theorem) *Every planar graph is 5-colorable.*

Proof The proof is by induction on the order p of the graph. For $p \leq 5$, the result is obvious.

Assume that all planar graphs with $p-1$ vertices, $p > 5$, are 5-colorable, and let G be a plane graph of order p. By Corollary 4.2b, G contains a vertex v of degree 5 or less. By deleting v from G, we obtain the plane graph $G-v$. Since $G-v$ has order $p-1$, it is 5-colorable by the inductive hypothesis. Let there be given a 5-coloring of $G - v$, denoting the colors by 1, 2, 3, 4, and 5. If some color is not used in coloring the vertices adjacent with v, then v may be assigned that color, producing a 5-coloring of G itself. Otherwise, $\deg v = 5$ and all five colors are used for the vertices adjacent with v.

Without loss of generality, we assume that v_1, v_2, v_3, v_4, v_5 are the five vertices adjacent with and arranged cyclically about v and that v_i is assigned the color i, $1 \leq i \leq 5$. Now consider any two colors assigned to nonconsecutive vertices v_i, say 1 and 3, and let H be the subgraph of $G - v$ induced by all those vertices colored 1 or 3. If v_1 and v_3 belong to different components of H, then by interchanging the colors assigned to vertices in the component of H containing v_1, for example, a 5-coloring of $G - v$ is produced in which no vertex adjacent with v is assigned the color 1. Thus if we color v with 1, a 5-coloring of G results.

Suppose then that v_1 and v_3 belong to the same component of H, so that there exists a v_1-v_3 path P, all of whose vertices are colored 1 or 3. The path P, together with the path v_3, v, v_1, produces a cycle C in G that encloses v_2, or v_4 and v_5. Hence there exists no v_2-v_4 path in G, all of whose vertices are colored 2 or 4. Denote by F the subgraph of G induced by all those vertices colored 2 or 4. Interchanging the colors of the vertices in the component of F containing v_2, we arrive at a 5-coloring of $G - v$ in which no vertex adjacent with v is assigned the color 2. If we color v with 2, a 5-coloring of G results. ■

In the 86 years that followed the appearance of Heawood's paper, numerous attempts were made to unlock the mystery of the Four Color Problem. Then on June 21, 1976, Kenneth Appel and Wolfgang Haken announced that they, with the aid of John Koch, had verified the Four Color Conjecture.

Appel and Haken's proof, described in [AHK1], was logically quite simple; in fact, many of the essential ideas were the same as those used (unsuccessfully) by Kempe and, then, by Heawood. However, their proof was combinatorially complicated by the extremely large number of necessary case distinctions, and nearly 1200 hours of computer time were required to perform extensive computations. Later refinements in the proof have resulted in a significant reduction in the amount of computer time needed.

Theorem 10.20 (The Four Color Theorem) *Every planar graph is 4-colorable.*

One interesting consequence of the Four Color Theorem provides a bound on the edge chromatic number of cubic plane blocks. In establishing this result, it is convenient to make use of the Klein four-group K from algebra. Using $\oplus$ as the binary operation for K and denoting its elements by (0, 0), (0, 1), (1, 0), and (1, 1), we define addition by

$$(i, j) \oplus (m, n) = (i + m, j + n),$$

where "+" denotes addition modulo two; hence, (0, 0) is the zero element of K.

Corollary 10.20 *Every cubic plane block is 3-edge colorable.*

Proof Let G be a cubic plane block. By Theorems 10.18 and 10.20, G is 4-region colorable. Let the regions of G be colored with the elements of the Klein four-group K. Since G is a block, each edge of G belongs to the boundary of two (adjacent) regions. Define the color of an edge to be the sum of the colors of those two regions bounded, in part, by the edge. Since every element of K is self-inverse, no edge of G is assigned the color (0, 0). However, since K is a group, it follows that the three edges incident with a vertex are assigned the colors (0, 1), (1, 0), and (1, 1). Hence G is 3-edge colorable. ■

More information on the history of the Four Color Problem can be found in Biggs, Lloyd and Wilson [BLW1]. Summaries of the proof of the Four Color Theorem are to be found in Appel and Haken [AH1], Haken [H1], and F. Bernhart [B9].

The *chromatic number of a surface* (where, as always, a surface is a compact orientable 2-manifold) S_n of genus n, denoted $\chi(S_n)$, is the maximum chromatic number among all graphs that can be embedded on S_n. The surface S_0 is the sphere and the Four Color Theorem states that $\chi(S_0) = 4$. Heawood [H9] showed that $\chi(S_1) = 7$; that is, the chromatic number of the torus is 7. Moreover, Heawood was under the impression that he had proved

$$\chi(S_n) = \left\lfloor \frac{7 + \sqrt{1 + 48n}}{2} \right\rfloor$$

for all $n > 0$. However, Heffter [H11] pointed out that Heawood had only established the upper bound:

$$\chi(S_n) \leq \left\lfloor \frac{7 + \sqrt{1 + 48n}}{2} \right\rfloor. \tag{10.4}$$

The statement that $\chi(S_n) = \lfloor (7 + \sqrt{1 + 48n})/2 \rfloor$ for all $n > 0$ eventually became known as the Heawood Map Coloring Conjecture. In 1968, Ringel and Youngs [RY1] completed a remarkable proof of the conjecture, which has involved a number of people. This result is now known as the Heawood Map Coloring Theorem. The proof we present assumes inequality (10.4).

Theorem 10.21 (The Heawood Map Coloring Theorem) *For every positive integer n,*

$$\chi(S_n) = \left\lfloor \frac{7 + \sqrt{1 + 48n}}{2} \right\rfloor.$$

Proof Because of inequality (10.4), it remains only to verify that

$$\chi(S_n) \geq \left\lfloor \frac{7+\sqrt{1+48n}}{2} \right\rfloor$$

for all $n > 0$. Define

$$p = \left\lfloor \frac{7+\sqrt{1+48n}}{2} \right\rfloor$$

so that $p \leq (7+\sqrt{1+48n})/2$. From this, it follows that $n \geq (p-3)(p-4)/12$. Therefore,

$$n \geq \left\lceil \frac{(p-3)(p-4)}{12} \right\rceil. \tag{10.5}$$

Since the right-hand expression of (10.5) equals the genus of K_p (by Theorem 4.26), $\gamma(K_p) \leq n$ so that

$$\chi(S_{\gamma(K_p)}) \leq \chi(S_n).$$

Clearly K_p is embeddable on $S_{\gamma(K_p)}$; consequently, $\chi(S_{\gamma(K_p)}) \geq p$, implying that $\chi(S_n) \geq p$. ■

Note that as a consequence of the Four Color Theorem, Theorem 10.21 also holds for $n = 0$. A thorough discussion of the Heawood Map Coloring Problem can be found in Ringel [R6] and White [W5].

Exercises 10.3

10.40 Use a proof similar to that of Theorem 10.19 to show that $a(G) \leq 3$ for every planar graph G.

10.41 Prove that every cubic planar block is 1-factorable.

10.42 Give an example of a graph G for which $\gamma(G) = 2$ and $\chi(G) = \chi(S_2)$. Verify that your example has these properties.

10.43 Use the result given in Theorem 10.3 to establish an upper bound for the chromatic number of the class of graphs embeddable on the torus. Discuss the sharpness of this upper bound.

Chapter Eleven

Extremal Graph Theory and Ramsey Theory

Next we investigate several topics and problems that belong to an area referred to as extremal graph theory. A number of problems in this area (although certainly not all) involve, for a given property P and a parameter f defined on some class $\mathscr{C}$ of graphs, determining the least integer n such that every graph $G \in \mathscr{C}$ with $f(G) \geq n$ has property P. Those graphs $G \in \mathscr{C}$ not having property P and with $f(G) = n - 1$ are then called the extremal graphs in this case. Much of the major activity in extremal graph theory, however, lies in the field known as ramsey theory, which will receive the main emphasis in this chapter.

11.1 Extremal Graph Theory

Extremal graph theory is considered to have begun in 1941 when Turán [T10] proposed and then solved the following problem: For given positive integers p and n, determine the minimum positive integer $T(p, n)$ such that every graph of order p and size $T(p, n)$ contains K_n as a subgraph. In order to establish Turán's Theorem, we begin with a theorem of Erdös [E6]. The following notation will be useful in the proof of Theorem 11.1 and throughout this section.

For a vertex v of a graph G, recall that the neighborhood $N(v)$ of v consists of the vertices of G adjacent with v. The *closed neighborhood* $N[v]$ of v is defined by $N[v] = N(v) \cup \{v\}$.

Theorem 11.1 *Let G be a graph with vertex set $V(G)=\{v_1, v_2, \ldots, v_p\}$ that does not contain $K_n (n \geq 2)$ as a subgraph. Then there exists an ℓ-partite graph H of order p, where $\ell = \min(p, n-1)$, with vertices $w_1, w_2, \ldots, w_p$ such that $\deg_H w_i \geq \deg_G v_i$ for each i, $1 \leq i \leq p$.*

Proof We employ induction on n. If $n=2$, then $G \cong \overline{K}_p$, so that we take $H=\overline{K}_p$. Assume the result is true for $n-1$ $(n \geq 3)$, and let G be a graph of order p satisfying the hypothesis of the theorem. If $G \cong \overline{K}_p$, then again we take $H=\overline{K}_p$. Thus, without loss of generality, we may assume that $N(v_p)=\{v_1, v_2, \ldots, v_d\}$, where $d=\Delta(G)=\deg v_p \geq 1$. Let $G_1=\langle N(v_p)\rangle$. Since G does not contain K_n as a subgraph, G_1 does not contain K_{n-1} as a subgraph. By the inductive hypothesis, there exists a k-partite graph H_1 whose vertices can be labeled $w_1, w_2, \ldots, w_d$, where $k=\min(d, n-2)$, such that $\deg_{H_1} w_i \geq \deg_{G_1} v_i$ for each i, $1 \leq i \leq d$. Define the graph $H=H_1+\overline{K}_{p-d}$, where the vertices of the subgraph $\overline{K}_{p-d}$ are labeled $w_{d+1}, w_{d+2}, \ldots, w_p$. Then H is a $(k+1)$-partite graph of order p, where $k+1=\min(d+1, n-1)$. Since $\min(d+1, n-1) \leq \min(p, n-1) \leq p$, the graph H is ℓ-partite, where $\ell=\min(p, n-1)$. It remains to show that $\deg_H w_i \geq \deg_G v_i$ for each i, $1 \leq i \leq p$. For $i=1, 2, \ldots, d$,

$$\deg_H w_i = \deg_{H_1} w_i + (p-d) \geq \deg_{G_1} v_i + (p-d) \geq \deg_G v_i. \qquad (11.1)$$

Also, for $i=d+1, d+2, \ldots, p$,

$$\deg_H w_i = d = \Delta(G) \geq \deg_G v_i. \quad \blacksquare$$

A corollary to the proof of Theorem 11.1 will prove useful in the proof of Theorem 11.2.

Corollary 11.1 *If the degrees of the vertices of the graph H constructed in the proof of Theorem 11.1 satisfy $\deg_H w_i = \deg_G v_i$ for each i, $1 \leq i \leq p$, then $H \cong G$.*

Proof As in the proof of Theorem 11.1, we employ induction on n. If $\deg_H w_i = \deg_G v_i$ for $i=1, 2, \ldots, p$, then from (11.1) we see that $\deg_{H_1} w_i = \deg_{G_1} v_i$ for each i, $1 \leq i \leq d$. By the inductive hypothesis, then, $H_1 \cong G_1$. Furthermore, it must be the case that $\deg_G v_i = \deg_{G_1} v_i + (p-d)$ so that $v_i v_j \in E(G)$ for $1 \leq i \leq d$ and $d+1 \leq j \leq p$. This, of course, implies that $\{v_{d+1}, v_{d+2}, \ldots, v_p\}$ is an independent set of vertices of G, and that $H \cong G$. $\blacksquare$

We are now prepared to present the classical theorem of Turán [T10].

Theorem 11.2 (Turán) *For positive integers p and n, with $3 \leqslant n \leqslant p$, let $T(p, n)$ be the least positive integer such that every graph of order p and size $T(p, n)$ contains K_n as a subgraph. Then*

$$T(p, n) = 1 + \binom{p}{2} - \frac{t(p - n + 1 + r)}{2},$$

where $p = t(n-1) + r$, $0 \leqslant r < n - 1$. Furthermore, there is only one extremal graph, namely the complete $(n-1)$-partite graph $K(p_1, p_2, \ldots, p_{n-1})$, where r of the numbers p_i equal $t+1$ and $n-1-r$ of the numbers p_i equal t.

Proof First observe that the graph $K(p_1, p_2, \ldots, p_{n-1})$, where r of the numbers p_i equal $t+1$ and $n-1-r$ of the numbers p_i equal t, is a $(p, \binom{p}{2} - t(p-n+1+r)/2)$ graph that fails to contain K_n as a subgraph. Therefore $T(p, n) \geqslant 1 + \binom{p}{2} - t(p-n+1+r)/2$. In order to show that $T(p, n) \leqslant 1 + \binom{p}{2} - t(p-n+1+r)/2$, we show that if G is a graph of order p that does not contain K_n as a subgraph, then G has at most $\binom{p}{2} - t(p-n+1+r)/2$ edges. Label the vertices of G as $v_1, v_2, \ldots, v_p$. By Theorem 11.1, there exists an $(n-1)$-partite graph H with vertices $w_1, w_2, \ldots, w_p$ so that $\deg_H w_i \geqslant \deg_G v_i$ for each i, $1 \leqslant i \leqslant p$. In particular, $q(G) \leqslant q(H)$. Add edges between the partite sets of H (if necessary) to obtain a complete $(n-1)$-partite graph, say $K(m_1, m_2, \ldots, m_{n-1})$, where $m_1 \leqslant m_2 \leqslant \cdots \leqslant m_{n-1}$. Then

$$q(G) \leqslant q(H) \leqslant q(K(m_1, m_2, \ldots, m_{n-1})). \tag{11.2}$$

If $K(m_1, m_2, \ldots, m_{n-1}) \not\cong K(p_1, p_2, \ldots, p_{n-1})$, then for some $j > k$, we have $m_j > m_k + 1$. The complete $(n-1)$-partite graph $K(m_1, \ldots, m_k + 1, \ldots, m_j - 1, \ldots, m_{n-1})$ has $m_j - m_k - 1$ more edges than does $K(m_1, m_2, \ldots, m_{n-1})$. This implies that $K(p_1, p_2, \ldots, p_{n-1})$ has the maximum size among all $(n-1)$-partite graphs of order p. It follows from (11.2), then, that $T(p, n) = 1 + \binom{p}{2} - t(p - \mathrm{n} + 1 + r)/2$.

In order to complete the proof, assume that $q(G) = T(p, n) - 1$. Then it follows from (11.2) that $q(G) = q(H) = q(K(m_1, m_2, \ldots, m_{n-1})) = q(K(p_1, p_2, \ldots, p_{n-1}))$. From Corollary 11.1, we have $G \cong H$. By the way in which $K(m_1, m_2, \ldots, m_{n-1})$ was constructed, $H \cong K(m_1, m_2, \ldots, m_{n-1})$. Finally, it was shown above that if $q(K(m_1, m_2, \ldots, m_{n-1})) = q(K(p_1, p_2, \ldots, p_{n-1}))$, then $K(m_1, m_2, \ldots, m_{n-1}) \cong K(p_1, p_2, \ldots, p_{n-1})$. Thus, $G \cong K(p_1, p_2, \ldots, p_{n-1})$. ∎

The special case of Theorem 11.2 in which $n = 3$ is of added interest.

Corollary 11.2 *For $p \geqslant 3$, the smallest positive integer $T(p, 3)$ such that every $(p, T(p, 3))$ graph contains a triangle is given by*

$$T(p, 3) = 1 + \left\lfloor \frac{p^2}{4} \right\rfloor.$$

Moreover, the only extremal graph is the complete bipartite graph $K(\lfloor p/2 \rfloor, \lceil p/2 \rceil)$.

Turán also proposed the following problem: Determine the minimum number q so that for a fixed integer p, every (p, q) graph contains a prescribed subgraph H. Accordingly, then, there exists at least one $(p, q-1)$ graph G that fails to contain H; such a graph G is then an extremal graph for this problem. Turán's Theorem therefore qualifies as a solution to this type of problem.

While Corollary 11.2 provides a rather simple expression for the minimum size necessary for a graph of order p to possess a triangle, it may be somewhat unexpected to learn that the problem of finding a corresponding expression for 4-cycles appears to be hopeless.

The minimum size required for a graph of order p to contain K_4 as a subgraph is, of course, the Turán number $T(p, 4)$. A related result, by Dirac [D6], is the following.

Theorem 11.3 *For* $p \geq 4$, *every* $(p, 2p-2)$ *graph contains a subgraph homeomorphic to* K_4. *Furthermore, the number* $2p-2$ *cannot be reduced.*

It has been conjectured by Dirac that for $p \geq 5$, every $(p, 3p-5)$ graph contains a subgraph homeomorphic to K_5; however, it has only been verified, by Thomassen [T1], that every $(p, 4p-10)$ graph contains such a subgraph.

Since every (p, q) graph satisfying $q \geq p \geq 3$ contains a cycle and every tree of order p has size $p-1$, the minimum size required for a graph of order p (≥ 3) to contain a cycle is p. The following extremal result, due to Pósa (see [E5]), gives the minimum size for a graph to contain two disjoint cycles.

Theorem 11.4 *For* $p \geq 6$, *the smallest positive integer* $s(p)$ *so that every* $(p, s(p))$ *graph contains two disjoint cycles is*

$$s(p) = 3p - 5.$$

Proof First we use induction on p to show that $s(p) \leq 3p-5$. There are only two (6, 13) graphs—one obtained by removing two nonadjacent edges from K_6 and the other obtained by removing two adjacent edges from K_6. In both cases, the graph has two disjoint triangles. Thus, $s(6) \leq 13$.

