Polytopes, Graphs and Optimisation

This is the first book devoted to the combinatorial theory of polyhedra, an important branch of applied mathematics. The basis of the work is an examination of combinatorial, geometrical and algebraic properties of polyhedra in close connection with optimisation problems. There is a full account of the classic results (the Euler-Poincaré formula, Minkowski and Weyl duality theorems, Dehn-Sommerville equations) as well as an interesting presentation of new questions emerging from optimisation problems: polyhedral aspects of the theory of matroids and polymatroids, the structure of integer polyhedra from various combinatorial problems, minimax theorems of combinatorics and the connections between linear programming and combinatorial topology. Four basic problems of the combinatorial theory of polyhedra are isolated and examined in detail: (1) The classification and enumeration of polyhedra. (2) The study of the meaning of polyhedron vector functions, the components of which give the number of faces of relative size. (3) The determination of graphical characteristics of polyhedra. (4) Constructing convex hulls of discrete sets. There are a great number of challenging exercises provided throughout the text. This book will be an essential purchase for all those working in the areas of combinatorics, operations research and computer science with an interest in optimisation and linear programming.

Polytopes, Graphs and Optimisation

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INTRODUCTION

Polytopes have been studied since ancient times. The thirteenth book of Euclid is devoted to the five regular polytopes known as the platonic solids. In his work 'On Polyhedra', Archimedes described all semi-regular polyhedra (Archimedean solids).

The first result in the combinatorial theory of polytopes, and one of the classical results of mathematics, is the formula connecting the number of vertices, edges and faces of a three dimensional polytope obtained by Descartes and later, independently, by Euler in 1736. Poincaré gave the generalization of this result for convex polytopes of any dimension. This was a fundamental result of combinatorial topology.

The study of the figures formed by the vertices and edges of any three-dimensional polytope led to another discipline - Graph Theory.

Graph Theory and Combinatorial Topology established themselves as independent branches of mathematics and supplied the apparatus for studying the problems which arose at the end of the last century in the combinatorial theory of polytopes as a result of studying multidimensional parallelotopes (convex polytopes). The characterization of convex bodies by means of convex polytopes remains a basic research technique even today. The effectiveness of such an approach is due to the fact that polytopes are characterized by a finite set of data. We always understand a convex polytope to be the convex linear hull of a finite number of points in n-dimensional Euclidean space.

The work of Voronoi, Zolotarev, Korkin and Minkowski (1910) on the geometry of numbers led to the study of new classes of problems on polytopes, namely problems concerning the distribution of integral points in polytopes. Some of these problems arose from the geometrical investigations of the well-known Russian crystallographer Fyedorov. The classical theorems of Minkowski and Kronecker give criteria for the existence of an integral point in a convex body which is symmetrical relative to the origin of coordinates. The problem of the existence of an integral point in a polytope is equivalent to the existence of a solution to a system of linear diophantine inequalities.

The work of Alexandrov (1950) in the middle of the present century brought to completion the study of the metrical theory of polytopes initiated by Cauchy and was followed by the creation of the general theory of convex surfaces by Pogorelov (1969).

In the nineteen forties discrete mathematics emerged in the forefront of mathematical science as a foundation of cybernetics (Yablonski & Lupanov 1974 ; Yablonski 1979). This led to the establishment of combinatorial and discrete geometry as an independent scientific discipline which studied problems of finding the 'best' configuration of a finite system of points or geometrical figures (Hadwiger & Debrunner 1964 ; Soltan 1976 ; Boltyanski & Soltan 1978). The problems of combinatorial geometry are typically concerned with calculating the number of geometrical figures which can adopt a configuration which is acceptable for the problem. Most problems in combinatorial geometry involve convex bodies and in particular the properties of polytopes are used in the solution of many such problems. This stimulated research into the combinatorial and metrical properties of polytopes and their interelationships. As a result of this a new branch of the theory of convex polytopes, the combinatorial theory of polytopes, emerged in the forefront of study in the early nineteen fifties.

The combinatorial theory of polytopes is concerned with extremal properties of polytopes and studies the set of faces of all dimensions as a single complex.

The contemporary trends in the new problem areas of the combinatorial theory of polytopes were partly reflected in the monographs of Grünbaum (1967), McMullen & Shephard (1971), Bartels (1973) and in papers which were read at the Vancouver Mathematical Congress (Klee 1974 ; McMullen 1974).

However these monographs, as well as many survey papers (Grünbaum 1970), do not deal with many problems of current interest in the study of polytopes which arose at the end of the nineteen fifties under the mutual influence of two important areas of applied mathematics - the theory of systems of inequalities and optimization theory. We believe that a most important problem now is the solution of combinatorial problems which are presented in analytical form using systems of linear inequalities rather than in purely geometrical or topological form. Some of these problems have been solved completely - for instance, the problem of determining the maximum number of inequalities in an analytic description of a polytope with a specified number of vertices. Other problems are still under current study - for instance, the problem of determining the diameter of a polytope or of finding the 'best' simplex method in linear programming.

The basic objects of study in this book are combinatorial problems in the theory of linear inequalities with both real and integer variables and coefficients. General as well as special systems of linear inequalities are studied. Almost all of the results in this book which are given a geometrical interpretation can be reformulated in terms of the theory of linear inequalities.

The systematic study of polytopes as solution sets of finite systems of linear inequalities began at the end of the last century, although isolated properties of systems of linear inequalities can be found in earlier works due to Fourier (method of elimination), Ostrogradsky (a connection with analytical mechanics) and Farkas. However, the general problem of studying the geometrical properties of polytopes defined by finite systems of linear inequalities apparently only emerged after the work of Voronoi (1908a). In particular, Voronoi (1908b,1909) obtained a criterion for the consistency of a system of strict inequalities and for the dimension of the polytope defined by its feasible points. Subsequently many prominent mathematicians, such as Minkowski (1911) and Weyl (1935), were attracted to the study of systems of linear inequalities. Soviet mathematicians made a significant contribution to the development of this theory. In particular, mention should be made of Chernikov (1968), Yeremin (1976,1979) and Charin (1978).

A feature of the present work is that the study of the combinatorial properties of polytopes (sets of solutions to systems of linear inequalities) is closely interrelated with optimization problems which have important applications. Both the classical works of Kantorovich (1939) and Dantzig (1963) and the more recent works of Klee (1965) and Khachian (1980) reveal the role of combinatorial characteristics of feasible sets in constructing effective methods for solving linear programming problems. Thus, in presenting the material of this book we have emphasized the connection between the combinatorial and topological aspects of polytope theory and the analytical aspects and, ultimately, with the theory of linear and discrete programming.

A central problem in the combinatorial theory of polytopes is the enumeration and classification of polytopes with a given face structure. The combinatorial properties of polytopes can be characterized more precisely by means of the concept of combinatorial equivalence (isomorphism of polytopes).

Euler solved a number of enumeration problems for certain types of triangulated polytopes in the plane. Nevertheless, there are still important problems in polytope enumeration which remain unsolved. The greatest efforts have been directed to the enumeration of 3-dimensional polytopes with a given number of vertices for this is the case with the most numerous applications. The enumeration of 3-dimensional polytopes was studied by Kirkman(1855), Steiner (1881), Brückner (1900) and others. By a theorem of Steinitz (§2.1), this problem is equivalent to the problem of counting the number of triply-connected planar graphs. But even in this case the problem has not been completely solved (Tutte 1973). At the present time only the d-dimensional polytopes whose number of vertices does not exceed $d + 3$ have been enumerated (§3.2), where d is arbitrary.

The problem of enumerating and classifying the combinatorial types of polytopes given in analytical form is studied here for the first time. When such a form is used, the use of the traditional apparatus of marked face-complexes of polytopes leads to a number of difficulties. To overcome them we introduce a new technique of enumerating and classifying polytopes by means of semi-matroids of polytopes which yields information about the incidence relations between vertices and faces of maximal dimension (facets). By using this technique we obtain criteria for the combinatorial equivalence of marked polytopes (§3.1). The use of semi-matroids of polytopes enables us to establish the combinatorial type of the polytope of a set of constraints in many important applied problems such as the standardization problem and extremal problems on permutations (Chapter 5). Another fundamental concept which is useful for identifying the combinatorial type of a marked polytope is that of the spectrum of a pair of polytopes (§3.1). This idea was particularly useful in determining the various combinatorial properties of transportation polytopes (§§6.6-6.9).

Since the time of Euler, a second important problem in combinatorial polytope theory has been the determination of the range of values of the vector function $f(M)$, where the components of the vector f are the numbers of faces of the polytope M of each dimension. As we have already noted, the Euler-Poincaré formula was the first result which showed that the f-vectors of all polytopes of a given dimension lay in a particular hyperplane. It was later shown that there are no linear relations, other than the Euler-Poincaré formula, which are satisfied by the components of all f-vectors of polytopes.

Attempts were made to find non-linear relations or to find linear relations which were satisfied for special subclasses of polytopes. Of these the best known are the Dehn-Sommerville equations for simplicial polytopes (§1.5).

The study of polytopes was greatly stimulated by two conjectures about the maximum and minimum numbers of faces in the class of all d-dimensional polytopes with a fixed number of vertices which were proposed in about 1957. Both of these conjectures gave rise to an extensive literature (for a survey see Grünbaum 1970). The first of these was completely solved by McMullen in 1970, while the second was partially solved by Barnette in 1971 (for the case of simplicial polytopes) (§§3.3,3.4) The range of values of f-vectors for special classes of polytopes is still being studied. It should be noted that the range of the function $f(M)$ is only completely known for d-dimensional polytopes whose number of vertices does not exceed $d + 3$ and also for certain special combinatorial types of polytopes : simplexes, prisms, pyramids, etc. The transportation polytopes, which have important applications, are studied closely in this book. In particular, the so-called classical transportation polytopes (Chapter 6) are classified according to their numbers of faces : we distinguish classes with extremal values of the f-vector and we also find criteria for a transportation polytope with a fixed number of faces to belong to the class of polytopes having minimum or maximum numbers of vertices. Using these criteria we are able to solve a series of well-known problems and conjectures in the combinatorial theory of transportation polytopes. Some of the results obtained for the classical transportation polytopes can be extended to the case of multi-indexed transportation polytopes (planar and axial). Transportation polytopes with additional constraints and with bounded flow conditions are studied separately.

A third problem is the study of properties of the graphs (1-skeletons) of polytopes (Chapter 2). The theorem of Steinitz and Balinski are fundamental here. The first of these states that a graph is a 1-skeleton of a 3-dimensional polytope if and only if it is planar and 3-connected, while the second states that the graph of a d-dimensional polytope is d-connected.

The most interesting graph-theoretic characteristics of polytopes are the diameter, radius and height of a polytope. The diameter $D(M)$ of a polytope is the smallest integer k such that there is a chain of length not greater than k joining any pair of its vertices. Let $\Delta(d,n)$ denote the maximal diameter in the class of d-dimensional polytopes with n facets. It has been conjectured that $\Delta(d,n) \leq n-d$; this is the *maximal diameter conjecture.* It has not been proved in the general case. The following bounds are known for $\Delta(d,n)$:

$$[(n-d)-(n-d)/[5d/4]] + 1 \leq \Delta(d,n) \leq 2^{d-3}n.$$

This shows how little is still known about the maximal diameter. It has been shown that it suffices to verify the conjecture for the case $n=2d$. There are a number of special results which evaluate $\max D(M)$ for the case where M is restricted to lie in certain special classes of polytopes such as the bi-stochastic matrix polytopes (§5.1), the travelling salesman polytopes (§5.2), the standardization polytopes (§5.5) and the permutation polytopes (§5.3). This raises the important question of isolating new classes of polytopes for which either the maximum diameter conjecture can be verified or which yield lower estimates of $\Delta(d,n)$. Significant progress has been made in studying the maximum diameter conjecture in the case of the transportation polytopes (§6.4).

The converse problem of characterizing the set of polytopes with a given diameter or radius is also of considerable interest. This problem has only been completely solved in the case of polytopes whose radius or diameter is equal to two.

We note that there is a close connection between the metrical properties of the graph of a polytope and the estimation of the number of iterations and the effectiveness of simplex-type algorithms for solving linear-programming problems. If it is required to extremize a linear function on a polytope M with n facets, then the maximum number of vertices in the class of polytopes with n facets is an upper bound on the number of iterations. The diameter and the radius of a polytope give the maximum number of iterations for the 'best' simplex algorithm using the worst and the best starting points respectively. The most accurate characterization of the effectiveness of simplex algoritms is given by the 'height' of a polytope. The height $\eta(M)$ of a polytope M is defined to be the length (number of edges) of the longest chain in the graph $G(M)$

such that there is a linear function which is strictly monotonic along it. Thus the height of a polytope can be interpreted as the exact number of iterations required by the worst simplex algorithm using the worst initial vertex. Klee and Minty showed that

$$\alpha n^{[d/2]} < \max \eta(M) < \beta n^{[d/2]} ,$$

where the maximum is taken over all d-dimensional polytopes with n facets and α, β are constants depending on d. In particular it is shown that $\eta(M) \geqq 2^d - 1$ and an example is given of a linear programme for which this bound is attained.

The classical theorem of Weyl and Minkowski asserts that a set $M \subset E_n$ is a polytope if and only if it is bounded and is the intersection of a finite number of closed half-spaces. A minimal family of closed half-spaces whose intersection is M is determined by the set of hyperplanes which are the affine hulls of the facets of M. The Weyl-Minkowski theorem implies that there are two ways of specifying a polytope; the first, as the convex hull of a finite set of its points (a parametric representation), and the second, as the solution set of a finite system of inequalities (an analytic representation). The smallest set of points whose convex hull is M is precisely the set of vertices of M, and an irreducible system of inequalities determining M corresponds to the facets of M.

A fourth problem area in the combinatorial theory of polytopes is concerned with finding an effective way of passing from one type of polytope specification to another. To pass from an analytic specification of a polytope to a parametric specification it is necessary to find all the vertices of the polytope. In some cases this can be done explicitly, but more often only certain properties of the verticies are studied. A particularly important case is where it can be estblished that all vertices of a polytope have integer coordinates ; such a polytope is called *integral*. Integral polytopes play a fundamental rôle in integer programming. The problem of describing all systems of linear inequalities which determine integral polytopes is unsolved. However a deep connection has already been revealed between integral polytopes and many important problems of graph theory and hypergraph theory, such as the strong conjecture of Berge concerning perfect graphs (§4.5). Any result to do with integral polytopes automatically implies a series of results in graph theory. Thus in

Chapter 4 almost all of the important theorems about coverings and matchings in graphs, such as the theorems of König, Whitney, Menger, Gale and others, are derived from properties of integral polytopes. Many well-knowm theorems about matroids and polymatroids are also derived from the integrality properties of the corresponding polytopes. The concept of α-modular matrices, introduced in Chapter 4, enables us to extend known classes of integral polytopes. In Chapter 4 we also systematically study classes of polytopes some of whose integral vertices have properties which enable us to solve integer programming problems using simplex-type algorithms. Among such polytopes are those encountered in such important applied problems as the p-median problem, the problem of packing the edges of a hypergraph and the location problem.

The transformation of a parametric specification of a polytope into an analytic specification has great significance for problems of discrete optimization, for it enables us to formulate them as a linear programming problem. To do this it is necessary to describe all the facets of a polytope. For most discrete optimization problems an explicit representation of all the facets has not yet been found. The most interesting results have been obtained for the polytopes which occur in the packing problem, the maximum matching problem in a graph, the travelling salesmen problem and the knapsack problem (these are considered in Chap.4). Of great theoretical interest is the problem of describing analytically the convex hull of the integral solutions of a system of linear inequalities, that is, of the integral points in a polytope. Hilbert's theorem on the finite basis of a ring of polynomials shows that such a description should be possible in principle but efficient means of finding such a description have not yet been obtained. In §4.1 we present a method of constructing analytical and parametric specifications of the integral points of a polytope based on the determination of generating sets of semigroups.

In Chapter 4, besides considering general approaches to the construction of convex hulls we also use the specialized theorems of Birkhoff and Rado on permutation matrices to obtain an analytical specification of polytopes whose vertices have components which are permutations of a given vector. Such polytopes arise in the theory of scheduling.

Another feature of the book lies in the connections established between polytopes and combinatorial analysis (see Todd 1976). In particular the relations between multi-indexed assignment problems and orthogonal systems of latin cubes are studied. Similarly, properties of polytopes are related to finite geometries (§8.3)

Thus in this book an extensive body of work on the combinatorial properties of the feasible sets of a variety of optimization problems is systematized and presented from a unified viewpoint.

Each chapter ends with a list of auxiliary and more specialized results. They are formulated as exercises whose solutions can mostly be found in the reference literature. Among the problems are some (indicated by a star) whose solution is not known to the authors.

The book ends with a list of unsolved problems and conjectures. Some of these are well known but most of them are presented for the first time.

1 CONVEX POLYTOPES

§1 CONVEX SETS

The purpose of this section is to recall certain properties of convex sets and also to enable the reader to appreciate the place of convex polytopes in the context of convex sets in general. For proofs of the classical results given here the reader is referred to Fenchel 1953, Maltsev 1970, Rockafellar 1970, Stoer & Witzgall 1970, Karmanov 1975 or Pshenichny 1980.

1.1 Affine Sets

A subset A of the real d-dimensional Euclidean space E_d is called an *affine set* if it contains the line passing through any two of its distinct points, i.e. if $x,y \varepsilon A$ then $\lambda x + (1-\lambda)y \; \varepsilon \; A$ for all $\lambda \varepsilon E_1$. Affine sets which contain the origin (denoted by 0) are linear spaces.

The mapping $\alpha: E_d \to E_k$ defined by the rule

$$\alpha(x) = Ax + a \qquad , \; x \varepsilon E_d$$

where A is a (kxd)-matrix and $a \varepsilon E_k$, is called an *affine mapping*. If A is a nonsingular matrix the map α is called a *nonsingular mapping*, otherwise it is called *singular*.

If $A \subset E_d$ and $a \varepsilon E_d$, the set $A+a = \{x+a : x \varepsilon A\}$ is called a *translation of the set* A *through the vector* a. Two affine sets are called *parallel sets* if one of the sets is a translation of the other set or of one of its subsets through some vector. Two sets A and A' are called *affinely equivalent* if a nonsingular affine mapping α exists such that $\alpha(A) = A'$. Every non-empty affine set is parallel to a unique subspace, namely

$$L = \{x-a : x \varepsilon A\} \qquad , \; a \varepsilon A.$$

A linear combination $\sum_{i=1}^{n} \lambda_i x^i$ of points $x^1,..,x^n$ in E_d is called an *affine combination* if $\sum_{i=1}^{n} \lambda_i = 1$ where $\lambda_i \varepsilon E_1$. A finite set of points is called *linearly (affinely) independent* if none of its points can be expressed as a linear (affine) combination of the others. It is clear that the set of points $\{x^1,..,x^n\}$ is linearly (affinely) dependent if the origin can be expressed in the form $\sum_{i=1}^{n} \lambda_i x^i$ with some $\lambda_i \neq 0$ (and with $\sum_{i=1}^{n} \lambda_i = 0$ in the case of affine dependence). The maximum number of linearly independent points in the set $\{x^1,..,x^n\}$, where $x^j = (x_{1j},..,x_{dj}) \varepsilon E_d$, is equal to the rank of the (dxn)-matrix (x_{ij}). The maximum number of affinely independent points equals the rank of the ((d+1)xn)-matrix whose columns consist of the vectors $\bar{x}^j = (x_{1j},..,x_{dj},1)$. The maximum number of linearly independent points in E_d is d while the maximum number of affinely independent points in E_d is $d + 1$.

A linear space (affine set) is called d-*dimensional* if the maximum number of linearly independent (affinely independent) points in it is d (or $d+1$). The dimension of a set A is denoted by dim A. It is clear that the dimension of an affine set A equals the dimension of the linear space whose translation yields A The empty set is defined to have dimension -1. Affine sets of dimension 0,1 and 2 correspond to points, lines and planes respectively.

Let S be an arbitrary nonempty subset of E_d. Then the set of all affine combinations of points taken from S is an affine set called the *affine hull* of S and denoted by aff S. Clearly, if S is an affine set then aff S = S.

The following theorem shows that in constructing affine hulls it is not necessary to take affine combinations of all possible subsets. It suffices to consider only certain subsets called *generating sets*.

Theorem 1.1 *An affine* k-*dimensional set* $A \subseteq E_d$ *is the affine hull of any subset* $S \subseteq A$ *consisting of* k+1 *affinely independent points and conversely.*

An affinely independent set of points S which generate the set A, as in Theorem 1.1, is called an *affine basis* of A.

Given any two affine bases there is a unique nonsingular affine mapping which maps one basis onto the other.

A $(d-1)$-dimensional affine set in E_d is called a *hyperplane.* Every hyperplane $H \subset E_d$ may be represented by an equation of the type $ax = \beta$, where $a \in E_d$, $a \neq 0$, $\beta \in E_1$; the vector a is called the *normal vector* to the hyperplane H. Hyperplanes are called *linearly independent hyperplanes* if their normal vectors are linearly independent.

Theorem 1.2 *Every affine set of dimension* $d-k$ *in* E_d *may be represented as the intersection of* k *linearly independent hyperplanes. Conversely, the intersection of* k *linearly independent hyperplanes in* E_d *is an affine set of dimension* $d-k$.

Let the vector $b \in E_m$ and the $(m \times d)$-matrix A of rank m be given. Then, the non-empty solution set of the system of linear equations

$$Ax = b \tag{1.1}$$

is a $(d-m)$-dimensional affine set in E_d.

1.2 Convex Sets

The *line segment* joining the points $x, y \in E_d$ consists of the set of points $\lambda x + (1-\lambda)y$, where λ ranges over all real numbers between 0 and 1 inclusive. We will denote such a line segment by $[x,y]$.

The set $W \subseteq E_d$ is called a *convex set* if it contains the line segment joining any two of its points.

The following are examples of convex sets.

1 Affine sets.

2 The *ray* $\{x \in E_d : x = a + bt, t \geq 0\}$ with endpoint $a \in E_d$ and direction $b \neq 0$.

3 The *closed half-spaces* $H^+ = \{x \in E_d : ax \geq \beta\}$ and $H^- = \{x \in E_d : ax \leq \beta\}$, defined by the hyperplane $H = \{x \in E_d : ax = \beta\}$.

4 The *closed sphere* $S(a,r) = \{x\in E_d : \|x-a\| \leq r\}$ with centre at the point a and radius $r \geq 0$.

Since the intersection of any number of convex sets is a convex set, the set of solution points of any (finite or infinite) system of linear inequalities $a_i x \leq \beta_i$, $i=1,2,3,\ldots$ is either convex or empty (if the system is inconsistent).

The solution set of a finite system of linear inequalities is called a *polyhedron*.

Two convex sets $W_1, W_2 \subset E_d$ are called *separable* if a hyperplane $H \subset E_d$ exists such that W_1 lies in one and W_2 lies in the other of the two closed halfspaces defined by H ; the hyperplane H is called a *separating hyperplane*. Further, the two convex sets $W_1, W_2 \subset E_d$ are called *strongly separable* if a separating hyperplane H exists such that W_1 and W_2 are contained in the corresponding open half-spaces. If W_1 is contained in an open half-space corresponding to the hyperplane H and W_2 is contained in the other half-space (possibly closed), then we say that the *hyperplane* H *strictly separates the set* W_1 *from the set* W_2.

We will formulate an important assertion about convex sets. This assertion plays a fundamental rôle in the proofs of many of the basic facts in the theory of convex sets.

We recall that a set is called *bounded* if it is contained within some sphere.

<u>Theorem 1.3</u> (Separation Theorem). *Let* W_1, W_2 *be any two closed convex sets in* E_d *with no points in common and such that at least one of them is bounded. Then* W_1 *and* W_2 *are strongly separable.*

<u>Corollary 1.4</u> *Let* W_1 *and* W_2 *be arbitrary convex sets with no points in common, then they are separable.*

The Separation Theorem has important corollaries known as supporting hyperplane theorems.

Let W be a non-empty set in E_d. The hyperplane H is called a *supporting hyperplane* to the set W if H has at least one point in common with W and if W is contained in one of the two closed half-spaces H^+ and H^- defined by H. The half-space containing W is called a *supporting half-space* to W.

<u>Corollary 1.5</u> *For every closed bounded convex set* $W \subset E_d$ *there exists a supporting hyperplane to* W *with any given normal vector.*

A *projection of a point* x *on the convex set* W is a point x' at which the infimum, $\inf_{y \in W} \|x-y\|$, is attained. Such a point is unique.

<u>Corollary 1.6</u> *Let* W *be a closed bounded convex set and let* x *be a point,* $x \notin W$. *Then there exists a supporting hyperplane* H *to* W *which strictly separates* x *from* W *and for which* $x' \in H \cap W$, *where* x' *is the projection of the point* x *on the set* W.

<u>Theorem 1.7</u> *Every closed convex set* $W \neq E_d$ *can be represented as the intersection of a family of closed half-spaces. To define the family it suffices to take all the supporting half-spaces to* W.

Thus any closed bounded convex set in E_d can be defined by means of a (possibly infinite) system of linear inequalities.

The *dimension of a convex set* $W \subseteq E_d$ is defined to be the dimension of its affine hull. The set of all interior points of a set $W \subseteq E_d$ is denoted by int W and is called the *interior of the set* W. Clearly, if a convex set W has a non-empty interior in E_d then the dimension of W equals d. If W has dimension less than d then W does not have any interior points in E_d. But, relative to its affine hull aff W, a convex set W does have interior points. The interior of a set $W \subset E_d$ relative to its affine hull, whose dimension is less than d, is denoted by rel int W. If a set $W \subset E_d$ is convex, then its closure $\overline{W}$ and its relative interior rel int W are also convex sets. The set $\overline{W} \setminus$ rel int W, i.e. the boundary of the convex set W, is denoted by rel bd W. If $\dim W = d$, then rel bd W is the boundary of the set W in E_d and is written bd W.

The linear combination $\sum_{i=1}^{m} \lambda_i x^i$ of points $x^1,\ldots,x^m \in E_d$ is called *convex* if $\lambda_i \geqq 0$, $\sum_{i=1}^{m} \lambda_i = 1$. Let S be a non-empty set in E_d. The set of all convex combinations of points taken from S is a convex set called the *convex hull* of S, denoted by conv S. The convex hull conv S of a set $S \subseteq E_d$ is the smallest convex set containing S. A set W is convex if and only if $W = \text{conv } W$.

The following classical result, due to Carathéodory (1907,1911) shows that in constructing the convex hull of $S \subset E_d$ it is not necessary to take combinations of more than $d + 1$ points.

Theorem 1.8 *The convex hull of a set* $S \subseteq E_d$ *is the union of all convex combinations of all subsets of* S *containing no more than* $d + 1$ *points.*

A point x of a convex set W is called an *extreme point* if it is not an interior point of any line segment with distinct endpoints in W.

Theorem 1.9 *A non-empty closed bounded convex set in* E_d *possesses extreme points and is the convex hull of the set of its extreme points.*

1.3 Convex Cones

A subset $K \subseteq E_d$ is called a *cone* if $\lambda x \in K$ for all $x \in K$ and $\lambda \geqq 0$. A *convex cone*, is a cone which is also a convex set. Thus every linear space in E_d is also a convex cone. A half-space in E_d defined by a hyperplane passing through the origin is also a cone. Since any intersection of convex cones is also a convex cone, the set of solutions of a finite system of homogeneous linear inequalities is also a convex cone called a *polyhedral cone*.

A point $x = \sum_{i=1}^{m} \lambda_i x^i$, $\lambda_i \geqq 0$, $i = 1,\ldots,m$, is called a *conical combination* of the points $x^1,\ldots,x^m$.

Let S be a non-empty set in E_d. The set con S of conical combinations of all subsets of S is a convex cone called the *cone generated by* S. A convex cone generated by a finite set of vectors is called *multifaceted*. A convex cone is called a *pointed cone* if it does not contain any non-null sub-spaces. A pointed cone does not contain any lines. A pointed multifaceted cone has a unique (to within positive scalar multiples) generating set whose elements are called the *basis of the cone.*

The following fundamental result in the theory of convex cones and the theory of linear inequalities is due to Weyl (1935).

<u>Theorem 1.10</u> *A convex cone* K *is polyhedral if and only if it is multifaceted.*

§2 <u>CONVEX POLYTOPES</u>

In this section we list some elementary facts about polytopes (Grünbaum 1967, McMullen & Shephard 1971) and present the classical Weyl-Minkowski theorem.

<u>Definition 2.1</u> The convex hull of a finite set of points V in E_d is called the *convex polytope* generated by the points of V.

Since, in the sequel, we will be considering only convex polytopes and cones, we will omit the word 'convex' from now on.

2.1 <u>Vertices</u>

Let H be a supporting hyperplane of the polytope M.

<u>Definition 2.2</u> The set $F = M \cap H$ is a *face* of the polytope M, generated by H. If $\dim F = i$ then F is an *i-face* of the polytope M. The 0-faces are called the *vertices* of M. The set of all vertices of M is denoted by vert M. 1-faces are called *edges* of M. The empty set $\emptyset$ and M itself are called *improper faces*; all other faces are *proper faces* of M. If $\dim M = d$ then (d-1)-faces of M are called *facets* of M. They are proper faces of maximal dimension.

<u>Theorem 2.1</u> *A polytope has a finite number of distinct faces and each of its faces is itself a polytope.*

Proof. Suppose that the polytope $M = \text{conv}\ V$, where $V = \{x^1,\ldots,x^n\}$.

Let $H = \{x \in E_d \ : \ ax = \beta\}$ be a supporting hyperplane to M generating the face F. For simplicity let $H \cap V = \{x^1, \ldots, x^s\}$.

We show that F is a polytope. To this end, we show that $F = \text{conv}\,(x^1, \ldots, x^s)$. For any point $x \in M$ we have

$$x = \sum_{i=1}^{n} \lambda_i x^i \quad , \quad \sum_{i=1}^{n} \lambda_i = 1 \quad , \quad \lambda_i \geqq 0 \ \forall i \in N_n \ .$$

Thus

$$ax = \sum_{i=1}^{n} \lambda_i a x^i = \beta \sum_{i=1}^{n} \lambda_i + \sum_{i=1}^{n} \lambda_i \delta_i = \beta + \sum_{i=1}^{n} \lambda_i \delta_i \ ,$$

where $\delta_i = ax^i - \beta$. Let $M \subset H^+$. Then $x^i \in H^+$, $\forall i \in N_n$. Clearly $x^i \notin H$ for $i = s+1, \ldots, n$. Thus $\delta_i = 0 \ \forall i \in N_s$, while $\delta_i > 0$, $i = s+1, \ldots, n$. But $x \in F$ if and only if $ax = \beta$. This last equation is possible if and only if $\lambda_i = 0$ for $i = s+1, \ldots, n$. Hence $F = \text{conv}\,(x^1, \ldots, x^s)$ i.e. F is a polytope. The set V is finite, and since every face is generated by a subset of V we see that the number of faces is finite. //

The following theorem may be proved similarly.

Theorem 2.2 *A polytope is the convex hull of its vertices.*

Corollary 2.3 *Every face F of a polytope is the convex hull of its vertices*, i.e. $F = \text{conv vert } F$.

Corollary 2.4 *The vertices of a polytope are its only extreme points.*

Theorem 2.2 and Corollary 2.4 allow us to give an equivalent definition of the vertices of a polytope.

Definition 2.3 A point of a polytope M is a *vertex* if it cannot be represented as a convex combination of any other two distinct points of M.

2.2 The Weyl-Minkowski Theorem

Theorem 2.5 (Minkowski 1897, Weyl 1935) *The set* M *is a polytope if and only if* M *is a bounded polyhedron.*

Proof. (i) Let M be a polytope. Without loss of generality suppose that M is a d-polytope in E_d. This prevents us from having to consider rel int M. Let $\{F_1,\dots,F_s\}$ be the set of all facets of M and let $H_1,\dots,H_s$ be the supporting hyperplanes to M generating the facets $F_1,\dots,F_s$. Let $H_1^+,\dots,H_s^+$ be the supporting half-spaces to M corresponding to the hyperplanes $H_1,\dots,H_s$. We show that

$$M = \bigcap_{i=1}^{s} H_i^+ \ . \tag{2.1}$$

It is clear that M is contained in the intersection of the half-spaces H_i^+, $i = 1,\dots,s$.

We establish the reverse inclusion. Suppose this inclusion is false, i.e. suppose there is a point $x \in \bigcap_{i=1}^{s} H_i^+$ such that $x \notin M$. Consider the affine hulls A_ω of each $(d-1)$-subset ω of the set vert M and of the point x. Let $A = \bigcup_\omega A_\omega$. Since $\dim M = d$ and since $\dim A_\omega \leqq d-1$, there exists a point y such that $y \in \operatorname{int} M$ and $y \notin A$. Since $x \notin M$ there is a unique point z in the intersection of the segment $[x,y]$ with bd M. We show that z belongs to some facet F_i. Indeed, if z belongs to a j-face of smaller dimension, then Theorems 2.1 and 2.2 imply $z \in \operatorname{conv}(x^1,\dots,x^s)$ where $x^1,\dots,x^s$ are certain vertices of this face, and where, by Carathéodory's Theorem (Th. 1.8) $s \leqq j + 1$ and so $s \leqq d - 1$. Thus, $z \in A$. But, by the way the set A was constructed, $x \in A$ and hence the entire segment $[x,y] \in A$. But this contradicts the choice of the point y. Hence, z belongs to some facet F_i, but then $z \in H_i$. Since $y \in \operatorname{int} M \subset H_i^+$, we have $x \notin H_i^+$. This contradiction shows that $x \in M$ and hence equation (2.1) holds.

(ii) Let $M = \bigcap_{i=1}^{s} H_i^+$ be a bounded polyhedron, where $H_1^+, \ldots, H_s^+$ are closed half-spaces. Without loss of generality assume that $\dim M = d$ and that there are no redundant H_i^+ among the half-spaces.

Let $F_i = M \cap H_i$. Then

$$F_i = \left(\bigcap_{j=1}^{s} H_j^+ \right) \cap H_i = \bigcap_{j \neq i} \left(H_j^+ \cap H_i \right) . \tag{2.2}$$

Since M is a bounded set and because of (2.2), F_i is a bounded polyhedron for each i.

The proof uses induction on d. A bounded polyhedron $M \subset E_1$ is obviously either a point or a line segment. In the first case $M = \{x\} = \text{conv } \{x\}$, while in the second case $M = [x^1, x^2] = \text{conv } \{x^1, x^2\}$. Suppose the theorem is true for the space E_{d-1}. Since $\dim F_i \leqq d-1$, the inductive assumption shows that F_i is a polytope. By Th. 2.2 $F_i = \text{conv vert } F_i$. Let $V = \bigcup_{i=1}^{s} \text{vert } F_i$. Since $V \subseteq M$ and since M is convex, $\text{conv } V \subseteq M$.

We establish the reverse inclusion. Let $x \in M$. Suppose first that $x \in \text{bd } M$. Every point in $\text{bd } M$ lies in the boundary of one of the H_i^+. It is clear that $\text{bd } M = \bigcup_{i=1}^{s} F_i$. Thus $x \in F_i$ for some i, $1 \leqq i \leqq s$. By the inductive assumption $x \in \text{conv vert } F_i$ and hence $x \in \text{conv } V$. Next suppose that $x \in \text{int } M$. Consider a line passing through x. This line intersects $\text{bd } M$ (M is a bounded convex set) in two points x' and x''. By the above $x', x'' \in \text{conv } V$ and hence $x \in \text{conv } (x', x'')$ also belongs to the set $\text{conv } V$. Thus, $\text{conv } V = M$. //

<u>Corollary 2.6</u> *Every* d-*polytope in* E_d *with* m *facets is an intersection of* m *closed half-spaces.*

<u>Corollary 2.7</u> *Let* M *be a polytope in* E_d *and let* A *be an affine set in* E_d. *Then* $A \cap M$ *is also a polytope.*

2.3 Faces

Proposition 2.8 *Let* M_1, M_2 *be polytopes such that* $M_2 \subset M_1$. *If* F *is a face of the polytope* M_1 , *then* $F \cap M_2$ *is a face of the polytope* M_2 *(possibly improper).*

Proof. The proposition is obvious if F is an improper face of M_1 . Otherwise, let H be the supporting hyperplane of the polytope M_1 which generates the face F . Then either $M_2 \cap H = \emptyset$, or H is a supporting hyperplane to M_2 . In the first case $H \cap M_2 = \emptyset$ is an improper face of M_2 . In the second case $H \cap M_2 = F \cap M_2$ is a proper face of M_2 . //

Theorem 2.9 *Let* F_1, F_2 *be faces of the polytope* M, $F_2 \subset F_1$. *Then* F_2 *is a face of the polytope* F_1 . *Conversely, if* F_1 *is a face of the polytope* M *and* F_2 *is a face of the polytope* F_1 , *then* F_2 *is a face of the polytope* M .

Proof. The first part of the theorem follows directly from Proposition 2.8 . We prove the second part. Without loss of generality suppose $0 \in F_2$ and that M is a d-polytope in E_d . Let a_1 be the normal vector to the hyperplane H_1 generating the face F_1 , with $M \subset H_1^+$. Let the vector $a_2 \in H_1$ be such that $F_1 \subset \{x \in H_1 : a_2 x \geqq 0\}$ and $F_2 = F_1 \cap H_2$, where $H_2 = \{x \in H_1 : a_2 x = 0\}$ is an affine set of dimension $d-2$. Let $H_\delta = \{x \in E_d : (a_1 + \delta a_2)x = 0\}$. Then $H_\delta \supset H_2 \supset F_2$ for all δ . Let $\alpha = \max\{|a_2 x| : x \in \text{vert } M \setminus \text{vert } F_1\}$, $\beta = \min\{a_1 x : x \in \text{vert } M \setminus \text{vert } F_1, a_1 x > 0\}$. We show that if $0 < \delta < \beta/2\alpha$ ($0 < \delta$ if $\alpha = 0$), then H_δ is a supporting hyperplane to M and $F_2 = M \cap H_\delta$. Indeed, if $x \in \text{vert } M \setminus \text{vert } F_1$, then $(a_1 + \delta a_2)x \geqq \beta - \delta\alpha > \beta/2 > 0$, and if $x \in \text{vert } F_1 \setminus \text{vert } F_2$, then $(a_1 + \delta a_2)x = \delta a_2 x > 0$; finally, if $x \in \text{vert } F_2$, then $(a_1 + \delta a_2)x = 0$, i.e. $x \in H_\delta$. //

Theorem 2.10 *Let* $F_1,\dots,F_s$ *be a family of faces of a polytope* M . *Then* $F = \bigcap_{i=1}^{s} F_i$ *is also a face of* M *(possibly improper).*

Proof. If $F = \emptyset$ or $s = 1$ the theorem is obvious. Let $F \neq \emptyset$ and $s \neq 1$. Without loss of generality suppose that the F_i are proper faces of M and that the coordinate system is chosen so that $0 \in F$. Let $H_i = \{x \in E_d : a_i x = 0\}$ be the supporting hyperplane of M generating the face F_i with $M \subset H_i^+$ $i=1,\dots,s$. Let $H = \{x \in E_d : ax = 0\}$ where $a = \sum_{i=1}^{s} a_i$. Then $M \subset H^+$ and, since $0 \in H \cap M$, H is a supporting hyperplane of the polytope M . It remains to show that $F = M \cap H$. For every $x \in F$ we have $a_i x = 0$, $i=1,\dots,s$, so $x \in M \cap H$. Hence $F \subseteq M \cap H$. On the other hand, if $x \in M \setminus F$ then $a_i x > 0$ for at least one i , so $ax > 0$. Hence $x \notin M \cap H$ and therefore $M \cap H \subseteq F$. //

Corollary 2.11 *Every* (d-2)- *face of a* d-*polytope is the intersection of two of its* (d-1)-*faces.*

Theorem 2.12 *Let* F^j *be a proper* j-*face of the* d-*polytope* M *and let* $j \leq k \leq d-1$. *Then* F^j *is the intersection of at least* $k-j+1$ k-*faces of* M *which contain* F^j .

Proof. We show first that there exist j-,(j+1)-,...,(d-1)-faces F^j, $F^{j+1},\dots,F^{d-1}$ of M satisfying the inclusions $F^j \subset F^{j+1} \subset \dots \subset F^{d-1}$. To do this we note that if F^j is a proper face of M then F^j is also a face of some (d-1)-face. For, take a point $x \in \operatorname{rel\,int} F^j$. Then $x \in \operatorname{bd} M$ since $F^j \subset \operatorname{bd} M$. But in the proof of Theorem 2.5 we saw that $\operatorname{bd} M$ is the union of all (d-1)-faces of the polytope M. Hence there is a (d-1)-face F^{d-1} which contains x . Let H be the supporting hyperplane of M which generates F^{d-1}. Since $x \in \operatorname{rel\,int} F^{d-1} \cap H$ and H is a supporting hyperplane of M , it follows that $F^j \subset H$. Hence $F^j \subset F^{d-1}$ and by Theorem 2.9 F^j is a face of the polytope F^{d-1} . Continuing by induction we deduce that F^j is a face of some face F^{d-2} of the polytope F^{d-1} and so on .

Let F^{k+1} be a (k+1)-face containing F^j (if $k=d-1$ then $F^{k+1}=M$). Then every (k-1)-face, including F^{k-1}, of the polytope F^{k+1} is the intersection of two of its k-faces (Cor.2.11). Every (k-2)-face, including F^{k-2} , is in turn an intersection of two (k-1)-faces of F^{k+1} and so on. Eventually we find that the face F^j is the intersection of not less than $k-j+1$ k-faces of F^{k+1} , which by Theorem 2.10, are faces of M . //

<u>Corollary 2.13</u> *Every* j-*face of a* d-*polytope is the intersection of not less than* $d-j$ *of its facets.*

<u>Proposition 2.14</u> *Let* F *be a* j-*face of a* d-*polytope* M . *Then there exists a* $(d-j-1)$-*face* F' *of* M *such that*
$\dim \operatorname{conv} (F \cup F') = d$.

Proof. First note that from the definition of dimensionality it follows that $F \cap F' = \emptyset$. If F is a (d-1)-face of M then for F' we select a vertex not belonging to F . If $\dim F = j \leq d-2$, then F is contained in some (d-1)-face G of M and by the inductive assumption there exists a (d-j-2)-face G' of G such that $\dim(F \cup G') = d-1$. Let F' be a (d-j-1)-face of M which contains G' but which is not contained in G . The existence of such a face is obvious, since G' is contained in some (d-1)-face of M , distinct from G . Then F' is the required face of M since $\dim \operatorname{conv} (F \cup F') > \dim \operatorname{conv} (F \cup G') = d - 1$. //

<u>Proposition 2.15</u> *Let* M *be a polytope and let* $W \subseteq V = \operatorname{vert} M$. *Then* $\operatorname{conv} W$ *is a face of* M *if and only if*

$$(\operatorname{aff} W) \cap \operatorname{conv} (V \setminus W) = \emptyset .$$

The proof of this simple proposition is left to the reader.

2.4 Examples of Polytopes

The simplest type of polytopes are the simplexes.

Definition 2.4 The convex hull of an affinely independent set of points is a *simplex* . A d-simplex is denoted by T_d .

Every face of a polytope is the convex hull of some subset of its vertices. Since a subset of an affinely independent set is also affinely independent, it follows that every face of a simplex is also a simplex of some dimension. Let $T_d \subset E_d$. Then every d-subset W of vert T_d determines a hyperplane in E_d which obviously supports T_d . Therefore conv W is a facet of T_d . Since, by Theorem 2.9 , every face of the simplex conv W is a face of T_d , we obtain the following proposition by induction.

Proposition 2.16 *Let* $0 \leq k \leq d-1$. *Every* $(k+1)$*-subset of the vertices of a* d*-simplex determines a* k*-face. The number* $f_k(T_d)$ *of* k*-faces of a simplex* T_d *equals* $\binom{d+1}{k+1}$.

Clearly, any two k-simplexes in E_d , $k \leq d$, are affinely equivalent. Indeed, if T'_k and T''_k are two k-simplexes in E_d then the set of their vertices $x^0,\ldots,x^k,y^0,\ldots,y^k$ may be extended to affine bases $x^0,\ldots,x^d$ and $y^0,\ldots,y^d$ in E_d . There exists a non-singular affine map α which maps one basis onto the other : $\alpha(x^i) = y^i$, $i=0,\ldots,d$. Hence, the simplexes T'_k and T''_k are affinely equivalent.

The simplex T_d has $d+1$ facets and a coordinate system can be chosen in E_d such that T_d is given by the following inequalities :

$$\sum_{i=1}^{d} x_i \leq 1 , \qquad x_i \geq 0 \qquad i=1,2,\ldots,d .$$

Such a simplex is called a *regular simplex* .

Definition 2.5 A polytope is called *simplicial* if all its proper faces are simplexes.

A subset of points $V \subset E_d$ is said to be *in general position* if all of its $(d+1)$-subsets consist of affinely independent points. If the set of vertices of a d-polytope is in general position, then there is no hyperplane in E_d containing more than d vertices of M.

It follows that every facet of such a polytope is a simplex. Thus the convex hull of a set of points in general position is a simplicial polytope. Note that there exist simplicial polytopes whose vertex set is not in general position : more than d vertices may lie in the same hyperplane H provided that H is not a supporting hyperplane of M. (Fig.1)

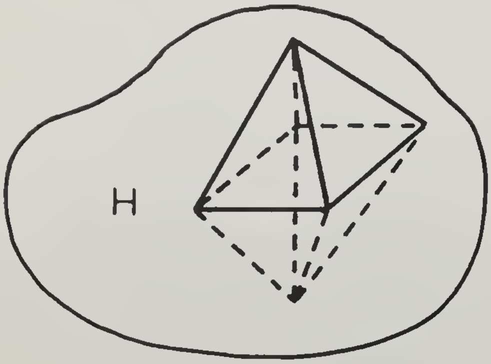


Fig. 1.

Every j-face of a d-polytope is, by Theorem 2.12, the intersection of not less than $d-j$ facets. We consider the class of d-polytopes whose facets are in general position. This implies that every j-face is determined by the intersection of exactly $(d-j)$ facets.

Definition 2.6 A d-polytope is a *simple polytope* if each of its vertices lies in exactly d facets.

We will see later that simple and simplicial polytopes are closely related and are the duals of each other in a certain sense.

We will consider one of the most interesting examples of simplicial polytopes. In E_d define a curve by the parametric equations $x(\tau) = (x_1(\tau),\dots,x_d(\tau))$, where $x_i(\tau) = \tau^i$.

Definition 2.7 The convex hull of n distinct points in E_d lying on the curve $x(\tau)$ is called a *cyclic polytope*, denoted by $C(d,n)$.

Cyclic polytopes play an important part in the combinatorial theory of polytopes. They were introduced in 1907 by Carathéodory (1907) and were rediscovered in 1956 by Gale (1956, 1963) ; see also Shashkin (1963).

<u>Proposition 2.17</u> *A cyclic polytope is simplicial.*

Proof We show that the vertices of a cyclic polytope $C(d,n)$ are in general position. Thus, take $d+1$ vertices $x(\tau_1),\dots,x(\tau_{d+1})$. Then

$$\Delta = \begin{vmatrix} \tau_1 & \tau_2 & \dots & \tau_{d+1} \\ \tau_1^2 & \tau_2^2 & \dots & \tau_{d+1}^2 \\ . & . & \dots & . \\ \tau_1^d & \tau_2^d & \dots & \tau_{d+1}^d \\ 1 & 1 & \dots & 1 \end{vmatrix}$$

is Vandermonde's determinant and hence $\Delta \neq 0$. Hence, the points $x(\tau_i)$ are affinely independent and every proper face of a cyclic polytope is a simplex //

<u>Definition 2.8</u> A polytope M is called k-*neighbourly* if every k-subset of its vertices is the vertex set of some proper face of M.

For example, a d-simplex is d-neighbourly and every polytope is 1-neighbourly.

<u>Proposition 2.18</u> *The cyclic polytope* $C(d,n)$ *is* $[d/2]$-*neighbourly.*

Proof. Let $m = [d/2]$. For an arbitrary m-subset $V_m = \{x(\tau_i^*) : i=1,\dots,m\}$, where $\tau_1^* < \dots < \tau_m^*$, of vertices of a cyclic polytope $C(d,n)$ we introduce the polynomial

$$g(\tau) = \prod_{i=1}^{m} (\tau-\tau_i^*)^2 = \beta_0 + \beta_1\tau + \dots + \beta_{2m}\tau^{2m} .$$

Let $H = \{x \varepsilon E_d : ax=-\beta_0\}$ be the hyperplane with normal vector $a = (\beta_1,\dots,\beta_d)$, where $\beta_d = 0$ if $d=2m+1$. Clearly, $x(\tau_i^*) \varepsilon H$, $i=1,\dots,m$, while for all other vertices of $C(d,n)$ we have

$$ax(\tau') = -\beta_0 + \prod_{i=1}^{m} (\tau'-\tau_i^*)^2 > -\beta_0 .$$

Thus, H is a supporting hyperplane of $C(d,n)$ and $H \cap \operatorname{vert} C(d,n) = V_m$. This means that the vertices V_m generate the face $F = H \cap C(d,n)$. //

Corollary 2.19

$$f_i(C(d,n)) = \binom{n}{i+1} \qquad \forall i \in N_{[d/2]} .$$

§3 OPERATIONS ON POLYTOPES

3.1 The Simplest Operations

Let M_1 and M_2 be polytopes in E_d. The set

$$M = \{x \in E_d : x = x_1 + x_2 , \; x_1 \in M_1 , \; x_2 \in M_2\}$$

is the *sum of the polytopes* M_1 and M_2 denoted by $M_1 + M_2$. Evidently the operation of summation of polytopes can be generalized to the case of arbitrary convex sets. Such sums will also be convex sets.

We can use this operation to generalize Theorem 1.10 as follows

Theorem 3.1 *The polyhedron* P *of solutions of a non-homogeneous system of simultaneous linear inequalities can be represented as a sum* $P = M + K$ *of a polytope* M *and a multifaceted cone* K *which is identical with the polyhedral cone of solutions of the corresponding system of homogeneous inequalities.*

The set

$$M_1 \otimes M_2 = \{(x_1, x_2) : x_i \in M_i , \; i=1,2\}$$

is called the *product of the polytopes* $M_1 \subset E_{d_1}$ and $M_2 \subset E_{d_2}$.

A *projective mapping* of a space E_d into E_k is a map τ given by the rule

$$\tau(x) = \frac{\alpha(x)}{ax + \beta} \qquad x \in E_d$$

where α is an affine map from E_d to E_k, a is a d-vector and β is a real number. A projective map τ is called *non-singular* if the associated affine map $\alpha : E_{d+1} \to E_{k+1}$ defined by the rule

$$\alpha(x,1) = (\alpha(x) , ax+\beta)$$

is non-singular. The projective map τ is called *feasible for the set* W if $W \cap H = \emptyset$, where $H = \{x \in E_d : ax+\beta = 0\}$.

Let τ be a projective map feasible for a polytope $M \subset E_d$, then the set $\tau(M)$ is called a *projective image* of M .

Proposition 3.2 *The following are polytopes :*

1) *The sum of a finite set of polytopes;* 2) *The convex hull of a finite set of polytopes;* 3) *The non-empty intersection of a finite set of polytopes;* 4) *The product of a finite set of polytopes;* 5) *The affine image of a polytope;* 6) *The projective image of a polytope.*

The first four assertions are obvious. The fifth and sixth follow from the definitions of affine and projective maps and from the obvious properties $\alpha(M) = \text{conv}\, \alpha(\text{vert}\, M)$, $\tau(M) = \text{conv}\, \tau(\text{vert}\, M)$.

3.2 Polars

According to Theorem 2.2 a polytope is completely determined by its vertices. Thus it is natural to associate with a given polytope a second polytope given by the intersection of a set of closed half-spaces whose normal vectors correspond to the vertices of the first polytope.

Definition 3.1 Let W be a non-empty set in E_d . The set W^* given by

$$W^* = \{y \in E_d : xy \leqq 1 , x \in W\}$$

is called the *polar* of W .

We consider some examples of polars of certain convex sets.

1 Let a be a point in E_d , then $a^* = \{y \in E_d : ay \leqq 1\}$ is a halfspace in E_d . Also $0^* = E_d$.

2 The polar of the halfspace $H^- = \{x \in E_d : ax \leqq \beta\}$ is the segment $(H^-)^* = \{y \in E_d : y = at , 0 \leqq t \leqq 1/\beta\}$ if $\beta > 0$, and the ray $(H^-)^* = \{y \in E_d : y = at , t \geqq 0\}$, if $\beta \leqq 0$.

3 The polar of the sphere $S(0,r)$ with centre 0 and radius r is the sphere $S(0,1/r)$ with the same centre and radius $1/r$.

According to definition 3.1 the polar W^* is the intersection of the closed half-spaces $H_x^- = \{y \varepsilon E_d : xy \leqq 1\}$, $\forall x \varepsilon W$, i.e. $W^* = \bigcap_{x \varepsilon W} H_x^-$.

Consequently, the polar of any non-empty set (not necessarily convex) is a closed convex set. The following Lemma also follows directly from defintion 3.1 .

<u>Lemma 3.3</u> *Let* $\emptyset \neq W_1 \subseteq W_2$. *Then* $W_1^* \supseteq W_2^*$.

<u>Theorem 3.4</u> *Let* M *be a polytope such that* $0 \varepsilon \text{ int } M$, *then the polar* M^* *is also a polytope.*

Proof Let $\text{vert } M = \{x^1,\dots,x^s\}$. We show first that

$$M^* = \bigcap_{i=1}^{s} (x^i)^* = \bigcap_{i=1}^{s} \{y \varepsilon E_d : x^i y \leqq 1\} . \tag{3.1}$$

Since $\text{vert } M \subset M$, by Lemma 3.3

$$M^* \subset (\text{vert } M)^* = \bigcap_{i=1}^{s} (x^i)^* .$$

Conversely, let $y \varepsilon (\text{vert } M)^*$, that is, the inequalities

$$x^i y \leqq 1 \qquad \forall i \varepsilon N_s$$

are satisfied. Let $a \varepsilon M$. Then

$$a = \sum_{i=1}^{s} \lambda_i x^i , \quad \sum_{i=1}^{s} \lambda_i = 1 , \quad \lambda_i \geqq 0 , \quad 1 \leqq i \leqq s$$

and so

$$ay = \sum_{i=1}^{s} (\lambda_i x^i) y = \sum_{i=1}^{s} \lambda_i x^i y \leqq 1 .$$

Thus $y \varepsilon M^*$ and so $M^* \supset (\text{vert } M)^*$. This proves (3.1).

It remains to show that M^* is a bounded polyhedron. Since $0 \varepsilon \text{ int } M$, $\exists r>0$ such that the sphere $S(0,r)$ is contained in M . The polar of $S(0,r)$ is the sphere $S(0,1/r)$. By Lemma 3.3 $M^* \subset S(0,1/r)$ and so, by Theorem 2.5, M^* is a polytope. //

<u>Lemma 3.5</u> *Let the polytope* M *contain the origin* 0 *(not necessarily as an interior point). Then* $M^{**} = M$.

Proof We show first that $M \subseteq M^{**}$. Let $x \varepsilon M$. Then for all $y \varepsilon M^*$ we have $yx \leqq 1$. Thus $x \varepsilon M^{**}$. Hence $M \subseteq M^{**}$.

Next we show that $M^{**} \subseteq M$. Suppose, for contradiction, that $\exists\, a \varepsilon M^{**}$ such that $a \notin M$. Let a' be the projection of the point a on M . Then the hyperplane

$$H = \{x \varepsilon E_d : (a-a')x = \alpha\}$$

passing through a' is a supporting hyperplane of M (Corollary 1.5) . We have $(a-a')x \leqq \alpha$ for all $x \varepsilon M$, but $(a-a')a > \alpha$. If α_1 is such that $\alpha < \alpha_1 < (a-a')a$ then, since $0 \varepsilon M$, we have $\alpha_1 > 0$. So, for any $x \varepsilon M$ we have $cx < 1$, where $c = (a-a')/\alpha_1$. Thus $c \varepsilon M^*$. But for any $y \varepsilon M^*$ we must have the inequality $ay \leqq 1$, whereas $ac > 1$.
Hence $M^{**} \subseteq M$. //

<u>Lemma 3.6</u> *Let* F *be a face of the polytope* M *and let* $0 \,\varepsilon\, \mathrm{int}\, M$. *Then the set*

$$\phi(F) = \{y \varepsilon M^* : xy = 1 \ \ \forall x \varepsilon F\} \tag{3.2}$$

is a face of the polytope M^* .

Proof It follows from (3.2) that $\phi(\emptyset) = M^*$ and that $\phi(M) = \emptyset$. Now suppose F is a proper face of M . Let $x^o \,\varepsilon$ rel int F . Then the hyperplane $H = \{y \varepsilon E_d : x^o y = 1\}$ is a supporting hyperplane of M^* so that $F^* = M^* \cap H$ is a face of M^* . Also $\phi(F) \subseteq F^*$. We will show that $\phi(F) \supseteq F^*$. Let $y^o \,\varepsilon\, M^* \setminus \phi(F)$. Then $\exists\, x^1 \varepsilon F$ such that $x^1 y^o < 1$.
Since $x^o \,\varepsilon$ rel int F , $\exists\, x^2 \varepsilon F$ such that $x^o = (1-\lambda)x^1 + \lambda x^2$, $0 < \lambda < 1$.
Since $y^o \varepsilon M^*$, $y^o x^2 \leqq 1$ and so

$$y^o x^o = (1-\lambda) y^o x^1 + \lambda y^o x^2 < 1 .$$

Thus $y^o \notin F^*$, Hence $F^* = \phi(F)$ and $\phi(F)$ is a face of M^* . //

3.3 Duality

Duality is one of the fundamental concepts in the theory of convex sets and, in particular, in polytope theory. Different aspects of duality have been investigated by many authors, notably by Motzkin (1933), Weyl (1935), Fenchel (1953) and Kutateladze & Rubinov (1976).

Definition 3.2 The polytope M^o is said to be *dual* to the polytope M if there is a bijective map ϕ between the sets of faces of all dimensions of M and M^o respectively with the property :

$$F_1 \subset F_2 \quad \Leftrightarrow \quad \phi(F_1) \supset \phi(F_2) \quad .$$

Such a map between the faces of the two polytopes is called an *anti-isomorphism.*

An example of a dual pair of 3-polytopes is given by the cube and the octahedron (Figure 2). Two other examples are given in Figures 3 and 4. A simplex is clearly self-dual.

It follows from definition 3.2 that

$$\dim M = \dim M^o = \dim F + \dim \phi(F) + 1$$

for any face F of the polytope M .

The following theorem answers the question : does every polytope have a dual?

Theorem 3.7 *Let* $M \subset E_d$ *be a polytope and let* $0 \varepsilon \operatorname{int} M$. *Then the polar* M^* *is a polytope which is dual to* M .

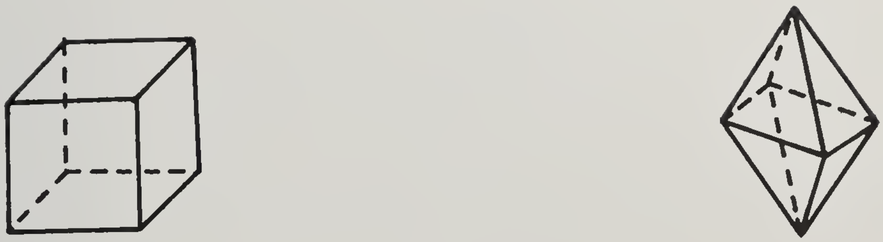
Proof We show that the map ϕ , defined by equation (3.2) is the anti-isomorphism required by definition 3.2 . If $F_1 \subset F_2$ then Lemmas 3.3 and 3.6 imply that $\phi(F_1) \supset \phi(F_2)$, If we now show that $\phi(\phi(F)) = F$, the theorem will be established.

By definition

$$\phi(\phi(F)) = \{x \varepsilon M^{**} : yx=1 \;\; \forall\, y \varepsilon \phi(F)\} \quad .$$

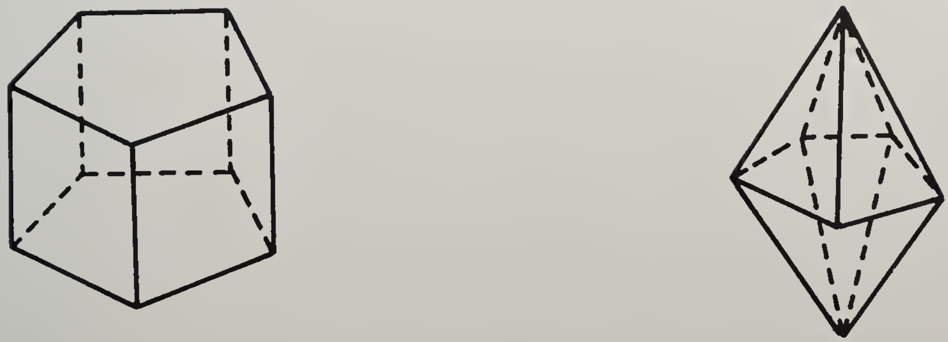
Since $M^{**}=M$ (Lemma 3.5) we have $F \subseteq \phi(\phi(F))$.

Let the face F be generated by the supporting hyperplane $H = \{x \varepsilon E_d : ax=1\}$ with $M \subset H^-$, Clearly $a \varepsilon \phi(F)$. If $x^o \varepsilon M \setminus F$ then $ax^o < 1$ and $x^o \notin \phi(\phi(F))$. Consequently $\phi(\phi(F)) \subseteq F$. //



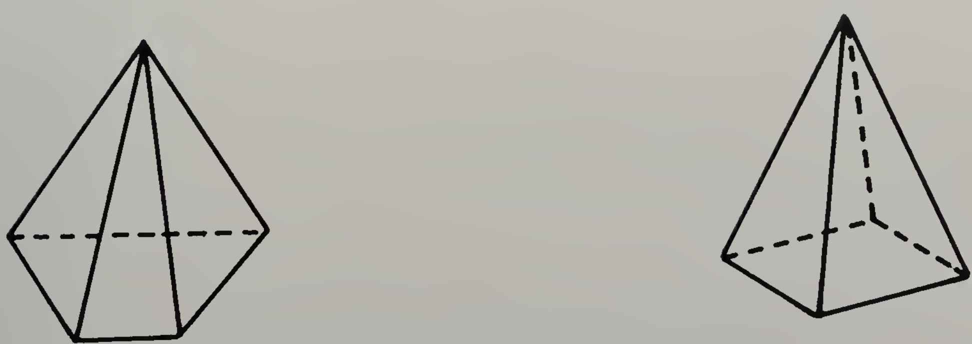
cube (a) octahedron (b)

Fig. 2.



(a) (b)

Fig. 3.



(a) (b)

Fig. 4.

Corollary 3.8 *The dual of a simplicial polytope is simple. The dual of a simple polytope is simplicial.*

3.4 Construction of Supporting Hyperplanes

Lemma 3.5 yields an algorithm for representing a polytope as the intersection of a finite number of closed half-spaces. Indeed, let the d-polytope $M \subset E_d$ be given by

$$M = \operatorname{conv}\{v^1, \ldots, v^n\} \quad . \tag{3.3}$$

We consider the auxiliary polytope

$$M_1 = \operatorname{conv}\{v^1 - v^0, \ldots, v^n - v^0\}$$

where the vector v^0 is chosen so that $0 \in \operatorname{int} M_1$. Such a vector exists because of the condition $\operatorname{int} M \neq \emptyset$. The polar M_1^* of M_1 is given by the following system of inequalities :

$$(v^i - v^0)x \leqq 1 \qquad , i = 1, \ldots, n \quad .$$

By Theorem 3.4 , M_1^* is a polytope and so

$$M_1^* = \operatorname{conv}\{u^1, \ldots, u^s\}$$

where $\{u^1, \ldots, u^s\}$ is the set of vertices of M_1^* . The polytope M_1^{**} is now given by the constraints

$$u^i x \leqq 1 \qquad i = 1, \ldots, s \quad .$$

By Lemma 3.5 $M_1^{**} = M_1$. Thus the polytope $M = M_1 + v^0$ will be given by the following system of inequalities :

$$u^i x \leqq 1 + u^i v^0 \qquad i = 1, \ldots, s \quad . \tag{3.4}$$

Thus in order to obtain the analytical representation (3.4) of the polytope M from the parametric representation (3.3), one requires a means of constructing the vertices of its polar or the vertices of the polar of the auxiliary polytope M_1 if M does not contain 0 in its interior.

§4 THE SOLUTION POLYTOPE OF A SYSTEM OF LINEAR INEQUALITIES

The Weyl-Minkowski Theorem (Theorem 2.5) shows that every polytope in a fixed coordinate system can be specified by means of a finite system of linear inequalities. This allows us, on the one hand, to utilise the well developed apparatus of the theory of linear inequalities to study polytopes and on the other hand, to give an algebraic interpretation of the geometrical properties of polytopes. In this section we examine ways of specifying polytopes by means of various systems of linear inequalities.

4.1 Specifications of Polytopes

Let M be a d-polytope and let $f_{d-1}(M) = n$. Then by Theorem 2.5 M is the intersection of n half-spaces H_i^+ in E_d given by the inequalties

$$A_i x \geqq b_i \quad , \quad A_i \varepsilon E_d \quad , \quad i=1,\dots,n \, . \tag{4.1}$$

If the dimension d of M is less than the dimension of the space E_m in which it is defined, M is the intersection of $m-d$ linearly independent hyperplanes

$$A_i x = b_i \quad , \quad A_i \varepsilon E_m \quad , \quad i=1,\dots,m-d \, , \tag{4.2}$$

and of n half-spaces

$$A_i x \geqq b_i \quad , \quad A_i \varepsilon E_m \quad , \quad i=m-d+1,\dots,m-d+n \, . \tag{4.3}$$

In linear programming a d-polytope M is usually viewed as a subset of a space whose dimension equals the number n of its (d-1)-faces. Moreover the chosen basis of the space E_n is taken to be the set of vectors orthogonal to the hyperplanes which generate the polytopes (d-1)-faces. Using such a coordinate system in E_n the polytope is given by the following set of constraints

$$A_i x = b_i \quad , \quad A_i \varepsilon E_n \quad , \quad i=1,\dots,n-d \tag{4.4}$$

$$x \geqq 0 \, . \tag{4.5}$$

Such a polytope may be denoted by $M(A,b)$. On the other hand, any bounded subset of E_d determined by n linear inequalities (4.1) in d variables is a polytope M such that $\dim M \leqq d$ and $f_{d-1}(M) \leqq n$.

Definition 4.1 If a polytope is given by a system of inequalities (4.1) we call this system a *normal specification of the polytope.* If a polytope is given by a system (4.4),(4.5) we call the system a *canonical specification of the polytope.*

It is easy to transform a canonical polytope specification into a normal polytope specification by means of a singular affine map,and conversely. Let a polytope M be given by a system of constraints (4.4), (4.5). Then we can use equations (4.4) to solve for r of the variables in terms of the others, where r is the rank of the system (4.4). Let these be the first r variables, then we obtain a system of the special form

$$x_i = \bar{b}_i + \sum_{j=r+1}^{n} \bar{a}_{ij}x_j \qquad \forall i\varepsilon N_r$$

which gives the normal form of the affine set determined by (4.4) which in matrix form is

$$x_B = B^{-1}b - B^{-1}Hx_H \quad ,$$

where B is a basis of column-vectors for the column space of the matrix A and x_B, x_H are vectors made up of the components of x corresponding to the indices of the columns of B and H respectively.

If we then replace the inequalities $x_B \geqq 0$ by

$$\sum_{j=r+1}^{n} \bar{a}_{ij}x_j \geqq -\bar{b}_i \quad , \forall i\varepsilon N_r \qquad (-B^{-1}Hx_H \geqq -B^{-1}b) \; ,$$

and adjoin the remaining inequalities

$$x_j \geqq 0 \quad , \quad j=r+1,\dots,n \qquad (x_H \geqq 0) \quad ,$$

we obtain a normal specification of the polytope in E_{n-r} .

To transform from a normal specification (4.1) of a polytope M to a canonical specification it suffices to write $x_j = x_j^+ - x_j^-$, $\forall j\varepsilon N_d$ and to introduce n slack variables x_{d+i} $\forall i\varepsilon N_n$. The system

$$A_i x^+ - A_i x^- - x_{d+i} = b_i \qquad \forall i \varepsilon N_n$$

$$x_j^+ \geqq 0 \quad , \quad x_j^- \geqq 0 \quad x_{d+i} \geqq 0 \qquad \forall j \varepsilon N_d \; , \forall i \varepsilon N_n$$

is clearly a canonical specification of the same polytope M .

Definition 4.2 The i-th constraint in the system (4.2),(4.3) is called a *rigid constraint* for the polytope M if the coordinates of every point of M satisfy the constraint as an exact equality.

Clearly, all the constraints (4.2) and (4.4) are rigid constraints. However, some of the constraints (4.3) or (4.5) may also be rigid constraints. To show that the i-th constraint is not rigid it suffices to exhibit a point of the polytope whose coordinates satisfy this inequality strictly.

The matrix whose rows are the normal vectors A_i of the hyperplanes (supporting or non-supporting) is called the *constraint matrix* of the polytope.

The following proposition is a consequence of Theorem 1.2 and the definition of the dimension of a convex set.

Proposition 4.1 *The dimension of the polytope* $M \subset E_n$ *in both normal and canonical specifications is* $n-r$, *where* r *is the rank of the matrix of rigid constraints of the polytope.*

The number of facets of a d-polytope M defined by some system of constraints is not necessarily equal to the number of non-rigid constraints since some of these may be redundant. A *redundant constraint* is a constraint (either equality or inequality) which may be omitted from the constraint set without altering the solution set (the polytope). It is difficult to search analytically for such constraints. Therefore, in studying polytopes given by concrete systems of equations and inequalities we will allow them to include redundant constraints. A redundant constraint $A_k x \geqq b_k$ defines geometrically a hyperplane $H_k = \{x \varepsilon E_n : A_k x = b_k\}$ which either has zero intersection with the polytope M or has a non-zero intersection such that $\dim(M \cap H_k) < d-1$. Clearly if the rank of a system of rigid constraints equals the number of such constraints then none of them are redundant. We give without proof a fundamental criterion for the existence of redundant constraints due to Minkowski (1897) and Farkas (1902).

Theorem 4.2 *The inequality* $A_k x \leqq b_k$ *is redundant in the system* $A_i x \leqq b_i$, $\forall i \in N_m$, *if and only if there exist non-negative numbers* λ_i *such that*

$$A_k = \sum_{i \neq k} \lambda_i A_i \quad , \quad b_k \geqq \sum_{i \neq k} \lambda_i b_i \quad .$$

Definition 4.3 The system (4.2),(4.3) is called *irreducible* if the rank of the system of rigid constraints equals the number of such constraints and if there are no redundant inequalities in (4.3).

Clearly, if (4.2),(4.3) is an irreducible system, then the polytope it defines has dimension d , while the number of its facets equals the number of non-rigid constraints. On the other hand, if the polytope has dimension equal to the dimension of the underlying space, then in this case the polytope has a unique irreducible specifying system.

Let the system (4.2),(4.3) be irreducible. According to Corollary 2.13 every j-face of a d-polytope $M \subset E_d$ is an intersection of some of its facets. Thus a system specifying a j-face can be obtained by changing some of the inequalities (4.3) into equalities in such a way that the number of linearly independent constraints equals equals $m-j$. For ease of reference we formulate this fact as a Proposition.

Proposition 4.3 *A subset* F *of the solution set of the system* (4.2),(4.3) *defining a* d-*polytope* M *is a* j-*face of* M *if and only if there are* $m-j$ *linearly independent constraints in* (4.2),(4.3) *which are satisfied by all points* $x \varepsilon F$ *as equalities.*

In particular, a point $x \varepsilon F$ is a vertex of a polytope M if and only if among the constraints (4.2),(4.3) defining M there are m linearly independent constraints which are satisfied by x as equalities. Each vertex corresponds to a combination of m linearly independent equations in m variables and each distinct vertex corresponds to a distinct combination of equations. Take the rigid constraints of a polytope and replace some of the inequalities by equalities so as to obtain a system of m linearly independent equations. If the unique solution of this system satisfies the remaining constraints (inequalities) we have found a vertex of the polytope.

4.2 Bases, Feasible Bases

We examine in more detail how the vertices of a polytope given in canonical form (4.4),(4.5) are determined. Let A be the matrix of equality constraints (4.4). We assume that there are no rigid constraints in the set (4.5). In order to obtain a system determining the coordinates of a vertex of the polytope M the equality constraints (4.4) must be supplemented by equations

$$x_j = 0 \qquad j \in J_H \tag{4.6}$$

where J_H is a d-subset of the set N_n such that the rank of the constraint matrix (4.4),(4.6) equals n. This clearly occurs when the rank of the submatrix B of matrix A consisting of columns with indices $j \in J_B$, where $J_B = N_n \setminus J_H$, equals $n-d$.

Let the rank of the matrix A of equality constraints of the polytope M equal m.

<u>Definition 4.4</u> A set of m linearly independent columns of A is called a *basis of the polytope* M.

Every basis B of a polytope defines a system of n linearly independent equations

$$Bx_B = b \tag{4.7}$$

$$x_j = 0 \quad , \quad j \in J_H \tag{4.8}$$

A solution $(x_B^o, 0)$ of this system is called a *basic solution*. It is a point at which a system of linearly independent hyperplanes intersect. A basic solution is a vertex of the polytope M if and only if the components of x_B^o (the basic variables) satisfy the remaining constraints, that is $x_B^o \geqq 0$.

We denote the class of polytopes given by a system (4.4),(4.5) by $\mathcal{M}(m,n)$ where $m=n-d$. Let $\beta(A,b)$ be the number of bases of the polytope

$$M(A,b) = \{x \in E_n : Ax=b , x \geqq 0\}$$

in the class $\mathcal{M}(m,n)$, assuming that the system (4.4),(4.5) is irreducible.

At first sight it appears that $\beta(A,b)$ depends only on the matrix A. But this is not quite true since for some b the system (4.4),(4.5) may be reducible or even inconsistent.

The following problem arises : to describe the range of the function $\beta(A,b)$ on the set $\mathcal{M}(m,n)$ and to characterize the class of polytopes $M(A,b)$ in $\mathcal{M}(m,n)$ with a fixed number of bases (Kowaljow & Milanow 1976, Kovalev , Milanov & Isachenko 1978, Kowaljow & Milanow 1979)

The problem is sometimes posed in a more general context. To do this we introduce the concept of a matroid, introduced by Whitney (1933).

Definition 4.5 A *matroid* $\mathcal{M}$ is a pair $(J,\mathcal{B})$, where J is a non-empty finite set and $\mathcal{B}$ is a non-empty collection of subsets of J (called bases) satisfying the following conditions :

1) No basis contains another basis as a proper subset ;
2) If J' and J'' are bases and $e \in J'$ then $\exists\, f \in J''$ such that $(J' \setminus e) \cup \{f\}$ is a basis

It is easily shown that any two bases of a matroid $\mathcal{M}$ contain the same number of elements. This number is called the *rank of the matroid* $\mathcal{M}$.