For $n \geq 7$, we assume that $s(p) \leq 3p-5$ for all $p \leq n-1$ and let G be an $(n, 3n-5)$ graph. Since

$$\sum_{v \in V(G)} \deg v = 6n - 10,$$

there exists a vertex v_0 of G such that $\deg v_0 \leq 5$. Assume first that $\deg v_0 = 5$, and $N(v_0) = \{v_i | i = 1, 2, \ldots, 5\}$. If $\langle N[v_0] \rangle$ contains 13 or more edges, then we have already noted that $\langle N[v_0] \rangle$ has two disjoint cycles, implying that G has two disjoint cycles. If, on the other hand, $\langle N[v_0] \rangle$ contains 12 or fewer edges, then, since $\deg v_0 = 5$, some vertex, say v_1, is not adjacent with two other elements of $N(v_0)$, say v_2 and v_3. Add to G the edges v_1v_2 and v_1v_3 and delete the vertex v_0, obtaining the graph G', that is, $G' = G + v_1v_2 + v_1v_3 - v_0$. The graph G' is an $(n-1, 3n-8)$ graph and, by the inductive hypothesis, contains two disjoint cycles C_1 and C_2. At least one of these cycles, say C_1, does not contain the vertex v_1 and thus contains neither the edge v_1v_2 nor the edge v_1v_3. Hence C_1 is a cycle of G. If C_2 contains neither v_1v_2 nor v_1v_3, then C_1 and C_2 are disjoint cycles of G. If C_2 contains v_1v_2 but not v_1v_3, then by removing v_1v_2 and adding v_0, v_0v_1, and v_0v_2, we produce a cycle of G that is disjoint from C_1. The procedure is similar if C_2 contains v_1v_3 but not v_1v_2. If C_2 contains both v_1v_2 and v_1v_3, then by removing v_1 from C_2 and adding v_0, v_0v_2, and v_0v_3, a cycle of G disjoint from C_1 is produced.

Suppose next that $\deg v_0 = 4$, where $N(v_0) = \{v_1, v_2, v_3, v_4\}$. If $\langle N[v_0] \rangle$ is not complete, then some two vertices of $N(v_0)$ are not adjacent, say v_1 and v_2. By adding v_1v_2 to G and deleting v_0, we obtain a $(n-1, 3n-8)$ graph G', which by hypothesis contains two disjoint cycles. We may proceed as before to show now that G has two disjoint cycles. Assume then that $\langle N[v_0] \rangle$ is a complete graph of order 5. If some vertex of $V(G) - N[v_0]$ is adjacent with two or more elements of $N(v_0)$, then G contains two disjoint cycles. Hence we may assume that no element of $V(G) - N[v_0]$ is adjacent with more than one element of $N(v_0)$. Remove the vertices v_0, v_1, v_2 from G, and note that the resulting graph G'' has order $n-3$ and contains at least $(3n-5)-(n-5)-9 = 2n-9$ edges. However, $n \geq 6$ implies that $2n-9 \geq n-3$, so that G'' contains at least one cycle C. The cycle C and the cycle v_0, v_1, v_2, v_0 are disjoint and belong to G.

Finally, we assume that $\deg v_0 \leq 3$. The graph $G - v_0$ is an $(n-1, q)$ graph, where $q \geq 3n-8$. Hence by the inductive hypothesis, $G - v_0$ (and therefore G) contains two disjoint cycles.

This establishes the fact that $s(p) \leq 3p-5$. To prove that $s(p) = 3p-5$, we need only observe that for each $p \geq 6$, the graph $K(1, 1, 1, p-3)$ is a $(p, 3p-6)$ graph that fails to contain two disjoint cycles. This follows because each cycle of $K(1, 1, 1, p-3)$ contains at least two of the three vertices having degree $p-1$. ■

For two edge-disjoint cycles only $p+4$ edges are required, as another theorem of Pósa (see [E5]) shows.

Theorem 11.5 *For $p \geq 6$, every $(p, p+4)$ graph contains two edge-disjoint cycles. Furthermore, the number $p+4$ cannot be reduced.*

A detailed discussion of extremal graph theory is given in Bollobás [B12].

Exercises 11.1

11.1 Illustrate Theorem 11.1 for the graph $G \cong C_5$ that does not contain K_3 as a subgraph.

11.2 For $3 \leq n \leq p$, let $p = t(n-1) + r$, where $0 \leq r < n-1$. Show that the size of the graph $K(p_1, p_2, \ldots, p_{n-1})$, where r of the numbers p_i equal $t+1$ and $n-1-r$ of the numbers p_i equal t, is $\binom{p}{2} - t(p-n+1+r)/2$.

11.3 Use Theorem 11.2 to give a proof of Corollary 11.2.

11.4 Prove Theorem 11.3.

11.5 For positive integers $p \geq 9$, define $s'(p)$ as the least positive integer so that every $(p, s'(p))$ graph contains three pairwise disjoint cycles. Determine a formula for $s'(p)$.

11.6 Prove Theorem 11.5.

11.7 Let $n(\geq 3)$ be a fixed integer. For $p > n$, let $g(p)$ denote the least positive integer so that every $(p, g(p))$ graph contains the star graph $K(1, n)$ as a subgraph. Determine a formula for $g(p)$.

11.8 For positive even integers p, define $F(p)$ as the least positive integer so that every $(p, F(p))$ graph contains a 1-factor. Determine a formula for $F(p)$.

11.2 Ramsey Numbers

Probably the best known and most studied area within extremal graph theory is ramsey theory. We shall discuss this subject in this section and the next. We begin with the classical ramsey numbers.

For positive integers m and n, the *ramsey number* $r(m, n)$ is the least positive integer p such that for every graph G of order p, either G contains K_m as a subgraph or $\overline{G}$ contains K_n as a subgraph; that is, G contains either m mutually adjacent vertices or an independent set of n vertices. The ramsey number is named for Frank Ramsey [R1], who studied this concept in a set theoretic framework and essentially verified the existence of ramsey numbers. Since $\overline{(\overline{G})} = G$ for every graph G, it follows that the ramsey number $r(m, n)$ is symmetric in m and n in the sense that $r(m, n) = r(n, m)$.

It is rather straightforward to show that $r(m, n)$ exists if at least one of m and n does not exceed 2, and that

$$r(1, n) = 1 \quad \text{and} \quad r(2, n) = n.$$

The degree of difficulty in determining the values of other ramsey numbers increases sharply as m and n increase, and no general values like the above are known.

It is sometimes convenient to investigate ramsey numbers from an "edge coloring" point of view. For every graph G of order p, the edge sets of G and $\bar{G}$ partition the edges of K_p. Thus, $r(m, n)$ can be thought of as the least positive integer p such that if every edge of K_p is arbitrarily colored red or blue (where, of course, adjacent edges may receive the same color), then there exists either a complete subgraph of order m, all of whose edges are colored red, or a complete subgraph of order n, all of whose edges are colored blue. In the first case, we say that G contains a red K_m; in the second case, G contains a blue K_n. For example, for $n \geq 2$, $r(2, n) > n - 1$ since if all $\binom{n-1}{2}$ edges of K_{n-1} are colored blue, then K_{n-1} contains neither a red K_2 nor a blue K_n. However, $r(2, n) \leq n$ since if we arbitrarily color the edges of K_n red or blue, then either all the edges are blue and we have a blue K_n, or at least one edge is red and we have a red K_2. Thus, $r(2, n) = n$.

The proof of Theorem 11.6 illustrates some common proof techniques in ramsey theory.

Theorem 11.6 *The ramsey number* $r(3, 3) = 6$.

Proof Since neither C_5 nor $\bar{C}_5$ contains K_3 as a subgraph, $r(3, 3) \geq 6$.

Let G be any graph of order 6, and let v be a vertex of G. Clearly, v is incident with at least three edges of G or at least three edges of $\bar{G}$. Without loss of generality, we assume that vv_1, vv_2, and vv_3 are edges of G. If any of v_1v_2, v_1v_3, and v_2v_3 is an edge of G, then G contains K_3 as a subgraph; otherwise, v_1, v_2, v_3 are mutually adjacent vertices of $\bar{G}$, so that $\bar{G}$ contains K_3 as a subgraph. Thus, $r(3, 3) \leq 6$. Combining the two inequalities, we have $r(3, 3) = 6$. ∎

Before proceeding further, we show that the ramsey numbers exist and, at the same time, establish an upper bound for $r(m, n)$, which was discovered originally by Erdös and Szekeres [ES2].

Theorem 11.7 *For every two positive integers m and n, the ramsey number* $r(m, n)$ *exists; moreover,*

$$r(m, n) \leq \binom{m + n - 2}{m - 1}.$$

Proof We proceed by induction on $k = m + n$. Note that we have equality for $m = 1$ or $m = 2$, and arbitrary n; and for $n = 1$ or $n = 2$, and arbitrary m. Hence the result is true for $k \leq 5$. Furthermore, we may assume that $m \geq 3$ and $n \geq 3$.

Assume that $r(m', n')$ exists for all positive integers m' and n' with $m' + n' < k$, where $k \geq 6$, and that $r(m', n') \leq \binom{m'+n'-2}{m'-1}$. Let m and n be positive integers such that $m + n = k$, $m \geq 3$, and $n \geq 3$. By the inductive hypothesis, it follows that $r(m-1, n)$ and $r(m, n-1)$ exist, and that

$$r(m-1, n) \leq \binom{m+n-3}{m-2} \quad \text{and} \quad r(m, n-1) \leq \binom{m+n-3}{m-1}.$$

Since

$$\binom{m+n-3}{m-2} + \binom{m+n-3}{m-1} = \binom{m+n-2}{m-1},$$

we have that

$$r(m-1, n) + r(m, n-1) \leq \binom{m+n-2}{m-1}. \tag{11.3}$$

Let G be a graph of order $r(m-1, n) + r(m, n-1)$. We show that either G contains K_m as a subgraph or $\bar{G}$ contains K_n as a subgraph. Let $v \in V(G)$; we consider two cases.

Case 1: Assume $\deg_G v \geq r(m-1, n)$. Thus if S is the set of vertices adjacent to v in G, either $\langle S \rangle_G$ contains K_{m-1} as a subgraph or $\overline{\langle S \rangle_G} = \langle S \rangle_{\bar{G}}$ contains K_n as a subgraph. If $\langle S \rangle_{\bar{G}}$ contains K_n as a subgraph, then so does $\bar{G}$. If $\langle S \rangle_G$ contains K_{m-1} as a subgraph, then G contains K_m as a subgraph since in G, the vertex v is adjacent to each vertex in S. Hence in this case, $K_m \subset G$ or $K_n \subset \bar{G}$.

Case 2: Assume $\deg_G v < r(m-1, n)$. Then $\deg_{\bar{G}} v \geq r(m, n-1)$. Thus if T denotes the set of vertices adjacent to v in $\bar{G}$, then $|T| \geq r(m, n-1)$ and either $\langle T \rangle_G$ contains K_m as a subgraph or $\langle T \rangle_{\bar{G}}$ contains K_{n-1} as a subgraph. It follows, as in Case 1, that either $K_m \subset G$ or $K_n \subset \bar{G}$.

Since G was an arbitrary graph of order $r(m-1, n) + r(m, n-1)$, we conclude that $r(m, n)$ exists and that

$$r(m, n) \leq r(m-1, n) + r(m, n-1). \tag{11.4}$$

Combining (11.3) and (11.4), we obtain the desired result. ∎

Corollary 11.7 *For integers $m \geq 2$ and $n \geq 2$,*

$$r(m, n) \leq r(m-1, n) + r(m, n-1). \tag{11.5}$$

Moreover, if $r(m-1, n)$ and $r(m, n-1)$ are both even, then strict inequality holds in (11.5).

Proof The inequality in (11.5) follows from the proof of Theorem 11.7.

In order to complete the proof of the corollary, assume that $r(m-1, n)$ and $r(m, n-1)$ are both even, and let G be any graph of order $r(m-1, n)+r(m, n-1)-1$. We show that either G contains K_m as a subgraph or $\bar{G}$ contains K_n as a subgraph.

Since G has odd order, some vertex v of G has even degree. If $\deg_G v \geq r(m-1, n)$, then, as in Case 1 of Theorem 11.7, either G contains K_m as a subgraph or $\bar{G}$ contains K_n as a subgraph. If, on the other hand, $\deg_G v < r(m-1, n)$, then $\deg_G v \leq r(m-1, n)-2$ since $\deg_G v$ and $r(m-1, n)$ are both even. But then $\deg_{\bar{G}} v \geq r(m, n-1)$, and we may proceed as in Case 2 of Theorem 11.7. ■

As we have already noted, the bound given in Theorem 11.7 for $r(m, n)$ is exact if one of m and n is 1 or 2. The bound is also exact for $m=n=3$. By Theorem 11.7,

$$r(3, n) \leq \frac{n^2+n}{2}.$$

An improved bound for $r(3, n)$ is now presented.

Theorem 11.8 *For every integer $n \geq 3$,*

$$r(3, n) \leq \frac{n^2+3}{2}. \tag{11.6}$$

Proof We proceed by induction on n. For $n=3$, $r(3, n)=6$ while $(n^2+3)/2=6$, so that (11.6) holds if $n=3$. Assume that $r(3, n-1) \leq ((n-1)^2+3)/2$, for some $n \geq 4$, and consider $r(3, n)$. By Corollary 11.7,

$$r(3, n) \leq n + r(3, n-1). \tag{11.7}$$

Moreover, strict inequality holds if n and $r(3, n-1)$ are both even.

Combining (11.7) and the inductive hypothesis, we have

$$r(3, n) \leq n + \frac{(n-1)^2+3}{2} = \frac{n^2+4}{2}. \tag{11.8}$$

To complete the proof, it suffices to show that the inequality given in (11.8) is strict.

If n is odd, then $r(3, n) < (n^2+4)/2$, since n^2+4 is odd. Thus we may assume that n is even. If $r(3, n-1) < ((n-1)^2+3)/2$, then clearly the inequality in (11.8) is strict. If, on the other hand, $r(3, n-1) = ((n-1)^2+3)/2 = n^2/2 - n + 2$, then $r(3, n-1)$ is even since n is even. Therefore the inequality in (11.7) is strict, which implies the desired result. ■

According to Theorem 11.8, $r(3, 4) \leq 9$ and $r(3, 5) \leq 14$. Actually equality holds in both these cases. The equality $r(3, 5) = 14$ follows since there exists a graph G of order 13 containing neither a triangle nor an independent set of five vertices; that is, $K_3 \not\subseteq G$ and $K_5 \not\subseteq \bar{G}$. The graph G is shown in Figure 11.1.

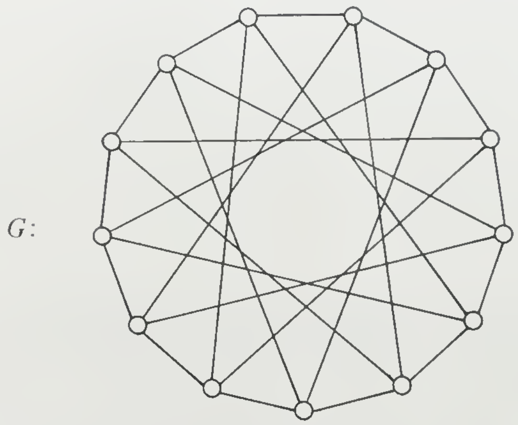


Figure 11.1 *An extremal graph showing* $r(3, 5) \geq 14$

Theorem 11.7 gives an upper bound for the "diagonal" ramsey number $r(n, n)$, namely $r(n, n) \leq \binom{2n-1}{n-1}$. There are two ways in which lower bounds for $r(n, n)$ have been obtained—the constructive method and the probabilistic method. In the constructive method, a lower bound for $r(n, n)$ is established by explicitly constructing a graph G of an appropriate order such that neither G nor $\bar{G}$ contains K_n as a subgraph. Better lower bounds, however, have been obtained using the probabilistic method, which we describe briefly. Suppose we wish to prove that there exists a graph G of order p having some given property P. If we can estimate the number of graphs of order p that do not have property P and we can show that this number is strictly less than the total number of graphs of order p, then there must exist a graph G of order p having property P. Of course, this procedure offers no method for constructing G. In 1947, in one of the first applications of the probabilistic method, Erdös [E3] established the following bound.

Theorem 11.9 *For every integer* $n \geq 3$,

$$r(n, n) > \lfloor 2^{n/2} \rfloor.$$

Proof Let $p = \lfloor 2^{n/2} \rfloor$. We demonstrate the existence of a graph G of order p such that neither G nor $\bar{G}$ contains K_n as a subgraph.

There are $2^{\binom{p}{2}}$ nonidentical graphs of order p with the same vertex set V. For each subset S of V with $|S| = n$, the number of these graphs in which S induces a complete graph is $2^{\binom{p}{2} - \binom{n}{2}}$. Thus, if M denotes the number of nonidentical graphs with vertex set V that contain a subgraph isomorphic to K_n, then

$$M \leq \binom{p}{n} 2^{\binom{p}{2}-\binom{n}{2}} < \frac{p^n}{n!} 2^{\binom{p}{2}-\binom{n}{2}}. \tag{11.9}$$

By hypothesis, $p \leq 2^{n/2}$. Thus, $p^n \leq 2^{n^2/2}$. Since $n \geq 3$, we have $2^{n^2/2} < (\frac{1}{2}) n! 2^{\binom{n}{2}}$, and so

$$p^n < (\tfrac{1}{2}) n! 2^{\binom{n}{2}}. \tag{11.10}$$

Combining (11.9) and (11.10), we conclude that

$$M < (\tfrac{1}{2}) 2^{\binom{p}{2}}.$$

If we list the M nonidentical graphs with vertex set V that contain a subgraph isomorphic to K_n, together with their complements, there are at most $2M < 2^{\binom{p}{2}}$ nonidentical graphs in the list. Since there are $2^{\binom{p}{2}}$ nonidentical graphs with vertex set V, we conclude that there is a graph G with vertex set V such that neither G nor $\bar{G}$ appears in the aforementioned list, i.e., neither G nor $\bar{G}$ contains a subgraph isomorphic to K_n. ■

By Theorems 11.7 and 11.9, we have $4 < r(4, 4) \leq 20$. Actually $r(4, 4) = 18$ (see Exercise 11.13); in fact, the only known ramsey numbers $r(m, n)$ for $3 \leq m \leq n$ are

$r(3, 3) = 6$	$r(3, 6) = 18$	$r(4, 4) = 18$
$r(3, 4) = 9$	$r(3, 7) = 23$	
$r(3, 5) = 14$	$r(3, 9) = 36$	

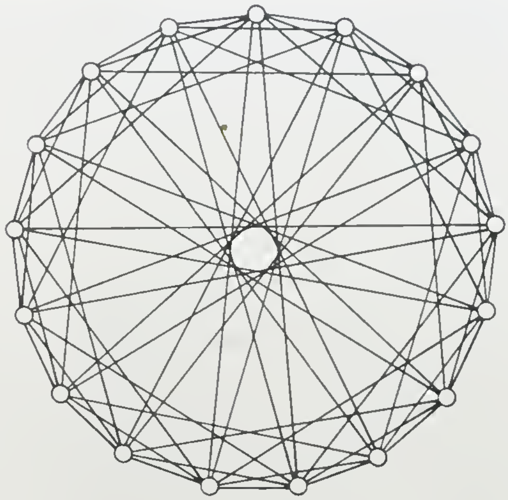
Exercises 11.2

11.9 Show that $r(m, n) = r(n, m)$ for all positive integers m and n.

11.10 Show that if G is a graph of order $r(m, n) - 1$, then
(a) $K_{m-1} \subset G$ or $K_n \subset \bar{G}$,
(b) $K_m \subset G$ or $K_{n-1} \subset \bar{G}$.

11.11 If $2 \leq m' \leq m$ and $2 \leq n' \leq n$, then prove that $r(m', n') \leq r(m, n)$. Furthermore, prove that equality holds if and only if $m' = m$ and $n' = n$.

11.12 Show that $r(3, 4) = 9$.



11.13 The accompanying graph has order 17 and contains neither four mutually adjacent vertices nor an independent set of four vertices. Thus, $r(4, 4) > 17$. Show that $r(4, 4) = 18$.

11.14 The value of the ramsey number $r(5, 5)$ is unknown. Establish upper and lower bounds (with explanations) for this number.

11.3 Generalized Ramsey Theory

For positive integers n_1 and n_2, the classical ramsey number $r(n_1, n_2)$, discussed in Section 11.2, may be defined as the least positive integer p such that for any factorization $K_p = F_1 \oplus F_2$ (therefore, $F_2 = \overline{F_1}$), either $K_{n_1} \subset F_1$ or $K_{n_2} \subset F_2$. Defining the ramsey number in this manner suggests a variety of interesting generalizations. In this section we consider a sample of the many directions of investigation in the rapidly growing field of ramsey theory.

Let $G_1, G_2, \ldots, G_k$ $(k \geq 2)$ be graphs. The (generalized) *ramsey number* $r(G_1, G_2, \ldots, G_k)$ is the least positive integer p such that for any factorization

$$K_p = F_1 \oplus F_2 \oplus \cdots \oplus F_k,$$

the graph G_i is a subgraph of F_i for at least one $i = 1, 2, \ldots, k$. Hence, $r(K_{n_1}, K_{n_2}) = r(n_1, n_2)$. Furthermore, we denote $r(K_{n_1}, K_{n_2}, \ldots, K_{n_k})$ by $r(n_1, n_2, \ldots, n_k)$. The existence of such "ramsey numbers" is guaranteed by the existence of the classical ramsey numbers, as we now see.

Theorem 11.10 *Let the graphs $G_1, G_2, \ldots, G_k$ $(k \geq 2)$ be given. Then the ramsey number $r(G_1, G_2, \ldots, G_k)$ exists.*

Proof It suffices to show that if $p_1, p_2, \ldots, p_k$ are positive integers, then $r(p_1, p_2, \ldots, p_k)$ exists; for suppose that $G_1, G_2, \ldots, G_k$ have orders $p_1, p_2, \ldots, p_k$, respectively, and that $r(p_1, p_2, \ldots, p_k)$ exists. If $F_1 \oplus F_2 \oplus \cdots \oplus F_k$ is any factorization of the complete graph of order $r(p_1, p_2, \ldots, p_k)$, then $K_{p_i} \subset F_i$ for some i, $1 \leq i \leq k$. Since $G_i \subset K_{p_i}$, it follows that $G_i \subset F_i$. Thus $r(G_1, G_2, \ldots, G_k)$ exists and $r(G_1, G_2, \ldots, G_k) \leq r(p_1, p_2, \ldots, p_k)$.

We proceed by induction on k, where $r(p_1, p_2)$ exists for all positive integers p_1 and p_2 by Theorem 11.7. Assume that $r(n_1, n_2, \ldots, n_{k-1})$ exists $(k \geq 3)$ for any $k-1$ positive integers $n_1, n_2, \ldots, n_{k-1}$, and let $p_1, p_2, \ldots, p_k$ be k positive integers. We show that $r(p_1, p_2, \ldots, p_k)$ exists. By the inductive hypothesis, $r(p_1, p_2, \ldots, p_{k-1})$ exists; say $r(p_1, p_2, \ldots, p_{k-1}) = p_0$. Let $r(p_0, p_k) = p$. We now verify that $r(p_1, p_2, \ldots, p_k) \leq p$, thereby establishing the required existence.

Let $K_p = F_1 \oplus F_2 \oplus \cdots \oplus F_k$ be an arbitrary factorization of K_p into k factors. We show that $K_{p_i} \subset F_i$ for at least one i, $1 \leq i \leq k$. Let $H = F_1 \oplus F_2 \oplus \cdots \oplus F_{k-1}$; hence, $K_p = H \oplus F_k$. Since $r(p_0, p_k) = p$, it follows that $K_{p_0} \subset H$ or $K_{p_k} \subset F_k$.

Suppose that $K_{p_0} \subset H$. Let V_0 be a set of p_0 mutually adjacent vertices of H, and define $F_i' = \langle V_0 \rangle_{F_i}$ for $i = 1, 2, \ldots, k-1$. Since $H = F_1 \oplus F_2 \oplus \cdots \oplus F_{k-1}$, it follows that $K_{p_0} = F_1' \oplus F_2' \oplus \cdots \oplus F_{k-1}'$. However, $r(p_1, p_2, \ldots, p_{k-1}) = p_0$, so that $K_{p_i} \subset F_i'$ for some i, $1 \leq i \leq k-1$. Because $F_i' \subset F_i$ for all i, $1 \leq i \leq k-1$, the graph K_{p_i} is a subgraph of F_i for at least one i, $1 \leq i \leq k-1$.

Hence, we may conclude that $K_{p_i} \subset F_i$ for some i, $1 \leq i \leq k$. ■

While it is known that $r(3, 3, 3) = 17$, no other nontrivial numbers of the type $r(n_1, n_2, \ldots, n_k)$, $k \geq 2$, have been evaluated except those mentioned in the preceding section. It is perhaps a little surprising that there has been considerably more success in evaluating the numbers $r(G_1, G_2, \ldots, G_k)$ when not all the graphs G_i are complete. One of the most interesting results in this direction is due to Chvátal [C5], who determined the ramsey number $r(T_m, K_n)$, where T_m is an arbitrary tree of order m. This very general result has a remarkably simple proof.

Theorem 11.11 (Chvátal) *Let T_m be any tree of order $m \geq 1$ and let n be a positive integer. Then*

$$r(T_m, K_n) = 1 + (m-1)(n-1).$$

Proof For $m = 1$ or $n = 1$, $r(T_m, K_n) = 1 = 1 + (m-1)(n-1)$. Thus, we may assume $m \geq 2$ and $n \geq 2$.

The graph $F \cong (n-1)K_{m-1}$ does not contain T_m as a subgraph since each component of F has order $m-1$. The complete $(n-1)$-partite graph $\overline{F} \cong K(m-1, m-1, \ldots, m-1)$ does not contain K_n as a subgraph. Therefore, $r(T_m, K_n) \geq 1 + (m-1)(n-1)$.

Let F be any graph of order $1 + (m-1)(n-1)$. We show that $T_m \subset F$ or $K_n \subset \overline{F}$, implying that $r(T_m, K_n) \leq 1 + (m-1)(n-1)$ and completing the proof. If K_n is not a subgraph of $\overline{F}$, then $\beta(F) \leq n-1$. Therefore, since F has order $1 + (m-1)(n-1)$ and $\beta(F) < n-1$, it follows that $\chi(F) \geq m$ (see Exercise 10.1). Let H be a subgraph of F that is critically m-chromatic. By Corollary 10.1b, $\delta(H) \geq m-1$. Now applying Theorem 3.6, we have that $T_m \subset H$, so that $T_m \subset F$. ■

For $k \geq 3$, the determination of ramsey numbers $r(G_1, G_2, \ldots, G_k)$ has proved to be quite intractable, for the most part. For only a very few classes of graphs has any real progress been made. One such example, however, is where each G_i, $1 \leq i \leq k$, is a star graph. The following result is by Burr and Roberts [BR2].

Theorem 11.12 *Let $n_1, n_2, \ldots, n_k$ $(k \geq 2)$ be positive integers, t of which are even. Then*

$$r(K(1, n_1), K(1, n_2), \ldots, K(1, n_k)) = \sum_{i=1}^{k} (n_i - 1) + \theta_t,$$

where $\theta_t = 1$ if t is positive and even and $\theta_t = 2$ otherwise.

Proof Let $r(K(1, n_1), K(1, n_2), \ldots, K(1, n_k)) = p$, and let $\Sigma_{i=1}^{k} n_i = N$. First, we show that $p \leq N - k + \theta_t$. Since each vertex of K_{N-k+2} has degree $N - k + 1 = \Sigma_{i=1}^{k}(n_i - 1) + 1$, any factorization

$$K_{N-k+2} = F_1 \oplus F_2 \oplus \cdots \oplus F_k$$

necessarily has $K(1, n_i) \subset F_i$ for at least one i, $1 \leq i \leq k$. Thus, $p \leq N - k + 2$. To complete the proof of the inequality $p \leq N - k + \theta_t$, it remains to show that $p \leq N - k + 1$ if t is positive and even. Observe that, in this case, $N - k + 1$ is odd. Suppose, to the contrary, that there exists a factorization

$$K_{N-k+1} = F_1 \oplus F_2 \oplus \cdots \oplus F_k$$

such that $K(1, n_i)$ is not a subgraph of F_i for each $i = 1, 2, \ldots, k$. Since each vertex of K_{N-k+1} has degree $N - k = \Sigma_{i=1}^{k}(n_i - 1)$, this implies that F_i is an $(n_i - 1)$-factor of K_{N-k+1} for each $i = 1, 2, \ldots, k$. However, $N - k + 1$ is odd and $n_j - 1$ is odd for some j $(1 \leq j \leq k)$; thus, F_j contains an odd number of odd vertices, which is impossible.

Next we show that $p \geq N - k + \theta_t$. If $t = 0$, then each integer n_i is odd as is $N - k + 1$. By Theorem 8.12, the complete graph K_{N-k+1} is the edge sum of $(N - k)/2$ hamiltonian cycles. For each $i = 1, 2, \ldots, k$, let F_i be the edge sum of $(n_i - 1)/2$ of these cycles, so that F_i is an $(n_i - 1)$-factor of K_{N-k+1}. Hence there exists a factorization $K_{N-k+1} = F_1 \oplus F_2 \oplus \cdots \oplus F_k$ such that $K(1, n_i)$ is not a subgraph of F_i, for each i $(1 \leq i \leq k)$. This implies that $p \geq N - k + 2$ if $t = 0$.

Assume that t is odd. Then $N - k + 1$ is even. By Theorem 8.11, K_{N-k+1} is 1-factorable and is therefore the edge sum of $N - k$ 1-factors. For $i = 1, 2, \ldots, k$, let F_i be the edge sum of $n_i - 1$ of these 1-factors, so that each F_i is an $(n_i - 1)$-factor of K_{N-k+1}. Thus, there exists a factorization $K_{N-k+1} = F_1 \oplus F_2 \oplus \cdots \oplus F_k$ such that $K(1, n_i)$ is not a subgraph of F_i for each i. Thus $p \geq N - k + 2$ if t is odd.

Finally, assume that t is even and positive, and suppose that n_1, say, is even. Then there is an odd number of even integers among $n_1 - 1, n_2, \ldots, n_k$, which implies by the previous remark that

$$p \geq r(K(1, n_1 - 1), K(1, n_2), \ldots, K(1, n_k)) \geq N - k + 1.$$

Hence, in all cases $p \geq N - k + \theta_t$, so that $p = N - k + \theta_t$. ■

For $k = 2$ in Theorem 11.12, we have the following.

Corollary 11.12 *Let m and n be positive integers. Then*

$$r(K(1, m), K(1, n)) = \begin{cases} m + n - 1, & \text{if } m \text{ and } n \text{ are both even,} \\ m + n, & \text{otherwise.} \end{cases}$$

Given graphs $G_1, G_2, \ldots, G_k$, where $k \geq 2$, we know (as a result of Ramsey's Theorem) that if G is a complete graph of sufficiently large order, then for every factorization $G = F_1 \oplus F_2 \oplus \cdots \oplus F_k$, the graph G_i is a subgraph of F_i for at least one i, $1 \leq i \leq k$. This suggests the following. For graphs $G, G_1, G_2, \ldots, G_k$ $(k \geq 2)$, we say G *arrows* $G_1, G_2, \ldots, G_k$, written $G \to (G_1, G_2, \ldots, G_k)$, if it is the case that for every factorization $G = F_1 \oplus F_2 \oplus \cdots \oplus F_k$, we have $G_i \subset F_i$ for at least one i, $1 \leq i \leq k$. The natural problem, then, is to determine those graphs G for which $G \to (G_1, G_2, \ldots, G_k)$ for given graphs $G_1, G_2, \ldots, G_k$.

In a few special cases of pairs of graphs G_1 and G_2, the aforementioned problem has been solved. In general, however, the problem is extremely difficult. Therefore most attention has been centered on the case $k = 2$, and, for given graphs G_1 and G_2, on the properties a graph G can possess if $G \to (G_1, G_2)$. For example, if $G \to (K_m, K_n)$ where $m, n \geq 2$, then clearly $\omega(G) \geq \max(m, n)$. Folkman [F4] has shown that this is a sharp bound on $\omega(G)$; specifically, given integers $m, n \geq 2$, there is a graph G' with clique number $\max(m, n)$ for which $G' \to (K_m, K_n)$. Nešetřil and Rödl [NR1] have extended this result by showing that for any graph H and integer $k \geq 2$, there exists a graph G with clique number $\omega(H)$ for which $G \to (H_1, H_2, \ldots, H_k)$, where $H_i \cong H$ for $i = 1, 2, \ldots, k$.

If $G \to (K_m, K_n)$, then it is easily seen that the order of G is at least $r(m, n)$; that is, if $G \to (K_m, K_n)$, then $p(G) \geq p(K_r)$, where $r = r(m, n)$. Burr, Erdös, and Lovász [BEL1] have shown that a similar result holds in the case of chromatic numbers.

Theorem 11.13 *For all positive integers m and n, if $G \to (K_m, K_n)$, then $\chi(G) \geq \chi(K_r)$, where $r = r(m, n)$.*

Proof The result holds if $m = 1$ or if $n = 1$. Thus we may assume that $m \geq 2$ and $n \geq 2$, so that $r(m, n) \geq 2$.

Suppose that $\chi(G) \leq r - 1$. Since $p(G) \geq r$, there is an $(r-1)$-coloring of G with resulting color classes $U_1, U_2, \ldots, U_{r-1}$.

By definition of $r = r(m, n)$, there is a factorization $K_{r-1} = F_1 \oplus F_2$ such that $K_m \not\subset F_1$ and $K_n \not\subset F_2$. Label the vertices of K_{r-1} as $v_1, v_2, \ldots, v_{r-1}$.

We construct a factorization of G as follows. Let $V(G_1) = V(G_2) = V(G)$. Each edge e of G is of the form $e = u_j u_k$ where $u_j \in U_j$ and $u_k \in U_k$ $(1 \leq j < k \leq r-1)$. Since $v_j v_k \in E(K_{r-1})$, either $v_j v_k \in E(F_1)$ or $v_j v_k \in E(F_2)$ in the factorization $K_{r-1} = F_1 \oplus F_2$. Let $u_j u_k \in E(G_i)$ if $v_j v_k \in E(F_i)$, $i = 1, 2$. Then $G = G_1 \oplus G_2$.

Suppose $K_m \subset G_1$. Thus G_1 contains m mutually adjacent vertices, say $w_1, w_2, \ldots, w_m$, and there are distinct color classes $U_{i_1}, U_{i_2}, \ldots, U_{i_m}$ such that $w_j \in U_{i_j}$ for $j = 1, 2, \ldots, m$. By the way G_1 was constructed, this implies that $\langle \{v_{i_1}, v_{i_2}, \ldots, v_{i_m}\} \rangle_{F_1} \cong K_m$, which is impossible. Therefore, $K_m \not\subset G_1$. Similarly, $K_n \not\subset G_2$, so that $G \not\to (K_m, K_n)$. This is a contradiction. Thus, $\chi(G) \geq r$, and the proof is complete. ■

Corollary 11.13 *For all positive integers m and n, if $G \to (K_m, K_n)$, then $q(G) \geq q(K_r)$, where $r = r(m, n)$.*

For arbitrary graphs G_1 and G_2, if $G \to (G_1, G_2)$ then $p(G) \geq p(K_r)$, where $r = r(G_1, G_2)$. However, it is not true in general that $\chi(G) \geq \chi(K_r)$ or that $q(G) \geq q(K_r)$.

The concept of arrowing can be extended quite naturally in the following manner. Given a graph G and classes of graphs $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k$, $k \geq 2$, we write $G \to (\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k)$ if for every factorization $G = F_1 \oplus F_2 \oplus \cdots \oplus F_k$, it is the case that F_i contains a member of $\mathcal{G}_i$ for at least one i, $1 \leq i \leq k$. For a positive integer n, let $\mathcal{K}_n$ denote the class of graphs with chromatic number n. It follows that if $n_1, n_2, \ldots, n_k$ are positive integers, then $G \to (\mathcal{K}_{n_1}, \mathcal{K}_{n_2}, \ldots, \mathcal{K}_{n_k})$ if and only if for every factorization $G = F_1 \oplus F_2 \oplus \cdots \oplus F_k$, we have $\chi(F_i) \geq n_i$ for at least one i, $1 \leq i \leq k$. Burr and Erdös [BE1] characterized such graphs G.