If J is a finite set of vectors in E_m , for example - the columns of an (mxn)-matrix A , then taking as our bases all possible maximal linearly independent subsets of J , which span E_m , we obtain a matroid which is usually called a *vector matroid* .

Thus a more general problem consists in characterising the range of values of $\beta(\mathcal{M})$ (the number of bases of the matroid $\mathcal{M}$) in the class of all matroids of rank m over an n-set J and in enumerating the non-isomorphic matroids of fixed rank. Note that a basis of a vector matroid given by the columns of a matrix A is the same thing as a basis of a polytope $M(A,b)$.

It is clear that the number of bases of a polytope of class $\mathcal{M}(m,n)$ cannot exceed $\binom{n}{m}$. Let $n>m$. We indicate a method of constructing an (mxn)-matrix A of rank m with exactly $\binom{n}{m}$ bases. With $m=1$ and any n such a matrix is given by any (1xn)-matrix with non-zero components. Suppose the ((m-1)xn)-matrix A has $\binom{n}{m-1}$ bases. Let

$$A^{\ell} = \sum_{j \in J_B} \lambda^B_{j\ell} A^j$$

be the expansion of column A^{ℓ} relative to the basis B .

Consider the row-vector $A_m = (a_{m1},\dots,a_{mn})$ whose first $m-1$ components are arbitrary nonzero numbers and whose remaining components $a_{m\ell}$ are distinct from each of the numbers

$$\left\{ \sum_{j\varepsilon J_B} \lambda^B_{j\ell} a_{mj} \right\}$$

Here B can be any basis constructed from the first $\ell-1$ columns and $m-1$ rows of the matrix A. Let

$$\overline{A} = \begin{pmatrix} A \\ A_m \end{pmatrix} .$$

We show that $\beta(\overline{A}) = \binom{n}{m}$. Suppose the contrary. Let there exist a singular $(m\text{x}m)$-submatrix $\overline{B}$ of $\overline{A}$. Let $\overline{A}^s$ be the cloumn of $\overline{B}$ with largest index. Since $\overline{B}$ is singular we have

$$A^s = \sum_{j\varepsilon J_{\overline{B}}\setminus s} \mu_j A^j \quad , \quad a_{ms} = \sum_{j\varepsilon J_{\overline{B}}\setminus s} \mu_j a_{mj} \quad . \tag{4.9}$$

Here A^s, A^j are vectors consisting of the first $(m-1)$ components of the vectors $\overline{A}^s$, $\overline{A}^j$. Since the expansion of a vector A^s relative to the basis B consisting of the columns A^j with indices in $J_{\overline{B}}\setminus s$ is unique, we have $\mu_j = \lambda^B_{js}$. Hence, equation (4.9) contradicts the choice of the number a_{ms}.

Another way of describing a matrix with the maximum possible number of bases is obtained by carrying out an induction on n. Let the $(m\text{x}n)$-matrix A possess the required property. Consider the set of $\binom{n}{m-1}$ $(m-1)$-dimensional linear subspaces of E_m generated by all possible combinations of $(m-1)$ columns of A. It is obvious that it is always possible to choose an m-vector A^{n+1} which does not belong to any of these subspaces. For instance, it suffices to put

$$A^{n+1} = \sum_{j=1}^{m} \lambda_j A^j \quad , \quad \lambda_j = \left(\frac{pm}{q}\right)^j \quad j\varepsilon N_m \quad ,$$

where p and q are the minors of order m of A of greatest and least absolute value.

Definition 4.6 A *simplex array* of a matrix A is a matrix

$$\Lambda_B = (\lambda_{ij}^B)_{m\times(n-m)}$$

of expansion coefficients, relative to a basis B, of the column vectors of A which are not in B. That is $\Lambda_B = B^{-1}H$ where $A = (B,H)$.

Proposition 4.4 *The (mxn)-matrix* A *has the maximum possible number* $\binom{n}{m}$ *of bases if and only if at least one of its simplex arrays has all of its minors distinct from zero.*

The proof follows from the fact that a set of m-vectors $\{A^1,\dots,A^m\}$ is linearly independent if and only if the set $\{BA^1,\dots,BA^m\}$ is linearly independent for any nonsingular matrix B.

The range of values of $\beta(A,b)$ on $\mathcal{M}(m,n)$ has only been described in the simplest cases (Kowalyow & Milanov 1976). Thus, for $\mathcal{M}(2,n)$ the function $\beta(A,b)$ can only take values in the set

$$\left\{ \tfrac{1}{2}(n^2 - \sum_{i=1}^{n} u_i^2) : \sum_{i=1}^{n} u_i = n \ , \ u_1 \geqq \dots \geqq u_n \geqq 0 \ , \ u_i\text{-integer} \right\}$$

This result follows from the fact that the columns of any (2xn)-matrix can be partitioned into groups of collinear vectors, the i-th of which consists of u_i vectors

It is not difficult to calculate the number of bases of a unimodular matrix, that is, a matrix for which $\det B = \pm 1$ for any basis B.

Proposition 4.5 *If* A *is a unimodular matrix, the number of its bases equals* $\det(AA^T)$.

The proof follows from the well known Binet-Cauchy formula

$$\det AB = \sum_{\substack{J \subset N_n \\ |J|=m}} \det A^J_{N_m} \det B^{N_m}_J \quad ,$$

where A is an (mxn)-matrix and B is an (nxm)-matrix with $n \geqq m$.

Definition 4.7 A basis of a polytope M is a *feasible basis* if the basic solution satisfies the inequalities (4.5), i.e. if the basic variables are non-negative. If B is a feasible basis, the corresponding solution of the system (4.4),(4.5) is called a *basic feasible solution.*

The enumeration of the feasible bases of a polytope of class $\mathcal{M}(m,n)$ is a complex problem. As will be seen in the sequel, this problem is not always identical with the problem of enumerating the vertices of a polytope. We will examine one possible method of enumerating the feasible bases.

Let B be a basis. Consider the cone con B generated by the column-vectors A^j which comprise B. The basis B is feasible if and only if $b \in$ con B. Thus the problem of counting the number of feasible bases of a polytope $M(A,b)$ (denoted by $\beta^*(A,b)$) is equivalent to the problem of counting all the cones con B containing the vector b.

We illustrate a method of calculating $\beta^*(A,b)$ for the case $m = 3$ when all the vectors A^j have non-negative components (Kowaljow & Milanow 1976 , Kovalev , Milanov & Isachenko 1978). Consider the intersection of the cone con A in E_m with the plane

$$\sum_{j=1}^{m} x_j = 1 \quad . \tag{4.10}$$

Then every cone con (A^p,A^q,A^s) will correspond to a triangle whose vertices are the intersections of the vectors A^p,A^q,A^s with the plane (4.10). If $b \in$ con B , then the point of intersection of b with the plane (4.10) will lie within this triangle.

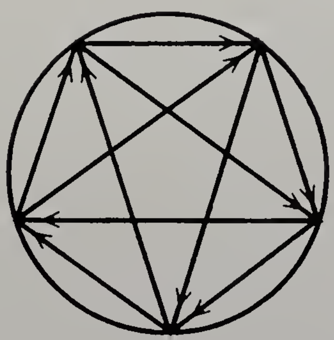


Fig. 5.

The problem of determining the range of values of $\beta^*(A,b)$ can now be given the following geometrical interpretation : n points $A_1,\ldots,A_n$ are given in the plane ; it is required to find the number of triangles $A_pA_qA_s$ which contain a given point B . Clearly, this number will not change if we move each of the points A_j along the ray originating at B and passing through A_j . We may therefore assume that all of the points A_j lie on a unit circle with centre B . On each chord A_jA_k we define an orientation such that the point B lies on the right-hand side of the chord. The set of points and oriented chords obtained in this way (Fig.5) is called the *diagram of the polytope* $M(A,b)$.

Proposition 4.6 *The number of feasible bases* $\beta^*(A,b)$ *of the polytope* $M(A,b)$ *of class* $\mathcal{M}(3,n)$ *is given by the formula*

$$\binom{n}{3} - \sum_{i=1}^{n} \binom{s_i}{2} \qquad ,$$

where s_i *is the number of directed chords leaving* A_i *in the polytope diagram* .

The proof is based on the well known formula for the number of cyclic triplets in a tournament (Beinecke 1974).

4.3 Degenerate Polytopes

Definition 4.8 A basic feasible solution of a polytope is called *nondegenerate* if the number of constraints which it satisfies as an equality is equal to n , (the dimension of the underlying space). If a basic feasible solution satisfies more than n constraints as an equality it is called a *degenerate solution* . A vertex of a polytope M corresponding to a degenerate basic feasible solution is also called a *degenerate vertex* . A system which possesses at least one degenerate feasible solution is a *degenerate specification of the polytope* M and such a polytope is a *degenerate polytope* .

Degeneracy of a canonical specification of a polytope corresponds to the case in which a basic feasible solution exists in which at least one of the basic variables is zero.

A polytope M may have a degenerate specification in two ways. Firstly, the number of supporting hyperplanes of M which intersect the vertex may exceed the dimension of the polytope. Thus any non-simple polytope has only degenerate specifications. Also, every non-degenerate, irreducible system of constraints defines a simple polytope.

Secondly, if a system of constraints is reducible, that is, if it contains superfluous constraints, then it is degenerate. Thus we may assert the following proposition.

Proposition 4.7 *A polytope* M *is degenerate if and only if, either* M *is not a simple polytope, or the system of constraints specifying the polytope contains a superfluous constraint.*

If a polytope is degenerate then more than one feasible basis corresponds to each degenerate vertex .

4.4 Polytopes with few faces

The class $\mathcal{M}(m,n)$ can be divided into subclasses $\mathcal{M}(m,n,k)$ of polytopes with a fixed number of facets :

$$M(A,b) \in \mathcal{M}(m,n,k) \quad \Leftrightarrow \quad f_{d-1}(M(A,b)) = d + k \quad .$$

The problem of finding an irreducible system specifying a polytope $M(A,b)$ reduces to the problem of finding the class $\mathcal{M}(m,n,k)$ to which a particular polytope $M(A,b)$ belongs (Kovalev & Isachenko 1978).

Let B be a feasible basis of the polytope $M(A,b)$.

Definition 4.9 The simplex-array Λ_B is called k-*regular* ($k \in N_m$), if the minimum

$$\min_i \left\{ \frac{\lambda^B_{i0}}{\lambda^B_{ij}} : i \in J_B,\ \lambda^B_{ij} > 0 \right\} , \qquad (\lambda^B_{i0})_{i \in J_B} = B^{-1} b \tag{4.11}$$

is attained at exactly k distinct i's for all values of $j \in J_H$.

Lemma 4.8 *If there exists a feasible basis* B *of a non-degenerate polytope* $M(A,b)$ *for which* Λ_B *is a* k-*regular simplex array, then there is an* $\ell \geq k$ *such that* $M(A,b) \in \mathcal{M}(m,n,\ell)$.

Proof Let the minima in (4.11) for all values of $j \in J_H$ be attained at indices i in the set $J^o_B \subset J_B$. Then it is clear that in the feasible basis B we may interchange every column-vector A^i for $i \in J^o_B$ with some column-vector A^j $\forall j \varepsilon J_H$ and we again obtain a feasible basis. Consequently, the sets

$$F_i = \{x \varepsilon M(A,b) : x_i = 0\} \qquad i \varepsilon J_B^o$$

are nonempty and together with the sets F_j for $j \varepsilon J_H$ are facets of $M(A,b)$. //

Definition 4.10 The simplex arrays $\Lambda_{B'}$ and $\Lambda_{B''}$ are called k-*similar* if they are both k-regular and if $J^o_{B'} = J^o_{B''}$.

The following theorem follows from Lemma 4.8.

Theorem 4.9 *If the simplex arrays of all feasible bases of the polytope* $M(A,b)$ *are* k-*similar, then* $M(A,b) \; \varepsilon \; \mathcal{M}(m,n,k)$.

It follows quickly from Theorem 4.9 that the polytope $M(A,b)$ is a simplex if and only if the simplex arrays of all feasible bases B of the matrix A are 1-similar. It is easy to see that the last statement holds if the simplex array of at least one feasible basis is 1-regular. For example, let B be such a basis and let $J_B^o = \{k\}$. Then an irreducible system specifying the polytope $M(A,b)$ is given by

$$x_k + \sum_{j \varepsilon J_H} \lambda^B_{kj} x_j = \lambda^B_{k0} \quad ,$$

$$x_i = \lambda^B_{i0} \;, \quad i \varepsilon J_B \setminus k \quad ; \quad x_j \geqq 0 \quad , \quad j \varepsilon J_H \cup k \quad .$$

Theorem 4.9 implies that if a simplex array of a feasible basis B of A is 1-regular then all simplex arrays are 1-similar. It follows that the polytope $M(A,b) \; \varepsilon \; \mathcal{M}(m,n,2)$ if and only if the simplex arrays of all feasible bases are 2-similar.

§5 THE f-VECTOR OF A POLYTOPE

Let M be a d-polytope and let i be an integer, $i \varepsilon N_{d-1}$. As usual, we denote the number of i-faces of M by $f_i(M)$. When it is clear which polytope we are considering we write simply f_i . Thus, with each d-polytope M we have associated a d-dimensional vector $f(M) = (f_0, f_1, \ldots, f_{d-1})$. We will call this the f-*vector of a polytope* .

<u>Definition 5.1</u> Two polytopes M and M' are *f-equivalent* if their f-vectors are identical ; $f(M) = f(M')$.

The problem naturally arises of determining the classes of f-equivalent polytopes and of describing the range of values of the function f for different classes of d-polytopes.

5.1 <u>The Euler-Poincaré Formula</u>

In 1752 Euler (1752) published a formula connecting the components of the f-vector of a 3-polytope :

$$f_o - f_1 + f_2 = 2 \quad .$$

It is interesting to note that this formula was already known to Descartes about a hundred years earlier ; however, his manuscript was lost and a partial copy of it was only found in 1860 among Leibnitz's papers. Poincaré (1893) generalized Euler's formula to arbitrary d-polytopes. Poincaré used a topological proof. The elementary geometrical proof of the Euler-Poincaré formula given here is due to Grünbaum (1967). Other proofs of the formula can be found in Hilbert & Cohn-Vossen (1952) and Ashkinuze (1963).

<u>Theorem 5.1</u> (Euler-Poincaré) *Let* M *be a* d-*polytope. Then*

$$\sum_{i=0}^{d-1} (-1)^i F_i(M) = 1 + (-1)^{d-1} \quad .$$

Proof We use induction on d . The theorem is true when $d = 1$ since $f_0(M) = 2$. Suppose the theorem is true for all polytopes whose dimension is no greater than $d-1$ $(d \geq 2)$.

Let M be a d-polytope in E_d with n vertices. Let $a \in E_d$ be any vector which is not perpendicular to any of the 1-faces of M . Let H be a hyperplane with normal vector a . Construct n hyperplanes $H_1, H_3, \ldots, H_{2n-1}$, each of which is parallel to H and contains exactly one of the vertices of M . (This is possible from the way H was chosen). Let $H_2, H_4, \ldots, H_{2n-2}$ be hyperplanes parallel to H and such that for all $k \in N_{n-1}$ the hyperplane H_{2k} lies between H_{2k-1} and H_{2k+1} . It is clear that the hyperplanes H_1 and H_{2n-1} are supporting hyperplanes of M, while for each $i = 2,4,\dots,2n-2$ the set $M_i = M \cap H_i$ is a $(d-1)$-polytope (Fig.6).

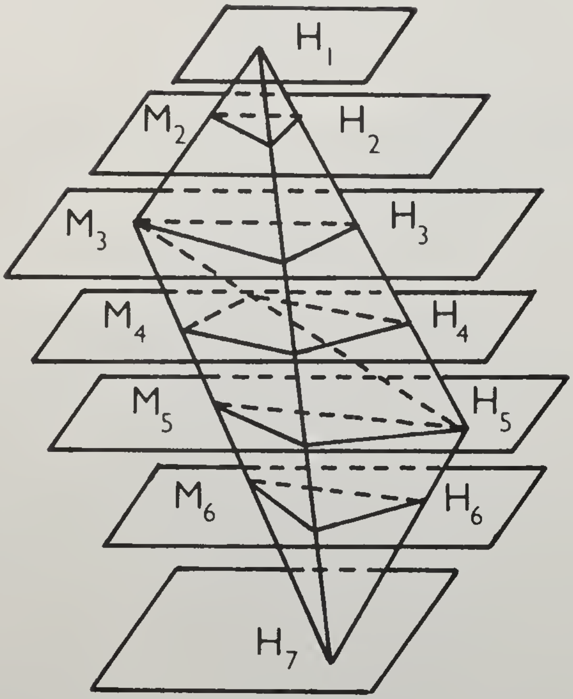


Fig. 6.

For each j-face F of M, $j \in N_{d-1}$, and for each polytope M_i, $i = 2,4,\dots,2n-2$, we define the function

$$\Psi(F,M_i) = \begin{cases} 0 & \text{if } M_i \cap \text{rel int } F = \emptyset \\ 1 & \text{if } M_i \cap \text{rel int } F \neq \emptyset \end{cases} .$$

The first and last hyperplanes intersecting a given j-face F have odd indices. Let these indices be $2\ell-1$ and $2m-1$ respectively where $\ell \neq m$ when $j \neq 0$. Consequently, for $i = 2\ell,\dots,2m-2$ we have $M_i \cap \text{rel int } F \neq \emptyset$ and so $M_i \cap F$ is a $(j-1)$-face of M_i (Proposition 2.8). Hence, for every j-face F of M, if $\Psi(F,M_i) = 1$ for r even indices i, then $\Psi(F,M_i) = 1$ for $(r-1)$ odd indices i. Thus

$$\sum_{i=2}^{2n-2} (-1)^i \Psi(F,M_i) = 1 ,$$

or alternatively

$$\sum_{F} \sum_{i=2}^{2n-2} (-1)^i \Psi(F,M_i) = f_j(M) \quad , \quad j \in N_{d-1} \tag{5.1}$$

Here, the sum is taken over all j-faces F of M. Consequently

$$\sum_{j=1}^{d-1} (-1)^j \sum_{F} \sum_{i=2}^{2n-2} (-1)^i \Psi(F,M_i) = \sum_{j=1}^{d-1} (-1)^j f_j(M) \quad . \tag{5.2}$$

We will find another expression for the left-hand side of equation (5.1) by changing the order of summation. Note that if i is even or if $j > 1$, then every $(j-1)$-face of the polytope M_i is an intersection of a j-face of M with the hyperplane H_i; if i is odd and $j = 1$, then one of the vertices of M_i is a vertex of M and the other vertices of M_i are intersections of 1-faces of M with H_i. We obtain

$$\sum_{F} \Psi(F,M_i) = \begin{cases} f_0(M_i) - 1 & \text{if } j=1 \text{ and } i \text{ is odd ,} \\ f_{j-1}(M_i) & \text{otherwise .} \end{cases}$$

It follows that

$$\sum_{j=1}^{d-1} (-1)^j \sum_{F} \Psi(F,M_i) = \begin{cases} \sum_{j=1}^{d-1} (-1)^j f_{j-1}(M_i) + 1 & \text{i-odd} \\ \sum_{j=1}^{d-1} (-1)^j f_{j-1}(M_i) & \text{i-even} \end{cases}$$

$$= \begin{cases} (-1)^{d-1} & \text{i-odd} \\ (-1)^{d-1} - 1 & \text{i-even} \end{cases}$$

The last equality follows from the inductive assumption applied to the (d-1)-polytopes M_i, $i=2,\ldots,2n-2$.

Thus we have

$$\sum_{i=2}^{2n-2} (-1)^i \sum_{j=1}^{d-1} (-1)^j \sum_{F} \Psi(F,M_i) = (-1)^{d-1} - 1 - (n-2) \quad .$$

Substituting the last expression in (5.2) and replacing n by $f_0(M)$ we obtain finally

$$\sum_{j=1}^{d-1} (-1)^j f_j(M) = 1 + (-1)^{d-1} - f_0(M) \quad .$$

This concludes the proof. //

Note : If we define $f_{-1}(M) = 1$, $f_d(M) = 1$ to count the improper faces of M , then the Euler-Poincaré formula takes the simple form

$$\sum_{j=-1}^{d} (-1)^j f_j(M) = 0 .$$

The Euler-Poincaré formula establishes a linear dependence between the components of the f-vector of any d-polytope. The following theorem shows that there are no other linear relations between the f-vector components of an arbitrary polytope of fixed dimension.

<u>Theorem 5.2</u> *The affine hull of the set of* f-*vectors of all* d-*polytopes has dimension* $d-1$.

According to the Euler-Poincaré formula the f-vectors of all d-polytopes lie in a (d-1)-dimensional hyperplane. We must show that every linear equation

$$\sum_{j=0}^{d-1} \alpha_j f_j(M) = \beta \tag{5.3}$$

which holds for all d-polytopes M , is equivalent to the Euler-Poincaré equation. Before proving Theorem 5.2 we will establish the form of the f-vectors for two special classes of polytopes.

<u>Definition 5.2</u> A *pyramid* is the convex hull of a polytope Q , called the *base of the pyramid* , and a point $x \notin \operatorname{aff} Q$, called the *vertex* of the pyramid.

<u>Proposition 5.3</u> *Let* M *be a* d-*pyramid with base* Q *and vertex* v , *then*

$$f_k(M) = f_k(Q) + f_{k-1}(Q) \qquad k \in N_{d-1}$$

where $f_{d-1}(Q) = 1$.

Proof Let F be a k-face of M generated by the supporting hyperplane H , so that $F = M \cap H$. Since $\operatorname{vert} M \subset \operatorname{vert} (Q \cup v)$ there are two possibilities.

1. $v \notin \text{vert } F$. Then by Corollary 2.3 F is a k-face of the base Q.

2. $v \in \text{vert } F$. Then $\text{vert } F \setminus v \subset \text{vert } Q$ is the vertex set of a (k-1)-face $Q \cap H = F \cap H$ of Q (Proposition 2.8).

Conversely, by Theorem 2.9 every face of Q, including Q itself, is a proper face of the pyramid M. //

Definition 5.3 A d-*bipyramid* is the convex hull of a (d-1)-polytope Q (the base) and a line segment $[a,b]$ such that $\text{rel int } Q \cap \text{rel int } [a,b]$ is a single point. (Fig.7)

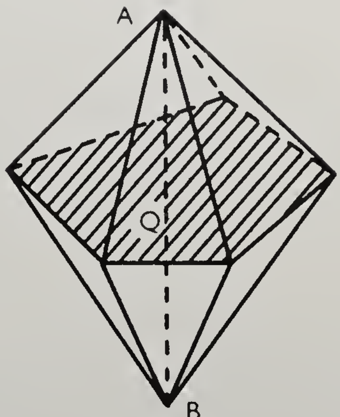


Fig. 7.

We may verify that every face of a bi-pyramid M is either a proper face of Q, or a pyramid with a face of Q as a base and either a or b as a vertex, or one of the vertices a or b on its own.

This gives the proposition :

Proposition 5.4 *Let* M *be a d-bipyramid with base* Q *and* $\dim Q = d-1$. *Then*

$$f_j(M) = f_j(Q) + 2f_{j-1}(Q) \qquad j \in N_{d-2}$$

$$f_{d-1}(M) = 2f_{d-2}(Q) .$$

We return to the proof of Theorem 5.2. Suppose the theorem is true for f-vectors of polytopes of dimension not greater than d-1 (the theorem is trivial for d=1). Let M^* be a d-pyramid with base Q

and let M^{**} be a d-bipyramid with the same base Q. Their f-vectors satisfy

$$\sum_{j=0}^{d-1} \alpha_j f_j(M^*) = \beta \tag{5.4}$$

$$\sum_{j=0}^{d-1} \alpha_j f_j(M^{**}) = \beta \quad . \tag{5.5}$$

According to Propositions 5.3 and 5.4 the f-vectors of the pyramid M^* and the bipyramid M^{**} take the form

$$f(M^*) = (1+f_0(Q), f_0(Q)+f_1(Q), .., f_{d-3}(Q)+f_{d-2}(Q), f_{d-2}(Q)+1) ,$$

$$f(M^{**}) = (2+f_0(Q), 2f_0(Q)+f_1(Q), .., 2f_{d-3}(Q)+f_{d-2}(Q), 2f_{d-2}(Q)).$$

Subtracting equation (5.4) from (5.5) we obtain

$$\sum_{j=0}^{d-2} \alpha_{j+1} f_j(Q) = \alpha_{d-1} - \alpha_0 \quad .$$

Excluding the trivial case, when (5.3) is an identity and using the inductive assumption on the (d-1)-polytope Q, we have

$$\alpha_j = (-1)^j \alpha_0 \qquad j \in N_{d-1} \quad \text{and} \quad \alpha_0 = (-1)^{d-1} \alpha_{d-1} \quad .$$

Substituting in (5.3) the values of $f_j(M)$ for a d-simplex, we find $\beta = (1 - (-1)^d)\alpha_0$. Hence, equation (5.3) is equivalent to the Euler-Poincaré formula.

5.2 The Dehn-Sommerville Equations

Theorem 5.2 established that there are no linear relations, other than the Euler-Poincaré formula, connecting the components of f-vectors of arbitrary d-polytopes. However the f-vectors of certain special classes of polytopes may satisfy other linear relations. Of these the most important are the Dehn-Sommerville equations for Simplicial polytopes. Dehn (1905), a pupil of Hilbert, showed that the f-vector of a simplicial polytope satisfied two linearly independent equations when $d=4$ and three such equations when $d=5$. He made the hypothesis that the number of such equations for arbitrary d was $[(d+1)/2]$. Subsequently, the English geometer Sommerville (1927) found the equations for any d.

Theorem 5.5 (Dehn-Sommerville) *Let* M *be a simplicial polytope. Then*

$$\sum_{j=k}^{d-1} (-1)^j \binom{j+1}{k+1} f_j(M) = (-1)^{d-1} f_k(M) \quad , \quad k=0,1,..,d-2. \qquad (5.6)$$

Proof For every k-face F^k and j-face F^j of M $(0 \leq k \leq j \leq d-1)$ we define the function

$$\delta(F^k,F^j) = \begin{cases} 0 & \text{if } F^k \not\subseteq F^j \\ 1 & \text{if } F^k \subseteq F^j \end{cases} .$$

To calculate the sum

$$\sum_{j=k}^{d-1} (-1)^j \sum_{F^k} \sum_{F^j} \delta(F^k,F^j) \qquad (5.7)$$

(the sum is taken over all k and j faces of M, with k fixed) we need the following Lemma.

Lemma 5.6 *Let* F_1,F_2 *be faces of the polytope* M *and let* $\mathcal{F}(F_1,F_2)$ *be the set of all faces* F *of* M *such that* $F_1 \subseteq F \subseteq F_2$. *Then there exists a polytope (denoted by* $M(F_1,F_2)$*) of dimension* $\dim F_2 - \dim F_1 - 1$ *such that there is a bijection* ψ *between the faces of* $M(F_1,F_2)$ *and the set* $\mathcal{F}(F_1,F_2)$ *with the property*

$$F' \subset F'' \quad \Leftrightarrow \quad \psi(F') \subset \psi(F'') \quad .$$

Proof of Lemma By Theorem 2.9 F_1 is a face of F_2. Let F_2^* be the polytope dual to F_2. Then by Lemma 3.6, the face $\phi(F_1) = \{y \in F_2^* : xy=1 \ \forall x \in F_1\}$ of F_2^* has dimension $\dim F_2 - \dim F_1 - 1$. We also have $\phi(F_2) \subseteq \phi(F_1) \subseteq \phi(\emptyset) = F_2^*$. Thus, if we transform from the polytope $\phi(F_1)$ to its dual $(\phi(F_1))^*$, we obtain the required polytope $M(F_1,F_2)$, (Fig.8). //

Thus, to each face F^j of M for which $\delta(F^k,F^j) = 1$, there corresponds a (j-k-1)-face of the (d-k-1)-polytope $M(F^k,M)$ and conversely. Hence $\sum_{F^j} \delta(F^k,F^j)$ gives the number of (j-k-1)-faces of the polytope $M(F^k,M)$ and so, by the Euler-Poincaré equation

$$\sum_{j=k}^{d-1} (-1)^j \sum_{F^j} \delta(F^k,F^j) = (-1)^{d-1} \quad .$$

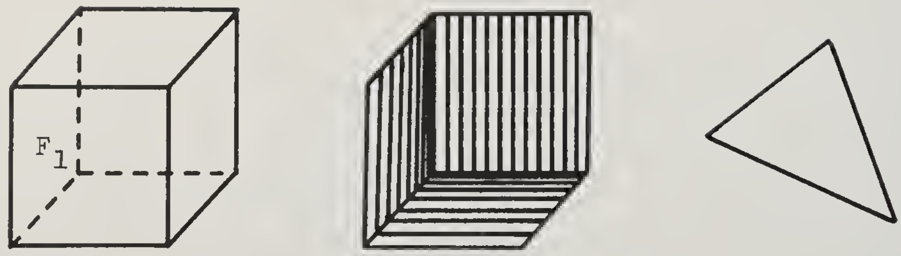


Fig. 8.

Note that the case $j=k$ corresponds to the improper face $\emptyset = M(F^k,F^k)$. Hence, summing over the k-faces of M,

$$\sum_{F^k} \sum_{j=k}^{d-1} (-1)^j \sum_{F^j} \delta(F^k,F^j) = (-1)^{d-1} f_k(M) \quad . \tag{5.8}$$

On the other hand, $\sum_{F^k} \delta(F^k,F^j)$ equals the number of k-faces of the j-polytope F^j and since F^j is a simplex

$$\sum_{F^k} \delta(F^k,F^j) = \binom{j+1}{k+1} \quad .$$

Thus

$$\sum_{F^j} \sum_{F^k} \delta(F^k,F^j) = \binom{j+1}{k+1} f_j(M) \; . \tag{5.9}$$

Substituting (5.9) into (5.7) and using (5.8) we obtain equation (5.6) . //

From now on equations (5.6), together with the Euler-Poincaré formula (k=-1), will be called the *Dehn-Sommerville equations* and will be written in the form

$$\sum_{j=k}^{d-1} (-1)^j \binom{j+1}{k+1} f_j(M) = (-1)^{d-1} f_k(M) \tag{5.10_k}$$

where $k=-1,0,1,\ldots,d-2$ with $f_{-1}(M) = 1$. Among the Dehn-Sommerville equations not less then $[(d+1)/2]$ equations are linearly independent. Indeed, if d is even then for $j=0,1,\ldots,d/2 - 1$ the term $f_{2j}(M)$ only occurs in the first $j+1$ of the equations $(5.10_{-1}),(5.10_1),\ldots(5.10_{d-3})$. Thus all of these equations are linearly independent. Similarly, if d is odd then the inhomogeneous equation (5.10_{-1}) and the equations (5.10_1), $\ldots,(5.10_{d-2})$ in which the term f_{2j-1} $\forall j \in N_{(d-1)/2}$ occurs only in the first $j+1$ equations, are linearly independent. We show that there are exactly $[(d+1)/2]$ linearly independent equations. To this end we construct $[d/2]+1$ simplicial polytopes with affinely independent f-vectors. Consider the cyclic polytopes $C(d,n),C(d,n+1),\ldots,C(d,n+k)$, where $k=[d/2]$. Their f-vectors are affinely independent, because the determinant

$$D = \begin{vmatrix} 1 & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{k} \\ 1 & \binom{n+1}{1} & \binom{n+1}{2} & \cdots & \binom{n+1}{k} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot \\ 1 & \binom{n+k}{1} & \binom{n+k}{2} & \cdots & \binom{n+k}{k} \end{vmatrix},$$

constructed from the first k components of their f-vectors, is non-zero (in fact, $D = 1$). This is easily verified by successively subtracting each row from its successor. Combining this result with the earlier result that not less than $[(d+1)/2]$ Dehn-Sommerville equations are linearly independent, we obtain the following result

Theorem 5.7 *For* $d \geq 1$ *exactly* $[(d+1)/2]$ *Dehn-Sommerville equations are linearly independent. Geometrically, this implies that the affine hull of the set of* f-*vectors of simplicial* d-*polytopes has dimension* $[d/2]$.

5.3 Solutions of the Dehn-Sommerville Equations

The rank of the Dehn-Sommerville system equals $[(d+1)/2]$ so that we can solve for $[(d+1)/2]$ of the variables f_i in terms of the others. Obviously there are many possible sets of linearly independent columns of the matrix of this system of equations, so that there are a variety of different schemes for solving these equations in the literature. The following variant, in which the second half of the list of variables f_i is expressed in terms of the first half, is presented in McMullen & Shephard (1971). We adopt the convention that the binomial coefficient $\binom{a}{b}$ is zero if $b<0$ or $b>a$.

Theorem 5.8 *The f-vector of any simplicial d-polytope satisfies the following equations*

$$f_{m+p} = \begin{cases} \sum_{q=0}^{m-1} (-1)^q \dfrac{m-q}{p+q+1} \chi(m-1,p,q)\, f_{m-q-1} & \textit{when} \quad d=2m \\ \sum_{q=0}^{m-1} (-1)^q \dfrac{m+p+2}{p+q+1} \chi(m,p,q)\, f_{m-q-1} & \textit{when} \quad d=2m+1 \end{cases}$$

where $p=0,1,\ldots,m-1$ *and*

$$\chi(m,p,q) = \sum_{s\geq 0} \binom{m-s}{p}\binom{m-s+q+1}{m+1} \quad .$$

We remark that Stanley (1980) has recently given a complete characterization of the f-vectors of simplicial polytopes.

5.4 3-Polytopes

For simplicial 3-polytopes the Dehn-Sommerville equations take the form

$$f_0 - f_1 + f_2 = 2 \quad , \qquad -2f_1 + 3f_2 = 0 \quad ,$$

or $f_1 = 3f_0-6$, $f_2 = 2f_0-4$.

Theorem 5.9 *The vector* (f_0,f_1,f_2) *is an f-vector of a simplicial 3-polytope if and only if*

$$f_1 = 3f_0 - 6 \quad \textit{and} \quad f_2 = 2f_0 - 4 \quad .$$

Proof It suffices to show that for any vector (f_0,f_1,f_2) satisfying the conditions of the theorem, there exists a simplicial 3-polytope with this f-vector. We use induction. If $f_0=4$ we have a 3-simplex with vector $(4,6,4)$. If $f_0>4$ suppose M is a simplicial polytope such that $f(M) = (f_0-1, 3(f_0-1)-6, 2(f_0-1)-4)$. Let $v^*\epsilon E_3$ be a point which does not lie in any of the planes generated by the vertices of M and which is strictly separated from M by precisely one of the supporting planes which generate the 2-faces of M . Then it is easily verified that the simplicial polytope $M^* = \text{conv}\,(M \cup v^*)$ has the f-vector

$$f(M^*) = (f_0 \,,\, 3f_0-6 \,,\, 2f_0-4) \quad . \qquad //$$

Transforming the simplicial polytopes into their duals, namely, the simple 3-polytopes, we obtain the following Corollary.

Corollary 5.10 *The f-vectors of simple 3-polytopes take the form*

$$(2f_2-4\ ,\ 3f_2-6\ ,\ f_2), \qquad f_2 = 4,5,6,\ldots.$$

In the general case we have the following theorem due to Steinitz (1922).

Theorem 5.11 *The vector* (f_0,f_1,f_2) *is an f-vector of a 3-polytope if and only if the whole numbers* f_0,f_1,f_2 *satisfy the following conditions :*

$$f_1 = f_0 + f_2 - 2 \quad , \quad 4 \leqq f_0 \leqq 2f_2 - 4 \quad , \quad 4 \leqq f_2 \leqq 2f_0 - 4.$$

EXERCISES

1. Show that the following is an equivalent alternative definition of a vertex of a polyhedron : the point $x^0 \varepsilon P$ is a vertex of the polyhedron P if there exists a vector c such that $\max\{cx : x\varepsilon P\}$ is attained only at the point x^0.

2. Let v be a vertex of a polytope M and let H^+ be a closed half-space such that $v \varepsilon \operatorname{bd} H^+$ and such that all edges incident to v belong to H^+. Show that H^+ is a supporting hyperplane of M.

3. Let M be a d-polytope in E_d and let τ be a projective map in E_d (not necessarily non-degenerate). Is it true that $f_i(\tau(M)) \leqq f_i(M)$, $i\varepsilon N_d$?

4. Let $F_{k-1} \subset F_{k+1}$, where F_{k-1} and F_{k+1} are faces of a d-polytope M of dimension $k-1$ and $k+1$ respectively. Show that : 1) there are exactly two k-faces of M which contain F_{k-1} and which are contained in F_{k+1} ; 2) for any $k\varepsilon \dot{N}_d$ there is a (d-k)-face of M which does not contain k specified vertices of M ; 3) an i-face of a simple d-polytope is contained in exactly $\binom{d-i}{d-j}$ j-faces, where $0\leqq i\leqq j\leqq d-1$.

5. A d-*parallelepiped* is the sum of d non-parallel line segments with a common endpoint. The simplest d-parallelepiped is the unit d-*cube* (denoted by K_d). The cube K_d is a polytope which is the sum of d mutually orthogonal line segments of unit length, i.e.

$$\mathcal{K}_d = \mathrm{conv}(0, e_1, \dots, e_d) = \{e \in E_d : 0 \leqq x_i \leqq 1 \ , \forall i \in N_d\} .$$

Here $e_1, \dots, e_d$ is an orthogonal basis of E_d. Show that

$$f_k(\mathcal{K}_d) = 2^{d-k}\binom{d}{k}, \qquad k = 0,1,\dots,d-1 .$$

6. Let Q be a $(d-1)$-polytope in E_d and let the line segment $I = [0,a]$ be not parallel to the hyperplane aff Q. Then the sum $M = Q + I$ is called a d-*prism* with base Q. It is easy to see that the prism M is the convex hull of the base Q and the set $a + Q$. Every k-face of the prism M either coincides with a k-face of Q or of $a + Q$, or is the sum of the segment I and a $(k-1)$-face of Q. Conversely, every proper face of Q and of $a + Q$ is a face of M; the sum of I and any face of Q is also a face of M. Show that

$$f_k(M) = 2f_k(Q) + f_{k-1}(Q) \quad , \quad k = 0,1,\dots,d-1 .$$

The simplest example of a d-fold bi-pyramid is the d-*octahedron* Q_d which is the convex hull of d line segments which are mutually orthogonal and which have a common interior point. Show that

$$f_k(Q_d) = 2^{k+1}\binom{d}{k+1}, \qquad k = 0,1,\dots,d-1 .$$

7. (Grünbaum 1967). Let k and s be integers such that $1 \leqq r, s \leqq d-1$. The d-polytope M is called r-*simplicial* if all its r-faces are simplexes, and s-*simple* if every $(d-1-s)$-face is contained in exactly $s+1$ facets of M. We say that M is of type (r,s) if it is r-simplicial and s-simple. Show that:

(1) a simplicial d-polytope is of type $(d-1,1)$ and a simple d-polytope is of type $(1,d-1)$;

(2) an i-simple polytope is also j-simple for all $j \leqq i$;

(3) if M is a d-polytope of type (r,s) with $r+s \geqq d+1$ then M is a simplex;

(4) the polytope given by the intersection of a $(d+1)$-cube in E_{d+1} with the hyperplane $\sum_{i=1}^{d+1} x_i = k$ is of type $(2,d-2)$;

(5) the d-polytope given by the conditions

$\sum_{i=1}^{d} \eta_i x_i \leqq d-2$, $\eta_i = \pm 1$, $\forall i \in N_d$, where there are an odd number of η_i equal to 1, is of type $(3,d-3)$.

8. Let T_r and T_{d-r} be simplexes located in E_d so that the intersection T_r T_{d-r} is a single point belonging to rel int $T_r \cap$ rel int T_{d-r}, where $r \leqq [d/2]$. Show that the polytope $T_d^r = \text{conv}(T_r \cup T_{d-r})$ is simplicial and calculate $f_i(T_d^r)$ for all i.

9. Let $A \subseteq E_n$ and $D \subseteq E_n$ be non-empty sets and let A^* and D^* be their polars. The set $\bar{A} = A^* \cap D$ is called *antiblocking relative to* D. In studying minimax relations in integer programming it is important to know conditions under which $A = \bar{\bar{A}}$, i.e. under which A and $\bar{A}$ are a pair of antiblocking sets (Fulkerson 1971). Note that relative to E_n the antiblocking set of A coincides with the polar A^*. Show that $A = \bar{\bar{A}}$ relative to a closed convex set D with $0 \in D$ if and only if there is a closed convex set $C \subseteq E_n$ such that $A = C \cap D$ and $D^* \subseteq C$. Let $A = \{x \in E_n^+ : Ax \leqq e\}$, where A is a matrix with non-negative elements which does not contain a zero column. Show that $A = \bar{\bar{A}}$ relative to the set $D = E_n^+$ and that $\bar{A} = \{x \in E_n^+ : Bx \leqq e\}$ where B is a matrix whose rows are the coordinates of the vertices of the polytope A.

10. Let the polytope M be defined as the intersection of closed halfspaces: $M = \{x \in E_d : A_i x \leqq 1, \forall i \in N_n\}$. Show that $M^* = \text{conv}(A_1, \ldots, A_n)$.

11. Specify the dual polytope of the cyclic polytope $C(d,n)$ as an intersection of supporting half-spaces.

12. Let Q be a self-dual $(d-1)$-polytope. Show that a d-pyramid with base Q is also self-dual.

13. Generalize tha polytope duality theorem (Th. 3.7) to the case of polyhedra.

14. Does there exist a polytope all of whose non-rigid constraints are redundant?

15. A *section of a polytope* M is a set $M \cap A$ where A is some affine set. Show that any d-polytope with n facets $(n \geqq d+1)$ is a section of an $(n-1)$-simplex.

16. Let the polytope $M^*(A,b)$ be given in E_n by an irreducible system $Ax \leqq b$, where $A \in E_{m \times n}$. An *r-dimensional basis set* of $M^*(A,b)$ is a set of solutions of a system consisting of $n-r$ linearly independent equations of the form $A_i x = b_i$, $\forall i \in I \subset N_m$. A *basis point* is a 0-dimensional basis set. If an r-dimensional basis set has a non-empty intersection with the polytope, then this intersection is a face of the polytope.

Bartels (1973) proved that:

(1) the minimum number of basis points of $M^*(A,b)$ is given by

$$2^{m-1}(n-m+2) \qquad \text{if } m \leqq n\ ;$$
$$2^{n-3}(m-n+2)(m-n+4) \qquad \text{if } m > n \text{ and } m-n \equiv 0 \pmod 2\ ;$$
$$2^{n-3}(m-n+3)^2 \qquad \text{if } m > n \text{ and } m-n \equiv 1 \pmod 2\ ;$$

(2) if $M^*(A,b)$ has the minimum number of basis points, then the components of its f-vector are given by

$$f_0 = \begin{cases} 2^{m-1}(n-m+2) & \text{if } m \leqq n\ , \\ 2^{n-2}(m-n+4) & \text{if } m \geqq n\ , \end{cases}$$

$$f_r = \begin{cases} \sum_{i=0}^{r} 2^{m-i-1} \; {m-1 \atop i} \binom{n-m+2}{r-i+1} & \text{if } m \leqq n\ , \\ 2^{n-r-2}(m-n+4)\binom{n-2}{r} + 2^{n-r-1}(m-n+4)\binom{n-2}{r-1} + \\ \quad + 2^{n-r}\binom{n-r}{r-2}\ , & \text{if } n \geqq m\ ; \end{cases}$$

(3) the maximum number of r-dimensional basis sets of $M^*(A,b)$ is equal to $\binom{n}{n-m-r}$;

(4) the minimum number of r-dimensional basis sets of $M^*(A,b)$ is given by

$$\sum_{i=0}^{r} 2^{m-i-1}\binom{m-1}{i}\binom{n-m+2}{r-i+1} \qquad \text{if } m \leqq n\ ;$$

$$2^{n-r-3}(m-n+4)\binom{n-2}{r} + 2^{n-r-1}(m-n+4)\binom{n-2}{r-1} + 2^{n-r}\binom{n-2}{r-2}$$
$$\text{if } m > n\ ,\ m-n \equiv 1 \pmod 2;$$

$$2^{n-r-3}(m-n+3)\binom{n-2}{r} + 2^{n-r-1}(m-n+4)\binom{n-2}{r-1} + 2^{n-r}\binom{n-2}{r-2}$$
$$\text{if } m > n\ ,\ m-n \equiv 0 \pmod 2).$$

17. Let $\mathcal{M}^k(m,n)$ be the class of polytopes $M(a,b) \in \mathcal{M}(m,n)$ for which there exists a basis B such that the rank of the simplex-matrix Λ_B is equal to k. Show that:

(1) if $M(A,b) \in \mathcal{M}(m,n)$, then $\beta(A,b) = 1 + \sum_{s=1}^{\min(m,n-m)} i_s(\Lambda_B)$,

where $i_s(\Lambda_B)$ is the number of nonzero minors of order s of the simplex-matrix Λ_B ;

(2) if $M(A,b) \in \mathcal{M}^k(m,n)$, then $\beta(A,b) \leqq 1 + \sum_{s=1}^{k}\binom{m}{s}\binom{n-m}{s}$

where the bound can be attained;

(3) if $M(A,b) \in \mathcal{M}^1(m,n)$ then the function $\beta(A,b)$ can only take the values $1 + s(n-m)$, $\forall s \in N_m$.

18. Let $\mathcal{M}$ be a matroid on the set N_n of rank r. Let $i(\mathcal{M})$ be the number of independent sets of the matroid $\mathcal{M}$ and let $\beta(\mathcal{M})$ be the number of bases of the matroid. Show that:

(1) if the integer t is such that $\binom{n}{t} \leqq \beta(\mathcal{M}) \leqq \binom{n}{t+1}$, then

$$1 + \binom{n}{1} + \ldots + \binom{n}{t} \leqq i(\mathcal{M}) \leqq 1 + \binom{n}{1} + \ldots + \binom{n}{t+1} ;$$

(2) every matroid of rank 2 is a vector matroid.

19. Find a condition which ensures that the set of columns of an $(m\times n)$-matrix of rank m can be partitioned into k pairwise non-intersecting linearly independent subsets. Generalize the condition to the case if matroids.

20. Let x be a non-degenerate vertex of a polytope M in E_n for which the rank of the system of rigid constraints is equal to m. Then the number of edges which are incident to x equals $n - m$. Give examples which show that if x is a degenerate vertex of M then the number of edges incident to x can be greater than $n - m$. An i-face F of a polytope M is called *degenerate* if all points in F satisfy more than $n - i$ of the inequalities defining M as equalities. If the d-polytope M has a degenerate face, does it follow that M is a degenerate polytope? If M has a degenerate i-face, does it follow that M has a degenerate q-face for $q \leqq i$?

21. (Chernikov 1968). When $n \leqq 5$ or $m \leqq 2$ the number of vertices of a non-degenerate polytope in E_n given by the constraints $Ax \leqq b$, $A \in E_{n\times m}$ uniquely determines the remaining components of the f-vector. Give examples with $m > 2$ or $n > 5$ for which this assertion is not true.

22. (Klee 1964, Grünbaum 1970). Verify the following non-linear relations satisfied by the f-vectors of simplicial d-polytopes:

$$\binom{k+1}{r} f_k \leqq \binom{f_0+r-1-k}{r} f_{k-r} ;$$

$$(k+1)\binom{k}{r} f_k \leqq \binom{f_0+r-k}{r}((k+1-r)f_{k-r} - rf_{k-r-1}) ,$$

where $r = 0,1,\ldots,k$ and $k = 0,1,\ldots,d$.

23. (McMullen 1977). The following is a generalization of the Euler-Poincaré formula. Let F be a k-face of the d-polytope M. Then

$$\sum_{j=k}^{d-1} (-1)^j h_j(F) = (-1)^{d-1} ,$$

where $h_j(F)$ is the number of j-faces of M which contain F.

24. (Grünbaum 1967). Let P be a polyhedron of dimension d and let $f_k^0(P)$ be the number of bounded k-faces and $f_k^\infty(P)$ be the number of unbounded k-faces. Then the following formulae are true:

$$\sum_{i=0}^{d-1} (-1)^i f_i^0(P) = 1 , \quad \sum_{i=0}^{d} (-1)^i f_i^\infty(P) = 1 ,$$

$$\sum_{i=0}^{d} (-1)^i f_i(P) = 0 , \quad f_i(P) = f_i^0(P) + f_i^\infty(P) .$$

25. (Grünbaum 1967). Obtain the following relations from the Dehn-Sommerville equations:

$$f_m = \sum_{i=0}^{m-1} (-1)^{m-i+1} \frac{i+1}{m+1} \binom{2m-i}{m} f_i ,$$

$$f_{2m-1} = \sum_{i=0}^{m-1} (-1)^{m-i+1} \frac{i+1}{m} \binom{2m-2-i}{m-i} f_i ,$$

if $d = 2m$, and

$$f_m = \sum_{i=-1}^{m-1} (-1)^{m-i-1} \binom{2m-i+1}{m+1} f_i ,$$

$$f_{2m} = 2 \sum_{i=-1}^{m-1} (-1)^{m-i+1} \binom{2m-i-1}{m} f_i ,$$

if $d = 2m+1$.

26. The Dehn-Sommerville equations yield the following relations for the f-vector components of simplicial 4- and 5-polytopes:

$d = 4$, $f_2 = 2f_1 - 2f_0$, $f_3 = f_1 - f_0$;

$d = 5$, $f_2 = 4f_1 - 10f_0 + 20$, $f_3 = 5f_1 - 15f_0 + 30$, $f_4 = 2f_1 - 6f_0 + 12$.

27. The Dehn-Sommerville equations are equivalent to the following:

$$\sum_{i=-1}^{k-1} (-1)^{d+i} \binom{d-i-1}{d-k} f_i = \sum_{i=-1}^{d-k-1} (-1)^i \binom{d-i-1}{k} f_i ,$$

for $k = 0, 1, \ldots, [(d-1)/2]$.

28. If $d = 2m$ then the affine hull of the f-vectors of simplicial d-polytopes is identical with the affine hull of the $m+1$ vectors h^k, where $h^k = (h_0^k, \dots, h_{2m-1}^k)$ and

$$h_i^k = \binom{k}{1+i-k}, \quad i = 0,1,\dots,2m-1 \quad , \quad k = 0,1,\dots,m .$$

29. For the case of simple d-polytopes the Dehn-Sommerville equations take the form

$$\sum_{i=0}^{r} (-1)^{i+d-r-1} \binom{i+d-r}{d-r} f_{r-i} = (-1)^{d-1} f_r ,$$

or

$$\sum_{i=1}^{r} (-1)^{i+d-r-1} \binom{i+d-r}{d-r} f_{r-i} = f_r((-1)^{d-1} - (-1)^{d-r-1}) ,$$

$$\text{for} \quad r \in N_{[d/2]} .$$

30. The f-vector of any simplicial d-polytope satisfies the relations :

$$f_{m+p} = \sum_{i=-1}^{d-m-2} \sum_{k=0}^{d-m-1} (-1)^{k+i+1} \binom{k}{d-m-1-p} \binom{d-1-i}{d-k} f_i +$$

$$+ \sum_{i=1}^{m-1} (-1)^{m+i+1} \binom{d-1-i}{d-m-1-p} \binom{m+p-i-1}{p} f_i ,$$

$$\text{for} \quad p = 0,\dots,d-m-1 .$$

2 GRAPHS OF POLYTOPES

The pair consisting of the set of vertices and the set of edges (1-faces) of a polytope M is called the *graph of the polytope* and is denoted by $G(M)$.

Polytope graphs have may interesting properties. In studying them we encounter many problems of interest in graph theory, combinatorial theory, topology, geometry and also in the theory of linear programming.

A graph G is called a d-*polyhedral graph* if and only if it is isomorphic to the graph of some d-polytope M. In this case, we say that the polytope M is a realization of the polytope graph $G(M)$. The first and most basis problem in the theory of polyhedral graphs consists in describing the properties of these graphs. The case in which $d=2$ is trivial. Indeed, the graph G is 2-polyhedral if and only if G is a cycle with $n\geq 3$ vertices. The 3-polyhedral graphs have also been characterized completely by Steinitz (1922). In the general case a notable result is the theorem concerning the number of non-intersecting chains in a d-polyhedral graph due to Balinski (1961) which was obtained independently by Remesh & Steinberg (1967) and Medyanik (1972).

A second problem, which has greatly influenced the theory of polyhedral graphs, is connected with the problem of determining the efficiency of linear programming methods and consists in finding upper and lower bounds for such metric characteristics of polyhedral graphs as the diameter, radius, height, etc.

§1 CONNECTEDNESS OF POLYHEDRAL GRAPHS

Many results are known concerning characterizations of d-polyhedral graphs but for the case $d\geq 4$ a complete solution of the basic problem has so far not been found. The only general result is due to Balinski and concerns the connectivity of polyhedral graphs.

1.1 Definitions

To avoid terminological confusion we present some of the standard definitions of Graph Theory (Harary 1969). A *graph* is a pair (V,E) consisting of a finite, non-empty set V, whose elements are called *vertices* (or *nodes*) and a set E of unordered pairs of distinct vertices of V. Each pair of vertices $e = (i,j)$, $i,j \in V$, in E is called an *edge* (or *arc*) *of the graph* G. We say that i and j are *adjacent vertices* which are *incident to the edge* e.

In a polytope graph, neighbouring vertices belong to the 1-faces (edges) of the polytope. We call such vertices *adjacent vertices of the polytope*. By considering different specifications of a polytope we can obtain different criteria for vertices to be adjacent. Thus, if a polytope M is given in E_n in canonical form (the most common method) by

$$Ax = b \quad , \quad x \geq 0 \tag{1.1}$$

then an edge is a non-empty set of points of the polytope which satisfy the additional conditions

$$x_j = 0 \qquad \forall j \in \omega \tag{1.2}$$

where ω is a subset of N_n such that the number of linearly independent equations in (1.1) and (1.2) is $n-1$. This leads to the following definition of adjacent vertices of a polytope, which is equivalent to the previous definition.

Definition 1.1 Two vertices of a polytope M, given in canonical form, are *adjacent* if their corresponding feasible bases differ by only one column.

A *subgraph* of a graph G is a graph all of whose vertices and edges belong to G. A subgraph which contains all the vertices of a graph G is a *spanning subgraph* of G. If S is a subset of the vertices of a graph G then the maximal subgraph of G having vertices S is called the *graph generated by* S (written $G(S)$). A *chain* L in a graph G between the vertices u and v is a subgraph with vertices $u=v_0,v_1,\dots,v_n=v$ and edges (v_{i-1},v_i) $i \in N_n$ where all the edges are distinct. If in a chain $v_0=v_n$ then the chain is called a *cycle*. If all the vertices in a chain L are distinct, then L is a *simple chain* . If, in addition, $v_0 = v_n$ then L is a *simple cycle* . The graph G is *connected* if there is a simple chain connecting any two vertices of G . Two simple chains between the vertices u and v are called *non-vertex-intersecting* if they have no vertices in common, other than u and v . A graph G is d-*connected* if there are d non-vertex-intersecting chains connecting any pair of its vertices.

If we remove a vertex v from a graph G we obtain a subgraph G_v which contains all the vertices of G other than v and all the edges of G except for those which are incident to v . The following theorem due to Whitney (1933) gives a criterion for the d-connectedness of a graph.

Theorem 1.1 *A graph* G *is* d-*connected if and only if the subgraph of* G *obtained by removing any* d-1 *vertices is connected.*

The *degree of a vertex* v of a graph G is the number of edges which are incident to v (written deg v). Clearly, in any d-connected graph, every vertex has degree not less than d .

1.2 Balinski's Theorem

Theorem 1.2 *The graph of a* d-*polytope is* d-*connected.*

Proof By Theorem 1.1 it suffices to show that the removal of any d-1 vertices does not destroy the connectedness of the polytope graph. Let $\{x^1,\dots,x^{d-1}\}$ be any set of d-1 vertices of a d-polytope M and let G*(M) be the subgraph of the graph G(M) obtained by removing the vertices $x^1,\dots,x^{d-1}$.

We show that G*(M) is a connected graph. Let $Q = \text{aff}\,(x^1,\dots,x^{d-1})$. There are two cases : a) $Q \cap \text{int } M = \emptyset$, b) $Q \cap \text{int } M \neq \emptyset$.

Case a). Let $F = Q \cap M$ be a face of M (dim Q < d-1) and let H be a supporting hyperplane which generates F . Consider the opposite supporting hyperplane H' which is parallel to H . Every vertex x of the graph G*(M) either lies in H' , or there exists a vertex x' adjacent to it such that the distance of x' from H' (in the Euclidean metric) is strictly less than the distance of x from H'.

Indeed, if $x \notin H'$ then the required vertex x' exists, otherwise the hyperplane passing through x which is parallel to H' would be supporting to M, which is impossible (see Problem 2, Ch.1). If $x' \notin H'$ then there is a vertex x'' adjacent to x', whose distance from H' is less than that of x', etc. Eventually we construct a chain in $G^*(M)$ from the vertex x to some vertex $x^* \in H'$. Similarly, for any other vertex $y \neq x$ in $G^*(M)$, there is a chain which connects it to some vertex $y^* \in H'$. Since $H' \cap M$ is a polytope, its graph is connected and there is a chain in the graph $G(H' \cap M)$ connecting y^* and x^*. Thus, there is a chain connecting any two vertices y and x in the graph $G^*(M)$; that is, $G^*(M)$ is connected.

Case b). Let H be a hyperplane which contains the affine set Q and some other vertex u of M. Such a hyperplane exists, since $\dim Q < d-1$. Consider the two supporting hyperplanes H' and H'' to the polytope M which are parallel to H. Let x and y be arbitrary vertices of M distinct from $x^1, \ldots, x^{d-1}$. The case in which x and y both belong to one of the closed half-spaces generated by H ia analagous to case a). So, let x and y belong to distinct half-spaces generated by H. As in case a) we can find a chain in $G^*(M)$ connecting x and u and another chain connecting y and u. Their union is a chain connecting x and y. Hence $G^*(M)$ is a connected graph. //

Corollary 1.3 *The graph obtained from a polytope graph by removing all the vertices of an arbitrary face is connected.*

To prove the corollary we construct two supporting hyperplanes H and H' to M which are parallel and one of which generates the face F containing the vertices which have been removed. A chain connecting any two vertices x and y may then be found as in the proof of case a) in Theorem 1.2.

1.3 Steinitz's Theorem

A graph is called *planar* if it can be represented in the plane in such a way that no two of its edges intersect.

Theorem 1.4 (Steinitz (1922)) *A graph is 3-polyhedral if and only if it is planar and 3-connected.*

The significance of Steinitz's Theorem is that it enables us to replace the syudy of 3-polytopes by the study of 3-connected planar graphs.

Proof The necessity of Steinitz's conditions is obvious. The assertion that every graph of a 3-polytope is 3-connected is a special case of Theorem 1.2. A realization of a 3-polytope graph on the plane can be obtained as follows. Excise one of the faces of the polytope and deform the remaining faces such that they are brought to lie in the plane of the excised face. The regions delineated by the graph on the plane are called the *faces of the graph* . The unbounded region is called the *exterior face* , while the other regions are *interior faces* . It is clear that the faces of the graph of a 3-polytope M are bijectively related to the facets of M .

The sufficiency of Steinitz's conditions is the hard part of the theorem. The known proofs use induction on the number of edges e of the 3-polytope which corresponds to the given 3-connected graph. The assumption that G is 3-connected implies that $e \geq 6$ and equality is only possible when $G=K_4$, the complete graph on 4 vertices. (In a *complete graph* any pair of vertices are connected by an edge). In this case G corresponds to a 3-simplex. The common step in the inductive proof is divided into two stages. First, it is shown how any 3-connected planar graph with more than 6 edges can be associated with a graph G^* of the same type but having fewer edges than G . Second, it is shown how to construct a 3-polytope M corresponding to G by using a 3-polytope corresponding to G^* . The known proofs differ in the methods used in the second stage.

We now examine the details of the proof of sufficiency. For brevity we introduce the notation

$$v = f_0(M) , \quad e = f_1(M) , \quad p = f_2(M) .$$

Let v_k be the number of vertices of a polytope M of degree k and let p_k be the number of facets of M which have k vertices (k-polygons). Then

$$v = \sum_{k \geq 3} v_k \quad \text{and} \quad p = \sum_{k \geq 3} p_k$$

and Euler's formula takes the form $v-e+p = 2$. Since each edge is the intersection of two facets, we have $2e = \sum_{k\geq 3} kp_k$ and similarly, since each edge is incident on two vertices, we have $2e = \sum_{k\geq 3} kv_k$. This gives

$$\sum_{k\geq 3} kp_k + \sum_{k\geq 3} kv_k = 4e = 4v + 4p - 8 = 4\sum_{k\geq 3} v_k + 4\cdot\sum_{k\geq 3} p_k - 8 \ .$$

Hence

$$v_3 + p_3 = 8 + \sum_{k\geq 5} (k-4)(v_k+p_k) \geq 8 \ ,$$

that is, every 3-polytope M has at least eight trivalent elements (triangles or vertices of degree 3).

The *reduction of a graph* G is a procedure for obtaining, from a given planar, 3-connected graph G and a fixed trivalent element in it, another graph G^* which is also planar and 3-connected. If the fixed trivalent element of G contains a vertex of degree 3 which is incident to a triangle, then the reduction will reduce the number of edges of G by at least one. If the graph does not contain such a vertex, then we show that there is a finite sequence of reductions which will lead to a graph having a vertex of degree 3 which is incident to a triangle. The reduction of the graph in the case where a vertex of degree 3 is fixed is shown in Figure 10. The case where a triangle is fixed is shown in Figure 9. It is clear that any graph G^* ,obtained by reducing a planar, 3-connected graph G, is also planar and 3-connected. Also, after using the reductions ω_i , $i\varepsilon N_3$, or η_i , $i\varepsilon N_3$, the graph G contains i fewer edges than G .

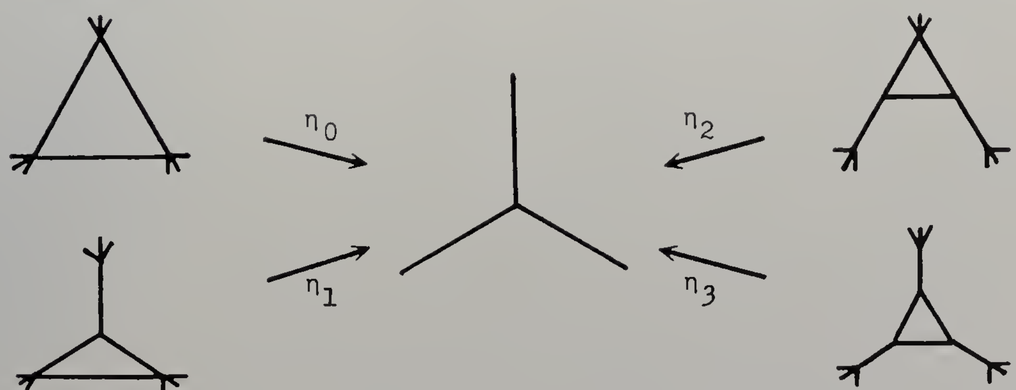


Fig. 9.

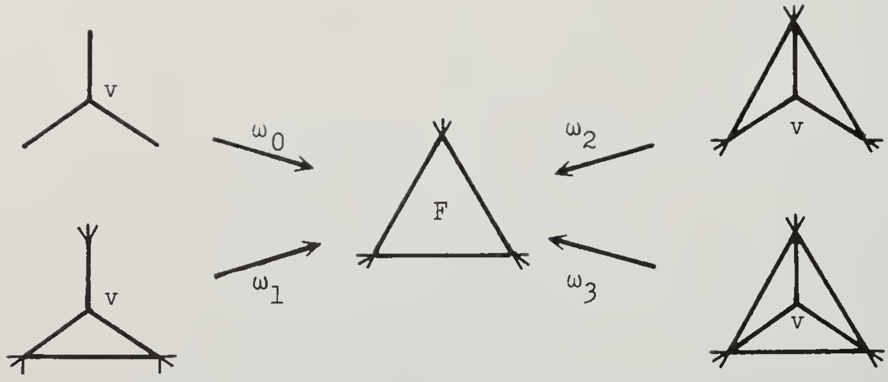


Fig. 10.

We describe a general method of reconstituting M from its 'reduced' image. Let M^* be a 3-polytope whose graph $G(M^*)$ is isomorphic to G^*. We show how to construct a polytope M corresponding to the graph G. For the reductions η_i the polytope M is obtained from from M^* by cutting away the vertex v^* with a plane. For the reductions $\omega_1, \omega_2, \omega_3$, M is the convex hull of M^* and a point v which is strongly separated from M^* by a single plane aff F. In the case of the reduction ω_0, the point v is chosen such that it coincides with the point of intersection of the planes generating the faces of M^* adjacent to the new triangle (if such a point does not exist, as in the case of parallel planes, then we apply a preliminary projective transformation to M^*). In the case of reduction ω_1, the point v lies in the intersection of only two of the planes, and for the reduction ω_2 v lies in only one such plane, while for ω_3, v lies in none of these planes.

Let G be a planar, 3-connected graph. We define a graph $I(G)$ as follows : the vertices if $I(G)$ are the edges of G. Two vertices of $I(G)$ are connected by an edge if and only if their corresponding edges in G have a common vertex and are incident to a common face in the realization of G on the plane. It is clear that the graph $I(G)$ is also planar and 3-connected and that each vertex has degree 4. There is a bijective correspondence between the faces of the graph $I(G)$ and the set of vertices and faces of G, that is, $p(I(G)) = p(G) + v(G)$. Two faces of $I(G)$ have a common edge if and only if their corresponding vertex and face in G are incident. A k-polygon in $I(G)$ corresponds to either a k-polygon in G or to a vertex of degree k .

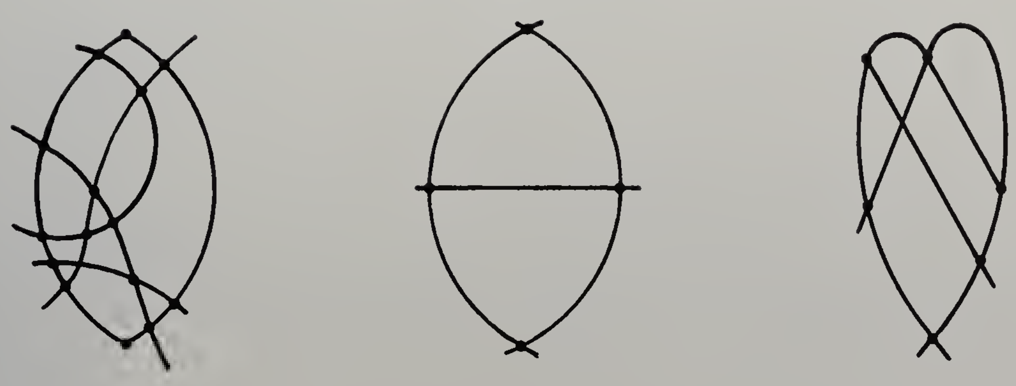
Let G be a 3-connected, planar graph all of whose vertices have degree 4. We say that the edge (i,j) has a *direct continuation* (j,k) in G if the edges (i,j) and (j,k) in a planar realization of G separate the other two edges incident to vertex j . A chain $j_0, j_1, \ldots, j_n$ in G is a *geodesic line* if $\forall k \in N_{n-1}$ the edge (j_k, j_{k+1}) is a direct continuation of (j_{k-1}, j_k). It is a *closed geodesic line* if, in addition, $j_0 = j_n$ and (j_n, j_0) is a direct continuation of (j_{n-1}, j_n).

A subgraph L of G is a *lens* if the following conditions are satisfied :

1) L consists of a cycle ℓ , called the *boundary of the lens* L , and of the vertices and edges which lie in the interior of ℓ in a plane realization of G ;

2) The cycle ℓ consists of two geodesic lines $i_0, i_1, \ldots, i_n, j_0$ and $j_0, j_1, \ldots, j_m, i_0$ such that in the subgraph L the only edges incident to the vertices i_0 and j_0 are $(i_0, i_1), (j_m, i_0), (i_n, j_0), (j_0, j_1)$.

In Figures 11a and 11b the diagrams represent lenses. The diagram in Figure 11c is not a lens.



(a) (b) (c)

Fig. 11.

A lens L is an *irreducible lens* if it does not contain a lens as a proper subgraph. Every graph G contains at least one irreducible lens. If L is an irreducible lens, then $n=m$ and every vertex i_k, $k \in N_n$, is connected to a unique vertex j_s, $s \in N_n$, by a geodesic line ℓ_k lying in L and called a *section of the lens*. Two sections ℓ_k and ℓ_r, $k \neq r$, intersect in at most one of the interior points of the lens L. Every interior point belongs to exactly two sections.

We consider the class of all subgraphs of G which contain a simple cycle composed of at most two geodesic lines and of the vertices and edges of G in the interior of the cycle. Clearly, the members of this class which are minimal with respect the numbers of vertices are irreducible lenses.

Lemma 1.5 *Every irreducible lens L contains a triangle which is incident to its boundary.*

Proof If L has no interior vertices, then the face of L incident to i_0 is a triangle. Let the lens L have interior vertices $d_1,\dots,d_r$ each of which is adjacent to some vertex i_k. Let $h(d_i)$ be the number of faces of L contained in the region i_s,i_k,d_i bounded by the two sections ℓ_s and ℓ_k which intersect at d_i and by the boundary of the lens. Let $h(d_n) = \min\{h(d_1),\dots,h(d_r)\}$. Then the vertices d_n,i_s,i_k determine a triangle which is incident to the boundary of L. //

Let $g(G)$ be the minimum number of faces in any irreducible lens L of $I(G)$. We have

$$2 \leqq g(G) \leqq \tfrac{1}{2}p(I(G)) = \tfrac{1}{2}(p(G)+v(G)) < e(G) .$$

If $g(G)=2$ then the corresponding irreducible lens is as depicted in Fig. 11b. In this case the graph G contains a triangle which is incident to a vertex of degree 3, so that one of the reductions ω_i or η_i, $i \in N_3$ may be applied to G.

To complete the proof of Theorem 1.4 it remains to show that for a graph for which $g(G) > 2$ we may use reductions ω_0 or η_0 to transform G into a graph G^* such that $g(G^*) < g(G)$.

In $I(G)$ consider a lens L with $g(G)$ faces. By Lemma 1.5, there is a triangle T in L which is incident to the boundary of L. According to whether T corresponds to a triangle in G or to a vertex of degree 3, we use one of the reductions η_0 or ω_0. In both cases, it is easily verified that $g(G^*) < g(G)$. This completes the proof of Steinitz's Theorem. //

A *maximal planar graph* is a graph which ceases to be a planar graph if any one edge is added to the graph. Whitney showed that any maximal planar graph with $v \geqq 4$ vertices is 3-connected, (see Harary 1969).

Corollary 1.6 *Every maximal planar graph with at least four vertices is 3-polyhedral.*

Let G be a 3-connected, planar graph. Its dual graph G^* may be constructed as follows : in each region of the planar realization of G (including the unbounded exterior) we locate a vertex of G^*. If two regions have a common edge e, join the corresponding vertices by an edge e^* which intersects e only. In this way we obtain a planar graph which is also 3-connected. This gives the following corollary.

Corollary 1.7 *Dual 3-connected, planar graphs* G *and* G^* *are realized by dual polytopes.*

§2 DIAMETERS OF POLYTOPES

An interest in the study of metric properties of polyhedral graphs has arisen quite recently and has been evoked by the wide interest in the techniques of Linear Programming.

Recall that the distance $r(u,v)$ between the vertices u and v of a connected graph G is the length (i.e. the number of edges) of the shortest chain connecting u and v.

Definition 2.1 The *diameter of a graph* G is the smallest integer k such that the distance between any two of its vertices is no greater than k. The *diameter of a polytope* (diam M) is the diameter of its graph $G(M)$.

2.1 The Maximum Diameter Conjecture

We denote by $\Delta(d,n)$ the *maximum diameter* of any polytope in the class of all d-polytopes with n facets. The problem of determining $\Delta(d,n)$ is closely connected with the estimation of the number of iterations of the simplex algorithm in Linear Programming. A *simplex algorithm* is an algorithm based on the construction of some chain between an initial vertex (assumed to be chosen arbitrarily) and an optimal vertex. In such an algorithm, every iteration consists of a choice (according to a variety of rules) of the following vertex in the chain from among those vertices which are adjacent to the current vertex. If x and y are vertices of the polytope M such that $r(x,y) = \text{diam } M$, then if we take a linear function cx whose maximum is attained at y and if we use the vertex x as our initial vertex, we find that the number of iterations of the simplex algorithm in solving the problem of finding $\max\{cx : x \in M\}$ can not be less than $\text{diam } M$. In this sense, the quantity $\Delta(d,n)$ gives the number of iterations required to solve the 'worst' linear programming problem using the 'best' simplex algorithm (Klee & Minty 1972). We are assuming, of course, that the 'best' algorithm can be constructed.

The conjecture that $\Delta(d,n) \leq n-d$ has become widely known. It has been proved only in special cases. First, it is obvious that

$$\Delta(2,n) = [n/2] .$$

It has also been shown by Klee and Walkup (1967) that

$$\Delta(3,n) = [2n/3] - 1 .$$

It has also been proved (Klee & Walkup 1967) that the conjecture is true in the case $n \leq d+5$. We show that it suffices to prove the maximum diameter conjecture for simple polytopes. We begin by introducing a construction which permits us to transform any polytope into a simplicial polytope with the same number of vertices and with no less faces of any dimension than the original. The construction was proposed by Eggleston, Grünbaum & Klee (1964).

We say that a point v^0 is *separated (strictly separated) from* M by a facet F of a d-polytope M, if the hyperplane $\text{aff } F$ separates (strictly separates) the point v^0 from M.

Definition 2.2 Let v be a vertex of a d-polytope M and let $v^0 \notin M$ be strictly separated from M by the facets of M which are incident to v. We say that the polytope $M^0 = \text{conv } (M \cup v^0)$ has been obtained from M by means of a *right displacement of the vertex* v (Figure 12).

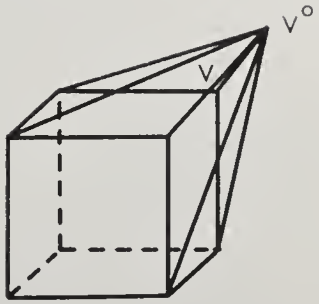


Fig. 12.

<u>Lemma 2.1</u> *Let the* d*-polytope* $M^o \subset E_d$ *be obtained from the* d*-polytope* M *by a right displacement of the vertex* v *to the vertex* v^o. *Then, every* i*-face of* $M^o = \mathrm{conv}(M \cup v^o)$ *is of one of the two following types :*

1) *An* i*-face* F *of* M *is a face of* M^o *if and only if* F *is a face of some facet not containing the vertex* v .

2) *An* i*-pyramid* F^o *with vertex* v^o *and base* F *is a face of* M^o *if and only if* F *is an* $(i-1)$*-face which does not contain* v *, but which is a face of a facet of* M *which does contain* v .

Proof Clearly, every face of M^o is either a face of M or the convex hull of the point v^o and some face of M . It is also obvious that the face F of M is a face of M^o if and only if case 1) of the lemma holds.