Theorem 11.14 *Let $n_1, n_2, \ldots, n_k$ $(k \geq 2)$ be positive integers, and let G be a graph. Then $G \to (\mathcal{K}_{n_1}, \mathcal{K}_{n_2}, \ldots, \mathcal{K}_{n_k})$ if and only if $\chi(G) \geq 1 + \prod_{i=1}^{k} (n_i - 1)$.*

Proof The result is immediate if $n_i = 1$ for some i; hence, we assume that $n_i \geq 2$ for each $i = 1, 2, \ldots, k$.

Assume first that $G \not\to (\mathcal{K}_{n_1}, \mathcal{K}_{n_2}, \ldots, \mathcal{K}_{n_k})$. Thus, there is a factorization $G = F_1 \oplus F_2 \oplus \cdots \oplus F_k$ such that $\chi(F_i) \leq n_i - 1$ for each $i = 1, 2, \ldots, k$. By Theorem 10.9, it follows that

$$\chi(G) \leq \prod_{i=1}^{k} \chi(F_i) \leq \prod_{i=1}^{k} (n_i - 1).$$

For the converse, we show that if $\chi(G) \leq \prod_{i=1}^{k} (n_i - 1)$, then $G \not\to (\mathcal{K}_{n_1}, \mathcal{K}_{n_2}, \ldots, \mathcal{K}_{n_k})$. Let $N = \prod_{i=1}^{k} (n_i - 1)$, and let the vertices of G be colored with the N colors $(c_{j_1}, c_{j_2}, \ldots, c_{j_k})$, where $1 \leq j_i \leq n_i - 1$ for $i = 1, 2, \ldots, k$. We construct a factorization $G = F_1 \oplus F_2 \oplus \cdots \oplus F_k$ with $\chi(F_i) \leq n_i - 1$ for $i = 1, 2, \ldots, k$.

Let $V(F_i) = V(G)$ for $i = 1, 2, \ldots, k$. Now, for each edge uv of G, we have $uv \in E(F_i)$ if and only if the ith coordinate is the first coordinate in which the colors assigned to u and v differ. Then the vertices of F_i can be colored with the $n_i - 1$ colors $c_1, c_2, \ldots, c_{n_i - 1}$, and so $\chi(F_i) \leq n_i - 1$ for $i = 1, 2, \ldots, k$. ■

In [CP1], the *chromatic ramsey number* $\chi(n_1, n_2, \ldots, n_k)$ was defined as the least integer p such that if $K_p = F_1 \oplus F_2 \oplus \cdots \oplus F_k$ is any factorization of K_p into k factors, then $\chi(F_i) \geq n_i$ for at least one i, $1 \leq i \leq k$. A formula was established for $\chi(n_1, n_2, \ldots, n_k)$; this result follows immediately from Theorem 11.14.

Corollary 11.14 *Let $n_1, n_2, \ldots, n_k$ $(k \geq 2)$ be positive integers. Then the chromatic ramsey number*

$$\chi(n_1, n_2, \ldots, n_k) = 1 + \prod_{i=1}^{k} (n_i - 1).$$

We note in closing that Graham, Rothschild, and Spencer [GRS1] have written a book on ramsey theory.

Exercises 11.3

11.15 Show that $r(3, 3, 3) \leq 17$.

11.16 Show for graphs $G_1, G_2, \ldots, G_k (k \geq 2)$ that

$$r(G_1, G_2, \ldots, G_k, K_2) = r(G_1, G_2, \ldots, G_k).$$

11.17 Show for positive integers $n_1, n_2, \ldots, n_k$ $(k \geq 2)$ that $r(K_{n_1}, K_{n_2}, \ldots, K_{n_k}, T_m) = 1 + (r-1)(m-1)$, where T_m is any tree of order $m \geq 1$ and $r = r(n_1, n_2, \ldots, n_k)$.

11.18 Let m and n be integers with $m \geq 3$ and $n \geq 1$. Show that

$$r(C_m, K(1, n)) = \begin{cases} 2n+1, & \text{if } m \text{ is odd and } m \leq 2n+1, \\ m, & \text{if } m \geq 2n. \end{cases}$$

(Note that this does not cover the case where m is even and $m < 2n$.)

11.19 Let G_1 be a graph whose largest component has order m, and let G_2 be a graph with $\chi(G_2) = n$. Prove that $r(G_1, G_2) \geq 1 + (m-1)(n-1)$.

11.20 Show for positive integers ℓ and n, that $r(K_\ell + \overline{K}_n, T_m) \leq \ell(m-1) + n$, where T_m is any tree of order $m \geq 1$.

11.21 Let m and n be positive integers, and recall that $a(G)$ denotes the vertex-arboricity of a graph G. Determine a formula for $a(m, n)$, where $a(m, n)$ is the least positive integer p such that for any factorization $K_p = F_1 \oplus F_2$, either $a(F_1) \geq m$ or $a(F_2) \geq n$.

11.22 Show that if G, G_1, and G_2 are graphs such that $G \to (G_1, G_2)$, then $p(G) \geq p(K_r)$, where $r = r(G_1, G_2)$.

11.23 Prove Corollary 11.13.

11.24 (a) Let m and n be positive integers. Show that if $G \to (K(1, m), K(1, n))$, then $q(G) \geq m + n - 1$.

(b) Give an example of a graph G for which $G \to (K(1, m), K(1, n))$ and $q(G) = m + n - 1$.

11.25 (a) Give an example of graphs G, G_1, and G_2 such that $G \to (G_1, G_2)$ but $\chi(G) < \chi(K_{r(G_1, G_2)})$.

(b) Give an example of graphs G, G_1, and G_2 such that $G \to (G_1, G_2)$ but $q(G) < q(K_{r(G_1, G_2)})$.

11.26 Prove Corollary 11.14.

11.27 (a) Prove that there exists no triangle-free graph G of order $4n+3$ $(n \geq 1)$ for which $\delta(G) \geq 2n+2$.

(b) Let $G_1, G_2, \ldots, G_{n+1}$ be $n+1$ (≥ 2) graphs such that $G_1 \cong K_3$ and $G_i \cong K(1, 3)$ for $2 \leq i \leq n+1$. Prove that $r(G_1, G_2, \ldots, G_{n+1}) = 4n+3$.

11.28 (a) Let n_1, n_2, n_3 (≥ 2) be integers. Prove that $r(n_1, n_2, n_3) \leq r(n_1, n_2, n_3 - 1) + r(n_1, n_2 - 1, n_3) + r(n_1 - 1, n_2, n_3) - 1$.

(b) Generalize the result in part (a).

Chapter Twelve

Enumeration of Graphs and Digraphs

Lastly, we turn to the enumeration of graphs and digraphs. In the first section, we prove Pólya's classical enumeration theorem. Applying this theorem in Section 12.2, we determine the number of nonisomorphic (p, q) graphs and (p, q) digraphs for fixed integers p and q.

12.1 Pólya's Theorem

In Chapter 1 we saw that the number of nonidentical graphs of order p (with the same vertex set) is $2^{p(p-1)/2}$. Moreover, it is not difficult to see, for fixed integers p and q with $p \geq 1$ and $0 \leq q \leq \binom{p}{2}$, that the number of nonidentical (p, q) graphs is $\binom{p(p-1)/2}{q}$. Analogously, the number of nonidentical digraphs of order p is $2^{p(p-1)}$, and the number of nonidentical (p, q) digraphs is $\binom{p(p-1)}{q}$ for integers p and q with $p \geq 1$ and $0 \leq q \leq p(p-1)$.

The corresponding problems of counting the number of nonisomorphic (p, q) graphs and (p, q) digraphs are considerably more difficult; indeed, in order to present solutions we must first develop a substantial amount of background material.

Let us consider the problem of counting the number of nonisomorphic graphs of order 3. By inspection we see that there are four such graphs, one $(3, q)$ graph for each $q = 0, 1, 2, 3$. This information can be conveniently represented by the "pattern inventory"

$$b^3 + b^2w + bw^2 + w^3,$$

where the coefficient of the term $b^q w^{3-q}$ is the number of graphs of order 3 with q edges (that is, the number of graphs of order 3 with q pairs of distinct adjacent vertices and $3-q$ pairs of distinct nonadjacent vertices). Our goal is to develop a method of generating pattern inventories for this and other counting problems.

Returning to our problem of determining the number of nonisomorphic graphs of order 3, we recall that there are $2^{3(3-1)/2} = 8$ nonidentical graphs with vertex set $\{1, 2, 3\}$. It will be helpful to represent the collection of these graphs by the collection of functions from the set $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ to the set $\{b, w\}$. For convenience, we write ij (or, ji) to mean $\{i, j\}$. A graph G will be represented by the function f defined by

$$f(ij) = \begin{cases} b, & \text{if } ij \in E(G), \\ w, & \text{if } ij \notin E(G). \end{cases}$$

It is readily verified that graphs G and $\hat{G}$ are isomorphic if and only if there is a permutation π on the set $\{12, 13, 23\}$ such that $\hat{f}(ij) = f(\pi(ij))$ for every $ij \in \{12, 13, 23\}$, where f and $\hat{f}$ are the functions representing G and $\hat{G}$, respectively. If such a permutation π exists, we say that f and $\hat{f}$ are related. Since the relation "is isomorphic to" is an equivalence relation on the collection of nonidentical graphs with vertex set $\{1, 2, 3\}$, the aforementioned relation on the corresponding collection of functions is also an equivalence relation. Furthermore, the number of distinct equivalence classes of functions determined by this relation equals the number of nonisomorphic graphs of order 3.

The eight functions from $\{12, 13, 23\}$ to the set $\{b, w\}$, which we denote f_i, $i = 1, 2, \ldots, 8$, are defined in the table in Figure 12.1. For each i, the graph G_i that is represented by the function f_i is also illustrated in Figure 12.1.

We see, for example, that the graphs G_2 and G_3 of Figure 12.1 are isomorphic. If π is the permutation on $\{12, 13, 23\}$ given by $\pi = \binom{12\ 13\ 23}{12\ 23\ 13}$, then $f_3(ij) = f_2(\pi(ij))$ for all ij; that is, f_2 and f_3 are related.

In general, let D and R be nonempty finite sets and let R^D denote the set of all functions from D to R. Let A be a group of permutations on D. For each $\pi \in A$ we define a mapping $\pi^*: R^D \to R^D$ by $(\pi^*(f))(d) = f(\pi(d))$ for $d \in D$. Then π^* is a permutation on R^D. For $f_1, f_2 \in R^D$, the function f_1 is said to be *related* to f_2 if there exists a permutation $\pi \in A$ such that $\pi^*(f_1) = f_2$. It is not difficult to verify that this relation is an equivalence relation on R^D; this relation is said to be the equivalence relation *induced by the action of A on* R^D. Under this equivalence relation, the functions from D to R are divided into equivalence classes called *patterns*. The problem in general, then, is to determine the number of distinct patterns.

For example, let $D = \{12, 13, 23\}$ and $R = \{b, w\}$. Then R^D consists of the functions f_i, $1 \leq i \leq 8$, defined in the table in Figure 12.1. If A is taken to be the symmetric group S_3, then $A = \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6\}$, where the permuta-

	$f_i(12)$	$f_i(13)$	$f_i(23)$
f_1	b	b	b
f_2	b	b	w
f_3	b	w	b
f_4	w	b	b
f_5	b	w	w
f_6	w	b	w
f_7	w	w	b
f_8	w	w	w

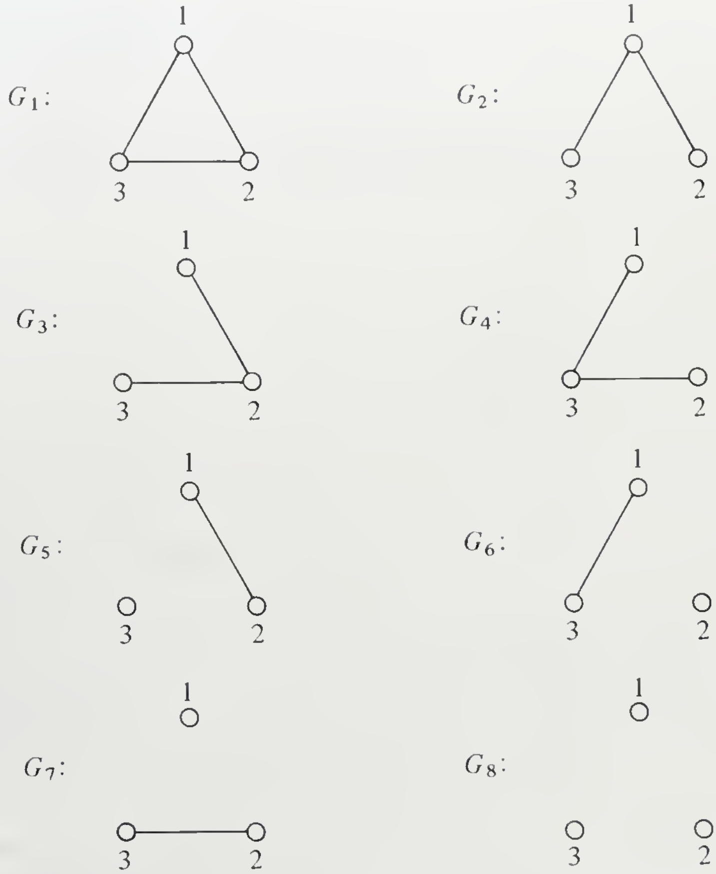


Figure 12.1 *Nonidentical graphs of order* 3

tions π_i on D and the corresponding permutations π_i^* on R^D are given in Figure 12.2. The equivalence classes of functions in the equivalence relation on R^D induced by the action of A on R^D are $\{f_1\}$, $\{f_2, f_3, f_4\}$, $\{f_5, f_6, f_7\}$, and $\{f_8\}$. These four equivalence classes correspond to the four nonisomorphic graphs of order 3.

Our first theorem, a modified form of a result due to Burnside [B16] known as Burnside's Lemma, gives a solution to the problem of determining the number of equivalence classes of functions.

Two observations will be useful in the proof of Theorem 12.1. First, if π_1 and π_2 are permutations in the group A, then their composition $\pi_1 \circ \pi_2$ is in A and $(\pi_1 \circ \pi_2)^* = \pi_2^* \circ \pi_1^*$. Second, if $\pi \in A$ and $f_1, f_2 \in R^D$ such that $\pi^*(f_1) = f_2$, then $\pi^{-1} \in A$ and $(\pi^{-1})^*(f_2) = f_1$.

Theorem 12.1 (Burnside) *Let D and R be finite nonempty sets and let A be a permutation group on D. Then the number N of equivalence classes in the equivalence relation on R^D induced by the action of A on R^D is given by the formula*

$$\pi_1 = \begin{pmatrix} 12 & 13 & 23 \\ 13 & 23 & 12 \end{pmatrix} \quad \pi_2 = \begin{pmatrix} 12 & 13 & 23 \\ 23 & 12 & 13 \end{pmatrix} \quad \pi_3 = \begin{pmatrix} 12 & 13 & 23 \\ 12 & 23 & 13 \end{pmatrix}$$

$$\pi_4 = \begin{pmatrix} 12 & 13 & 23 \\ 23 & 13 & 12 \end{pmatrix} \quad \pi_5 = \begin{pmatrix} 12 & 13 & 23 \\ 13 & 12 & 23 \end{pmatrix} \quad \pi_6 = \begin{pmatrix} 12 & 13 & 23 \\ 12 & 13 & 23 \end{pmatrix}$$

$$\pi_1^* = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 \\ f_1 & f_3 & f_4 & f_2 & f_7 & f_5 & f_6 & f_8 \end{pmatrix} \quad \pi_2^* = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 \\ f_1 & f_4 & f_2 & f_3 & f_6 & f_7 & f_5 & f_8 \end{pmatrix}$$

$$\pi_3^* = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 \\ f_1 & f_3 & f_2 & f_4 & f_5 & f_7 & f_6 & f_8 \end{pmatrix} \quad \pi_4^* = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 \\ f_1 & f_4 & f_3 & f_2 & f_7 & f_6 & f_5 & f_8 \end{pmatrix}$$

$$\pi_5^* = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 \\ f_1 & f_2 & f_4 & f_3 & f_6 & f_5 & f_7 & f_8 \end{pmatrix} \quad \pi_6^* = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 \\ f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 \end{pmatrix}$$

Figure 12.2 *Permutations on D and R^D*

$$N = \frac{1}{|A|}\sum_{\pi \in A} \psi(\pi^*), \tag{12.1}$$

where $\psi(\pi^*)$ is the number of elements $f \in R^D$ for which $\pi^*(f) = f$.

Proof For $h \in R^D$, let $[h]$ be the equivalence class containing h in the equivalence relation on R^D induced by the action of A on R^D. For $g \in [h]$, let $T(h, g)$ denote the set of permutations $\pi \in A$ such that $\pi^*(h) = g$. We first show that $|T(h, g)| = |T(g, g)|$.

Let $\pi \in T(h, g)$ and suppose $T(g, g) = \{\pi_1, \pi_2, \ldots, \pi_k\}$. Then for $1 \leq i \leq k$, $\pi \circ \pi_i \in T(h, g)$ since $(\pi \circ \pi_i)^*(h) = (\pi_i^* \circ \pi^*)(h) = \pi_i^*(g) = g$; furthermore, if $i \neq j$, then $\pi \circ \pi_i \neq \pi \circ \pi_j$. Thus, $\pi \circ \pi_1, \pi \circ \pi_2, \ldots, \pi \circ \pi_k$ are distinct elements of $T(h, g)$. Let $\pi_0 \in T(h, g)$. Then $(\pi^{-1} \circ \pi_0)^*(g) = (\pi_0^* \circ (\pi^{-1})^*)(g) = (\pi_0)^*(h) = g$. Thus, $\pi^{-1} \circ \pi_0 \in T(g, g)$, say $\pi^{-1} \circ \pi_0 = \pi_j$; therefore, $\pi_0 = \pi \circ \pi_j$. Hence, $T(h, g) = \{\pi \circ \pi_1, \pi \circ \pi_2, \ldots, \pi \circ \pi_k\}$ and $|T(h, g)| = |T(g, g)|$.