We prove assertion 2). (i) Let F be an $(i-1)$-face of M and let $F^o = \mathrm{conv}\,(F \cup v^o)$ be an i-face of M^o . Then $F = M \cap \mathrm{aff}\, F^o$. Let $x^o \in \mathrm{rel\,int}\, F$, $y^o \in \mathrm{int}\, M$ and let $E = \mathrm{aff}\,(v^o, x^o, y^o)$ be the plane containing the points v^o, x^o, y^o . Then $\overline{M} = E \cap M$ is a 2-polytope (polygon). The line $L = \mathrm{aff}\,(x^o, v^o)$ is the intersection of the plane E with $\mathrm{aff}\, F^o$ and $\overline{F} = L \cap \overline{M}$ is either an edge (Fig.13), or a vertex (Fig.14) of $\overline{M}$.

The case in which $\overline{F} = L \cap \overline{M}$ is an edge of $\overline{M}$ is impossible since then $v^o \in \mathrm{aff}\, \overline{F} \subset \mathrm{aff}\, F$ and consequently $i = \dim F^o = \dim F = i-1$.

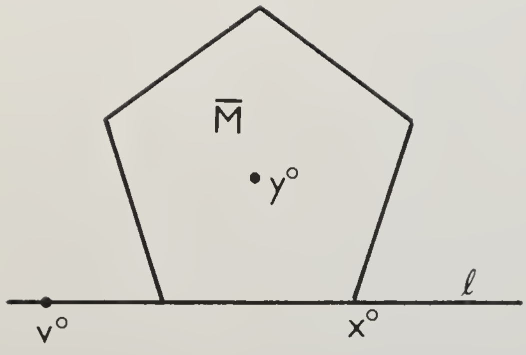


Fig. 13.

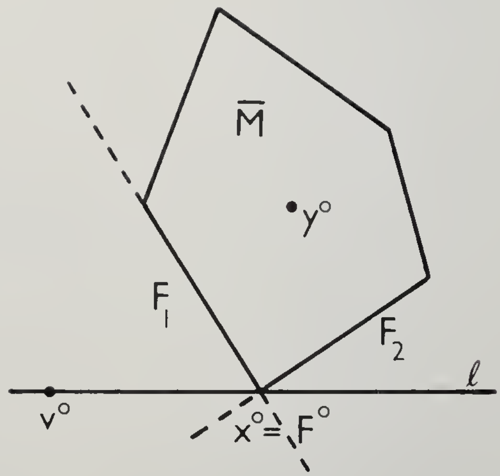


Fig. 14.

If $\bar{F}$ is a vertex it coincides with x^o and the point v^o in the plane E is strictly separated by one edge and not separated by the other edge of the polygon $\bar{M}$ which are incident at x^o. Hence, if F_1 and F_2 are facets of M containing respectively these edges, then the point v^o in E_d is strictly separated from M by one of the facets F_1 or F_2, say F_1, and not separated by the other. By definition 2.2 the facet F_1 contains the point v.

(ii) Now, let F be a face of M satisfying condition 2) of the lemma. We show that $F^o = \text{conv}\,(F \cup v^o)$ is a face of M^o. If $v^o \in \text{aff}\,F$, this is certainly true. Let $v^o \notin \text{aff}\,F$ and let F_1 and F_2 be facets of M such that $F \subset F_1 \cap F_2$ and F_1 strictly separates v^o from M, while F_2 does not separate v^o from M. Let $H_i = \text{aff}\,F_i$ and let H_0 be a supporting hyperplane which generates the face F. We rotate the hyperplanes H_i about $H_i \cap H_0$ towards H_0 until the new hyperplanes H_i^* satisfy the condition $H_i^* \cap M = F$ and the point v^o is strictly separated from M by H_1^* but not separated from M by H_2^*. More precisely, if

$$H_i = \{x \in E_d : c^i x = 0\} \quad i=0,1,2$$

(assuming for simplicity that $0 \in F$) and $M \subset H_i^+$, then, putting

$$\lambda_1 = \sup \{ \lambda : (c^1+\lambda c^0)v^o \leqq 0 \} ,$$

$$\lambda_2 = \sup \{ \lambda : (c^2+\lambda c^0)v^o \geqq 0 \}$$

we obtain the desired hyperplanes

$$H_i^* = \{x\varepsilon E_d : (c^i+c^0\lambda_i/2)x = 0\} , \ i=1,2.$$

The hyperplane $H_0^* = \text{aff } (v^o \cup (H_1^* \cap H_2^*))$ contains the point v^o and generates the face F . Further, since

$$H_o^* \cap M^o = H_0^* \cap \text{conv } (M \cup v^o) = \text{conv } (F \cup v^o) = F^o$$

F^o is a face of M^o . //

Theorem 2.2 *There exists a simple* d-*polytope with* n *facets having diameter* $\Delta(d,n)$.

Proof We begin with a definition. The *face-diameter* of a d-polytope is the smallest integer k such that given any two facets F and G there is a sequence of facets $F = F_0, F_1, \ldots, F_k = G$ such that $F_{i-1} \cap F_i$ is a (d-2)-face for $i\varepsilon N_k$. Such a sequence is called a *face-chain* . Clearly, the face-diameter of a polytope equals the diameter of the dual polytope.

Let M be a d-polytope such that $\text{diam } M = \Delta(d,n)$. Let M^* be the dual of M . The d-polytope M^* may be transformed into a simplicial polytope M^o by means of right displacements of its vertices (Lemma 2.1). Moreover, the face-diameter of M^o is easily seen to be no less than that of M^* . Thus $(M^o)^*$ is a simple polytope with n facets for which $\text{diam } (M^o)^* \geq \text{diam } M$. //

We now calculate the diameter of the product of two polytopes (see Chapter 1, §3 for the definition). We use the symbol $r_M(x,y)$ when we need to emphasize which polytope M the distance $r(x,y)$ refers to.

Lemma 2.3

$$\text{diam } (M_1 \otimes M_2) = \text{diam } M_1 + \text{diam } M_2 .$$

Proof Let $(v_1,v_2),(v_1',v_2')$ be two vertices of $M_1 \otimes M_2$, where v_i, v_i' are vertices of M_i, $i=1,2$. If $v_i=v_i^0,\dots,v_i^{k_i}=v_i'$ is the shortest chain between the vertices v_i and v_i' of M_i, $i=1,2$, then

$$(v_1,v_2)=(v_1^0,v_2),\dots,(v_1^{k_1},v_2)=(v_1',v_2^0),\dots,(v_1',v_2^{k_2})=(v_1',v_2')$$

is a chain of length k_1+k_2 between the vertices (v_1,v_2) and (v_1',v_2') of $M_1 \otimes M_2$. Consequently

$$r_{M_1\otimes M_2}((v_1,v_2),(v_1',v_2')) \leqq r_{M_1}(v_1,v_1') + r_{M_2}(v_2,v_2') .$$

Further, if $(u_1,u_2),(u_1',u_2')$ is a pair of adjacent vertices in $M_1 \otimes M_2$, where $u_i,u_i' \in M_i$, $i=1,2$, then either $u_1=u_1'$ and u_2,u_2' are adjacent vertices of M_2, or $u_2=u_2'$ and u_1,u_1' are adjacent vertices of M_1. Hence

$$r_{M_1\otimes M_2}((v_1,v_2),(v_1',v_2')) \geqq r_{M_1}(v_1,v_1') + r_{M_2}(v_2,v_2') .$$

Consequently

$$\operatorname{diam}(M_1 \otimes M_2) = \operatorname{diam} M_1 + \operatorname{diam} M_2 . \quad //$$

Lemma 2.4

$$\Delta(d_1+d_2,n_1+n_2) \geqq \Delta(d_1,n_1)+\Delta(d_2,n_2) .$$

In particular

$$\Delta(d+1,n+2) \geqq \Delta(d,n) + 1 .$$

Proof Let M_i be a d-polytope with n_i facets and let $\operatorname{diam} M_i = \Delta(d_i,n_i)$, $i=1,2$. Since $\dim M_1 \otimes M_2 = d_1+d_2$, $f_{d_1+d_2-1}(M_1 \otimes M_2) = n_1+n_2$ we have, by Lemma 2.3

$$\Delta(d_1+d_2,n_1+n_2) \geqq \operatorname{diam}(M_1 \otimes M_2)$$

$$= \operatorname{diam} M_1 + \operatorname{diam} M_2 = \Delta(d_1,n_1) + \Delta(d_2,n_2) .$$

Since $\Delta(1,2) = 1$ we have $\Delta(d+1,n+2) \geq \Delta(d,n) + 1$. //

Definition 2.3 A *wedge on a d-polytope* M *relative to its k-face* F $(0 \leq k < d)$ is a $(d+1)$-polytope $W = H^+ \cap (M \otimes L)$, where $L = [0,\infty)$, H^+ is a halfspace containing M and such that $H \cap M = F$, and H is a hyperplane which intersects the interior of $M \otimes L$. An example of a wedge is shown in Figure 15.

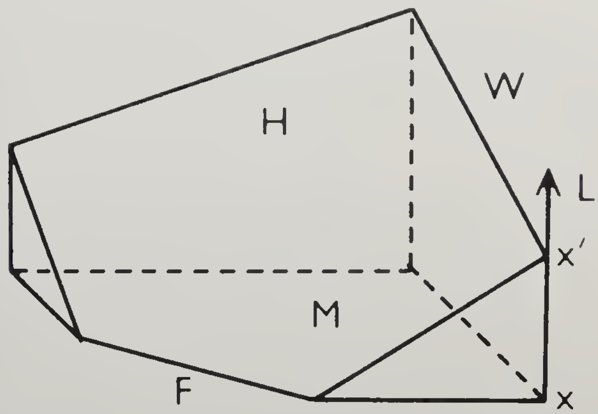


Fig. 15.

The d-faces M and $H \cap (M \otimes L)$ are called the *lower* and *upper bases* respectively. With each vertex $x \in M$, $x \notin F$, we may associate a vertex x' of the upper base which is the intersection of the upper base with $x \otimes L$. If M is a simple d-polytope and F is one of its facets, then the wedge W is also a simple polytope. By Lemma 2.3 and considering the manner of construction of the wedge W , we have the equality diam W = diam M .

Definition 2.4 A *non-returning chain* in the graph G(M) of a polytope M is a sequence of adjacent vertices $v^1,\ldots,v^s$ with the property that if a vertex v^j belongs to a facet F and the vertex $v^{j+1} \notin F$ then also $v^{j+k} \notin F$, $k \in N_{s-j}$.

Definition 2.5 A *Dantzig d-figure* is a triple (M,x,y) where M is a d-polytope with 2d facets of which d are incident to vertex x and the other d are incident to vertex y .

Klee and Walkup (1967) gave a number of equivalent formulations of the maximum diameter conjecture.

Theorem 2.5 *The following statements are equivalent :*

1. *Any two vertices of any simple polytope can be joined by a non-returning chain ;*

2. $\Delta(d,n) \leq n-d$ *for any* d,n, $1 \leq d < n$;

3. $\Delta(d,2d) \leq d$ for all d ;

4. *For all Dantzig* d-*figures* (M,x,y), *we have* $r(x,y) = d$.

Proof 1) ⇒ 2). Let arbitrary vertices x and y of a simple d-polytope be incident to k $(0 \leq k \leq d-1)$ common facets. Then, from the definition of a non-returning chain it follows that $r(x,y) \leq n-d-k$. Hence diam $M \leq n-d-k \leq n-d$. Since M is arbitrary and using Theorem 2.2 we deduce that assertion 2) follows.

2) ⇒ 3). It suffices to put $n = 2d$.

3) ⇒ 4). Assertion 3) implies that $r(x,y) \leq d$. On the other hand, since vertices x and y are not incident to a common facet, $r(x,y) \geq d$. Hence $r(x,y) = d$.

4) ⇒ 1). Let M be a simple d-polytope with $d+m$ facets and let x and y be any two of its vertices. Put $y=y_0$ and consider a face F_0 of smallest possible dimension containing both x and y . Let

$$\dim F_o = d' \quad , \quad f_{d'-1}(F_0) = d'+m' \quad , \quad d' \leq d \quad , \quad m' \leq m \quad .$$

Since there are no $(d'-1)$-faces of F_0 containing both x and y , we have $m' = d'+k$, where k is the number of $(d'-1)$-faces which are not incident to either x or y . If $k>0$, let G be a $(d'-1)$-face of F_0 not containing either x or y . Construct the wedge F_1 on F_0 relative to its face G . The $(d'+1)$-polytope F_1 has $k-1$ facets which are not incident to x or to y_1 - a vertex in the upper base of the wedge F_1 . By repeating this process of replacing a polytope by a wedge with a smaller number of faces not incident to the selected vertices we obtain after at most k steps a Dantzig $(d'+k')$-figure (F_k, x, y_k).

By assumption $r_{F_k}(x,y_k) = m' = d'+k$. From definition 2.5 it follows that there exists a non-returning chain between the vertices x and y_k of the Dantzig figure F_k . It is easily seen that to each non-returning chain C of the wedge W on M there corresponds a non-returning chain in M obtained from C by replacing each vertex on an upper base by its corresponding vertex on a lower base. Thus, to each non-returning chain between the vertices x , y_k of the corresponding Dantzig figure there corresponds a non-returning figure between x , y in M . //

2.2 An Upper Bound for the Diameter

The bound given below is due to Larman (1970).

Theorem 2.6 *If* $d \geqq 3$ *, then* $\Delta(d,n) \leqq 2^{d-3}n$.

Note that the slightly improved bound

$$\Delta(d,n) \leqq \frac{2^{d-3}}{3}(n-d+\frac{5}{2})$$

which was announced in 1974 (Barnette 1974a) has been queried (Barnette 1974b).

Lemma 2.7 *The maximum diameter in the class of* d-*polytopes with* n *vertices does not exceed* $[(n-2)/d] + 1$.

Proof Choose any two vertices in a d-polytope with n vertices. By Theorem 1.2 there exist at least d non-intersecting edge chains connecting them. Hence the length of the shortest edge chain connecting the chosen vertices cannot exceed $[(n-2)/d] + 1$. //

Lemma 2.8

$$\Delta(3,n) \leqq \left[\frac{2}{3}n\right] - 1 . \qquad (2.1)$$

Proof By Theorem 2.2 it suffices to consider only simple 3-polytopes. By Corollary 5.10 of Ch. 1 , the number of vertices of a simple 3-polytope with n facets is given by $f_0 = 2n - 4$. Hence, using Lemma 2.7 we have the following chain of inequalities :

$$\Delta(3,n) = \Delta(3,f_2=n) = \Delta(3,f_0=2n-4) \leqq \left[\frac{2n-4-2}{3}\right]+1 \leqq \left[\frac{2n}{3}\right] - 1 . \quad //$$

We note, although we will not use this in the sequel, that the bounds in these Lemmas are exact. Indeed there are even simplicial polytopes for which the bound in Lemma 2.7 is attained.

Consider the following extremal construction (Klee & Walkup 1967). The d-polytopes P(d,j) , j=1,2,..., are convex hulls of j+1 (d-1)-simplexes which are located in parallel hyperplanes in such a way that successive simplexes are anti-homothetic and their relative boundary lies on the boundary of P(d,j) Thus, Fig.16(a) shows P(3,2) and

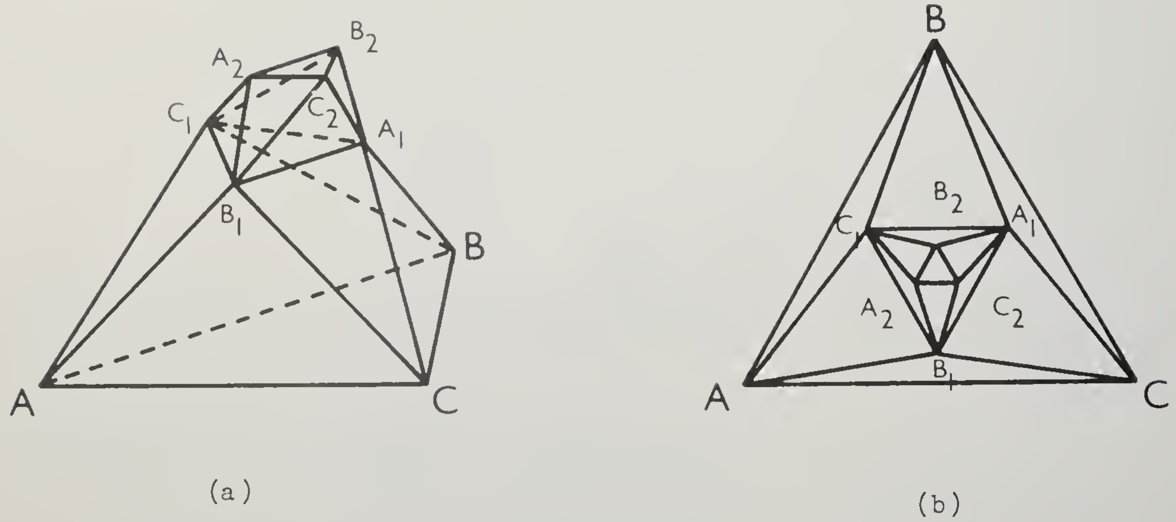


Fig. 16.

Fig.16(b) shows $P(3,2)$ from above.

Note that the d-polytope $P(d,j)$ has $d(j+1)$ vertices and $(2^d-2)j+2$ facets. It is easily seen that on $P(d,j)$ the bound of Lemma 2.7 is attained. Also, the polytopes $P(d,j)$ have *facet diameter* (i.e. the diameter of the corresponding dual polytope) equal to $(d-1)j+2$. (see also Exercise 11).

Let M be a d-polytope and let $C = \{v^1, v^2, \ldots, v^s\}$ be a chain in the graph $G(M)$. A *visit of a chain* C *to a face* F is a subchain $v^i, v^{i+1}, \ldots, v^{j-1}, v^j$ of C such that $v^t \in F$, $i \leq t \leq j$, $v^{i-1}, v^{j+1} \notin F$. We say that *the chain* C *visits the face* F k *times* if C contains exactly k different visits to F.

Let F be a face of M. The distance $r(x,F)$ from the vertex x to the face F is defined to be the quantity $\min\{r(x,y) : y \in F\}$. We use the symbol $r_M(x,F)$ when we wish to emphasize that the distance is measured on the polytope M.

<u>Lemma 2.9</u> *Between any two vertices* x *and* y *of a simple d-polytope* M, $d \geq 3$, *there is a chain which visits each facet not more than* 2^{d-3} *times*.

Proof We use induction on d . For $d=3$ from (2.1) we have $\operatorname{diam} M \leq \frac{2}{3}n - 1$. Hence, by Theorem 2.5, the Lemma is true.

Let the length of the shortest face-chain between vertices x and y be k and let F_1 be the first facet of one of these chains. Among all the face-chains from x to y of length k and beginning with F_1 we select a face-chain with second facet F_2 such that $r_{F_1}(x, F_1 \cap F_2)$ is minimal. Let x^1 be a vertex of the $(d-2)$-face $F_1 \cap F_2$ such that $r_{F_1}(x, x^1) = r_{F_1}(x, F_1 \cap F_2)$. Further, among all the face-chains from x to y of length k which begin with the pair F_1, F_2 we select a face-chain with third face F_3 such that $r_{F_2}(x^1, F_2 \quad F_3)$ is minimal. Let x^2 be a vertex of $F_2 \cap F_3$ such that $r_{F_2}(x^1, x^2) = r_{F_2}(x^1, F_2 \cap F_3)$. Continuing this process, we construct a face chain $F_1, F_2, \ldots, F_k$ such that $x \varepsilon F_1$, $y \varepsilon F_k$. Let C_i , $i \varepsilon N_k$ denote the shortest chain between x^{i-1} and x^i , which are vertices of F_i $(x^0 = x, x^k = y)$. The union of the chains C_i, $i \varepsilon N_k$, yields a chain C in the polytope M . It is clear that

$$C_i \cap C_{i+1} = x^i \qquad\qquad \forall i \, \varepsilon \, N_{k-1} .$$

Consider any facet F of M . If F coincides with one of the facets F_i , $i \varepsilon N_k$, the chain C visits F once along the chain C_i and, by the inductive assumption, since $\dim(F_i \cap F_{i+1}) = d-2$, not more than 2^{d-4} times along the chain C_{i+1} .

Suppose F does not coincide with one of the F_i , $i \varepsilon N_k$. If F intersects only one of these faces, say F_i , then by the inductive assumption, the chain C_i visits the face $F \quad F_i$ of F_i not more than 2^{d-4} times, since $\dim(F \cap F_i) = d-2$. This follows from the fact that the polytope M is simple.

Clearly, the facet F can intersect at most three facets among the shortest face-chain $F_1, F_2, \ldots, F_k$. Also, if F does intersect three facets, then these must form a triple F_{i-1}, F_i, F_{i+1} . This implies that $F \cap C$ is contained in $C_{i-1} \cup C_i \cup C_{i+1}$.

We now show that $F \cap C$ is contained either in $C_{i-1} \cup C_i$ or in $C_i \cup C_{i+1}$. Indeed, if F has points in common with C_{i-1} and C_{i+1} (it will meet C_{i-1} in some vertex other than x^{i-1}), then replacing F_i by F in the face chain $F_1, F_2, \ldots, F_k$ we find a contradiction of the rule whereby F_i was chosen.

Thus, for example, let $F \cap C \subseteq C_i \cup C_{i+1}$. Then, if $F \cap F_i \neq \emptyset$, then $\dim (F \cap F_i) = d-2$ and by the inductive assumption the chain C_i visits the face $F \cap F_i$ of F_i not more than 2^{d-4} times. Thus, the chain C visits face F not more than 2^{d-3} times. //

Proof of Theorem 2.6. By Theorem 2.2 we may restrict our attention to simple d-polytopes.

Let x,y be arbitrary vertices of the d-polytope M with n facets. By Lemma 2.9, there is a chain $C = \{x^0, x^1, \ldots, x^p\}$, $x^0 = x$, $x^p = y$ which visits each facet of M not more than 2^{d-3} times. Moreover as we pass from vertex x^i to vertex x^{i+1}, the chain C terminates its visit to some facet and begins its visit to some other facet. Thus, if $p > 2^{d-3}n$, there is at least one facet of M which is visited by the chain C more than 2^{d-3} times and this contradicts the choice of C. Hence, $p \leq 2^{d-3}n$ and since the vertices x and y are arbitrary, $\Delta(d,n) \leq 2^{d-3}n$. //

2.3 A Lower Bound for the Maximal Diameter

We refer to Adler (1974). Let M_1 and M_2 be simple d-polytopes.

Definition 2.6 The *join* of these polytopes *by the vertices* v^1, v^2 (denoted by $M_1 \oplus M_2$) is the polytope obtained by carrying out the following procedure.

1. Select the vertices v^1 and v^2 in M_1 and M_2.
2. Remove the vertices v^i by means of right cuts (see Definition 1.3, Ch.3) thereby forming the polytopes M_i' with simplicial faces $F_i = M_i \cap H_i$, $i=1,2$, where H_i are the hyperplanes used to remove the vertices v^i.
3. Let τ_i be a projective map which maps the hyperplane H_i', which is parallel to H_i and which passes through v^i, to infinity. Construct the polytope $\tau_i(M_i')$ such that all of its edges which intersect $\tau_i(F_i)$ are parallel.
4. By means of suitable affine maps α_i construct polytopes $M_i^2 = \alpha_i(\tau_i(M_i'))$, in which every facet intersecting $\alpha_i(\tau_i(F_i))$ is orthogonal to it.
5. Find an affine map α_3 of M_1^2 which transforms the face $\alpha_1(\tau_1(F_1))$ into $\alpha_2(\tau_2(F_2))$ but which leaves the faces which intersect $\alpha_1(\tau_1(F_1))$ perpendicular to it.

6. Position M_2^2 and $\alpha_3(M_1^2)$ so that $\alpha_3(\alpha_1(\tau_1(F_1)))$ and $\alpha_2(\tau_2(F_2))$ coincide and so that the interior of M_2^2 does not intersect the interior of $\alpha_2(M_1^2)$.

Note that all faces of M_i which do not contain v^i will be faces of $M_1 \oplus M_2$ and that the d facets of M_1 containing v^1 together with the d facets of M_2 containing v^2 form (after the transformation) the remaining d facets of $M_1 \oplus M_2$ (Figure 17).

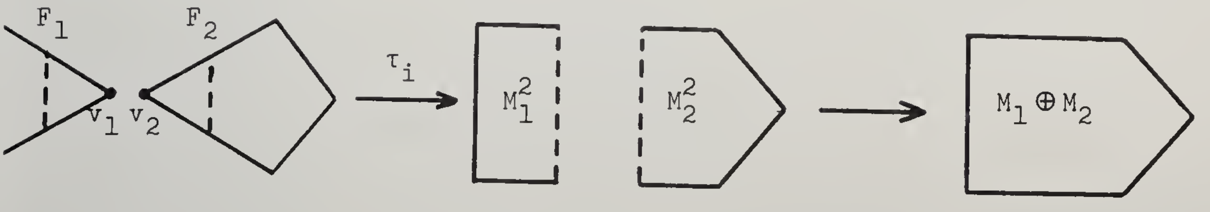


Fig. 17.

It follows directly from the definitions that

$$\dim(M_1 \oplus M_2) = d \quad , \quad f_{d-1}(M_1 \oplus M_2) = f_{d-1}(M_1) + f_{d-1}(M_2) - d \, .$$

Note also that the polytope $M_1 \oplus M_2$ is uniquely defined and depends on the choice of the vertices v^1, v^2.

Lemma 2.10 *There is a join of the two d-polytopes* M_1 *and* M_2 *such that*

$$\operatorname{diam} M_1 + \operatorname{diam} M_2 - 1 \leqq \operatorname{diam}(M_1 \oplus M_2) \leqq \operatorname{diam} M_1 + \operatorname{diam} M_2 \, .$$

Proof Let v_i, $v_i' \in M_i$ be vertices such that $r_{M_i}(v_i, v_i') = \operatorname{diam} M_i$, $i=1,2$. Let $M_1 \oplus M_2$ be the join of M_1 and M_2 by the vertices v_1, v_2. Let v_i^o, v_i be adjacent vertices of M_i, then since $r_{M_i}(v_i, v_i') = \operatorname{diam} M_i$, we have $r_{M_i}(v_i^o, v_i') = \operatorname{diam} M_i$ or $\operatorname{diam} M_i - 1$, $i=1,2$. Every vertex of M_1 adjacent to v_1 is adjacent in $M_1 \oplus M_2$ to exactly one vertex of M_2. The result of the Lemma follows. //

Lemma 2.11

$$\Delta(d,n_1+n_2-d) \geqq \Delta(d,n_1) + \Delta(d,n_2) - 1 .$$

Proof Let M_i be a d-polytope with n_i facets having maximal diameter, that is, diam $M_i = \Delta(d,n_i)$, $i=1,2$. By Lemma 2.10, if we join these polytopes suitably we have

$$\Delta(d,n_1+n_2-d) \geqq \operatorname{diam}(M_1 \oplus M_2)$$

$$\geqq \operatorname{diam} M_1 + \operatorname{diam} M_2 - 1 = \Delta(d,n_1) + \Delta(d,n_2) - 1$$

This establishes the Lemma. //

Theorem 2.12

$$\Delta(d,n) \geqq \left[n - d - \frac{n-d}{[5d/4]}\right] + 1 .$$

Proof Denote the right-hand side of the inequality of the Theorem by $Z(d,n)$. We use induction on d. For $d \leqq 2$ we have $\Delta(d,n) = [n/2] \geqq Z(d,n)$. Suppose that $\Delta(d-1,n) \geqq Z(d-1,n)$ for some $d-1 \geqq 2$ and all $n \geqq d$. By Lemma 2.4 and the inductive assumption

$$\Delta(d,n) \geqq \Delta(d-1,n-2) + 1 \geqq Z(d-1,n-2) + 1 .$$

Suppose $d \not\equiv 0 \pmod 4$. Then

$$Z(d-1,n-2) + 1 = \left[n - d - \frac{n-d-1}{[5d/4]-1}\right] + 1 .$$

Thus, in this case, for $1 \leqq n-d \leqq [5d/4]$ we have that $Z(d-1,n-2) + 1 \geqq Z(d,n)$.

Suppose $d \equiv 0 \pmod 4$. Then

$$Z(d-1,n-2) + 1 = \left[n - d - \frac{n-d-1}{[5d/4]-2}\right] + 1 .$$

Thus, in this case, for $1 \leqq n-d \leqq 5d/4$ we have that $Z(d-1,n-2)+1 \geqq Z(d,n)$.

Suppose that $d \equiv 0 \pmod 4$. Then

$$Z(d-1,n-2)+1 = \left[n-d-\frac{n-d-1}{[5d/4]-2}\right]+1 .$$

and, as in the previous case,

$$\Delta(d,n) \geqq Z(d,n) \quad \text{when} \quad n-d \leqq [5d/4]-1 .$$

Further, since $d \equiv 0 \pmod 4$ and $\Delta(4,9)=5$ (Exercise 15), we have, using Lemma 2.4, that

$$\Delta(d,d+\tfrac{5d}{4})=\Delta(\tfrac{d}{4}\cdot 4,\tfrac{d}{4}\cdot 9) \geqq \tfrac{d}{4}\Delta(4,9)=\tfrac{5d}{4}=Z(d,d+\tfrac{5d}{4}) .$$

We have thus shown that when $n-d \leqq [5d/4]$ the theorem is true.

Now suppose that $\Delta(d,n) \geqq Z(d,n)$ for all $n \leqq n_o$, where $n_o \geqq d+[5d/4]$. Let $n_o - d \equiv b \pmod{[5d/4]}$, that is

$$n_o-d-b = k[5d/4] \quad , \quad 0 \leqq b < [5d/4] .$$

By Lemma 2.11 and the inductive assumption

$$\Delta(d,n_o+1) \geqq \Delta(d,n_o-b)+\Delta(d,b+d+1)-1$$

$$\geqq Z(d,n_o-b)+Z(d,b+1+d)-1$$

$$=\left[n_o-b-d-\frac{n_o-b-d}{[5d/4]}\right]+1+\left[b+1+d-d-\frac{b+1+d-d}{[5d/4]}\right]+1-1$$

$$=k[5d/4]-k\frac{[5d/4]}{[5d/4]}+\left[b+1-\frac{b+1}{[5d/4]}\right]+1$$

$$=\left[n_o+1-d-\frac{n_o+1-d}{[5d/4]}\right]+1 = Z(d,n_o+1) . \qquad //$$

2.4 Thickness

Definition 2.7 The *thickness of a polytope* M (denoted by $\lambda(M)$ is the number of vertices in the longest simple chain in the graph of a polytope. Equivalently, $\lambda(M)=\mu(M)+1$, where $\mu(M)$ is the length of the longest simple chain in the polytope graph.

If the graph $G(M)$ has a simple spanning cycle C, then G is called a *Hamiltonian graph* and C is a *Hamiltonian cycle* . Thus, if M is a Hamiltonian graph, the thickness $\lambda(M)$ is equal to the number of vertices of the polytope M . We remark, that the study of Hamiltonian graphs actually originated in the study of graphs of polytopes. Hamilton constructed simple cycles which contained every vertex of a 3-polytope (dodecahedron). In 1880, Tait conjectured that every 3-polyhedral graph is Hamiltonian. The truth of Tait's conjecture would imply the truth of the four-colour theorem. This led to a large amount of work directed towards proving the Hamiltonian property of polyhedral graphs. The first counterexample to Tait's conjecture was constructed by Tutte in 1964. Tutte's graph is shown in Figure 18.

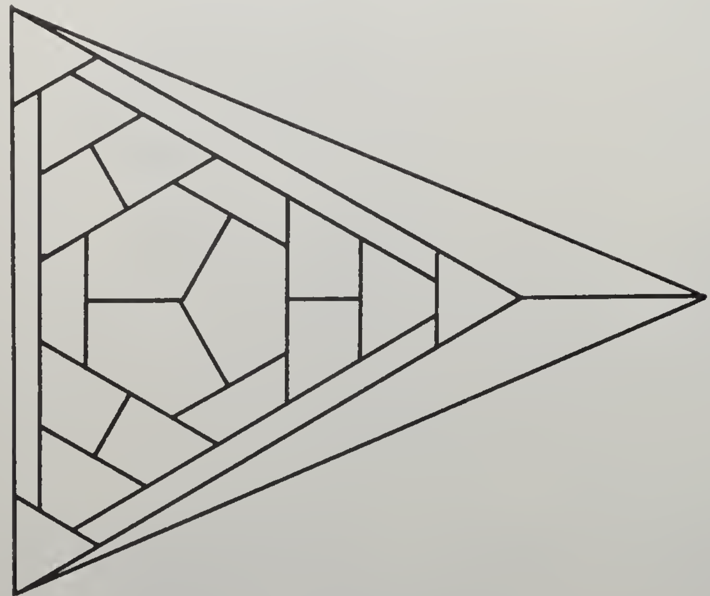


Fig. 18.

We note that for any d and n , with $n \geq d+1$, there exists a d-polytope with n vertices whose graph is Hamiltonian. An example of such a polytope is the cyclic polytope $C(d,n)$. For $d > 3$ every cyclic d-polytope is 2-neighbourly (i.e. any two vertices are connected by an edge) and so its graph is clearly Hamiltonian. When $d = 3$, it is also easily shown that $C(3,n)$ is Hamiltonian using a characterization of the edges. Thus, the maximum thickness of a d-polytope with n vertices is exactly n and this thickness is attained on the simplicial polytopes.

Proposition 2.13 *Let* M *be a* d-*polytope with* n *facets. Then*

$$\lambda(M) \leq \binom{n - \left[\frac{d+1}{2}\right]}{n-d} + \binom{n - \left[\frac{d+2}{2}\right]}{n-d}$$

This bound is attained on the simple d-*polytopes*

The proof will follow from results in §3, Ch.3.

Proposition 2.14 *The minimum thickness in the class of simple* d-*polytopes with* n *facets is no greater than* $(d-1)(n-d) + 2$.

Proof Let x be a vertex of M and let the hyperplane H strongly separate x from the set $\mathrm{conv}(\mathrm{vert}\, M \setminus x)$. We say that the polytope $M \cap H^+$ has been obtained from M by a *right cut* at the vertex x . It is clear that if M is a simple polytope, the face $M \cap H$ of $M \cap H^+$ is a simplex. Consider the polytope $Q(d,n)$ which has been obtained from a d-simplex after $n-d-1$ successive right cuts at vertices. It should be noted that if the graph of the simple d-polytope $Q(d,n-1)$ is Hamiltonian then the graph of $Q(d,n)$ obtained by taking a right cut at a vertex of $Q(d,n-1)$ is also Hamiltonian. It is easily verified that $Q(d,n)$ has n facets and $(d-1)(n-d) + 2$ vertices. This proves the proposition. //

Definition 2.8 The *ℓ-thickness of a polytope* M is the number of edges in the longest chain $x^0,\ldots,x^s$ of the graph $G(M)$ for which there exists a linear function $\ell(x) = cx$ such that $cx^0 < cx^1 < \ldots < cx^s$. The largest ℓ-thickness in the class of d-polytopes with n facets will be denoted by $H(d,n)$.

Klee and Minty (1972) obtained the following bounds :

$$\alpha_d \, n^{[d/2]} < H(d,n) < \beta_d \, n^{[d/2]} , \qquad n > d \qquad (2.2)$$

$$\frac{1}{2^{[d/2]^2}} < \liminf_{n\to\infty} \frac{H(d,n)}{n^{[d/2]}} \leq \limsup_{n\to\infty} \frac{H(d,n)}{n^{[d/2]}} \leq \frac{2}{[d/2]!} \qquad (2.3)$$

In applications, it is interesting to know not only the ℓ-thickness of a polytope but also the closely associated *simplex thickness* which is defined to be the maximum number of iterations of the standard simplex method in solving any linear programming problem on the polytope M . Let $\theta(d,n)$ denote the maximum simplex thickness in the class of all d-polytopes with n facets. Clearly $\theta(d,n) \leqq H(d,n)$. It turns out that $\theta(d,n)$ satisfies the same bounds as $H(d,n)$ in (2.2) and (2.3).

Finally we present an example of a polytope, obtained by slightly deforming a d-cube ($0<\varepsilon<\frac{1}{2}$), whose ℓ-thickness is equal to 2^d-1 , where $\ell(x) = x_d$ (Fig.19) : the defining inequalities are

$$
\begin{array}{lcccl}
0 & \leqq & x_1 & \leqq & 1 , \\
\varepsilon x_1 & \leqq & x_2 & \leqq & 1 - \varepsilon x_1 , \\
\varepsilon x_2 & \leqq & x_3 & \leqq & 1 - \varepsilon x_2 , \\
\cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \\
\varepsilon x_{d-1} & \leqq & x_d & \leqq & 1 - \varepsilon x_{d-1} .
\end{array}
$$

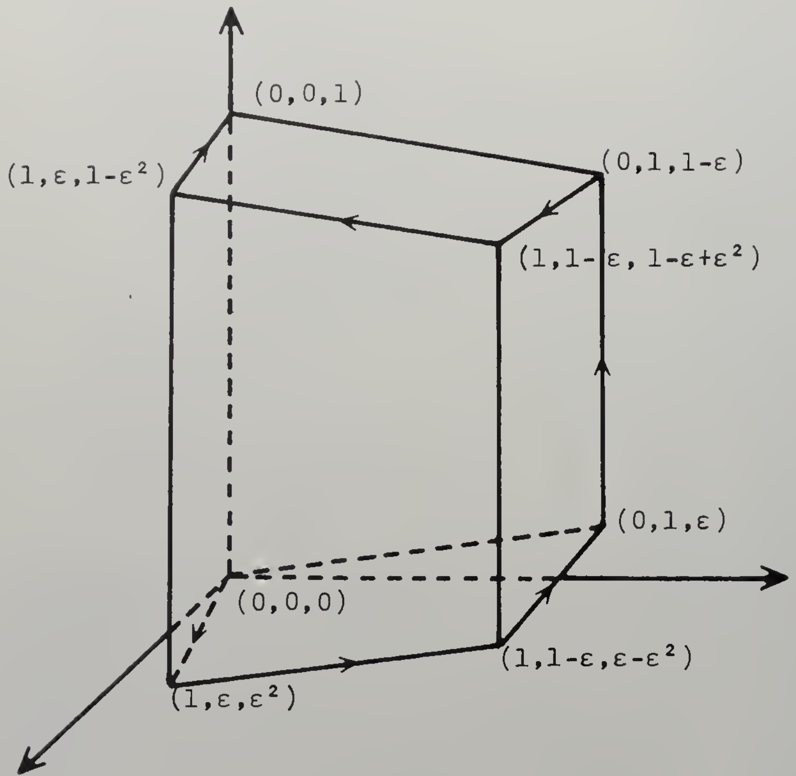


Fig. 19.

EXERCISES

1. Let $(a_1,b_1), \dots, (a_m,b_m)$ be any m distinct pairs of vertices in the graph of a d-polytope, where $m = [(d+1)/3]$. Show that there exist non-vertex-intersecting chains joining these vertex pairs. Show that in the case of simplicial d-polytopes one can take $m = [(d+1)/2]$.

2. Any graph of a simplicial d-polytope has the complete graph K_{d-i}, $i = 0,1,\dots,d-1$ as a subgraph.

3. A three-connected graph with $p \geq 6$ vertices is planar if and only if it does not contain a subgraph which is homeomorphic to the bi-partite graph $K_{3,3}$.

4. (Barnette & Grünbaum 1969). Every 3-polyhedral graph contains a spanning tree whose vertices have degree not exceeding 3.

5*. Prove or disprove the following two conjectures of Barnette (see Barnette & Grünbaum 1969):

(1) the graph of any simple 4-polytope is Hamiltonian; (the following theorem due to Tutte partially vertifies the conjecture: every 4-connected planar graph is Hamiltonian).

(2) if all the facets of a simple 3-polytope have an even number of edges, then its graph is Hamiltonian.

6. For every $d \geq 3$ there is a d-polyhedral graph which is not Hamiltonian.

7. (Grünbaum 1975). Show that:

(1) the minimum number of vertices or edges of a 3-polytope whose graph does not contain a hamiltonian cycle is equal to 11 or 18 respectively;

(2) the minimum number of facets in a 3-polytope whose graph does not contain a hamiltonian cycle is equal to 9;

(3) if the 3-polytope is simplicial then the corresponding numbers in (1) are 11 and 27 respectively.

8. There are many results, going back to Eberhard's Theorem (Eberhard 1891), concerning the problem of the existence of 3-polytopes with a specified number p_k of facets with k edges and a specified number v_k of vertices of degree k (for further details see Grünbaum (1975)). We list a number of such results:

(1) Euler's formula implies directly that the sequences p_k, v_k satisfy the conditions

$$\sum_{k\geq 3}(6-k)p_k + 2\sum_{k\geq 3}(3-k)v_k = 12 ,$$

$$\sum_{k\geq 3}(4-k)(p_k+v_k) = 8 ; \qquad (*)$$

(2) let the non-negative integers $p_5, p_6, \dots, p_n$ satisfy the conditions $p_6 \geqq 8$ and

$$\sum_{k\geq 5}(6-k)p_k = 12 , \qquad (**)$$

then there is a simple 3-polytope M such that $p_k(M) = p_k$ for $k \geqq 5$;

(3) let the non-negative integers $p_3, p_5, p_6, \dots, p_n$ satisfy the condition $\sum_{k\geq 3}(4-k)p_k = 8$, then there is a 3-polytope M such that $p_k(M) = p_k$ for $k \neq 4$ and $v_4(M) = f_0(M)$;

(4) if M is a 3-polytope such that $\sum_{k\geq 7} p_k(M) \geqq 3$, then

$$p_6(M) \geqq 2 + \tfrac{1}{2}p_3(M) - \tfrac{1}{2}p_5(M) - \sum_{k\geq 7} p_k(M) ,$$

$$3p_6(M) > 12 - 2p_4(M) - 3p_5(M) + \sum_{k\geq 7}([\tfrac{1}{2}(k+1)] - 6)p_k(M) ;$$

(5) let $p_3, \dots, p_n, v_3, \dots, v_m$ be non-negative integers which satisfy conditions (*) and for which $\sum_{k\geq 3} kv_k \equiv 0 \pmod 2$, then there is a 3-polytope M such that $p_k(M) = p_k$ and $v_k(M) = v_k$ for all $k \neq 4$;

(6) let the sequence $\{p_k\}$ satisfy the condition (**), then there is a number $m_0 \leqq 3\sum_{k\neq 6} p_k$ such that for all $p_6 = m_0 + 2m$, where m is a positive integer, there is a simple 3-polytope M such that $p_k(M) = p_k$.

9. (Grünbaum 1975). The graph of every 3-polytope has at least three edges $e^i = (v_1^i, v_2^i)$ for each of which $\deg v_1^i + \deg v_2^i \leqq 13$. A simplicial 3-polytope has at least 6 such edges.

10. Let e_{ij} be the number of edges in the graph of a polytope whose end-vertices have degrees i and j respectively ($i \leqq j$). Then, for a simplicial 3-polytope we have the following inequality:

$$120 \leqq e_{33} + 25e_{34} + 16e_{36} + 2013e_{37} + 5e_{38} + 512e_{39} + 2e_{3\,10} + 20e_{44} + 11e_{45} + 5e_{46} + 5e_{47} + 5e_{48} + 3e_{49} + 8e_{55} + 2e_{56} + 2e_{57} + 2e_{58} .$$

In particular, if $e_{jk} = 0$ for $j + k \leqq 12$, then $e_{3\,10} \geqq 60$.

11. The maximum diameter in the class of d-polytopes with n vertices is equal to $[(n-2)/d]+1$ and there exists a simplicial d-polytope with n vertices having this diameter. When $d=3$ the maximum diameter is attained by triangular prism with four-faced caps on the upper and lower base triangles when $n \equiv 2 \pmod 3$. In the remaining cases one or both caps can be omitted.

12. Most of the propositions formulated below have been proved by V.Klee (Klee & Walkup 1967):

(1) the maximum diameter in the class of simple d-polytopes with n vertices equals $[(n-2)/d]+1$ for $d \leqq 3$ and is not less than $(d-1)[(n-2)/(2^d-2)]+1$ for $n \geqq 2^d$;

(2) the maximum diameter in the class of simplicial d-polytopes with n facets equals $[(n-2d)/(2^d-2)]+2$ for $d \leqq 3$. When $d \geqq 4$ the maximum diameter is not less than this quantity and is not greater than $\min\{n-d, (n+2d(d-1))/d(d-1)\}$;

(3) the minimum diameter in the class of simplicial d-polytopes with n vertices is equal to 2 when $d=3$ and is equal to 1 when $d \geqq 4$.

13. The *radius* $R(M)$ of a polytope M is defined to be the radius of its graph $G(M)$, that is, the smallest integer r such that the chain length from some vertex of $G(M)$ to any other vertex is no greater than r. Show that:

(1) $R(M) \leqq \operatorname{diam} M \leqq 2R(M)$;

(2) the minimum radius in the class of d-polytopes with n vertices is not less than $]\log_{d-1}((d-2)n+2)/d[$ and is equal to this number if $n \equiv 2 \pmod{(d-1)}$;

(3) the minimum radius in the class of d-polytopes with n vertices equals $]\log_{d-1}((d-1)(d-2)n-d^3-3d^2-2)/d[$;

(4*) the maximum radius in the class of 3-polytopes with $n \geqq 6$ vertices is greater or equal to $[(n+4)/4]$. (Yukovich-Moon conjecture).

14. (Grünbaum 1975). Show that:

(1) the maximum thickness in the class of simplicial d-polytopes with n facets is bounded above by $[(n-2)/(d-1)]+d$ and is equal to this bound if $n \equiv 2 \pmod{d-1}$;

(2) the minimum thickness in the class of d-polytopes with n facets is bounded below by $2\log_2(n+4)/2$;

(3) the number $3\log_2(2n+1) - 6$ is a lower bound for the minimum thickness in the class of simple 3-polytopes with n edges;

(4) the minimum thickness in the class of 3-polytopes with n vertices is bounded below by $2\log_2 n - 5$;

(5) there exist constants $\alpha < 1$, c and a simple 3-polytope M with n vertices such that there is a simple chain in the graph $G(M)$ which contains at least cn^α vertices.

15. The *graph of a polyhedron* (an unbounded polytope) is a graph generated by the vertices and the bounded edges of the polyhedron (it is assumed that the polyhedron has at least one vertex). The maximum diameter of the graph of a d-polyhedron with n facets is denoted by $\Delta^*(d,n)$. The following relations hold (Klee 1974):

(1) $\Delta^*(2,n) = n - 2$, $\Delta^*(3,n) = n - 3$, $\Delta^*(4,4) = 5$, $\Delta^*(4,8) = \Delta(4,9) = 5$;

(2) $\Delta^*(d+1,n+1) \geqq \Delta^*(d,n)$, $\Delta^*(d,n+1) > \Delta^*(d,n)$;

(3) $\Delta^*(d,2d) \geqq d + [d/4]$;

(4) $\Delta^*(d,n) \geqq n - d + \min\{[d/4], [(n-d)/4]\}$, thus the maximum diameter hypothesis is not true for d-polyhedra when $d \geqq 4$.

16. The equivalence of the following assertions was established by A.N.Isachenko:

(1) the simple d-polytope M has $d+2$ facets;

(2) $\operatorname{diam} M = 2$;

(3) $M = T_k \otimes T_{d-k}$, $k \in N_d$.

17. Every simple d-polytope of radius 2 is a wedge whose base is a (d-1)-polytope of radius 2.

3 COMBINATORIAL PROPERTIES OF THE FACE COMPLEX OF A POLYTOPE

§1 COMBINATORIAL TYPES OF POLYTOPES

In addition to the analytical study of polytopes, in which they are defined by means of inequalities, there has also developed a topological interpretation of polytopes as complexes. The well developed apparatus of combinatorial topology enables us to solve certain classification problems of polytopes (Alexandroff 1956, Pontryagin 1976). The first section introduces the basic definitions and concepts.

1.1 Combinatorial Equivalence

Definition 1.1 A *complex* is a finite set $\mathcal{K}$ of polytopes in E_d satisfying the conditions :

1) If the polytope M is in $\mathcal{K}$ then any face of M is also in $\mathcal{K}$.

2) The intersection of any two polytopes in $\mathcal{K}$ is a face of both of them.

The maximal dimension of any polytope in $\mathcal{K}$ is called the *dimension of the complex* , a k-dimensional complex is a k-*complex.* If every element of $\mathcal{K}$ is a simplex, then $\mathcal{K}$ is a *simplicial complex.*

Let M be a d-polytope in E_d and let k be an integer in the range $0 \leqq k \leqq d$. The set of all faces of M whose dimension does not exceed k is a complex. It is called the k-*skeleton of the polytope* M and is denoted by skel_k M . The (d-1)-skeleton of M is called the *face complex* of M , denoted by $\mathcal{F}(M)$. The 1-skeleton of M is, of course, its graph.

Two complexes $\mathcal{K}$ and $\mathcal{K}'$ are *isomorphic complexes* if there is a bijective map ϕ between them such that

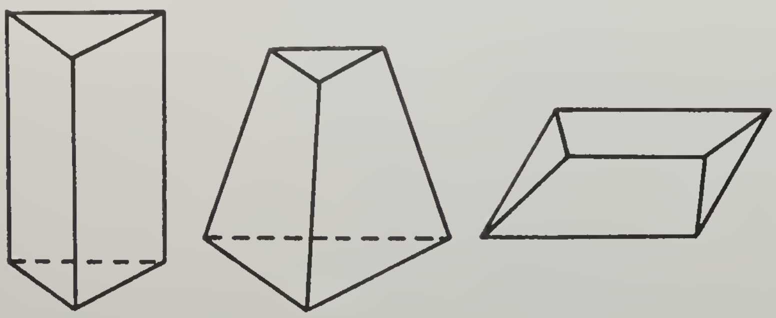
$$F_1 \subset F_2 \quad \Leftrightarrow \quad \phi(F_1) \subset \phi(F_2) \ .$$

Given a complex K in E_n, we can ask if there exists a polytope M in E_n whose face-complex $\mathcal{F}(M)$ is isomorphic to K? If such a polytope exists, we say that the *complex* K *is realized by the polytope* M.

The characterization of complexes which are realizable by 2-polytopes is trivial. Clearly, such a complex must be 1-dimensional. A 1-complex is realizable by a 2-polytope if and only if it consists of s distinct points $v^1,\dots,v^s$, $s\geq 3$, and s line segments $[v^{i-1},v^i]$ $i\in N_s$ with $v^0=v^s$.

There are not many results known concerning the realization problem. These are mainly necessary conditions which a complex must satisfy if it can be realized.

Definition 1.2 Two polytopes M and M' are *combinatorially equivalent* (written $M \cong M'$), if their face complexes $\mathcal{F}(M)$ and $\mathcal{F}(M')$ are isomorphic.



(a) (b) (c)

Fig. 20.

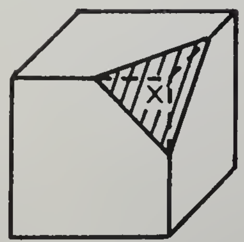


Fig. 21.

In other words, two polytopes M and M' are combinatorially equivalent when there is a bijective map ϕ between their face complexes which preserves inclusions : $F_1 \subset F_2 \Leftrightarrow \phi(F_1) \subset \phi(F_2)$. Thus (Fig.20) a triangular prism, a truncated triangular pyramid and a wedge are combinatorially equivalent.

We also say that two combinatorially equivalent polytopes are polytopes of the same type. The most important problem in the combinatorial theory of polytopes is the determination of all possible combinatorial types of d-polytopes with a given number of vertices or a given number of facets. The number of combinatorial types of polytopes was studied by Euler, Steiner (1882) and Cayley (1862). For d=2 the problem is trivial :

two polygons are combinatorially equivalent if and only if they have the same number of vertices. However, despite its great practical significance in crystallography, the problem of enumerating all combinatorial types of 3-polytopes is still not fully solved. We will be returning to this enumeration problem throughout this chapter.

The following facts about combinatorially equivalent polytopes are easily established.

1. If $M \cong M'$ then $\dim F = \dim \phi(F)$ and $F \cong \phi(F)$.
2. If $M \cong M'$ and $\{F_1,\dots,F_n\}$ is a set of faces of M , then

$$\phi\left(\bigcap_{i=1}^{n} F_i\right) = \bigcap_{i=1}^{n} \phi(F_i) .$$

3. Let α be a non-singular affine map of E_d into itself and let M be a polytope in E_d , then $M \cong \alpha(M)$, that is, if two polytopes are affinely equivalent then they are combinatorially equivalent.
4. If τ is a non-singular projective map then $M \cong \tau(M)$.

In particular, since all d-simplexes are affinely equivalent, they are all of the same combinatorial type.

We will require later two elementary theorems about the realizability of sub-complexes of the face-complex of a polytope. First note an obvious fact. If $M \cong M_1$, $M^* \cong M_1^*$ and if M and M^* are dual, then the polytopes M_1 and M_1^* are dual. Conversely, if the polytopes M_1 and M_2 are both dual to a polytope M^* , then $M_1 \cong M_2$.

Let M be a polytope with face complex $\mathcal{F}(M)$. Let F_1 and $F_2 \in \mathcal{F}(M)$ be such that $F_1 \subseteq F_2$. Then the sub-complex of $\mathcal{F}(M)$ consisting of all faces F of M such that $F_1 \subseteq F \subseteq F_2$ is denoted by $\mathcal{F}(M,F_1,F_2)$. In our new terminology, Lemma 5.6 of Chapter 1 takes the form :

<u>Theorem 1.1</u> *The subcomplex* $\mathcal{F}(M,F_1,F_2)$ *is isomorphic to the face complex of the polytope* $M(F_1,F_2)$ *of dimension* $\dim F_2 - \dim F_1 - 1$.

<u>Corollary 1.2</u> *If* $F_1 \subseteq F_2 \subseteq F_3$ *are faces of a polytope* M , *then the polytope* $M(F_1,F_2)$ *is combinatorially equivalent to some face of the polytope* $M(F_1,F_3)$.

Let x be a vertex of a polytope M and let the hyperplane H strongly separate x from the set $\mathrm{conv}(\mathrm{vert}\, M \setminus x)$.

Definition 1.3 The polytope $M \cap H$ is called a *section of* M *at the vertex* x and is written M_x (see Fig.21 - the polytope M_x is hatched).

The concepts of section and right cut at a vertex are widely used in proving many theorems about polytopes.

Theorem 1.3 *The section at* x *of a polytope* M *is a polytope which is combinatorially equivalent ot the polytope* $M(x,M)$.

Proof The hyperplane H required in definition 1.3 is constructed as follows. Let x' be the projection of the vertex x on the polytope $\operatorname{conv}(\operatorname{vert} M \setminus x)$. Then H is a hyperplane with normal vector $x-x'$ which intersects the line segment $[x,x']$ at an interior point. H intersects the relative interior of every face containing x and for any pair of such faces $F_1 \subset F_2$ we have $\emptyset \neq F_1 \cap H \subset F_2 \cap H$. In this way we establish a bijection between the faces of M containing x and the faces of the section at x. (Fig.21). //

Corollary 1.4 *The section of a simple* d*-polytope at any vertex is a* $(d-1)$*-simplex.*

Theorem 1.3 enables us to construct the polytope $M(F_1,F_2)$ by taking successive cuts at vertices. In fact, if F_1 is a j-face of M then, as in the proof of Theorem 2.12 of Chapter 1, we construct a sequence of 0-,1-,...,(j-1)-faces $F^0,F^1,\dots,F^{j-1}$ of M such that $F^0 \subset F^1 \subset \dots \subset F^{j-1}$. Now, if F_2 is a k-face containing F_1, then for every $i = 0,1,\dots,j$ the polytope $M(F^i,F_2)$ is combinatorially equivalent to a section of $M(F^{i-1},F_2)$ at the vertex corresponding to F^i, with $F^{-1} = \emptyset$, $F^j = F_1$. In this way we inductively construct the polytope $M(F_1,F_2)$.

A polytope is completely defined by its vertices. Hence, in order to specify a complex K in E_d, it suffices to exhibit the vertices of all of its polytopes and to indicate those vertex subsets whose convex hulls give the polytopes of K. In establishing an isomorphism, the geometrical position of the vertices is of no importance. By neglecting the positions of the vertices of a complex, we arrive at the following definition.

Definition 1.4 An *abstract complex* is a family Ψ of subsets of a finite set V (also called *abstract polytopes*) with the properties :

1) all single element subsets of V lie in Ψ and are called *vertices* .

2) if F and $F' \in \Psi$ then $F \cap F' \in \Psi$.

Clearly, every complex K corresponds to a unique abstract complex Ψ . Indeed, the face complex of a polytope M corresponds to the abstract complex $\Psi(M)$ for which $V = \text{vert } M$ and the family Ψ consists of all subsets vert F where F is any face of M . Clearly, two polytopes M and M' are combinatorially equivalent if and only if their abstract complexes are isomorphic.

An abstract complex Ψ is realized by a d-polytope M if $\Psi \cong \Psi(M)$. The problem of identifying whether or not a given abstract complex is realizable by a polytope and the enumeration of all combinatorial types of d-polytopes with a given number of vertices is algorithmically solvable (Grünbaum 1967)

1.2 Pontryagin's Theorem

An abstract complex Ψ is called an *abstract simplicial complex* if, given any abstract polytope in Ψ (called an *abstract simplex*) all of its subsets are also in Ψ .

If an abstract simplex $\Psi_s = \{v_0, v_1, \ldots, v_s\}$ has $s+1$ vertices, then s is called its *dimension* . The largest dimension of all abstract simplexes contained in an abstract simplicial complex Ψ is called the *dimension of the complex* Ψ .

The question of the realizability of abstract simplicial complexes is easily resolved. Let Ψ be an abstract simplicial complex with vertices $v_0, v_1, \ldots, v_n$ and let T_n be an n-simplex in E_n with vertices $x^0, x^1, \ldots, x^n$. With each abstract simplex $\Psi_s = \{v_{i_0}, \ldots, v_{i_s}\}$ of Ψ we associate the face $T_s = \text{conv}(x^{i_0}, \ldots, x^{i_s})$ of T_n . It is clear that the collection K of simplexes so obtained constitutes a complex, for faces of T_n satisfy condition 2) of definition 1.1 . The geometrical realization K so obtained for the abstract simplicial complex is called its *natural realization* .

The following theorem, due to Pontryagin (1976), shows that an abstract simplicial complex also admits other realizations, distinct from the natural one.

Theorem 1.5 *Any abstract simplicial* n*-complex can be realized as a complex in* E_{2n+1} . *Moreover, the vertices of the complex can be chosen arbitrarily in* E_{2n+1} *, provided that they are in general position.*

Proof Let $v_0, v_1, \ldots, v_r$ be the vertices of an abstract simplicial n-complex Ψ . With each vertex v_i we associate a point $x^i \varepsilon E_{2n+1}$ such that the system $x^0, x^1, \ldots, x^r$ is in general position in E_{2n+1} . With each abstract simplex $\Psi_s = \{v_{i_0}, v_{i_1}, \ldots, v_{i_s}\}$ in Ψ we associate the simplex $T(\Psi_s) = \mathrm{conv}(x^{i_0}, x^{i_1}, \ldots, x^{i_s})$. We need to show that the set K of simplexes in E_{2n+1} so obtained is a simplicial complex, i.e. K satisfies conditions 1) and 2) of definition 1.1 . The fact that each element of K is a simplex is clear from the manner of construction. The condition 1) of definition 1.1 is satisfied because of the definition of an abstract simplicial complex. We show that condition 2) is also satisfied.

Let Ψ_r and Ψ_s be two abstract simplexes in Ψ and let $T(\Psi_r)$ and $T(\Psi_s)$ be their corresponding simplexes in K . Let $x^0, x^1, \ldots, x^t$ be the set of all vertices in E_{2n+1} which are vertices of the simplexes $T(\Psi_r)$ or $T(\Psi_s)$. Since the dimension of Ψ is equal to n , $r \leqq n$ and $s \leqq n$ so that $t \leqq 2n+1$. Thus the simplex $T_t = \mathrm{conv}(x^0, \ldots, x^t)$ exists in E_{2n+1} , although, of course, it does not necessarily belong to the set K . The simplexes $T(\Psi_r)$ and $T(\Psi_s)$ are faces of the simplex T_t and so they satisfy condition 2) of definition 1.1 . Thus K is a complex which realizes the abstract complex Ψ . //

The following theorem shows that any n-complex, not just simplicial n-complexes, are realizable in E_{2n+1} .

Theorem 1.6 *To every* n*-complex in* E_d *there corresponds an isomorphic* n*-complex in* E_{2n+1} .

Note that the question of the realizability of a complex $K \subset E_d$ in a space of different dimension is not a trivial one. For example, the n-skeleton of the simplex T_{2n+2} is not realizable even in E_{2n} .

Proof Assume that $d > 2n+1$, otherwise there is nothing to prove. For any $F_i, F_j \in K$ the affine set $H_{ij} = \mathrm{aff}(F_i \cup F_j)$ has dimension not greater than $2n+1 \leq d-1$. Hence the space E_d contains a one-dimensional space L which is not contained in any of the H_{ij} and which is not parallel to any H_{ij}. Let H be a subspace of dimension $d-1$ in E_d which does not contain L, and let τ be the projective map of E_d onto H parallel to L. Then the complex $\{\tau(F) : F \subset K\}$ is isomorphic to K and is contained in E_{d-1}. The proof is completed by induction. //

1.3 Semi-Matroids

The k-skeletons of d-polytopes when k is close to d are rather cumbersome objects to study. The 1-skeletons (polytope graphs) only take into account the incidence relations between vertices and edges and when $d>3$ this is insufficient for identifying the combinatorial type of a polytope. We examine a new method of studying polytopes by means of semi-matroids - a sort of net based on the incidence relations between vertices and facets of a polytope (Kovalev 1979, Kowaljow & Isatchenko 1979).

Definition 1.5 A *semi-matroid* of rank d is a pair $(\mathcal{F}, \mathcal{V})$, where $\mathcal{F}$ is a non-empty finite set whose elements are called *abstract faces* and $\mathcal{V}$ is a family of nonempty subsets of $\mathcal{F}$ called *vertices*. Faces and vertices must satisfy the following axioms :

1) every vertex v contains exactly d abstract faces $\{F_1, \ldots, F_d\}$. We say that the faces F_i, $i \in N_d$, and the vertex v are incident to each other.

2) given any abstract face F, incident to a vertex $v \in \mathcal{V}$, there is a unique abstract face $F' \in \mathcal{F} \setminus v$ such that $\{v \setminus \{F\}\} \cup \{F'\}$ is also a vertex.

Property 2) in definition 1.5 can be replaced by the equivalent property : 2') given any subset of $d-1$ abstract faces, either there exist two vertices to which they are all incident or there exists no such vertex.

We show the equivalence of axioms 2) and 2'). Suppose 2) holds. An arbitrary $(d-1)$-subset $G \subset \mathcal{F}$ cannot belong to a single vertex v, which in this case will be incident to the faces in G and to some other face F. Otherwise, for the face $F \in v$ there would not exist a face F' such that $(v \setminus \{F\}) \cup \{F'\} = G \cup \{F'\}$ is a vertex. If G is contained in more than two vertices, say $v_1 = G \cup \{F\}$, $v_2 = G \cup \{F'\}$, $v_3 = G \cup \{F''\}$, then this contradicts axiom 2) which guarantees the uniqueness of the face F'. Hence, either G belongs to exactly two vertices or it does not belong to any.

Conversely, let v be a vertex having properties 1) and 2') and let F be an arbitrary face incident to v. Then, the (d-1)-subset $G = v\setminus\{F\}$ is, by property 2'), contained in v and in precisely one other vertex v_1. Thus the face $F' = v_1\setminus G$ is the unique face associated with F and v as in property 2).

Two semi-matroids $\mathcal{P} = (\mathcal{F},\mathcal{V})$ and $\mathcal{P}' = (\mathcal{F}',\mathcal{V}')$ are called *isomorphic* if there is a bijection ψ between $\mathcal{F}$ and $\mathcal{F}'$, $\mathcal{V}$ and $\mathcal{V}'$ preserving incidence.

Let M be a simple d-polytope, i.e. all its vertices are incident to exactly d facets. The pair $(\mathcal{F},\mathcal{V})$, where $\mathcal{F}$ is the set of facets of M and $\mathcal{V}$ is the set of its vertices, is a semi-matroid. We call it the *semi-matroid of the polytope* M and denote it by $\mathcal{P}(M)$.

<u>Theorem 1.7</u> *The simple polytopes* M *and* M' *are combinatorially equivalent if and only if their semi-matroids* $\mathcal{P}(M)$ *and* $\mathcal{P}(M')$ *are isomorphic.*

We will give another formulation of Theorem 1.7 in more concrete terminology.

<u>Definition 1.6</u> We call a polytope M a *marked polytope* if each of its facets is given a mark, say, for example, the numbers $1,2,..,f_{d-1}(M)$. Two marked polytopes M and M' are *equivalent* (written $M \sim M'$) if there is an isomorphism of their face-complexes which preserves the marks on their facets.

Theorem 1.7 implies that two simple polytopes are equivalent if and only if there is an isomorphism of their semi-matroids preserving marks. Hence the equivalence of two simple polytopes implies their combinatorial equivalence. Usually the concept of equivalence of polytopes is used in situations in which the polytope is given in canonical form. In this case the facets of the polytope $M(A,b)$ are non-empty sets $F_j = \{x\in M(A,b) : x_j=0\}$, $j\in N_n$. We mark each of them with the number j. Thus, combinatorial equivalence of polytopes $M(A,b)$ is invariant with respect to non-singular affine maps of the space E_n, but equivalence is not invariant with respect to such maps.

We will establish an equivalence criterion for polytopes in the class $\mathcal{M}(A)$ of non-singular polytopes $M(A,b)$ in E_n with a fixed matrix A (we denote such polytopes by $M(b)$). Let the rank of the $(m\times n)$-matrix A be m and let the constraints $x\geq 0$ be non-rigid.

Consequently, $\dim M(b) = d = n-m$. We assume, without loss of generality, that $F_j \neq \emptyset$, $j\in N_n$. Let J_H be a d-subset of N_n and let $J_B = N_n \setminus J_H$. Let B be the submatrix consisting of the columns of the matrix A with indices in J_B and let H denote the remaining columns of A. The set J_H defines a vertex $(x_B, x_H) = (B^{-1}b, 0)$ of the polytope $M(b)$ if and only if $\det B \neq 0$ and the vector b belongs to the cone $\operatorname{con} B$, generated by the columns of B, that is, when B is a feasible basis of $M(b)$. Hence, using Theorem 1.7, we have proved the Lemma :

<u>Lemma 1.8</u> *The polytopes* $M(b), M(b')$ *in the class* $\mathcal{M}(A)$ *are equivalent if and only if a feasible basis of one polytope is also a feasible basis of the other.*

Proof of Theorem 1.7. We will prove sufficiency only, that is, we will show that the face complexes $\mathcal{F}(M)$ and $\mathcal{F}(M')$ are isomorphic. Let ψ be an isomorphism of the semi-matroids $\mathcal{P}(M)$ and $\mathcal{P}(M')$, that is, a bijection between $\mathcal{F}$ and $\mathcal{F}'$, $\mathcal{V}$ and $\mathcal{V}'$ such that for every vertex $v = \{F_{i_1}, \ldots, F_{i_d}\} \in \mathcal{V}$ we have $\psi(v) = (\psi(F_{i_1}), \ldots, \psi(F_{i_d})) \in \mathcal{V}'$. Every proper face F of M can be represented either as an intersection of a certain set ω of facets (Corollary 2.13, Ch.1) : $F = \bigcap_{i\in\omega} F_i$, or as the convex hull of its vertices $\operatorname{vert} F$ (Corollary 2.4, Ch.1) : $F = \operatorname{conv} \operatorname{vert} F$. We define a map ϕ of the face complex $\mathcal{F}(M)$ such that for any proper face $F \in \mathcal{F}(M)$

$$\phi(F) = \bigcap_{i\in\omega} \psi(F_i) \quad , \quad (\phi(\emptyset) = \emptyset \ , \ \phi(M) = M') .$$

Since ψ is an isomorphism of the semi-matroids $\mathcal{P}(M)$ and $\mathcal{P}(M')$, we have

$$\psi(\operatorname{vert} \bigcap_{i\in\omega} F_i) = \operatorname{vert} \bigcap_{i\in\omega} \psi(F_i) \qquad \omega \subseteq \mathcal{F},$$

$$\psi^{-1}(\operatorname{vert} \bigcap_{i\in\omega} F_i) = \operatorname{vert} \bigcap_{i\in\omega} \psi^{-1}(F_i) \qquad \omega \subseteq \mathcal{F}'.$$

These equalities imply that ϕ is a bijection between the face complexes $\mathcal{F}(M)$ and $\mathcal{F}(M')$ which preserves inclusion relations. Thus M and M' are combinatorially equivalent polytopes. //

Definition 1.7 The *spectrum* $S(b_1,b_2)$ *of the polytopes* $M(b_1)$, $M(b_2) \in \mathcal{M}(A)$ is the set of all numbers $\lambda \in (0,1)$ such that the polytope $M(b_\lambda)$ is degenerate. Here $b_\lambda = \lambda b_1 + (1-\lambda)b_2$.

The polytope $M(b_\lambda)$ is degenerate when the vector b_λ belongs to a cone generated by less than m column vectors of the matrix A.

Theorem 1.9 *Two polytopes in the class* $\mathcal{M}(A)$ *are equivalent if and only if their spectrum is empty.*

Proof 1) Sufficiency. Let the spectrum $S(b_1,b_2)$ of $M(b_1)$ and $M(b_2) \in \mathcal{M}(A)$ be empty. Suppose, for contradiction, that the marked polytopes $M(b_1), M(b_2)$ are not equivalent. Then Lemma 1.8 implies the existence of a basis B of the matrix A which is feasible for $M(b_1)$ and infeasible for $M(b_2)$. We show that the segment $\lambda b_1 + (1-\lambda)b_2$, $0 \leq \lambda \leq 1$, has a point of intersection b_λ with a face of the cone con B.

Let $(\beta_1',\ldots,\beta_m')$, $(\beta_1'',\ldots,\beta_m'')$ be the components of the vectors $B^{-1}b_1$ and $B^{-1}b_2$ respectively. By assumption all $\beta_i'>0$ while among the β_i'' there are negative numbers. Let $J^- = \{i : \beta_i''<0\}$ and $J^+ = \{i : \beta_i''>0\}$. Since $M(b_2)$ is non-singular, none of the β_i'' are equal to zero. Clearly we have for all $\lambda>0$, that $\lambda\beta_i' + (1-\lambda)\beta_i'' > 0$, $i \in J^+$.

Let $\lambda_0 = \min\{-\beta_i''/(\beta_i'-\beta_i'') : i \in J^-\}$ and let the minimum be attained at $i=s$. It is easily seen that $0<\lambda_0<1$. From the way in which λ_0 was chosen and from the non-singularity of $M(b_1), M(b_2)$ it follows that

$$\lambda_0\beta_i' + (1-\lambda_0)\beta_i'' \begin{cases} > 0 & i \in J+ \\ = 0 & i = s \\ \geq 0 & i \in J^-\setminus s . \end{cases} \qquad (1.1)$$

The inequalities (1.1) show that B is a feasible basis for the polytope $M(b_{\lambda_0})$, and that for the vertex determined by the basis B, the s-th coordinate, at least, is equal to zero. Thus $M(b_{\lambda_0})$ is a degenerate polytope. This contradiction establishes the sufficiency of the conditions in the theorem.

2) Necessity. Let $M(b_1) \sim M(b_2)$. By Lemma 1.8, every feasible basis B of $M(b_1)$ is feasible for $M(b_2)$ and conversely. This means that the vectors b_1 and b_2 belong to precisely the same

cones con B which are generated by the matrices B consisting of m columns of the matrix A . Thus, the vector b_λ lies within these cones and does not belong to any other cone. //

§2 GALE DIAGRAMS

The method of Gale diagrams is one of the few general methods for studying the combinatorial structure of polytopes (Gale 1964). In this section we present the basis of the method and we illustrate how it enables us to enumerate the combinatorial types of polytopes. In particular we obtain enumeration results for d-polytopes with d+2 and d+3 vertices.

2.1 Gale Sets

Let M be a d-polytope in E_d and let $V = \text{vert}\, M = \{v^1,\dots v^n\}$. We examine the space $L(V)$ of all solutions $(\lambda_1,\dots,\lambda_n)$ of the following system of linear homogeneous equations :

$$\sum_{i=1}^{n} \lambda_i v^i = 0 \quad , \quad \sum_{i=1}^{n} \lambda_i = 0 \ . \tag{2.1}$$

Let $a^1,\dots,a^{n-d-1}$ be a basis of $L(V)$. Here $a^i = (\alpha_{i1},\dots,\alpha_{in})$, $i \in N_{n-d-1}$. Let $A(V)$ be a $((n-d-1)\mathrm{x}n)$-matrix whose rows are the vectors $a^1,\dots,a^{n-d-1}$. For each $j \in N_n$, denote the j^{th}-column of the matrix $A(V)$ by $\bar{v}^j$ and let it have components $(\alpha_{1j},\dots,\alpha_{n-d-1,j})$. For every subset $Z \subset V$ we denote by $\Gamma(Z)$ the set $\{\bar{v}^j : v^j \in Z\}$.

<u>Definition 2.1</u> The set $\Gamma(V)$ is called the *Gale set* of the polytope M . Different vertices of M may correspond to the same point in the Gale set. Thus, with each point $\bar{v}^j \in \Gamma(V)$ we associate the multiplicity $m_j = |\Gamma^{-1}(\bar{v}^j)|$.

Clearly, the Gale set is not uniquely defined. If we choose different bases of the space $L(V)$, we will obtain different Gale sets (related by a linear map).

<u>Definition 2.2</u> The set $Z \subset V$ is a *co-face* of the polytope M , if $F = \text{conv}(V \setminus Z)$ is a face of M .

<u>Theorem 2.1</u> *The set* $Z \subset V$ *is a co-face of the polytope* M *if and only if* $0 \in \text{rel int conv}\, \Gamma(Z)$.

Proof By Proposition 2.15 ,Ch.1 , Z is a co-face of M if and only if

$$\operatorname{aff}(V\setminus Z) \cap \operatorname{conv} Z = \emptyset \ . \tag{2.2}$$

Suppose that $Z = \{v^1,\dots,v^s\}$ is not a co-face of M , that is $\operatorname{conv}(v^1,\dots,v^s) \cap \operatorname{aff}(v^{s+1},\dots,v^n) \neq \emptyset$. Then, there is a point x such that

$$x = \sum_{i=1}^{s} \lambda_i v^i \quad , \quad \sum_{i=1}^{s} \lambda_i = 1 \quad , \quad \lambda_i \geqq 0 \quad \forall\, i \varepsilon N_s \ . \tag{2.3}$$

and

$$x = \sum_{i=s+1}^{n} (-\lambda_i) v^i \quad , \quad \sum_{i=s+1}^{n} (-\lambda_i) = 1 \ . \tag{2.4}$$

From (2.3) and (2.4) we obtain

$$\sum_{i=1}^{n} \lambda_i v^i = 0 \quad , \quad \sum_{i=1}^{n} \lambda_i = 0 \ . \tag{2.5}$$

But (2.5) implies that the vector $\lambda = (\lambda_1,\dots,\lambda_n) \ \varepsilon\ L(V)$. Hence $\lambda = \sum_{i=1}^{n-d-1} \gamma_i a_i$. Let $\gamma = (\gamma_1,\dots,\gamma_{n-d-1})$. Then, by definition 2.1, $\lambda_i = \gamma\Gamma(v^i)$, $i\varepsilon N_n$, so that, by (2.3), $\gamma\Gamma(v^i) \geqq 0$, $i \varepsilon N_s$. Since at least one of the λ_i , $i\varepsilon N_s$, is strictly positive, it follows that at least one of the points $\Gamma(v^i)$ lies in the open half-space $\gamma x>0$, while the others lie in the closed half-space $\gamma x \geqq 0$. Thus $0 \notin \operatorname{rel\,int} \operatorname{conv}(\Gamma(v^1),\dots,\Gamma(v^s))$.

Sufficiency is proved by reversing the argument. //

<u>Corollary 2.2</u> *The* d-*polytope* $M \subset E_d$ *is simplicial if and only if for each hyperplane* $H \subset E_{n-d-1}$ *containing the origin, we have*

$$0 \notin \operatorname{rel\,int} \operatorname{conv}(H \cap \Gamma(V))$$

or equivalently

$$\dim \operatorname{conv} \Gamma(Z) = \dim \operatorname{conv} \Gamma(V)$$

for every non-empty co-face Z .

The following theorem characterizes certain point sets in E_{n-d-1} which are Gale sets of some polytope.

Theorem 2.3 *Let* $\overline{V} = \{\overline{v}^1, \ldots, \overline{v}^n\}$ *be a set of points in* E_s *with the properties :*

1) $\sum_{i=1}^{n} \overline{v}^i = 0$;

2) *Any open half-space* H^+ *, generated by a hyperplane* H *containing* 0 *, contains at least two points of* $\overline{V}$.

Then $\overline{V}$ *is a Gale set of some* (n-s-1)*-polytope.*

Proof Let $\overline{A} = (\overline{v}^1, \ldots, \overline{v}^n)$ be the (s×n)-matrix whose columns are the vectors $\overline{v}^1, \ldots, \overline{v}^n$. The system of equations $\overline{A}y = 0$ has $n-s-1$ affinely independent solutions, say $y^1, \ldots, y^{n-s-1}$. By condition 1) the system certainly has the solution $e = (1, \ldots, 1)$. Let A be the $n \times (n-s-1)$ matrix whose columns are the vectors y^i , and let $V = \{v^1, \ldots, v^n\}$ be the rows of A . By condition 2) and Theorem 2.1 we see that every point in V is a vertex of the polytope conv V . Thus $\overline{V}$ is a Gale set for conv V . //

Theorem 2.4 *The polytope* M *is a pyramid with apex* v *if and only if* $\Gamma(v) = 0$. *Further, if* M *is a pyramid with base* Q *, then* $\Gamma(Q) = \Gamma(V) \setminus \Gamma(v)$

Proof Let M be a pyramid with base Q and apex v . Then $v \notin$ aff Q , that is, the coefficient of v in (2.1) is always zero. Thus, the matrix $A(\text{vert } M)$ is obtained from the matrix $A(\text{vert } Q)$ by the addition of a zero column corresponding to the point v . Sufficiency is shown similarly by reversing the argument. //

Definition 2.3 An r-*fold* d-*pyramid* is a pyramid M whose base Q is an (r-1)-fold (d-1)-pyramid ; a 1-fold d-pyramid is a d-pyramid.

Theorem 2.4 generalizes in an obvious manner to the case of an r-fold pyramid : a polytope M is an r-fold pyramid when the multiplicity of 0 in the Gale diagram $\Gamma(M)$ is equal to r .

We give one of the possible geometric interpretations of Gale sets. Let E_d and E_{n-d-1} be orthogonal subspaces of E_{n-1} and let

T_{n-1} be an (n-1)-simplex centred at the origin. If V is the orthogonal projection of the set vert T_{n-1} on E_d and $\overline{V}$ is the orthogonal projection of vert T_{n-1} on E_{n-d-1}, then $V = \Gamma(\overline{V})$ and conversely $\overline{V} = \Gamma(V)$.

2.2 Gale Diagrams

Among the various Gale sets of a polytope it is convenient to select one.

Definition 2.4 Two point sets $\overline{V} = \{\overline{v}^1,\ldots,\overline{v}^n\}$ and $\overline{U} = \{\overline{u}^1,\ldots,\overline{u}^n\}$ in E_{n-d-1}, such that $0 \in \text{int conv } \overline{V}$, $0 \in \text{int conv } \overline{U}$, are called *isomorphic* if the correspondence $\phi : \overline{v}^i \to \overline{u}^i$ has the property that for any pair of subsets $Z \subseteq \overline{V}$, $\phi(Z) \subseteq \overline{U}$ either $0 \in \text{rel int conv } Z$ and $0 \in \text{rel int conv } \phi(Z)$, or $0 \notin \text{rel int conv } Z$ and $0 \notin \text{rel int conv } \phi(Z)$.

For example, all the Gale sets of a polytope generated by different bases of the space $L(V)$ are isomorphic. In particular, if $\mu_i > 0$, then the set $\overline{U} = \{\mu_1\overline{v}^1,\ldots,\mu_n\overline{v}^n\}$ is isomorphic to the set $\overline{V}$.

Definition 2.5 A *Gale Diagram* $\mathcal{D}(M)$ of a d-polytope $M \subset E_d$ with n vertices $v^1,\ldots,v^n$ is a set of points $\hat{v}^1,\ldots,\hat{v}^n \in E_{n-d-1}$ defined by the rule : $\hat{v}^i=0$ if $\Gamma(v^i) = 0$; $\hat{v}^i = \Gamma(v^i)/\|\Gamma(v^i)\|$ if $\Gamma(v^i) \neq 0$. Each point $\hat{v}^i \in \mathcal{D}(M)$ is given the mark $m_i = |\Gamma^{-1}(\overline{v}^i)|$.

Thus, a Gale diagram consists of a subset of points in the set $S^{n-d-2} \cup \{0\}$, where S^{n-d-2} is the unit sphere in E_{n-d-1} with centre at the origin.

The following important result follows from Theorem 2.1 and Definitions 2.4 and 2.5 .

Theorem 2.5 *Two polytopes* M *and* M' *are combinatorially equivalent if and only if their Gale diagrams are isomorphic.*

The concepts of isomorphism of Gale sets and of Gale diagrams of a given polytope enable us to reformulate all the results on Gale sets in terms of Gale diagrams. We will combine these results in one theorem.

Let $Z \subset \text{vert } M$, then the set of points in the Gale diagrams $\mathcal{D}(M)$ which correspond to points in Z will be denoted by $\hat{Z}$.

Theorem 2.6

1) *The set* $Z \subset \text{vert}\, M$ *is a co-face of the polytope* M *if and only if* $0 \in \text{rel int conv}\, \hat{Z}$;

2) *The set* $\hat{V} \subset E_{n-d-1}$, *consisting of* n *points, is a Gale diagram of some* d*-polytope* M *with* n *vertices if and only if every open half-space, generated by a hyperplane which contains the origin, contains at least two points of* $\hat{V}$;

3) *Let* F *be a facet of the polytope* M *and let* Z *be its corresponding co-face, then* $\hat{Z}$ *is the vertex set of a simplex which contains the origin in its relative interior.*

4) *The polytope* M *is simplicial if and only if for every hyperplane* H *containing* 0 , $0 \notin \text{rel int conv}(\hat{V} \cap H)$;

5) *The polytope* M *is an* r*-fold pyramid if and only if the origin has multiplicity* r *in its Gale diagram.*

2.3 Polytopes with d+2 vertices

The Gale diagram of a d-polytope with $d+2$ vertices in E_1 $(n-d-1 = 1)$ lies in the set $\{-1,0,1\}$. Let the points $-1,0,1$ have multiplicities m_{-1}, m_0, m_1 respectively (Fig.22). By Theorem 2.6

$$m_0 \geqq 0 \ , \ m_1 \geqq 2 \ , \ m_{-1} \geqq 2 \ , \ m_0 + m_1 + m_{-1} = d+2 \ . \qquad (2.6)$$

Conversely, any triple $\{m_{-1}, m_0, m_1\}$ satisfying the conditions of (2.6) is obtained, by the second part of Theorem 2.6 , from some d-polytope with $d+2$ vertices. According to Theorem 2.5 , two d-polytopes M and M' with $d+2$ vertices are combinatorially equivalent if and only if $(m_{-1}, m_0, m_1) = (m'_{-1}, m'_0, m'_1)$ or $(m_1, m_0, m_{-1}) = (m'_{-1}, m'_0, m'_1)$, where (m_{-1}, m_0, m_1) and (m'_{-1}, m'_0, m'_1) are the multiplicities of the points $(-1,0,1)$ in the Gale diagrams of M and M' respectively.

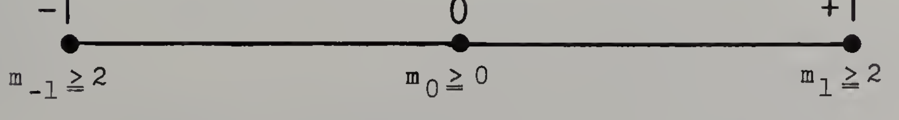


Fig. 22.

By part 4 of Theorem 2.6 , M is a simplicial polytope if and only if $m_0 = 0$. Thus, the number of partitions of d into two positive integral parts gives the number of combinatorial types of simplicial polytopes. Hence there are $[d/2]$ types of simplicial d-polytopes with $d+2$ vertices. The simplicial d-polytope whose Gale diagram has multiplicities $r+1, 0$ and $d-r+1$, $r \varepsilon N_{[d/2]}$, is denoted by T_d^r .

If $m_0 > 0$, the polytope M is an m_0-fold pyramid whose base is the $(d-m_0)$-polytope $T^r_{d-m_0}$, where $r \varepsilon N_{[(d-m_0)/2]}$. Summarizing the above we have :

<u>Theorem 2.7</u> *There are* $[d^2/4]$ *different combinatorial types of* d-*polytopes with* $d+2$ *vertices. Of these,* $[d/2]$ *are the simplicial polytopes* T^r_d , $r \varepsilon N_{d/2}$, *and the remainder are the* t-*fold pyramids* $T_d^{t,r}$ *whose bases are the simplicial polytopes* T^r_{d-r} , $r \varepsilon N_{[(d-t)/2]}$.

We remark that the number of simplicial d-polytopes with $d+2$ vertices was established by Schlegel (1891).

Let us calculate the number of k-faces of the simplicial d-polytope T_d^r . A k-face of T_d^r is a k-simplex and its coface has $d-k+1$ vertices of which at least one corresponds to -1 and one corresponds to $+1$ in the Gale diagram , (statement 3 in Th. 2.6). Thus for each $k \varepsilon N_{d-1}$ we have

$$f_k(T_d^r) = \sum_{\substack{u+v=d-k+1 \\ u,v\geq 1}} \binom{r+1}{u}\binom{d+1}{v} = \binom{d+2}{d-k+1} - \binom{r+1}{d-k+1} - \binom{d-r+1}{d-k+1} .$$

From this we find that the number of k-faces of the t-fold pyramid $T_d^{t,r}$ with base T_d^r is given by the formula

$$\begin{aligned} f_k(T_d^{t,r}) &= \sum_i \binom{t}{i} f_{k-i}(T_d^r) \\ &= \binom{d+2}{d-k+1} - \binom{r+t+1}{d-k+1} - \binom{d-r+1}{d-k+1} + \binom{t+1}{d-k+1} . \end{aligned}$$

For any d-polytope M with $d+2$ vertices, it is easy to establish the following inequalities

$$f_k(T_d^{d-2,1}) \leqq f_k(M) \leqq f_k(T_d^{0,[d/2]}) = f_k(T_d^{[d/2]}) \quad , \quad k \varepsilon N_{d-1} .$$

We will see later (§3) that the polytope $T_d^{[d/2]}$ is combinatorially equivalent to the cyclic polytope $C(d,d+2)$.

2.4 Polytopes with $d+3$ vertices

The Gale diagram of a d-polytope with $d+3$ vertices consists of points located on the unit circle in E_2 and at its centre. Draw the diameters through every point in a Gale diagram. There are a number of operations which may be carried out on a Gale diagram without changing the isomorphism class of the diagram. Firstly, we can alter the angles between the diameters provided we do not alter their relative ordering. Secondly, if two neighbouring diameters have points of $\hat{V}$ at only one end of the diameter, then these diameters can be coalesced, provided we increase the resultant multiplicity correspondingly (Fig. 23).

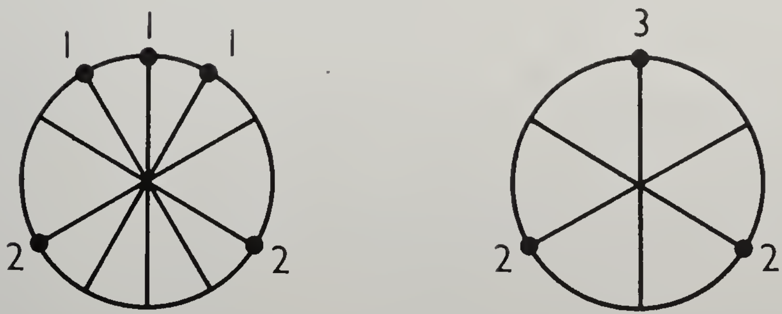


Fig. 23.

Definition 2.6 A *standard Gale diagram* of a d-polytope with $d+3$ vertices is a diagram consisting of the vertices of a regular polygon inscribed in a unit circle and labelled according to the following rules :

1) every label is a non-negative number and the sum of the labels is $d+3-t$, where t is the label of the centre of the circle;

2) no two diametrically opposite vertices of the polygon are both labelled zero;

3) no two neighbouring vertices are both labelled zero;

4) the sum of the labels of the vertices lying in any open half-space, whose boundary passes through the origin, is not less than two.

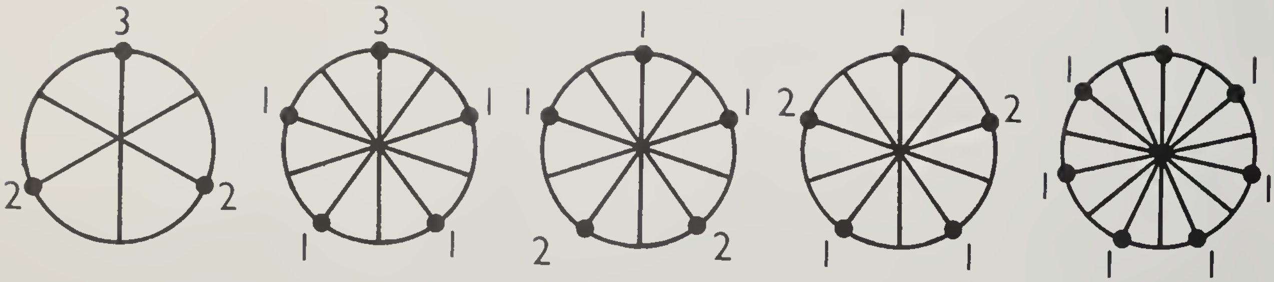


Fig. 24.

Note that rule 4) is automatically satisfied when $n \geq 5$. Figure 24 shows all possible Gale diagrams for simplicial 4-polytopes with 7 vertices.

Two d-polytopes with $d+3$ vertices are combinatorially equivalent if and only if their standard Gale diagrams are isomorphic (that is, they coincide after a suitable rotation or reflection). Thus the problem of enumerating the number of combinatorial types of such polytopes reduces to the problem of enumerating all standard Gale diagrams.

In the standard Gale diagram of a simplicial d-polytope with $d+3$ vertices, the centre of the circle has label zero and on each diameter only one end-point has a non-zero label. This somewhat simplifies the problem of counting Gale diagrams of simplicial d-polytopes. The methods of enumerating non-isomorphic standard Gale diagrams in the plane are the same as those used in the problem of counting graphs. The group of permutations acting on the vertices of a Gale diagram is introduced and then, using Polya-Burnside enumeration theory, the number of permutations invariant under every possible symmetry (reflections and rotations) is counted.

Using such techniques it has been found (Grünbaum 1967) that the number of combinatorial types of simplicial d-polytopes with $d+3$ vertices is given by the number

$$2^{\lfloor d/2 \rfloor} - \left\lfloor \frac{d+4}{2} \right\rfloor + \frac{1}{4(d+3)} \sum_{h} \phi(h) 2^{(d+3)/h} ,$$

where the summation is carried out over all odd divisors of $d+3$, and

$\phi(h) = h \prod_{p|h} (1 - \frac{1}{p})$ is Euler's totient fuction.

In 1970 Lloyd (1970) calculated the number of all combinatorial types of d-polytopes with $d+3$ vertices.

§3 MAXIMUM NUMBER OF FACES

The problem of describing the range of values of the f-vectors of polytopes in the general case is unsolved. Attempts have been made to find upper and lower bounds on particular components of the f-vector when the values of the other components are fixed. An abundant literature has been devoted to the problem of finding the exact upper bound $\phi_k(d,n)$ for the number of k-faces of a d-polytope M when the number of vertices is equal to n :

$$\phi_k(d,n) = \max\left\{f_k(M) : \dim M = d \ , \ f_0(M) = n\right\} \quad , \quad 1 \leq k < d < n \quad .$$

In 1957, Motzkin (1957) conjectured that $\phi_k(d,n) = f_k(C(d,n))$, $k \in N_{d-1}$. In other words, among all d-polytopes with a fixed number of vertices, the corresponding cyclic polytope has the largest number of faces of all dimensions. The conjecture was proved to be true by McMullen (1970). The conjecture had been proved for special choices of the parameters d and n in many previous works (they are listed in Grünbaum (1967) and in McMullen & Shephard (1971)).

It suffices to prove the conjecture for simplicial polytopes.

Theorem 3.1 *Let the polytope* M^o *be obtained from the* d-*polytope* M *by a right displacement of each of its vertices. Then* M^o *is a simplicial polytope with the properties*

$$f_0(M^o) = f_0(M) \quad , \quad f_i(M^o) \geq f_i(M) \qquad i \in N_{d-1} \ .$$

The proof follows directly from the definitions and properties of a pyramid and Lemma 2.1, Ch.2 .

3.1 Transformation of the Dehn-Sommerville Equations.

Let M be a simplicial d-polytope. Consider the polynomial

$$f(M,t) = \sum_{j=-1}^{d-1} (-1)^{j+1} f_j(M) t^{j+1} \quad .$$

It is clear that the Dehn-Sommerville equations are equivalent to the identity

$$f(M,1-t) = (-1)^d f(M,t) . \tag{3.1}$$

In addition to the polynomial $f(M,t)$ we introduce another polynomial in t of degree d

$$g(M,t) = (1-t)^d f(M,t/(t-1)) . \tag{3.2}$$

The coefficients of this polynomial are denoted by $g_k(M)$;

$$g(M,t) = \sum_{k=-1}^{d-1} g_k(M) t^{k+1} . \tag{3.3}$$

Lemma 3.2 *To prove the upper bound conjecture it suffices to prove the inequalities*

$$g_k(M) \leqq \frac{n-d+k}{k+1} g_{k-1}(M) \qquad k \in N_{d-1} \tag{3.4}$$

for every simplicial d-*polytope with* n *vertices.*

Proof We establish some relations between the coefficients of the polynomials $f(M,t)$ and $g(M,t)$. Equating coefficients of equal powers of t in (3.2) and (3.3) we find that

$$g_k(M) = \sum_{j=-1}^{k} (-1)^{k-j} \binom{d-j-1}{d-k-1} f_j(M) . \tag{3.5}$$

On the other hand, from (3.1) and (3.2) we have

$$t^d g(M,t^{-1}) = t^d (1-t^{-1})^d f(M,t^{-1}/(t^{-1}-1))$$

$$= (t-1)^d f(M,1-t/(t-1)) = (1-t)^d f(M,t/(t-1)) = g(M,t) .$$

Hence

$$t^d g(M,t^{-1}) = g(M,t) . \tag{3.6}$$

From (3.3) and (3.6) we conclude that

$$g_k(M) = g_{d-k-2}(M) \quad , \quad k=-1,0,1,\ldots,[d/2]-1 \ . \tag{3.7}$$

It is easy to verify that $f(M,t) = (1-t)^d g(M,t/(t-1))$ so that

$$f_j(M) = \sum_{k=-1}^{j} \binom{d-k-1}{d-j-1} g_k(M) \ . \tag{3.8}$$

We have established a correspondence between the numbers $f_j(M)$ and $g_k(M)$ and obtained a system of equations (3.7) which are equivalent to the Dehn-Sommerville equations. The equations (3.7) are independent for odd d ; when $d=2m$ is even, the $(m-1)$-st equation is clearly redundant. Using (3.7), equations (3.8) can be written in the form

$$f_j(M) = \sum_{k=-1}^{m-1} \left\{ \binom{d-k-1}{d-j-1} + (1-\delta_{k,d-m-1}) \binom{k+1}{d-j-1} \right\} g_k(M) \quad , \tag{3.9}$$

where δ_{ij} is the Kronecker delta and $m = [d/2]$. The coefficients of $g_k(M)$ in (3.9) are non-negative for all j and for $j \geqq m-1$ are positive for each k .

For cyclic polytopes we have (Corollary 2.19, Ch.1) that $f_j(C(d,n)) = \binom{n}{j+1}$, $j=-1,0,1,\ldots,m-1$, so that

$$g_k(C(d,n)) = \sum_{j=-1}^{k} (-1)^{k-j} \binom{d-j-1}{d-k-1} \binom{n}{j+1} = \binom{n-d-k}{k+1} \ . \tag{3.10}$$

Equation (3.10) is most simply proved by noticing that $f(M,t)$ and $(1-t)^n$ are polynomials which only differ in terms of degree higher than m . The same is true for the polynomials $g(M,t)$ and $(1-t)^d(1-t/(t-1))^n = (1-t)^{-(n-d)}$, and the coefficient of t^{k+1} in the latter expression equals $\binom{n-d+k}{k+1}$.

Equations (3.9) and (3.10) imply that the inequalities $f_j(M) \leqq f_j(C(n,d))$, $j \varepsilon N_{d-1}$, are a consequence of the inequalities

$$g_k(M) \leqq \begin{matrix} n-d+k \\ k+1 \end{matrix} \quad , \qquad k \ \varepsilon \ N_{d-1} \ . \tag{3.11}$$

In addition, $g_0(M) = n-d$. Hence (3.11) is true if the inequalities (3.4) are true. //

3.2 Shelling the Boundary Complex

A *shelling of the boundary complex* $\mathcal{F}(M)$ of a polytope M is a listing of its facets, say $F_1,\ldots,F_u$ $(u=f_{d-1}(M))$ with the following property : for $s=2,\ldots,u-1$, the set

$$F_s \cap \left(\bigcup_{t=1}^{s-1} F_t \right)$$

is homeomorphic to a (d-2)-ball.

It follows from this definition that for $s\varepsilon N_{u-1}$, $\bigcup_{t=1}^{s} F_t$ is homeomorphic to a (d-1)-ball.

Bruggesser & Mani (1971) showed that the boundary complex of any polytope can be shelled. An outline of their method can be described as follows. Take a curve L which intersects in distinct points all the supporting hyperplanes which generate the facets of the polytope M and which also intersects the interior of M , Let a point z move along the curve L , beginning from a point in $L \cap \text{int } M$ and successively intersecting the supporting hyperplanes $H_1,\ldots,H_u$ generating the faces $F_1,\ldots,$ F_u . It may then be shown that $F_1,\ldots,F_u$ is a shelling of M (Fig.25).

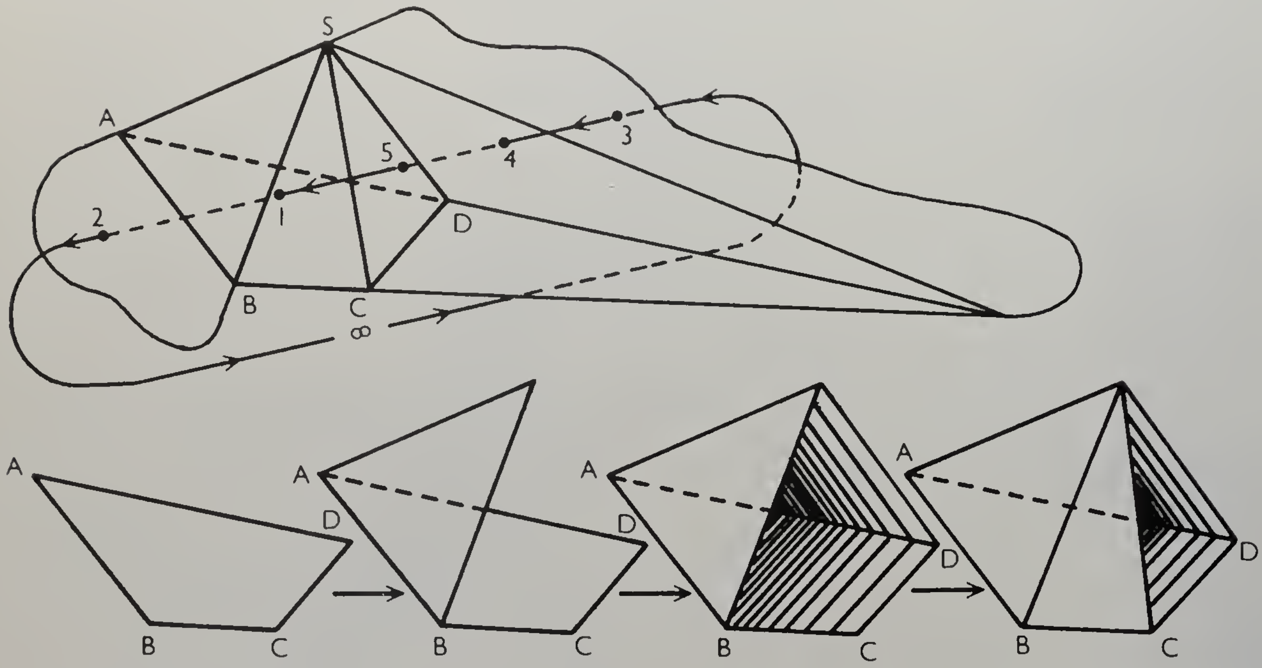


Fig. 25.

Let $F_1, \dots, F_u$ be a shelling of the simplicial d-polytope M. Let $M_s = \bigcup_{t=1}^{s} F_t$ and let $f_j(M_s)$ be the number of j-faces of M belonging to M_s. Further, let

$$g_k(M_s) = \sum_{j=-1}^{k} (-1)^{k-j} \binom{d-j-1}{d-k-1} f_j(M_s) \quad , \quad k=-1,0,1,\dots,d-1 .$$

We calculate the quantity $g_k(M_s) - g_k(M_{s-1})$, putting $g_k(M_0) = 0$. The set $F_s \cap M_{s-1}$ is topologically a (d-2)-ball in the boundary of the (d-1)-simplex F_s, and so $F_s \cap M_{s-1}$ is the union of some (d-2)-faces of F_s. Let the intersection of these faces be a (d-r-2)-face F'. By Proposition 2.14, Ch.1, the polytope F_s has an r-face F such that $F \cap F' = \emptyset$. When we add the face F_s to the shelling M_{s-1} we are adding faces which contain the r-face F. The number of such j-faces is equal to $f_{j-r-1}(M(F,F_s)) = \binom{d-r-1}{d-j-1}$, since $M(F,F_s)$ is a (d-r-2)-simplex (Corollary 1.2). Hence

$$\begin{aligned} g_k(M_s) - g_k(M_{s-1}) &= \sum_{j=-1}^{k} (-1)^{k-j} \binom{d-j-1}{d-k-1} \left[f_j(M_s) - f_j(M_{s-1}) \right] \\ &= \sum_{j=-1}^{k} (-1)^{k-j} \binom{d-j-1}{d-k-1} \binom{d-r-1}{d-j-1} = \sum_{j=-1}^{k} (-1)^{k-j} \binom{d-r-1}{d-k-1} \binom{k-r}{j-r} = \delta_{kr} \end{aligned} \tag{3.12}$$

where δ_{kr} is the Kronecker delta. Equation (3.12) is also satisfied in the extreme cases $s=1, r=-1$ and $s=u, r=d-1$. Thus in passing from M_{s-1} to M_s the coefficient g_r increases by unity, whereas the remaining g_k, $k \neq r$, remain unchanged. From (3.12) we find that $g_k(M) = g_k(M_u) \geqq 0$.

<u>Lemma 3.3</u> *Let* M *be a simplicial* d-*polytope. Then*

$$g_k(M) \leqq \frac{n-d+k}{k+1} g_{k-1}(M) \qquad k = -1,0,1,\dots,d-1$$

Proof Let x be a vertex of M and let M_x be a section of M at the vertex x. We prove the Lemma by evaluating the sum $\sum_{x \in \mathrm{vert} M} g_{k-1}(M_x)$ in two ways. First, using the relation

$$\sum_{x \in \mathrm{vert} M} f_{j-1}(M_x) = (j+1) f_j(M) ,$$

and calculating the value of $g_{k-1}(M_x)$ by (3.5) we obtain

$$\sum_{x\varepsilon \mathrm{vert} M} g_{k-1}(M_x) = (k+1)\, g_k(M) + (d-k)\, g_{k-1}(M) . \qquad (3.13)$$

The truth of (3.13) may also be seen from geometrical considerations. A shelling of the complex $\mathcal{F}(M)$ will induce a shelling of each of the complexes $\mathcal{F}(M_x)$. Let $F_1,\ldots,F_u$ be a shelling of $\mathcal{F}(M)$ and, by adding F_s to M_{s-1} to obtain M_s, let us increase $g_r(M)$ by unity, leaving all the other $g_k(M)$ unchanged. By examining the polytopes M_x, let us see what happens in this process to the quantities $g_k(M_x)$. For $r+1$ of them we clearly add a unit to $g_{r-1}(M_x)$, thanks to the $(r-1)$-face of the r-face which does not contain x, the point at which the cut was taken : for the remaining $d-r-1$ of them we add a unit to $g_r(M_x)$. Summing over all $x \varepsilon$ vert M we arrive at (3.13).

Next, we establish the inequality

$$\sum_{x\varepsilon \mathrm{vert} M} g_{k-1}(M_x) \leqq n g_{k-1}(M) . \qquad (3.14)$$

To do this we consider a shelling of M in which for some s, M_s consists of all the faces containing the vertex x and only then are the remaining faces added. We see easily that in the induced shelling of $\mathcal{F}(M_x)$ the addition of unity to $g_{k-1}(M_x)$ gives rise also to the addition of unity to $g_{k-1}(M)$ for $\mathcal{F}(M)$. Thus $g_{k-1}(M_x) \leqq g_{k-1}(M)$, and summing over all vertices of M we obtain the inequality (3.14). Comparing (3.13) and (3.14) we obtain the inequality of the lemma. //

Theorem 3.1 and Lemmas 3.2 and 3.3 yield the solution of the Upper Bound Conjecture.

Theorem 3.4 *The cyclic polytopes have the maximum number of faces of all dimensions in the class of* d-*polytopes with a fixed number of vertices.*

3.3 The f-vector of a Cyclic Polytope

Some of the components of the f-vector of a cyclic polytope $C(d.n)$ were found in §2, Ch.1 .

$$f_k(C(d,n)) = \binom{n}{k+1} \qquad k \varepsilon N_{[d/2]}.$$

The remaining components can be found by substituting f_k, $k\varepsilon N_{[d/2]}$ in the Dehn-Sommerville equations (Theorem 5.8, Ch.1). However, it turns out that it is very tedious to simplify the expressions obtained. Below we give a method for calculating the number of k-faces of a cyclic polytope $C(d,n)$ for all k, based on a set of necessary and sufficient conditions satisfied by subsets of vertices which generate faces. The method was proposed by McMullen & Shephard (1971).

Theorem 3.5 *The number of* k-*faces* ($1\leq k\leq d-1$) *of a cyclic* d-*polytope* $C(d,n)$ *is given by the expressions*

$$f_k(C(d,n)) = \begin{cases} \sum_{j=1}^{m} \frac{n}{n-j}\binom{n-j}{j}\binom{j}{k+1-j}, & \text{if } d=2m \\ \sum_{j=0}^{m} \frac{k+2}{n-j}\binom{n-j}{j+1}\binom{j+1}{k+1-j}, & \text{if } d=2m+1 . \end{cases} \tag{3.15}$$

The proof consists of two parts. In the first part we establish properties of subsets of vertices which generate k-faces and in the second part we enumerate such subsets.

Let the vertices $x^i=x(\tau_i)$, $i\varepsilon N_n$, have the same ordering as the parameter values τ_i. Let $W\subset \text{vert } C(d,n)$. The subset $V\subseteq W$ is called *connected* if $\exists\ i,j\varepsilon N_n$, $i<j$, such that $V=\{x^i,x^{i+1},\dots,x^j\}$, $x^{i-1}\notin W$, $x^{j+1}\notin W$.

Subsets $Y_1,Y_2\subseteq W$ of the form

$$Y_1 = \{x^1,\dots,x^i\}, \qquad x^{i+1}\notin W ,$$

$$Y_2 = \{x^j,\dots,x^n\}, \qquad x^{j-1}\notin W$$

are called *terminal*. Clearly, every proper subset $W\subset \text{vert } C(d,n)$ may be represented uniquely in the form $W = Y_1\cup V_1\cup\dots\cup V_t\cup Y_2$. where $0\leqq t\leqq[(n-1)/2]$, the V_i are connected sets and Y_1,Y_2 are terminal sets. The set W is called an (r,s)-*set* if $|W|=r$ and exactly s of its connected subsets contain an odd number of elements.

Lemma 3.6 *Let* $W\subset \text{vert } C(d,n)$, $n\geqq d+1$. *Then* conv W *is a* k-*face of the cyclic polytope if and only if* W *is a* $(k+1,s)$-*set for some* s, $0\leqq s\leqq d-k-1$.

Proof By Proposition 2.17, Ch.1, $C(d,n)$ is a simplicial polytope. Thus if conv W is a k-face of the polytope $C(d,n)$ then $|W| = k+1$.

Consider first the case $k=d-1$. Let $|W| = d$. Then the points of W are affinely independent. So $H = \text{aff } W$ is a hyperplane in E_d. Since the curve $x(\tau) \subset E_d$, the points of W divide it into $d+1$ arcs which lie successively on opposite sides of H. Further, conv W is a face of $C(d,n)$ if and only if the hyperplane H is supporting to $C(d,n)$; that is, when the points vert $C(d,n)\setminus W$ all lie in one of the half-spaces generated by H (Proposition 2.15, Ch.1). Clearly this is the case if and only if there are an even number of points of W between every pair of points of vert $C(d,n)\setminus W$. In turn, this is equivalent to the statement that W is a $(d,0)$-set, that is, it does not contain any connected subsets with an odd number of elements.

Now consider the general case. Let $W \subset \text{vert } C(d,n)$ and let $|W| = k+1$. If W has no more than $d-k-1$ connected subsets with an odd number of elements, then it is possible to find a subset T of points on the curve $x(\tau)$ such that $T \cap C(d,n) = \emptyset$, $|T| = d-k-1$ and $T \cup W$ as a subset of the $(n+d-k-1,0)$-set $T \cup \text{vert } C(d,n)$ has only connected subsets with an even number of elements. Then the hyperplane $H = \text{aff } (T \cup W)$ is is supporting to the cyclic polytope $C(d,n+d-k-1) = \text{conv } (T \cup \text{vert } C(d,n))$. Consequently $H \cap \text{vert } C(d,n) = W$ and $C(d,n) \subseteq C(d,n+d-k-1)$. By Theorem 2.2, Ch.1, the hyperplane H generates a face of $C(d,n)$.

The conditions are also necessary, for by Theorem 2.12, Ch.1, if conv W is a face of $C(d,n)$, then it is also a face of some facet conv W', where $W \subseteq W' \subseteq \text{vert } C(d,n)$. Since W' has no connected subsets with an odd number of elements, it is clear that W cannot have more than $d-k-1$ connected subsets with an odd number of elements. //

We now try to count the number of different $(k+1,s)$-sets $W \subset \text{vert } C(d,n)$ where $s \leq d-k-1$.

We introduce an auxiliary concept. An n-*cycle* is a set of n distinct points taken on a closed oriented curve. Every point in an n-cycle has a unique successor and a unique predecessor. The n^{th}-successor of each point is itself. Connected subsets of an n-cycle are defined in the same way as connected subsets of the vertices of a cyclic polytope. We say that W is an (r,s)-set of an n-cycle V, if $W \subset V$, $|W| = r$ and W contains exactly s connected subsets with an odd number of elements.

Let $d=2m$, $V = \text{vert } C(d,n)$ and let W be a $(k+1,s-1)$-set or a $(k+1,s)$-set where $s \equiv k+1 \pmod 2$. We convert the set V into an n-cycle V_1 by identifying the points $x(\tau_1-\varepsilon)$ and $x(\tau_n+\varepsilon)$, $\varepsilon>0$, of the curve $x(\tau)$. In other words, we consider x_1 to be the successor of x_n in V_1. Then W becomes a $(k+1,s)$-set W_1 of the n-cycle V_1 (if W is a $(k+1,s-1)$-set of V, then, the condition $s \equiv k+1 \pmod 2$ means that the union of the two terminal subsets yields a connected subset with an odd number of elements). Let $z(n,k+1,s)$ be the number of distinct $(k+1,s)$-sets W_1 of an n-cycle V_1. If to each of the s connected sets with an odd parity we add its first successor, then the given $(k+1,s)$-set W_1 becomes a $(k+s+1,0)$-set W_2. Since $s \equiv k+1 \pmod 2$, the number $k+s+1$ is even. Let $k+s+1=2j$. Partition the set W_2 into j pairs of neighbouring points of V_1. To each set W_2 there corresponds $\binom{j}{s}$ distinct subsets W_1 which are obtained by removing the second point in each of s pairs, arbitrarily chosen from the given n pairs. Since the number of subsets W_2 is $z(n,2j,0)$, we obtain the relation

$$z(n,k+1,s) = \binom{j}{s} z(n,2j,0) , \qquad 2j=k+s+1 . \tag{3.16}$$

Let us calculate $z(n,2j,0)$. If we remove one point from each of the j pairs in W_2, we obtain a subset W_3 of an $(n-j)$-cycle V_2, with $|W_3|=j$. The number of such subsets W_3 is clearly equal to $\binom{n-j}{j}$. The correspondence between the number of distinct subsets W_2 and W_3 is obtained as follows.

Let r be the number of cyclic permutations acting on V_2 which leave the subset W_3 invariant. It is clear that the number of cyclic permutations acting on V_1 which leave W_2 invariant is also r. Thus, the cyclic permutations of V_2, applied to W_3 give $(n-j)/r$ distinct subsets V_2, and the cyclic permutations of V_1, applied to W_2, give n/r distinct $(2j,0)$-subsets V_1. Consequently

$$z(n,2j,0) = \frac{n}{n-j}\binom{n-j}{j} \tag{3.17}$$

From (3.16) and (3.17) we obtain

$$z(n,k+1,s) = \frac{n}{n-j}\binom{n-j}{j}\binom{j}{s} , \qquad 2j=k+s+1 . \tag{3.18}$$

From Lemma 3.6 we have

$$f_k(C(2m,n)) = \sum_{\substack{s=0 \\ s\equiv(k+1)\bmod 2}}^{2m-k-1} z(n,k+1,s) . \tag{3.19}$$

Substituting for $z(n,k+1,s)$ from (3.18) into (3.19) and summing with respect to j rather than s, we obtain (3.15) for the case of even d.

Now let $d=2m+1$. In contrast to the case of even d we construct a $(n+1)$-cycle V_1 by adding a fictitious point x^{n+1} between the vertices x^n and x^1, so that x^{n+1} is the successor of x^n and x^1 is the successor of x^{n+1}. For a given $(k+1,s-1)$-set or $(k+1,s)$-set (with $s \equiv k \pmod 2$) of vertices of the cyclic polytope, we define, as in the previous case, a $(k+2,s)$-set W_1 of the $(n+1)$-cycle V_1 where W_1 contains the additional point x^{n+1}. From (3.19) it follows that the number of such subsets is equal to

$$\sum_{\substack{s=0 \\ s\equiv k \bmod 2}}^{2m-k} z(n+1,k+2,s) = f_{k+1}(C(2m+2,n+1)) . \qquad (3.20)$$

For each $(k+2,s)$-set W_1 $(s\equiv k \pmod 2)$ let r be the number of cyclic permutations acting on V_1 which leave W_1 invariant. The cyclic permutations, acting on V_1 relative to W_1, give $(n+1)/r$ distinct $(k+2,s)$-subsets of the $(n+1)$-cycle V_1. The removal of one of the $k+2$ points of W_1 transforms the cycle V_1 into an n-set V, and each subset W_1 yields $(k+2)/r$ distinct $(k+1,s-1)$-subsets (or $(k+1,s)$-subsets depending on the position at which the point is removed) W of the vertices of $C(d,n)$. Thus the total number of distinct $(k+1,s)$-sets $(s \leqq 2m-k)$ is, using (3.20),

$$f_k(C(2m+1,n)) = \frac{k+1}{n+1} f_{k+1}(C(2m+2,n+1)) .$$

Substituting the value of $f_{k+1}(C(2m+2,n+1))$ already found and changing the summation index from j to $j-1$, we obtain (3.15) for odd d. //

The following important result follows from these theorems.

<u>Theorem 3.7</u> *Let* $f_k(M)$ *be the number of* k-*faces* $(1 \leqq k \leqq d-1)$ *of an arbitrary* d-*polytope* M *with* n *vertices. Then*

$$f_k(M) \leqq \begin{cases} \displaystyle\sum_{j=1}^{m} \frac{n}{n-j} \binom{n-j}{j}\binom{j}{k+1-j} & ,\text{if } d = 2m , \\[2ex] \displaystyle\sum_{j=0}^{m} \frac{k+2}{n-j} \binom{n-j}{j+1}\binom{j+1}{k+j-1} & ,\text{if } d = 2m+1 . \end{cases}$$

§4 MINIMUM NUMBER OF FACES

4.1 The Lower Bound Conjecture

Not much is known about the lower bound $\mu_k(d,n)$ of the number of k-faces of an arbitrary d-polytope with n vertices. First, using the upper bound for the number of facets of a d-polytope, it is easy to obtain the relation

$$\mu_{d-1}(d,n) \geqq \min\{r : f_{d-1}(C(d,r)) \geqq n\} .$$

Also, the following inequalities are proved in Grünbaum (1967)

$$\mu_k(d,d+s) \geqq \mu_k(d,d+r) \geqq \binom{d+1}{k+1} + \binom{d}{k+1} - \binom{2d+1-r}{k+1} ,$$

where $k \in N_{d-1}$, $r \in N_{\min\{4,d\}}$, $s>r$.

Let $\mu_k^s(d,n)$ be the lower bound of the numbers of k-faces of simplicial d-polytopes with n vertices. V.Klee suggested that

$$\mu_{d-1}^s(d,n) = (d-1)n - (d+1)(d-2) . \tag{4.1}$$

Grünbaum conjectured further that

$$\mu_k^s(d,n) = \binom{d}{n}n - \binom{d+1}{k+1}k , \quad k\in N_{d-2} . \tag{4.2}$$

The relations (4.1) and (4.2) are known as the Lower Bound Conjecture. They can be stated in dual form as follows : the f-vector of a simple d-polytope satisfies the inequalities

$$f_0 \geqq (d-1)f_{d-1} - (d+1)(d-2) \tag{4.1'}$$

$$f_{d-k} \geqq \binom{d}{k}f_{d-1} - \binom{d+1}{k+1}k , \qquad k\in N_{d-2} \tag{4.2'}$$

The Lower Bound Conjecture was proved in 1973 by Barnette (1973). In this paper there are references to earlier work in which the conjecture was proved for special cases. An outline of Barnette's proof is the following : the conjecture is first proved for the number of vertices of a simple polytope (inequality (4.1')) or, equivalently, for the number of facets of a simplicial d-polytope. Next, the lower bound is proved for the (d-2)-faces of a simple polytope and this proves the conjecture for the number of edges of a simplicial polytope. Finally, it is shown that if the conjecture is true for a d-polytope in the cases $k=1$, $k=d-1$, then it is true for the remaining k.

4.2 Lower Bound for the Number of Vertices of a Simple Polytope

We establish a lower bound for the number of vertices of a simple d-polytope with a fixed number of facets.

We begin by carrying out some auxiliary constructions. The sub-complex $\mathcal{F}'$ of the face-complex $\mathcal{F}(M)$ of a simple d-polytope M is a *connected sub-complex* if there is an enumeration $F_1,\dots,F_n$ of the facets in $\mathcal{F}'$ such that $F_i \cap F_{i+1} \neq \emptyset$ for each i, that is, it is a (d-2)-face of the polytope M. We say that the vertex x is an *outer vertex of the connected complex* $\mathcal{F}'$ if it belongs to only one of the facets of the complex $\mathcal{F}'$ (Figure 26)

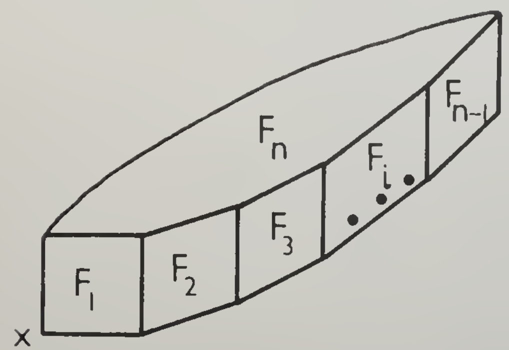


Fig. 26.

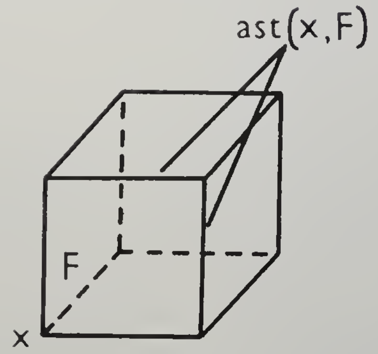


Fig. 27.

Let x be a vertex of the polytope M. Let $\mathrm{ast}(x,M)$ be the set of all k-faces ($0 \leq k \leq d-1$) which do not contain x. This is called an *anti-star of the face complex*.

Let $\mathcal{F}'$ be a connected complex and let x be a vertex belonging to some facet $F \in \mathcal{F}'$. By $\mathcal{F}'/\mathrm{ast}(x,F)$ we mean the connected complexes into which the set $\mathrm{ast}(x,F)$ topologically separates the complex $\mathcal{F}'$ (Figure 27).

<u>Lemma 4.1</u> *Let* $\mathcal{F}'$ *be a connected sub-complex of the face complex of a simple* d-*polytope* M, *and let* $\mathcal{F}'$ *have at least one outer vertex. Then, there exists an outer vertex* x *belonging to some facet* F

in $\mathcal{F}'$ *such that the set* $\mathcal{F}'/\mathrm{ast}(x,F)$ *consists precisely of two connected complexes, one of which consists precisely of the facet* F .

Proof Suppose the contrary. Suppose that for any outer vertex x of the complex $\mathcal{F}$ the set $\mathcal{F}/\mathrm{ast}(x,F)$ consists of three connected complexes $\{F\}$, $\mathcal{B}$ and $\mathcal{B}_1$. Then, for x we choose that outer vertex of $\mathcal{F}'$ for which the connected complex $\mathcal{B}$ has the maximum number of facets. If $\mathcal{B}_1$ has no outer points other than those contained by F , then the removal of these vertices makes the graph of the polytope M disconnected, which is a contradiction. If $\mathcal{B}_1$ has an outer vertex x^1 which does not belong to F , then this is also an outer vertex for the connected complex $\mathcal{F}$. Let $x^1 \varepsilon F_1$, where F_1 is a facet of $\mathcal{F}$. Since $\mathcal{B} \cup \{F\}$ is a connected complex consisting of the faces of $\mathcal{F}/\mathrm{ast}(x^1,F_1)$ we have obtained a contradiction to the definition of x . //

Let x be a vertex of a simple polytope M of dimension d and let $\mathcal{F} = \mathrm{ast}(x,M)$. Clearly, $\mathcal{F}$ is a connected subcomplex of the face complex of M . We show that $\mathcal{F}$ must have an outer vertex. Since there are d non-intersecting chains connecting x with any arbitrary vertex in $\mathcal{F}$, there are exactly d edges not in $\mathcal{F}$ each of which is determined by the intersection of exactly $d-1$ facets of M . These edges are incident to exactly d vertices of $\mathcal{F}$, each of which is an outer vertex of $\mathcal{F}$.

Because of Lemma 4.1 we may carry out the following procedure. Select an outer vertex x^1 in $\mathcal{F}$ such that the set $\mathcal{F}/\mathrm{ast}(x^1,F_1)$ consists of two connected subcomplexes $\mathcal{F}_1$ and a facet F_1 containing x^1 . The set $S_1 = \mathcal{F}_1 \cap F_1$ is called *separating* . Further, we choose an outer vertex x^2 of the connected complex $\mathcal{F}_1$ belonging to a face F_2 of $\mathcal{F}_1$ such that the set $\mathcal{F}_1/\mathrm{ast}(x^2,F_2)$ consists of two connected subcomplexes $\mathcal{F}_2$ and F_2 ; the separating set $F_2 \cap \mathcal{F}_2$ is denoted by S_2 . Continuing this process we obtain three sequences : the connected complexes $\mathcal{F}=\mathcal{F}_0$, $\mathcal{F}_1,\dots,\mathcal{F}_n$, the separating sets $S_1,\dots,S_n$ and the facets $F_1, \dots, F_n$ where $n = f_{d-1}-d-1$ (Figure 28).

Lemma 4.2 *For any two connected complexes* $\mathcal{F}_i$, $\mathcal{F}_{i+1}$ *of the sequence constructed, there exist* $d-1$ *distinct vertices* $v^i_1,\dots,v^i_{d-1}$ *with the properties :* a) v^i_k *is an outer vertex of the complex* $\mathcal{F}_i$; b) v^i_k *is not an outer vertex of the complex* $\mathcal{F}_{i-1}$.

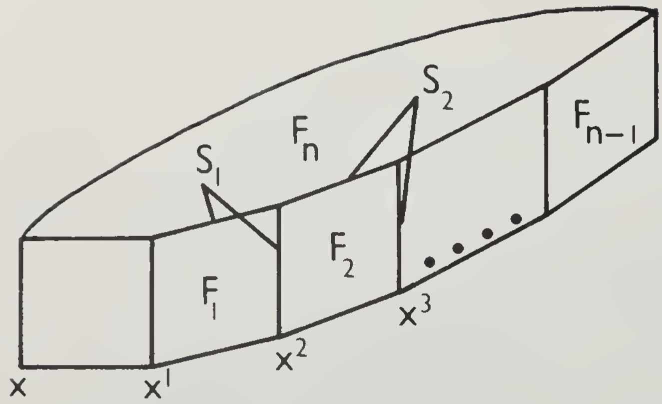


Fig. 28.

Proof Let F_i^o be a facet in $\mathcal{F}_i$ having a non-empty intersection with the facet F_i. Let x_o^i be a vertex in F_i^o not belonging to S_i. Then there are d-1 non-intersecting chains between the vertices x^i and x_o^i in the graph of the (d-1)-complex $\mathcal{F}_{i-1}$. Let v_k^i be the first vertex in the k-th chain from x^i to x_o^i which belongs to the separating set S_i. Then the preceding edge in the k-th chain is formed by the intersection of d-1 facets not belonging to $\mathcal{F}_i$ Thus v_k^i is an outer vertex of the complex $\mathcal{F}_i$. Since $v_k^i \varepsilon S_i$, it cannot be an outer vertex of the complex $\mathcal{F}_{i-1}$. //

Theorem 4.3 *Let* M *be a simple* d-*polytope, then*

$$f_0(M) \geqq (d-1)f_{d-1}(M) - (d+1)(d-2) .$$

Proof The result follows directly from Lemma 4.2 . Indeed

$$f_0(M) \geqq (d+1) + (d-1)n = (d+1) + (d-1)(f_{d-1}(M)-d-1)$$

$$= (d-1)f_{d-1}(M) - (d+1)(d-2) . \quad //$$

4.3 Lower Bound for the Number of (d-2)-faces of a Simple Polytope

Theorem 4.4 *Let* M *be a simple* d-*polytope, then*

$$f_{d-2}(M) \geqq d\, f_{d-1}(M) - d^2 - d .$$

Proof We continue to use the notation of the last section. Consider an arbitrary facet F_i in the sequence $F_1,\dots,F_n$ constructed above. Let F_i^* be a (d-1)-polytope dual to F_i and let $x_1',\dots,x_k'$ be the vertices of F_i^* which are dual to the (d-2)-faces of F_i belonging to S_i. Let G_i be the (d-2)-face of the polytope F_i^* dual to the vertex x^i. Take a point w^i strictly separated from F_i^* by the hyperplane which generates G_i such that w^i is close to the centroid of the face G_i. Then the graph consisting of the union of the graph of the polytope F_i^* and the vertex w^i connected by edges with the vertices of the face G_i, is the graph of the (d-1)-polytope $\mathrm{conv}(F_i^* \cup w^i)$. In this graph there are d-1 non-intersecting chains $\Gamma_1,\dots,\Gamma_{d-1}$ between the vertices w^i and x_1'. For each chain Γ_k, let x_0^k be the last vertex encountered in traversing the chain from w^i to x_1' before meeting the first of the vertices $x_1',\dots,x_k'$. The vertices x_0^k will be distinct and so their dual (d-2)-faces in M are also distinct. We note also that the intersection of the separating set S_i with each of these (d-2)-faces is a (d-3)-face of M. A (d-2)-face with this property is called *facial*. Thus, the polytope M has at least $(d-1)(f_{d-1}-d-1)$ facial (d-2)-faces.

The next step in the proof consists in counting the (d-2)-faces of M which belong to separating sets. We show that every separating set S_i has at least one (d-2)-face which is not facial. Note that a (d-2)-face of a separating set S_i is regular only if its intersection with some other separating set is a (d-3)-face. Consider successively all separating sets S_j, $j>i$, whose intersection with S_i contains a (d-3)-face Let the set $S_j \cap S_i$ contain a (d-3)-face F. Since M is a simple d-polytope, the face F is the intersection of precisely three (d-1)-faces F_i, F_j and some F_k. The set $F_k \cap F_i$ is not empty, so it is a (d-2)-face. Since the set S_j topologically separates the connected complex $\mathcal{F}_j$, S_j also separates S_i. Select a connected complex $\mathcal{B}$ from S_i which is separated by the separating set S_j and which contains a (d-2)-face which does not become facial when we remove F_j. Then there is some other separating set S_ℓ which intersects the complex $\mathcal{B}$ and the intersection of three (d-1)-faces F_i, F_ℓ and F_m, say, gives a (d-3)-face F' in $\mathcal{B}$. Clearly, $F_m \cap F_i$ is a (d-2)-face in S_i.

If $F_m \cap F_i$ does not belong to the complex $\mathcal{B}$ then the (d-2) -face $F_i \cap F_\ell$ of $\mathcal{B}$ intersects the (d-2)-face $F_m \cap F_i$ in a face not belonging to $\mathcal{B}$ and then F' is contained in the separating set S_j which intersects S_i .

Different separating sets have different (d-2)-faces. Thus we have at least four (d-2)-faces which contain F' : two in S_i and one each in S_ℓ and S_j . This is a contradiction.

Hence the set $F_i \cap F_m$ belongs to $\mathcal{B}$. Then S_m separates the complex $\mathcal{B}$ and at least one of its subcomplexes contains a (d-2)-face which does not become facial when F_m is removed. Repeating this process we eventually arrive at a (d-2)-face in the separating set S_i which is not facial. Thus a lower bound for the number of (d-2)-faces is $(d-1)(f_{d-1}-d-1) + f_{d-1} - d - 1 = d\, f_{d-1} - d^2 - d$. //

4.4 Lower Bound for the Number of Edges

Lemma 4.5 *If* $f_1(M) \geqq dn - k$ *for any simplicial* d-*polytope* M *with* n *vertices, where the constant* k *depends only on* d , *then*

$$f_1(M) \geqq dn - \frac{d(d+1)}{2} \quad . \tag{4.3}$$

Proof Suppose that for a simplicial polytope M with n vertices the relation (4.3) does not hold, that is

$$f_1(M) = dn - \frac{d^2+d}{2} - r \quad ,$$

where r is a positive integer. Let the facet F of M be generated by the supporting hyperplane H . Let M' be the union of the polytope M and its mirror image relative to H . The set M' is not necessarily convex, but if we first deform M by means of a suitable non-degenerate projective transformation then M' will be a simplicial d-polytope (Figure 29). Since F is a simplex, the number of edges of M' is equal to $2dn-d^2-d-2r-(d^2-d)/2 = (2n-d)d-(d^2+d)/2 - 2r$.

Since the number of vertices of the d-polytope M' is equal to 2n-d , the relation (4.3) does not hold for M' either. Similarly, if we take the mirror image of M' relative to a hyperplane H which generates some facet F , we obtain a simplicial d-polytope M" for which (4.3) does not hold and such that the right hand side differs from the required number by 4r . Continuing this process we obtain a contradiction to the assertion that the constant k depends only on d . //

Fig. 29.

The following theorem is proved by taking the dual of M in Theorem 4.4 and then applying Lemma 4.5 .

Theorem 4.6 *The lower bound conjecture is true in the case of edges of simplicial polytopes.*

4.5 Minimum Number of Faces of a Simplicial Polytope

Lemma 4.7 *If the lower bound conjecture is true for the 1-faces (edges) of a simplicial* d*-polytope, it is true for the faces of all dimensions.*

Proof We use induction on d . The lemma is certainly true for $d=3$. Suppose that (4.2) is true for simplicial $(d-1)$-polytopes with $k=1$. Then it is also true for $k=2,\dots,d-2$. Let M be a simplicial d-polytope with n vertices and let x^i be any vertex of M . Suppose that the vertex x^i is formed by the intersection of n_i facets. By the inductive assumption the number of $(k-1)$-faces in a section of M at x^i is not less than $\mu^s_{k-1}(d-1,n_i)$. Hence the vertex x^i is incident to at least $\mu^s_{k-1}(d-1,n_i)$ k-faces of M . The number of incidences of vertices and k-faces of M is equal to

$$\sum_{i=1}^{n}\mu^s_{k-1}(d-1,n_i) \geqq \sum_{i=1}^{n}\left[\binom{d-1}{k-1}n_i - \binom{d}{k}(k-1)\right]$$

$$= \binom{d-1}{k-1}\sum_{i=1}^{n} n_i - n\binom{d}{k}(k-1) \ . \tag{4.4}$$

From relation (4.2) with $k=1$ we have

$$\sum_{i=1}^{n} n_i = 2f_1 \geqq 2dn - d^2 - d \ .$$

Substituting this in equation (4.4) we obtain

$$\sum_{i=1}^{n} \mu_{k-1}^{s}(d-1,n_i) \geqq \binom{d-1}{k-1}(2dn-d^2-d) - n\binom{d}{k}(k-1)$$

$$\geqq 2k\binom{d}{k}n - k\binom{d}{k} + n\binom{d}{k} - \binom{d+1}{k+1}k(k+1)$$

$$\geqq \binom{d}{k}n(k+1) - \binom{d+1}{k+1}k(k+1) \ .$$

On the other hand, since M is a simplicial polytope the number of incidences of vertices with k-faces is equal to $(k+1)f_k$. Hence

$$f_k \geqq \binom{d}{k}n - \binom{d+1}{k+1}k \ , \qquad k = 2,\dots,n-2 \ .$$

This completes the proof. //

Theorem 4.8 *The minimum number of* k-*faces in the class of simplicial* d-*polytopes with* n *vertices is given by the formula*

$$\mu_k^s(d,n) = \begin{cases} (d-1)n - (d+1)(d-2) & \text{for } k=d-1 \\ \binom{d}{k}n - \binom{d+1}{k+1}k & \text{for } k\varepsilon N_{d-2} \end{cases}$$

Proof Combining the results of Theorems 4.3, 4.4, Lemma 4.7 and Theorem 4.6 we see that the number $\mu_k^s(d,n)$ is a lower bound for the number $f_k(M)$ in the class of simplicial d-polytopes with n vertices.

We show that these bounds are attained. To do this we exhibit a simple d-polytope with n facets for which the inequalities (4.1'),(4.2') are satisfied as equalities. Such a polytope is given by applying $n-d-1$ successive right sections of vertices. //

EXERCISES

1. (Grünbaum 1967). If the k-skeletons of the polytopes M and

M' are isomorphic for $k \geqq [d/2]$ where $d = \dim M$, then $\dim M' = d$. If the $(d-2)$-skeletons of the polytopes M and M' are isomorphic then M and M' are combinatorially equivalent.

2. A connected d-complex K in E_n $(1 \leqq d \leqq n)$ is called *simple* if all of its i-faces $(0 \leqq i \leqq d)$ are contained in precisely $d - i + 1$ distinct d-faces. Show that K is a simple d-complex if and only if K is isomorphic to the complete face complex of a simple $(d+1)$-polytope.

3. (Grünbaum 1967). The d-complex K is called *dimensionally undetermined* if there are two polytopes M' and M'' of different dimension such that K is isomorphic to both of their d-skeletons. Give examples of dimensionally undefined complexes. Show that:

(1) the i-skeleton of a d-polytope with $i \geqq [d/2]$ is not dimensionally undetermined;

(2) for all i and d $(1 \leqq i < [d/2])$ there is a d-polytope whose i-skeleton is dimensionally undetermined.

4. Let M' and M'' be two polytopes and let ψ be a bijection between $\operatorname{vert} M'$ and $\operatorname{vert} M''$ with the following property: the set $A \subset \operatorname{vert} M'$ generates a face F' of M' (that is $A = \operatorname{vert} F'$) if and only if there is a face F'' of M'' such that $\psi(A) = \operatorname{vert} F''$. Show that $M' \cong M''$.

5. Let M' and M'' be combinatorially equivalent polytopes in E_n and let $\phi(F)$ be the face of M'' which corresponds to the face F of M' under this equivalence. Show that there is an affine map α such that $\alpha(F) = \phi(F)$ for every face F of M'.

6. The d-polytope M is called projectively unique if every d-polytope M' which is combinatorially equivalent to M is projectively equivalent to M. Using Gale diagrams show that a 3-polytope M is projectively unique if and only if M has at most 9 edges.

7. (Grünbaum 1967). Using Gale diagrams show that for any d-polytope M with at most $d + 3$ vertices there is a polytope combinatorially equivalent to M all of whose vertices in E_d have rational coordinates. Construct an 8-polytope with 12 vertices such that there is no polytope combinatorially equivalent to it all of whose vertices in E_8 have rational coordinates.

8. Establish the following relations:

$$f_k(T_d^1) < f_k(T_d^2) < \ldots < f_k(T_d^k) = f_k(T_d^{k+1}) = \ldots = f_k(T_d^{[d/2]}),$$

where $k < [d/2]$, $f_{d-1}(T_d^{[d/2]}) = [(d+2)^2/4]$.

Let M be a d-polytope with $d+3$ vertices $(d=2n)$ such that $f_k(M)=f_k(C(d,d+3))$, $k=n-1,\ldots,2n-1$. Show that $M \cong C(d,d+3)$.

9. (Altschuler & McMullen 1973, McMullen 1974). Use Gale diagrams to derive the following formula for the number of simplicial n-neighbourly (2n+1)-polytopes with $2n+4$ vertices :

$$2^{[(n-1)/2]} + \frac{1}{4(n+2)} \sum \phi(h)\, 2^{(n+2)/h} ,$$

where the summation is taken over all odd h which divide $n+2$ and $\phi(h)$ is Euler's function.

Verify that the number of general (not necessarily simplicial) n-neighbourly (2n+1)-polytopes with $2n+4$ vertices is given by

$$\tfrac{1}{4}\{(5+(-1)^n)3^{[(n+1)/2]}+6\} + \frac{1}{4(n+2)} \sum \phi(h)(3^{(n+2)/h}-1) ,$$

where again the sum is taken over all odd h which divide $n+2$. In proving these formulae it should be verified that in a Gale diagram of an n-neighbourly (2n+1)-polytope with 2n+4 vertices the sum of the multiplicities of points at the ends of a diameter is not greater than two and that the points on the circle S^2 are uniformly distributed.

10. Simplify the expressions for the number of k-faces of a cyclic polytope for selected values of k, for example, show that

$$f_m(C(d,n)) = \begin{cases} \binom{n}{m+1}-\binom{n-m-2}{m+1} & \text{for } d=2m+1 , \\ \binom{n}{m+1}-\binom{n-m-2}{m} & \text{for } d=2m ; \end{cases}$$

$$f_{d-1}(C(d,n)) = \begin{cases} \binom{n-m}{m}\frac{n}{n-m} & \text{for } d=2m , \\ 2\binom{n-m-1}{m} & \text{for } d=2m+1 , \end{cases}$$

or

$$f_{d-1}(C(d,n)) = \binom{n-[(d+1)/2]}{n-d} + \binom{n-[(d+2)/2]}{n-d} .$$

11. The facets of the cyclic d-polytope $C(d,n)$ can be ordered in a sequence $F_1,F_2,\ldots,F_u$, $u=f_{d-1}(C(d,n))$ so that $F_{i-1}\cap F_i$ is a (d-2)-face of $C(d,n)$.

12. Show that $\mu_1(2,n)=n$ if $n \geq 3$.

13. Use the known characteristics of the f-vectors of 3 polytopes to show that:

$$\mu_1(3,n) = [(3n+1)/2]\ , \quad \mu_2(3,n) = [(n+3)/2]\ , \quad n \geqq 4\ .$$

14. Find a simplicial d-polytope M with n vertices such that $f_k(M) = \mu_k^s(d,n)$, where $n > d > k > 0$.

15. (Klee 1974). Construct a d-polyhedron with n facets and $n-d+1$ vertices and show that $n-d+1$ is a lower bound for the number of vertices in the class of simple polyhedra P of dimension d with n facets (vert $P \neq \emptyset$). A 'simple' polyhedron P is a polyhedron such that every vertex of P is given by the intersection of d facets. The minimum number of vertices in the class of simple d-polyhedra with n facets and v unbounded facets is equal to $(v-n-2)(d-1)+2$.

16. An *abstract polytope* is defined to be a semi-matroid $(\mathcal{F},\mathcal{V})$ with the property that to every pair of vertices V^*, V^{**} there corresponds a sequence of vertices $V_1 = V^*, V_2, \ldots, V_k = V^{**}$ such that $|V_i \cap V_{i+1}| = d-1$, $V^* \cap V^{**} \subset V_i$, $\forall i \in N_{k-1}$.

The *graph of a semi-matroid* is a graph whose vertex set is in one-to-one correspondence with the set of vertices of the semi-matroid and where two vertices V', V'' are adjacent if and only if $|V' \cap V''| = d-1$.

An *abstract face* of dimension $d-k$, $k \leq d$, of the semi-matroid $(\mathcal{F},\mathcal{V})$ is a pair $(\mathcal{F}(w),\mathcal{V}(w))$ where w is an arbitrary k-subset of $\mathcal{F}$, $\mathcal{V}(w) = \{\bar{V} : \bar{V} \subset \mathcal{F}\setminus w, \bar{V} \cup w \in \mathcal{V}\}$, $\mathcal{F}(w) = \bigcup_{\bar{V} \in \mathcal{V}(w)} \bar{V}$.

Show that a semi-matroid is an abstract polytope if and only if the graph of each of its faces is connected.

17. The maximum diameter conjecture for abstract polytopes is formulated in Altschuler & McMullen (1973); namely

$$\Delta_a(d,n) \leqq n-d\ .$$

In Adler, Dantzig & Murty (1974) and in Altschuler & McMullen (1973) it is shown that the conjecture is true for the case $\Delta_a(2,n) = [n/2]$. Theorem 2.5 of Ch.2 generalizes to the case of abstract polytopes. For arbitrary complexes the maximum diameter conjecture is false. A counter-example was constructed by Walkup (1978).

18. Give examples of abstract d-polytopes which are not realizable as simple d-polytopes. Show that all abstract d-polytopes on $d+k$ ($k \leqq 3$) symbols are realizable.

19. The number of pairwise combinatorially non-equivalent polytopes of dimension d ($d \geqq 2$) of radius 2 is given by the formula

$$[d/2] + \sum_{i=0}^{d-3} [(d-i)/2] + \sum_{n=d+4}^{2d+1} \sum_{j=0}^{2d-n+1} \gamma_{n-d-j}(d-j) ,$$

where $\gamma_{n-d-j}(d-j)$ is the number of partitions of the number $d - j$ into $n - d - j$ positive integers. This result is due to A.N.Isachenko.

4 INTEGRAL POINTS OF POLYHEDRA

Every point $x \in E_n$ all of whose coordinates are whole numbers is called an *integral point* or an *integral vector*. The set of all integral points in E_n is denoted by Z_n and is called the *integral lattice* (Cassels 1959).

Several classical problems are connected with the distribution of integral points in polyhedra. The first problem consists in finding criteria for the existence of integral solutions of systems of linear inequalities. If integral points do exist in a polyhedron, the problem arises of counting them and of finding conditions for their uniform distribution. The classical theorems of Kronecker and Minkowski yield partial solutions to this problem. The first problem is the subject matter of the Geometry of Numbers and of Mathematical Crystallography and is only partially concerned with the qualitative theory of Integer Programming (Belousov 1977). The other two problems are directly connected with Integer Programming.

The second problem includes a range of problems connected with the construction of convex hulls of integral points of polyhedra. The main problem is to develop methods of constructing systems of linear inequalities which define the convex hull of the integral points for special classes of polyhedra, for ultimately this allows us to reduce the problem of integer linear programming to ordinary linear programming. Further, the duality theorem of linear programming enables us to obtain important combinatorial and graph-theoretic theorems.

The third problem consists in characterizing systems of linear inequalities which determine polyhedra having integral points as vertices. We remark that not every combinatorial type of polytope in E_n can be specified so that all its vertices are integral points (see Exercise 7, Ch.3).

The second and third problems are, in a certain sense, dual to each other. In one of them a system of inequalities is fixed and we wish to establish the integrality of the vertices of the polytope determined by this system : in the other, integral points of a polytope are given and we wish to find, in explicit form, a set of linear inequalities which specify the polytope. Chapters 4 and 5 are concerned with these last two problems.

§1 INTEGRAL SOLUTIONS OF SYSTEMS OF LINEAR INEQUALITIES

This section is concerned with an algebraic characterization of sets which are intersections of polyhedra and the integral lattice. The set of integral points in a set W is denoted by W_Z .

1.1 The Polyhedral Semigroup

The integral lattice Z_n forms a semigroup with respect to addition.

Definition 1.1 The semigroup K_z consisting of the integral points of the polyhedral cone $K = \{x \varepsilon E_n : Ax \geq 0\}$ is called a *polyhedral semigroup* . A semigroup $\mathcal{B}$ of integral vectors is called *finitely generated* , if

$$\mathcal{B} = \{x : x = \sum_{j=1}^{t} z_j q^j , z_j \varepsilon Z^+ , j \varepsilon N_t\} .$$

Here $q^1,\dots,q^t$ are given integral vectors, called a *generating set of the semigroup* $\mathcal{B}$, which, in this case, is denoted by $\mathcal{B}(q^1,\dots,q^t)$.

Not every polyhedral semigroup is finitely generated. For example, the semigroup of integral points in E_2^+ which are located in the first quadrant between two half-lines drawn from the origin and having irrational angular coefficients is not finitely generated. Conversely, not every finitely generated semigroup is polyhedral (see Exercise 5).

The results which follow originate in Hilbert's theorem on the existence of a basis of polynomials (Hilbert 1890). These results have been repeatedly rediscovered and generalized (Presburger 1930, Fiorot 1972, Petrova 1976, Jeroslow 1978, Petrova 1978). The proofs of most of the theorems given are due to Shevchenko & Ivanov (Shevchenko 1970, Ivanov & Shevchenko 1975, Shevchenko & Ivanov 1976).

Theorem 1.1 *Let* $K = \{x \varepsilon E_n : Ax \geq 0\}$ *be a polyhedral cone, and let* A *be a matrix with rational elements. Then the polyhedral* *semigroup* K_Z *is finitely generated.*

Proof Without loss of generality we can assume that A is a matrix with integral elements. By Theorem 1.10. Ch.1 . on the representation of polyhedral cones, we have $K = \mathrm{con}(q^1,\dots,q^t)$ where, because A is integral, we may chose the vectors $q^1,\dots,q^t$, which generate the cone, to have integral components. We show that the set $\{q^1,\dots,q^t\}$, augmented by the integral points of the half-open 'parallelepiped'

$$Q = \left\{ y\varepsilon E_n : y=\sum_{i=1}^{t} \lambda_i q^i \ , \ 0\leq\lambda_i<1 \ , \ i\varepsilon N_t \right\}$$

is a generating set of the semigroup K_Z . Let x be an arbitrary element in K_Z . Since $q^1,\dots,q^t$ generate the cone K , there are $\lambda_i\geq 0$, $i\varepsilon N_t$ such that

$$x = \sum_{i=1}^{t} \lambda_i q^i \ .$$

Consider the vector

$$x' = \sum_{i=1}^{t} \{\lambda_i\} q^i = x - \sum_{i=1}^{t} [\lambda_i] q^i \ .$$

Then $x'\varepsilon Q_Z$. We have

$$x = x' + \sum_{i=1}^{t} [\lambda_i] q^i$$

which shows that any element in K_Z can be represented as a linear combination, with non-negative integral coefficients, of the vectors $q^1,\dots,q^t$ and an integral point of the 'parallelepiped' Q . It is easily seen that the set Q contains a finite number of integral points. Thus $\{q^1,\dots,q^t\} \cup Q_Z$ is a finite generating set of the semigroup K_Z . //

We extend the results of Theorem 1.1 to the case of a system of inhomogeneous linear inequalities.

<u>Theorem 1.2</u> *Let* $M = \{x\varepsilon E_n^+ : Ax\geq b\}$ *be a polyhedron, where the matrix* A *has rational elements. Then there exists a finite set of integral vectors* G *and a finitely generated semigroup* $B(p^1,\dots,p^s)$ *such that any integral point of the polyhedron can be represented in the form*

$$x = g + \sum_{j=1}^{s} z_j p^j \;, \qquad z_j \in Z^+ \;, \; j \in N \tag{1.1}$$

where $g \in G$.