If g_1, g_2 are distinct elements of $[h]$, then $T(h, g_1) \cap T(h, g_2) = \varnothing$. Furthermore, $\cup_{g \in [h]} T(h, g) = A$. Therefore, $|A| = \Sigma_{g \in [h]} |T(h, g)| = \Sigma_{g \in [h]} |T(g, g)|$ and

$$\sum_{f \in R^D} |T(f, f)| = \sum_{\substack{\text{distinct} \\ \text{equivalence} \\ \text{classes } [h]}} \sum_{g \in [h]} |T(g, g)| = N \cdot |A|.$$

It remains to show that $\Sigma_{f \in R^D} |T(f, f)| = \Sigma_{\pi \in A} \psi(\pi^*)$. Define a function $\gamma\colon A \times R^D \to \{0, 1\}$ by

$$\gamma(\pi, h) = \begin{cases} 1, & \text{if } \pi^*(h) = h, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\Sigma_{f\in R^D}|T(f, f)| = \Sigma_{f\in R^D}(\Sigma_{\pi\in A}\gamma(\pi, f)) = \Sigma_{\pi\in A}(\Sigma_{f\in R^D}\gamma(\pi, f)) = \Sigma_{\pi\in A}\ \psi(\pi^*)$. ■

In a manner completely analogous to that used in establishing Theorem 12.1, one can verify the following result, which will prove useful in subsequent material.

Theorem 12.2 *Let D and R be finite nonempty sets and let A be a permutation group on D. If S is a subset of R^D that is the union of N_S equivalence classes in the equivalence relation on R^D induced by the action of A on R^D, then*

$$N_S = \frac{1}{|A|}\sum_{\pi\in A}\psi(\pi^*|S), \tag{12.2}$$

where $\psi(\pi^*|S)$ is the number of elements $f\in S$ for which $\pi^*\ (f) = f$.

We have seen that if $D = \{12, 13, 23\}$, $R = \{b, w\}$, and A is the symmetric group S_3, then there are four equivalence classes of functions in the equivalence relation on R^D induced by the action of A on R^D. If we apply the formula in (12.1), we have

$$N = \frac{1}{6}\sum_{i=1}^{6}\psi(\pi_i^*),$$

where according to Figure 12.2,

$$\psi(\pi_1^*) = 2, \qquad \psi(\pi_2^*) = 2, \qquad \psi(\pi_3^*) = 4,$$
$$\psi(\pi_4^*) = 4, \qquad \psi(\pi_5^*) = 4, \qquad \psi(\pi_6^*) = 8.$$

Thus, $N = \frac{1}{6}(24) = 4$.

In general, applying the formula given in Theorem 12.1 is quite difficult if we must first determine the permutations π^* and then explicitly compute $\psi(\pi^*)$. However, this work can be simplified by calculating $\psi(\pi^*)$ from the cycle structure of π. Consider, for example, the permutation $\pi_3 = \binom{12\ 13\ 23}{12\ 23\ 13}$ on the set $D = \{12, 13, 23\}$. Then π_3 can be expressed uniquely (except for order) as a product of disjoint (permutation) cycles $\pi_3 = (12)(13\ 23)$. We wish to determine which functions $f\in R^D$, where $R = \{b, w\}$, satisfy $\pi_3^*(f) = f$; that is, which functions $f\in R^D$ satisfy $f(\pi_3(ij)) = f(ij)$ for all $ij\in D$. Since π_3 interchanges 13 and 23, it must be the case that $f(13) = f(23)$ if $f(ij) = f(\pi_3(ij))$ for all ij. Conversely, if $f(13) = f(23)$, then since π_3 fixes 12 and interchanges 13 and 23, we have that $f(ij) = f(\pi_3(ij))$ for all ij. Thus $\pi_3^*(f) = f$ if and only if $f(13) = f(23)$. The number of such functions from $\{12, 13, 23\}$ to $\{b, w\}$ is $2\cdot 2 = 4$, since there are two possible images for 12 and two possible images for 13 and 23. This, of course, agrees with our earlier statement that $\psi(\pi_3^*) = 4$. We note that $\pi_3^*(f) = f$ for the functions $f = f_1, f_4, f_5, f_8$ (see Figure 12.2) and that each of these functions satisfies $f(13)=f(23)$ (see Figure 12.1).

The discussion in the preceding paragraph is easily generalized. Suppose π is a permutation on a finite nonempty set D and π^* is the corresponding permutation on the set of all functions from D to some finite nonempty set R. Then, by definition, $\pi^*(f)=f$ if and only if $f(d)=f(\pi(d))$ for all $d \in D$. However, $f(d)=f(\pi(d))$ for all $d \in D$ if and only if $f(d_1)=f(d_2)$ whenever d_1 and d_2 are in the same cycle in the disjoint cycle decomposition of π. Thus $\psi(\pi^*)$ equals the number of functions $f \in R^D$ such that the elements in D that are in the same cycle in the disjoint cycle decomposition of π have the same image under f. We conclude that $\psi(\pi^*)=|R|^m$, where m equals the number of cycles in the disjoint cycle decomposition of π.

It is evident that the cycle structures of the elements of a permutation group A on a finite set D play an important role in the problem at hand. Let $\pi \in A$, and for each integer k, where $1 \leqslant k \leqslant n=|D|$, let $j_k(\pi)$ denote the number of cycles of length k in the disjoint cycle decomposition of π. Then the *cycle structure representation* of π is the polynomial $x_1^{j_1(\pi)} x_2^{j_2(\pi)} \cdots x_n^{j_n(\pi)}$ in the variables $x_1, x_2, \ldots, x_n$. The *cycle index* P_A of A is defined to be the sum of the cycle structure representations of the permutations in A divided by the number of permutations in A; that is,

$$P_A(x_1, x_2, \ldots, x_n) = \frac{1}{|A|} \sum_{\pi \in A} (x_1^{j_1(\pi)} x_2^{j_2(\pi)} \cdots x_n^{j_n(\pi)}).$$

We have, then, the following result.

Theorem 12.3 *Let D and R be finite sets and let A be a permutation group on D. Then the number of equivalence classes in the equivalence relation on R^D induced by the action of A on R^D is $P_A(|R|, |R|, \ldots, |R|)$, where $P_A(x_1, x_2, \ldots, x_{|D|})$ is the cycle index of A.*

Proof By Theorem 12.1, the number N of equivalence classes is given by the formula

$$N = \frac{1}{|A|} \sum_{\pi \in A} \psi(\pi^*),$$

where $\psi(\pi^*)$ is the number of elements $f \in R^D$ such that $\pi^*(f)=f$. Let π be a permutation in A with cycle structure representation $x_1^{j_1(\pi)} x_2^{j_2(\pi)} \cdots x_n^{j_n(\pi)}$, where $n=|D|$. We have seen that $\psi(\pi^*)=|R|^m$, where m is the number of disjoint cycles in the disjoint cycle decomposition of π. Since $m=j_1(\pi)+j_2(\pi)+\cdots+j_n(\pi)$, we conclude that $\psi(\pi^*)=|R|^{j_1(\pi)}|R|^{j_2(\pi)} \cdots |R|^{j_n(\pi)}$. Therefore,

$$N = \frac{1}{|A|} \sum_{\pi \in A} (|R|^{j_1(\pi)}|R|^{j_2(\pi)} \cdots |R|^{j_n(\pi)})$$

$$= P_A(|R|, |R|, \ldots, |R|). \blacksquare$$

Let $A = \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6\}$, where the permutations π_i are as given in Figure 12.2. Then the cycle structure representation of π_i is x_3^1, if $i = 1, 2$, the cycle structure representation of π_i is $x_1^1 x_2^1$, if $i = 3, 4, 5$, and the cycle structure representation of π_i is x_1^3 if $i = 6$. Thus the cycle index P_A of A is $P_A(x_1, x_2, x_3) = \frac{1}{6}(2x_3^1 + 3x_1^1 x_2^1 + x_1^3)$. By Theorem 12.3, the number of equivalence classes in the equivalence relation on R^D, where $D = \{12, 13, 23\}$ and $R = \{b, w\}$, induced by the action of A on R^D, is $P_A(|R|, |R|, |R|) = \frac{1}{6}(2 \cdot 2 + 3 \cdot 2 \cdot 2 + 2^3) = 4$. This agrees with previous results.

Theorem 12.3 gives us an applicable formula for counting, for example, the number of nonisomorphic graphs of order 3. We now turn to the more difficult problem of determining the number of nonisomorphic (3, q) graphs for each possible q. In general, we will be interested in dividing the set of equivalence classes of functions into subsets according to certain properties the functions in these classes possess and obtaining a count of the number of classes in each subset. In order to do so, we introduce some new notation and definitions.

Let D and R be finite nonempty sets, where $R = \{r_1, r_2, \ldots, r_k\}$. With each function f in R^D, we associate an element of the polynomial ring in $r_1, r_2, \ldots, r_k$ over the integers. Specifically, for $i = 1, 2, \ldots, k$, let c_i be the number of elements d of D such that $f(d) = r_i$. Then the *weight* of the function f, denoted by $\omega(f)$, is defined to be $r_1^{c_1} r_2^{c_2} \cdots r_k^{c_k}$. Equivalently, $\omega(f) = \Pi_{d \in D} f(d)$, where the product is in the aforementioned ring. The *inventory of a set of functions* from D to R is defined to be the sum of the weights of the functions in the set.

For example, let $D = \{12, 13, 23\}$ and $R = \{b, w\}$. Then the weight of the function f_1 of Figure 12.1 is $\omega(f_1) = b^3$. The inventory of the set of all functions from D to R is

$$\sum_{i=1}^{8} \omega(f_i) = b^3 + b^2 w + b^2 w + b^2 w + bw^2 + bw^2 + bw^2 + w^3$$
$$= b^3 + 3b^2 w + 3bw^2 + w^3.$$

We observe that since the image of each of the three elements of D can be (independently) b or w, the inventory of the set of all functions from R to D is given by $(b + w)^3 = b^3 + 3b^2 w + 3bw^2 + w^3$. Similarly, let $D_1 = \{12, 13\}$ and $D_2 = \{23\}$, and let S be the set of all functions f from D to R such that elements in the same subset D_i, $i = 1, 2$, have the same image under f. Then $S = \{f_1, f_2, f_7, f_8\}$, so that the inventory of the functions in S is $\omega(f_1) + \omega(f_2) + \omega(f_7) + \omega(f_8) = b^3 + b^2 w + bw^2 + w^3$. We once again observe that the inventory of the functions in S can be obtained without specifically determining the functions in S and the sum of their weights. Since the images of the two elements of D_1 must be the same and can be either b or w, and since the image of the single element of D_2 can be b or w, the inventory of the functions in S is given by $(b^2 + w^2)(b + w) = b^3 + b^2 w + bw^2 + w^3$.

The two previous examples of inventories of sets of functions illustrate a more general result. Let $\{D_1, D_2, \ldots, D_m\}$ be a partition of a set D. Then the inventory of the set of all functions f from D to a set R such that the elements in the same subset of the partition have the same image under f is

$$\prod_{i=1}^{m}\left(\sum_{r\in R} r^{|D_i|}\right),$$

since the images of the $|D_i|$ elements of D_i must be the same and can be any element of R.

Let D and R be finite nonempty sets and let A be a permutation group on D. If f_1 and f_2 are functions in the same equivalence class in the equivalence relation on R^D induced by the action of A on R^D, then, by definition, there is a permutation $\pi \in A$ such that $f_2(d) = f_1(\pi(d))$ for every $d \in D$. Therefore,

$$\omega(f_2) = \prod_{d\in D} f_2(d) = \prod_{d\in D} f_1(\pi(d)) = \prod_{d\in D} f_1(d),$$

the last equality holding because π is a permutation on D. Since $\Pi_{d\in D} f_1(d) = \omega(f_1)$, we see that functions in the same equivalence class have the same weight (although two functions with the same weight might not be in the same equivalence class). This common weight is called the *weight of the equivalence class* (*pattern*). The *inventory of a set of equivalence classes* (*patterns*) is then defined to be the sum of the weights of the equivalence classes in the set. Our problem, which will be solved by a theorem of Pólya [P4], is to find the inventory of the set of all equivalence classes of functions; that is, the *pattern inventory*.

Let $W_1, W_2, \ldots, W_t$ denote the distinct weights of the functions in R^D, and for $1 \leq i \leq t$, let $F_i = \{f \in R^D \mid \omega(f) = W_i\}$. By definition, then, the pattern inventory is

$$\sum_{i=1}^{t} m_i W_i,$$

where m_i is the number of equivalence classes that have weight W_i. As we shall see, the problem of finding a formula for the pattern inventory consists of several counting subproblems whose solutions employ formula (12.2).

The following theorem is a somewhat restricted form of Pólya's Theorem. For the complete result and generalizations of Pólya's Theorem, see [L3, Chapter 5].

Theorem 12.4 (Pólya) *Let D and R be finite nonempty sets and let A be a permutation group on D that acts to induce an equivalence relation on R^D. Then the inventory of the set of all equivalence classes of functions is*

$$P_A\left(\sum_{r\in R} r, \sum_{r\in R} r^2, \ldots, \sum_{r\in R} r^{|D|}\right);$$

that is, the pattern inventory is obtained by substituting $\Sigma_{r\in R} r^i$ *for* x_i, $1 \leq i \leq |D|$, *in the cycle index* $P_A(x_1, x_2, \ldots, x_{|D|})$ *of* A.

Proof Employing the notation of the preceding paragraph, we see that the pattern inventory is

$$\sum_{i=1}^{t} m_i W_i,$$

where m_i is the number of equivalence classes that have weight W_i. However, for each i, the union of these m_i equivalence classes is precisely the set F_i. Thus, according to (12.2),

$$m_i = \frac{1}{|A|} \sum_{\pi \in A} \psi(\pi^* | F_i),$$

where $\psi(\pi^* | F_i)$ is the number of elements $f \in F_i$ for which $\pi^*(f) = f$; that is, the number of functions $f \in R^D$ having weight W_i such that $\pi^*(f) = f$. Therefore,

$$\begin{aligned}\sum_{i=1}^{t} m_i W_i &= \sum_{i=1}^{t} \left(\frac{1}{|A|} \sum_{\pi \in A} \psi(\pi^* | F_i) \right) W_i \\ &= \frac{1}{|A|} \sum_{\pi \in A} \left(\sum_{i=1}^{t} \psi(\pi^* | F_i) W_i \right).\end{aligned}$$

For each $\pi \in A$, the term $\Sigma_{i=1}^{t} \psi(\pi^* | F_i) W_i$ is the inventory of the set of all functions $f \in R^D$ such that $\pi^*(f) = f$. We have seen that $\pi^*(f) = f$ if and only if the elements in D that are in the same cycle in the disjoint cycle decomposition of π have the same image under f. Thus, the term $\Sigma_{i=1}^{t} \psi(\pi^* | F_i) W_i$ is the inventory of the set of all functions f from D to R such that the elements of D that are in the same cycle in the disjoint cycle decomposition of π have the same image under f. Let $n = |D|$. Therefore, if the cycle structure representation of π is $x_1^{j_1(\pi)} x_2^{j_2(\pi)} \cdots x_n^{j_n(\pi)}$, then it follows that

$$\sum_{i=1}^{t} \psi(\pi^* | F_i) W_i = \left(\sum_{r \in R} r \right)^{j_1(\pi)} \left(\sum_{r \in R} r^2 \right)^{j_2(\pi)} \cdots \left(\sum_{r \in R} r^n \right)^{j_n(\pi)};$$

that is, $\Sigma_{i=1}^{t} \psi(\pi^* | F_i) W_i$ can be obtained by substituting $\Sigma_{r\in R} r^i$ for x_i, $1 \leq i \leq n$, in the cycle structure representation of π. Thus,

$$\begin{aligned}\sum_{i=1}^{t} m_i W_i &= \frac{1}{|A|} \sum_{\pi \in A} \left(\left(\sum_{r \in R} r \right)^{j_1(\pi)} \left(\sum_{r \in R} r^2 \right)^{j_2(\pi)} \cdots \left(\sum_{r \in R} r^n \right)^{j_n(\pi)} \right) \\ &= P_A\left(\sum_{r \in R} r, \sum_{r \in R} r^2, \ldots, \sum_{r \in R} r^n \right). \quad \blacksquare\end{aligned}$$

If $D = \{12, 13, 23\}$, $R = \{b, w\}$, and $A = \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6\}$, where the permutations π_i are those defined in Figure 12.2, then in the resulting equivalence relation on R^D, there are four equivalence classes of functions, namely $[f_1]$, $[f_2]$, $[f_5]$, and $[f_8]$ (see Figure 12.1). The weights of these equivalence classes, or patterns, are b^3, b^2w, bw^2, and w^3, respectively. Hence the pattern inventory is

$$b^3 + b^2w + bw^2 + w^3.$$

If we apply Pólya's Theorem to this situation, we have that the pattern inventory is

$$P_A(b + w,\ b^2 + w^2,\ b^3 + w^3)$$

where $P_A(x_1, x_2, x_3) = \frac{1}{6}(2x_3^1 + 3x_1^1x_2^1 + x_1^3)$. Thus the pattern inventory is

$$\tfrac{1}{6}(2(b^3 + w^3) + 3(b + w)(b^2 + w^2) + (b + w)^3) = b^3 + b^2w + bw^2 + w^3.$$

Recall that the four equivalence classes of functions in this example correspond to the four nonisomorphic graphs of order 3. Furthermore, an equivalence class has weight b^qw^{3-q}, $0 \le q \le 3$, if and only if the corresponding graph has q edges. Thus we have obtained the desired result that there is one $(3, q)$ graph for each $q = 0, 1, 2, 3$.

Exercises 12.1

12.1 (a) Show that the number of nonidentical (p, q) graphs with vertex set $V = \{v_1, v_2, \ldots, v_p\}$ is $\binom{p(p-1)/2}{q}$, where p and q are integers satisfying $p \ge 1$ and $0 \le q \le \binom{p}{2}$.