Proof As usual, if $b \neq 0$, we convert the inhomogeneous system $Ax \geqq b$ into a homogeneous system :

$$(A,-b)\bar{x} \geqq 0 \;, \qquad \bar{x} \in Z_{n+1} \tag{1.2}$$

$$x_{n+1} = 1 \;. \tag{1.3}$$

The set of integral vectors $\bar{x}$ satisfying (1.2) forms a polyhedral semigroup which, by Theorem 1.1, has a finite generating set $\{\bar{q}^1,\ldots,\bar{q}^k\}$. Thus, any integral point in the cone given by $(A,-b)\bar{x} \geqq 0$ may be written in the form

$$\bar{x} = \sum_{j=1}^{k} z_j \bar{q}^j \;, \qquad z_j \in Z^+ \;, \; j \in N_k \tag{1.4}$$

and, conversely, any vector of the form (1.4) is a solution of (1.2). To find the solution of the system (1.2),(1.3) we have to add the constraint

$$x_{n+1} = \sum_{j=1}^{k} z_j q_{n+1}^j = 1 \tag{1.5}$$

where q_{n+1}^j is the (n+1)-th component of the vector $\bar{q}^j$. Since $q_{n+1}^j \geqq 0$, $j \in N_k$, it follows from (1.5) that when $q_{n+1}^j=0$ the coefficient z_j can assume arbitrary non-negative integral values, and when $q_{n+1}^j=1$ then $z_j=1$, but this is true only for one such j - for the others $z_j=0$. From the generating set $\{\bar{q}^1,\ldots,\bar{q}^k\}$ select all the vectors with components $q_{n+1}^j=0$ or 1 . Let

$$G = \{(q_1^j,\ldots,q_n^j) : q_{n+1}^j=1 \;, \; j \in N_k\} \;, \tag{1.6}$$

$$P = \{(q_1^j,\ldots,q_n^j) : q_{n+1}^j=0 \;, \; j \in N_k\} \;. \tag{1.7}$$

Then, every integral point of the polyhedron M , where $b \neq 0$ and the sets G and $P = \{p^1,\ldots,p^s\}$ are defined by (1.6) and (1.7), takes the form (1.1). Note that P is a generating set of the polyhedral semigroup

$$K_Z = \{x\varepsilon Z_n^+ : Ax\geq 0\} .$$

If $b=0$ then $G=\emptyset$ and by Theorem 1.1 we have that $x = \sum_{j=1}^{s} z_j p^j$, $z_j \varepsilon Z^+$, $j\varepsilon N_s$ for all $x\varepsilon M_Z$. //

<u>Corollary 1.3</u> *Let*

$$\mathcal{B}_g = \left\{x : x=g+\sum_{j=1}^{s} z_j p^j , z_j\varepsilon Z^+ , j\varepsilon N_s\right\} .$$

Under the assumptions made in Theorem 1.2 *we have that*

$$M_z = \bigcup_{g\varepsilon G} \mathcal{B}_g .$$

<u>Remarks</u> 1) The integral points of the polytope $M(A,b) = \{x\varepsilon E_n^+ : Ax=b\}$ also may be represented parametrically in the form (1.1). To show this, it suffices to represent the set $M(A,b)$ in the form $\{x\varepsilon E_n^+ : Ax\geq b , -Ax\geq -b\}$.

2) If the elements of the matrix A and the components of the vector b are real numbers, then Theorem 1.2 remains true except that the sets G and P may be infinite.

3) Theorem 1.2 also follows from the results of Presburger (1930) on the solvability of arithmetic systems.

1.2 <u>Convex Hulls of Integral Points of Polyhedra</u>

<u>Theorem 1.4</u> *Let* $M = \{x\varepsilon E_n^+ : Ax\geq b\}$ *be a polyhedron and let* A *be a matrix with rational elements. Then, if the set of integral points of* M *is non-empty, their convex hull is also a polyhedron.*

If M is a polytope, then since the set M_z is finite, Theorem 1.4 is true even when the elements of A are real. If M is unbounded and if some of the elements of M are irrational, then, generally speaking, the set M_Z cannot be characterized by a finite system of inequalities. For example, the set $\{(x,y)\varepsilon Z_2 : x-\sqrt{2}y\geq 0, x\geq -1\}$ is not a polyhedron.

Proof of Theorem 1.4 By Corollary 1.3 we have $M_Z = \bigcup_{g\varepsilon G} \mathcal{B}_g$. Hence

$$\text{conv } M_Z = \text{conv} \bigcup_{g\varepsilon G} \mathcal{B}_g = \text{conv} \bigcup_{g\varepsilon G} \text{conv } \mathcal{B}_g .$$

We show that the set conv $\mathcal{B}_g$ coincides with the set of points $x \varepsilon E_n$ representable in the form

$$x = g + \sum_{j=1}^{t} \lambda_j p^j \qquad \lambda_j \geqq 0 \ , \ \forall j \ \varepsilon \ N_t . \qquad (1.8)$$

Clearly, every point $x \ \varepsilon$ conv $\mathcal{B}_g$ is representable in the form (1.8). Now let x be given by (1.8). Then

$$x = t^{-1} \sum_{j=1}^{t} x^j$$

where $x^j = g + t\lambda_j p^j$, $\forall\, j\varepsilon N_t$. We show that $x^j \ \varepsilon$ conv $\mathcal{B}_g$, $\forall\, j\varepsilon N_t$. To do this consider the points

$$\bar{x}^j = g + t[\lambda_j]p^j \ , \quad \bar{\bar{x}}^j = g + t([\lambda_j]+1)p^j$$

which belong to $\mathcal{B}_g$, Clearly $x^j = (1-\{\lambda_j\})\bar{x}^j + \{\lambda_j\}\bar{\bar{x}}^j$. Thus, $x^j \ \varepsilon$ conv $\mathcal{B}_g$ which implies that $x \ \varepsilon$ conv $\mathcal{B}_g$. Hence

$$\text{conv } M_Z = \{x : x = \sum_{g\varepsilon G} \mu_g g + \sum_{j=1}^{t} \lambda_j p^j , \sum_{g\varepsilon G} \mu_g = 1, \mu_g \geqq 0, \lambda_j \geqq 0, \forall i\varepsilon N_t\}$$

and, by Theorem 3.1, Ch.1, this means that conv M_Z is a polyhedron. //

We remark that even though the proof of Theorem 1.4 is constructive, there are as yet no efficient algorithms for finding a system of inequalities generating conv M_Z.

1.3 Solvability of Linear Diophantine Equations

We consider one of the methods of solving systems of linear equations in whole numbers, based on the reduction of a matrix A to normal diagonal form (Smith 1861).

Definition 1.2 An integral matrix $D = (d_{ij})_{m\times n}$ is called *normal diagonal* if for some $r \leq \min(m,n)$ the diagonal elements d_{ii} are positive integers ($\forall i\varepsilon N_r$) and all other elements $d_{ij}=0$, where also

$$d_{i+1,i+1} \equiv 0 \pmod{d_{ii}} \qquad \forall\, i \ \varepsilon \ N_{r-1} . \qquad (1.9)$$

that is, every d_{jj} divides d_{ii} when $j<i$.

Theorem 1.5 *Given any integral matrix* A , *there exist unimodular matrices* U *and* V *such that the matrix* $D = UAV$ *is normal diagonal. Moreover, the matrix* D *is unique.*

Proof To prove the existence of D we describe a constructive procedure using a sequence of three types of transformations (called *elementary row operations*) : a) interchange of rows, b) adding an integral multiple of one row to another, c) multiplication of a row by -1 . Similar operations on columns are called *elementary column operations* .

Every elementary row (column) operation can be carried out by multiplying the matrix on the left (right) by the corresponding *elementary matrix* U . U is obtained from the identity matrix I by applying to I the same operation. Note that any elementary matrix has determinant equal to ± 1 , that is, such a matrix is *unimodular* .

The method of constructing the normal diagonal form of A is carried out in two stages.

In the first stage we diagonalize A in r steps where $r = \text{rank } A$. The first step is to define two unimodular matrices U_1 and V_1 such that

$$U_1 A V_1 = \begin{pmatrix} b_{11} & 0 & \dots & 0 \\ 0 & \cdot & \dots & \cdot \\ \vdots & \cdot & \dots & \cdot \\ 0 & \cdot & \dots & \cdot \end{pmatrix} \tag{1.10}$$

By using row and column operations of type a) we make a_{11} equal to that element in the first row or first column of A which has least absolute magnitude. We then subtract $\lambda_j = [a_{1j}/a_{11}]$ times the first column from the j^{th} column $(j \neq 1)$ and subtract $\mu_i = [a_{i1}/a_{11}]$ times the first row from the i^{th} row $(i \neq 1)$. This yields a matrix in which all elements in the first row or the first column (other than a_{11}) are either zero or have absolute magnitude less than $|a_{11}|$. This procedure is continued until we obtain a matrix of type (1.10). Let the matrices $P_1, \dots, P_s$ and $Q_1, \dots, Q_p$ correspond to the successive row and column operations respectively. Let $U_1 = P_s \dots P_1$ and $V_1 = Q_1 \dots Q_p$. Then the matrix $U_1 A V_1$ has the form (1.10). The remaining steps are analagous. Note that at each stage of the diagonalization of A we are actually finding the highest common factor of the elements in the corresponding row and column. Let δ be the highest common factor of the elements in the first row of

A $(\delta = \text{h.c.f.}(a_{11},\ldots,a_{1n}))$. Let $\alpha_{1j} = a_{1j}/\delta$, $j \in N_n$. Then there exist coprime integers γ_j such that $\sum_{j=1}^{n} \alpha_{1j}\gamma_j = 1$ and there is a unimodular matrix V_1 whose first column consists of the numbers $\gamma_1,\ldots,\gamma_n$. Thus, the matrix AV_1 has a first row of the form $(\delta, \delta\beta_{12},\ldots,\delta\beta_{1n})$. Now putting

$$V_2 = \begin{bmatrix} 1 & -\beta_{12} & \cdots & -\beta_{1,n-1} & -\beta_{1n} \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

we find that the matrix AV_1V_2 will have a first row of the form $(\delta,0,\ldots,0)$. Repeating these operations on the rows of the matrix so obtained, we construct a matrix of type (1.10).

In the second stage we normalize the diagonal matrix

$$B = \begin{bmatrix} b_{11} & & & & 0 \\ & b_{22} & & & 0 \\ & & \ddots & & \\ & & & b_{rr} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

obtained in the first stage, that is, we transform to a matrix satisfying condition (1.9). If some number b_{jj} does not divide b_{ii} , $i>j$, so that $b_{ii} = \lambda b_{jj} + q$, $0<q<b_{jj}$, then we carry out a sequence of elementary operations on the rows and columns numbered i and j , which for the corresponding (2x2)-submatrix $B^{\{i,j\}}_{\{i,j\}}$ takes the following form :

$$\begin{pmatrix} b_{jj} & 0 \\ 0 & \lambda b_{jj}+q \end{pmatrix} \to \begin{pmatrix} b_{jj} & 0 \\ \lambda b_{jj} & \lambda b_{jj}+q \end{pmatrix} \to \begin{pmatrix} b_{jj} & -b_{jj} \\ \lambda b_{jj} & q \end{pmatrix} \to \begin{pmatrix} q & \lambda b_{jj} \\ -b_{jj} & b_{jj} \end{pmatrix} \qquad (1.11)$$

The matrix so obtained can be diagonalized as in the first stage. As a result we will either have a diagonal matrix B' with b'_{jj} dividing b'_{ii} , or the element b'_{jj} will be less than q and the sequence of operations (1.11) can be repeated. This procedure is repeated until condition (1.9) is satisfied.

Thus, by means of a finite sequence of elementary operations we obtain a normal diagonal matrix D and unimodular matrices U and V

which correspond to these operations.

We will establish some invariants of the elementary operations which will ensure the uniqueness of the normal diagonal form of the matrix A .

Let $\Delta_\nu(A)$ denote the highest common factor of all the ν^{th} order minors of the matrix A .

<u>Lemma 1.6</u> *Let* $D = UAV$ *where* U , V *are unimodular matrices. Then* $\Delta_\nu(A) = \Delta_\nu(D)$, $\forall \nu \varepsilon N_r$, $r = \text{rank } A$.

Proof It suffices to examine the case $D = UA$. Let D_I^J be a square sub-matrix, with $|I| = |J| = \nu$. By the Cauchy-Binet formula

$$\det D_I^J = \sum \det U_I^{I'} \det A_{I'}^J,$$

where the sum is taken over all subsets $I' \subset N_m$ with the property $|I'| = \nu$. Hence $\Delta_\nu(A)$ divides $\det D_I^J$ for all subsets I and J of order ν . It follows that $\Delta_\nu(A)$ divides $\Delta_\nu(D)$. Further, since $A = U^{-1}D$, we see that $\Delta_\nu(D)$ divides $\Delta_\nu(A)$. Hence $\Delta_\nu(A) = \Delta_\nu(D)$. //

The uniqueness of the normal diagonal form D of the matrix A follows from the fact that Lemma 1.6 implies that its elements are uniquely expressible in terms of the common factors $\Delta_\nu(A)$ according to the formulae :

$$d_{11} = \Delta_1(A) \quad , \quad d_{ii} = \Delta_i(A)/\Delta_{i-1}(A) \quad , \qquad i=2,\ldots,r.$$

This completes the proof of Theorem 1.5 . //

Using the normal diagonal form of the matrix A we obtain the general form of the integral solutions of the system

$$Ax = b \quad , \quad A \varepsilon Z_{m,n} \quad , \quad b \varepsilon Z_m . \tag{1.12}$$

Let us multiply the system (1.12) on the left by the unimodular matrix U and then make a change of variables $x = Vy$: this gives the system

$$Dy = Ub . \tag{1.13}$$

Since V is a unimodular matrix, the affine map $x = Vy$ is a bijection between the integral solutions of (1.13) and the integral solutions of (1.12). Thus a necessary and sufficient condition for the integral solvability of (1.12) is that d_{ii} should divide the i^{th} component $(Ub)_i$ of the vector Ub. Using Lemma 1.6 and Theorem 1.5 we have the following criterion.

Theorem 1.7 *The system of linear equations*

$$Ax = b \quad , \quad A \in Z_{m,n} \quad , \quad b \in Z_m \tag{1.12}$$

has an integral solution if and only if

$$\Delta_\nu(A) = \Delta_\nu((A,b)) \quad , \qquad \forall\, \nu \in N_r$$

where $r = \text{rank } A$.

Let $R = N_r$. Then the system (1.13) defines a vector $y_R^o = (y_1^o,\ldots,y_r^o)$, where $y_i^o = (Ub)_i/d_{ii}$, $i \in N_r$. Then the general form of the integral solutions of (1.12) is

$$x = V^R y_R^o + V^{\bar{R}} y_{\bar{R}} = x^o + V^{\bar{R}} y_{\bar{R}}$$

where $x^o = V^R y_R^o$ is a particular integral solution of (1.12) and $V^{\bar{R}} y_{\bar{R}}$ is the general solution of the homogeneous system $Ax=0$ which depends on $n-r$ integer parameters $y_{\bar{R}} = (y_{r+1},\ldots,y_n)$. Thus, the set of integral points of the polyhedron $M(A,b)$ is in one-to-one correspondence $x = Vy$ with the set of integral points of the polyhedron given by the following system of inequalities :

$$V^{\bar{R}} y_{\bar{R}} \geqq -x^o \ .$$

We remark that efficient (polynomial) algorithms have recently been proposed for finding the general integral solutions of a system of equations with integer coefficients ; see, for example, Votyakov & Frumkin (1976).

1.4 Aggregates

The problem of the existence of and the search for a vector $t \varepsilon E_m$ such that the simplex $T(t,A,b) = \{x \varepsilon E_n^+ : tAx = tb\}$ has the same integral points as the polyhedron $M(A,b)$ is called the *Aggregate Problem*. In other words, we wish to find a linear combination of the integral system (1.12), called the *aggregated equation*, which will have the same solution set in the non-negative integers as the original system. The first results on the aggregate problem were obtained by Mathews (1897). The interest in the problem grew in connection with the reduction of the integer linear programming problem to the knapsack problem (Ivanov 1975, Padberg 1979). The aggregate problem was fully solved in Shevchenko (1976) and Veselov & Shevchenko (1978).

Theorem 1.8 *Let* con A *be a pointed cone. Then there is a vector* $t \varepsilon Z_m$ *such that*

$$M_Z(A,b) = T_Z(t,A,b) .$$

Note that for all $t \varepsilon E_m$ we have the inclusion $M_Z(A,b) \subseteq T_Z(t,A,b)$. The reverse inequality is proved using the following result.

Lemma 1.9 *The equality* $M_Z(A,b) = T_Z(t,A,b)$ *holds if and only if the hyperplane* $H = \{u \varepsilon E_m : tu = tb\}$ *does not contain any points* $u \varepsilon \mathcal{B}(A)$, *except, possibly, for* b .

Proof of Lemma (i) Let there exist a point u^o , $u^o \neq b$, such that $u^o \varepsilon H$ and $u^o \varepsilon \mathcal{B}(A)$. Since $u^o \varepsilon \mathcal{B}(A)$ there is an $x \varepsilon M_Z(A,u^o)$ where $x \notin M_Z(A,b)$ since $u^o \neq b$. On the other hand, $x \varepsilon T_Z(t,A,b)$ since $u^o \varepsilon H$ and $M_Z(A,u^o) \subseteq T_Z(t,A,u^o)$.

(ii) If there is a vector y such that $y \notin M_Z(A,b)$ but $y \varepsilon T_Z(t,A,b)$ then construct the vector $v=Ay$. Then $v \neq b, v \varepsilon \mathcal{B}(A)$ and $tb = (tA)y = t(Ay) = tv$. This proves the Lemma. //

In place of a proof of the Theorem we will examine some possible methods of finding an aggregate. According to Lemma 1.9 , to do this it is necessary to find an equation $tu = tb$ which has the unique solution $u = b$ on $\mathcal{B}(A)$. It is difficult to construct such an equation, so instead one usually finds equations with a unique integral solution on some set $\Omega \supset \mathcal{B}(A)$. The set Ω is usually taken to be the lattice Z_m^+ .

In practice all methods of aggregation are based on the following two principles which we illustrate for the case of aggregating two equations.

First Principle Let h.c.f.$(t_1,t_2) = 1$ and let t_1 not divide $y_2(x) = \sum_{j=1}^{n} a_{2j}x_j - b_2$ for any $x \varepsilon Z_n^+$, and let t_2 not divide $y_1(x) = \sum_{j=1}^{n} a_{1j}x_j - b_1$ for any $x \varepsilon Z_n^+$. Then the equation

$$\sum_{j=1}^{n} (t_1 a_{1j} + t_2 a_{2j})x_j = t_1 b_1 + t_2 b_2 \quad . \tag{1.14}$$

is equivalent on Z_n^+ to the system :

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \quad , \qquad i=1,2. \tag{1.15}$$

Second Principle Let h.c.f.$(t_1,t_2) = 1$ and let $t_1 > \sup\{y_2(x) : x \varepsilon Z_n^+\}$, $t_2 > \sup\{y_1(x) : x \varepsilon Z_n^+\}$. Then (1.14) and (1.15) are equivalent on Z_n^+ .

From Theorem 1.8 we can obtain the following discrete analogue of the Farkas-Minkowski Lemma.

Corollary 1.10 *Let* con A *be a pointed cone. Then* $M_Z(A,b) \neq \emptyset$ *if and only if* $tb \, \varepsilon \, T_Z(t,A,b) \ \forall t$.

Clearly, the boundedness of the set $M(A,b)$ is necessary in order that an aggregate should be possible. Indeed, if $M_Z(A,b) \neq \emptyset$, rank $A \geq 2$ and if the cone con A is not pointed, then, as is shown by Shevchenko (1976), there is no vector $t \varepsilon Z_m$ such that $M_Z(A,b)=T_Z(t,A,b)$.

Additional information about the range of values of the functions $y_i(x)$ will naturally enable us to reduce the values of the coefficients t_1,t_2 . Nonetheless, all known aggregation methods lead to a rapid increase in the values of the m coefficients in the aggregation equation with m . The following theorem accounts for this phenomenon.

Theorem 1.11 *Given any integers* $m,d \varepsilon Z^+$, $m \geq 2$, *there is a system of type* (1.12) *such that any aggregation equation has* m *coefficients, each of which is greater than or equal to* $(d+1)^{m-1}$.

Proof Let the system (1.12) have the property that there exist m linearly independent columns, say, $A^1,\dots,A^m$, such that

$$b = \sum_{j=1}^{m} \lambda_j A^j \quad , \quad \lambda_j \geqq d \ , \ \lambda_j \varepsilon Z^+ \ \forall j \varepsilon N_m \ .$$

We show that in this case the coefficients $\alpha_j = \sum_{i=1}^{m} t_i a_{ij}$ in the aggregate equation satisfy the inequalities

$$\alpha_j \geqq \prod_{i \neq j} (\lambda_i + 1) \geqq (d+1)^{m-1} \ , \qquad \forall j \varepsilon N_m \ . \quad (1.16)$$

Since the system $\sum_{j=1}^{m} A^j x_j = b$ has the unique solution $p = (\lambda_1,\dots,\lambda_m)$, the equation

$$\sum_{j=1}^{m} \alpha_j x_j = \alpha_0 = \sum_{i=1}^{m} t_i b_i \ . \quad (1.17)$$

must also have the unique integral solution $(\lambda_1,\dots,\lambda_m)$. We show that this is possible only in the case when the inequalities (1.16) hold. Suppose, on the contrary, that for some k, $\alpha_k < \prod_{i \neq k}(\lambda_i+1)$. Then we can prove that equation (1.17) has an integral solution, distinct from p. For every integral point of the parallelepiped $H = \{(y_1,\dots,y_{k-1},y_{k+1},\dots,y_n) : 0 \leqq y_i \leqq \lambda_i\}$ in E_{n-1}, we find integers $h(y)$ and $r(y)$ such that

$$\sum_{i \neq k} \alpha_i y_i = h(y)\alpha_k + r(y) \ , \quad 0 \leqq r(y) \leqq \alpha_k - 1 \ .$$

Since $|H_Z| = \prod_{i \neq k} (\lambda_i + 1)$ and since the function $r(y)$ can take no more than α_k distinct values, there must exist $y' \varepsilon H_Z$, $y'' \varepsilon H_Z$ such that $y' \neq y''$ but $r(y') = r(y'')$. Hence

$$\sum_{i \neq k} \alpha_i y_i' = h(y')\alpha_k + \sum_{i \neq k} \alpha_i y_i'' - h(y'')\alpha_k \ .$$

Suppose, for definiteness, that $h(y') \geqq h(y'')$. It is easy to see that $q = (\lambda_1 - y_1' + y_1'',\dots,\lambda_{k-1} - y_{k-1}' + y_{k-1}'', \lambda_k + h(y') - h(y''), \lambda_{k+1} - y_{k+1}' + y_{k+1}'',\dots,\lambda_m - y_m' + y_m'')$ is a solution of equation (1.17) and that $q \neq p$. This contradiction proves the theorem. //

§2 CONDITIONS FOR A POLYHEDRON TO BE INTEGRAL

In this section we investigate the problem of recognizing, from its algebraic formulation, the property that the coordinates of all vertices of a polyhedron are integers. Polyhedra, all of whose vertices have integral coordinates, are called *integral*. The problem of characterizing systems of linear inequalities which determine integral polyhedra is, as yet, unsolved. A simpler problem is to describe classes of constraint matrices for which the corresponding polyhedra are integral for any right hand side. We give such conditions for integral vertices of polyhedra in terms of unimodular matrices.

2.1 The Dantzig-Veinott Criterion

The first integral polyhedron criterion was obtained by Hoffman & Kruskal (1958). To facilitate the proof we begin the exposition with a criterion proposed by Veinott & Dantzig (1968).

Theorem 2.1 *Let* $A \in Z_{m,n}$. *The polyhedron*

$$M(A,b) = \{x \in E_n : Ax=b,\ x \geqq 0\}$$

is integral for all vectors $b \in Z_m$ *if and only if the matrix* A *is unimodular.*

Proof (i) Sufficiency. Each vertex (basic feasible solution) $x = (x_1,\dots,x_n)$ of the polyhedron $M(A,b)$ is uniquely determined by specifying the indices $j_1,\dots,j_m$ of the basic variables (without loss of generality we can assume that rank $A = m$). Let B be a feasible basis containing columns with numbers $j_1,\dots,j_m$. Then the components $x_B = (x_{j_1},\dots,x_{j_m})$ of the basic feasible solution x are defined by $Bx_B = b$. By assumption $\det B = \pm 1$ and b is an integral vector. Then Cramer's rule implies that x_B is an integral vector. Since the remaining components of x are zero, x is an integral vector.

(ii) Necessity. It is required to prove that if B is a basis and x_B is an integral vector for any $b \in Z_m$, then $\det B = \pm 1$. Let the vector $y \in Z_m$ have the property

$$y + B^{-1}e_i \geqq 0, \qquad i \in N_m. \tag{2.1}$$

Consider the system

$$Az = b^o \quad , \quad b^o = By + e_i \quad . \tag{2.2}$$

Since B is a basis of A, the system (2.2) is consistent. The basic solution of (2.2) with non-zero components given by $z_B = B^{-1}(By+e_i) = y + B^{-1}e_i$ is non-negative by (2.1) and is therefore a vertex of the polyhedron $M(A,b^o)$. By assumption, the polyhedron $M(A,b)$ is integral for any b, including $b=b^o$. Thus z_B is an integral vector. Since the left hand side of the equation $z_B-y = B^{-1}e_i$ is an integral vector, so is $B^{-1}e_i$ which is the i^{th} column of B^{-1}. Thus B^{-1} is an integral matrix. Then, since the determinants B and B^{-1} are integers and since $\det B \cdot \det B^{-1} = 1$, we have $\det B = \pm 1$. //

Definition 2.2 A matrix is called *absolutely unimodular* if all of its non-zero minors are equal to 1 or -1.

Theorem 2.2 *Let* $A \in Z_{m,n}$. *The polytope*

$$M(A,b^1,b^2,d^1,d^2) = \{x \in E_n \; ; \; b^1 \leq Ax \leq b^2 \; , \; d^1 \leq x \leq d^2\}$$

is integral for any vectors $b^1,b^2 \in Z_m$, $d^1,d^2 \in Z_n$ *if and only if the matrix* A *is absolutely unimodular.*

Theorem 2.2 follows from Theorem 2.1 if we transform from the normal specification of the polytope $M(A,b^1,b^2,d^1,d^2)$ to the canonical specification by introducing artificial variables and using the obvious assertion that : a matrix A is absolutely unimodular if and only if the matrix (A,I_m) is unimodular, where I_m is the $(m \times m)$-identity matrix.

2.2 α-Modular Matrices

Definition 2.4 A matrix A of rank m is called *α-modular* if all of its m^{th}-order non-zero minors are equal to $\pm\alpha$, (where α is positive).

The following result is contained in Kowaljow, Nguen Ngia & Kühn (1977) and Kovalev, Isachenko & Nguen Ngia (1978).

Theorem 2.3 *Let* A *be a α-modular matrix. Then the polyhedron* $M(A,b)$ *is integral if and only if* $M(A,b)$ *has at least one integral vertex.*

The necessity of the condition is obvious. Before demonstrating its sufficiency we list some properties of α-modular matrices.

<u>Lemma 2.4</u> *The following statements are equivalent :*

(1) A *is an α-modular matrix;*

(2) *Let* B *be any basis of the matrix* A . *Then* $B^{-1}A$ *is a unimodular integral matrix.*

(3) $B^{-1}H$ *is an absolutely unimodular matrix for any basis* B *of the matrix* A *,where* H *is the submatrix formed by the columns of* A *which are not included in the basis* B .

(4) *The matrix* A *may be represented as the product of a non-singular matrix* D $(\det D = \alpha)$ *and a unimodular matrix* V .

Proof of the Lemma (2)⇔(3). The matrix $B^{-1}A$ is the matrix of coefficients of the expansions of the columns of A relative to the basis B . After suitable interchanges of columns, it takes the form $(B^{-1}H, I_m)$ where I_m is the (mxm)-identity matrix. Hence (2) and (3) are equivalent.

(1)⇒(2). Note first that if B is a basis of an α-modular matrix, then every element of the matrix $B^{-1}A$ is either 0 or ± 1 . Indeed, let $B = (A^{j_1},\dots,A^{j_m})$ and let A^j be a column not in the basis. Then solving the system of equations

$$A^j = \sum_{i=1}^{m} \lambda_i A^{j_i} ,$$

we obtain $\lambda_i = -1, 0$ or 1 . The unimodularity of an arbitrary basis $B^{-1}(A^{j_1},\dots,A^{j_m})$ of the matrix $B^{-1}A$ follows from the relations $\det(B^{-1}A^{k_1},\dots,B^{-1}A^{k_m}) = \det B^{-1} . \det(A^{k_1},\dots,A^{k_m}) = \pm 1$, which depend upon the definition of an α-modular matrix and the theorem on the determinant of a product of matrices.

(2)⇒(4). If $B^{-1}A$ is a unimodular matrix, then putting $D=B$, we have $A = DB^{-1}A = DV$, where $V = B^{-1}A$.

(4)⇒(1). Any basis B of a matrix A representable in the form $A=DV$, takes the form $(DV^{i_1},\dots,DV^{i_m})$. By the unimodularity of V we have $\det B = \det D . \det(V^{i_1},\dots,V^{i_m}) = \det D$, that is, A is an α-modular matrix. //

Thus to establish the α-modularity of a matrix A it suffices to establish the unimodularity of $B^{-1}A$ or the absolute unimodularity of

$B^{-1}H$ for any basis B of A. As the proof of Lemma 2.4 shows, if there is a basis B such that the matrix of components of the columns of A expanded relative to the basis B is unimodular, then for any basis U of A, $U^{-1}A$ is unimodular.

Proof of sufficiency in Theorem 2.3. Let B be a basis such that $(B^{-1}b,0)$ is an integral vertex of the polyhedron $M(A,b)$. Consider the system $Ax=b$ and the equivalent system

$$B^{-1}Ax = B^{-1}b . \tag{2.3}$$

By Lemma 2.4 $B^{-1}A$ is an integral unimodular matrix and by the conditions of the theorem $B^{-1}b$ is an integral vector. Further, by Theorem 2.1 all basic solutions of (2.3) are integral. It follows that all basic solutions of $Ax=b$ are integral, that is the polyhedron $M(A,b)$ is integral. //

Corollary 2.5 *Let* A *be an* α*-modular matrix and let* B *be an arbitrary basis of* A. *Then, the polyhedron* $M(A,b)$ *is integral for all vectors* b *such that* $B^{-1}b$ *is an integral vector.*

For example, if the augmented matrix (A,b) is α-modular, then $B^{-1}b$ is an integral vector and so $M(A,b)$ is an integral polyhedron.

2.3 (±1)-matrices

A matrix A all of whose elements are equal to ± 1 is called a (± 1)-*matrix* and denoted by $A_{\pm 1}$. We consider for what vectors b the polyhedron $M(A_{\pm 1},b)$ is integral.

We say that the components of the vector $b = (b_1,\ldots,b_m)$ have the *same parity* if they are all either odd or even, that is, $b_1 \equiv b_2 \equiv \ldots \equiv b_m \pmod 2$.

Theorem 2.6 i). *If* $M(A_{\pm 1},b)$ *is an integral polyhedron then the components of* b *have the same parity.* ii). *If the components of* b *have the same parity and if* $A_{\pm 1}$ *is a* 2^{m-1}*-modular* $(m\times n)$*-matrix, then* $M(A_{\pm 1},b)$ *is an integral polyhedron.*

The proof depends on the following Lemma, which follows from the matrix diagonalization procedure described in §1.

Lemma 2.7 *Let* $B_{\pm 1}$ *be a non-singular* (mxm)*-matrix whose elements are equal to* ± 1 . *Then, there is a unimodular* (mxm)*-matrix* V *such that the matrix* $H = B_{\pm 1}V$ *(the Hermitian form of* $B_{\pm 1}$*) takes the form :*

$$h_{i1} = \pm 1 \ , \ \forall i \varepsilon N_m \ ,$$
$$h_{ij} \equiv 0 \pmod 2 \ , \qquad \forall (i,j) \ \varepsilon \ N_m \times N_m \ , \quad i \geqq j \ , \ j \neq 1$$
$$h_{ij} = 0 \ , \qquad \forall (i,j) \ \varepsilon \ N_m \times N_m \ , \quad i < j \ .$$

Proof of Theorem 2.6. Since the matrix V is unimodular, for any basis B the system $Bx_B = b$ has an integral solution if and only if the system

$$Hy = b \tag{2.4}$$

has an integral solution. These solutions are connected by the rule $x_B = Vy$, $y = V^{-1}x_B$. The structure of the matrix H , given by Lemma 2.7 , demands the equal parity of the components of b in order that (2.4) should be solvable in integers.

If A is a 2^{m-1}-modular matrix then $h_{ii} = \pm 2$ for all $i > 2$. Thus in this case the condition that the vector components are of the same parity is sufficient for the solvability of (2.4).

Corollary 2.8 *If all the elements of the nonsingular matrix* B *of order* $m \geqq 2$ *are equal to* ± 1 . *then* $\det B \equiv 0 \pmod{2^{m-1}}$.

§3 ABSOLUTELY UNIMODUALAR MATRICES

Clearly, an absolutely unimodular matrix can only have components equal to 0 , +1 or -1 . The class of all such (m×n)-matrices is denoted by $C_{m,n}$.

3.1 Criteria for Absolute Unimodularity

Definition 3.1 A matrix is called an *Eulerian matrix* if the sum of the elements in each of its rows and each of its columns is even.

Theorem 3.1 *The following statements are equivalent :*

1) A *is an absolutely unimodular matrix.*

2) *Given a vector* x *with components* 0 , ± 1 , *there is a vector* y *with components* 0 , ± 1 *such that*

$$y \equiv x \pmod 2 , \tag{3.1}$$

$$A_i y = \begin{cases} 0 , & \text{if} \quad A_i x \equiv 0 \pmod 2 \\ \pm 1 , & \text{if} \quad A_i x \equiv 1 \pmod 2 \end{cases} \tag{3.2}$$

for all rows A_i *of* A.

3) *Every square, eulerian sub-matrix of* A *is singular.*

4) *Every minor of* A *is either zero or is an odd number.*

5) *For any non-singular sub-matrix* A_I^J *of* A *the following condition holds :* $\text{h.c.f.}\{\sum_{j\in J} \lambda_j a_{ij} : i \in I\} = 1$ *for all* $\lambda_j \in \{0, \pm 1\}$ *not all zero.*

Proof We follow the scheme : (1)⇒(2)⇒(3)⇒(1) , (1)⇒(4)⇒(3) , (1)⇒(5)⇒(3). The criteria (2)-(5) for absolute unimodularity were proposed respectively by Padberg (1976) , Camion (1965) , Gomory (1969) and Chandrasekaran (1969).

(1)⇒(2). Let A be an absolutely unimodular matrix. Then, if x is an arbitrary vector with components $0, \pm 1$, define

$$d_i^\nu = \begin{cases} 0 , & \text{if} \quad x_i \equiv 0 \pmod 2 , \\ (x_i - 1)/2 , & \text{if} \quad x_i \equiv 1 \pmod 2 , \ \nu = 1 , \\ (x_i + 1)/2 , & \text{if} \quad x_i \equiv 1 \pmod 2 , \ \nu = 2 , \end{cases}$$

$$b_i^\nu = \begin{cases} a_i/2 , & \text{if} \quad a_i \equiv 0 \pmod 2 , \\ (a_i - 1)/2 , & \text{if} \quad a_i \equiv 1 \pmod 2 , \ \nu = 1 , \\ (a_i + 1)/2 , & \text{if} \quad a_i \equiv 1 \pmod 2 , \ \nu = 2 , \end{cases}$$

where $a_i = \sum_{j=1}^{n} a_{ij} x_j$. By Theorem 2.2 , the non-empty polyhedron $M = M(A, b^1, b^2, d^1, d^2)$ is integral. Hence, there is an integral vector $x' \in M$. Then, the vector $y = x - 2x'$ satisfies conditions (3.1),(3.2).

(2)⇒(3). Let A_I^J be an eulerian sub-matrix of A . Let x be a vector with components $x_j = 1$, $j \in J$ and $x_j = 0$, $j \notin J$. Then there exists a vector $y \neq 0$ such that $A_i y = 0$ for $i \in I$ and this means that the rows of the matrix A_I^J are linearly dependent , so that $\det A_I^J = 0$.

(3)⇒(1). Suppose that the matrix A is not absolutely unimodular. Further, let B be the minimal $(s\times s)$-submatrix of A such that $\det B \neq 0, \pm 1$. Consider the matrix $\overline{B} = B^{-1}\det B$. Take any of its columns, say the k^{th}, $\overline{B}^k$. Then $B\overline{B}^k = \det B\, e_k$. From the rule for calculating inverse matrices and from the assumption that all proper submatrices of B are absolutely unimodular, it follows that $\overline{B} \in C_{s,s}$. Permute the rows so that the vector $\overline{B}^k$ takes the form $(d,0)$, where all components of d are ± 1. For the submatrix D of B consisting of the columns of B corresponding to the vector d we have $Dd = B\overline{B}^k = e_k.\det B$.

Since $\det B \neq 0$ there is a non-singular submatrix D' of D such that $D'd = e_k' \det B$, where e_k' are the components of e_k corresponding to the submatrix D'. Since the components of d equal ± 1, then, if we replace one of the columns of D' by $e_k' \det B$ and expand the determinant of the matrix D'' so obtained by this column, we have $0 \neq \det D' = \pm\det D'' = \pm\det B$. The final equality follows from the absolute unimodularity of any proper submatrix of B.

By the minimality property of B we deduce that $D' = B$, $d = \overline{B}^k$. Hence, since k is arbitrary, the components of every column of $\overline{B}$ are ± 1. Thus $\overline{B}$ is a ± 1-matrix. Then, by Corollary 2.8, $\det \overline{B} \equiv 0 \pmod{2^{s-1}}$. On the other hand, $\det \overline{B} = (\det B)^s \det B^{-1} = (\det B)^{s-1}$. So, $\det B \equiv 0 \pmod 2$

At the same time $B\overline{B}^k = e_k \det B$, from which it follows that the sum of the elements in the k^{th} row, and hence in any row, is even. It can be shown similarly that the sum of the elements in any column of B is even. Thus, B is an eulerian matrix. But this contradicts the supposition that $\det B \neq 0$.

(1)$\Rightarrow$(4). Obvious.

(4)$\Rightarrow$(3). Proof by contradiction. Let an eulerian submatrix B of A be non-singular. Then, adding together all the rows of B we obtain a nonzero row all of whose elements are even numbers. Thus $\det B$ is even. This contradicts (3). It follows that (4)$\Rightarrow$(3).

(1)$\Rightarrow$(5). Suppose that for a non-singular submatrix A_I^J and for some $\lambda_j \in \{0,+1,-1\}$ the statement in (5) is not satisfied. In A_I^J replace a column s, for which $\lambda_s \neq 0$, by a column with elements $\sum_{j\in J} \lambda_j a_{ij}$. In so doing we obtain a matrix B such that

$$0 \neq \det A_I^J = \pm\det B = \det B'(\text{h.c.f.}\{\sum_{j\in J} \lambda_j a_{ij} : i\in I\}),$$

where B' is an integral matrix. This contradicts the assumption that $\det A_I^J = \pm 1$.

(5)⇒(3). Suppose there is a non-singular eulerian submatrix A_I^J of A. The statement in (5) is true for all choices of $\lambda_j \varepsilon \{0,1,-1\}$, and in particular for $\lambda_j = 1$. Then $\text{h.c.f.}\{\sum_{j\varepsilon J} a_{ij} : i \varepsilon I\} = 1$ which contradicts the condition $\sum_{j\varepsilon J} a_{ij} \equiv 0 \pmod 2$. //

3.2 Eulerian Matrices

In Theorem 3.1, a central place is taken by the algebraic characterization of absolutely unimodular matrices by means of eulerian submatrices. We give an additional such characterization in the next theorem.

Theorem 3.2 *The following statements are equivalent :*

1) A *is an absolutely unimodular matrix*

2) *For every square eulerian submatrix of* $A \varepsilon C_{m,n}$, *the sum of all the elements is a multiple of four.*

Proof (1)⇒(2). Let A_I^J be an eulerian submatrix of the absolutely unimodular matrix A. Then $\sum_{j\varepsilon J} a_{ij} x_j \equiv 0 \pmod 2$, $i \varepsilon I$, where all $x_j = 1$. By (2) of Theorem 3.1 there is a vector y with components 1 or -1 such that $\sum_{j\varepsilon J} a_{ij} y_j = 0$, $i \varepsilon I$. Let the vector w be obtained from y by replacing all the elements -1 by $+1$. Since the number of non-zero components in any column of an eulerian submatrix is even, we have

$$0 = \sum_{i\varepsilon I} \sum_{j\varepsilon J} a_{ij} y_j \equiv \sum_{i\varepsilon I} \sum_{j\varepsilon J} a_{ij} w_j \pmod 4$$

which gives

$$\sum_{i\varepsilon I} \sum_{j\varepsilon J} a_{ij} \equiv 0 \pmod 4. \qquad (3.3)$$

(2)⇒(1). By (3) of Theorem 3.1 it suffices to show that if equation (3.3) is satisfied for an eulerian submatrix A_I^J, then $\det A_I^J$ is zero. Suppose that an eulerian submatrix B of A exists for which (3.3) is satisfied, but that $\det B \neq 0$. Let B have minimal order among all matrices having the stated property. Then, every proper eulerian submatrix of B is singular and so all proper submatrices of B are absolutely unimodular (Theorem 3.1).

Let $b = A_I^J e$, where e is a vector of ones. Replace any column, say the last, of the matrix A_I^J by the column b . Then, the matrix so obtained, B , will be absolutely unimodular. Hence, by (2) of Theorem 3.1 , there is a vector y such that $yB = (0,0,\dots,0,4k)$, where k is an integer. Thus $\det A_I^J = \det B = \pm 4k$. Since, by assumption, $\det A_I^J \neq 0$, we have $|\det A_I^J| \geq 4$.

To complete the proof we need the following Lemma, due to R.Gomory (see Camion (1965)).

Lemma 3.3 *Let* $A \varepsilon C_{n,n}$ *and let* $|\det A| > 2$ *, then there exists a square submatrix* Q *of* A *such that* $|\det Q| = 2$.

Proof of the Lemma. Let $D = (A, I_n)$ and let $K \subset C_{n,2n}$ be the class of matrices which can be obtained from D by multiplication on the left by a unimodular matrix and contains I_n as a submatrix.

Let $F \varepsilon K$ have the property that in the first n columns of F there are the maximum possible number of unit column vectors. At least one column vector F^k , $k \leq n$, is not a unit vector since there are no matrices in K of the form (I_n, G) because $|\det A| > 2$.

Let this be the column F^1 and let the set J contain 1 and the indices j , $j<n$, of unit column vectors F^j of F .

Consider the $(n \times n)$-submatrix of F consisting of the columns A^j , $j \varepsilon J$, and of any of the remaining unit column vectors of the remaining part of F . Clearly, there are $n - |J|$ such submatrices. Among them there is at least one non-singular matrix, for otherwise the matrix determined by the first n columns of F is singular, which contradicts the assumption $|\det A| > 2$. Without loss of generality suppose that the submatrix

$$U = \begin{pmatrix} \eta_1 & 0 & \dots & 0 \\ \eta_2 & 1 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ \eta_n & 0 & \dots & 1 \end{pmatrix}$$

is non-singular. Because U is non-singular $\eta_1 \neq 0$, so $\eta_1 \varepsilon \{1,-1\}$. Suppose for definiteness that $\eta_1 = 1$. Then the matrix U^{-1} has the form

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ -\eta_2 & 1 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ -\eta_n & 0 & \dots & 1 \end{pmatrix}$$

The matrix $U^{-1}F \notin K$ since it contains in its first n columns one unit vector more than F. Hence, among the elements of the matrix $U^{-1}F$ there is at least one which is not equal to 0 or ± 1. We represent the unimodular matrix U^{-1} as a product of elementary matrices $U_n, \dots, U_2$, where

$$U_2 = \begin{bmatrix} 1 & 0 & \dots & 0 \\ -\eta_2 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & 1 \end{bmatrix}, \dots, U_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ -\eta_n & 0 & \dots & 1 \end{bmatrix}$$

Let q be the smallest integer such that one of the elements of the matrix $W = U_q \dots U_2 F$ is not equal to 0, ± 1. The operation corresponding to the matrix U_q is the addition (or subtraction, depending on the sign of η_q) of the first row of the matrix $U_{q-1} \dots U_2 F$ to the q^{th} row. Thus the elements of W which are not equal to 0, ± 1 must be equal to ± 2 and they are all in the q^{th} row, moreover the elements corresponding to these in the first row must be non-zero. Let $W_q^j = \pm 2$. Note that in the first column of W, the element $W_q^1 = 0$. Thus W contains a submatrix

$$\begin{pmatrix} 1 & \pm 1 \\ 0 & \pm 2 \end{pmatrix}$$

which lies in the intersection of the 1^{st} and j^{th} columns with the 1^{st} and q^{th} rows.

Let S be the set whose elements are the numbers k, j and the numbers of the $(n-2)$ unit columns which do not have units in their 1^{st} and q^{th} rows. Then $\det D^S = \det W^S$ since the matrix W^S is obtained by multiplication on the left by some unimodular matrix. Since $D = (A, I_n)$, $\det D^S$ coincides with the determinant of some sub-matrix Q of A. Thus $\det Q = \pm 2$. This proves the Lemma. //

3.3 Unimodular Hypergraphs

We consider conditions for absolute unimodularity of Boolean matrices. Every Boolean Matrix is an incidence matrix of a hypergraph. The terminology of hypergraphs is taken from Berge (1970) and Zykov (1974).

A pair $H = (I,E)$ is a *hypergraph* if I is a finite set and E is a family of non-empty subsets of I. The elements of I are called *vertices* and the subsets E_j of the family E are *edges of the hypergraph*.

Let $I' \subset I$. The *hypergraph generated by the vertices of* I' is the pair $H_{I'} = (I', E_{I'})$, where $E_{I'} = \{E_j \cap I' : E_j \cap I' \neq \emptyset\}$

We say that the hypergraph H is *bichromatically balanced* if its vertices are coloured in two colours in such a way that for each edge the number of vertices coloured in the first colour is either equal to the number coloured in the second colour or differs from it by one.

Definition 3.2 The hypergraph $H = (I,E)$ is called *unimodular* if the hypergraph $H_{I'} = (I', E_{I'})$ generated by any subset of vertices $I' \subseteq I$ is bichromatically balanced

The following criterion was apparently first obtained by Ghouilla-Houri (1962).

Theorem 3.4 *The following statements are equivalent :*

1) A *is an absolutely unimodular Boolean matrix.*

2) A *is an incidence matrix of a unimodular hypergraph.*

3) *Every subset* I *of the row indices of a Boolean matrix* A *can be partitioned into two subsets* I', I'' *such that*

$$\left| \sum_{i \in I'} A_i - \sum_{i \in I''} A_i \right| \leqq e \quad ,$$

where A_i *is the* i^{th} *row of* A .

Proof The equivalence of 2) and 3) is obvious.

(1)$\Leftrightarrow$(2). Let H be the hypergraph generated by the matrix A by regarding A as an incidence matrix. (That is, the rows correspond to the vertices of H and the columns correspond to the edges of H and $a_{ij}=1$ means that the i^{th} vertex lies in the j^{th} set, while $a_{ij}=0$ means that the i^{th} vertex does not lie in the j^{th} edge set). By Theorem 3.1, it suffices to show the equivalence of statement (2) of Th.3.1 and statement (2) of the above theorem. We reformulate statement (2) of Theorem 3.1 in the following form : for any subset I of the row indices of the matrix A, there is a vector y satisfying

$$y_i = 0 \qquad \forall i \notin I , \tag{3.4}$$

$$y_i = \pm 1 \qquad \forall i \varepsilon I , \tag{3.5}$$

$$yA^j = 0 , \pm 1 \qquad \forall j . \tag{3.6}$$

With each vector y whose components satisfy the relations (3.4)-(3.6), we associate a colouring of the vertices of the hypergraph H_I in two colours, according to the values of the components y_i, $i \varepsilon I$. The relation (3.6) guarantees that the hypergraph H_I is bichromatically balanced. Conversely, every bichromatically balanced hypergraph generated by the vertices of I corresponds to a vector y satisfying conditions (3.4)-(3.6). //

Example : On a line L let there be given a set of points I and a finite family of subsets of I

$$E_{a_i,b_i} = \{x \varepsilon I : a_i \leqq x \leqq b_i\} \qquad i \varepsilon N_n .$$

The hypergraph $H = (I,E)$, where $E = \{E_{a_i,b_i} : i \varepsilon N_n\}$, is called a *hypergraph of intervals.* Clearly, a hypergraph of intervals is unimodular. The required colouring can be obtained by colouring the points of I successively in order as one moves along the line. By Theorem 3.4 the incidence matrix of the hypergraph is absolutely unimodular.

A subset I' of vertices of a hypergraph $H = (I,E)$ is called *inner-stable* if $|I' \cap E_j| \leqq 1$ for every edge $E_j \varepsilon E$. The *inner stability number* $\alpha(H)$ of a hypergraph H is defined to be the largest number of vertices in any inner stable set of H. The *outer stability number* $\rho(H)$ of H is the smallest number of edges which cover all vertices of H.

Corollary 3.5 (Berge 1970) *Let H be a unimodular hypergraph, then its inner and outer stability numbers are equal.*

Proof Let $x = (x_1,\dots,x_n)$ be the characteristic vector of an inner-stable set. Then, if A is the incidence matrix of the hypergraph H, we have, by Theorems 3.2 and 3.4 that $\alpha(H) = \max\{ex : xA \leqq e , x \geqq 0\}$. The dual problem $\min\{ey : Ay \geqq e , y \geqq 0\}$, by the unimodularity of the matrix A, also has an integral solution y^*, whose components are 0 or 1. Hence, by the duality theorem of Linear Programming, we have $\alpha(H) = \rho(H)$. //

A *chain of length* q in a hypergraph H is a sequence of distinct vertices and edges of the form $v_1, E_1, v_2, E_2, \dots, E_q, v_{q+1}$, such that $v_k, v_{k+1} \in E_k$ $\forall k \in N_q$. If $v_{q+1} = v_1$, then the chain is called a *cycle of length* q. An *odd cycle of a hypergraph* is a cycle whose length is odd.

<u>Proposition 3.6</u> *If a hypergraph does not contain any odd cycles, then it is unimodular.*

Proof Let the edge E_i of the hypergraph H consist of the vertices $\{v_i^1, \dots, v_i^{n_i}\}$, where $n_i = |E_i|$. Consider the graph $G = (V, E)$ where

$$E = \left\{ (v_i^1, v_i^2), \dots, (v_i^{2[n_i/2]-1}, v_i^{2[n_i/2]}) , i \in N_n \right\}.$$

Suppose the graph G contains an odd cycle. Then, consider an odd cycle μ of minimal length. If the cycle μ contains two edges (v_s, v_{s+1}) and (v_t, v_{t+1}) belonging to some edge E_i of the hypergraph H, then we break it up into two chains of odd length of the form $v_1, v_2, \dots, v_s, v_t, v_{t+1}, \dots, v_1$; $v_1, \dots, v_s, v_{t+1}, \dots, v_1$. In this way we obtain a sequence of chains of odd length which together define an odd cycle of the hypergraph H, which contradicts the hypothesis. Thus, the graph G contains no odd cycles and hence is bichromatic (see Corollary 4.3). Since G has a bichromatic colouring it follows that the hypergraph H and any hypergraph generated by a subset of its vertices is also bichromatically balanced. Thus, H is a unimodular hypergraph. //

§4 <u>UNIMODULAR INCIDENCE MATRICES</u>

If the incidence matrix of a graph is absolutely unimodular, then an extremal problem on such a graph reduces to a linear programming problem which implicitly guarantees the existence of simple and effective algorithms for finding an extremum. Moreover, the duality theorem of linear programming is true for such problems and this gives rise to many important combinatorial theorems.

4.1 Criteria for Absolute Unimodularity

Let $C^*_{m,n}$ be the set of all matrices of class $C_{m,n}$ which contain exactly two nonzero elements in each column. It is clear that every matrix $A \in C^*_{m,n}$ is an incidence matrix (vertices-edges) of some mixed graph $G(A)$ which may have both oriented and non-oriented edges. Without loss of generality we assume that A has no columns in which both non-zero elements are negative. Such columns could be multiplied by -1 ; this would only change the signs of some of the minors of A .

Theorem 4.1 *Let* $A \in C^*_{m,n}$. *The following statements are equivalent :*

1) A *is an absolutely unimodular matrix ;*

2) A *is the incidence matrix of a mixed graph* $G(A)$ *in which every cycle, consisting of both oriented (disregarding orientation) and unoriented edges, has an even number of unoriented edges ;*

3) (Heller & Tompkins 1958) *The rows of* A *can be partitioned into two disjoint sets* J_1 *and* J_2 *such that if the two non-null elements in a given column have the same sign then they belong to different sets, while if they have opposite signs then they belong to the same set.*

Proof 1) $\Rightarrow$ 2). Let $G(A)$ contain a cycle C in which there is an odd number ℓ of unoriented edges. Let B be the (kxk)-submatrix of A corresponding to the cycle C . We calculate the determinant of B . To do this, add the p^{th}-row, containing the element $a_{pe} = 1$ to the i^{th}-row, containing the element $a_{ie} = -1$. The e^{th}-column now contains only one non-zero element, namely a_{pe} . Now expand the determinant by this column and obtain $\det B = \pm\det B'$, where B' is the $(k-1)x(k-1)$-matrix which corresponds to the cycle C' which is obtained from C by removing the vertex i and shrinking the edge e . Continuing this process, we eventually obtain an $(\ell x \ell)$-matrix which is the incidence matrix of a cycle consisting of ℓ non-oriented edges. This matrix, after suitable row and column permutations, takes the form

$$B^* = \begin{bmatrix} 1 & 0 & 0 & . & . & . & 1 \\ 1 & 1 & 0 & . & . & . & 0 \\ 0 & 1 & 1 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 1 \end{bmatrix} .$$

It is easily seen that $\det B^* = 1 + (-1)^{\ell-1}$, so that if ℓ is odd, then $\det B = \pm\det B^* = \pm 2$. This contradiction shows that 1) $\Rightarrow$ 2).

2) $\Rightarrow$ 3). Let $G(A)$ be a connected graph (the connectivity is understood to ignore orientation), otherwise carry out all constructions on the connected components of G and assume, in this case, that the matrix A has a block structure. Select an arbitrary vertex i of the graph $G(A)$. Construct two sets of vertices J_1 and J_2 as follows : J_1 consists of the vertex i and all vertices which can be connected to i by a chain with an even number of non-oriented edges; J_2 consists of all vertices which can be connected to i by a chain with an odd number of non-oriented edges. By condition 2) and since $G(A)$ is connected, each vertex lies in one and only one of the subsets J_1 and J_2. Also, there are no non-oriented edges connecting vertices in the same set J_1 or J_2 and any oriented edge must connect vertices in the same set, otherwise there would be a cycle with an odd number of non-oriented edges. Thus, we have obtained a partition of the rows into the required sets J_1 and J_2.

3) $\Rightarrow$ 1). We use induction on the dimension of an arbitrary $(k\times k)$-submatrix B of the matrix A. Statement 1) is true for $k=1$ since all $a_{ij} = 0,\pm 1$. Suppose that all minors of order k are equal to $0,\pm 1$ and consider any $(k+1)\times(k+1)$-submatrix B. If B contains a zero column, then $\det B = 0$. If B has a column with a single nonzero element then, expanding the determinant of B along this column, we have $\det B = \beta'$, where β' is plus or minus the cofactor of the nonzero element. By the inductive assumption $\beta' = \pm 1, 0$. It remains to consider the case where each column of B has two non-null elements. Then, by statement 3)

$$\sum_{i\in J_1} A_i = \sum_{i\in J_2} A_i .$$

Thus, the rows of B are linearly dependent, so that $\det B = 0$. All conclusions remain valid in the case where one of the sets J_k is empty.//

Corollary 4.2 *The incidence matrix of any oriented graph is absolutely unimodular.*

This result was already formulated by Poincaré (1901).

A *colouring of a graph* is an assignment of colours to its vertices such that no two adjacent vertices receive the same colour. Thus, a colouring of a graph with k colours partitions the set of vertices into k disjoint classes in each of which there are no adjacent vertices. The *chromatic number* $\chi(G)$ of a graph G is the smallest k for which the graph G has a colouring in k colours. A graph which can be coloured in two colours is called *bichromatic*.

Corollary 4.3 *A graph is bichromatic if and only if it does not contain any odd cycles.*

A sufficient condition for the unimodularity of a matrix, due to Heller (1957), follows from Theorem 4.1.

Corollary 4.4 *A matrix whose columns are the coordinates of the edges of a simplex relative to a basis consisting of a subset of the edges of the simplex is unimodular.*

Proof Let the simplex T_{n-1} be given by the conditions

$$\sum_{i=1}^{n} x_i = 1 \quad , \quad x_i \geq 0 \qquad i \varepsilon N_n .$$

Any edge ℓ of T_{n-1} is the intersection of T_{n-1} with hyperplanes $x_i = 0$, $i \neq p,s$ where $p,s \varepsilon N_n$. Any normal to ℓ has components $\alpha_p = \alpha_s = 1$ and the other α_i, $i \neq p,s$, arbitrary. The direction vector $a^\ell = (a_1,\ldots,a_n)$ of the edge ℓ (of either orientation) satisfies the conditions $\alpha a^\ell = 0$. Hence $a_i = 0$, $i \neq p,s$, $a_p = \pm 1$, $-a_s = \pm 1$. The matrix A consisting of the vectors a^ℓ for each of the sides ℓ of the simplex T_{n-1} is absolutely unimodular by Theorem 4.1. By Lemma 2.6, the matrix of coefficients in the expansions of the non-basic vectors relative to any basis B is absolutely unimodular. This proves the corollary. //

4.2 Bipartite Graphs

Non-oriented graphs with absolutely unimodular incidence matrices play an important rôle in a number of applications of graph theory.

Definition 4.1 A *bipartite graph* is a graph $G = (U,V,E)$ in which the set of vertices is partitioned into two disjoint subsets U and V such that every edge $(i,j) \varepsilon E$ joins some vertex $i \varepsilon U$ with a vertex $j \varepsilon V$.

If any two vertices $i \varepsilon U$, $j \varepsilon V$ of G are joined by an edge (i,j) then the graph is called a *complete bipartite graph* and is denoted by $K_{m,n}$, where $m = |U|$, $n = |V|$.

The equivalence of statements 2) and 3) of Theorem 4.1 implies that a graph G is bipartite if and only if all of its simple cycles have even length.

The incidence matrix R of a complete bipartite graph $K_{m,n}$ takes the form

$$R = \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots & \cdot \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & 1 \\ 1 & \cdots & 0 & 1 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdots & \cdot & \cdots & \cdot \\ 0 & \cdots & 1 & 0 & \cdots & 1 & \cdots & 0 & \cdots & 1 \end{bmatrix} .$$

From the equivalence of statements 1) and 3) of Theorem 4.1 we obtain :

<u>Corollary 4.5</u> *The incidence matrix of a non-oriented graph G is absolutely unimodular if and only if G is a bipartite graph.*

In the theory of bipartite graphs a fundamental rôle is played by König's Theorem which we present here in a matrix interpretation.

A *line* in a matrix is a row or a column of the matrix. Two elements of a matrix are *non-collinear* if they do not lie in any one line.

<u>Theorem 4.6</u> (König's Theorem) *The maximum number of pairwise non-collinear units of any Boolean matrix is equal to the minimum number of lines which cover all the units in the matrix.*

Proof To find the maximum number of pairwise non-collinear units of the $(m \times n)$-Boolean matrix (c_{ij}) , it suffices to find

$$\max \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to the constraints :

$$\sum_{i=1}^{m} x_{ij} \leqq 1 \qquad \forall j \varepsilon N_n , \tag{4.1}$$

$$\sum_{j=1}^{n} x_{ij} \leqq 1 \qquad \forall i \varepsilon N_m , \tag{4.2}$$

$$x_{ij} = 0 \text{ or } 1 \qquad \forall (i,j) \varepsilon N_m \times N_n .$$

The minimum number of lines covering all units of the matrix (c_{ij}) may be found by solving the problem :

$$\begin{aligned} &\min \left\{ \sum_{i=1}^{m} u_i + \sum_{j=1}^{n} v_j \right\} , \\ &u_i + v_j \geqq c_{ij} \qquad \forall (i,j) \varepsilon N_m \times N_n , \\ &u_i , v_j = 0 \text{ or } 1 \qquad \forall (i,j) \varepsilon N_m \times N_n . \end{aligned} \tag{4.3}$$

The optimal solution (u_i^*, v_j^*) of the latter problem corresponds to the minimum covering consisting of the set I of rows for which $u_i^* = 1$ and the set J of columns for which $v_j^* = 1$.

The coefficient matrices A and A^T of ~~the~~ left hand sides of (4.1),(4.2) and (4.3) are absolutely unimodular since they are incidence matrices of a bipartite graph (see Corollary 4.5). Thus we can replace the requirement that the variables be integral by the requirement that the variables be non-negative. We then have a pair of dual linear programming problems and by the duality theorem we have

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}^* = \sum_{i=1}^{m} u_i^* + \sum_{j=1}^{n} v_j^* .$$

This proves the theorem. //

It should be noted that König's Theorem remains true if instead of Boolean matrices we consider non-negative matrices, i.e. matrices with non-negative elements. Of course, in this case we consider sets of lines covering all the positive elements of the matrix.

A *diagonal of the* $(n \times n)$*-matrix* (c_{ij}) is a sequence of elements $c_{1\pi_1}, \ldots, c_{n\pi_n}$, any two of which are non-collinear. The following result, due to Frobenius, will be used later.

<u>Corollary 4.7</u> *Let* A *be an* $(n \times n)$*-non-negative matrix. Then every diagonal of* A *includes a zero element if and only if* A *has a zero* $(s \times t)$*-submatrix, where* $s+t \geqq n+1$.

Proof (i) Suppose that every diagonal includes a zero element. Then, by König's theorem, the minimal number of lines, ρ, covering the positive elements of A, satisfies the inequality $\rho < n$. Among these lines let there be r rows and ℓ columns. Then, using the remaining lines, there is a null submatrix which has $s = n-r$ rows and $t = n-\ell$ columns. Hence $s+t = n-r+n-\ell = 2n-\rho > n$.

(ii) Now suppose that the matrix A has a zero $(s\times t)$-submatrix A_I^J with $|I|+|J| = s+t > n$. Suppose, further, that the matrix A has a diagonal which contains no zero element. Then, in the columns with indices in J, the diagonal elements must lie in the $((n-s)\times t)$-submatrix $A^J_{N_n\setminus I}$ and, consequently, $t \leqq n-s$. But $t > n-s$ and we have a contradiction. //

4.3 Theorems on Flows

Let $G = (V,E)$ be an oriented graph (*digraph*) with two distinguished vertices s and t called, respectively, the *source* and the *sink*. To each directed edge $(i,j)\varepsilon E$ of G we associate a weight $d_{ij} \geqq 0$ called the *flow capacity*. A *flow* of magnitude θ in the digraph G is a set of numbers $\{(x_{ij}):(i,j)\varepsilon E\}$ satisfying the following conditions.

$$\sum_{i:(i,j)\varepsilon E} x_{ij} - \sum_{k:(j,k)\varepsilon E} x_{jk} = \begin{cases} -\theta & \text{for } j=s \\ \theta & \text{for } j=t \\ 0 & \text{for } j\neq s,t. \end{cases} \tag{4.4}$$

$$0 \leqq x_{ij} \leqq d_{ij} \qquad \forall (i,j)\varepsilon E . \tag{4.5}$$

The flow (x_{ij}) is a *maximal flow* if the quantity θ assumes its largest possible magnitude.

Let (S,T) be a partitioning of the vertex set of the digraph G into two disjoint subsets S and T such that $s\varepsilon S$ and $t\varepsilon T$. Then the set of directed edges $(i,j)\varepsilon E$ with $i\varepsilon S, j\varepsilon T$ is called a *cutset of the digraph* G. The *capacity of the cutset* (S,T) is the sum of the flow capacities of the edges in the cutset. The following theorem, due to Ford and Fulkerson (1962), is widely known as the Max.Flow-Min.Cut Theorem.

Theorem 4.8 (Ford-Fulkerson) *The maximum flow,* θ, *in a digraph* G *is equal to the capacity of the minimum cutset in* G.

Proof We introduce dual variables u_i , $i\varepsilon V$, corresponding to the constraints (4.4) and variables w_{ij} , $(i,j)\varepsilon E$, corresponding to constraints (4.5). Then, the dual of the maximum flow problem takes the form

$$\min \sum_{(i,j)\varepsilon E} d_{ij} w_{ij}$$

$$-u_s + u_t \geqq 1 ,$$

$$-u_i + u_j + w_{ij} \geqq 0 \qquad \forall (i,j)\varepsilon E ,$$

$$w_{ij} \geqq 0 \qquad \forall (i,j)\varepsilon E .$$

The constraint matrix in (4.4) is clearly the incidence matrix of the digraph G and, by Corollary 4.2, it is absolutely unimodular. Hence, the optimal solution of the dual problem is integral. Furthermore, from Cramer's Rule it follows that the components of the optimal solution are equal to 1 or 0 . From this optimal solution we define a cut (S,T) by $i\varepsilon S$ if $u_i = 1$ and $i\varepsilon T$ if $u_i = 0$. It follows that $w_{ij} = 1$ if $i\varepsilon S, j\varepsilon T$ and $w_{ij} = 0$ otherwise. The statement of the theorem follows from the duality theorem. //

<u>Corollary 4.9</u> *If the flow capacities* d_{ij} *are whole numbers then there is an integral maximum flow.*

The following corollary, known as Menger's Theorem (Ford & Fulkerson 1962) follows from Theorem 4.8 and Corollary 4.9.

<u>Corollary 4.10</u> (Menger) *Let* S *and* T *be two disjoint subsets of the vertex set of a graph* G . *The maximum number of disjoint chains from* S *to* T *is equal to the minimum number of vertices in any* (S,T)-*separating set, that is, a set of vertices which blocks all paths from* S *to* T .

A second variant of Menger's Theorem was published by Whitney (Ford & Fulkerson 1962)

<u>Corollary 4.10'</u> (Whitney's Theorem) *Any pair of vertices in a graph* G *can be joined by at least* n *disjoint paths if and only if*

the smallest number of vertices whose removal would lead to an unconnected graph is n .

A consequence of Theorem 4.8 is the Theorem of Supply and Demand. Suppose that, instead of two vertices s and t as source and sink, we designate two subsets of vertices S and T . To each vertex $i\varepsilon S$ there corresponds a number $a_i \geqq 0$ (the supply at source i) and to each vertex $j\varepsilon T$ there corresponds a number $b_j \geqq 0$ (the demand at sink j). The question arises : is it possible, using the flow capacities of the directed edges, to satisfy the demand at the sinks using the supply at the sources. This is equivalent to determining whether or not the following system of constraints is consistent :

$$\sum_{j:(i,j)\varepsilon E} x_{ij} - \sum_{j:(j,i)\varepsilon E} x_{ji} \begin{cases} \leqq a_i, & \text{if } i\varepsilon S, \\ = 0, & \text{if } i\notin S, i\notin T, \\ \leqq -b_i, & \text{if } i\varepsilon T, \end{cases} \tag{4.6}$$

$$0 \leqq x_{ij} \leqq d_{ij} \qquad \forall (i,j)\varepsilon E . \tag{4.7}$$

This question is answered by the following theorem, due to Gale (Ford & Fulkerson 1962).

<u>Theorem 4.11</u> *The constraints* (4.6),(4.7) *are consistent if and only if, for any set* $I \subset V$ *it is true that*

$$\sum_{i\varepsilon T\cap\overline{I}} b_i - \sum_{i\varepsilon S\cap\overline{I}} a_i \leqq \sum_{(i,j)\varepsilon I\times\overline{I}} d_{ij} \tag{4.8}$$

where $\overline{I} = V\setminus I$.

Proof (i) Suppose that the system (4.6),(4.7) has a solution (x_{ij}) , then multiplying each of the inequalitites in (4.6) with $i\varepsilon T$ by -1 and then summing them over $i\varepsilon\overline{I}$, we obtain

$$\sum_{i\varepsilon T\cap\overline{I}} b_i - \sum_{i\varepsilon S\cap\overline{I}} a_i \leqq \sum_{(i,j)\varepsilon V\times\overline{I}} x_{ij} - \sum_{(i,j)\varepsilon\overline{I}\times V} x_{ij}$$

$$= \sum_{(i,j)\varepsilon I\times\overline{I}} x_{ij} - \sum_{(i,j)\varepsilon\overline{I}\times I} x_{ij} \leqq \sum_{(i,j)\varepsilon I\times\overline{I}} x_{ij}$$

$$\leqq \sum_{(i,j)\varepsilon I\times\overline{I}} d_{ij} .$$

Thus conditions (4.8) are necessary.

(ii) To prove sufficiency, consider the digraph G^* with vertex set $V^* = V \cup \{s,t\}$ and with directed edges $E^* = E \cup \{(s,i): i \in S\} \cup \{(i,t): i \in T\}$. Define the flow capacities of the edges of the new graph by

$$d^*_{ij} = \begin{cases} a_j, & \text{if } i=s,\ j \in S, \\ b_i, & \text{if } i \in T,\ j=t, \\ d_{ij}, & \text{if } (i,j) \in E. \end{cases}$$

We show that the set (T,t) defines a minimal cut in the graph G^*. Indeed, let $(U,\overline{U})$ be any cut in G^*. Then, putting $I = U \setminus s$, $\overline{I} = \overline{U} \setminus t$, we obtain a cut $(I,\overline{I})$ in the graph G, for which

$$\sum_{(i,j) \in U \times \overline{U}} d^*_{ij} - \sum_{(i,j) \in T \times t} d^*_{ij}$$

$$= \sum_{(i,j) \in I \times t} d^*_{ij} + \sum_{(i,j) \in s \times \overline{I}} d^*_{ij} + \sum_{(i,j) \in I \times \overline{I}} d^*_{ij} - \sum_{(i,j) \in T \times t} d^*_{ij}$$

$$= \sum_{i \in T \cap I} b_i + \sum_{i \in S \cap \overline{I}} a_i + \sum_{(i,j) \in I \times \overline{I}} d_{ij} - \sum_{j \in T} b_j$$

$$= -\sum_{j \in T \cap \overline{I}} b_j + \sum_{i \in S \cap \overline{I}} a_i + \sum_{(i,j) \in I \times \overline{I}} d_{ij} \geqq 0 .$$

From the Ford-Fulkerson Theorem applied to the digraph G^*, there exists a flow x^* of magnitude $\sum_{(i,j) \in T \times t} d^*_{ij}$ whose restriction to the set E clearly satisfies the constraints (4.6),(4.7). //

In particular, if G is a bipartite graph and if $\sum_{i \in S} a_i = \sum_{j \in T} b_j$, then Theorem 4.11 takes the following form :

<u>Corollary 4.12</u> *The system*

$$\sum_{j=1}^{n} x_{ij} = a_i , \quad \sum_{i=1}^{m} x_{ij} = b_j , \qquad 0 \leqq x_{ij} \leqq d_{ij} , \qquad (4.9)$$

in which a_i, b_j, d_{ij} *are given nonzero numbers, is consistent if and only if*

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

and one of the following equivalent conditions is satisfied :

$$1) \quad \sum_{j=1}^{n} \max(0, b_j - \sum_{i \in I} d_{ij}) \leqq \sum_{i \in \overline{I}} a_i \qquad \forall I \subset N_m ;$$

2) $\sum_{i=1}^{m} \min(a_i, \sum_{j\varepsilon J} d_{ij}) \geqq \sum_{j\varepsilon J} b_j \qquad \forall J \subset N_n$.

If a_i, b_j, d_{ij} are all integers and if the conditions of Corollary 4.12 are satisfied, then, since the constraint matrix of (4.9) is unimodular, there is an integral solution of system (4.9). Thus, when $d_{ij} = 1$, Corallary 4.12 implies the following result, known as Ryser's Theorem (Ford & Fulkerson 1962), on the existence of Boolean matrices with given sums of elements in each line.

Corollary 4.13 *The system of linear inequalities*

$$\sum_{j=1}^{n} x_{ij} \leqq a_i \quad \forall i\varepsilon N_m, \qquad \sum_{i=1}^{m} x_{ij} \geqq b_j \quad \forall j\varepsilon N_n,$$

$$0 \leqq x_{ij} \leqq 1 \quad \forall (i,j)\varepsilon N_m x N_n$$

has an integral solution if and only if

$$\sum_{j=1}^{s} b_j \leqq \sum_{j=1}^{s} a_j^* \quad \forall s\varepsilon N_n,$$

where a_j^* *is the number of elements* $a_1,\dots,a_m$ *not less than* j *and* $b_1 \geqq b_2 \geqq \dots \geqq b_n$.

Let $I = \{v_1,\dots,v_m\}$ be a given finite set and let there be given a family E of subsets of I , $E_1,\dots,E_n$, not necessarily distinct. The set $\{e_1,\dots,e_n\} \subseteq I$ is called a *set of distinct representatives* (s.d.r) of the family E if $e_j \varepsilon E_j$, $\forall j\varepsilon N_n$ and all the e_j are distinct.

Corollary 4.14 (Hall's Theorem) *A set of distinct representatives exists for the subsets* $E = \{E_1,\dots,E_n\}$ *of the set* I *if and only if for any subset* $J \subset N_n$

$$\left|\bigcup_{j\varepsilon J} E_j\right| \geqq |J| . \qquad (4.10)$$

Proof The necessity of (4.10) is obvious. To prove sufficiency we use Corollary 4.13 with $a_i = b_j = 1$ and

$$d_{ij} = \begin{cases} 1, & \text{if } e_i \varepsilon E_j, \\ 0, & \text{if } e_i \notin E_j . \end{cases} \quad //$$

§5 COVERING, PARTITIONING AND PACKING POLYTOPES

Partitioning, covering and packing problems are mathematical models for many theoretical and applied problems such as graph colouring, the construction of perfect codes and of minimal disjuctive normal forms, the formation of block-schemes, information retrieval, the organisation of train, ship and airplane timetables and of regional administration (Ford & Fulkerson 1962, Balas & Padberg 1972 & 1975 , Sapozhenko et al. 1977). In this section we consider the basic properties of polytopes which are convex hulls of the characteristic vectors of partitions, coverings and packings.

5.1 Problem Formulation

Let $I = \{v_1,\ldots,v_m\}$ be a given finite set and let $E = \{E_1,\ldots,E_n\}$ be a family of subsets of I . Let $E' = \{E_{j_1},\ldots,E_{j_s}\}$ be a subfamily of the family E . If every element v_i is contained in not more (not less) than one of the subsets E_j of E' , then E' is called a *packing* (*covering*) *of* I . A covering which is also a packing of I is called a *partition of* I . Let $A = (a_{ij})_{m\times n}$ be the incidence matrix of the elements I and the subsets E_j : $a_{ij} = 1$ if $v_i \varepsilon E_j$, and $a_{ij} = 0$ if $v_i \notin E_j$. Every subfamily E' of the family E can be given by means of a *characteristic vector* for which $x_j = 1$ if E_j lies in E' and $x_j = 0$ otherwise. Thus, there is a one-to-one correspondence between the coverings, partitions and packings of a set and the integral solutions of one of the following systems of linear inequalities :

$$1)\ Ax \geqq e\ , \quad 2)\ Ax = e\ , \quad 3)\ Ax \leqq e\ , \tag{5.1}$$

$$\text{where} \quad 0 \leqq x \leqq e\ . \tag{5.2}$$

The polytopes of solutions to each of the systems (5.1),(5.2) are denoted respectively by $M^{\geqq}(A,e)$, $M^{=}(A,e)$, $M^{\leqq}(A,e)$ according to which sign is chosen in (5.1). We have the following three problems :

$\max\{ex : x \varepsilon M_Z^{\leqq}(A,e)\}$ - the *packing problem* ;

$\min\{ex : x \varepsilon M_Z^{=}(A,e)\}$ - the *partitioning problem* ;

$\min\{ex : x \varepsilon M_Z^{\geqq}(A,e)\}$ - the *covering problem* .

Definition 5.1 The convex hull of the set $M^{\leq}_{Z}(A,e)$ is called the *packing polytope*. The *partitioning polytope* and the *covering polytope* are similarly defined. The convex hulls of the sets $M^{\geq}(A,e)$, $M^{=}(A,e)$ and $M^{\leq}(A,e)$ are called the *relaxed polytopes* of coverings, partitionings and packings respectively.

We study coverings and packings in graphs. Usually, in covering (packing) problems in graphs, the set I to be covered (packed) consists either of the set of vertices or of the set of edges.

Let the graph $G = (V,E)$ be given with m vertices and n edges. We explain some terminology. A packing made up of edges of G is called a *matching in the graph*, in other words, a matching is a set of non-adjacent edges. A matching which covers all the vertices in a graph G is called a *complete matching*. A packing made up of vertices of a graph G is called an *inner stable set* and a covering is an *outer stable set*. In other words, an inner stable set of a graph G is a subset of its vertices no two of which are adjacent, and an outer stable set is a subset of vertices which cover all edges.

Let A_G be the $(m\times n)$-incidence matrix of G and let A^T_G be its transpose. Then the most important graph characteristics are defined by the relations :

$$\nu(G) = \max\{ex : x\in M^{\leq}_{Z}(A_G,e)\} \quad - \text{ the matching number;}$$

$$\rho(G) = \min\{ex : x\in M^{\geq}_{Z}(A_G,e)\} \quad - \text{ the edge covering number;}$$

$$\alpha(G) = \max\{ex : x\in M^{\leq}_{Z}(A^T_G,e)\} \quad - \text{ the inner stability number;}$$

$$\tau(G) = \min\{ex : x\in M^{\geq}_{Z}(A^T_G,e)\} \quad - \text{ the vertex covering number.}$$

When the relaxed polytopes of coverings, packings or partitionings are integral, the duality theorem applied to these graph characteristics leads to important relations in graph theory.

Note that, by Corollary 4.5, the polytopes $M^{\geq}(A_G,e)$ and $M^{\leq}(A_G,e)$ are integral if G is a bipartite graph.

Thus, König's theorem may be reformulated as follows :

Theorem 5.1 *If G is a bipartite graph, then* $\nu(G) = \tau(G)$.

We now describe other important classes of integral polytopes $M^{\leq}(A,e)$.

5.2 Cliques in an Intersection Graph

We formulate conditions for the relaxed packing polytope $M^{\leq}_{=}(A,e)$ to be integral. We introduce the *intersection graph* G_A of the Boolean matrix A as follows : the vertices of G_A correspond to the columns of A and two vertices k and j, corresponding to columns A^k, A^j are joined by an edge if the scalar product $A^kA^j \geq 1$, that is, if the two columns A^k and A^j have a non-zero component in the same row, say $a_{ik} = a_{ij} = 1$. Let A_G be the incidence matrix of the intersection graph G_A. It is easily checked that $M^{\leq}_{\overline{Z}}(A,e) = M^{\leq}_{\overline{Z}}(A_G,e)$. Thus, the packing problem for an arbitrary incidence matrix A is equivalent to the packing problem for the vertices of the intersection graph G_A. One of the first results related to the problem of constructing the packing polytope is the following theorem due to Fulkerson (1971).

A *clique* in a graph is a maximal complete subgraph.

Theorem 5.2 *The inequality*

$$\sum_{j \in K} x_j \leqq 1 \qquad K \subseteq N_n \tag{5.3}$$

defines a facet of the n-*packing polytope* conv $M^{\leq}_{\overline{Z}}(A,e)$ *if and only if* K *is the set of vertices of some clique of the intersection graph of the matrix* A.

Proof (i) Sufficiency. Let K be the set of vertices of a clique in G_A. Then, in the graph G_A for any $i,j \in K$ there is an edge (i,j). Thus, inequality (5.3) is satisfied for all $x \in M^{\leq}_{\overline{Z}}(A,e)$. We show that the dimension of the face (5.3) equals $n-1$. Consider $|K|$ points $x^i = e^i$, $i \in K$, and $n-|K|$ points $x^{i,j}$, $i \in K$, $\forall j \in N_n \setminus K$, $(i,j) \notin E$ for each of which the components $x_i = x_j = 1$ and the other components are zero. The vertex i exists since, by definition, a clique is a maximal complete subgraph. The points so constructed satisfy (5.3) as an equality and are linearly independent.

(ii) Necessity. Note first, that the subgraph generated by the vertices of K is complete. Suppose that this subgraph is not a clique of the graph G_A. Then we can suppose that the vertices of the set $K \cup \{i\}$ form a complete subgraph. Then the inequality $\sum_{j \in K \cup \{i\}} x_j \leqq 1$ is satisfied as an equality at all points which satisfy (5.3) as an equality and by at least one extra point $x = e_i$. Thus (5.3) cannot be an $(n-1)$-face. //

The *clique matrix of a graph* is the incidence matrix of graph vertices and cliques, that is $a_{ij} = 1$ if the vertex v_j belongs to the clique K_i and $a_{ij} = 0$ otherwise. It follows from Theorem 5.2 that for the polytope $M^{\leq}(A,e)$ to be integral, assuming the irreducibility of the system, it is necessary that the matrix A coincide with the clique matrix of its intersection graph. If the defining system of the polytope $M^{\leq}(A,e)$ contains redundant constraints, then, in addition to rows of A corresponding to the cliques of the intersection graph, there may be rows which dominate them (in the vector sense). We will describe a class of matrices A for which these conditions are also sufficient.

5.3 Perfect Graphs

The maximum number of elements in a clique of a graph G is called the *plumpness* of G, denoted by $\omega(G)$. The graph G is called *perfect* if, for any subgraph G' induced by G, the chromatic number $\chi(G')$ equals the plumpness $\omega(G')$.

Let $\overline{G}$ be the graph which has the same vertices as G and which is such that two vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in G. Then $\overline{G}$ is called the *complement* of G.

Note that since every inner stable set of G corresponds to a clique of $\overline{G}$ and conversely, then $\alpha(G) = \omega(\overline{G})$.

Theorem 5.3 *Let* A *be the clique matrix of a graph* G. *The following statements are equivalent :*

(1) G *is a perfect graph;*

(2) *The relaxed packing polytope* $M^{\leq}(A,e)$ *is integral;*

(3) $\overline{G}$ *is a perfect graph.* (Berge's Conjecture)

Proof We follow the scheme (1)⇒(2)⇒(3)⇒(2)⇒(1). The equivalence of (1) and (2) was proved by Fulkerson (1971) and the equivalence of (1) and (3) by Lovász (1972). Our proof follows that of Lovász (1979).

(1)⇒(2). Let G be a perfect graph and let A be its clique matrix. Let x be a vertex of the polytope $M^{\leq}(A,e)$. The components of x are clearly rational numbers so that there is an integer k such that the vector $kx = (p_1,\dots,p_n)$ is integral. Let v_i be any vertex of the perfect graph G for which $p_i > 0$. Replace each such vertex v_i by the complete graph K_{p_i} and join each vertex of K_{p_i} by means of an edge with each of the vertices of G which were adjacent to

v_i . It is easily verified that the graph G' so obtained is perfect. Let K' be a clique of the graph G' and let K be the subgraph of G which corresponds to K' . Then $|K'| \leqq \sum_{i\varepsilon K} p_i = k \sum_{i\varepsilon K} x_i \leqq k$. Since G' is a perfect graph, $\chi(G') = \omega(G') \leqq k$. Let $\{V_1', \ldots, V_k'\}$ be a colouring of the vertices of G' in k colours, that is, all the vertices in a subset V_i' are all given the same colour. Let V_i be the subset of vertices of G corresponding to V_i' and let y^i be its characteristic vector. Each vertex v_k belongs to exactly p_k of the subsets V_i . Therefore

$$k^{-1} \sum_{i=1}^{k} y^i = x . \tag{5.4}$$

Since x is a vertex of the polytope $M^{\leqq}(A,e)$ and $y^i \varepsilon M^{\leqq}(A,e) \ \forall i \varepsilon N_k$, equations (5.4) can hold only if $x = y^1 = \ldots = y^k$, that is, x is an integral point.

(2)$\Rightarrow$(3). Let $M^{\leqq}(A,e)$ be an integral polytope. It is clear that each of its faces is also an integral polytope. Thus, it suffices to show that if $M^{\leqq}(A,e)$ is an integral polytope then $\chi(\overline{G}) = \omega(\overline{G})$. Since $M^{\leqq}(A,e)$ is an integral polytope and A is the clique matrix of the graph G , we have

$$\alpha(G) = \max \{ex : x \varepsilon M^{\leqq}(A,e)\} . \tag{5.5}$$

All optimal solutions of problem (5.5) belong to the face F generated by the supporting hyperplane $\sum_{i=1}^{n} x_i = \alpha(G)$. The face F is the intersection of a certain collection of facets, among which there must be at least one facet generated by the hyperplane : $\sum_{i\varepsilon K} x_i = 1$, where K is some clique of G . The clique K has exactly one vertex in common with every inner stable set of maximum cardinality $\alpha(G)$. Hence $\alpha(G \setminus K) = \alpha(G) - 1$. Continuing in this manner we see that there are $\alpha(G)$ cliques which form a covering of all the vertices in the graph G . This is evidently the smallest number of cliques which cover the vertices of G . We call this the *clique number* and denote it by $\theta(G)$. Thus, we have shown that $\alpha(G) = \theta(G)$. Every clique in the graph G corresponds to an inner stable set in the complementary graph $\overline{G}$ and conversely. Hence $\chi(\overline{G}) \leq \theta(G)$ and $\alpha(G) = \omega(\overline{G})$. But also $\chi(\overline{G}) \geqq \omega(\overline{G})$ and consequently $\chi(\overline{G}) = \omega(\overline{G})$ which was to be proved.

The implications $(3)\Rightarrow(2)\Rightarrow(1)$ are proved analagously by replacing the perfect graph G by the perfect graph $\bar{G}$. //

The following characterization of perfect graphs follows from the proof of Theorem 5.3.

Corollary 5.4 *The graph G is perfect if and only if the clique number $\theta(G')$ of any subgraph G' of G is equal to its inner stability number $\alpha(G')$.*

A characterization of the class of integral matrices $M^{\leq}(A,e)$ in terms of forbidden submatrices has been proposed by Padberg (1973, 1974, 1979).

Let B be a non-degenerate Boolean $(m\times k)$-matrix $(m\geq k)$.

Definition 5.2 We say that the matrix B has *property* $\pi_{\beta,k}$ if the following conditions are satisfied : 1) B contains a non-singular $(k\times k)$-submatrix B' such that the sum of the elements in every line of B is equal to β ; 2) every row which is not included in B' either has an element sum which is strictly less than β or is identical with one of the rows of B' .

Theorem 5.5 *The polytope $M^{\leq}(A,e)$ is integral if and only if the matrix A does not contain an $(m\times k)$-submatrix B with property $\pi_{\beta,k}$ with $\beta \geqq 2$, $3 \leqq k \leqq n$.*

The proof of the theorem is based on a characterization of the bases of the matrix (A,J_n) which generate the non-integral vertices of the polytope $M^{\leq}(A,e)$ (see Ex.16)

The conditions formulated in Theorem 5.5 for the relaxed packing polytope do not suffice to ensure the integrality of the relaxed covering polytope $M^{\geq}(A,e)$. Such conditions were obtained by Berge (1972). A Boolean matrix is called a *balanced matrix* if it does not contain any square submatrices of odd order such that the sum of the elements in each row and each column is equal to two. If A is a balanced matrix then $M^{\geq}(A,e)$ is an integral polytope (see problem 18).

Finally we present a well-known conjecture of graph theory about perfect graphs which is known as the Strong Berge Conjecture (Berge 1970). A *chordless cycle* of a graph G is a cycle each of whose vertices in G is incident to only two vertices of the cycle. Berge's conjecture states that a graph G is perfect if and only if neither it nor its complement contain any chordless cycles of odd length besides triangles. As yet, Berge's conjecture has only been proved for planar graphs (Tucker 1973).

A partial verification of Berge's conjecture is given by the following result proved by Padberg (1973).

<u>Theorem 5.6</u> *For every* $J \subseteq N_n$ *let*

$$T(J) = \{j \in N_n \setminus J : A^k A^j \geqq 1 \text{ for some } k \in J\}.$$

Then if in the intersection graph of the matrix A J *is* 1) *an odd chordless cycle other than a triangle, or* 2) *the complement of such a cycle, then there exist integers* β_j , $0 \leqq \beta_j \leqq s$, *such that*

$$\sum_{j \in J} x_j + \sum_{j \in T(J)} \beta_j x_j \leqq s$$

defines a facet of the polytope conv $M_Z^{\leqq}(A,e)$. *Here* $s = (|J|-1)/2$ *in case* 1) *and* $s=2$ *in case* 2).

5.4 The Matching Polytope

We recall that the *matching polytope* of a given graph G is the convex hull of the characteristic vectors of all matchings of the graph G (the sets of non-intersecting edges). Such a polytope will be denoted by $M(G)$. In other words $M(G) = \text{conv } M_Z^{\leqq}(A_G,e)$, where A_G is the incidence matrix of G (rows correspond to vertices and columns to edges). The polytope $M^{\leqq}(A_G,e)$ contains the matching polytope $M(G)$ but only coincides with it when G is a bi-partite graph (Corollary 4.5). Otherwise, as can be seen from the proof of theorem 4.1 , the polytope $M^{\leqq}(A_G,e)$ contains vertices with coordinates equal to $\frac{1}{2}$. By carefully examining the proof of theorem 4.1 we can establish the following fact.

<u>Proposition 5.7</u> *Let* $C_1,\ldots,C_p$ *be distinct odd cycles and let* P *be a matching in the subgraph generated by the vertices which do not occur in* C_i , $\forall i \in N_p$. *Then the vector* x *with components*

$$x_i = \begin{cases} 1/2, & \text{if } e_i \in C_1 \cup \ldots \cup C_p, \\ 1, & \text{if } e_i \in P, \\ 0 & \text{otherwise} \end{cases} \tag{5.6}$$

is a vertex of the polytope $M^{\leq}(A_G, e)$ *and all of its vertices take the form* (5.6).

Thus, in order to construct a set of inequalities which specify $M(G)$ we need to construct hyperplanes which cut off the vertices (5.6) with $p \geq 1$ from $M^{\leq}(A_G, e)$. Such a description of the convex hull $M(G)$ was first given by Edmonds (1970).

Theorem 5.8 *The matching polytope* $M(G)$ *of a graph* G *is given by the following set of inequalities :*

$$x_j \geq 0 \qquad \forall j \in N_n, \tag{5.7}$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq 1 \qquad \forall i \in N_m, \tag{5.8}$$

$$\sum_{j: e_j \in G(S)} x_j \leq \frac{|S|-1}{2} \qquad \forall S \subseteq V,\ |S|\text{-odd}. \tag{5.9}$$

Proof We note first that the characteristic vector of every matching in the graph G satisfies inequalities (5.7)-(5.9). We will show further that every facet of the n-polytope is generated by one of the supporting hyperplanes whose equation is obtained by converting one of the inequalities (5.7)-(5.9) into an equality. Thus, let the hyperplane H with equation

$$\sum_{j=1}^{n} a_j x_j = b, \tag{5.10}$$

generate the facet F of $M(G)$ and let $M(G)$ lie in the half-space H^-.

Case 1. Let there be an index j_0 such that $a_{j_0} < 0$. Then every matching x which belongs to F must satisfy $x_{j_0} = 0$, otherwise H would not be a supporting hyperplane. Consequently, dim $F < n-1$ so that case 1 is impossible.

Case 2. Let $a_j \geq 0$ for all j and let there exist a vertex v_i of G such that every matching lying in F contains an edge incident to v_i. Then every such matching satisfies the equation

$$\sum_{j=1}^{n} a_{ij} x_j = 1, \tag{5.11}$$

which corresponds to one of the inequalities (5.8).

Case 3. Let $a_j \geqq 0$ $\forall j \varepsilon N_n$ and for each vertex of G let there exist a matching which does not contain it but which belongs to the facet F. Let G' be the graph formed by the edges e_j for which $a_j > 0$. We will assume that G' is connected, otherwise our constructions are to be carried out on each connected component of G'. We show that every matching which belongs to F fails to cover exactly one vertex of G'. Suppose that $\mathcal{P}_1$ is a matching belonging to F which does not cover vertices $u, v \varepsilon V$. We use induction on the distance between u and v. If u and v are adjacent in G' we have an immediate contradiction. If u and v are not adjacent, choose a vertex z on the shortest chain connecting u and v. Let $\mathcal{P}_2$ be a matching belonging to F which does not cover z. By the inductive assumption $\mathcal{P}_1$ covers z and $\mathcal{P}_2$ covers both u and v. Consider the two connected components of the graph $\mathcal{P}_1 \cup \mathcal{P}_2$ which contain the vertices u and v respectively. These will be chains. One of them, say chain C, does not contain z. Let

$$\mathcal{P}' = (\mathcal{P}_1 - (\mathcal{P}_1 \cap C)) \cup (\mathcal{P}_2 \cap C)$$

$$\mathcal{P}'' = (\mathcal{P}_2 - (\mathcal{P}_2 \cap C)) \cup (\mathcal{P}_1 \cap C) .$$

Then $\mathcal{P}'$ and $\mathcal{P}''$ are matchings which satisfy

$$2b \geqq \sum_{i:e_i \varepsilon \mathcal{P}'} a_i x_i + \sum_{i:e_i \varepsilon \mathcal{P}''} a_i x_i = \sum_{i:e_i \varepsilon \mathcal{P}_1} a_i x_i + \sum_{i:e_i \varepsilon \mathcal{P}_2} a_i x_i$$
$$= 2b .$$

Thus the matching $\mathcal{P}''$ lies in the facet F and does not cover z and either u or v. This contradiction shows that the graph G' has an odd number of vertices and that every matching belonging to F satisfies the equation

$$\sum_{i:e_i \varepsilon G(S)} x_i = \frac{|S|-1}{2} \tag{5.12}$$

where S is the set of vertices of the graph G'. Thus equations (5.12) and (5.9) will be identical. //

§6 <u>POLYMATROIDS</u>

In this section we study a special class of integral polytopes - the polymatroids. The simplicity of construction of the face complex of a polymatroid enables one to solve efficiently the problems of maximizing (minimizing) linear and convex functions on the set of integral points of a polymatroid. Polymatroids were introduced by Edmonds (1970) who also obtained the basic results about their structure.

6.1 Submodular Functions

We introduce a partial ordering on the set E_n^+ by defining the order $x \leqq y$ to mean the coordinatewise inequalities $x_i \leqq y_i$, $\forall i \in N_n$. Let $D \subseteq E_n^+$. The element $x^0 \in D$ is called a *minimal (maximal) element of the partially ordered set* D if there is no distinct element $x \in D$ such that $x \leqq x^0$ $(x \geqq x^0)$. Let $x, y \in E_n^+$, then $x \vee y$ denotes the vector with coordinates $\max(x_i, y_i)$ and the symbol $x \wedge y$ is the vector with coordinates $\min(x_i, y_i)$.

Definition 6.1 A *bounded polymatroid* in E_n^+ is a polytope M with the properties : (1) if $0 \leqq y \leqq x$, $x \in M$ then $y \in M$; (2) for any vector $a \in E_n^+$, all maximal elements of the set $M_a = \{x \in M : x \leqq a\}$ have the same component sum.

A maximal element of the set M_a is called a *base of the vector* a, and the sum of its components is called the *rank* of the vector, denoted by $r(a)$. The function $r(a)$, defined on E_n^+, is called the *rank function of the polymatroid*.

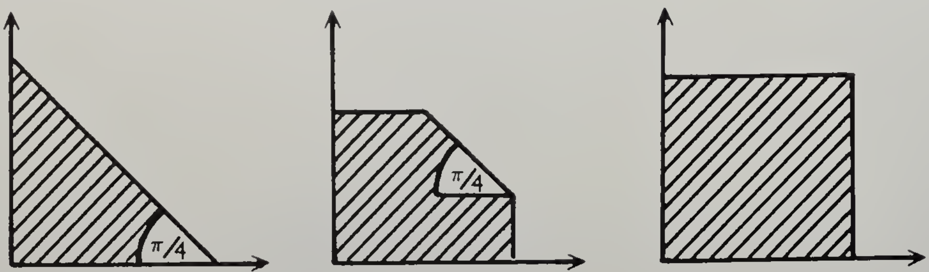


Fig. 30.

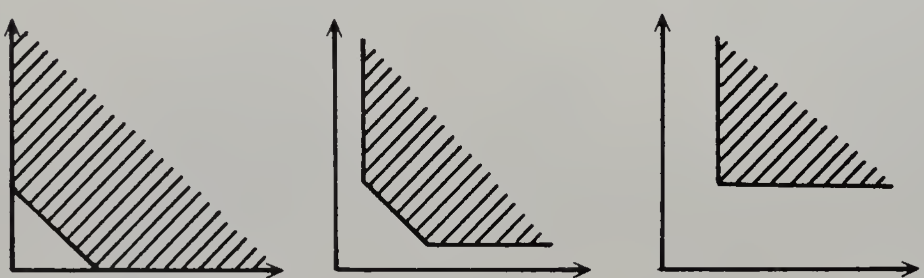


Fig. 31.

Definition 6.2 An *unbounded polymatroid* in E_n^+ is a polyhedron Q with the properties : (1) if $y \geqq x$ and $x \varepsilon Q$, then $y \varepsilon Q$; (2) for any vector $a \varepsilon E_n^+$, every minimal vector in the set $Q_a = \{x \varepsilon Q : x \geqq a\}$ has the same component sum, called the rank $r(a)$ of the vector a.

Bounded and unbounded polymatroids are shown in E_2^+ in Figures 30 and 31 respectively.

We investigate how polymatroids may be described by means of linear inequalities.

Definition 6.3 A real function ρ defined on 2^{N_n} is called *submodular* if it satisfies the inequality

$$\rho(U) + \rho(V) \geqq \rho(U \cup V) + \rho(U \cap V) \qquad \forall\, U,V \subseteq N_n$$

and *supermodular* if the reverse inequality is satisfied.

Theorem 6.1 *The polytope* $M \subset E_n^+$ *is a bounded polymatroid if and only if there exists a non-decreasing submodular function* $\rho(\omega)$ *defined on* 2^{N_n}, *with* $\rho(\emptyset) = 0$, *such that* $M = M(\rho)$, *where*

$$M(\rho) = \left\{x \varepsilon E_n^+ : \sum_{i \varepsilon \omega} x_i \leqq \rho(\omega) \quad \forall \omega \subseteq N_n\right\} . \tag{6.1}$$

Proof (i) Necessity. Let M be a polymatroid with rank function r. Define the function $\rho(\omega)$ by the rule $\rho(\emptyset) = 0$ and $\rho(\omega) = \max\left\{\sum_{i \varepsilon \omega} x_i : x \varepsilon M\right\}$. It is clear that $\rho(\omega)$ is a non-decreasing, non-negative function. We show that $\rho(\omega)$ is a submodular function. For $U,V \subseteq N_n$, let u and v be defined by

$$u_j = \begin{cases} \rho(j), & \text{if } j \varepsilon U, \\ \rho(N_n), & \text{if } j \notin U, \end{cases}$$

$$v_j = \begin{cases} \rho(j), & \text{if } j \varepsilon V, \\ \rho(N_n), & \text{if } j \notin V. \end{cases}$$

It is easily seen that $\rho(U) = r(u)$, $\rho(V) = r(v)$, $\rho(U \cup V) = r(u \vee v)$ and $\rho(U \cap V) = r(u \wedge v)$. To show that $\rho(\omega)$ is submodular, it suffices to show that

$$r(u) + r(v) \geqq r(u \vee v) + r(u \wedge v) . \tag{6.2}$$

Let a be a base of the vector $u \wedge v$. Then, from the definition of a polymatroid it follows that there is a vector $b \varepsilon M$ such that $a \leqq b \leqq u \vee v$ with $r(b) = \sum_{i=1}^{n} b_i = r(u \vee v)$. Hence $a = b \wedge (u \wedge v)$ which gives $a + b = (b \wedge u) + (b \wedge v)$. But $b \wedge u$, $b \wedge v \ \varepsilon\ M$ and $b \wedge u \leqq u$, $b \wedge v \leqq v$. Hence

$$r(u \wedge v) + r(u \vee v) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} (b \wedge u)_i + \sum_{i=1}^{n} (b \wedge v)_i$$

$$\leqq r(u) + r(v) .$$

This proves inequality (6.2). It remains to show that $M = M(\rho)$. From the definition of $\rho(\omega)$ it follows that $M \subseteq M(\rho)$. We show that $M(\rho) \subseteq M$. Let $x \varepsilon M(\rho)$ but $x \notin M$. Then choose a base u of the vector x with the largest number of components u_i less than x_i . Let $w = (u+x)/2$ and let $J(u) = \{i \varepsilon N_n : u_i < x_i\}$. Then $u \leqq w \leqq x$. Clearly, u is also a base of the vector w . Then $r(x) = r(w)$ and every base w is also a base of the vector x . For $\omega \subseteq N_n$ define the vector x^{ω} with coordinates $x_i^{\omega} = x_i$, $\forall i \varepsilon \omega$ and $x_i^{\omega} = 0$, $\forall i \varepsilon N_n \setminus \omega$. Since

$$\sum_{i \varepsilon J(u)} u_i < \sum_{i \varepsilon J(u)} w_i \leqq \rho(J(u)) ,$$

the vector $u \wedge x^{J(u)}$ cannot be a base of the vector $w \wedge x^{J(u)}$. Extending it to a base $w \wedge x^{J(u)}$, we obtain a base $\hat{u}$ of the vector w . Since

$$\sum_{i \varepsilon J(u)} \hat{u}_i > \sum_{i \varepsilon J(u)} u_i \quad , \quad \sum_{i=1}^{n} \hat{u}_i = \sum_{i=1}^{n} u_i ,$$

then $\hat{u}_i < u_i$ for some $i \notin J(u)$. The last inequalities contradict the choice of the vector u as a vector of maximal cardinality in the set $J(u)$. Hence, $M = M(\rho)$.

(ii) Sufficiency. Let the polytope $M(\rho)$ be given by a submodular non-decreasing function $\rho(\omega)$, with $\rho(\emptyset) = 0$. Property (1) of Definition 6.1 is clearly satisfied. Suppose that for some vector $z \ \varepsilon\ E_n^+$ there are bases u,v for which property (2) does not hold, i.e. $r(u) < r(v)$. Let $J(u) = \{i \varepsilon N_n : u_i < v_i\}$. Then for $e \ \varepsilon\ J(u)$ there is a set $\omega_e \subseteq N_n$ such that $e \varepsilon \omega_e$ and

$$\sum_{i \varepsilon \omega_e} u_i = \rho(\omega_e) . \tag{6.3}$$

Let ω be a maximal subset N_n with the property (6.3). By the submodularity of $\rho(\omega)$ it follows that

$$\sum_{i\varepsilon\omega\cup\omega_e} u_i = \rho(\omega\cup\omega_\varepsilon) .$$

From this it is clear that if $e\notin\omega$, then ω is not a maximal set with property (6.3). Hence $e\varepsilon\omega$. Since e is an arbitrary element of $J(u)$, $J(u)\subset\omega$. But then $\rho(\omega) = \sum_{i\varepsilon\omega} u_i < \sum_{i\varepsilon\omega} v_i$, and this contradicts the fact that $v \varepsilon M(\rho)$. //

If the function $\rho(\omega)$ is submodular, the function $\rho'(\omega) = \rho(N_n) - \rho(N_n\setminus\omega)$ is supermodular. Thus, the set

$$Q(\rho') = \{x\varepsilon E_n : \sum_{i\varepsilon\omega} x_i \geqq \rho'(\omega) \ \forall\omega\subseteq N_n\}$$

is an unbounded polymatroid if and only if the set $M(\rho)$ is a bounded polymatroid.

Theorem 6.2 *The polyhedron* $Q\subseteq E_n^+$ *is an unbounded polymatroid if and only if there is a non-decreasing supermodular function* $\rho'(\omega)$, $\rho'(\emptyset)=0$, *such that* $Q = Q(\rho')$.

6.2 Vertices of a Polymatroid

For every permutation $(\pi_1,\ldots,\pi_n) \varepsilon S_n$ we define the sets $\omega_\pi^0 = \emptyset$, $\omega_\pi^s = \{\pi_1,\ldots,\pi_s\}$ $\forall s\varepsilon N_n$.

Theorem 6.3 *The point* x *is a vertex of the (bounded or unbounded) polymatroid* $M(\rho)$ *if and only if there is a permutation* $\pi \varepsilon S_n$ *and an integer* $0 \leqq k \leqq n$ *such that the components of* x *are given by*

$$\begin{aligned} x_{\pi_s} &= \rho(\omega_\pi^s) - \rho(\omega_\pi^{s-1}) && \forall s\varepsilon N_k, \\ x_{\pi_s} &= 0 && \forall s\varepsilon N_n\setminus N_k . \end{aligned} \tag{6.4}$$

Proof It suffices to prove that for an arbitrary vector $c = (c_1,\ldots,c_n)$ the maximum in the linear programming problem

$$\max \sum_{i=1}^n c_i x_i , \tag{6.5}$$

$$\sum_{i\varepsilon\omega} x_i \leqq \rho(\omega) \qquad \forall\,\omega \subseteq N_n, \tag{6.6}$$

$$x_i \geqq 0 \qquad \forall\, i\varepsilon N_n \tag{6.7}$$

is attained at a point given by (6.4) (see Problem 1, Ch.1).

Consider the problem

$$\min \sum_{\omega \subseteq N_n} \rho(\omega) y_\omega, \tag{6.8}$$

$$y_\omega \geqq 0 \qquad \forall\,\omega \subseteq N_n \tag{6.9}$$

$$\sum_{\substack{\omega \subseteq N_n \\ i\varepsilon\omega}} y_\omega \geqq c_i \qquad \forall\, i\varepsilon N_n \tag{6.10}$$

which is the dual of (6.5)-(6.7) and where the sum in (6.10) is taken over all subsets $\omega \subseteq N_n$ which contain the number i. Let $(\pi_1,\dots,\pi_n)$ be a permutation with the property $c_{\pi_1} \geqq \dots \geqq c_{\pi_k} > 0 \geqq c_{\pi_{k+1}} \geqq \dots \geqq c_{\pi_n}$. Define the components of the vector y^* by

$$y^*_{\omega^s_\pi} = c_{\pi_s} - c_{\pi_{s+1}} \qquad \forall\, s\varepsilon N_{k-1},$$

$$y^*_{\omega^k_\pi} = c_{\pi_k},$$

$$y^*_\omega = 0, \quad \omega \neq \omega^i_\pi \qquad \forall\, i\varepsilon N_k.$$

It can be verified directly that y^* satisfies the constraints (6.9),(6.10) and that

$$\sum_{\omega \subseteq N_n} \rho(\omega) y^*_\omega = \sum_{i=1}^{n} c_i x^*_i$$

where the components of the vector x^* are given by (6.4). By the duality theorem, the vectors x^* and y^* are optimal solutions of the primal and dual programmes respectively. This proves the theorem. //

If the function $\rho(\omega)$ on 2^{N_n} takes only integral values, then the polytope $M(\rho)$, is an integral polymatroid.

6.3 The Facets of a Polymatroid

For definiteness we suppose that $M = M(\rho)$ is a bounded polymatroid. The extension to the case of an unbounded polymatroid is obvious.

Definition 6.4 The subset $\omega^0 \subset N_n$ is called *ρ-closed* if for all $\omega \supset \omega^0$, $\omega \subseteq N_n$, we have $\rho(\omega^0) < \rho(\omega)$. The subset $\omega^0 \subseteq N_n$ is *ρ-separable* if $\rho(\omega^0) = \rho(\omega_1^0) + \rho(\omega_2^0)$ where $\omega_1^0 \cup \omega_2^0 = \omega^0$, $\omega_1^0 \cap \omega_2^0 = \emptyset$. Otherwise ω^0 is *ρ-nonseparable.*

It may easily be verified that the polymatroid $M(\rho)$ in E_n^+ has dimension n if and only if the empty set is ρ-closed. If the dimension of a polytope M in E_n is equal to n , there is a unique irreducible system of linear inequalities such that M is its solution set. Each of these inequalities determines a facet of M .

Theorem 6.4 *Let the empty set be ρ-closed for the function* $\rho(\omega)$. *Then, the facets of the* n-*polymatroid* $M(\rho)$ *are sets of the type*

$$F_j = \{x \varepsilon M(\rho) : x_j = 0\} ,$$

for any $j \varepsilon N_n$ *and of the type*

$$F_\omega = \{x \varepsilon M(\rho) : \sum_{i \varepsilon \omega} x_i = \rho(\omega)\}$$

for any ρ-closed and ρ-nonseparable subset $\omega \subseteq N_n$.

6.4 Intersections of Polymatroids

The following theorem describes a class of integral polytopes whose constraint matrix is not absolutely unimodular.

Theorem 6.5 *Let* M_1 *and* M_2 *be two integral polymatroids in* E_n^+ . *Then the polytope* $M_1 \cap M_2$ *is integral.*

Proof We show first that if the vector $x^0 \varepsilon M(\rho)$ satisfies the equation

$$\sum_{i \varepsilon U} x_i^0 = \rho(U) \quad , \quad \sum_{i \varepsilon V} x_i^0 = \rho(V) ,$$

then either $U \cap V = \emptyset$, or

$$\sum_{i \varepsilon U \cap V} x_i^0 = \rho(U \cap V) . \tag{6.11}$$

Suppose for definiteness that $M(\rho)$ is a bounded polymatroid, that is, ρ is a submodular function. From the submodularity of ρ it follows that

$$\rho(U\cup V) + \rho(U\cap V) \leqq \rho(U) + \rho(V) = \sum_{i\in U} x_i^0 + \sum_{i\in V} x_i^0$$

$$= \sum_{i\in U\cup V} x_i^0 + \sum_{i\in U\cap V} x_i^0 \leqq \rho(U\cup V) + \rho(U\cap V) .$$

The equality (6.11) follows.

Let x^0 be an arbitrary vertex of the polytope $M_1 \cap M_2$, where $M_i = M(\rho_i)$, $i=1,2$. Then its nonzero coordinates are solutions of the system of equations

$$\sum_{i\varepsilon\omega} x_i = \rho_1(\omega) \qquad \forall \omega\varepsilon V_1, \tag{6.12}$$

$$\sum_{i\varepsilon\omega} x_i = \rho_2(\omega) \qquad \forall \omega\varepsilon V_2, \tag{6.13}$$

where V_1, V_2 are families of subsets of N_n. By the property proved above, any two subsets in V_i are either disjoint or their intersection also lies in V_i. Thus, if A is the matrix of coefficients of the system (6.12),(6.13) then, by subtracting suitable rows from other rows in each group V_i, we can obtain a matrix A' of the type described in statement (3) of Theorem 4.1. Thus A' is absolutely unimodular. Thus, system (6.12),(6.13) has an integral solution. //

We remark that the intersection of three or more integral polymatroids may have non-integral vertices.

6.5 Matroid Polytopes

The theory of matroids generalizes many results of graph theory, projective geometry and the theory of electrical networks. Many different optimization problems, and especially optimization problems on networks, can be formulated as extremal problems on matroids. We will show that most constraint polytopes for extremal problems on matroids are integral.

Definition 6.5 A *matroid* $\mathcal{M}$ is a pair $(J,\mathcal{F})$ in which J is a finite set and $\mathcal{F}$ is a family of subsets of J, called *independent sets*, having the properties : 1) any subset of an independent set is independent; 2) if ω is any subset of J, then all independent sets contained in ω, which are maximal with respect to inclusion, have the same number of elements.

Independent sets in J which are maximal with respect to inclusion are called *bases* of the matroid. The *rank* $r(\omega)$ of the set $\omega \subseteq J$ is the (unique) cardinality of a maximal independent subset of ω.

We leave it to the reader to show the equivalence of definition 6.4 and of definition 4.5 of Chapter 1.

We give some examples of the most important types of matroids. In §4 of Chapter 1 we studied the so-called *vector matroid* in which the set J consisted of the column vectors of a matrix A and $\mathcal{F}$ consisted of all linearly independent subsets of these vectors.

Let J be the set of edges of a graph G and let $\mathcal{F}$ consist of subsets of the edges which constitute an acyclic subgraph of G (*forests*). The pair $\mathcal{M} = (J,\mathcal{F})$ is a matroid, called a *graphical matroid*. The bases of a graphical matroid are the maximal forests, or, when G is a connected graph, the maximal trees.

Let the finite set J be partitioned into m distinct subsets $E_1,\dots,E_m$ and let a non-negative integer d_i be associated with each of these sets. Consider the family $\mathcal{F} \subset 2^J$ such that every $I \in \mathcal{F}$ contains not more than d_i elements of the set E_i, $\forall i \in N_m$. The pair $\mathcal{M} = (J,\mathcal{F})$ is a matroid called a *partition matroid*. In the case $d_i = 1$ $\forall i \in N_m$, the partition matroid is called a *transversal matroid* and the independent sets are called *partial transversals* and the bases are called *systems of distinct representatives.*

A *matching matroid* is defined on the set of vertices of a given graph. Here, the independent sets are those subsets of vertices for which there exists a complete matching in G.

Let $\mathcal{M} = (J,\mathcal{F})$ be a matroid and let $r(\omega)$ be its rank function. From the definition of a matroid it follows that $r(\emptyset) = 0$ and that $r(\omega)$ is a non-decreasing function.

We show that the rank function of a matroid is submodular :

$$r(U \cup V) + r(U \cap V) \leq r(U) + r(V) .$$

Let $\omega_{U\cap V}$ be a maximal independent subset of $U \cap V$. Since $\omega_{U\cap V}$ is an independent subset of U, it can be extended to a maximal independent subset ω_U of U. Similarly, ω_U may be extended to a maximal independent subset $\omega_{U\cup V}$ of $U \cup V$. Since the set $\omega_{U\cap V} \cup (\omega_{U\cup V} \setminus \omega_U)$ is an independent subset of V, it follows that

$$r(V) \geqq r(\omega_{U\cap V} \cup (\omega_{U\cup V} \setminus \omega_U))$$

$$= |\omega_{U\cap V}| + |\omega_{U\cup V}| - |\omega_U| = \rho(U\cap V) + \rho(U\cup V) - \rho(U) ,$$

which was to be proved. Hence, by Theorem 6.1 , the polytope

$$M(r) = \{x\varepsilon E_n^+ : \sum_{i\varepsilon\omega} x_i \leqq r(\omega) \quad \forall \omega \subseteq N_n\} , \quad n=|J|$$

is a polymatroid. We call this the *polytope of the matroid* $\mathcal{M}$.

Theorem 6.6 *The vertices of a matroid polytope are precisely the characteristic vectors of the independent sets of the matroid.*

Proof Let x^0 be the characteristic vector of any independent set F of a matroid $\mathcal{M}$. Then we have

$$\sum_{i\varepsilon\omega} x_i^0 \leqq r(\omega) \qquad \forall \omega \subseteq N_n,$$

since the set $\omega\cap F$ is an independent subset and so

$$\sum_{i\varepsilon\omega} x_i^0 = r(\omega\cap F) \leqq r(\omega) .$$

A vertex of the polytope $M(r)$ is the unique solution of a subsystem of rank n , obtained by replacing some of the inequalities defining $M(r)$ by equalities. It is easily seen that the vertex x^0 is a solution of the following system of rank n :

$$x_i = 0 \qquad i\notin F,$$

$$x_i = r(i) \qquad i\varepsilon F.$$

Thus, the vector x^0 is a vertex of the matroid polytope $M(r)$.

The converse follows from Theorem 6.3 about the characterization of the vertices of the polymatroid $M(r)$ and from the obvious property of the rank function : $r(i) = 1$ if $\{i\}\varepsilon\mathcal{F}$. //

From Theorems 6.5 and 6.6 we obtain the following description of sets which are independent in both of two given matroids.

<u>Theorem 6.7</u> *Let* M_1 *and* M_2 *be the polytopes defined by the matroids* $\mathcal{M}_1 = (J,\mathcal{F}_1)$ *and* $\mathcal{M}_2 = (J,\mathcal{F}_2)$. *Then the vertices of the polytope* $M_1 \cap M_2$ *are precisely the characteristic vectors of all sets which are independent in both* $\mathcal{M}_1$ *and* $\mathcal{M}_2$.

In particular, it follows from Theorem 6.7 that the vertices of the polytope

$$\sum_{i\varepsilon\omega} x_i \leqq r(\omega) \quad \forall \omega \subset N_n \ , \quad \sum_{i=1}^{n} x_i = r(N_n)$$

are in one-to-one correspondence with the bases of the matroid $\mathcal{M} = (J,\mathcal{F})$ with rank function $r(\omega)$.

6.6 <u>Duality Theorems</u>

The dual polytope to an integral polytope which is the intersection of two polymatroids is not, in general, integral. However, the following theorem shows that the linear programming problem which is dual to the problem on $M_1 \cap M_2$ has an integral optimum.

<u>Theorem 6.8</u> *Let* c_j *be integers,* $j \varepsilon N_n$, *and let* $r_1(\omega), r_2(\omega)$ *be the integral rank functions of the matroids* $\mathcal{M}_1, \mathcal{M}_2$. *Then the following dually related linear programming problems have optimal integral solutions* x^*, y^* *with* $cx^* = ry^*$:

$$(P) \quad \left\{ \begin{array}{ll} \max \sum_{j=1}^{n} c_j x_j & \\ \sum_{j\varepsilon\omega} x_j \leqq r_1(\omega) & \forall \omega \subseteq N_n, \\ \sum_{j\varepsilon\omega} x_j \leqq r_2(\omega) & \forall \omega \subseteq N_n, \\ x_j \geqq 0 & \forall j \varepsilon N_n. \end{array} \right. \tag{6.14}$$

$$(D) \quad \left\{ \begin{array}{ll} \min \sum_{\omega \subseteq N_n} (r_1(\omega)y_1(\omega) + r_2(\omega)y_2(\omega)) & \\ \sum_{j\varepsilon\omega\subseteq N_n} (y_1(\omega) + y_2(\omega)) \geqq c_j & \forall j \varepsilon N_n, \\ y_1(\omega) \geqq 0 \ , \ y_2(\omega) \geqq 0 & \forall \omega \subseteq N_n. \end{array} \right. \tag{6.15}$$

Proof It suffices to show that the constraint polytope given by (6.15) has an integral vertex at which ry is minimized. Then from the duality theorem and from theorem 6.5 on the integrality of the polytope given by the intersection of two polymatroids, we obtain the assertion of the theorem.

Let y^0 be an optimal solution of problem (6.15). Consider two linear programming problems for $s=1$ and $s=2$ as follows

$$\min \sum_{\omega \subseteq N_n} r_s(\omega) y_s(\omega) , \tag{6.16s}$$

$$\sum_{j\varepsilon\omega \subseteq N_n} y_s(\omega) \geqq c_j^s \quad \forall j\varepsilon N_n, \quad y_s(\varepsilon) \geqq 0 \quad \forall \omega \subseteq N_n,$$

where

$$c_j^s = \sum_{j\varepsilon\omega \subseteq N_n} y_s^0(\omega) .$$

Let y_s^* be an optimal solution in (6.16s). By theorem 6.3 there is a y_s^* such that the sets ω for which $y_s^*(\omega) \neq 0$ form a sequence

$$\omega_s^1 \subset \omega_s^2 \subset \ldots \tag{6.17}$$

Since y_s^0 satisfies the constraints (6.16s), we have $r_s y_s^* \leqq r_s y_s^0$ for $s=1,2$ so that $ry^* \leqq ry^0$. At the same time $c_j^1 + c_j^2 \geqq c_j$ for all $j\varepsilon N_n$. Consequently y^* satisfies the constraints (6.15) so that y^* is an optimal solution of the problem (6.15). Thus, there is an optimal solution of problem (6.15) satisfying condition (6.17). Its nonzero coordinates satisfy the system of equations

$$\sum_{j\varepsilon\omega_i} y(\omega_i^1) + \sum_{j\varepsilon\omega_i} y(\omega_i^2) = c_j \qquad \forall j\varepsilon N_n \tag{6.18}$$

The columns of the matrix of coefficients A of constraints (6.18) can be partitioned into two subsets V_1 and V_2 such that for any two columns A^q, A^p of one set we have either $A^q \geqq A^p$ or $A^p \geqq A^q$. Consequently (see Problem 11), the matrix A is absolutely unimodular so that y^* is an integral optimal solution of problem (6.15). //

When $c_j=1$ and $r_1(\omega), r_2(\omega)$ are rank functions of matroids, we obtain the following well known result from theorem 6.8.

Theorem 6.9 (Tutte, 1971). *Let* $\mathcal{M}_1 = (J,\mathcal{F}_1)$ *and* $\mathcal{M}_2 = (J,\mathcal{F}_2)$ *be matroids with rank functions* $r_1(\omega)$ *and* $r_2(\omega)$. *Then*

$$\max_{\omega\varepsilon\mathcal{F}_1\cap\mathcal{F}_2} |\omega| = \min_{\omega\subseteq J}\left[r_1(\omega) + r_2(J\setminus\omega)\right] .$$

Theorem 6.9 generalizes such dual assertions of combinatorial analysis as König's Theorem (Th.4.6), Hall's Theorem (Cor.4.14) and the Max-Flow – Min-Cut Theorem (Th.4.8).

Covering problems can be considered on an arbitrary matroid $\mathcal{M} = (J,\mathcal{F})$ in which the set to be covered is J and the covering sets are members of $\mathcal{F}$. Similarly, partitioning and packing problems can be considered on a matroid. We give some results which are essentially corollaries of theorem 6.8 and which are concerned with partitionings of a matroid.

Theorem 6.10 *Let* $\mathcal{M} = (J,\mathcal{F})$ *be a matroid with rank function* $r(\omega)$. *Then* J *can be partitioned into no more than* k *independent subsets if and only if* $|\omega| \leqq kr(\omega)$ *for any* $\omega\subseteq J$.

When the matroid is a graphical matroid, Theorem 6.10 is equivalent to the Nash-W lliams Theorem on the disection of a graph into non-intersecting forests. Theorem 6.10 has the following generalization due to Rado (Edmonds, 1970)

Theorem 6.11 *Let* $\mathcal{M}_i = (J,\mathcal{F}_i)$ *be matroids with rank functions* $r_i(\omega)$, $i\varepsilon N_k$. *Then the set* $I\subseteq J$ *can be partitioned into* k *subsets* I_i *such that* $I_i\varepsilon\mathcal{F}_i$ *if and only if for any subset* $\omega\subseteq I$ *we have*

$$|\omega| \leqq \sum_{i=1}^{k} r_i(\omega) .$$

§7 LOCALLY INTEGRAL POLYTOPES

The main aim of this section is to introduce classes of polytopes whose adjacent integral vertices remain adjacent after taking the convex hull of all the integral points of the polytope.

7.1 Quasi-integral Polytopes

Definition 7.1 A polytope M is called *quasi-integral* if every edge of conv M_Z is also an edge of M or, in other words, if the graph G (conv M_Z) is a subgraph of $G(M)$.

Quasi-integral polytopes are studied in Trubin (1969), Kovalev (1977), Kovalev et al. (1977,1978). An interest in studying such polytopes was aroused for the following reason : in solving an integer programming problem whose feasible set is the set of integral points of a quasi-integral polytope, we can use the simplex method with the following amendment. Since conv M_Z is an integral polytope, it is possible, using the standard simplex method, to attain the optimum by starting at any vertex and moving from vertex to vertex along the edges of conv M_Z. From the definition of a quasi-integral polytope, it follows that this path will also exist on the polytope M. Thus, our version of the simplex method, called the *integral simplex method*, consists of the following : choose any integral vertex as an initial basic feasible solution and, at each iteration step choose a new basic variable such that, 1) the cost function is reduced, and 2) the new basic solution is integral. The procedure terminates when there are no further candidates for a new basic variable satisfying conditions 1) and 2). In this way the optimal solution of the integer programming problem is obtained.

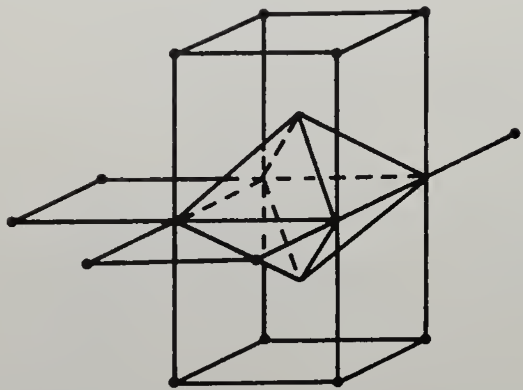


Fig. 32.

An example of a quasi-integral polytope is a polytope which has an integral face containing all of its integral points. Figure 32 shows a quasi-integral polytope whose integral points do not all lie in one face. Nevertheless, since the integral faces of the polytope M are faces of conv M_Z, the following is a sufficient (but not necessary, see Fig.32) condition for quasi-integrality.

Proposition 7.1 *Let* $M_Z \subseteq \text{vert } M$. *If, given any two integral vertices of the polytope* M *there is an integral face containing them, then* M *is a quasi-integral polytope.*

7.2 The Relaxed Partition Polytope

It was shown by Trubin (1969) that the relaxed partition polytope, introduced in §5, is quasi-integral. The relaxed partition polytope $M^{=}(A,e) \subset E_n$ is given by the constraints

$$Ax = e \quad , \quad x \geqq 0 \tag{7.1}$$

where A is a Boolean $(m \times n)$-matrix. Without loss of generality we may assume that A does not contain any zero columns or rows.

Theorem 7.2 *The relaxed partition polytope is quasi-integral.*

Proof Since A is a Boolean matrix without any zero columns, we have for any $x \in M^{=}(A,e)$ that $x_i \leqq 1 \ \forall i \in N_n$, that is, $M^{=}(A,e)$ is contained inside the (0,1)-cube. Thus, any integral point of $M^{=}(A,e)$ is a vertex of this cube and so, by Proposition 7.1, to prove the theorem it suffices to show that any two integral vertices x', x'' of $M^{=}(A,e)$ belong to an integral face. We partition the index set N_n into three mutually disjoint subsets

$$J_0 = \{j : x'_j = x''_j = 0\}, \quad J_1 = \{j : x'_j = x''_j = 1\}, \quad J_2 = \{j : x'_j \neq x''_j\}.$$

Consider the family of hyperplanes

$$x_j = 0 \qquad \forall j \in J_0, \tag{7.2}$$

$$x_j = 1 \qquad \forall j \in J_1, \tag{7.3}$$

each of which is supporting to the polytope $M^{=}(A,e)$. Thus, the set of points of $M^{=}(A,e)$ satisfying conditions (7.2) and (7.3) constitute a face (which may be all of $M^{=}(A,e)$, for example, if $J_0 \cup J_1 = \emptyset$). Denote this face by F . To show that F is integral it suffices to check that the matrix A^{J_2} is unimodular. The vector $x^0 = (x' + x'')/2 \in M^{=}(A,e)$. So

$$2 = \sum_{j\varepsilon J} a_{ij}(x'_j+x''_j) = \sum_{j\varepsilon J_2} a_{ij} ,$$

that is, every row of A^{J_2} contains exactly two elements equal to 1. Partition the columns of A^{J_2} into two disjoint sets : in the first, put those columns for which $x'_j = 1$ and in the second, put those for which $x''_j = 1$. Thus, the matrix A^{J_2} satisfies, after suitable transpositions, the conditions of Theorem 4.1 and is therefore absolutely unimodular. //

Corollary 7.3 *Let* D *be a non-negative, integral* (m×n)-*matrix. Then* $M(D,e) = \{x\varepsilon E_n : Dx=e , x\geqq 0\}$ *is a quasi-integral polytope.*

Proof Let J_0 be the set of indices of those columns of D which contain an element not less than two. Then, by Theorem 7.2 the face

$$F = \{x\varepsilon M(D,e) : x_i=0 \quad \forall i\varepsilon J_0\}$$

is a quasi-integral polytope. But a polytope which has a quasi-integral face containing all of its integral points is itself quasi-integral. //

7.3 The Simplest Location Problem

The problem is formulated as follows :

$$\min \sum_{i=1}^{m}\sum_{j=1}^{n}(c_{ij}x_{ij} + c_i y_i)$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall i\varepsilon N_m \qquad (7.4)$$

$$0 \leqq x_{ij} \leqq y_i \qquad \forall (i,j)\varepsilon N_m\times N_n \qquad (7.5)$$

$$y_i = 1,0 \qquad \forall i\varepsilon N_m \qquad (7.6)$$

In matrix form, the constraints (7.4),(7.5) take the form

$$A^*x \overset{=}{<} b^*,$$

where

$$x = (x_{11},\ldots,x_{m1},\ldots,x_{mn},y_1,\ldots,y_m), \quad b^* = (1,\ldots,1,0,\ldots,0),$$

$$A^* \;=\; \left(\begin{array}{c|ccc} J_m \; J_m \; \cdots \; J_m & & & \\ \hline & -e & & \\ J_{mn} & & -e & \\ & & & -e \end{array}\right)$$

Here J_k is a unit $(k\times k)$-matrix, $-e$ is a column vector of dimension n all of whose components are equal to -1. Denote by $M(A^*,b^*)$ the polytope of constraints (7.4),(7.5).

Theorem 7.4 *The polytope* $M(A^*,b^*)$ *for the simplest location problem is quasi-integral.*

All integral points of $M(A^*,b^*)$ are vertices. Thus, by Proposition 7.1, to prove the theorem it suffices to prove the following Lemma.

Lemma 7.5 *Any two integral vertices* x', x'' *of the polytope* $M(A^*,b^*)$ *belong to an integral face of the polytope.*

Proof Consider the face $F(x',x'')$ of the polytope given by the constraints (7.4),(7.5) and by $y_i = y_i'$ for those i for which $y_i' = y_i''$, and $x_{ij} = x_{ij}'$ for those i,j for which $x_{ij}' = x_{ij}''$.

After excluding the fixed variables and the constraints which are identities, we obtain a system

$$\bar{A}\,\bar{x} \leqq \bar{b},$$

$$\bar{x} \geqq 0,$$

defining the domain of variation of the non-fixed variables in the face $F(x',x'')$ (these variables make up the vector $\bar{x}$). The point $\bar{x}^0 = (\bar{x}'+\bar{x}'')/2$ augmented by the fixed components, belongs to the face $F(x',x'')$. Thus, in each of the first m rows of the matrix $\bar{A}$ there are exactly two non-zero elements which are equal to 1. In the remaining rows of $\bar{A}$ there are also exactly two nonzero elements, equal to 1 and -1. As in the proof of Theorem 7.2, we partition the columns of A into two subsets such that the conditions of Theorem 4.1 are satisfied, so that A is an absolutely unimodular matrix. Thus, $F(x',x'')$ is an integral polytope. This proves the Lemma and hence the Theorem. //

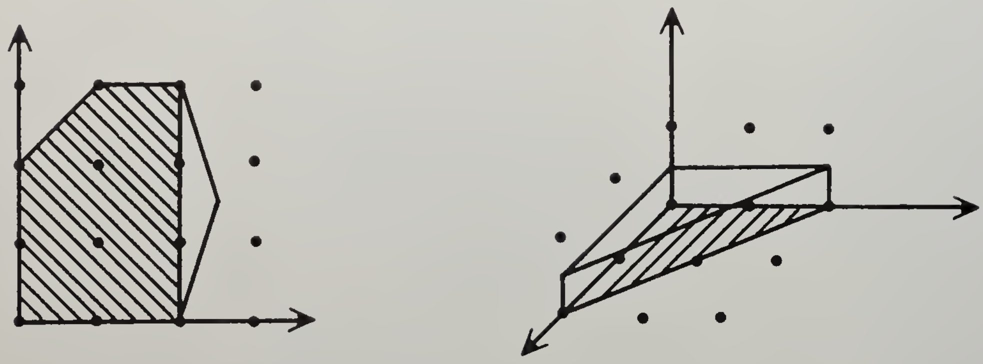
7.4 Connected Integral Polytopes

We study a class of polytopes which is wider than the class of quasi-integral polytopes but which, nevertheless, has properties which allow us to locate local extrema in integer programming problems by using the integer simplex method.

Definition 7.2 The polytope M is called *connected integral* if the subgraph of the graph $G(M)$ generated by its integral vertices is a spanning subgraph of the graph G (conv M_Z).

In other words, M is a connected integral polytope if : 1) vert conv $M_Z \subseteq$ vert M ; 2) the subgraph of the graph $G(M)$ generated by its integral vertices is a connected graph.

Note that if all the integral points of a polytope M are vertices of M, then condition 1) is automatically satisfied. Clearly, every quasi-integral polytope is connected integral (Fig.33b), but the converse is not generally true (Fig.33a). In both diagrams the polytope conv M_Z is shaded.



(a) (b)

Fig. 33.

Consider the polytope $M(A,e') = \{x\varepsilon E_n : Ax = e', x \geq 0\}$, where A is a Boolean matrix, $e'=(k,1,\ldots,1)$, k is an integer greater than or equal to 2. The polytope $M(A,e')$ is a generalization of the relaxed partition polytope.

Theorem 7.6 *If for every* k*-subset* w *of the set* $J_1 = \{j\varepsilon N_n : a_{1j}=1\}$ *the face* $F(w)=\{x\varepsilon M(A,e') : x_j=1 \quad j\varepsilon w\}$ *of the polytope* $M(A,e')$ *is non-empty, then* $M(A,e')$ *is connected integral.*

The proof uses the following two lemmas.

Lemma 7.7 *If the set* $F(w)$ *is not empty, then* $F(w)$ *is a quasi-integral polytope.*

Proof Putting $x_j=1$ $\forall j\varepsilon w$, $x_j=0$ $\forall j\varepsilon J_1\setminus w$ in the system $Ax=e'$ and removing those constraints which become identities, we obtain a system of constraints $\bar{A}x=\bar{e}$, $\bar{x}\geqq 0$, which, together with the fixed variables, determine the face $F(w)$. Here $\bar{A}$ is a Boolean matrix and $\bar{e}=(1,\ldots,1)$. By Theorem 7.2, $F(w)$ is a quasi-integral polytope. //

Lemma 7.8 *Let* $w',w'' \subset J_1$, $|w'\cap w''| = k-1$, *and let* $x'\varepsilon F(w')$ *and* $x''\varepsilon F(w'')$ *be integral vertices of the polytope* $M(A,e')$. *Then, the face* $F(w',w'') = \{x\varepsilon M(A,e') : x_j=1 \ \ \forall j\varepsilon w'\cap w''\}$ *is a quasi-integral polytope which contains the vertices* x',x'' .

Proof The face $F(w',w'')$ is given by the following system of constraints

$$x_j = 1 \qquad\qquad \forall j\varepsilon w'\cap w'',$$

$$x_j = 0 \qquad\qquad \forall j\varepsilon J_1\setminus(w'\cup w''),$$

$$\sum_{j\varepsilon v} x_j = 1,$$

$$\sum_{j\notin J_1\setminus(w'\cup w'')} a_{ij}x_j = 1, \qquad\qquad i=2,\ldots,m,$$

where $v = (w'\cup w'')\setminus(w'\cap w'')$. By Theorem 7.2, $F(w',w'')$ is a quasi-integral polytope. //

Proof of Theorem 7.6 All integral points of the polytope $M(A,e')$ are vertices. It is therefore sufficient to show that between any two integral vertices x',x'' of $M(A,e')$ there is a path in its graph which passes only through integral vertices. Consider the face $F(w')$, where $w' = \{j\varepsilon J_1 : x'_j=1\}$, which contains the vertex x' , and the analagous face $F(w'')$ which contains the vertex x'' . Let the sequence $w'=w_1,\ldots,$ $w_s=w''$ be such that $w'\cap w'' \subset w_i \subset J_1$, $|w_i|=k$ and $|w_i\cap w_{i+1}|=k-1$ for any $i\varepsilon N_s$. From the conditions of the theorem $F(w_i)\neq\emptyset$, $\forall i\varepsilon N_s$ and hence, by Lemma 7.7, $F(w_i)$ is a quasi-integral polytope. Also, by Lemma 7.8, the face $F(w_i,w_{i+1})$ $\forall i\varepsilon N_{s-1}$ is also a quasi-integral polytope. Thus, if we choose an integral vertex x^i on each face $F(w_i)$ then, in the graph generated by the integral vertices of $M(A,e')$, there is a path connecting the vertices x^i and x^{i+1}. Taking the union of these paths we obtain a path connecting the vertices x' and x''. //

7.5 Medians of Graphs

Let $M(k,n)$ be the polytope given by the conditions

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall i \varepsilon N_n, \tag{7.7}$$

$$\sum_{i=1}^{n} x_{ii} = k, \tag{7.8}$$

$$x_{ji} - x_{ii} \leqq 0 \qquad \forall (i,j) \varepsilon N_n \times N_n \ , \quad i \neq j \tag{7.9}$$

$$x_{ij} \geqq 0 \qquad \forall (i,j) \varepsilon N_n \times N_n \ . \tag{7.10}$$

The integral points of the polytope $M(k,n)$ constitute the feasible set for the problem of the location of the k-medians in a graph (Christofides 1975), which has important applications. We therefore call the polytope $M(k,n)$ the *graph medians polytope* . The k-median problem consists in specifying k vertices (median centres) in a given weighted graph such that the sum of the edge weights along chains connecting the specified vertices with the remaining vertices of the graph is minimized. Let $(c_{ij})_{n\times n}$ be the matrix of shortest distances between the vertices of a graph. Then, the k-median problem consists in finding

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to the constraints (7.7)-(7.10) and the additional condition that x_{ij} are integers. Equations (7.7) guarantee that the following condition is satisfied : every vertex j of the graph G is attached to only one median centre. The constraints (7.9) ensure that vertex j is not attached to vertex i if i is not a median centre. And finally, constraints (7.8) guarantee that there will be exactly k median centres.

We study the polytope $M(k,n)$ and show that it is connected integral. We rewrite the constraints (7.7)-(7.10) in the matrix form $Ax \leq b$, where $x = (x_{11},\ldots,x_{n1},\ldots,x_{1n},\ldots,x_{nn})$, $b = (1,\ldots,1,k,0,\ldots,0)$, and

$$A = \begin{bmatrix} J_n & J_n & \cdots & J_n \\ e_1 & e_2 & \cdots & e_n \\ U_1 & & & \\ & U_2 & & \\ & & \ddots & \\ & & & U_n \end{bmatrix}.$$

Here U_k is a $((n-1)\times n)$-matrix obtained from the unit matrix J_{n-1} by inserting a column vector, all of whose elements are equal to -1, between the $(k-1)$-th and k-th column vectors of J_{n-1}.

We note first that the polytope $M(k,n)$ is not integral. When $n>2$ and $k\neq n-1$, $M(k,n)$ has vertices with fractional coordinates of the following type. Fix two distinct indices $s,p\in N_n$. Now consider the system composed of the constraints (7.7)-(7.10) and the conditions $x_{ij}=0$ for all (i,j) such that $i\neq j$, $j\neq s$, $(i,j)\neq(s,p)$ and $x_{is} = x_{ss}$ for all $i\neq s$. This system can be written in the form :

$$\sum_{i=1}^{n} x_{ii} = k,$$

$$x_{ss} + x_{sp} = 1,$$

$$x_{ii} + x_{is} = 1, \qquad i\neq s,$$

$$x_{is} = x_{ss}, \qquad i\neq s,$$

$$x_{ij} = 0 \qquad \text{for the remaining } (i,j).$$

The rank of this system is n^2. Hence, its solution

$$x_{is} = \frac{n-k-1}{n-2}, \; x_{ii} = \frac{k-1}{n-2}, \; i\neq s, \; x_{sp} = \frac{k-1}{n-2},$$

$$x_{ij} = 0 \quad \text{for the remaining } (i,j),$$

is a vertex of $M(k,n)$.

Proposition 7.9 *The graph median polytope* $M(k,n)$ *is integral for* $k=1$ *and* $k=n-1$.

Proof i) Let $k=1$. If $x^0 = (x^0_{ij})$ is a non-integral point of $M(1,n)$ then, by (7.7)-(7.10), all the components in any column are equal. Hence $x^0 = \sum_{s=1}^{n} \lambda_s x^s$, where $\lambda_s = x^0_{ss}$ and x^s is an integral vector of the polytope $M(1,n)$ with nonzero components $x_{is}=1$, $i \in N_n$. Thus, any nonintegral point x^0 can be represented as a convex combination of vertices x^s of $M(1,n)$ and so cannot be a vertex. Thus $M(1,n)$ is an integral polytope.

ii) Let $k=n-1$. To show that $M(n-1,n)$ is integral we establish the relation $\text{conv}\, M_Z(n-1,n) = M(n-1,n)$. With each integral point of $M(n-1,n)$ we associate a pair of indices $x^{s,k}$, $(s,k) \in N_n \times N_n$, $s \neq k$. Here the index s indicates the number of the column with all components zero, and the index k is the number of the column containing two non-null components. In other words, the point $x^{s,k}$ has nonzero components

$$x_{ii} = 1 \qquad \forall\, i \in N_n \setminus s,$$

$$x_{sk} = 1 \ .$$

Every point $x \in \text{conv}\, M_Z(n-1,n)$ can be represented in the form

$$x = \sum_{s=1}^{n} \sum_{k \neq s} \lambda_{sk} x^{s,k} , \tag{7.11}$$

where

$$\sum_{s=1}^{n} \sum_{k \neq s} \lambda_{sk} = 1 \qquad \text{and} \qquad \lambda_{sk} \geqq 0 \ .$$

Now, considering the structure of the integral points of $M(n-1,n)$, we have

$$x = \begin{bmatrix} \sum_{s \neq 1} \sum_{k \neq s} \lambda_{sk} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \sum_{s \neq 2} \sum_{k \neq s} \lambda_{sk} & \cdots & \lambda_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{n1} & \lambda_{n2} & \cdots & \sum_{s \neq n} \sum_{k \neq s} \lambda_{sk} \end{bmatrix}$$

On the other hand, let x^0 be an arbitrary point of $M(n-1,n)$. Since $\dim M(n-1,n) = n^2-n-1$, from the specification of $M(n-1,n)$ we can list n dependent variables :

$$x^0_{ii} = 1 - \sum_{k\neq i} x^0_{ik} \qquad \forall i\in N_n .$$

The sum of all the diagonal elements of x^0 is equal to $n-1$. Hence

$$\sum_{s=1}^{n} \sum_{k\neq s} x^0_{sk} = 1 .$$

Hence, the diagonal elements of x^0 can be expressed in the form

$$x^0_{ii} = \sum_{s=1}^{n} \sum_{k\neq s} x^0_{sk} - \sum_{k\neq i} x^0_{ik} = \sum_{s\neq i} \sum_{k\neq s} x^0_{sk} .$$

Clearly, if we put $\lambda_{sk} = x^0_{ik}$ for $k\neq s$, then

$$\sum_{s=1}^{n} \sum_{k\neq s} \lambda_{sk} = \sum_{s=1}^{n} \sum_{k\neq s} x^0_{sk} = 1 .$$

It is also clear that $\lambda_{sk} \geqq 0$. Hence, any point $x^0 \in M(n-1,n)$ may be represented in the form (7.11), and so $x^0 \in \operatorname{conv} M_Z(n-1,n)$. Thus

$$M(n-1,n) = M_Z(n-1,n) . \qquad //$$

Theorem 7.10 *The graph medians polytope* $M(k,n)$ *is connected integral for any* $k\in N_n$.

This theorem could be proved by transforming the constraint system (7.7)-(7.10) into a system defining a polytope $M(A,e')$ and showing that the property of integrality is preserved by this transformation. However, it is simpler to obtain the analogues of Lemmas 7.7 and 7.8 for the polytope $M(k,n)$ itself and its connected integrality will follow.

Lemma 7.11 *All integral points of the polytope* $M(k,n)$ *are vertices of the integral faces* $F(w) = \{x\in M(k,n) : x_{ii}=0 \ \ \forall i\in N_n\setminus w\}$ *for each* k*-subset* w *of* N_n .

To prove this lemma it suffices to show that the matrix of the constraint system

$$\sum_{j\varepsilon w} x_{ij} = 1 , \qquad i \notin w ,$$

$$x_{ij} \geqq 0 , \qquad i \notin w , \ j \varepsilon w ,$$

which defines the domain of variation of the nonfixed variables in the face $F(w)$, is absolutely unimodular.

<u>Lemma 7.12</u> *Let* $w', w'' \subset N_n$, $|w' \cap w''| = k-1$ *and let* x' *and* x'' *be two integral points belonging respectively to the faces* $F(w')$ *and* $F(x'')$ *of the polytope* $M(k,n)$. *Then, there is an integral face* $F(w',w'')$ *which contains both* x' *and* x'' .

Proof $F(w',w'')$ will be defined as a face of $M(k,n)$ defined by some supplementary constraints. The domain of variation of the nonfixed variables in the face $F(w',w'')$ is the polytope $M(k,k+1)$. By Proposition 7.9 the face $F(w',w'')$ is an integral polytope. Also it is clear that x' and x'' belong to $F(w',w'')$. //

<u>EXERCISES</u>

1. (Minkowski 1910). Let $W \subset E_n$ be a convex set which is symmetric with respect to the origin of coordinates and whose volume is greater than 2^n . Then W contains at least one integral point distinct from the origin.

2. (Belousov 1977). The set W is called *uniform* if there exists a number $\eta > 0$ such that there is a point $x^0 \varepsilon W_Z$ in the η-neighbourhood of any given point $x \varepsilon W$. The straight line $L = \{x \varepsilon E_2 : x_2 = \sqrt{2}\, x_1\}$, which has only one integral point $(0,0)$, is not a uniform set.

An *integral basis of a subspace* is a basis of the space which consists of integral vectors. Each of the following two conditions is a necessary and sufficient condition for a linear subspace L to be uniform : (1) L has an integral basis; (2) L can be specified by means of a system of linear equations with integral coefficients.

Show that an affine set, which contains an integral point, is uniform if and only if it can be specified by means of a system of linear equations with constant coefficients. Let the polyhedron M be given in the form

$$M = \{x \varepsilon E_n : Ax \leqq b\} , \quad b \varepsilon E_m , \ A \varepsilon Z_{m,n} .$$

Then, if M contains an integral point, it is uniform.

3. The integer programming problem $\max\{cx : x \in M_Z\}$ is solvable if there is a vector $x^0 \in M_Z$ such that $cx^0 = \sup\{cx : x \in M_Z\}$. If the polyhedron N is specified by an integral constraint matrix then the condition $\sup\{cx : x \in M_Z\} < +\infty$ implies that the integer programming problem has a solution (Meyer 1974, Belousov 1977).

In order to analyse the efficiency of algorithms for the solution of this type of optimization problem it is necessary to estimate the number of vertices in the convex hull of the integral points of a polytope. Interesting results in this area were obtained by Shevchenko (1979):

(1) if every integral point x of a polytope $M \subset E_n$ satisfies $x_j \leqq \Delta_j - 1$, $\forall j \in N_{n-1}$, then

$$|\text{vert conv } M_Z| \leqq \prod_{j=1}^{n-1} (1 + \log_2 \Delta_j) ;$$

(2) let $a_0 \geqq \max\{a_1, \ldots, a_n\} = a$, $a_j \in Z^+$ and let ν be the number of vertices of the polytope $\text{conv}\left\{x \in Z_n^+ : \sum_{j=1}^{n} a_j x_j \leqq a_0\right\}$. Then

$$\nu \leqq \sum_{i=1}^{n} (1 + [\log_2(a_0/a_i + 1)]) ;$$

(3) if, in addition, $a_0 \geqq a(a-1)$, then

$$\nu \leqq 1 + \sum_{i=1}^{n} \binom{1 + \log_2 a_i + n - 2}{n - 1} .$$

4. The equation $cx = \alpha$ has an integral solution if and only if the highest common factor of the coefficients $c_1, \ldots, c_n$ divides α.

Several criteria for the solvability of a system of linear homogeneous diophantine equations in non-negative integers, going back to Stiemke's theorem of 1915, are given by Stanley (1974).

The number σ of integral non-negative points in the simplex $T_n = \{x \in E_n : cx \leqq \alpha, x \geqq 0\}$ lies between the bounds (Beged-Dov 1972)

$$\frac{\alpha^n}{n! \prod_{i=1}^{n} c_i} \leqq \sigma \leqq \frac{(\alpha + \sum_{i=1}^{n} c_i)^n}{n! \prod_{i=1}^{n} c_i} .$$

A necessary and sufficient condition for the system of linear equations $Ax = b$ to be solvable for any vector $b \in Z_m$ is given by:

$$\operatorname{rank} A = m \quad , \quad \Delta_m(A) = 1 .$$

Let $Ax = 0$, $A \in Z_{m,n}$, $m < n$ be a system of linear equations and let θ be a number such that $|a_{ij}| \leqq \theta$ for all i,j . Then there exists a non-trivial integral solution such that

$$|x_j| \leqq 2(n\theta)^{m/(n-m)} . \quad \text{(Siegel's Lemma)}$$

5. Not every finitely generated semigroup of integral vectors is polyhedral. For example, the semi-group B generated by the vectors $q^1 = (2,0)$, $q^2 = (0,2)$, $q^3 = (1,1)$ is not polyhedral. Conditions for an arbitrary (not necessarily polyhedral) semi-group B of integral vectors to be finitely generated are given by the following theorem: the semi-group $B \subset Z_n$ is finitely generated if and only if there is a matrix A with rational elements such that $\operatorname{con} B = \{x \in E_n : Ax \geq 0\}$ and if for any rational vector $v \in \operatorname{con} B$ there is an integer d such that $dv \in B$ (Trubin 1969, Petrova 1976). A generating set of a semigroup is called *irreducible* if no proper subset of it is generating. If $\operatorname{con} B$ is a pointed cone then B has a unique irreducible generating set. The irreducible generating set of the polyhedral semigroup K_Z (where K is a pointed cone) consists of the integral points of a half-open parallelepiped Q which are minimal elements relative to the following partial order : $x \succ x'$ if $\lambda_i \geqq \lambda_i'$, $\forall i \in N_t$ where at least one of these inequalities is strict. Here, λ_i and λ_i' are the coefficients in the expansions $x = \sum_{i=1}^{t} \lambda_i q^i$, $x' = \sum_{i=1}^{t} \lambda_i' q^i$ in terms of the generators $q^1, \ldots, q^t$ of the cone K . This theorem may be generalized (Shevchenko & Ivanov 1976). If the convex cone C (not necessarily polyhedral) in E_n is pointed, then the set of minimal elements of C_Z is the unique irreducible generating set of the semigroup C_Z .

6. The $(n \times n)$ integral matrix T is called *Hermitian* if $t_{ij} = 0$, $1 \leqq i < j \leqq n$ and $t_{ii} > t_{ij} \geqq 0$. Show that every non-singular integral matrix A can be uniquely represented in the form $T = AV$ ($A = TV^{-1}$), where V is a unimodular matrix and T is Hermitian.