(b) Show that the number of nonidentical (p, q) digraphs with vertex set $V = \{v_1, v_2, \ldots, v_p\}$ is $\binom{p(p-1)}{q}$, where p and q are integers satisfying $p \ge 1$ and $0 \le q \le p(p-1)$.

12.2 Let $R = \{a, b\}$, $D = \{1, 2, 3, 4\}$, and let $A = \{\alpha, \beta, \gamma, \delta\}$ be the permutation group on D where

$$\alpha = (1)\ (2)\ (3)\ (4), \qquad \beta = (12)\ (34),$$
$$\gamma = (13)\ (24), \qquad \delta = (14)\ (23).$$

(a) Determine the elements of R^D.

(b) Find α^*, β^*, γ^*, δ^*; find $\psi(\alpha^*)$, $\psi(\beta^*)$, $\psi(\gamma^*)$, $\psi(\delta^*)$.

(c) Use Theorem 12.1 to determine the number of equivalence classes in the equivalence relation on R^D induced by the action of A on R^D.

(d) Determine the equivalence classes in the equivalence relation on R^D.

(e) Find the cycle index of A.

(f) Use Theorem 12.3 to answer part (c).

(g) Find the pattern inventory without using Pólya's Theorem.

(h) Apply Pólya's Theorem to find the pattern inventory.

12.3 Prove Theorem 12.2.

12.4 Let D be a finite nonempty set and let A be a permutation group on D.

(a) Show that every term of the cycle index of A is of the form $Cx_1^{s_1}x_2^{s_2}\cdots x_n^{s_n}$, where C is a constant, $n=|D|$, and

$$\sum_{i=1}^{n} is_i = n.$$

(b) Using only the definition of cycle index, show that if $P_A(x_1, x_2, \ldots, x_n)$ is the cycle index of A, then every term of

$$P_A\left(\sum_{i=1}^{k} r_i, \sum_{i=1}^{k} r_i^2, \ldots, \sum_{i=1}^{k} r_i^n\right)$$

is of the form $Kr_1^{\ell_1}r_2^{\ell_2}\cdots r_k^{\ell_k}$, where K is a constant, $\ell_i \geq 0$ for $1 \leq i \leq k$, and $\sum_{i=1}^{k} \ell_i = n$.

12.2 Graphical and Digraphical Enumeration

For a finite set D, with $|D| \geq 2$, we define $D^{(2)}$ to be the set of all 2-element subsets of D; that is,

$$D^{(2)} = \{\{i, j\} \mid i, j \in D, i \neq j\}.$$

If A is a group of permutations on D, then by the *pair group* $A^{(2)}$ we mean the permutation group on $D^{(2)}$ that is defined according to the following rule. For each $\alpha \in A$, there exists $\alpha' \in A^{(2)}$ such that $\alpha'(\{i, j\}) = \{\alpha(i), \alpha(j)\}$. It is routinely verified that if A is not the symmetric group S_2, then the groups A and $A^{(2)}$ are isomorphic.

We are now in a position to employ Pólya's Theorem to establish a formula for the number of nonisomorphic (p, q) graphs for any fixed integers p and q, where $p \geq 2$ and $0 \leq q \leq \binom{p}{2}$.

Theorem 12.5 *For fixed integers p and q, where $p \geq 2$ and $0 \leq q \leq \binom{p}{2}$, the number of nonisomorphic (p, q) graphs is given by the coefficient of r^q in $P_A(1+r, 1+r^2, \ldots, 1+r^{\binom{p}{2}})$, where A is the group $S_p^{(2)}$ and $P_A(x_1, x_2, \ldots, x_{\binom{p}{2}})$ is the cycle index of $S_p^{(2)}$.*

Proof We first note that by Exercise 12.4(b), every term of $P_A(1+r, 1+r^2, \ldots, 1+r^{\binom{p}{2}})$ is of the form Kr^q, where K is some constant and $0 \leq q \leq \binom{p}{2}$.

Let $D = \{v_1, v_2, \ldots, v_p\}$ and $R = \{1, r\}$. For each function f from $D^{(2)}$ to R, let G_f be that graph with vertex set D such that $v_iv_j \in E(G_f)$, $1 \leq i \neq j \leq p$, if and only if $f(\{v_i, v_j\}) = r$. This, then, gives a one-to-one correspondence between the set $R^{D^{(2)}}$ and the set of nonidentical graphs with vertex set D.

Since $S_p^{(2)}$ is a permutation group on $D^{(2)}$, we can consider the equivalence relation on $R^{D^{(2)}}$ induced by the action of $S_p^{(2)}$ on $R^{D^{(2)}}$; that is, $f_1, f_2 \in R^{D^{(2)}}$ are in the same equivalence class of functions if and only if for some $\pi' \in S_p^{(2)}$ we have that $f_2(\{v_i, v_j\}) = f_1(\pi'(\{v_i, v_j\}))$ for each $\{v_i, v_j\} \in D^{(2)}$. Thus f_1 and f_2 are in the same equivalence class if and only if for some $\pi \in S_p$ and for each $\{v_i, v_j\} \in D^{(2)}$,

$$v_iv_j \in E(G_{f_2}) \Leftrightarrow \pi(v_i)\pi(v_j) \in E(G_{f_1}).$$

We conclude that f_1 and f_2 are in the same equivalence class if and only if G_{f_1} and G_{f_2} are isomorphic. Thus there is a one-to-one correspondence between the set of equivalence classes of functions and the set of nonisomorphic graphs of order p. Furthermore, the weight of an equivalence class is $1^{\binom{p}{2}-q}r^q = r^q$ if and only if the corresponding graph has size q.

It follows from Pólya's Theorem that the number of equivalence classes of functions from $D^{(2)}$ to R that have weight r^q is given by the coefficient of r^q in $P_A(1+r, 1+r^2, \ldots, 1+r^{\binom{p}{2}})$, where A is the group $S_p^{(2)}$ and $P_A(x_1, x_2, \ldots, x_{\binom{p}{2}})$ is the cycle index of $S_p^{(2)}$. This completes the proof. ■

The problem of determining a formula for the number of nonisomorphic (p, q) digraphs for fixed integers p and q, where $p \geq 2$ and $0 \leq q \leq p(p-1)$, can be solved in a similar fashion. For a finite set D, with $|D| \geq 2$, define $D^{[2]}$ as the set of all ordered pairs of distinct elements of D; that is,

$$D^{[2]} = \{(i, j) \mid i, j \in D, i \neq j\}.$$

If A is a group of permutations on D, then the *ordered pair group* $A^{[2]}$ is the permutation group on $D^{[2]}$ defined as follows. For each $\alpha \in A$, there exists $\alpha' \in A^{[2]}$ such that $\alpha'((i, j)) = (\alpha(i), \alpha(j))$. We then have the following.

Theorem 12.6 *For fixed integers p and q, where $p \geq 2$ and $0 \leq q \leq p(p-1)$, the number of nonisomorphic (p, q) digraphs is given by the coefficient of r^q in*

$P_A(1+r, 1+r^2, \ldots, 1+r^{p(p-1)})$, where A is the group $S_p^{[2]}$ and $P_A(x_1, x_2, \ldots, x_{p(p-1)})$ is the cycle index of $S_p^{[2]}$.

Theorems 12.5 and 12.6 consequently provide formulas for the number of nonisomorphic graphs of order p and the number of nonisomorphic digraphs of order p in terms of the cycle indexes of $S_p^{(2)}$ and $S_p^{[2]}$, respectively. Because the determination of these cycle indexes entails calculations of quantities not easily calculated, there may be some question as to whether Theorems 12.5 and 12.6 constitute solutions to the problems of enumerating graphs and digraphs. However, in a certain sense they do, for it is possible to express the cycle indexes of $S_p^{(2)}$ and $S_p^{[2]}$ in an alternative manner. In order to do this, we introduce some additional notation.

By a partition of a positive integer p, we mean a summation $\Sigma p_i = p$, where p_i is a positive integer (the order of the summands is of no consequence). For example, the 11 partitions of 6 are:

$$\begin{array}{l} 1+1+1+1+1+1, \\ 1+1+1+1+2, \\ 1+1+2+2, \\ 1+1+1+3, \\ 1+2+3, \\ 2+2+2, \\ 3+3, \\ 1+1+4, \\ 2+4, \\ 1+5, \\ 6. \end{array}$$

In a given partition of p, we denote by j_i $(1 \le i \le p)$ the number of summands of the partition having value i. Obviously, $\Sigma i j_i = p$ for any partition of p. Finally, let $d(s, t)$ and $m(s, t)$ denote the greatest common divisor and least common multiple, respectively, of integers s and t. We state the following, verifications of which appear in [HP1, pp. 84, 121].

Theorem 12.7 *If A is the group $S_p^{(2)}$, then the cycle index $P_A(x_1, x_2, \ldots, x_{\binom{p}{2}})$ is*

$$\frac{1}{p!} \sum \frac{p!}{\prod_{k=1}^{p} j_k!\, k^{j_k}} \cdot \prod_{k=1}^{\lfloor p/2 \rfloor} (x_k x_{2k}^{k-1})^{j_{2k}} \cdot \prod_{k=1}^{\lfloor p/2 \rfloor} x_k^{k\binom{j_k}{2}}$$

$$\cdot \prod_{k=1}^{\lfloor (p-1)/2 \rfloor} x_{2k+1}^{k \cdot j_{2k+1}} \cdot \prod_{1 \le s < t \le p-1} x_{m(s,t)}^{d(s,t) j_s j_t},$$

where the sum is taken over all partitions of p.

Theorem 12.8 *If A is the group $S_p^{[2]}$, then the cycle index $P_A(x_1, x_2, \ldots, x_{p(p-1)})$ is*

$$\frac{1}{p!}\sum \frac{p!}{\prod_{k=1}^{p} j_k!\,k^{j_k}} \prod_{k=1}^{p} x_k^{(k-1)j_k+2k\binom{j_k}{2}} \prod_{1\leq s<t\leq p-1} x_{m(s,t)}^{2d(s,t)j_sj_t},$$

where the sum is taken over all partitions of p.

We conclude this chapter by determining the number of nonisomorphic $(4, q)$ graphs for $q=0, 1, 2, 3, 4, 5, 6$. By Theorem 12.5, the number of nonisomorphic $(4, q)$ graphs, $0\leq q\leq 6$, is given by the coefficient of r^q in $P_A(1+r, 1+r^2, \ldots, 1+r^6)$, where A is the group $S_4^{(2)}$ and $P_A(x_1, x_2, \ldots, x_6)$ is the cycle index of $S_4^{(2)}$. According to Theorem 12.7, the cycle index of $S_4^{(2)}$ is

$$\tfrac{1}{24}(x_1^6 + 9x_1^2x_2^2 + 8x_3^2 + 6x_2^1x_4^1).$$

Thus,

$$\begin{aligned}
&P_A(1+r, 1+r^2, \ldots, 1+r^6)\\
&=\tfrac{1}{24}((1+r)^6 + 9(1+r)^2(1+r^2)^2 + 8(1+r^3)^2 + 6(1+r^2)(1+r^4))\\
&=\tfrac{1}{24}(24r^6 + 24r^5 + 48r^4 + 72r^3 + 48r^2 + 24r + 24)\\
&= r^6 + r^5 + 2r^4 + 3r^3 + 2r^2 + r + 1.
\end{aligned}$$

We see that there is one $(4, q)$ graph for each $q=0, 1, 5, 6$, that there are two $(4, q)$ graphs for $q=2$ or 4, and that there are three $(4, 3)$ graphs. Thus the total number of nonisomorphic graphs of order 4 is 11. We note that according to Theorem 12.3, the total number of graphs of order 4 can be obtained by evaluating $P_A(2, 2, 2, 2, 2, 2)$.

Exercises 12.2

12.5 Show that if A is a permutation group (not the symmetric group S_2) on a set D, $|D|\geq 2$, then $A\cong A^{(2)}\cong A^{[2]}$.

12.6 Use Theorem 12.5 to determine the number of nonisomorphic $(5, q)$ graphs, $0\leq q\leq 10$.

12.7 Prove Theorem 12.6.

12.8 Use Theorem 12.6 to determine the number of nonisomorphic $(4, q)$ digraphs, $0\leq q\leq 12$.

Glossary of Symbols

Symbol	*Meaning*	*Page*
$A(G)$	adjacency matrix	4
$A^{(2)}$	pair group	329
$\mathcal{A}(G)$	automorphism group	250
$\mathcal{A}_1(G)$	edge-group	252
$\mathcal{A}^*(G)$	induced edge-group	252
$a(G)$	vertex-arboricity	83
$a_1(G)$	arboricity	85
$\bar{a}_1(G)$	cycle multiplicity	115
$\mathcal{B}(G)$	Betti number	141
$b(v)$	block index	51
$b(G)$	number of blocks	51
$bc(G)$	block-cut-vertex graph	51
C_n	cycle of order n	28
C	degree matrix	74
$C(G)$	closure	184
$C_{p+1}(G)$	$(p + 1)$-closure	190
$c(a)$	capacity of an arc	164
cap K	capacity of a cut	166
D	digraph	14
$D_\Delta(\Gamma)$	Cayley color graph	257
$\mathcal{D}_G$	degree set	23
$d(u, v)$	distance	29
deg v	degree	7
diam G	diameter	29

Symbol	*Meaning*	*Page*
$E(G)$	edge set of graph	4
$E(D)$	arc set of digraph	14
$e(v)$	eccentricity	29, 212
$f(a)$	flow in an arc	165
$f(r, n)$	smallest order of an r-regular graph with girth n	35
G	graph	4
$g(G)$	girth	35
$h(T)$	height	78
id v	indegree	15
K_p	complete graph	9
$K(m, n)$	complete bipartite graph	10
$K(p_1, p_2, \ldots, p_n)$	complete n-partite graph	10
$k(G)$	number of components	29
$k_0(G)$	number of odd components	230
$L(G)$	line graph	263
$N(v)$	neighborhood	130
$N[v]$	closed neighborhood	301
$N^+(x)$	vertices adjacent from x	164
$N^-(x)$	vertices adjacent to x	164
od v	outdegree	15
P_A	cycle index	324
P_n	path of order n	28
p	order	4
(p, q)	order p and size q	4
Q_n	n-cube	12
q	size	4
r	number of regions	88
rad G	radius	29
$r(m, n)$	ramsey number	306
$r(G_1, G_2, \ldots, G_k)$	(generalized) ramsey number	312
$[r, n]$-graph	r-regular graph with girth n	35
$T(G)$	total graph	270
$T(p, n)$	"Turan number"	301
$V(G)$	vertex set of graph	4
$V(D)$	vertex set of digraph	14
val f	value of a flow	166
$w(e)$	weight of an edge	30
$w(T)$	weight of a spanning tree	71
$Z(G)$	center	29

Symbol	*Meaning*	*Page*
$\alpha(G)$	vertex covering number	243
$\alpha_1(G)$	edge covering number	244
$\beta(G)$	independence number	188
$\beta_1(G)$	edge independence number	243
Γ	finite group	257
$\gamma(G)$	genus	117
$\gamma_M(G)$	maximum genus	141
$\Delta(G)$	maximum degree	61, 85
$\delta(G)$	minimum degree	72
$\theta(G)$	vertex-thickness	115
$\theta_1(G)$	thickness	115
$\bar{\theta}_1(G)$	coarseness	115
$\kappa(G)$	connectivity	152
$\kappa_1(G)$	edge-connectivity	152
$\nu(G)$	crossing number	107
$\bar{\nu}(G)$	rectilinear crossing number	109
$\xi_0(G)$	number of components of odd size	147
$\xi(G)$	min $\xi_0(G)$	147
$\rho_k(G)$	vertex partition number	285
$\sigma(G)$	vertex dominating number	248
$\sigma_1(G)$	edge dominating number	248
$\chi(G)$	chromatic number of a graph	271
$\chi(S_n)$	chromatic number of a surface	299
$\chi_1(G)$	edge chromatic number	286
$\chi_2(G)$	total chromatic number	293
$\tilde{\chi}(G)$	co-chromatic number	285
$\omega(G)$	clique number	279
$\bar{G}$	complement	9
G^n	*n*th power	196
G^*	symmetric digraph	17
$G_1 \cong G_2$	isomorphic	5
$G_1 = G_2$	identical	6
$G_1 \cup G_2$	union	11
$G_1 + G_2$	join	11
$G_1 \times G_2$	cartesian product	11
$G_1 \oplus G_2$	edge sum, factorization, decomposition	229, 239
$H \subset G$	subgraph	8
$H \prec G$	induced subgraph	8

Symbol	*Meaning*	*Page*
$\langle U \rangle$	subgraph induced by U	8
$G - v$	deletion of a vertex	8
$G - e$	deletion of an edge	8
$G + f$	addition of an edge	8
$G \rightarrow (G_1, G_2, \ldots, G_k)$	arrows	315
$\lceil x \rceil$	ceiling of x	80
$\lfloor x \rfloor$	floor of x	84

Bibliography

Each item in the bibliography is followed by one or more numbers in square brackets indicating the pages on which the item is referenced. The starred entries in the bibliography denote books.

Alaví, Y.

[ACLO1] (with G. Chartrand, L. Lesniak, and O. R. Oellermann) Bipartite regulation numbers. Preprint. [10]

Alekseev, V. B.

[AG1] (with V. S. Gonchakov) Thickness of arbitrary complete graphs. *Mat. Sbornik* **101** (143) (1976) 212–230. [115]

Alpert, S. R.

See [GA1], [GA2], and [GA3].

Anderson, I.

[A1] Perfect matchings of a graph. *J. Combinatorial Theory* **10B** (1971) 183–186. [229]

Appel, K.

[AH1] (with W. Haken) The solution of the Four-Color Map Problem. *Sci. Amer.* 237 (1977) 108–121. [299]

[AHK1] (with W. Haken and J. Koch) Every planar map is four-colorable. *Illinois J. Math.* 21 (1977) 429–567. [298]
See also p. 298.

Babai, L.

[B1] Doctoral thesis. Eötvös Loránd Tudományegyetem (1976). [260]

Battle, J.