7. (Veselov & Shevchenko 1978). Let $x_i \in \{0,1\}$ and let the equation $\sum_{i=1}^{2n} \alpha_i x_i = \alpha_0$ have the same solution set as the system $x_{2i-1} + x_{2i} = 1$, $\forall i \in N_n$. Then $\alpha_0 \geqq 2^{n-1}$.

8. Let M be an integral d-polytope and let n be a positive integer. Let nM denote the sum (§3, Ch.1) of n polytopes M. Also, let $\nu(M)$ and $\nu(\text{int } M)$ be the number of integral points in M and $\text{int } M$ respectively. The following assertions are true (Ehrhart 1977):

(1) $\nu(nM)$ is a polynomial $P_M(n)$ of degree d in n;

(2) $\nu(\text{int } nM) = (-1)^d P_M(-n)$. (reflexive law)

9. Show that:

(1) the expansion coefficients of any column of a matrix A relative to any basis of A are equal to $0, \pm 1$ if and only if A is an α-modular matrix;

(2) the matrix A is unimodular if and only if A has a unimodular basis B and $B^{-1}A$ is absolutely unimodular.

10. (Heller 1963). A subset U of a linear space is called a *unimodular set* if any two of its bases are related by a unimodular transformation. In other words, the coordinates of a vector $a \in U$ relative to any basis of U are integers. Partially order the family of unimodular sets by means of the inclusion relation $\subset$. Show that the number of distinct elements in a maximal unimodular set $U \subset E_n$ is at most $n(n+1)$, where the bound is attained only by maximal sets formed by the edges of an n-simplex taken with all possible orientations.

11. (Heller 1957). Suppose that the rows of a Boolean matrix A of dimension $m \times n$ can be partitioned into two disjoint classes I_1 and I_2 with the following property: if rows i and q belong to the same class and if there is a column with index k such that $a_{ik} = a_{qk} = 1$ then either $a_{ij} \geqq a_{qj}$, $\forall j \in N_n$, or $a_{ij} \leqq a_{qj}$, $\forall j \in N_n$. Then A is an absolutely unimodular matrix.

12. (Brown 1977). Let A be an absolutely unimodular $(n \times n)$-matrix. Let $A^{(k)}$ be the matrix of order k which is composed of all the k-th order minors of the matrix A. Then $A^{(n-1)}$ is an absolutely unimodular matrix.

13. Let $A \in C_{m,n}$. If the polytope $M(A, b^1, b^2, d^1, d^2) \neq \emptyset$, then for any pair of vectors $w, v \in C_{1,n}$ such that $wA = v$, we have

$$\sum_{i: w_i = -1} b_i^1 + \sum_{j: v_j = 1} d_j^1 \leqq \sum_{i: w_i = 1} b_i^2 + \sum_{j: v_j = -1} d_j^2 .$$

The following assertions are equivalent:

(1) A is an absolutely unimodular matrix;

(2) for all $b^1 \leqq b^2$ and $d^1 \leqq d^2$ the given inequality implies that $M(A, b^1, b^2, d^1, d^2) \neq \emptyset$;

(3) if $M(A,b^1,b^2,d^1,d^2) \neq \emptyset$ for integral vectors $b^1 \leqq b^2$, $d^1 \leqq d^2$ then $M_Z(A,b^1,b^2,d^1,d^2) \neq \emptyset$.

14. (Pulleyblank & Edmonds 1974). The inequality

$$\sum_{i:e_i \varepsilon G(S)} x_i \leqq (|S|-1)/2 \quad \text{for} \quad S \subset V \ , \ |S| \equiv 1 \pmod 2 \ ,$$

determines a facet of the matching polytope $M(G)$ if and only if the subgraph $G = (V,E)$ generated by the set S is 2-connected and the graph $G(S)$ has a perfect matching for any $v \varepsilon S$.

15. (Padberg 1974, Lovász 1972). A *critically non-perfect graph* is a non-perfect graph such that all subgraphs which it generates are perfect. Show that the strong Berge conjecture is equivalent to the following: every critically non-perfect graph is either an odd cycle without chords of length greater than or equal to five, or the complement of such a graph. Show that:

(1) a critical non-perfect graph has $1+\alpha(G)\omega(G)$ vertices;

(2) if A_G is the clique matrix of a critically non-perfect graph G, then the polytope $M^{\leqq}(A_G,e)$ has only one non-integral vertex all of whose coordinates are equal to $1/\omega(G)$.

16. (Chvatal 1975). An edge e of a graph G is called α-*critical* if $\alpha(G\setminus e) > \alpha(G)$. Let G be a graph with n vertices whose α-critical edges form a spanning connected subgraph. Then the inequality $\sum_{i=1}^{n} x_i \leqq \alpha(G)$ determines a facet of the polytope $\operatorname{conv} M_Z^{\leqq}(A_G,e)$.

17. Let A_G be the incidence matrix (vertices-edges) of a graph $G = (V,E)$ with m vertices and n edges. Show that the following four assertions are equivalent:

(1) G is a bipartite graph;

(2) $M^{\leqq}(A_G,e)$ is an integral polytope;

(3) $M^{\geqq}(A_G^T,e)$ is an integral polytope;

(4) $M^{\leqq}(A_G^T,e)$ is an integral polytope.

The assertion

(5) $M^{\geqq}(A_G,e)$ is an integral polytope

is not equivalent to (1) but is equivalent to

(1') either G is a bipartite graph or for every odd cycle C of G we have $M^{\geqq}(A_{G\setminus C},e) = \emptyset$.

18. (Fulkerson, Hoffman & Oppenheim 1974). Let A be a Boolean matrix. Then the following conditions are equivalent:

(1) A is a balanced matrix;

(2) for every proper submatrix A' of A the polytopes $M^{\leqq}(A',e)$, $M^{=}(A',e)$, $M^{\geqq}(A',e)$ are integral;

(3) A is the incidence matrix of a balanced hypergraph (a hypergraph is called *balanced* if every odd cycle has an edge which containes at least three vertices of the cycle);

(4) the intersection graph G_A of A does not contain odd chordless cycles.

Show that every absolutely unimodular matrix is balanced and that every balanced matrix is perfect.

19. Suppose that the Boolean matrix A is such that the linear programming problem $\min\{ex : x \in M^{\geqq}(A,b)\}$ has an integral solution for any Boolean vector b. Then this problem has an integral solution for any non-negative integral vector b. Use this fact to prove the implication (1)$\Rightarrow$(2) in Theorem 5.3.

20. (Ivanov 1974). The set of integral points in any polytope $M(a,b)$, $A \in Z_{m,n}$, $b \in Z_m$ coincides with the set of integral points of the relaxed partition polytope $M^{=}(B,e)$, where B is the submatrix obtained from the matrix $\bar{A}Q$ by removing its first row. Here

$$\bar{A} = \left[\begin{array}{c|c} & -b_1 \\ & -b_2+1 \\ A & \vdots \\ & -b_m+1 \\ \hline 0 \;\ldots\; 0 & 1 \end{array}\right]$$

and Q is a matrix whose column vectors form a generating set of the polyhedral semigroup $\mathcal{B} = \{x \in Z^+_{n+1} : \bar{A}x \geqq 0\}$, where the first of the inequalities in $\bar{A}x \geqq 0$ is taken as an equality. A bijection between the points $x \in M_Z(A,b)$ and $u \in M_Z(B,b)$ is given by the mapping $x = Qu$.

21. Let x^0 be a vertex of the polytope $M^{\leqq}(A^T,e)$ which maximizes the function ex. Then there is a vertex x' of the polytope $\operatorname{conv} M^{\leqq}_Z(A^T_G,e)$ which maximizes the same objective function and whose components, corresponding to the integral components of x^0, have the same values.

22. (Berge 1970). Let $H = (I,E)$ be a hypergraph. We say that the function $h(v)$ defined on I is *stochastic* if

$$0 \leqq h(v) \leqq 1 \qquad \forall v \in I,$$

$$\sum_{v \in E_i} h(v) = 1 \qquad \forall i \in N_m \ , \ m = |E| \ .$$

Show that (1) not every hypergraph H has a stochastic function;

(2) every stochastic function $h(v)$ defined on a unimodular hypergraph H can be represented in the form

$$h(v) = \sum_{i=1}^{m} \lambda_i h_i(v) \ , \text{ where } \lambda_i \geqq 0 \ , \ \sum_{i=1}^{m} \lambda_i = 1 \text{ and } h_i(v) \text{ is a stochastic}$$

function taking the values 0 or 1 .

23. Let $G = (V,E)$ be a graph with m vertices such that every vertex v_i is assigned a non-negative integral weight b_i . A b-*matching* in a graph G is a subset of edges such that no more than b_i are incident to the vertex v_i . The b-matching polytope can be defined by analogy with the matching polytope. Show that it is given by the following system of inequalities:

$$x_j \geqq 0 \ , \quad \forall j \in N_n \ , \ n = |E| \ ,$$

$$\sum_{j=1}^{n} a_{ij} x_j \leqq b_i \qquad \forall i \in N_m \ ,$$

$$\sum_{j : e_j \in G(S)} x_j \leqq \tfrac{1}{2}\Big(\sum_{i : v_i \in S} b_i - 1 \Big) \qquad \forall S \in \mathcal{F} \ ,$$

$$\text{where } \mathcal{F} = \Big\{ S \subseteq V \ , \ \sum_{i : v_i \in S} b_i \equiv 1 \ (\mathrm{mod}\ 2) \Big\} \ .$$

24. (Lovász 1979). A non-empty proper subset S of the vertices of a di-graph $G = (V,E)$ is called a *di-cut* if $(i,j) \notin E$, $\forall (i,j) \in \bar{S} \times S$. Let G be a di-graph with the property that if $(i,j) \in E$ then $(j,i) \notin E$. Then the maximum number of distinct di-cuts in G equals the minimum number of edges which cover all di-cuts.

25. (Saigal 1969). The *shortest-chain polytope* of the di-graph $G = (V,E)$ is given by the following system of inequalities:

$$\sum_{j=1}^{m} x_{1j} - \sum_{j=1}^{m} x_{j1} = 1 \ ,$$

$$\sum_{j=1}^{m} x_{ij} - \sum_{j=1}^{m} x_{ji} = 0 \ , \quad i = 2, \ldots, m-1 \ ,$$

$$\sum_{j=1}^{m} x_{mj} - \sum_{j=1}^{m} x_{jm} = -1 \ , \quad x_{ij} \geqq 0 \ , \quad \forall (i,j) \in N_m \times N_m \ ,$$

where m is the number of vertices in G and where the shortest chains are sought between the vertices numbered 1 and m. Show that the maximum distance conjecture is true for the shortest-chain polytope.

26. Let a be a positive integral n-vector and let α be a positive integer satisfying the conditions $a_j \leqq \alpha$ and $\sum_{j=1}^{n} a_j > \alpha$, $\forall j \in N_n$. The following two assertions are equivalent:

(1) $M(a,\alpha) = \{x \varepsilon E_n : ax \leqq \alpha \ , \ 0 \leqq x \leqq e\}$ is an integral polytope;

(2) $(a,\alpha) = \lambda(e,k)$, where λ and k are positive integers.

27. (Hoffman 1979). Let A be the incidence matrix of a graph G having the following property : every pair of odd cycles contains respectively vertices v_1 and v_2 such that either $v_1 = v_2$ or v_1 and v_2 are adjacent. Show that if the system $\bar{A}x = b$, where b is an integral vector and $\bar{A} = \left(\begin{array}{c|c} A & 0 \\ \hline \multicolumn{2}{c}{e} \end{array}\right)$, has both a non-negative solution and an integral solution, then it has a non-negative integral solution.

28. (Hoffman 1979). Let $G = (V,E)$ be a di-graph and let $\mathcal{F}$ be a family of subsets of V such that if $S,T \in \mathcal{F}$, $S \cap T \neq \emptyset$ and $S \cup T \neq V$ then $S \cap T \in \mathcal{F}$ and $S \cup T \in \mathcal{F}$. Let ρ be an integral supermodular function defined on $\mathcal{F}$. Then, if a_{ij}, d_{ij}, $\forall (i,j) \in E$ are integers, the polytope given by the conditions

$$a_{ij} \leqq x_{ij} \leqq d_{ij} \ , \quad \forall (i,j) \in E \ ,$$

$$\sum_{(i,j) \in S \times \bar{S}} x_{ij} - \sum_{(i,j) \in \bar{S} \times S} x_{ij} \geqq \rho(S) \quad , \quad \forall S \in \mathcal{F}$$

is integral.

29. (Welsh 1976). Let $\mathcal{M}$ be a matroid on $J = \{e, e_1, \ldots, e_n\}$. Let $(a_{ij})_{n \times m}$ be the incidence matrix of elements in the set $J \setminus e$ and cycles in the matroid which contain e. Define an *e-flow in the matroid* $\mathcal{M}$ to be a vector $u = (u_1, \ldots, u_m)$ which satisfies the conditions

$$\sum_{j=1}^{m} a_{ij} u_j \leqq d_i \ , \forall i \in N_n \quad , \quad u_j \geqq 0 \ , \forall j \in N_m \ ,$$

where d_i is the flow capacity of the element $e_i \in J$. The *magnitude of the flow* is defined to be $\sum_{i=1}^{m} u_i$. The set $C \subset J$ is a *cocycle* of the matroid $\mathcal{M}$ if C is a cycle in the dual matroid $\mathcal{M}^*$, that is, in the matroid whose bases are precisely the complements of the bases of $\mathcal{M}$. Let C^* be a cocycle of the matroid $\mathcal{M}$ which contains e. The *flow capacity of the cocycle* C^* is the number $\sum_{i:e_i \in C^*} d_i$. Show that if $\mathcal{M}$ is a regular matroid then the maximum flow magnitude of an e-flow equals the minimum flow capacity of cocycles containing e.

30. (Hoffman 1979). Let L be a partially ordered set with commutative binary operations $\wedge$ and $\vee$ such that:

$$a \prec b \Rightarrow a \wedge b = a \;,\; a \vee b = b \;;$$

$$a \wedge b \prec a \;,\; a \wedge b \prec b \;,\; a \prec a \vee b \;,\; b \prec a \vee b \;.$$

Let the map $\phi : L \to 2^J$, $J = N_n$ satisfy

(1) $a \prec b \prec c \Rightarrow \phi(a) \cap \phi(c) \subset \phi(b)$;

(2) $\phi(a \vee b) \cup \phi(a \wedge b) \subset \phi(a) \cup \phi(b)$;

or (2') $\phi(a \vee b) \cup \phi(a \wedge b) \supset \phi(a) \cup \phi(b)$;

(3) $\phi(a \vee b) \cap \phi(a \wedge b) \supset \phi(a) \cap \phi(b)$.

Show that if ρ is a non-negative supermodular (submodular) integral function, then the polytope determined by the inequalities

$$0 \leqq x_j \leqq d_j \;, \forall j \in J \quad (d_j \in Z^+),$$

$$\sum_{j \in \phi(a)} x_j \overset{(\leqq)}{\geqq} \rho(a) \;, \quad \forall a \in L \;,$$

is integral.

31. Let $G = (U,V,E)$ be a bipartite graph and let $\mathcal{M}_1$ and $\mathcal{M}_2$ be matroids on E whose independent sets are respectively the subsets ω which do not contain any edges incident to one vertex of U or V. Then the intersection of the matroid polytopes M_1, M_2 is a feasible set for an assignment problem.

32. Let M be an integral polymatroid all of whose vertices have coordinates consisting of 0's and 1's. Then there exists a matroid $\mathcal{M}$ whose corresponding polytope is M.

33. Find conditions on the functions $\rho_1(\omega)$ and $\rho_2(\omega)$ which are necessary to ensure that the polymatroids $M(\rho_1)$ and $M(\rho_2)$ are combinatorially equivalent. Characterize those polymatroids which have the maximum number of vertices.

34. (Ehrhart 1977). Let $\mathcal{M}_1 = (J, \mathcal{F}_1)$ and $\mathcal{M}_2 = (J, \mathcal{F}_2)$ be two given matroids where $J = N_n$, and let $r_1(\omega)$ and $r_2(\omega)$ be the rank functions of these matroids. Let $r(\omega) = \min\{r_1(\omega_1) + r_2(\omega_2) : \omega_1 \cup \omega_2 = \omega\}$.

Show that the vertices of the polyhedron $M = \{x \varepsilon E_n : \sum_{i \varepsilon \omega} x_i \geqq r(N_n) - r(N_n \setminus \omega)$, $\forall \omega \subseteq N_n\}$, and only these points, are the characteristic vectors of the common independent sets of maximum cardinality of the matroids $\mathcal{M}_1$ and $\mathcal{M}_2$.

35. Let $\mathcal{F}$ be a family of subsets of the set E_n. Then, if $\mathcal{F}$ contains the sets $\emptyset$ and J, and if it contains $U \cap V$ whenever it contains U and V, then the polytope in E_n^+ given by the conditions $\sum_{i \varepsilon \omega} x_i \leqq \rho(\omega)$, $\forall \omega \varepsilon \mathcal{F}$, is a polymatroid for any submodular non-negative non-decreasing function $\rho(\omega)$. Its rank function $r(a)$ is given by

$$r(a) = \min\{\sum_{i \varepsilon J} a_i z_i + \sum_{\omega \varepsilon \mathcal{F}} \rho(\omega) y(\omega) : z_j + \sum_{\omega \ni j} \rho(\omega) y(\omega) \geqq 1, \forall j \varepsilon J\}.$$

36. In the works of Kovalev (1977), Kovalev (1980), Kovalev & Yemelicheva (1975) there is constructed a theory of discrete-convex programming in which it is established that in convex integer programming polymatroids play the same rôle as do convex sets in the theory of convex programming.

Let M be a polymatroid in E_n^+. The separable function $f(x) = \sum_{i=1}^{n} f_i(x_i)$, defined on M_Z, is called *discretely convex*, if $\Delta_i(x) \geqq \Delta_i(y)$ when $x \leqq y$, where $\Delta_i(x) = f_i(x+1) - f_i(x)$. The quantity $\Delta_i(x)$ is called the *i-gradient of the function* $f(x)$.

Starting from the point $x^0 = 0$, the gradient algorithm generates a sequence of points x^k according to the rule $x^k = x^{k-1} + e_{i_k}$, where the index i_k corresponds to the largest positive i-gradient, among those i for which $x^{k+1} + e_i \varepsilon M_Z$. If such an index does not exist, then the algorithm terminates and the vector obtained is optimal for the problem of maximizing a separable discrete convex function on the polymatroid M.

If the set M is not a polymatroid but M satisfies the first defining condition of a polymatroid, then, if x^g is the vector obtained by the gradient algorithm, we have

$$\frac{f(x^g)}{f(x^*)} \geqq \min_{a \varepsilon Z_n^+} \frac{\ell(M_a)}{h(M_a)}, \quad \text{where}$$

$M_a = \{x \varepsilon M_Z : x \leqq a\}$, $h(M) = \max\left\{\sum_{i=1}^{n} x_i : x \varepsilon M_Z\right\}$, $\ell(M) = \min\left\{\sum_{i=1}^{n} x_i : x \varepsilon M_Z\right.$,

$x + e_i \notin M_Z, \forall i \in N_n \}$ are the maximum and minimum 'heights' of the set M_Z. It is clear that the maximum and minimum heights of each of the subsets M_a of the polymatroid M coincide.

If, in addition, the discrete-convex function $f(x)$ is not separable, then we have the estimate

$$\frac{f(x^*) - f(x^g)}{f(x^*) - f(0)} \leqq \left(1 - \frac{1}{h(M)}\right)^{\ell(M)} .$$

Further details on the maximization of non-linear functions on the intersection of polymatroids can be found in Kowaljow & Girlich (1978) and Girlikh & Kowaljow (1981).

5 PERMUTATION POLYTOPES

In the previous chapter we solved the problem of constructing the convex hull of the integral points of a polyhedron. Theorem 1.4, originating in the work of Hilbert, showed that the convex hull of the integral points could be represented as the solution set of a system of linear inequalities with rational coefficients and hence, as the intersection of a finite family of halfspaces. In this chapter we give methods for constructing such half-spaces for the case of polytopes associated with permutation matrices. In addition to the classical permutation polytopes introduced by Rado (1952), and the bistochastic polytope of Birkhoff (1946), we study newer classes of permutation matrices : the travelling salesman polytope, the standardization polytope and the assignment polytope. Permutation polytopes play an important part in combinatorial analysis (Sachkov 1977), scheduling theory (Tanaev & Shkurba 1975) and the theory of extremal problems on substitutions (Suprunenko & Metelski 1973).

§1 BISTOCHASTIC POLYTOPES

In this section we study a polytope defined by the constraints of the assignment problem and its generalizations, which is well known and has been widely studied.

1.1 Birkhoff's Theorem

Definition 1.1 A square matrix with real, non-negative elements is called *bistochastic* if the sum of the elemets in any line (row or column) is equal to one. Bistochastic Boolean Matrices are called *permutation matrices*.

Permutation $(n\times n)$-matrices and permutations $\pi\in S_n$ are connected as follows : every permutation $(\pi_1,\dots,\pi_n)$ corresponds to a permutation matrix (x_{ij}) whose components are defined by $x_{ij}=1$ if $i=\pi_j$ and $x_{ij}=0$ otherwise.

Theorem 1.1 (Birkhoff) *The set* M_n *of all bistochastic* $(n\times n)$-*matrices is a polytope in* E_{n^2} *which has the permutation matrices as its vertices.*

Proof Let $x\varepsilon M_n$. Tho theorem statement is cloarly truc if x is already itsolf a permutation matrix. Wo use induction on the number of positive elements of x to show that x is a convex combination of permutation matrices. We first show that the matrix x has at least one diagonal consisting entirely of positive olements. Indeed, if the matrix x contains a zero submatrix x_I^J , then all nonzero olements in the rows with indices in the set I lie in tho submatrix $x_I^{N_n\setminus J}$, so that the row sums of the matrix $x_I^{N_n\setminus J}$ are all 1 . Hence, the sum of all elements of this matrix is $|I|$. Similarly, the sum of all the elements of the matrix $x_{N_n\setminus J}^J$ is $|J|$. But, tho submatrices $x_I^{N_n\setminus J}$ and $x_{N_n\setminus I}^J$ have no elements in common, while the sum of all tho elements of x is n , hence $|I| + |J| \leq n$. Thus, by Frobenius' Theorem (Corollary 4.7, Ch. 4) tho matrix x contains a diagonal all of whose elements aro positive.

Let λ bo tho smallest element in one such diagonal of x and let P be the pormutation matrix with unit elements in tho positions corresponding to this diagonal. Cloarly $0<\lambda<1$ and $y = (x-\lambda P)/(1-\lambda)$ is a bistochastic matrix with at least one less positive olomont than x . By the inductive hypothesis, y is roprosentable as a convex combination of permutation matrices. Then $x = \lambda P + (1-\lambda)y$ is also a convex combination of pormutation matrices. Thus, the set M_n is a polytopo gonorated by the permutation matrices.

Obviously, no permutation matrix can bo represented as a convex combination of other permutation matrices, and so tho pormutation matrices are vertices of M_n . //

The constraint matrix R of tho constraints

$$x_{ij} \geq 0 \qquad \forall i,j\varepsilon N_n, \tag{1.1}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad \forall j\varepsilon N_n, \tag{1.2}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall i\varepsilon N_n, \tag{1.3}$$

written in the standard form $Rx = e$, where

$$x = (x_{11},\dots,x_{1n},\dots,x_{n1},\dots,x_{nn}) \in E_{n^2} ,$$

takes the form

$$R = \begin{bmatrix} 1 & \dots & 1 & & & & & & \\ & & & 1 & \dots & 1 & & & \\ . & \dots & . & . & \dots & . & . & \dots & . \\ & & & & & & 1 & \dots & 1 \\ 1 & \dots & . & 1 & \dots & . & 1 & \dots & . \\ . & & & . & & & . & & \\ & . & & & . & & & . & \\ . & \dots & 1 & . & \dots & 1 & . & \dots & 1 \end{bmatrix}$$

where R is a $(2n\times n^2)$-matrix in every column R^{ij} of which there are precisely two units, the other elements being zero.

The matrix R is the incidence matrix of a complete bipartite graph $K_{n,n}$. Thus, the permutation polytope M_n is identical with the partition polytope $\operatorname{conv} M_Z^=(R,e)$ of the edges of a complete bipartite graph $K_{n,n}$. Since the matrix R is absolutely unimodular (Corollary 4.5, Ch. 4), $M_n = \operatorname{conv} M_Z^=(R,e) = M^=(R,e)$. This is another proof of Birkhoff's theorem.

The polytope M_n can be regarded as the convex hull of the complete pairings in a complete bipartite graph. The characteristic vectors x of such pairings can be represented as a matrix in which the rows correspond to one vertex set of the graph $K_{n,n}$ and the columns correspond to the second vertex set.

The polytope M_n is the feasible set in the assignment problem, which has important applications, and it has therefore been studied by many authors. Most of the results obtained are elementary in character and follow from the theory of permutation matrices. Here we follow Balinski & Russakoff (1972) in giving a graph-theoretical proof of a basic theorem about polytopes M_n . This is the diameter theorem which is based on a simple criterion for testing for adjacency of vertices.

It is easily verified that $\operatorname{rank} R = 2n-1$. Since the point x^0 , all of whose coordinates are equal to $1/n$, is an interior point of M_n relative to the affine set defined by equations (1.2),(1.3), it follows that $\dim M_n = (n-1)^2$.

Proposition 1.2 *The bases (feasible bases) of the polytope M_n are in one-to-one correspondence with the spanning trees of the complete bipartite graph $K_{n,n}$ (the spanning trees containing a complete pairing of $K_{n,n}$).*

Proof With each basis B of M_n we associate a spanning subgraph $T(B)$ of the graph $K_{n,n}$ which contains edges (i,j) such that the columns R^{ij} lie in the basis B. If the graph $T(B)$ has a cycle, then the columns R^{ij} corresponding to the edges of the cycle form a submatrix $(B',0)^T$, where B' is the incidence matrix of the cycle. The determinant of B' is zero (see §4, Ch.4), and this contradicts the linear independence of the column vectors in the basis. It is clear, that if the spanning tree $T(B)$ does not contain a complete pairing of the graph $K_{n,n}$, then the basis B is not feasible.

Conversely, let T be a spanning tree of $K_{n,n}$ and let R^T be the set of columns R^{ij} of R corresponding to the edges $(i,j)\varepsilon T$. Consider the equation $\sum_{(i,j)\varepsilon T} R^{ij}\lambda_{ij} = 0$. Since T is a tree, it has a vertex, say i, of degree 1. Let the vertex i be incident to the unique edge (i,j). Then it is easily seen that $\lambda_{ij} = 0$. Removing the edge (i,j) with vertex i from the tree T and using the same argument for the remaining part of the graph T, we find that all $\lambda_{ij} = 0$. Thus the columns of R^T are linearly independent, that is R^T is a basis. If the tree T contains a complete pairing P, then

$$x_{ij} = \begin{cases} 1 & \text{if } (i,j)\varepsilon P, \\ 0 & \text{otherwise.} \end{cases}$$

This means that R^T is a feasible basis. //

Let $G(x)$ be the complete pairing corresponding to the permutation matrix x.

Theorem 1.3 *Two vertices $x \neq y$ of the polytope M_n are adjacent if and only if the subgraph $G(x) \cup G(y)$ contains only one cycle.*

Proof (i) Every vertex in the subgraph $G(x) \cup G(y)$ has degree equal to either 1 or 2. Thus, every connected component of $G(x) \cup G(y)$ is either an edge or a cycle with an even number of edges. Suppose that there is more than one cycle among the connected components. Let $T(x)$,

$T(y)$ be any spanning trees containing the pairings $G(x), G(y)$. Then the graph $T(x) \cup T(y)$ also contains more than one cycle. Thus the feasible bases $A^{T(x)}$ and $A^{T(y)}$ differ by more than one column-vector and so are not adjacent.

(ii) Now let the graph $G(x) \cup G(y)$ consist of a single cycle C and a set E of isolated edges. Let $|E|=p$; then $|C| = 2n - 2p$. Let Let F be a set of p edges of $K_{n,n}$ which do not belong to $G(x)\ G(y)$ but which connect together all the connected components of $G(x) \cup G(y)$. Choose two adjacent edges (s,r) and (r,t) from the cycle C . Let $(r,s) \in G(x)$, $(r,t) \in G(y)$. Add the edges of F to the subgraph $G(x) \cup G(y)$ and then form two graphs by removing the edge (r,s) in one case and the edge (r,t) in the second case. We thus obtain two spanning trees $T(x)$ and $T(y)$. The submatrices $A^{T(x)}$ and $A^{T(y)}$ are feasible bases corresponding to the vertices x and y respectively. These bases differ only by one column-vector and are therefore adjacent. //

Since every permutation matrix x corresponds uniquely to a permutation π , Theorem 1.3 can be reformulated as follows.

Corollary 1.4 *The vertices* σ *and* τ *of the assignment polytope are adjacent if and only if the permutation* $\pi = \sigma^{-1}\tau$ *contains exactly one cycle.*

1.2 Diameter of M_n

Let $G(M_n)$ be the graph of the polytope M_n . Two vertices x and y of a graph are called *similar* if there is an automorphism α of the graph such that $\alpha(x) = y$ (α is a permutation of the vertices of the graph which preserves adjacency relations). The graph is *vertex-symmetric* if any pair of its vertices are similar.

Proposition 1.5 $G(M_n)$ *is a vertex symmetric graph.*

Proof Let x , y be vertices of the graph $G(M_n)$, that is, permutation matrices. Let ϕ be the permutation of the columns of x which transforms x into y or, equivalently, ϕ is a permutation of one of the parts of the graph $K_{n,n}$ which transforms the pairing $G(x)$ into the pairing $G(y)$. Define the mapping α which maps the vertex (z_{ij}) of $G(M_n)$ to the vertex $\alpha(z) = (z_{i,\phi j})$. It is clear that α preserves adjacency of vertices in $G(M_n)$ and that $\alpha(x) = y$. Thus α is an automorphism of $G(M_n)$. //

Corollary 1.6 *The degree of each vertex of the graph* $G(M_n)$ (i.e. *the number of adjacent vertices to each vertex of* M_n) *is equal to*

$$\sum_{k=0}^{n-2} \binom{n}{k}(n-k-1)! \ .$$

Proof By Proposition 1.5 the degrees of all the vertices of $G(M_n)$ are equal. Thus it suffices to determine the number of adjacent vertices to the vertex (x_{ij}) with coordinates $x_{ii} = 1$ for all i and $x_{ij} = 0$ for $i \neq j$. By Theorem 1.3 , the number of vertices y adjacent to x such that the pairing $G(y)$ has no edges in common with $G(x)$ is $(n-1)!$, and the number such that $G(y)$ has one edge in common with $G(x)$ is $\binom{n}{1}(n-2)!$. The number such that $G(y)$ has k edges in common with $G(x)$ is $\binom{n}{k}(n-k-1)!$. Summing over $k \in N_{n-2}$ we obtain the desired formula . //

Theorem 1.7 *The diameter of the polytope* M_n *for* $n \geq 4$ *is* 2 .

Proof It suffices to prove that given any two non-adjacent vertices x and y of M_n , there is a vertex which is adjacent to both. We note that non-adjacent vertices x and y of M_n only exist for $n \geq 4$.

Let $G(x), G(y)$ be pairings in the graph $K_{n,n}$ corresponding to the vertices x and y . We examine the subgraph $G(x) \cup G(y)$. By Theorem 1.3 , $G(x) \cup G(y)$ contains at least two cycles. If the pairings $G(x), G(y)$ have at least one edge (i,j) in common then, putting $x_{ij}=1$ we obtain the assertion of the theorem by an induction on n . Thus, suppose that $G(x) \cup G(y)$ is the union of p $(p \geq 2)$ cycles $C_1, \ldots, C_p$. It is easy to see that these cycles have no edges in common and that each of them consists of edges which belong successively to $G(x)$ and $G(y)$. By removing one edge $e_i \in G(y)$ from each of the cycles C_i , we obtain p disjoint chains $\overline{C}_i$ and $G(x) \subset \bigcup_{i=1}^{p} C_i$. Let E be a set of p edges of the graph $K_{n,n}$ which connect all of the p chains $\overline{C}_1, \ldots, \overline{C}_p$ into one simple cycle. Note that the edges in E and the edges e_i which have been removed also form a simple cycle. Let $e \in E$. Then $T(x) = C \setminus e$ is a spanning tree of the graph $K_{n,n}$ which determines a feasible basis for the vertex $x \in M_n$.

We show that if an arbitrary edge $f \in G(x)$ is removed from the cycle C, then the spanning tree so obtained defines a feasible basis of a vertex z of M_n which is adjacent to the vertex x because their bases differ by one column. Indeed, the tree so constructed contains a complete pairing consisting of the edges of the set E augmented by $n-p$ edges of the pairing $G(y)$ belonging to the cycle C. Thus, the vertex z which we have constructed is adjacent to x but it is also adjacent to y, for the graph $G(y) \cup G(z)$ has a cycle formed by the edges $e_1, \dots, e_p$ and the edges in E, while the remaining edges in the pairings $G(y)$ and $G(z)$ are common to both. //

Noting that $\operatorname{diam} M_n = 1$ for $n \leqq 3$, we can assert that the diameter conjecture for polytopes of bi-stochastic matrices is true.

1.3 Symmetric Permutation Matrices

Birkhoff's theorem describes the convex hull of the permutation matrices. The following theorem, due to Cruse (1975) describes the convex hull of all symmetric permutation matrices $(x_{ij} = x_{ji})$.

Definition 1.2 The convex hull of the set of symmetric permutation $(n \times n)$-matrices is called the *symmetric permutation polytope* and is denoted by M_n^*.

Theorem 1.8 *The symmetric permutation polytope* M_n^* *is given by the constraint system*

$$\sum_{i \in S} \sum_{j \in S \setminus i} x_{ij} \leqq |S| - 1 \qquad \forall S \in \mathcal{F}, \tag{1.4}$$

$$x_{ij} \geqq 0 \qquad \forall i, j \in N_n, \tag{1.5}$$

$$x_{ij} = x_{ji} \qquad \forall i, j \in N_n, \tag{1.6}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad \forall j \in N_n, \tag{1.7}$$

where $\mathcal{F} = \{S \subseteq N_n : |S| \geqq 3,\ |S| \equiv 1 \pmod 2\}$.

Proof The proof consists in constructing a graph G, such that every symmetric permutation matrix is an adjacency matrix of some pairing, and conversely. It then only remains to use the theorem on the convex hull of the characteristic vectors of pairings in G and to replace the characteristic vectors of pairings by their adjacency matrices.

We define a graph G with vertex set $V = N_{2n}$ and edge set $E = \{(i,j) : i,j \in N_n \text{ or } i \equiv j \pmod{n}\}$. In other words, G consists of the complete graph K_n, with every vertex i adjacent, in addition, to the edge $(i,i+n)$.

Let $(x_{ij})_{n\times n}$ be a symmetric permutation matrix. We associate it with the complete pairing $G(x)$ of G, whose characteristic vector y has components

$$y_e = \begin{cases} x_{ij} = x_{ji} & \text{for } i,j \in N_n, \\ x_{ii} & \text{for } i \equiv j \pmod{n} \end{cases} \tag{1.8}$$

for $e = (i,j) \in E$. Analagously, if P is a complete pairing of G and y is its characteristic vector, then we associate it with a symmetric permutation matrix through the same formula. By Theorem 5.8, Ch.4, characteristic vectors of complete pairings satisfy the system of inequalities

$$\sum_{e \in G(S)} y_e \leqq (|S|-1)/2 \qquad \forall S \subseteq V, \quad |S| \equiv 1 \pmod{2} \tag{1.9}$$

where $G(S)$ is the subgraph of $G = (V,E)$ generated by the vertices in S. The inequalities in (1.9) which correspond to subsets S which generate connected subgraphs $G(S)$ of G are superfluous (see Prob. 14, Ch.4). Thus we may certainly exclude from (1.9) those inequalities corresponding to subsets S which contain vertices with indices greater than n. If we now transform the remaining inequalities, using (1.8), to inequalities in the variables x_{ij}, we obtain the system (1.7). //

§2 THE HAMILTONIAN CYCLE POLYTOPE

An important rôle in discrete optimization is played by the travelling salesman problem. In this section we examine the possibility of linearizing the problem, that is, of constructing the convex hull of its feasible solutions.

2.1 The Symmetric and Unsymmetric Travelling Salesman Problem

Let $G = (V,E)$ be a graph with n vertices. A simple spanning cycle C in G is called a *Hamiltonian cycle*, and a graph which contains such a cycle is called a *Hamiltonian graph*. Any hamiltonian cycle can be characterized by means of its adjacency matrix $(x)_{n\times n}$, where $x_{ij} = 1$ if the edge $(i,j) \in C$ and $x_{ij} = 0$ otherwise. Any adjacency matrix of a hamitonian cycle is a symmetric Boolean matrix.

Definition 2.1 The convex hull in E_{n^2} of the adjacency matrices of the hamiltonian cycles of a given graph G will be called the *Hamiltonian cycle polytope* of the graph G.

The hamiltonian cycle polytope of a graph G is a face of the hamiltonian cycle polytope of the complete graph K_n. Thus, in what follows we study only the hamiltonian cycle polytope of the complete graph K_n. We denote this polytope by M_n^s.

The polytope M_n^s is the convex hull of a certain subset of the integral points of the polytope

$$\sum_{i=1}^{n} x_{ij} = 2 \qquad \forall j \varepsilon N_n,$$

$$0 \leqq x_{ij} = x_{ji} \leqq 1 \qquad \forall i,j \varepsilon N_n,$$

$$x_{ii} = 0 \qquad \forall i \varepsilon N_n.$$

Taking account of the equalities $x_{ij} = x_{ji}$ we can associate every hamiltonian cycle with a point in E_m, where $m = n(n-1)/2$. Let W denote the incidence matrix of the complete graph on n vertices K_n. The set of hamiltonian cycles of K_n is in one-to-one correspondence with the integral points of the polytope of solutions of the following system of inequalities :

$$Wx = 2e \qquad 0 \leqq x \leqq e,$$

$$\sum_{\substack{i,j \varepsilon S \\ i<j}} x_{ij} \leqq |S| - 1 \qquad \forall\, S \subset N_n . \tag{2.1}$$

The constraints (2.1) serve to eliminate those cycles which are not spanning cycles.

Now let K_n^* be the complete digraph on n vertices (any pair of vertices i,j is connected by the directed edges (i,j) and (j,i)) A *Hamiltonian tour* is a spanning di-chain in which all the vertices are different, with the exception of the first and the last. The adjacency matrix of a hamiltonian tour is a permutation matrix. Clearly, not every permutation matrix corresponds to a hamiltonian tour. However, every permutation matrix which satisfies the conditions

$$\sum_{i,j\in S} x_{ij} \leqq |S| - 1 \qquad \forall\, S \subset N_n, \tag{2.2}$$

is easily seen to yield a hamiltonian tour. Also it may be verified that every hamiltonian tour satisfies the constraints (2.2). We remark that the inequalities (2.2) for the subsets S and $\overline{S} = N_n \setminus S$ are equivalent because of the equations (1.2),(1.3). From the relations

$$\sum_{i,j\in S} x_{ij} = \sum_{i\in S} \sum_{j=1}^{n} x_{ij} - \sum_{i\in S} \sum_{j\in \overline{S}} x_{ij}$$

it follows that conditions (2.2) are equivalent to the following

$$\sum_{i\in S} \sum_{j\in \overline{S}} x_{ij} \geqq 1 \qquad \forall\, S \subset N_n \ , \quad S \neq \emptyset \ .$$

<u>Definition 2.2</u> The convex hull in E_{n^2} of the adjacency matrices of the hamiltonian tours in the digraph K_n^* is called the *hamiltonian tour polytope* , denoted by M_n^{as} .

Hamiltonian tours may be characterized by a list of directed edges, by an adjacency matrix and by the permutation corresponding to this matrix. In what follows we will use whichever description is convenient and the hamiltonian tour so given will be called, simply, a *tour* . A permutation corresponding to a tour is called *cyclic* . A cyclic permutation π can be written in the form $\pi = \langle i_1,\dots,i_n \rangle$, which is interpreted to mean that $x_{i_k i_{k+1}} = 1 \ \ \forall k\in N_n$, where $i_{n+1}=i_1$.

One of the problems in polyhedral combinatorial theory is to construct a system of inequalities which defines the polytopes M_n^s and M_n^{as} . To do this it is necessary to find the equations of hyperplanes which cut away the vertices of the bi-stochastic polytope M_n which are not tours. The study of the faces of M_n^{as} and M_n^s was begun in the early 1950's by J.Heller who investigated the possibility of finding a minimal weighted cycle or tour using the techniques of linear programming. The problems are better known as the symmetrical and unsymmetrical travelling salesman problem.

The hamiltonian cycle polytope is the feasible set in the symmetric travelling salesman problem, while the hamiltonian tour polytope is the feasible set for the unsymmetric travelling salesman problem. Later, the problem of linearizing the travelling salesman problem was widely studied : Heller (1955), Kuhn (1955), Norman (1955), Heller (1956), Padberg & Rao (1974), Maurras (1975), Savage et al. (1976), Grötschel & Padberg (1977), Sarvanov (1977), Papadimitrou (1978), Grötschel & Padberg (1979).

Every hamiltonian cycle defines precisely two tours corresponding to the two distinct orientations of the cycle. Thus there is a connection between the two polytopes M_n^s and M_n^{as}. The following theorem establishes connections between the faces of M_n^s and the faces of M_n^{as}.

Theorem 2.1 1). *If the inequality*

$$\sum_{1 \leq i < j \leq n} a_{ij} x_{ij} \leq a_0$$

defines a face of the polytope M_n^s, *then the inequality*

$$\sum_{1 \leq i \neq j \leq n} a_{ij} x_{ij} \leq a_0$$

defines a face of the polytope M_n^{as} *in the case where* $a_{ij} = a_{ji}$;

2). (Heller 1955). *If the inequality*

$$\sum_{1 \leq i \neq j \leq n} a_{ij} x_{ij} \leq a_0$$

defines a (d-1)-*face of the polytope* M_n^{as}, *then the inequality*

$$\sum_{1 \leq i < j \leq n} (a_{ij} + a_{ji}) x_{ij}/2 \leq a_0$$

defines a (d'-1)-*face of the polytope* M_n^s *if and only if the matrix with elements*

$$b_{ij} = \begin{cases} \dfrac{\sum_{k=1}^{n} \sum_{t=1}^{n} a_{kt}}{(n-1)(n-2)} - \dfrac{(n-1) \sum_{k=1}^{n} (a_{ik} + a_{kj})}{n(n-2)} - \dfrac{\sum_{k=1}^{n} (a_{ki} + a_{jk})}{n(n-2)}, & i \neq j, \\ 0, & i = j, \end{cases}$$

is symmetric. Here $d = \dim M_n^{as}$, $d' = \dim M_n^s$.

2.2 Dimension

Many authors have found a basis for the affine hull of hamiltonian tours ; Maurras (1975), Grötschel & Padberg (1977), Sarvanov (1977). In proving the following theorem we follow Maurras.

Theorem 2.2 *The dimension of the hamiltonian tour polytope* M_n^{as} *is equal to* $n^2 - 3n + 1$, $n \geqq 3$.

Proof For every tour we have the equations

$$x_{ii} = 0 \qquad \forall i \varepsilon N_n. \tag{2.3}$$

The equations $Rx = e$ together with (2.3) form a system whose rank is $3n-1$. Hence $\dim M_n^{as} \leqq n^2 - 3n + 1 = d$.

We show that we have equality ; $\dim M_n^{as} = d$. For $n=3$ this can be checked directly. We assume $n \geq 4$.

Suppose the contrary, that is $\dim M_n^{as} < d$. Then, there must exist a hyperplane containing M_n^{as} and given by a linear equation

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_{ij} x_{ij} = b , \tag{2.4}$$

which is linearly independent of the system $Rx = e$. By subtracting suitable linear combinations of the equations of the system $Rx = e$ from (2.4) we can ensure that

$$a_{i1} = a_{1i} = a_{23} = 0 , \qquad i=2,3,\dots,n. \tag{2.5}$$

Now consider the tours <i213....> and <i123....>,i>3 . Here the dots stand for identical directed chains joining the vertex 3 to vertex i so that both become hamiltonian tours. Since both tours satisfy (2.4) we have : $a_{i2} + a_{21} + a_{13} = a_{i1} + a_{12} + a_{23}$, which implies that $a_{i2}=0$ for all $i>3$. Similarly, using the tours <31j2....> and <3j12....> we find $a_{3j}=0$ for all $j>3$. Then, from pairs of tours of the form <ij12....> and <i1j2....> we have $a_{ij}=0$, $i \neq j$, $i,j>3$ and so forth. Eventually we discover that all the coefficients in (2.4) are zero. This contradiction proves the theorem. //

The following theorem has a similar proof.

Theorem 2.3 *The dimension of the hamiltonian cycle polytope* M_n^s *is equal to* $n(n-3)/2$.

2.3 Diameter

The diameter of a hamiltonian tour polytope was established by Padberg & Rao (1974) even though a criterion for adjacency of vertices of M_n^{as} is still not known. Also, Papadimitriou (1978) showed that the problem of establishing adjacency is NP-complete.

Theorem 2.4 *The diameter of the polytope* M_n^{as} *is equal to* 2 *when* $n \geqq 6$ (diam $M_n^{as} = 1$ *for* $n \leqq 5$).

Proof Let x' and x'' be non-adjacent vertices of M_n^{as} . We show that there is a vertex y which is adjacent to both x' and x'' . We conduct the proof in the language of permutations. Let ρ and τ be cyclic permutations on the set N_n . We show that there exists a cyclic permutation σ such that the permutations $\phi = \rho^{-1}\sigma$ and $\psi = \sigma^{-1}\tau$ satisfy the conditions of Corollary 1.4, that is, the vertices ρ,σ and σ,τ are adjacent on M_n and so are also adjacent on M_n^{as} .

It is well known that any permutation can be represented uniquely as a product of disjoint cycles. Let $\rho^{-1}\tau$ be expressible as a product of t cycles : $\rho^{-1}\tau = \langle i_1 j_1 \ldots\rangle\langle i_2 j_2 \ldots\rangle \ldots \langle i_t j_t \ldots\rangle$. Since ρ and τ are not adjacent vertices, $t \geqq 2$. Also, since ρ and τ are cyclic permutations, the length of each cycle $\langle i_s j_s \ldots\rangle$ is greater than or equal to two. Without loss of generality suppose that $1 \leqq i_1 < i_2 < \ldots < i_t \leqq n$ Let

$$\phi = \langle i_1 j_1 \ldots\rangle , \quad \psi = \langle i_2 j_2 \ldots\rangle \qquad \text{if } t=2,$$

$$\phi = \langle i_1 j_2 \ldots i_2 j_3 \ldots i_{t-1} j_t \ldots i_t j_1 \ldots\rangle , \quad \psi = \langle i_t \ldots i_1\rangle \text{ if } t \geqq 3$$

By definition, ϕ and ψ are cycles satisfying the condition $\phi\psi = \rho^{-1}\tau$. Hence, putting $\sigma = \rho\phi = \tau\psi^{-1}$, we find that, by Corollary 1.4, the vertices ρ,σ and also σ,τ are adjacent on M_n^{as} . It remains to show that σ is a cyclic permutation. We consider separately the cases in which t is odd or even. Without loss of generality suppose that $\tau = \langle 12 \ldots n\rangle$.

Suppose t is odd. Then, putting $\psi = \langle i_t \ldots i_1\rangle$ and

$\phi = \langle i_1 j_2 \ldots i_2 j_3 \ldots i_t j_1 \rangle$ and $\sigma = \rho\phi$, we find that

$$\sigma = \tau\psi^{-1} = \langle 1 \ldots i_1 \ldots i_k \ldots i_t \ldots n \rangle \langle i_1 i_2 \ldots i_t \rangle$$

$$= \langle 1 .. i_1 i_2{+}1 .. i_3 i_4{+}1 .. i_t i_1{+}1 .. i_2 i_3{+}1 .. i_4 i_5{+}1 .. i_{t-1} i_t{+}1 .. n \rangle$$

which is a cyclic permutation.

Suppose t is even. Let i_{t+1} be the last component of the cycle $\langle i_t j_t \ldots i_{t+1} \rangle$ (if this cycle has length two we consider that $j_t = i_{t+1}$). Clearly $i_t < i_{t+1} \leqq n$. Define $\psi = \langle i_{t+1} i_t \ldots i_1 \rangle$ and $\phi = \langle i_1 j_2 \ldots i_2 j_3 \ldots i_{t-1} j_t \ldots i_{t+1} j_1 \ldots \rangle$. Putting $\sigma = \tau\psi^{-1} = \rho\phi$ we find

$$\sigma = \langle 1 \ldots i_1 \ldots i_k \ldots i_{t+1} \ldots n \rangle \langle i_1 i_2 \ldots i_t i_{t+1} \rangle$$

$$= \langle 1 .. i_1 i_2{+}1 .. i_3 i_4{+}1 .. i_{t+1} i_1{+}1 .. i_2 i_3{+}1 .. i_4 i_5{+}1 .. i_t i_t{+}1 .. n \rangle$$

which is a cyclic permutation.

Thus, $\operatorname{diam} M_n^{as} \leqq 2$ for $n \geqq 4$. We prove that $\operatorname{diam} M_n^{as} = 2$ for $n \geqq 6$. To do this we consider the cyclic permutations :

$$\tau = \langle 12 \ldots n \rangle, \quad \rho = \langle 1324657 \ldots n \rangle,$$

$$\sigma' = \langle 1324 \ldots n \rangle, \quad \sigma'' = \langle 1234657 \ldots n \rangle.$$

It can be checked directly that both ρ and τ are adjacent on M_n to both σ' and σ''. At the same time neither ρ and τ nor σ' and σ'' are adjacent on M_n. Also, the equation $\rho/2 + \tau/2 = \sigma'/2 + \sigma''/2$ shows that the minimal dimension of a face of M_n or of M_n^{as} which contains all four vertices is equal to 2. Hence, $\operatorname{diam} M_n^{as} = 2$ for $n \geqq 6$.

It remains to observe that the equation $\operatorname{diam} M_n^{as} = 1$ for $n \leqq 5$ can be checked directly. //

2.4 Faces

We will describe some classes of linear inequalities which define facets of the d-polytope M_n^{as}. To show that the inequality $ax \leqq a_0$ determines a face of M_n^{as} it suffices to show that M_n^{as} belongs to a half-space generated by this inequality and to exhibit a tour for which $ax^0 = a_0$. To show that $ax \leqq a_0$ defines a facet of M_n^{as} , we need to do some preliminary work. To prove the following theorem we use ideas in Maurras (1975).

Theorem 2.5 *Each of the inequalities*

$$x_{i_1i_2} + x_{i_2i_1} \leqq 1 \qquad \text{for } n \geqq 5, \qquad (2.6)$$

$$x_{i_1i_2} + x_{i_1i_3} + x_{i_2i_3} + x_{i_2i_4} + x_{i_4i_2} \leqq 2 \quad \text{for } n \geqq 6, \qquad (2.7)$$

$$2x_{i_1i_2} + x_{i_1i_4} + 2x_{i_2i_1} + x_{i_2i_1} + x_{i_2i_3} + x_{i_3i_1} + x_{i_4i_2} \leqq 3 \qquad \text{for } n \geqq 5, \qquad (2.8)$$

$$x_{i_1i_2} + x_{i_1i_3} + x_{i_3i_2} + \sum_{\substack{i \neq i_1, i_2 \\ j \neq i_1, i_2}} x_{ij} \leqq n-2 \quad \text{for } n \geqq 4, \qquad (2.9)$$

$$x_{i_1i_3} + x_{i_3i_1} + x_{i_1i_2} + x_{i_2i_1} + x_{i_2i_3} + x_{i_3i_2} \leqq 2 \qquad \text{for } n \geqq 5, \qquad (2.10)$$

defines a facet of the polytope M_n^{as} . *Here, the indices* $i_1, \ldots, i_4$ *can be any pairwise distinct values from* 1 *to* n .

Proof The fact that any tour $x \in M_n^{as}$ satisfies the inequalities (2.6)-(2.10) can be verified directly. It is rather more difficult to show for each of these equations that they determine a facet.

We illustrate how this can be done by using the procedure used in the proof of theorem 2.2.

Consider any inequality of the type (2.6). For example, consider the inequality

$$x_{12} + x_{21} \leqq 1 \qquad (2.11)$$

Suppose that it does not determine a facet of M_n^{as} . Then there is a hyperplane given by an equation of type (2.4) which is linearly independent of the equations $Rx = e$ and of equation (2.11). But every tour satisfying (2.11) must also satisfy (2.4). As in the proof of theorem 2.2 we suppose that $a_{i1} = a_{1i} = a_{23} = 0$, $i \neq 1$. Then, considering the appropriate pairs of tours we have

$$a_{i2}=0,\ i\geqq 4 \qquad (\text{tours } \langle i123\ldots\rangle, \langle i213\ldots\rangle);$$

$$a_{2i}=0,\ i\geqq 5 \qquad (\text{tours } \langle 421i\ldots\rangle, \langle 412i\ldots\rangle);$$

$$a_{32}=0 \qquad (\text{tours } \langle 3215\ldots\rangle, \langle 3125\ldots\rangle);$$

$$a_{24}=0 \qquad (\text{tours } \langle 5214\ldots\rangle, \langle 5124\ldots\rangle).$$

Finally, considering the tours $\langle ij12k\ldots\rangle$, $\langle i12jk\ldots\rangle$, $i\neq j\neq k$, $i,j,k\geqq 3$, we obtain $a_{ij}=a_{jk}$, that is, for all $p\neq q$, $p,q\geqq 3$, the coefficients a_{pq} are all equal. It is now easily seen that equation (2.4) is equivalent to the following

$$\sum_{\substack{i,j\geqq 3\\ i\neq j}} x_{ij} = n-3$$

which is linearly dependent on equation (2.11) and on $Rx = e$. This contradiction shows that equation (2.11) defines a facet. //

It should be noted that the classes of inequalities (2.6)-(2.10) given in the theorem do not give a complete linear description of the polytope M_n^{as} . At the same time, it is easily seen that the different classes of equations define distinct facets. Also, in each of the classes of inequalities (2.7),(2.8), each inequality generates a different facet.

Finally, we present some classes of linear inequalities which have recently been obtained by Grötschel & Padberg (1979) for describing the convex polytope M_n^s .

Theorem 2.6 *A facet of the polytope* M_n^s *is determined by each of the following inequalities :*

$$x_{ij} \leqq 1 \qquad \forall\, i,j \in N_n\ , \quad i<j,$$

$$x_{ij} \geqq 0 \qquad \forall\, i,j \in N_n\ , \quad i<j,$$

$$\sum_{\substack{i,j\in S\\ i<j}} x_{ij} \leqq |S|-1 \qquad \forall\, S\subset N_n\ , \quad |S|\geqq 3,$$

$$\sum_{s=0}^{k} \sum_{\substack{i,j\in V_s\\ i<j}} x_{ij} \leqq |V_0| + \sum_{s=1}^{k}(|V_s|-1) - \frac{k+1}{2}\ ,$$

where $(V_0,\dots,V_k)$ *is a 'toothed' system of sets, that is, a family of subsets of* N_n *such that :*

1) k *is odd ;*
2) $V_0 \cap V_i \neq \emptyset \quad \forall i \in N_k$;
3) $V_i \not\subset V_0 \quad \forall i \in N_k$;
4) $V_i \cap V_j = \emptyset \quad \forall i,j \in N_k\,,\quad i\neq j$

§3 THE PERMUTATION POLYTOPE

Let the vector $a = (a_1,\dots,a_n)$ be given. We assume that

$$a_1 > a_2 > \dots > a_n \geqq 0\,. \qquad (3.1)$$

As usual, let S_n be the set of permutations of the numbers N_n. With each permutation $\pi = (\pi_1,\dots,\pi_n) \in S_n$ we associate the point $a_\pi = (x_1,\dots,x_n)$ where $x_i = a_{\pi_i}$.

Definition 3.1 The convex hull of the points $\{a_\pi = (a_{\pi_1},\dots,a_{\pi_n}) : \pi \in S_n\}$ in E_n is called a *permutation polytope* and is denoted by $M_n(a)$. The polytope $M_4(a)$ with $a = (1,2,3,4)$ is depicted in Figure 34.

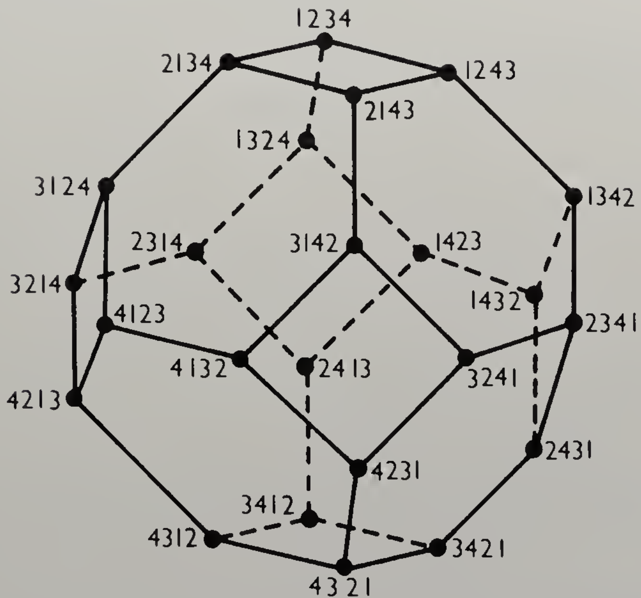


Fig. 34.

It is clear that the permutation polytope $M_n(a)$ is the image of the assignment polytope M_n under the singular affine mapping $\Lambda : E_{n^2} \to E_n$, where

$$\Lambda = \begin{bmatrix} a_1 & \dots & a_n & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & a_1 & \dots & a_n & \dots & 0 & \dots & 0 \\ . & \dots & . & . & \dots & . & \dots & . & \dots & . \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & a_1 & \dots & a_n \end{bmatrix}$$

3.1 Rado's Theorem

Theorem 3.1 *The permutation polytope* $M_n(a)$ *is given by the following system of constraints :*

$$\sum_{i\in\omega} x_i \leq \sum_{i=1}^{|\omega|} a_i \qquad \forall\, \omega \subset N_n \,, \tag{3.2}$$

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} a_i \,.$$

This theorem follows from a result of Rado (1952). Before formulating Rado's theorem we introduce the concept of the majorization of vectors.

Definition 3.2 The vector $x = (x_1,\dots,x_n)$ is *majorized* by the vector $y = (y_1,\dots,y_n)$ (written $x \prec y$), if

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \,, \tag{3.4}$$

and if there are permutations $\tau \in S_n$ and $\pi \in S_n$ such that

$$\sum_{i=1}^{\nu} x_{\tau_i} \leq \sum_{i=1}^{\nu} y_{\pi_i} \qquad \forall\, \nu \in N_{n-1} \,.$$

The following lemma, due to Schur (see Hardy, Littlewood & Polya, 1952) gives necessary and sufficient conditions for the majorization of vectors.

Lemma 3.2 *The vector* x *is majorized by the vector* y *if and only if there is a bi-stochastic matrix* $\Delta = (\delta_{ij})_{n\times n}$ *such that* $x = \Delta y$.

Proof 1) Sufficiency. Without loss of generality we may assume that $x_1 \geqq \cdots \geqq x_n$, $y_1 \geqq \cdots \geqq y_n$.

Let $w_j^t = \sum_{i=1}^{t} \delta_{ij}$. Since Δ is a bistochastic matrix we have

$$0 \leqq w_j^t \leqq 1 \quad , \quad t = \sum_{j=1}^{n} w_j^t .$$

Sufficiency is established by the following chain of relations :

$$\sum_{i=1}^{\nu} y_i - \sum_{i=1}^{\nu} x_i = \sum_{i=1}^{\nu} y_i - \sum_{i=1}^{\nu} \sum_{j=1}^{n} \delta_{ij} y_j$$
$$= \sum_{i=1}^{\nu} y_i - \sum_{j=1}^{n} w_j^{\nu} y_j = \sum_{i=1}^{\nu} y_i (1-w_i^{\nu}) - \sum_{i=\nu+1}^{n} w_i^{\nu} y_i$$
$$\geqq \sum_{i=1}^{\nu} y_{\nu}(1-w_i^{\nu}) - \sum_{i=\nu+1}^{n} w_i^{\nu} y_{\nu} = y_{\nu}(\nu - \sum_{i=1}^{n} w_i^{\nu}) = 0 .$$

2) Necessity. We use induction. When $n = 1$ the vectors x and y have one component each, $x_1 = y_1$, and the required matrix Δ clearly exists. Suppose that the theorem is true for (n-1)-vectors and consider two n-vectors x and y such that $x \prec y$. From the condition $x_1 \leqq y_1$ and equation (3.4) it follows that $y_n \leqq x_1 \leqq y_1$. Thus there is a k such that

$$y_{k+1} \leqq x_1 \leqq y_k . \tag{3.5}$$

Thus for some λ $(0 \leqq \lambda \leqq 1)$

$$x_1 = \lambda y_k + (1-\lambda) y_{k+1} . \tag{3.6}$$

Now use the vectors x and y to define two (n-1)-vectors $x' = (x_2, \ldots, x_n)$, $y' = (y_1, \ldots, y_{k-1}, y_k + y_{k+1} - x_1, y_{k+2}, \ldots, y_n)$.

It follows from (3.5) that the components of y' are in order of decreasing magnitude. We also have the relation $x' \prec y'$. Thus, by the inductive assumption, there is a bistochastic matrix $\Delta' = (\delta_{ij})_{(n-1)\times(n-1)}$ such that $x' = \Delta' y'$, or, in full

$$x_{s+1} = \delta_{s1} y_1 + \cdots + \delta_{s,k-1} y_{k-1} + \delta_{sk}(y_k + y_{k+1} - x_1)$$
$$+ \delta_{s,k+1} y_{k+2} + \cdots + \delta_{s,n-1} y_n \qquad \forall s \in N_{n-1} .$$

Substituting for x_1 from equation (3.6), we obtain

$$x_{s+1} = \delta_{s1}y_1 + \ldots + \delta_{sk}(1-\lambda)y_k + \delta_{sk}\lambda y_{k+1}$$
$$+ \delta_{s,k+1}y_{k+2} + \ldots + \delta_{s,n-1}y_n \qquad \forall s \in N_{n-1} .$$

Taking equations (3.6) into account, we find easily that the vectors x and y are connected by the bistochastic matrix

$$\Delta = \begin{pmatrix} 0 & 0 & \lambda & 1-\lambda & \ldots & 0 \\ \delta_{11} & \delta_{12} & \delta_{1k}(1-\lambda) & \delta_{1k}\lambda & \ldots & \delta_{1,n-1} \\ \cdot & \cdot & \cdot & \cdot & \ldots & \cdot \\ \delta_{n-1,1} & \delta_{n-1,2} & \delta_{n-1,k}(1-\lambda) & \delta_{n-1,k}\lambda & \ldots & \delta_{n-1,n-1} \end{pmatrix} .$$

This completes the proof of the Lemma. //

<u>Theorem 3.3 (Rado's Theorem)</u> *The point* $x \in M_n(a)$ *if and only if the vector* x *is majorized by the vector* a .

Proof Let $x \prec a$. Then, by Lemma 3.2, there is a bistochastic matrix Δ such that $x = \Delta a$. By Birkhoff's Theorem

$$\Delta = \sum_{\pi \in S_n} \lambda_\pi \Delta_\pi \quad , \quad \sum_{\pi \in S_n} \lambda_\pi = 1 \quad , \quad 0 \leqq \lambda_\pi \qquad \forall \pi \in S_n ,$$

where Δ_π is the permutation matrix corresponding to the permutation π . Consequently,

$$x = \sum_{\pi \in S_n} \lambda_\pi \Delta_\pi a = \sum_{\pi \in S_n} \lambda_\pi a_\pi$$

that is $x \in M_n(a)$. Necessity is proved by reversing the argument. //

3.2 <u>The f-vector</u>

We investigate the combinatorial properties of the permutation polytope. The most important of these is the fact that the combinatorial type of a permutation polytope does not depend on the vector a , provided all of its components are distinct.

Theorem 3.4 *A set of solutions of the system* (3.2),(3.3) *is an* i-*face* ($0 \leq i \leq n-2$) *of the permutation polytope* $M_n(a)$ *if and only if for each such solution, the inequalities* (3.2) *are satisfied as equalities only for subsets* $\omega_1, \omega_2, \ldots, \omega_{n-i-1}$ *such that*

$$\omega_1 \subset \omega_2 \subset \ldots \subset \omega_{n-i-1} \subset N_n \ . \tag{3.7}$$

Proof The system of inequalities (3.2) plus the equalities

$$\sum_{j \in \omega_k} x_j = \sum_{j=1}^{|\omega_k|} a_j \qquad \forall k \in N_{n-i} \ , \tag{3.8}$$

where $\omega_{n-i} = N_n$, is consistent and so determines a face of the polytope $M_n(a)$. The rank of the system (3.8) is equal to $n-i$ and also, all of the inequalities in (3.2) with $\omega \neq \omega_k \ \forall k \in N_{n-i}$ can be satisfied strictly (here, it should be remembered that the numbers a_j , $j \in N_n$ are all distinct). Hence, the face defined has dimension i .

We prove the converse. Let the face F of the permutation polytope be given by equations (3.8). Suppose, for contradiction, that the inclusions (3.7) do not hold for the sets ω_k , that is, there is a pair of index sets ω_p , ω_q such that neither of the inclusions $\omega_p \subset \omega_q$, $\omega_q \subset \omega_p$ hold. Then, for any arbitrary point $x^0 \in F$ we have

$$\sum_{i=1}^{|\omega_p|} a_i + \sum_{i=1}^{|\omega_q|} a_i = \sum_{i \in \omega_p} x_i^0 + \sum_{i \in \omega_q} x_i^0$$

$$= \sum_{i \in \omega_p \cup \omega_q} x_i^0 + \sum_{i \in \omega_p \cap \omega_q} x_i^0 \leq \sum_{i=1}^{|\omega_p \cup \omega_q|} a_i + \sum_{i=1}^{|\omega_p \cap \omega_q|} a_i \ .$$

On the other hand, because of (3.1) we have

$$\sum_{i=1}^{|\omega_p \cup \omega_q|} a_i + \sum_{i=1}^{|\omega_p \cap \omega_q|} a_i < \sum_{i=1}^{|\omega_p|} a_i + \sum_{i=1}^{|\omega_q|} a_i \ .$$

This contradiction shows the necessity of conditions (3.7) in the theorem.

Corollary 3.5 *For all permutations* $\pi \in S_n$ *the point* a_π *is a vertex of the polytope* $M_n(a)$.

It is easily verified that $\dim M_n(n) = n-1$.

Corollary 3.6 *All permutation polytopes of the same dimension are combinatorially equivalent.*

Theorem 3.7 *The components of the* f*-vector of the permutation polytope* $M_n(a)$ *are given by*

$$f_i(M_n(a)) = \sum \frac{n!}{t_1!\,t_2!\ldots t_{n-i}!} \qquad \forall i \in N_{n-1} \tag{3.9}$$

where the sum is carried out over all positive integral solutions of the equation $t_1+t_2+\ldots+t_{n-i} = n$.

Proof By theorem 3.4, every partition $Q_1,\ldots,Q_{n-i}$ of the set N_n into n-i non-empty subsets determines, through equations (3.8), an i-face of $M_n(a)$ with

$$\omega_k = \bigcup_{s=1}^{k} Q_s \ .$$

Let this face be denoted by $F(Q_1,\ldots,Q_{n-i})$. It is well known from combinatorial analysis, that the number of distinct partitions of a set of n elements into n-i subsets each of which contains t_s , $s \in N_{n-i}$, elements is given by

$$\frac{n!}{t_1!\,t_2!\ldots t_{n-i}!} \ .$$

This proves the theorem. //

Let $Q_1,\ldots,Q_{n-i}$ be a partition of N_n into n-i non-empty subsets. Let $S(Q_1,\ldots Q_{n-i})$ be the set of all permutations which permute the elements of each subset Q_s among themselves. Let $S^{-1}(Q_1,\ldots,Q_{n-i})$ be the set of permutations which are the inverses of permutations in $S(Q_1,\ldots,Q_{n-i})$.

Corollary 3.8 *The face* $F(Q_1,\ldots Q_{n-i})$ *is generated by the permutations* a_π *for all* $\pi \in S^{-1}(Q_1,\ldots,Q_{n-i})$.

Proof We show that the vertex a_π for $\pi \in S^{-1}(Q_1,\ldots,Q_{n-i})$ lies in the face $F(Q_1,\ldots,Q_{n-i})$. Indeed, let $x = a_\pi$. Then, from the definition of

$S^{-1}(Q_1,\dots,Q_{n-i})$, the sets $\{\pi_i : i \in \bigcup_{s=1}^{k} Q_s\}$ and $\{1,2,\dots,t_1+\dots+t_k\}$ coincide, that is, we have the equalities

$$\sum_{i\in\omega_k} x_i = \sum_{i\in\omega_k} a_{\pi_i} = \sum_{i=1}^{|\omega_k|} a_i \qquad \forall\, k \in N_{n-i} ,$$

where $\omega_k = \bigcup_{s=1}^{k} Q_s$. On the other hand, if $\pi \notin S^{-1}(Q_1,\dots,Q_{n-i})$, then the vertex a_π does not satisfy the inequalities (2.2) for $\omega = \omega_k$, $k \in N_{n-i}$, as equalities and so it does not lie in the face $F(Q_1,\dots,Q_{n-i})$. //

Corollary 3.9 *The vertices adjacent to the vertex* $a_\pi = (a_{\pi_1},\dots,a_{\pi_n})$ *are the vertices which correspond to the permutation* πk *which is obtained from* π *by transposing the* k^{th} *and* $k+1^{th}$ *components, for some* $k \in N_{n-1}$.

Proof Any 1-face (edge) $F = F(Q_1,\dots,Q_{n-1})$, where $Q_s = \{\pi_1,\dots,\pi_s\}$, $\forall s \in N_{k-1}$, and $Q_s = \{\pi_1,\dots,\pi_{s+1}\}$, $\forall s \in N_{n-1}\setminus N_{k-1}$, is, according to theorem 3.4, given by the constraints (3.2),(3.3) and by the following equations :

$$\sum_{i=1}^{s} x_{\pi_i} = \sum_{i=1}^{s} a_i \qquad s=1,\dots,k-1,k+1,\dots,n .$$

By corollary 3.8 , the vertices $a_{\pi^{-1}}$ and $a_{\pi_0^{-1}}$ belong to F , where $\pi = (\pi_1,\dots,\pi_n)$ and $\pi_0 = (\pi_1,\dots,\pi_{k-1},\pi_{k+1},\pi_{k+2},\dots,\pi_n)$. The permutations π^{-1} and π_0^{-1} clearly differ by a transposition of the k^{th} and $k+1^{th}$ components. //

Corollary 3.10 *The diameter of the permutation polytope* $M_n(a)$ *is equal to* $n(n-1)/2$.

The proof is by induction.

3.3 The Permutation Polymatroid

The constraints defining a permutation polytope are similar to the constraints used to define a polymatroid. We show that this is not fortuitous. Indeed, a permutation polytope is a face of some polymatroid.

First, note that if equation (3.3) is satisfied, the inequalities (3.2) are equivalent to the following :

$$\sum_{i\in\omega} x_i \geqq \sum_{i=1}^{|\omega|} a_{n-i+1} \qquad \forall\, \omega \subset N_n \ . \tag{3.11}$$

Theorem 3.11 *The polytope given by the constraints* (3.2) *and*

$$x_i \geqq 0 \qquad \forall\, i \in N_n \tag{3.12}$$

is a bounded polymatroid. The polyhedron, given by the constraints (3.11) *is an unbounded polymatroid.*

Proof We have to show that the functions

$$\rho(\omega) = \sum_{i=1}^{|\omega|} a_i \quad , \quad \rho'(\omega) = \sum_{i=1}^{|\omega|} a_{n-i+1}$$

are non-negative, non-decreasing and, respectively, submodular and supermodular functions. The first two properties hold because the numbers a_i $i \in N_n$ are non-negative. The submodularity of the function $\rho(\omega)$ follows from the following inequality :

$$\sum_{i=1}^{|I|} a_i + \sum_{i=1}^{|J|} a_i \geqq \sum_{i=1}^{|I\cup J|} a_i + \sum_{i=1}^{|I\cap J|} a_i \ ,$$

which follows from (3.1). The function $\rho'(\omega)$ can be represented in the form $\rho'(\omega) = \rho(N_n) - \rho(N_n \setminus \omega)$ and is therefore supermodular. //

Definition 3.3 The polytope defined by the constraints (3.2) and (3.12) is called a *bounded permutation polymatroid* . The polyhedron given by the constraints (3.11) is called an *unbounded permutation polymatroid* .

The following theorem is a consequence of theorems 3.1 and 3.11.

Theorem 3.12 *A permutation polytope is the intersection of a bounded and an unbounded polymatroid.*

A permutation polymatroid is shown in Fig. 35. The polytope $M_3(3,2,1)$ is shaded.

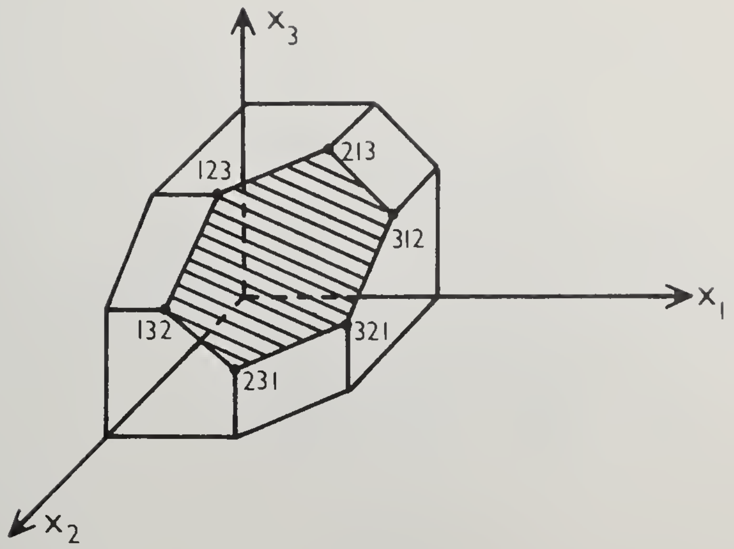


Fig. 35.

3.4 The Even Permutation Polytope

If, in a permutation, the number i lies to the left of the number j and if $i > j$ then these numbers are said to be inverted. If a permutation contains an even number of inversions it is called an *even permutation*, otherwise it is an *odd permutation*. Clearly, if a single transposition is carried out on a permutation then its parity is altered. The sets of even and of odd permutations in S_n are denoted by S_n^+ and S_n^- respectively.

Definition 3.4 The convex hull of the points $a_\pi = (a_{\pi_1}, \ldots, a_{\pi_n})$ for all $\pi \in S_n^+$ is called the *even permutation polytope*, $M_n^+(a)$. Fig. 36 shows the polytope $M_4^+(4,3,2,1)$.