[BHKY1] (with F. Harary, Y. Kodama, and J. W. T. Youngs) Additivity of the genus of a graph. *Bull. Amer. Math. Soc.* 68 (1962) 565–568. [122]

Beineke, L. W.

[B2] Decompositions of complete graphs into forests. *Magyar Tud. Adad. Mat. Kutató Int. Közl.* **9** (1964) 589–594. [85]

[B3] The decomposition of complete graphs into planar subgraphs. *Graph Theory and Theoretical Physics.* Academic Press, New York (1967) 139–154. [115]

[B4] Derived graphs and digraphs. *Beiträge zur Graphentheorie.* Teubner, Leipzig (1968) 17–33. [266]

[BH1] (with F. Harary) The genus of the n-cube. *Canad. J. Math.* **17** (1965) 494–496. [123]

[BH2] (with F. Harary) The thickness of the complete graph. *Canad. J. Math.* **17** (1965) 850–859. [115]

[BHM1] (with F. Harary and J. W. Moon) On the thickness of the complete bipartite graph. *Proc. Cambridge Philos. Soc.* **60** (1964) 1–5. [115]

[BM1] (with J. W. Moon) On bipartite tournaments and scores. *The Theory and Applications of Graphs.* Wiley, New York (1981) 55–71. [222]

[BR1] (with R. D. Ringeisen) On the crossing number of products of cycles and graphs of order four. *J. Graph Theory* **4** (1980) 145–155. [114]

[BW1] (with R. J. Wilson) On the edge-chromatic number of a graph. *Discrete Math.* **5** (1973) 15–20. [289]
See also [GB1] and [RB1].

Berge, C.

[B5] Two theorems on graph theory. *Proc. Nat. Acad. Sci. U.S.A.* **43** (1957) 842–844. [226]

[B6] Färbung von Graphen, deren sämtliche bzw. deren ungeraden Kreise starr sind. *Wissenschaftliche Zeitung, Martin Luther Univ., Halle Wittenberg* **114** (1961). [280]

*[B7] *The Theory of Graphs and its Applications.* Wiley, New York (1962). [38, 68]

*[B8] *Graphs and Hypergraphs.* North-Holland, London (1973). [280]

Bernhart, F.

[B9] A digest of the Four Color Theorem. *J. Graph Theory* **1** (1977) 207–225. [299]

Biggs, N. L.

*[B10] *Algebraic Graph Theory.* Cambridge University Press, London (1974). [40]

*[BLW1] (with E. K. Lloyd and R. J. Wilson) *Graph Theory 1736-1936.* Oxford University Press, London (1976). [52, 182, 299]

Blažek, J.

[BK1] (with M. Koman) A minimal problem concerning complete plane graphs. *Theory of Graphs and its Applications.* Academic Press, New York (1964) 113–117. [108]

Boesche, F. T.

[B11] The strongest monotone degree condition for n-connectedness of a graph. *J. Combinatorial Theory* **16B** (1974) 162–165. [153]

Bollobás, B.

*[B12] *Extremal Graph Theory*. Academic Press, New York (1978). [305]

Bondy, J. A.

[B13] Properties of graphs with constraints on degrees. *Studia Sci. Math. Hungar.* **4** (1969) 473–475. [153]

[B14] Pancyclic graphs. *J. Combinatorial Theory* **11B** (1977) 80–84. [192, 205]

[BC1] (with V. Chvátal) A method in graph theory. *Discrete Math.* **15** (1976) 111–136. [184, 190]

*[BM2] (with U. S. R. Murty) *Graph Theory with Applications*. North Holland, New York (1976). [57, 104]

Brooks, R. L.

[B15] On coloring the nodes of a network. *Proc. Cambridge Philos. Soc.* **37** (1941) 194–197. [275]

Bruck, R. H.

[BR2] (with H. J. Ryser) The nonexistence of certain finite projective planes. *Canad. J. Math.* **1** (1949) 88–93. [239]

Burnside, W.

*[B16] *Theory of Groups of Finite Order*. Cambridge Univ. Press, Cambridge (1911). [321]

Burr, S. A.

[BE1] (with P. Erdös) Generalizations of a Ramsey-theoretic result of Chvátal. *J. Graph Theory* **7** (1983) 39–51. [316]

[BEL1] (with P. Erdös and L. Lovász) On graphs of Ramsey type. *Ars Combinatoria* **1** (1976) 167–190. [316]

[BR2] (with J. A. Roberts) On Ramsey numbers for stars. *Utilitas Math.* **4** (1973) 217–220. [314]

Camion, P.

[C1] Chemins et circuits hamiltoniens des graphes complets. *C. R. Acad. Sci. Paris* **249** (1959) 2151–2152. [214]

Cayley, A.

[C2] On the colouring of maps. *Proc. Royal Geog. Soc.* **1** (1879) 259–261. [297]

[C3] A theorem on trees. *Quart. J. Math.* **23** (1889) 376–378. Collected papers, Cambridge, **13** (1892) 26–28. [72]
See also p. 297.

Chartrand, G.

[CGH1] (with D. Geller and S. Hedetniemi) Graphs with forbidden subgraphs. *J. Combinatorial Theory* **10B** (1971) 12–41. [115]

[CKK1] (with S. F. Kapoor and H. V. Kronk) A generalization of hamiltonian-connected graphs. *J. Math. Pures Appl.* **48** (1969) 109–116. [190]

[CK1] (with H. V. Kronk) The point arboricity of planar graphs. *J. London Math. Soc.* **44** (1969) 612–616. [84]

[CKW1] (with H. V. Kronk and C. E. Wall) The point arboricity of a graph. *Israel J. Math.* 6 (1968) 169–175. [115]

[CP1] (with A. D. Polimeni) Ramsey theory and chromatic numbers. *Pacific J. Math.* 55 (1974) 39–43. [282, 317]

[CPS1] (with A. D. Polimeni and M. J. Stewart) The existence of 1-factors in line graphs, squares, and total graphs. *Indag. Math.* 35 (1973) 228–232. [55]

[CW1] (with C. E. Wall) On the hamiltonian index of a graph. *Studia Sci. Math. Hungar.* 8 (1973) 43–48. [268]

[CW2] (with R. J. Wilson) The Petersen graph. *Graphs and Applications.* Wiley, New York (1985) 69–100. [40]

See also [ACLO1].

Chvátal, V.

[C4] Flip-flops in hypohamiltonian graphs. *Canad. Math. Bull* 16 (1973) 33–41. [195]

[C5] Tree-complete graph ramsey numbers. *J. Graph Theory* 1 (1977) 93. [313]

[CE1] (with P. Erdös) A note on hamiltonian circuits. *Discrete Math.* 2 (1972) 111–113 [188]

See also [BC1].

Clarke, L. E.

[C6] On Cayley's formula for counting trees. *J. London Math. Soc.* 33 (1958) 471–475. [72]

De Morgan, A.

See pp. 294, 295, and 297.

Demoucron, G.

[DMP1] (with Y. Malgrange and R. Pertuiset) Graphes planaires: reconnaissance et construction de représentation planaires topologiques. *Rev. Francaise Recherche Opérationnelle* 8 (1964) 33–47. [102]

Descartes, B.

[D1] A three colour problem. *Eureka* 9 (1947) 21. [281]

Dijkstra, E. W.

[D2] A note on two problems in connexion with graphs. *Numer. Math.* 1 (1959) 269–271. [30]

Dirac, G. A.

[D3] A property of 4-chromatic graphs and some remarks on critical graphs. *J. London Math. Soc.* 27 (1952) 85–92. [273]

[D4] Some theorems on abstract graphs. *Proc. London Math. Soc.* 2 (1952) 69–81. [187]

[D5] In abstrakten Graphen vorhande vollständige 4-Graphen und ihre unterteilungen. *Math. Nachr.* 22 (1960) 61–85. [160]

[D6] Homomorphism theorems for graphs. *Math. Ann.* 153 (1964) 69–80. [304]

[D7] Short proof of Menger's graph theorem. *Mathematika* 13 (1966) 42–44. [157]

[DS1] (with S. Schuster) A theorem of Kuratowski. *Nederl. Akad. Wetensch. Proc.* Ser. A 57 (1954) 343–348. [96]

Duke, R. A.

[D8] The genus, regional number, and Betti number of a graph. *Canad. J. Math.* 18 (1966) 817–822. [138]

Dyck, W.

[D9] Beiträge zur Analysis Situs. *Math. Ann.* 32 (1888) 457–512. [130]

Edmonds, J.

[E1] A combinatorial representation for polyhedral surfaces. *Notices Amer. Math. Soc.* 7 (1960) 646. [130]

[E2] The Chinese postman problem. *Oper. Res.* 13 Suppl. 1 (1965) B73. [60]

[EK1] (with R. M. Karp) Theoretic improvements in algorithmic efficiency for network flow problems. *J. Assoc. Comput. Mach.* **19** (1972) 248–264. [172]

Eggleton, R. B.

[EG1] (with R. K. Guy) The crossing number of the *n*-cube. *Notices Amer. Math. Soc.* **17** (1970) 757. [112]

Elias, P.

[EFS1] (with A. Feinstein and C. E. Shannon) A note on the maximum flow through a network. *IRE Trans. Inform. Theory* IT-2 (1956) 117–119. [161]

Erdös, P.

[E3] Some remarks on the theory of graphs. *Bull. Amer. Math. Soc.* 53 (1947) 292–294. [310]

[E4] Graph theory and probability II. *Canad. J. Math.* 13 (1961) 346–352. [282]

[E5] Extremal problems in graph theory. *A Seminar on Graph Theory.* Holt, Rinehart & Winston, New York (1967) 54–59. [304, 305]

[E6] On the graph theorem of Turán (Hungarian). *Mat. Lapok* 21 (1970) 249–251. [301]

[EG2] (with T. Gallai) Graphs with prescribed degrees of vertices (Hungarian). *Mat. Lapok* **11** (1960) 264–274. [23]

[ES1] (with H. Sachs) Reguläre Graphen gegebener Taillenweite mit minimaler Knotenzahl. *Wiss. Z. Univ. Halle, Math.-Nat.* **12** (1963) 251–258. [35]

[ES2] (with G. Szekeres) A combinatorial problem in geometry. *Compositio Math.* 2 (1935) 463–470. [307]

[EW1] (with R. J. Wilson) On the chromatic index of almost all graphs. *J. Combinatorial Theory* 23B (1977) 255–257. [288]

Euler, L.

[E7] Solutio problematis ad geometriam situs pertinentis. *Comment. Academiae Sci. I. Petropolitanae* 8 (1736) 128–140. [52, 54]

[E8] Demonstratio nonnullarum insignium proprietatum quibus solida hedris planis inclusa sunt praedita. *Novi Comm. Acad. Sci. Imp. Petropol.* 4 (1758) 140–160. [88]

Fáry, I.

[F1] On the straight line representations of planar graphs. *Acta Sci. Math.* **11** (1948) 229–233. [90]

Feinstein, A.

See [EFS1].

Fink, J. F.

[F2] Every graph is an induced isopart of a connected regular graph. Preprint. [239]

[FR1] (with S. Ruiz) Every graph is an induced isopart of a circulant. *Czechoslovak Math. J.* To appear. [240]

Fiorini, S.

*[FW1] (with R. J. Wilson) *Edge-colourings of Graphs.* Pitman, London (1977). [293]

Fleischner, H.

[F3] The square of every two-connected graph is hamiltonian. *J. Combinatorial Theory* **16B** (1974) 29–34. [196]

Folkman, J. H.

[F4] Graphs with monochromatic complete subgraphs in every edge coloring. *SIAM J. Appl. Math.* **18** (1970) 19–24. [315]

Ford, L. R., Jr.

[FF1] (with D. R. Fulkerson) Maximal flow through a network. *Canad. J. Math.* **8** (1956) 399–404. [161, 170, 172]

[FF2] (with D. R. Fulkerson) A simple algorithm for finding maximal network flows and an application to the Hitchcock problem. *Canad. J. Math.* **9** (1957) 210–218. [172]

*[FF3] (with D. R. Fulkerson) *Flows in Networks.* Princeton University Press, Princeton (1962). [176]

Fowler, P. A.

[F5] The Königsberg bridges . . . 250 years later. *Amer. Math. Monthly.* To appear. [54]

Frucht, R.

[F6] Herstellung von Graphen mit vorgegebener abstrakten Gruppe. *Compositio Math.* **6** (1938) 239–250. [259]

Fulkerson, D. R.

[F7] Zero-one matrices with zero traces. *Pacific J. Math.* **10** (1960) 831–836. [25]
See also [FF1], [FF2], and [FF3].

Gaddum, J. W.

See [NG1].

Gallai, T.

[G1] Über extreme Punkt- und Kantenmengen. *Ann. Univ. Sci. Budapest, Eötvös Sect. Math.* **2** (1959) 133–138. [244]

[G2] On directed paths and circuits. *Theory of Graphs.* Academic Press, New York (1968) 115–118. [277]

Gaudin, T.

See [HGR1].

Geller, D.

See [CGH1].

Ghouila-Houri, A.

[G3] Une condition suffisante d'existence d'un circuit Hamiltonien. *C. R. Acad. Sci. Paris* **156** (1960) 495–497. [202]

Gonchakov, V. S.

See [AG1].

Goodman, S. E.

[GH1] (with S. T. Hedetniemi) Eulerian walks in graphs. *SIAM J. Comput.* 2 (1973) 16–27. [60]

Graham, R. L.

*[GRS1] (with B. L. Rothschild and J. H. Spencer) *Ramsey Theory*. Wiley, New York (1980). [317]

Grinberg, E. J.

[G4] Plane homogeneous graphs of degree three without hamiltonian circuits. *Latvian Math. Yearbook* **4** (1968) 51–58. [197]

Gross, J. L.

[G5] Voltage graphs. *Discrete Math.* **9** (1974) 239–246. [150]

[GA1] (with S. R. Alpert) Branched coverings of graph imbeddings. *Bull. Amer. Math. Soc.* **79** (1973) 942–945. [149]

[GA2] (with S. R. Alpert) The topological theory of current graphs. *J. Combinatorial Theory* **17B** (1974) 218–233. [149]

[GA3] (with S. R. Alpert) Components of branched coverings of current graphs. *J. Combinatorial Theory* **20B** (1976) 283–303. [149]

Gustin, W.

[G6] Orientable embedding of Cayley graphs. *Bull. Amer. Math. Soc.* **69** (1963) 272–275. [149]

Guthrie, Francis

See pp. 295 and 297.

Guthrie, Frederick

[G7] Note on the colouring of maps. *Proc. Royal Soc. Edinburgh* **10** (1880) 727–728. [295]

See also pp. 294 and 295.

Guy, R. K.

[G8] A combinatorial problem. *Bull. Malayan Math. Soc.* 7 (1960) 68–72. [108]

[G9] Crossing number of graphs. *Graph Theory and Applications.* Springer-Verlag, New York (1972) 111–124. [108, 109]

[GB1] (with L. W. Beineke) The coarseness of the complete graph. *Canad. J. Math.* **20** (1966) 888–894. [115]

See also [EG1].

Häggkvist, R.

[HFS1] (with R. J. Faudree and R. H. Schelp) Pancyclic graphs—connected Ramsey number. *Ars Combinatoria* **11** (1981) 37–49. [194]

[HT1] (with C. Thomassen) On pancyclic digraphs. *J. Combinatorial Theory* **20B** (1976) 20–40. [206]

Haken, W.

[H1] An attempt to understand the Four Color Problem. *J. Graph Theory* **1** (1977) 193–206. [299]
See also [AH1], [AHK1], and p. 298.

Hakimi, S. L.

[H2] On the realizability of a set of integers as degrees of the vertices of a graph. *J. SIAM Appl. Math.* **10** (1962) 496–506. [20]

Halin, R.

[H3] Bemerkungen über ebene Graphen. *Math. Ann.* 53 (1964) 38–46. [100]

[H4] Unterteilungen vollständinger Graphen in Graphen mit unendlicher chromatischer Zahl. *Abh. Math. Seminar Univ. Hamburg* 31 (1967) 156–165. [276]

Hall, P.

[H5] On representation of subsets. *J. London Math. Soc.* **10** (1935) 26–30. [227]

Hamilton, W. R.

See pp. 182, 294, and 295.

Harary, F.

[H6] Graphs and matrices. *SIAM Review* **9** (1967) 83–90. [73]

*[H7] *Graph Theory*. Addison-Wesley, Reading, MA (1969). [68]

[HKS1] (with P. C. Kainen and A. J. Schwenk) Toroidal graphs with arbitrarily high crossing numbers. *Nanta Math.* **6** (1973) 58–67. [112]

[HM1] (with L. Moser) The theory of round robin tournaments. *Amer. Math. Monthly* 73 (1966) 231–246. [219]

[HN1] (with C. St. J. A. Nash-Williams) On eulerian and hamiltonian graphs and line graphs. *Canad. Math. Bull.* 8 (1965) 701–710. [267]

[HN2] (with R. Z. Norman) The dissimilarity characteristic of Husimi trees. *Ann. of Math.* 58 (1953) 134–141. [49]

*[HP1] (with E. M. Palmer) *Graphical Enumeration*. Academic Press, New York (1973). [331]

[HT2] (with W. T. Tutte) A dual form of Kuratowski's Theorem. *Canad. Math. Bull.* 8 (1965) 17–20, 373. [100]
See also p. 62.

Havel, V.

[H8] A remark on the existence of finite graphs (Czech.) *Časopis Pěst. Mat.* 80 (1955) 477–480. [20]

Heawood, P. J.

[H9] Map-colour theorem. *Quart. J. Math.* 24 (1890) 332–339. [297, 299]

Hedetniemi, S. T.

See [CGH1] and [GH1].

Hedman, B.

[H10] A sufficient condition for n-short-connectedness. *Math. Mag.* 47 (1974) 156–157. [161]

Heffter, L.

[H11] Über das Problem der Nachbargebiete. *Math. Ann.* 38 (1891) 477–508. [130, 299]

Herz, J. C.

[HDV1] (with J. J. Duby and F. Vigué) Recherche systématique des graphes hypohamiltoniens. *Theory of Graphs, International Symposium, Rome*, 1966. Gordon and Breach, New York (1967) 153–159. [195]

[HGR1] (with T. Gaudin and P. Rossi) Solution de Probleme No. 29. *Rev. Francaise Rech. Operationelle* 8 (1964) 214–218. [195]

Hierholzer, C.

[H12] Über die Möglichkeit, einen Linienzug ohne Wiederholung und ohne Unterbrechnung zu umfahren. *Math. Ann.* 6 (1873) 30–32. [54]

Hoffman, A. J.

[HS1] (with R. R. Singleton) On Moore graphs with diameters two and three. *IBM J. Res. Develop.* 4 (1960) 497–504. [38]

Isaacs, R.

[I1] Infinite families of nontrivial trivalent graphs which are not Tait colorable. *Amer. Math. Monthly* 82 (1975) 221–239. [289]

Izbicki, H.

[I2] Reguläre Graphen beliebigen Grades mit Vorgegebenen Eigenschaften. *Monatsh. Math.* 64 (1960) 15–21. [260]

Jackson, B.

[J1] Hamilton cycles in regular 2-connected graphs. *J. Combinatorial Theory* 29B (1980) 27–46. [187]

Jungerman, M.

[J2] A characterization of upper embeddable graphs. *Trans. Amer. Math. Soc.* 241 (1978) 401–406. [143]

Kainen, P. C.

See [HKS1] and p. 110.

Kapoor, S. F.

[KPW1] (with A. D. Polimeni and C. E. Wall) Degree sets for graphs. *Fund. Math.* 95 (1977) 189–194. [23]

See also [CKK1].

Karp, R. M.

See [EK1].

Kelly, J. B.

[KK1] (with L. M. Kelly) Paths and circuits in critical graphs. *Amer. J. Math.* 76 (1954) 786–792. [281]

Kelly, L. M.

See [KK1].

Kelly, P. J.

See p. 62.

Kempe, A. B.

[K1] On the geographical problem of the four colors. *Amer. J. Math.* 2 (1879) 193–200. [297]

[K2] How to color a map with four colors. *Nature* 21 (1880) 399–400. [297]
See also p. 297.

Kirchhoff, G.

[K3] Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme geführt wird. *Ann. Phys. Chem.* 72 (1847) 497–508. Gesammelte Abhandlungen, Leipzig (1882) 22–23. [73]

Kirkman, T. P.

[K4] On a problem in combinations. *Cambridge and Dublin Math. J.* 2 (1847) 191–204. [239]
See also p. 183.

Kleinert, M.

[K5] Die Dicke des n-dimensionalen Würfel-Graphen. *J. Combinatorial Theory* 3 (1967) 10–15. [115]

Kleitman, D. J.

[K6] The crossing number of $K_{5,n}$. *J. Combinatorial Theory* 9 (1970) 315–323. [110, 111]

Koch, J.

See [AHK1] and p. 298.

Kodama, Y.

See [BHKY1].

Koman, M.

See [BK1].

König, D.

[K7] Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre. *Math. Ann.* 77 (1916) 453–465. [235]

[K8] Graphen und Matrizen. *Math. Fiz. Lapok.* 38 (1931) 116–119. [227, 244]

*[K9] *Theorie der endlichen und unendlichen Graphen.* Leipzig (1936). Reprinted Chelsea, New York (1950). [259]

Krausz, J.

[K10] Démonstration nouvelle d'une théorème de Whitney sur les réseaux (Hungarian). *Mat. Fiz. Lapok* **50** (1943) 75–89. [262]

Kronk, H. V.

[KRW1] (with R. D. Ringeisen and A. T. White) On 2-cell imbeddings of complete n-partite graphs. *Colloq. Math.* **36** (1976) 295–304. [149]
See also [CKK1], [CKW1], and p. 282.

Kruskal, J. B., Jr.

[K11] On the shortest spanning subtree of a graph and the traveling salesman problem. *Proc. Amer. Math. Soc.* **7** (1956) 48–50. [71]

Kuratowski, K.

[K12] Sur le problème des courbes gauches en topologie. *Fund. Math.* **15** (1930) 271–283. [96]

Kwan, M. -K

[K13] Graphic programming using odd or even points. *Acta Math. Sinica* **10** (1960) 263–266; *Chinese Math.* **1** (1962) 273–277. [60]

Landau, H. G.

[L1] On dominance relations and the structure of animal societies. III. The condition for a score structure. *Bull. Math. Biophys.* **15** (1953) 143–148. [212, 218]

Lesniak, L.

See [ACLO1].

Lindgren, W. F.

[L2] An infinite class of hypohamiltonian graphs. *Amer. Math. Monthly* **74** (1967) 1087–1089. [195]

Liu, C. L.

*[L3] *Introduction to Combinatorial Mathematics.* McGraw-Hill, New York (1968). [326]

Lloyd, E. K.

See [BLW1].

Lovász, L.

[L4] On chromatic number of finite set-systems. *Acta Math. Acad. Sci. Hungar.* **79** (1967) 59–67. [282]

[L5] Normal hypergraphs and the perfect graph conjecture. *Discrete Math.* **2** (1972) 253–267. [280]

[L6] Three short proofs in graph theory. *J. Combinatorial Theory* **19B** (1975) 269–271. [275]
See also [BEL1].

Malgrange, Y.

See [DMP1].

Manvel, B.

[M1] *On Reconstruction of Graphs.* Doctoral Thesis, University of Michigan (1970). [64]

May, K. O.

[M2] The origin of the four-color conjecture. *Isis.* **56** (1965) 346–348. [294]

McKay, B. D.

[M3] Computer reconstruction of small graphs. *J. Graph Theory* **1** (1977) 281–283. [62]

McKee, T. A.

[M4] Recharacterizing Eulerian: intimations of new duality. *Discrete Math.* **51** (1984) 237–242. [56]

Menger, K.

[M5] Zur allgeminen Kurventheorie. *Fund. Math.* **10** (1927) 95–115. [157]

Meyniel, M.

[M6] Une condition suffisante d'existence d'un circuit Hamiltonien dans un graph oriente. *J. Combinatorial Theory* **14B** (1973) 137–147. [201]

Mirsky, L.

*[M7] *Transversal Theory.* Academic Press, New York (1971). [228]

Moon, J. W.

[M8] On subtournaments of a tournament. *Canad. Math. Bull.* **9** (1966) 297–301. [214]

[M9] Various proofs of Cayley's formula for counting trees. *A Seminar on Graph Theory.* Holt, Rinehart & Winston, New York (1967) 70–78. [72]

Moser, L.

See [HM1] and p. 219.

Murty, U. S. R.

See [BM2].

Mycielski, J.

[M10] Sur le coloriage des graphes. *Colloq. Math.* **3** (1955) 161–162. [281]

Nash-Williams, C. St. J. A.

[N1] Decomposition of finite graphs into forests. *J. London Math. Soc.* **39** (1964) 12. [85]

[N2] Hamilton arcs and circuits. *Recent Trends in Graph Theory.* Springer-Verlag, Berlin (1971) 197–210. [187]
See also [HN1] and p. 196.

Nešetřil, J.

[NR1] (with V. Rödl) The Ramsey property for graphs with forbidden complete subgraphs. *J. Combinatorial Theory* **20B** (1976) 243–249. [316]

Nijenhuis, A.

[N3] Note on the unique determination of graphs by proper subgraphs. *Notices Amer. Math. Soc.* 24 (1977) A-290. [62]

Nordhaus, E. A.

[NG1] (with J. W. Gaddum) On complementary graphs. *Amer. Math. Monthly* 63 (1956) 175–177. [282]

[NSW1] (with B. M. Stewart and A. T. White) On the maximum genus of a graph. *J. Combinatorial Theory* **11B** (1971) 258–267. [142, 149]

Norman, R. Z.

See [HN2].

Oellermann, O. R.

See [ACLO1].

Ore, O.

[O1] Note on Hamilton circuits. *Amer. Math. Monthly* 67 (1960) 55. [183]

*[O2] *Theory of Graphs.* Amer. Math. Soc. Publ., Providence (1962). [69]

[O3] Hamilton connected graphs. *J. Math. Pures. Appl.* 42 (1963) 21–27. [190]

Overbeck-Larisch, M.

[O4] Hamiltonian paths in oriented graphs. *J. Combinatorial Theory* **21B** (1976) 76–80. [203]

[O5] A theorem on pancyclic oriented graphs. *J. Combinatorial Theory* **23B** (1977) 168–173. [205]

Palmer, E. M.

See [HP1].

Parthasarathy, K. R.

[PR1] (with G. Ravindra) The strong perfect graph conjecture is true for $K_{1,3}$-free graphs. *J. Combinatorial Theory* **22B** (1976) 212–223. [280]

Pertuiset, R.

See [DMP1].

Petersen, J.

[P1] Die Theorie der regulären Graphs. *Acta Math.* 15 (1891) 193–220. [231, 235]

Plesník, J.

[P2] Bounds on chromatic numbers of multiple factors of a complete graph. *J. Graph Theory* 2 (1978) 9–17. [155]

[P3] Critical graphs of given diameter. *Acta Fac. Rerum Natur. Univ. Comenian. Math.* 30 (1975) 71–93. [284]

Plummer, M. D.

See p. 196.

Polimeni, A. D.

See [CP1], [CPS1], and [KPW1].

Pólya, G.

[P4] Kombinatorische Anzahlbestimmungen fur Gruppen, Graphen und chemische Verbindungen. *Acta Math.* 68 (1937) 145-254. [326]

Pósa, L.

[P5] A theorem concerning Hamilton lines. *Magyar Tud. Akad. Mat. Kutató Int. Közl.* 7 (1962) 225–226. [186]

Powell, M. B.

See [WP1].

Ramsey, F.

[R1] On a problem of formal logic. *Proc. London Math. Soc.* 30 (1930) 264–286. [306]

Ravindra, G.

See [PR1].

Rédei, L.

[R2] Ein kombinatorischer Satz. *Acta Litt. Szeged* 7 (1934) 39–43. [213]

Ringeisen, R. D.

[R3] Determining all compact orientable 2-manifolds upon which $K_{m,n}$ has 2-cell imbeddings. *J. Combinatorial Theory* **12B** (1972) 101–104. [135, 149]

[RB1] (with L. W. Beineke) The crossing number of $C_3 \times C_n$. *J. Combinatorial Theory* **24B** (1978) 134–136. [112]
See also [BR2] and [KRW1].

Ringel, G.

[R4] Uber drei kombinatorische Probleme am *n*-dimensionalen Würfel und Würfelgitter. *Abh. Math. Sem. Univ. Hamburg* 20 (1955) 10–19. [123]

[R5] Das Geschlecht des vollständigen paaren Graphen. *Abh. Math. Sem. Univ. Hamburg* 28 (1965) 139–150. [123]

*[R6] *Map Color Theorem.* Springer-Verlag, Berlin (1974). [123, 300]

[RY1] (with J. W. T. Youngs) Solution of the Heawood map-coloring problem. *Proc. Nat. Acad. Sci. USA* 60 (1968) 438–445. [123, 299]
See also p. 110.

Robacker, J. T.

[R7] On network theory. The RAND Corporation, Research Memorandum **RM-1498** (May 26, 1955). [180]

Robbins, H. E.

[R8] A theorem on graphs, with an application to a problem in traffic control. *Amer. Math. Monthly* 46 (1939) 281–283. [207]

Roberts, J. A.

See [BR2].

Rödl, V.

See [NR1].

Rosa, A.

[R9] On certain valuations of the vertices of a graph. *Theory of Graphs, Proc. Internat. Sympos. Rome 1966,* Gordon and Breach, New York (1967) 349–355. [240]

Rossi, P.

See [HGR1].

Rothschild, B. L.

See [GRS1].

Ruiz, S.

[R10] *On Isomorphic Decompositions of Graphs.* Doctoral Thesis, Western Michigan University (1983). [241]
See also [FR1] and p. 233.

Ryser, H.

*[R11] *Combinatorial Mathematics.* Wiley, New York (1963). [25, 239]
See also [BR1].

Schelp, R. H.

See [HFS1].

Schuster, S.

See [DS1].

Schwenk, A. J.

See [HKS1] and p. 234.

Sedláček, J.

[S1] Some properties of interchange graphs. *Theory of Graphs and its Applications.* Academic Press, New York (1962) 145–150. [269]

Seinsche, D.

[S2] On a property of the class of n-colorable graphs. *J. Combinatorial Theory* **16B** (1974) 191–193. [279]

Shannon, C. E.

See [EFS1].

Singleton, R. R.

See [HS1].

Spencer, J. E.

See [GRS1].

Stewart, B. M.

See [CPS1].

Szekeres, G.

[SW1] (with H. S. Wilf) An inequality for the chromatic number of a graph. *J. Combinatorial Theory* **4** (1968) 1–3. [276]
See also [ES2].

Szele, T.

[S3] Kombinatorische Untersuchungen über gerichtete vollständige Graphen. *Publ. Math. Debrecen.* **13** (1966) 145–168. [214]

Tait, P. G.

See p. 198.

Thomassen, C.

[T1] Some homeomorphism properties of graphs. *Math. Nachr.* **64** (1974) 119–133. [304]

[T2] Hypohamiltonian and hypotraceable graphs. *Discrete Math.* **9** (1974) 91–96. [195]

[T3] An Ore-type condition implying a digraph to be pancyclic. *Discrete Math.* **19** (1977) 85–92. [205]

[T4] Hamiltonian connected tournaments. *J. Combinatorial Theory* **28B** (1980) 142–163. [205]

[T5] Landau's characterization of tournament score sequences. *The Theory and Applications of Graphs.* Wiley, New York (1981) 589–591. [218]

[T6] A theorem on paths in planar graphs. *J. Graph Theory* **7** (1983) 169–176. [198]
See also [HT1] and pp. 192 and 214.

Toida, S.

[T7] Properties of an Euler graph. *J. Franklin Inst.* **295** (1973) 343–345. [56]

Tucker, A.

[T8] The strong perfect graph conjecture for planar graphs. *Canad. J. Math.* **25** (1973) 103–114. [280]

*[T9] *Applied Combinatorics.* Wiley, New York (1980). [189]

Turán, P.

[T10] Eine Extremalaufgabe aus der Graphentheorie. *Mat. Fiz. Lapok* **48** (1941) 436–452. [301, 302]

[T11] A note of welcome. *J. Graph Theory* **1** (1977) 7–9. [109]

Tutte, W. T.

[T12] On hamiltonian circuits. *J. London Math. Soc.* **2** (1946) 98–101. [198]

[T13] A short proof of the factor theorem for finite graphs. *Canad. J. Math.* **6** (1954) 347–352. [229]

[T14] A theorem on planar graphs. *Trans. Amer. Math. Soc.* **82** (1956) 99–116. [198]

Ulam, S. M.

See p. 62.

van Rooij, A.

[VW1] (with H. S. Wilf) The interchange graphs of a finite graph. *Acta Math. Acad. Sci. Hungar.* **16** (1965) 263–269. [266]

Vasak, J.

[V1] The thickness of the complete graph having $6m + 4$ points. Preprint. [115]

Vigué, F.

See [HDV1].

Vizing, V. G.

[V2] On an estimate of the chromatic class of a p-graph. *Diskret. Analiz.* 3 (1964) 25–30. [286]

[V3] Critical graphs with a given chromatic class. *Diskret. Analiz.* **5** (1965) 9–17. [290, 291, 292]

See also p. 361.

Wagner, K.

[W1] Bemerkungen zum Vierfarbenproblem. *Jber. Deutsch. Math. Verein.* **46** (1936) 21–22. [90]

[W2] Über eine Eigenschaft der ebene Komplexe. *Math. Ann.* **114** (1937) 570–590. [100]

Wall, C. E.

See [CKW1], [CW1], and [KPW1].

Weinstein, J. M.

[W3] On the number of disjoint edges in a graph. *Canad. J. Math.* **15** (1963) 106–111. [246]

Welsh, D. J. A.

[WP1] (with M. B. Powell) An upper bound for the chromatic number of a graph and its application to timetabling problems. *Computer J.* **10** (1967) 85–87. [278]

White, A. T.

*[W4] *Graphs, Groups and Surfaces,* Revised Edition. North-Holland, Amsterdam (1984). [111, 259]

[W5] The proof of the Heawood Conjecture. *Selected Topics in Graph Theory.* Academic Press, New York (1978) 51–81. [300]
See also [KRW1] and [NSW1].

Whitney, H.

[W6] Congruent graphs and the connectivity of graphs. *Amer. J. Math.* 54 (1932) 150–168. [152, 159, 254, 262]

Wilf, H. S.

See [SW1] and [VW1].

Williamson, J. E.

[W7] Panconnected graphs II. *Period. Math. Hungar.* 8 (1977) 105–116. [191]

Wilson, R. J.

See [CW2], [EW1], and [FW1].

Wilson, R. M.

[W8] Decompositions of complete graphs into subgraphs isomorphic to a given graph. *Proceedings of the Fifth British Combinatorial Conference* (1975) 647–659. [239]

Woodall, D. R.

[W9] Sufficient conditions for circuits in graphs. *Proc. London Math. Soc.* 24 (1972) 739–755. [201]

Xuong, N. H.

[X1] How to determine the maximum genus of a graph. *J. Combinatorial Theory* 26B (1979) 217–225. [143, 147]

Youngs, J. W. T.

[Y1] Minimal imbeddings and the genus of a graph. *J. Math. Mech.* 12 (1963) 303–315. [121, 130]

[Y2] The Heawood Map-Coloring conjecture. *Graph Theory and Theoretical Physics.* Academic Press, New York (1967) 313–354. [149]
See also [BHKY1] and [RY1].

Zaks, J.

[Z1] The maximum genus of cartesian products of graphs. *Canad. J. Math.* 26 (1974) 1025–1035. [149]

Zarankiewicz, K.

[Z2] On a problem of P. Turán concerning graphs. *Fund. Math.* 41 (1954) 137–145. [110]

See also p. 110.

Zykov, A. A.

[Z3] On some properties of linear complexes (Russian). *Mat. Sbornik* 24 (1949) 163–188. *Amer. Math. Soc. Translations* No. 79 (1952). [281]

Index

DATE DUE

ISBN 0-534-[illegible]-4