The polytope $M_n^+(a)$ can be obtained from the permutation polytope $M_n(a)$ by cutting out the vertices a_π for all $\pi \in S_n^-$ and ensuring that no new vertices are formed. This is easily done. Indeed, by corollary 3.9, if $\pi \in S_n^-$ then its adjacent vertices a_σ (there are $n-1$ of them) are associated with the permutations $\sigma \in S_n^+$. Thus, the hyperplane H_π which passes through the vertices a_σ strictly separates a_π from the polytope $\operatorname{conv}\{a_\tau : \tau \in S_n \setminus \pi\}$ which it supports. Thus, the intersection of the polytope $M_n(a)$ and all the half spaces H_π^+, $\pi \in S_n^-$ is precisely the polytope $M_n^+(a)$. The hyperplane H_π is uniquely defined by the vertices a_σ which are adjacent to a_π, for the points a_σ are

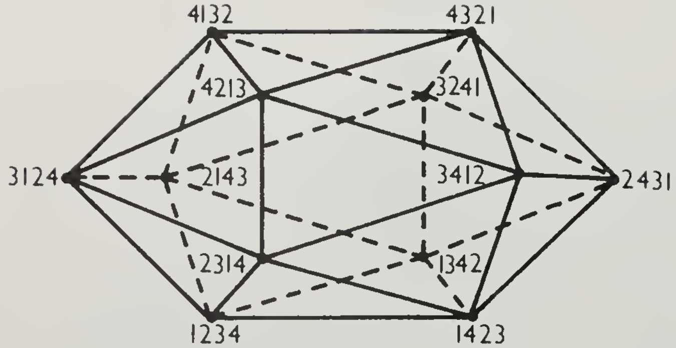


Fig. 36.

affinely independent. Putting

$$c_1 = a_1 \ , \ c_2 = a_2 \ , \ c_i = c_{i-1} - \frac{(a_{n-1}-a_n)(a_1-a_2)}{a_{n-i+1} - a_{n-i+2}}$$

we find that the desired hyperplane is given by the equation

$$\sum_{i=1}^{n} c_{\pi_i} x_i = \sum_{i=1}^{n} c_i a_{n-i+1} + (a_{n-1}-a_n)(a_1-a_2) \ .$$

We have thus established the following theorem.

Theorem 3.13 *The even permutation polytope* $M_n^+(a)$ *is given by the inequalities* (3.2),(3.3) *and*

$$\sum_{i=1}^{n} c_{\pi_i} x_i \geqq \sum_{i=1}^{n} c_i a_{n-i+1} + (a_{n-1}-a_n)(a_1-a_2) \qquad \forall \pi \in S_n^- \ . \qquad (3.13)$$

If $n > 4$ *then every inequality* (3.13) *defines a face .*

§4. THE ARRANGEMENT POLYTOPE

In this section we study the projection of the permutation polytope onto a space of lower dimension, in fact, projections onto an intersection of coordinate planes.

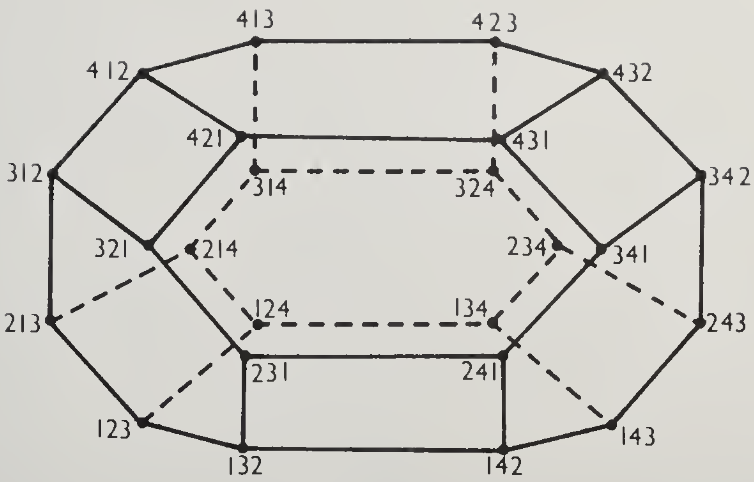


Fig. 37.

4.1 Algebraic Description

Let the vector $a = (a_1, \dots, a_n)$ be given whose components satisfy

$$a_1 > a_2 > \dots > a_n > 0 \, . \tag{4.1}$$

An m-*arrangement* is an ordered choice of m distinct elements ($m \leq n$) from the set N_n. With each m-arrangement $\pi = (\pi_1, \dots, \pi_m)$ we associate the point $x = (x_1, \dots x_m)$ by the rule $x_i = a_{\pi_i}$. This point is denoted by a_π.

Definition 4.1 The convex hull in E_m of the points a_π for all m-arrangements π of N_n is called an *arrangement polytope* and is denoted by $M_n^m(a)$.

Fig. 37 depicts $M_4^3(4,3,2,1)$.

Theorem 4.1 *The arrangement polytope* $M_n^m(a)$ *is the solution set of the following system of inequalities :*

$$\sum_{i=1}^{|\omega|} a_{n-i-1} \leq \sum_{i \in \omega} x_i \leq \sum_{i=1}^{|\omega|} a_i \qquad \omega \in N_m \, . \tag{4.2}$$

We first reformulate the theorem in terms of majorization of vectors. To do this we generalize the concept of majorization, introduced in the previous section, to the case of vectors of different dimensions. We say that the m-vector $x = (x_1,\ldots,x_m)$ is majorized by the n-vector $y = (y_1,\ldots,y_m,\ldots,y_n)$ if there are permutations $\tau \in S_m$ and $\pi \in S_n$ such that $x_{\tau_1} \geqq \cdots \geqq x_{\tau_m}$, $y_{\pi_1} \geqq \cdots \geqq y_{\pi_m} \geqq \cdots \geqq y_{\pi_n}$,

$$\sum_{i=1}^{\nu} x_{\tau_i} \leqq \sum_{i=1}^{\nu} y_{\pi_i} \qquad \forall \nu \in N_m , \tag{4.3}$$

$$\sum_{i=1}^{\nu} x_{\tau_{m-i+1}} \geqq \sum_{i=1}^{\nu} y_{\pi_{n-i+1}} \qquad \forall \nu \in N_m . \tag{4.4}$$

Definition 4.2 The matrix $\Delta = (\delta_{ij})_{m\times n}$ is called *sub-stochastic* if it satisfies the constraints

$$\sum_{j=1}^{n} \delta_{ij} = 1 \quad , \quad \sum_{i=1}^{m} \delta_{ij} \leqq 1 \quad , \quad \delta_{ij} \geqq 0 \qquad \forall (i,j) \in N_m \times N_n .$$

Lemma 4.2 *The* m-*vector* x *is majorized by the* n-*vector* y *if and only if there is a substochastic matrix* $\Delta = (\delta_{ij})_{m\times n}$ *such that* $x = \Delta y$.

Proof (i) Sufficiency is proved in a similar way to that given in lemma 3.2.

(ii) Necessity. We extend the m-vector x with additional components

$$x_{m+j} = c \ , \ \forall j \in N_{n-m} \quad , \quad c = \left(\sum_{i=1}^{n} y_i - \sum_{i=1}^{m} x_i\right)/(n-m) .$$

Denote the new n-vector by x^0 . We show that if the m-vector x is majorized by the n-vector y , then the n-vector x^0 is majorized by the n-vector y . Clearly

$$\sum_{i=1}^{n} x_i^0 = \sum_{i=1}^{n} y_i . \tag{4.5}$$

It remains to verify the inequalities

$$\sum_{i=1}^{\nu} x_{\phi_i}^0 \leqq \sum_{i=1}^{\nu} y_{\pi_i} \qquad \forall \nu \in N_{n-1} , \tag{4.6}$$

where the permutation $\phi \in S_n$ is such that $x^0_{\phi_1} \geqq \dots \geqq x^0_{\phi_n}$. Let $x_{\phi_\nu} > c$, $\nu=1,\dots,s$, $x_{\phi_\nu} = c$, $\nu=s+1,\dots,p$ and $x_{\phi_\nu} < c$ for the remaining ν. Then, conditions (4.3) imply that the inequalities (4.6) are true for $\nu = 1,\dots,s$. For $\nu > p$ the inequalities (4.6) follow from (4.4) and (4.5). For $\nu = s+1,\dots,p$ we prove (4.6) by contradiction. Suppose there is a number ν $(s<\nu\leqq p)$ such that $\sum_{i=1}^{\nu} x^0_{\phi_i} > \sum_{i=1}^{\nu} y_{\pi_i}$. Then the equality (4.5) implies that

$$\sum_{i=\nu+1}^{n} x^0_{\phi_i} < \sum_{i=\nu+1}^{n} y_{\pi_i} .$$

Since inequalities (4.6) have been established for $\nu \leqq s$ and $\nu > p$ we obtain the inequalities

$$\sum_{i=s+1}^{\nu} x^0_{\phi_i} > \sum_{i=s+1}^{\nu} y_{\pi_i} , \quad \sum_{i=\nu+1}^{p} x^0_{\phi_i} < \sum_{i=\nu+1}^{p} y_{\pi_i} .$$

Then, taking into account the equations

$$\sum_{i=s+1}^{\nu} x^0_{\phi_i} = c(\nu-s) , \quad \sum_{i=\nu+1}^{p} x^0_{\phi_i} = c(p-\nu) ,$$

we have

$$\frac{1}{\nu-s} \sum_{i=s+1}^{\nu} y_{\pi_i} < c < \frac{1}{p-\nu} \sum_{i=\nu+1}^{p} y_{\pi_i} ,$$

whereas

$$\frac{1}{\nu-s} \sum_{i=s+1}^{\nu} y_{\pi_i} \geqq \frac{1}{p-\nu} \sum_{i=\nu+1}^{p} y_{\pi_i} .$$

This contradiction establishes the truth of (4.6) for all $\nu \in N_{n-1}$. By lemma 3.2 we can write $x^0 = \Delta^0 y$, where Δ^0 is a bistochastic $(n\times n)$-matrix. Consequently, $x = \Delta y$ where Δ is the substochastic matrix formed by the first m rows of Δ^0. //

We reformulate theorem 4.1 in terms of majorization of vectors.

<u>Theorem 4.3</u> *The point* $x \in M^m_n(a)$ *if and only if the vector* x *is majorized by the vector* a.

Proof First note that every m-arrangement $\pi = (\pi_1,\dots,\pi_m)$ corresponds to a substochastic Boolean $(m\times n)$-matrix $\Delta = (\delta_{ij})$ whose components are defined by the rule : $\delta_{ij} = 1$ if $i = \pi_j$ and $\delta_{ij} = 0$ otherwise. Conversely, every substochastic Boolean $(m\times n)$-matrix corresponds by the same rule to an m-arrangement. We will refer to substochastic Boolean $(m\times n)$-matrices as m-*arrangement matrices* . By Birkhoff's theorem it follows that the set of all substochastic matrices coincides with the convex hull of the m-arrangement matrices. Also, every vector a_π , where π is an m-arrangement, can be represented as $a_\pi = \Delta_\pi a$. Here Δ_π is the corresponding arrangement matrix. Now, as in the proof of theorem 3.3 for any point $x \in M_n^m(a)$ we have

$$x = \sum_\pi \lambda_\pi a_\pi = \sum_\pi \lambda_\pi \Delta_\pi a = \Delta a ,$$

where the sum is taken over all m-arrangements π of N_n . Hence, by lemma 4.2, the vector x is majorized by the vector a . //

The following is a consequence of theorems 4.1 and 3.11.

<u>Corollary 4.4</u> *The arrangement polytope* $M_n^m(a)$ *is the intersection of a bounded permutation polymatroid* $M_m(a^1)$ *and an unbounded permutation polymatroid* $M_m(a^2)$ *where the vectors* a^1 *and* a^2 *consist respectively of the first* m *and the last* m *components of* a .

This corollary indicates the possibility of describing the faces of the arrangement polytope by using the techniques of polymatroids. However, we will not dwell on that here since later we will show the combinatorial equivalence of the arrangement polytope and the combinatorial polytope. We merely note that when $m \neq n$ the dimension of $M_n^m(a)$ is equal to m .

4.2 Adjacent Vertices

<u>Theorem 4.5</u> *The vector* $x \in M_n^m(a)$ *is a vertex of the arrangement polytope if and only if it is a permutation of the numbers* $a_1,\dots,a_s,a_{n-r+1},\dots,a_n$, *where*

$$0 \le s \le m \quad , \quad 0 \le r \le m \quad , \quad s+r = m . \tag{4.7}$$

Proof i) Necessity. We use proof by contradiction. Suppose there is a vertex $a_\pi = (a_{\pi_1},\dots,a_{\pi_m})$ such that there is an index π_i with $s+1 \leqq \pi_i \leqq n-r$ where $s+1$ and $n-r$ are respectively the smallest and largest numbers in N_n which are not present in the arrangement $\pi = (\pi_1,\dots,\pi_m)$. Let π' and π'' be two arrangements which differ from π only in the component with number π_i; and let this number be respectively equal to $s+1$ and $n-r$. Then with $\lambda = (a_{n-r}-a_{\pi_i})/(a_{n-r}-a_{s+1})$ we have $a_\pi = \lambda a_{\pi'} + (1-\lambda)a_{\pi''}$, $0 < \lambda < 1$. Then, since $0 < \lambda < 1$, this equality means that the point a_π is not a vertex of the polytope $M_n^m(a)$.

ii) Sufficiency. Suppose that for the vector $a_\pi = (a_{\pi_1},\dots,a_{\pi_m})$ the conditions of the theorem are satisfied. Then the arrangement $(\pi_1,\dots,\pi_m)$ is composed of the numbers $1,2,\dots,s,n-r+1,\dots,n$ where the numbers s and r satisfy conditions (4.7). Define the hyperplane

$$\sum_{j=1}^m c_j x_j = c_0 \quad , \quad c_0 = \sum_{j=1}^m c_j a_{\pi_j} . \tag{4.8}$$

Here the coefficients c_j are arbitrary real numbers satisfying the conditions

$$c_{\tau_1} > c_{\tau_2} > \dots > c_{\tau_s} > 0 > c_{\tau_{n-r+1}} > \dots > c_{\tau_m} , \tag{4.9}$$

where $\tau = \pi^{-1}$. Using the relations (4.9) for the coefficients c_j and (4.1) for the numbers a_i and using the inequality of Problem 3 for any vector $a_\sigma = (a_{\sigma_1},\dots,a_{\sigma_m})$, $\sigma \neq \pi$, we have

$$\sum_{j=1}^m c_j a_{\sigma_j} > \sum_{j=1}^m c_j a_{\pi_j} .$$

This means that equation (4.8) defines a supporting hyperplane to the arrangement polytope $M_n^m(a)$ at a_π. Thus, a_π is a vertex of $M_n^m(a)$. //

The following theorem solves the problem of establishing that two vertices of the arrangement polytope are adjacent.

Theorem 4.6 *Let the numbers* s *and* r *have property* (4.7) *and let* a_π *be the vertex of the polytope* $M_n^m(a)$ *for which the arrangement* $\pi = (\pi_1,\dots,\pi_m)$ *is a permutation of the numbers* $1,2,\dots,s,n-r+1,\dots,n$. *Then every adjacent vertex to* a_π, *say* a_{π^i}, *is defined by an* *arrangement* π^i, $1 \leq i \leq s-1$, $(n-r+1 \leq i \leq n)$ *obtained from the arrangement* π *by transposing the components equal to* i *and* $i+1$ $(i$ *and* $i-1)$, *or if* $i = s$ $(i = n-r+1)$ *by replacing the component equal to* s $(n-r+1)$ *by* $n-r$ $(s+1)$.

Proof We show first that for any i the segment joining the vertices a_π and a_{π^i} is an edge of the polytope $M_n^m(a)$. In equation (4.8) let all coefficients c_{τ_i} be defined by the following rule, depending on the value of i:

$$c_{\tau_i} = \begin{cases} c_{\tau_{i+1}} & \text{for } i=1,\dots,s-1, \\ 0 & \text{for } i=s \text{ or } n-r+1, \\ c_{\tau_{i-1}} & \text{for } i=n-r+2,\dots,n. \end{cases}$$

Then the hyperplane defined by equation (4.8) is supporting to $M_n^m(a)$ and its intersection with the polytope is the edge joining the vertices a_π and a_{π^i}.

We now show that any segment $[a_\pi, a_\sigma]$, where σ is an arrangement distinct from π and π^i, is not an edge of $M_n^m(a)$. Consider the vector equation

$$\sum_{i=1}^{s} \alpha_i (a_{\pi^i} - a_\pi) + \sum_{i=n-r+1}^{n} \alpha_i (a_{\pi^i} - a_\pi) = a_\sigma - a_\pi$$

from which we find

$$\alpha_i = \frac{1}{a_{i+1} - a_i} \sum_{j=1}^{i} (a_{\sigma_{\tau_j}} - a_j) \qquad i \in N_s,$$

$$\alpha_{n-i} = \frac{1}{a_{n-i+1} - a_{n-i}} \sum_{j=n-i}^{n} (a_{\sigma_{\tau_j}} - a_j) \qquad i \in N_{n-r+1} \cup \{0\}.$$

From (4.1) it follows that all $\alpha_i \geqq 0$. Since $\sigma \neq \pi$ and $\sigma \neq \pi^i$, we have

$$\sum_{i=1}^{s} \alpha_i + \sum_{i=n-r+1}^{n} \alpha_i > 1.$$

Putting

$$\lambda = \frac{1}{\sum_{i=1}^{s} \alpha_i + \sum_{i=n-r+1}^{n} \alpha_i}, \qquad \beta_i = \lambda \alpha_i,$$

we obtain

$$\lambda a_\sigma + (1-\lambda)a_\pi = \sum_{i=1}^{n} \beta_i a_{\pi^i} + \sum_{i=n-r+1}^{n} \beta_i a_{\pi^i} .$$

Hence, the segment $[a_\sigma, a_\pi]$ is not an edge of $M_n^m(a)$, since there is a point of this segment which is a convex combination of the points a_{π^i}. //

Theorems 4.5 and 4.6 have the corollary:

Corollary 4.7 *The polytope* $M_n^m(a)$ *is simple.*

4.3 Combinatorial Description

Theorem 4.8 *The arrangement polytope* $M_n^m(a)$ *with* $m < n$ *and for any vector* a *is combinatorially equivalent to the permutation polytope of dimension* m .

Proof The proof is in two parts. We first demonstrate the combinatorial equivalence of the polytopes $M_n^m(a)$ and $M_{m+1}^m(b)$, where $b = (b_1,..,b_{m+1})$ with $b_1 > \ldots > b_{m+1}$, and we then show the combinatorial equivalence of $M_{m+1}^m(b)$ and $M_{m+1}(c)$. The faces of $M_n^m(a)$ are defined as follows :

$$F_I'(a) = \{x \in M_n^m(a) : \sum_{i\in I} x_i = \sum_{i=1}^{|I|} a_i\} \qquad \forall I \subseteq N_m,$$

$$F_J'(a) = \{x \in M_n^m(a) : \sum_{i\in J} x_i = \sum_{i=1}^{|J|} a_{n-i+1}\} \qquad \forall J \subseteq N_m .$$

The faces $F_I'(b)$ and $F_J''(b)$ of $M_{m+1}^m(b)$ are similarly defined. Every vertex a_π, where π is a permutation of the numbers $1,\ldots,s,n-r+1,\ldots$ $\ldots,n$ (s and r satisfy (4.7)), is the point of intersection of the faces $F_{I_1}'(a),\ldots,F_{I_s}'(a),F_{J_{s+1}}''(a),\ldots,F_{J_m}''(a)$, where $I_k=\{\tau_1,\ldots,\tau_k\}$, $k \in N_s$, $J_k=\{\tau_{k+1},\ldots,\tau_m\}$, $\forall k \in N_m \setminus N_s$. Here $\tau = \pi^{-1}$. Let b_σ be a vertex of $M_{m+1}^m(b)$, where σ is a permutation of the numbers $1,\ldots,s,$ $m-r+2,\ldots,m+1$, which has the property $\sigma^{-1} = \pi^{-1} = \tau$. Then, the faces of $M_{m+1}^m(b)$ which are incident to b_σ are defined by the same subsets I_k and J_k which were used to define the faces of $M_n^m(a)$. Thus, the map $\phi : a_\pi \to b_\sigma$, $F_{I_k}'(a) \to F_{I_k}'(b)$ $\forall k \in N_s$, $F_{J_k}''(a) \to F_{J_k}''(b)$ $\forall k \in N_m \setminus N_s$

sets up a one-to-one correspondence between the vertices and faces of of $M_n^m(a)$ and $M_{m+1}^m(b)$ which preserves incidence properties of faces. Thus, by Theorem 1.7 of Chapter 3, the polytopes $M_n^m(a)$ and $M_{m+1}^m(b)$ are combinatorially equivalent.

Consider the permutation polytope $M_{m+1}(c)$ for an arbitrary vector c whose components obey condition (4.1). The polytope $M_{m+1}(c)$ is given by the constraints

$$\sum_{i\varepsilon\omega} x_i \leqq \sum_{i=1}^{|\omega|} c_i \qquad \forall\, \omega \subset N_{m+1}\,, \tag{4.9'}$$

$$\sum_{i=1}^{m+1} x_i = \sum_{i=1}^{m+1} c_i\,. \tag{4.10}$$

Since $\dim M_{m+1}(c) = m$, we can solve for x_{m+1} from (4.10)

$$x_{m+1} = \sum_{i=1}^{m+1} c_i - \sum_{i=1}^{m} x_i$$

and substitute for x_{m+1} into the inequalities (4.9') which contain it. We obtain the inequalities

$$\sum_{i=1}^{|\omega|} c_{n-i+1} \leqq \sum_{i\varepsilon\omega} x_i \qquad \forall\, \omega \subseteq N_m\,,$$

which together with the inequalities

$$\sum_{i\varepsilon\omega} x_i \leqq \sum_{i=1}^{|\omega|} c_i \qquad \forall\, \omega \subseteq N_m\,,$$

not containing x_{m+1}, give an algebraic description of the polytope $M_{m+1}^m(c)$. Clearly, the exclusion of redundant constraints does not alter the combinatorial type of the polytope. Thus, the arrangement polytope is combinatorially equivalent to the permutation polytope of the same dimension. //

<u>Corollary 4.9</u> *The number of* i-*faces* $(0 \leqq i \leqq m)$ *of the arrangement polytope is given by*

$$f_i(M_n^m(a)) = \sum \frac{(m+1)!}{t_1!\,t_2!\ldots t_{m-i+1}!} \qquad \forall\, i \,\varepsilon\, N_m\,,$$

where the sum is taken over all positive integral solutions of the equation

$$t_1 + \ldots + t_{m-i+1} = m\,.$$

Corollary 4.10

diam $M_n^m(a) = m(m+1)/2$.

§5 THE STANDARDIZATION POLYTOPE

The first papers dealing with the optimal standardization problem appeared in 1968-1970. Current interest is centred on the problem of improving the efficiency of known methods (Beresnev et al. (1978)). A mapping of the feasible region of the standardization problem into a space of smaller dimension was given by Girlikh & Kovalev (1974) and this led to a significant reduction of dimension. The study of the affine image of a certain polytope led to a new class of problems in polyhedral combinatorics.

5.1 The Affine Image of Stochastic Matrices

Let

$$G_{m,n} = \{(x_{ij})_{m\times n} : \sum_{i=1}^{m} x_{ij}=1 \ , \ x_{ij} \geqq 0 \quad \forall (i,j) \varepsilon N_m \times N_n\}$$

be a set of stochastic matrices and let $F(\omega)$ be a non-empty face of the polytope $G_{m,n}$ defined by the relations $x_{ij} = 0 \ \forall (i,j) \ \varepsilon \ \omega$, where ω is some given subset of index pairs (i,j). Consider the polytope $H_m(\omega)$ which is the image of the polytope $F(\omega)$ under the singular affine map Λ given by the relations

$$y_i = \sum_{j=1}^{n} a_j x_{ij} \qquad \forall i \ \varepsilon \ N_m$$

where $a_j > 0 \quad \forall j \ \varepsilon \ N_n$.

It is shown in Girlikh & Kovalev (1974), Yemelichev & Kovalev (1970,1972) and Kowaljov & Girlich (1977) that the polytope $H_m(\omega)$ is a feasible set of the standardization problem for suitable choice of the set ω and of the map Λ . We study ways of describing the polytope $H_m(\omega)$ by means of linear inequalities. It is convenient to specify the set ω by means of a Boolean $(m\times n)$-matrix $Q = (q_{ij})$ for which $q_{ij} = 0$ if $(i,j) \ \varepsilon \ \omega$ and $q_{ij} = 1$ otherwise. We sometimes use the symbols $F(Q)$ and $H_m(Q)$ in place of $F(\omega)$ and $H_m(\omega)$. Note that since we assumed that the set $F(Q)$ was non-empty we have the inequalities

$$\sum_{i=1}^{m} q_{ij} \geqq 1 \qquad \forall j \ \varepsilon \ N_m \ . \tag{5.1}$$

<u>Theorem 5.1</u> *The polytope* $H_m(Q)$ *in* E_m *lies in the hyperplane*

$$\sum_{i=1}^{m} y_i = \sum_{j=1}^{n} a_j \tag{5.2}$$

and is given by one of the following systems of linear inequalities

$$\begin{aligned} \sum_{i\varepsilon\omega(J)} y_i &\geqq \sum_{j\varepsilon J} a_j && \forall J \subseteq N_n, \\ y_i &\geqq 0 && \forall i \,\varepsilon\, N_m; \end{aligned} \tag{5.3}$$

$$\sum_{i\varepsilon I} y_i \geqq \sum_{j\varepsilon v(I)} a_j \qquad \forall I \subseteq N_m; \tag{5.4}$$

$$\sum_{i\varepsilon I} y_i \leqq \sum_{j\varepsilon u(I)} a_j \qquad \forall I \subseteq N_m, \tag{5.5}$$

where $\omega(J)=\{i\varepsilon N_m : \sum_{j\varepsilon J} q_{ij} \geqq 1\}$, $v(I)=\{j\varepsilon N_n : \sum_{i\notin I} q_{ij}=0\}$, $u(I)=\{j\varepsilon N_n : \sum_{i\varepsilon I} q_{ij} \geqq 1\}$.

Proof We introduce auxiliary variables $z_{ij} = a_j x_{ij}$ $\forall (i,j)\,\varepsilon\, N_m \times N_n$. Consider the system of linear equations and inequalities

$$\sum_{i=1}^{m} y_i = \sum_{j=1}^{n} a_j , \tag{5.6}$$

$$y_i = \sum_{j=1}^{n} z_{ij} \qquad \forall i \,\varepsilon\, N_m, \tag{5.7}$$

$$a_j = \sum_{i=1}^{m} z_{ij} \qquad \forall j \,\varepsilon\, N_n, \tag{5.8}$$

$$0 \leqq z_{ij} \leqq \xi q_{ij} \qquad \forall (i,j)\,\varepsilon\, N_m \times N_n, \tag{5.9}$$

where ξ is some sufficiently large positive number. It is clear that the vector $y = (y_1,\dots,y_m)\,\varepsilon\, H_m(Q)$ if and only if the system (5.6)-(5.9) has a solution. Consistency conditions (Corollary 4.12 Ch. 4) and the fact that $q_{ij} = 0$ or 1 yield the equivalent system (5.3)-(5.5). //

The polytope $H_m(Q)$ lies in the hyperplane (5.2), thus its dimension is less than or equal to $m-1$. Note that if the matrix Q has a column with only one non-zero element $q_{ij} = 1$ then for all $y \,\varepsilon\, H_m(Q)$ we have $y_i \geqq a_i$. If also the element q_{ij} is the only non-zero element in the ith row then we have $y_i = a_i$. In the latter case $\dim H_m(Q) \leqq m-2$. In what follows we will assume that

$$\sum_{i=1}^{m} q_{ij} \geqq 2 \qquad \forall j \in N_n . \qquad (5.1')$$

The assumption (5.1') is an essential feature of the standardization problem.

Theorem 5.2

1) *The polytope defined by the inequalities* (5.5) *and by* $y_i \geqq 0$, $i \in N_m$, *is a bounded polymatroid.*

2) *The polyhedron defined by the inequalities* (5.4) *is an unbounded polymatroid.*

Proof To prove the first part of the theorem it suffices to show that the function $\rho(I) = \sum_{j\varepsilon u(I)} a_j$ is a non-negative, non-decreasing, submodular function (Theorem 6.1, Ch. 4). The first two properties of $\rho(I)$ follow from the positivity of the numbers a_j , $j \varepsilon N_n$, and from the fact that $u(I) \subset u(I')$ when $I \subset I'$. Further, from the definition of the sets $u(I)$, $I \subset N_m$, we have the relations

$$u(I' \cup I'') = u(I') \cup u(I''),$$

$$u(I' \cap I'') \subseteq u(I') \cap u(I'') .$$

The submodularity of $\rho(I)$ then follows from the following chain of inequalities :

$$\rho(I' \cup I'') + \rho(I' \cap I'') \leqq \sum_{j\varepsilon u(I') \cup u(I'')} a_j + \sum_{j\varepsilon u(I') \cap u(I'')} a_j$$

$$= \sum_{j\varepsilon u(I')} a_j + \sum_{j\varepsilon u(I'')} a_j = \rho(I') + \rho(I'') .$$

It can be shown similarly that the function $\rho'(I) = \sum_{j\varepsilon v(I)} a_j$ is non-negative, non-decreasing and supermodular. //

Corollary 5.3 (Kovalev & Girlikh 1980)

1) *The standardization polytope* $H_m(Q)$ *is the intersection of a bounded polymatroid* $P(\rho)$ *and an unbounded polymatroid* $P(\rho')$.

2) *The polytope* $H_m(Q)$ *is the face of each of the polymatroids* $P(\rho)$ *and* $P(\rho')$ *generated by the supporting hyperplane* (5.2).

The polymatroid structure of the standardization polytope enables us to identify quite easily its combinatorial type and to study the way its vertices are constructed.

5.2 Vertices

Theorem 6.3 (Ch. 4) and Corollary 5.3 enable us to give a constructive description of all the vertices of the polytope $H_m(Q)$. Indeed, from the system (5.2),(5.5) which defines $H_m(Q)$ we deduce that the vertex x is given by the following equations

$$\sum_{i \in \omega_s} x_i = \rho(\omega_s) \qquad \forall s \in N_m,$$

where $\omega_0 = \emptyset$, $\omega_s = \omega_{s-1} \cup \{\pi_s\}$, $s \in N_m$, and the permutation $(\pi_1, \ldots, \pi_m) \in S_m$.

<u>Theorem 5.4</u> *The vector* x *is a vertex of the polytope* $H_m(Q)$ *if and only if there is a permutation* $(\pi_1, \ldots, \pi_m) \in S_m$ *such that*

$$x_{\pi_s} = \sum_{j \in u(\omega_s)} a_j - \sum_{j \in u(\omega_{s-1})} a_j \qquad \forall s \in N_m .$$

We consider another means of constructing all the vertices of $H_m(Q)$ which does not involve calculating the function $\rho(\omega)$ but which uses only the matrix Q and the vector $(a_1, \ldots, a_n)$.

Define the operator $\oplus$ by the relations : $1 \oplus 0 = 1$, $1 \oplus 1 = 0$, $0 \oplus 1 = 0$, $0 \oplus 0 = 0$. Let $(\pi_1, \ldots, \pi_m) \in S_m$. We consider a procedure $\phi(\pi)$ whose k^{th} step $(1 \leqq k \leqq m)$ consists in the following : calculate the π_k-th component of the vector y according to the rule

$$y_{\pi_k} = \sum_{j=1}^{n} a_j q_{\pi_k j}^{(k-1)}$$

and transform the matrix $(q_{ij}^{(k-1)})$ into the matrix $(q_{ij}^{(k)})$ according to the rule

$$q_{ij}^{(k)} = q_{ij}^{(k-1)} \oplus q_{\pi_k j}^{(k-1)} \quad , \quad q_{ij}^{(0)} = q_{ij} .$$

The result of carrying out the procedure $\phi(\pi)$ is to obtain a vector y which we denote by $y(\pi)$.

Theorem 5.5 *The vector* $y = (y_1, \ldots, y_m)$ *is a vertex of the polytope* $H_m(Q)$ *if and only if there is a permutation* $\pi \in S_m$ *such that* $y = y(\pi)$.

Proof 1) Necessity. Let y be a vertex of $H_m(Q)$. It suffices to show that among the positive components of y there is a component

$$y_k = \sum_{j=1}^{n} a_j q_{kj} .$$

Then, putting $\pi_1 = k$ we construct, by induction, a permutation $\pi \in S_m$ with the property that $y(\pi) = y$. Suppose the opposite. Suppose that for any positive component y_k of y , for some pre-image $x = \Lambda^{-1}(y)$ there is a component $x_{kj} = 0$, even though $q_{kj} = 1$. This enables us to construct a cycle $(k_1, j_1), (k_2, j_1), (k_2', j_2), \ldots, (k_1, j_t)$ in which the components x_{ij} in the odd positions are zero and also $q_{ij} = 1$, while the components x_{ij} in the even positions are positive (more precisely, equal to one, for the pre-image of the vertex $y \in H_m(Q)$ under the affine map Λ must be a vertex of the polytope $F(Q)$). If we subtract a sufficiently small number from the numbers in the even positions and add the same number to the numbers in the odd positions, we obtain a new preimage of the vertex y which is not a vertex of $F(Q)$ and this is impossible.

2) Sufficiency. Let $y = y(\pi)$ be a vector obtained through the procedure $\phi(\pi)$. Clearly $y \in H_m(Q)$. We show that $y \in \text{vert } H_m(Q)$. Suppose the opposite, that is, suppose $y \in \lambda y' + (1-\lambda)y''$ where $y', y'' \in H_m(Q)$, $y' \neq y''$, $0 < \lambda < 1$. Let $x', x'' \in F(Q)$ be pre-images of y', y'' respectively. Then $x^0 = \lambda x' + (1-\lambda)x''$ is a pre-image of y . It follows from the definition of the procedure $\phi(\pi)$ that in the matrix Q we have $q_{\pi_k j} = 0$, $(k,j) \in N_{m-1} \times N(\pi_m)$, $N(\pi_m) = \{j \in N_m : q_{\pi_m j} = 1\}$. Hence, for all preimages x of y we have $x_{\pi_m j} = 1$, $j \in N(\pi_m)$. Using similar considerations for rows $\pi_{m-1}, \ldots, \pi_1$, we show the uniqueness of the preimage of y . Hence $x = x^0 = \lambda x' + (1-\lambda)x''$. But by construction x is a vertex of the polytope $F(Q)$. This is a contradiction. //

Corollary 5.6 *Every vertex of the polytope* $H_m(Q)$ *has a unique preimage in* $F(Q)$.

5.3 Maximum Number of Vertices

It follows from Theorem 5.5 that $m!$ is an upper bound for the number of vertices of a standardization polytope. We show that this bound is attainable and that all polytopes $H_m(Q)$ with $m!$ vertices are combinatorially equivalent to the permutation polytope.

Definition 5.1 A Boolean matrix $Q = (q_{ij})_{m \times n}$ is called *complete* if it has as a submatrix a matrix which is the incidence matrix of a complete graph with m vertices K_m.

A complete matrix contains at least $m(m-1)/2$ columns.

Theorem 5.7 *The polytope* $H_m(Q)$ *has the maximum number* m *of vertices if and only if the matrix* Q *is complete.*

Proof 1) Necessity. Let $f_0(H_m(Q)) = m!$. Suppose the opposite. Suppose that the matrix Q has only k $(k < m(m-1)/2)$ distinct columns each of which contains exactly two units. There are two cases $k = 0$ and $k \neq 0$.

Suppose that the matrix Q has no column with two non-zero elements. Assume that Q does not contain any column with a single non-zero element. But, if each column of Q contains not less than three units, then every set of $m-2$ rows of Q form a submatrix which has at least one unit in every column. Thus, the vectors $y(\pi)$ and $y(\pi')$ which are generated by the procedures $\phi(\pi)$ and $\phi(\pi')$, where $\pi = (\pi_1,\ldots,\pi_{m-1},\pi_m)$, $\pi' = (\pi_1,\ldots,\pi_m,\pi_{m-1})$ coincide. Thus, in this case $f_0(H_m(Q)) < m!$.

Let the distinct columns of Q with two units form a $(m\times k)$-submatrix Q'. Since $k < m(m-1)/2$, there are at least two rows Q_p and Q_s such that $Q_pQ_s = 0$. Thus, the vectors $y(\pi)$ and $y(\pi')$ where $\pi = (\pi_1,\ldots,\pi_{m-2},p,s)$, $\pi' = (\pi_1,\ldots,\pi_{m-2},s,p)$ coincide. Hence $f_0(H_m(Q)) < m!$. This establishes necessity.

2) Sufficiency. We use induction on m. If $m = 2$, there is a column j of the complete matrix Q such that $q_{1j} = q_{2j} = 1$. Then, the components of the vectors $y' = y(1,2)$ and $y'' = y(2,1)$ satisfy the following relations

$$y'_1 = b + a_j\ ,\ y'_2 = c\ ;\ y''_1 \leqq b\ ,\ y''_2 \geqq c + a_j\ , \tag{5.10}$$

which imply that $y' \neq y''$ since the numbers $a_1,\ldots,a_n$ are non-negative.

Thus $f_0(H_m(Q)) = 2$. If the $(m\times n)$-matrix Q is complete, then the matrix obtained from Q by removing the i^{th} row (i - arbitrary) and all columns j such that $\delta_{ij} = 1$ is also complete. Thus, if we fix i and put $y_i = \sum_{j=1}^{n} a_j q_{ij}$, then, by the induction hypothesis, we may use the procedure $\phi(\pi)$, where $\pi = (i,\alpha)$ and α is any permutation of the numbers $1,\dots,i-1,i+1,\dots,m$ to obtain a set $V(i)$ consisting of $(m-1)!$ distinct vertices of $H_m(Q)$ with the same i^{th} component. We show that if $y' \in V(s)$, $y \in V(r)$, $r \neq s$ then $y' \neq y''$. Since in a complete graph any two vertices are joined by an edge, there is a column j of Q such that $q_{sj} = q_{rj} = 1$, $q_{ij} = 0$, $i \neq s,r$. Thus, relations of the type (5.10) hold for the components y_s , y_r of the vectors $y(s,\alpha)$ and $y(r,\alpha)$. from which it follows that $y' \neq y''$. Thus, all $m!$ vertices constructed through the procedures $\phi(1,\alpha),\dots,\phi(m,\alpha)$ are distinct. //

<u>Theorem 5.8</u> *Let* Q *be a complete matrix, then the polytope* $H_m(Q)$ *is combinatorially equivalent to a permutation polytope.*

Proof By Theorem 5.1 the polytope $H_m(Q)$ is given by equation (5.2) and by the inequalities

$$\sum_{i\in I} y_i \leqq \sum_{j\in u(I)} a_j \qquad \forall I \subset N_m . \qquad (\alpha_I)$$

We show that if Q is a complete matrix then each of the constraints (α_I) determines a $(d-1)$-face of the d-polytope $H_m(Q)$. By Theorem 6.4 of Ch.4 and by Corollary 5.3 it suffices to show that each set $I \subset N_m$ is ρ-closed and ρ-nonseparable.

Let $I' \subset I'' \subset N_m$. Since the matrix Q contains a submatrix which is the incidence matrix of a complete graph, for a row $i_0 \in I''\setminus I'$ there is a column j_0 such that $q_{i_0 j_0} = 1$ and $q_{ij_0} = 0$ for all $i \in N_m\setminus I''$. Thus $u(I') \cup j_0 \subseteq u(I'')$ and, consequently

$$\sum_{j\in u(I'')} a_j \geqq a_{j_0} + \sum_{j\in u(I')} a_j > \sum_{j\in u(I')} a_j ,$$

which establishes the ρ-closure of the set I' . We show that any subset $I \subset N_m$ is ρ-nonseparable. Suppose that for some $I \subset N_m$ there are sets S and T such that

$$S \cup T = I \quad , \quad S \cap T = \emptyset ,$$

$$\sum_{j \in u(S)} a_j + \sum_{j \in u(T)} a_j = \sum_{j \in u(S \cup T)} a_j . \tag{5.11}$$

Since $u(I) = u(S) \cup u(T)$ and since for any $i' \in S$, $i'' \in T$ there is a column j_0 such that $q_{i'j_0} = q_{i''j_0} = 1$, we have $u(S) \cap u(T) \neq \emptyset$. Since $a_j > 0$ we see that equation (5.11) is impossible. Thus, each of the inequalities (α_I) determines a facet of the d-polytope $H_m(Q)$. By Theorem 3.1, every facet of the permutation polytope $M_m(a)$ is defined by one of the inequalities

$$\sum_{i \in I} x_i \leqq \sum_{i=1}^{|I|} a_i \qquad \forall I \subset N_m , \tag{β_I}$$

and conversely, for each $I \subset N_m$ the inequality (β_I) determines a facet of $M_m(a)$. Let $\phi : (\alpha_{I'}) \to (\beta_{I'})$ be a bijection between the set of facets of $M_m(a)$ and $H_m(Q)$. If we show that this map preserves incidence relations between faces and vertices then, by Theorem 4.7 (Ch.1), we will have established the equivalence of $H_m(Q)$ and $M_m(a)$. Indeed, since $H_m(Q)$ and $M_m(a)$ are faces of their associated polymatroids (Theorem 3.12 and Corollary 5.3), then Theorem 6.3 (Ch.4) on the characterization of the vertices of a polymatroid implies that the sets $I^1 = \{i_1\}$, $I^k = I^{k-1} \cup \{i_k\}$, $k=2,\dots,m-1$ determine faces which are incident to some vertex of both $H_m(Q)$ and $M_m(a)$. Thus, the intersection of the facets with indices $\beta_{I^1},\dots,\beta_{I^{m-1}}$ determine a vertex of $M_m(a)$ if and only if the intersection of the facets with indices $\alpha_{I^1},\dots,\alpha_{I^{m-1}}$ determine a vertex of $H_m(Q)$. This completes the proof. //

Since the f-vector and the diameter of a permutation polytope are known (Theorem 3.7, Corollary 3.10), these characteristics are also known for the polytope $H_m(Q)$.

<u>Corollary 5.9</u> *Let the matrix* Q *be complete, then*

1) $f_i(H_m(Q)) = \sum \dfrac{m!}{t_1! \dots t_{m-i}!}$

where the summation is carried out over all solutions of the equation $t_1 + \dots + t_{m-i} = m$ *in positive whole numbers.*

2) $\operatorname{diam} H_m(Q) = m(m-1)/2$.

5.4 A One-Parameter Problem

In the one-parameter standardization problem the feasible set is a polytope $H_m(Q)$ such that $Q = Q_1$, where Q_1 is a triangular matrix. In other words, $H_m(Q_1)$ is the image of the polytope

$$F(Q_1) = \left\{ (x_{ij})_{m\times n} : \sum_{i=1}^{m} x_{ij} = 1 \ , \ x_{ij} \geqq 0 \ \forall (i,j) \in N_m \times N_n \ , \ x_{ij} = 0 \ , \ i < j \right\}$$

under an affine map Λ . It follows from Theorem 6.3 (Ch.4), Theorem 5.1 and Corollary 5.3, that the polytope $H(Q_1)$ belongs to the hyperplane (5.1) and is given by the following irreducible system of inequalities

$$y_i \geqq 0 \qquad \forall i \in N_{m-1} \ , \qquad (\alpha_i)$$

$$\sum_{j=1}^{i} y_j \leqq \sum_{j=1}^{i} a_j \qquad \forall i \in N_{m-1} \ . \qquad (\beta_i)$$

<u>Theorem 5.10</u> *The polytope* $H_m(Q_1)$ *is combinatorially equivalent to an* $(m-1)$*-cube.*

Proof Let the cube K be given in E_{m-1} by the constraints

$$x_i \geqq 0 \qquad \forall i \in N_{m-1} \qquad (\alpha_i')$$

$$x_i \leqq 1 \qquad \forall i \in N_{m-1} \qquad (\beta_i')$$

Every vertex x of the cube can be given by specifying the indices α'_{i_1} , $\ldots,\alpha'_{i_k}, \beta'_{i_{k+1}}, \ldots, \beta'_{i_{m-1}}$ of the faces whose intersection defines x . Here $(i_1,\ldots,i_{m-1}) \in S_{m-1}$. Similarly, every vertex y of $H(Q_1)$ is determined by the indices $\alpha_{i_1}, \ldots, \alpha_{i_k}, \beta_{i_{k+1}}, \ldots, \beta_{i_{m-1}}$ of the faces whose intersection gives y . Thus the map $\phi : \alpha_i \to \alpha'_i \quad \beta_i \to \beta'_i$ establishes an isomorphism of the polytopes K and $H_m(Q_1)$. By Theorem 1.7 (Ch.3), the polytope $H_m(Q_1)$ is combinatorially equivalent to the cube. //

<u>Corollary 5.11</u>

1) $f_i(H_m(Q_1)) = 2^{m-i-1} \binom{m-1}{i} \qquad i = 0,\ldots,m-2;$

2) $\operatorname{diam} H_m(Q_1) = m-1$.

EXERCISES

1. (Balinski & Russakov 1972). Show that

(1) to each vertex of the bistochastic polytope M_n there correspond $2^{n-1}n^{n-2}$ feasible bases;

(2) it is possible to pass from any feasible basis of the polytope M_n to any other feasible basis in not more than $2n-1$ steps. Moreover, in the sequence of bases so constructed, every pair of neighbouring bases differ in only one column vector and all the bases will be feasible;

(3) the graph of the polytope M_n is Hamiltonian.

2. Every i-face ($i \in N_{d-1}$, $d = \dim M_n$) of the polytope M_n is representable in the form $F = \{x \in M_n : x_{ij} = 0 \ , \ \forall (i,j) \in \omega \subset N_n \times N_n\}$. Conditions for F to be non-empty are given by Hall's Theorem (see Cor. 4.14, Ch. 4). In Brualdi & Gibson (1976, 1977) the properties of the faces F of the polytope M_n are studied in detail. In particular the following results are obtained:

(1) every i-face of F has no more than three (i-1)-faces;

(2) if the face F is a 2-neighbourly polytope, then F is affinely equivalent to M_3;

(3) the i-face F has $i+2$ vertices if and only if one of the following conditions is satisfied: (i) $i = 2$ and F is a rectangle; (ii) $i \geq 3$ and either F is affinely equivalent to the polytope M_3 or F is a pyramid with a rectangular base;

(4) the i-face F is an i-parallelepiped if and only if F does not contain a 2-face which is a triangle.

3. (Suprunenko & Metelski 1973). Let $C = (c_{ij})_{n\times n}$ be a matrix with real elements. The assignment problem consists in finding

$$F_n = \min \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}x_{ij} \ : \ (x_{ij}) \in M_n \right\} .$$

Let r be the rank of the matrix C. Then, there exist r pairs of vectors $(a^1,b^1), \ldots, (a^r,b^r)$ such that the assignment problem takes the form

$$\text{minimize} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i^1 b_j^1 + \ldots . + a_i^r b_j^r)x_{ij} \ : \ (x_{ij}) \in M_n \right\} .$$

In particular, when the rank of the matrix C equals 1, then the assignment problem is easily solved: the optimal permutation matrix x^* is determined by the conditions $x^*_{i_k j_k} = 1$, $\forall k \in N_n$, and

$x_{ij} = 0$ for the remaining (i,j), where the permutations $(i_1,\ldots,i_n)$, $(j_1,\ldots,j_n)$ are determined by the conditions $a^1_{i_1} \leqq \ldots \leqq a^1_{i_n}$, $b^1_{j_1} \geqq \ldots \geqq b^1_{j_n}$. Moreover, if the components of the vectors a^1 and b^1 are all distinct, then x^* is the unique optimal solution.

4. (Suprunenko & Metelski 1973, Leontiev 1979). The following assertions are equivalent:

(1) $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}x_{ij} = \text{const}$ for all $x \in M_n$;

(2) $c_{ij} = \alpha_i + \beta_j$, $\forall i,j \in N_n$;

(3) $c_{ij} + \left(\sum_{i=1}^{n} u_i\right)/n^2 - (u_i+v_j)/n = 0$, $u_i = \sum_{j=1}^{n} c_{ij}$, $v_j = \sum_{i=1}^{n} c_{ij}$.

An analagous result for multi-dimensional matrices was obtained by Mikulski (1974).

5. (Leontiev 1975, 1979). Let $c_{ii} = 0$, $\forall i \in N_n$. The following statements are equivalent:

(1) $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}x_{ij} = \text{const}$ for all $x \in M_n^s$;

(2) $c_{ij} = \alpha_i + \beta_j$, $\forall i,j \in N_n$, $i \neq j$;

(3) $c_{ij} + \frac{1}{(n-1)(n-2)} \sum_{i=1}^{n} u_i - \frac{n-1}{n(n-2)}(u_i+v_j) - \frac{1}{n(n-2)}(u_j+v_i)$
$= 0$, $\forall i,j \in N_n$, $i \neq j$.

6. If, in the assignment problem, the matrix $(c_{ij})_{n\times n}$ is such that $|j-i| \in \{0,1\} \Rightarrow c_{ij} \geqq 0$, and $|j-i| \notin \{0,1\} \Rightarrow c_{ij} = 0$, then the optimal value of the cost function F_n can be found by means of the following recurrence relation:

$$F_t = F_{t-1} + \min\{c_{tt}, c_{t-1,t} - c_{t-1,t-1}x^{(t-1)}_{t-1,t-1}\},$$

where $t = 2,\ldots,n$, and (x^t_{ij}) is the permutation matrix which minimizes the function $\sum_{i=1}^{t} \sum_{j=1}^{t} c_{ij}x_{ij}$ on the polytope M_t.

7. (Sachkov 1975). Let M_n^{**} be the symmetric bistochastic polytope defined by $M_n^{**} = \{(x_{ij}) \in E_{n^2} : \sum_{i=1}^{n} x_{ij} = 1,\ x_{ij} = x_{ji} \geqq 0,\ \forall i,j \in N_n\}$.

It is clear that $M_n^{**} \subset M_n^{*}$. Show that the polytope M_n^{**} is the convex hull of the set of all matrices $(x + x^T)/2$, where x is an $(n\times n)$-permutation matrix. The number of vertices of M_n^{**} is given by:

$$f_0(M_n^{**}) = \sum_{k=0}^{n} \sum_{j=0}^{[(n-k)/2]} \frac{n!}{j!(n-k-2j)!2^{n-k-j}} \sum_{\ell=0}^{k} \binom{1/4}{k-\ell}\binom{1/4+\ell-1}{\ell}$$

or by the following asymptotic formula:

$$f_0(M_n^{**}) \Rightarrow \frac{2^{\frac{1}{4}}en!}{\Gamma(1/4)n^{\frac{3}{4}}} , \quad \text{as} \quad n \to \infty .$$

8. (Sachkov 1977). For the permanent of the linear hull of two permutation matrices Δ_1 and Δ_2 we have the equality

$$\operatorname{per}(\alpha\Delta_1 + \beta\Delta_2) = \prod_{i=1}^{k} (\alpha^{e_i} + \beta^{e_i}) ,$$

where α , β are real numbers and $e_1 , \dots , e_k$ are the lengths of the cycles of the permutation $\pi_1^{-1} \pi_2$. Here π_1 and π_2 are the permutations corresponding to the matrices Δ_1 , Δ_2 .

9. (Koontz 1978). Let the non-negative real numbers m,n,t satisfy the constraints $1/m \leq 1-t$, $1/t \geq 1/n$. Show that the polytope

$$Q_n = \{x \varepsilon M_n : 1/m \leq x_{ij} \leq 1/t \quad \forall i,j \varepsilon N_n\} ,$$

which is the intersection of a bistochastic polytope and a parallelepiped, is the convex hull of the set of all matrices whose elements are equal to $1/m$ or $1/t$ with the possible exception of the elements of a single row (or column) in which all elements are equal to the same number.

10. The vertices x and x^T of the polytope M_n^{as} are adjacent. If the vertices $(x_1+x_1^T)/2$ and $(x_2+x_2^T)/2$ are adjacent, then the following vertex pairs (x_1,x_2) , (x_1,x_2^T) , (x_1^T,x_2) , (x_1^T,x_2^T) , (x_1,x_1^T) , (x_2,x_2^T) of the polytope M_n^s are also adjacent. The number of adjacent vertices to any vertex of the polytope M_n^s is not less than $[(n-2)/2]!$ (Savage, Weiner & Bagchi 1976).

11. (Heller 1956). The adjacency dimension, $k(M)$, of the polytope M is defined to be the smallest number r such that any two vertices of M belong to a k-face with $k \leq r$. Show that the adjacency dimension of the bistochastic polytope M_n is equal to $[n/2]$, and of the hamiltonian tour polytope is equal to:

$$k(M_n^{as}) = \begin{cases} 2m & \text{if } n = 4m+2 \text{ and } n \geq 8 , \\ [n/2] & \text{if } n \neq 4m+2 \text{ and } n \geq 8 , \\ 2 & \text{if } n = 6, 7 , \\ 1 & \text{if } n = 3,4,5. \end{cases}$$

12. (Grötschel & Padberg 1979). The inequalities defined by the toothed systems of sets $(V_0, V_1, \dots, V_k)$ and $(N_n \setminus V_0, V_1, \dots, V_k)$ define exactly the same face of the hamiltonian cycle polytope M_n^s. Show that the number of distinct faces of M_n^s which are generated by toothed systems of sets is given by the formula:

$$\sum_{q=3}^{n-3} \frac{1}{2}\binom{n}{q} \sum_{j=3}^{n-q} \binom{n-q}{j} \sum_{k=3}^{\min(j,k)} \left(\sum_{s=0}^{k} (-1)^s \binom{k}{s} (k-s)^j \right) \sum_{p=k}^{q} \frac{1}{k!} \binom{q}{p} \sum_{s=0}^{k} (-1)^s \binom{k}{s} (k-s)^p .$$

Verify, that the inequality defined by the toothed system $V_0 = \dots = V_k$, when $k = 1$ coincides with Dantzig's inequality.

13. (Leontiev 1979). The following system of 510 inequalities and 14 equations determines the polytope M_5^{as} :

(1) $x_{ii} = 0$ (5 equations);

(2) $\sum_{i=1}^{5} x_{ij} = \sum_{i=1}^{5} x_{ji} = 1$ (9 independent equations);

(3) $x_{ij} \geqq 0$, $i \neq j$ (20 inequalities);

(4) $x_{ij} + x_{ji} \leqq 1$, $i < j$ (10 inequalities);

(5) $-x_{ij} - x_{ji} + x_{st} + x_{tr} - x_{rs} \geqq -1$ (60 inequalities);

(6) $x_{ir} + x_{si} + x_{rs} - x_{jr} - x_{sj} - 2x_{ij} - 2x_{ji} \geqq -2$ (120 inequalities);

(7) $x_{ij} + x_{ir} + x_{ji} + x_{js} + x_{ri} + x_{rt} + x_{tj} + x_{sr} \geqq 1$ (60 inequalities);

(8) $x_{ij} + 2x_{ir} + x_{is} + x_{tj} + x_{ts} + x_{ji} + x_{jt} + x_{rt} + x_{rs} + x_{ri} + 2x_{si} \geqq 2$ (120 inequalities);

(9) $x_{ij} + x_{js} + x_{sr} + x_{ri} + 2x_{ji} + 2x_{si} + x_{sj} \leqq 3$ (120 inequalities),

where $i, j, s, r, t \in N_5$.

14. (Leontief 1979). The polytope M_6^s is given by the following system of equations and inequalities:

(1) $x_{ij} = x_{ji}$, $i \neq j$ (15 equations);

(2) $x_{ii} = 0$ (6 equations);

(3) $\sum_{i=1}^{6} x_{ij} = 2$ (6 equations);

(4) $x_{ij} \geqq 0$ (15 inequalities);

(5) $x_{ij} \leqq 1$ (15 inequalities);

(6) $x_{ij}+x_{ik}+x_{ki}+x_{ip}+x_{jq}+x_{k\ell} \leqq 4$ (120 inequalities);

(7) $x_{ij}+x_{jk}+x_{ki} \leqq 2$ (10 independent inequalities).

The polytope M_7^s is completely specified by a system of 9 types of inequalities, the first seven of which are analagous to those given for M_6^s and the remaining two of which take the following form:

(8) $x_{ij}+x_{jk}+x_{ki}+x_{ip}+x_{iq}+x_{pq}+x_{pr}+x_{qr} \leqq 5$ (1260 constraints);

(9) $2x_{ij}+2x_{jk}+2x_{ki}+2x_{ip}+x_{pq}+x_{qi}+x_{pk}+x_{pj}+2x_{kr}+2x_{jr} \leqq 9$ (2520 constraints).

15. (Maurras 1975). The inequality $\sum_{i,j} x_{ij} \leqq 9$ defines a facet of the 35-polytope M_{10}^s where the sum is taken over all edges (i,j) of Petersens's Graph constructed on 10 vertices.

16. (Bowman 1972). A di-graph is called a *tournament* if, given any pair of vertices i, j it contains either the directed edge (i,j) or the directed edge (j,i) but not both. The tournament is called *acyclic* if it does not contain any tours (cycles). The convex hull in E_{n^2} of the adjacency matrices of all acyclic tournaments is called the *acyclic tournament polytope* . Such a polytope is given by the following system of constraints:

$$x_{ij} \geqq 0 , \ \forall i,j \in N_n \ ; \quad x_{ii} = 0 , \ \forall i \in N_n \ ;$$

$$x_{ij} + x_{ji} = 1 , \quad \forall i,j \in N_n , \ i \neq j \ ;$$

$$x_{ij} + x_{jk} + x_{ki} \leqq 2 , \quad \forall i,j,k \in N_n , \ i \neq j \neq k .$$

17. (Sarvanov 1977). Let A be an $(n \times n)$-matrix and let Δ_π be a permutation matrix. The convex hull of the set of points $\Delta_\pi A \Delta_\pi^{-1}$ for all $\pi \in S_n$ is called the *quadratic choice problem polytope* and is denoted by $W(A)$.

Let C be the permutation polytope corresponding to the cycle <12...n> , $L = (i-j)_{n\times n}$ and let R be the matrix

$R = \begin{pmatrix} 0 & E_k \\ E_i & 0 \end{pmatrix}$, where E_i is the $(i \times i)$-matrix all of whose elements are equal to unity. Then $W(C) = M_n^{as}$ and the polytopes $W(L)$ and $W(R)$ are the feasible sets for the problem of linear rearrangements of a graph and for the problem of disconnections of a graph respectively. For any matrix A the graph of the polytope W(A) is regular, i.e. all its vertices have the same degree. Moreover, for the polytope $W(C) = M_n^{as}$, the degree of every vertex is greater than $2[(n-2)/2]!$, while for the polytope W(L) the degree of every vertex is greater than $3\times 2^{n-3} - 1$ $(n \geq 3)$. The graph of the polytope W(R) is complete.

18. The set of hamiltonian directed chains in the directed graph G(V,E) is given by the intersection of three matroids: $\mathcal{M}_1 = (E,\mathcal{F}_1)$, the graph matroid, and $\mathcal{M}_2 = (E,\mathcal{F}_2)$, $\mathcal{M}_3 = (E,\mathcal{F}_3)$, which are partition matroids where the independent sets are subsets of directed edges which do not contain edges both pointing towards or both pointing away from the same vertex, respectively. Using this fact Kovalev & Kotov (1981) showed that in the maximum-distance travelling salesman problem, the gradient algorithm always leads to a tour whose length is not less than a third of the optimal length.

19. Let $F = F(Q_1,\dots,Q_{n-i})$ be an i-face of the permutation polytope. Then $f_0(F) = \prod_{j=1}^{n-i} t_j$, where $t_j = |Q_j|$.

20. The f-vector of the polytope $M_n^+(a)$ of even permutations is given by the relations:

$$f_1 = n!(n-2)(n+1)/8 ,$$

$$f_2 = f_2(M_n(a)) + n!(n-2)(n-3)(2n-5)/24 ,$$

$$f_i = f_i(M_n(a)) + \frac{n!}{2}\binom{n-i}{i+1} , \quad i = 3,\dots,n-3,$$

$$f_{n-2} = 2(2^{n-1} - 1) + (n!)/2 .$$

21. Let $S_n(j)$ be the set of permutations $\pi \in S_n$, for which $\pi_j \neq j$, $j \in N_n$. The convex hull of the points $a_\pi = (a_{\pi_1},\dots,a_{\pi_n})$, $\forall \pi \in S_n(j)$ is called the *permutation polytope with the j-th prohibition* and is denoted by $M_n(j)$. Each of the $(n-1)(n-1)!$ points a_π, $\pi \in S_n(j)$ is a vertex of the polytope $M_n(j)$. When $2 \leq j \leq n-1$ the polytope $M_n(j)$ has the analytical representation

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} a_i , \quad \sum_{i\in\omega} x_i \geq \sum_{i=1}^{|\omega|} a_{n-i+1} \quad \forall \omega \subset N_n ,$$

$$(a_{j-1}-a_{j+1})\sum_{i\in\omega} x_i + (a_j-a_{j+1})x_j \leqq (a_{j-1}-a_{j+1})\sum_{i=1}^{j-1} a_i + (a_j-a_{j+1})a_{j-1}\ , \forall\,\omega\subset N_n\ ,$$

$$j\notin\omega\ ,\quad |\omega| = j-1\ .$$

When $j=1$ or $j=n$ the last class of inequality should be replaced by the following:

$$x_1 \leqq a_2 \qquad \text{or} \qquad \sum_{i=1}^{n-1} x_i \leqq \sum_{i=1}^{n-2} a_i + a_n\ .$$

This result is due to A.N.Isachenko.

22. (Kovalev & Girlikh 1978). In the two-parameter standardization problem the $(n\times n)$-matrix Q has the form

$$\begin{bmatrix} Q_1 & 0 & 0 & \dots & 0 \\ Q_1 & Q_1 & 0 & \dots & 0 \\ Q_1 & 0 & Q_1 & \dots & 0 \\ . & . & . & \dots & . \\ Q_1 & 0 & 0 & \dots & 0 \end{bmatrix},$$

where Q_1 is a triangular $(q\times q)$-matrix, $n = pq$. The following recurrence relation holds:

$$f_0(H_{pq}(Q)) = 2^{(p-1)(q-1)} + (p-1)2^{(p-2)(q-1)} + \\ + \sum_{t=1}^{q-1} [(p-1)2^{(p-2)t} + 2^{(p-1)t}] f_0(H_{p,q-t}(Q))\ ,$$

from which, by means of generating functions, it is possible to obtain the explicit formula:

$$f_0(H_{pq}(Q)) = 2^{(p-3)(q-1)-1}\left[\left(p - \frac{p^2+2}{\sqrt{(p^2+8)}}\right)(p+4-\sqrt{(p^2+8)})^{q-1} + \right. \\ \left. + \left(p + \frac{p^2+2}{\sqrt{(p^2+8)}}\right)(p+4+\sqrt{(p^2+8)})^{q-1}\right]\ ,$$

in particular,

$$f_0(H_{2q}(Q)) = (1-\sqrt{3}/2)(3-\sqrt{3})^{q-1} + (1+\sqrt{3}/2)(3+\sqrt{3})^{q-1}\ .$$

23. (Girlikh & Kovalev 1974). Let A_{K_m} be the incidence matrix of the graph K_m and let $a = (a,\dots,a)$. Then all vertices of the polytope $H_m(A_{K_m})$ take the form $y_i = a(m-\pi_i)$, $\forall\, i\in N_m$, $\pi\in S_m$. Using this result, show that the standardization problem $\min\limits_{y\in H_m(A_{K_m})} \sum_{i=1}^{m} f_i(y_i)$, where

$f_i(y)$ are concave functions, is equivalent to the assignment problem with matrix $(f_i(a(m-j)))_{m\times m}$.

24. Show that the combinatorial type of the polytope $H(Q)$ does not depend on the vector $a=(a_1,\ldots,a_n)$, but depends solely on the matrix Q .

25. In the class of polytopes $H_m(Q)$ where the matrices Q have the property $\sum_{i=1}^{m} q_{ij} \geqq r+1$, the maximum number of vertices is given by $(m-r)!\binom{m}{r}$. Moreover, this maximum is attained by the polytopes $H_m(Q_0)$, where Q_0 is an r-complete Boolean matrix, that is, a matrix, which has as a submatrix the incidence matrix of an r-complete hypergraph with m vertices, i,e. a hypergraph, such that every subset of $r+1$ vertices is an edge and there are no other edges. Show that the polytope $H_m(Q_0)$ has $m-1+\sum_{i=1}^{m-r-1}\binom{m}{i}$ facets.

26. The *permanent* of the matrix $A=(a_{ij})_{n\times n}$ is the sum

$$\text{per } A = \sum_{\pi\in S_n} a_{1\pi_1}a_{2\pi_2}\cdots a_{n\pi_n} .$$

In 1926 van der Waerden conjectured that $\min_{A\in M_n} \text{per } A = (n!)/n^n$, where the the minimum is attained if and only if $a_{ij}=1/n$, $\forall i,j\in N_n$.

A proof of the truth of this conjecture is given in 'A solution of van der Waerden's conjecture' by G.P.Yegorychev, Krasnoyarsk, 1980. Another proof is given by D.L.Flikman, 'A proof of van der Waerden's conjecture on the permanent of a doubly stochastic matrix', Matematicheskie Zametki, 1981, 29 , No.6.

6 CLASSICAL TRANSPORTATION POLYTOPES

Among the polytopes associated with linear programming problems, the transportation polytopes have been extensively studied, especially the polytopes of the classical transportation problem.

There are dozens of names in the list of literature devoted to characterizing and counting the vertices and faces of the classical transportation polytope. The review of Klee & Witzgall (1968), which appeared over ten tears ago, should be noted first. This review contains the Simmonard-Hadley formula (Simmonard & Hadley 1959) for the number of bases of the transportation problem, Demuth's formula (Demuth 1961) for the minimal number of vertices of the transportation polytope in both the degenerate and the non-degenerate case, together with results due to the authors themselves : bounds on the number of faces and a qualitative (asymptotic) estimate of the maximum number of vertices.

Serious difficulties were encountered in trying to obtain a formula for the maximum number of vertices $\phi(m,n)$ in the class of transportation polytopes of order $m\times n$. Thus, in Klee & Witzgall (1968) a formula for $\phi(2,n)$ only is obtained, while in Likhachev & Yemelichev (1974) and in Likhachev (1975) upper bounds for $\phi(m,n)$ are obtained.

In 1968 Klee & Witzgall conjectured that if m and n are coprime then the so-called *central polytope* $M(a^*,b^*)$ of order $m\times n$, defined by the vectors $a^* = (n,n,\dots,n) \varepsilon E_m$ and $b^* = (m,m,\dots,m) \varepsilon E_n$, has the maximum possible number of vertices. For the cases $n = mq \pm 1$, they derived a formula for calculating the number of vertices of a central polytope. This conjecture was verified by Bolker (1972) who formulated two further interesting conjectures concerning $\phi(m,n)$ and the asymptotic behaviour of the class of transportation polytopes with the maximum possible number of vertices. The first of these was proved by Yemelichev, Kravtsov & Krachkovsky (1977i) while the second was shown to be false by Krachkovsky (1979).

Recently, criteria have been obtained for a transportation polytope to belong to the class of polytopes with the minimum number of vertices (Yemelichev, Kravtsov & Krachkovsky 1977iii) and with the maximum number of vertices (Yemelichev & Kravtsov 1976iii, 1978); polytopes have been classified according to the number of their faces (Yemelichev & Kravtsov 1977, Kravtsov & Yemelichev 1976); criteria have been found for a non-degenerate classical transportation polytope with a fixed number of faces to belong to the class of polytopes with the minimum or maximum number of vertices (Yemelichev, Kravtsov & Krachkovsky 1978iii, 1979); the asymptotic behaviour of a number of classes of classical transportation polytopes with increasing order has been clarified (Yemelichev, Kravtsov & Krachkovsky 1978ii, Yemelichev & Krachkovsky 1978, Krachkovsky 1979) and a number of results have been obtained connected with estimating the diameter of the classical transportation polytope (Yemelichev, Kravtsov & Krachkovsky 1978ii)

This chapter is devoted to describing these results. The criteria for maximality of the number of vertices of the transportation polytope and the apparatus for calculating this number are the central results of this chapter. To establish them we introduce the auxiliary concepts of equivalence, regularity and spectrum.

§1 BASIC DEFINITIONS AND PROPERTIES

Transportation type problems are among the most widely occuring of all linear programming problems. They arise in various domains of economics, technology and industry. The transportation problems associated with planning the carriage of goods are well known.

The classical transportation problem may be described as follows. There are m suppliers which can supply the same product and which must be delivered to n users. Let the i^{th} supplier produce a_i units of the product in unit time ($a_i > 0$) and let the j^{th} user require exactly b_j unit in unit time ($b_j > 0$). Suppose that it costs c_{ij} units to transport one unit of product from the i^{th} supplier to the jth user. The problem is to determine the quantity of product x_{ij} to be transported from the i^{th} supplier to the j^{th} user. It is required to minimize the total transportation cost. Thus, the classical transportation problem of order $m{\times}n$ requires the minimization of the linear function

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij}$$

subject to the constraints

$$\sum_{i=1}^{m} x_{ij} = b_j \quad , \quad j \in N_n , \qquad \sum_{j=1}^{n} x_{ij} = a_i \quad , \quad i \in N_m , \tag{1.1}$$

$$x_{ij} \geqq 0 \quad , \quad (i,j) \in N_m \times N_n \ . \tag{1.2}$$

Let $a = (a_1,\dots,a_m)$, $b = (b_1,\dots,b_n)$. We denote by $M(a,b)$ the set of matrices $x = (x_{ij})_{m\times n}$ whose elements satisfy constraints (1.1) and (1.2). It is easily seen that this set is non-empty if and only if

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

that is, the total supply equals the total demand.

The set $M(a,b)$ is clearly bounded. We will call it the *classical transportation polytope of order* $m\times n$, determined by the vectors $a = (a_1,\dots,a_m)$, $b = (b_1,\dots,b_n)$ or simply the *transportation polytope.*

Proposition 1.1 *The dimension of the transportation polytope of order* $m\times n$ *is* $(m-1)(n-1)$.

Proof Clearly, the inequalities (1.2) can be satisfied strictly. It is easily shown that the rank of the matrix R of constraints (1.1) is $m+n-1$. Thus the proposition follows directly from Proposition 4.1 of Chapter 1. //

Let R^{ij} be a column-vector of the constraint matrix R of (1.1), where R^{ij} has a unit element in its i^{th} and $m+j^{th}$ rows. Then the system (1.1) can be written in the form

$$\sum_{i=1}^{m} \sum_{j=1}^{n} R^{ij} x_{ij} = (a,b)^T \ .$$

Thus, the point $(x_{ij})_{m\times n} \in M(a,b)$ is a vertex of the polytope $M(a,b)$ if and only if the vectors R^{ij} for which $x_{ij} > 0$ are linearly independent (§4, Ch.1). Thus, the number of positive components at any vertex of the transportation polytope of order $m\times n$ does not exceed $m+n-1$.

Definition 1.1 A vertex of an $m\times n$ transportation polytope is called a *non-degenerate vertex* if it has exactly $m+n-1$ positive components, otherwise the vertex is *degenerate* . A transportation polytope is called *non-degenerate* if all its vertices are non-degenerate, otherwise it is *degenerate* .

It is clear that every non-degenerate transportation polytope is simple. The following example shows that the converse statement is false. Let $a^0 = (m(n-1), \underbrace{1,1,\ldots,1}_{m-1})$, $b^0 = (\underbrace{m,m,\ldots,m}_{n-1}, m-1)$. It can be checked directly that there are exactly $(m-1)(n-1)$ facets which are incident to any vertex of the transportation polytope $M(a^0,b^0)$. Thus $M(a^0,b^0)$ is a simple polytope. On the other hand, this polytope has a degenerate vertex x, whose components are defined as follows : $x_{1j} = m$, $\forall j \in N_{n-1}$, $x_{in} = 1$, $i=2,3,\ldots,m$, $x_{ij} = 0$ otherwise.

We formulate a widely known criterion for a transportation polytope to be non-degenerate. First we make the following definitions with respect to the polytope $M(a,b)$:

$$\mu_{I,J}(a,b) = \sum_{i\in I} a_i - \sum_{j\in J} b_j \quad , \quad I \subseteq N_m \quad , \quad J \subseteq N_n ;$$

$$A(a,b) = \{(I,J) \in A_{m\times n} : \mu_{I,J}(a,b) = 0\},$$

where

$$A_{m\times n} = \{(I,J) : 1 \in I \subset N_m \ , \ \emptyset \neq J \subset N_n\}$$

<u>Theorem 1.2</u> *The transportation polytope* $M(a,b)$ *of order* $m\times n$ *is non-degenerate if and only if* $A(a,b) = \emptyset$.

The proof will use the following definitions and a lemma.

Let $R^{i_1j_1}, R^{i_2j_2}, \ldots, R^{i_{m+n-1}j_{m+n-1}}$ be a set of linearly independent columns of the matrix R containing those columns of R which correspond to the positive components of the vertex x of $M(a,b)$ of order $m\times n$. The set $T(a,b,x) = \{(i_k,j_k) : k \in N_{m+n-1}\}$ is called the *basis set of the vertex* $x \in M(a,b)$.

When x is a non-degenerate vertex, $T(a,b,x)$ is the set of those pairs (i,j) for which $x_{ij} > 0$. Note that if x is a degenerate vertex, the set $T(a,b,x)$ is not uniquely defined.

<u>Lemma 1.3</u> *Let* $T(a,b,x)$ *be a basis set of the vertex* x *of the transportation polytope* $M(a,b)$ *of order* $m\times n$. *Then, for every pair of indices* $(k,r) \in T(a,b,x)$ *there is a pair of subsets* (I,J) , $\emptyset \neq I \subseteq N_m$, $J \subseteq N_n$, $J \neq N_n$, *such that* $x_{kr} = \mu_{I,J}(a,b)$.

Proof For the pair $(k,r) \in T(a,b,x)$ we introduce the following notation:

$$I^1 = \{k\} \quad , \quad J^1 = \emptyset \ ,$$

$$J^s = J^{s-1} \cup \{j : j \neq r \ , \ (i,j) \in T(a,b,x) \ , \ i \in I^{s-1}\} \ ,$$

$$I^s = I^{s-1} \cup \{i : (i,j) \in T(a,b,x) \ , \ j \in J^s\} \ , \quad s \geqq 2 \ .$$

Let t be an index such that $I^t = I^{t+1}$, $J^t = J^{t+1}$. Then the pair (I^t, J^t) is the desired pair. //

Proof of Theorem 1.2. Necessity. Suppose that there is a non-degenerate transportation polytope $M(a,b)$ for which $A(a,b) \neq \emptyset$. Let $(L,P) \in A(a,b)$. Without loss of generality we may assume that $L = N_k$, $P = N_r$.

Adopt the following notation :

$$a^1 = (a_1, a_2, \dots, a_k) \ , \qquad b^1 = (b_1, b_2, \dots, b_r),$$

$$a^2 = (a_{k+1}, a_{k+2}, \dots, a_m) \ , \qquad b^2 = (b_{r+1}, b_{r+2}, \dots, b_n) \ .$$

Since the vectors a and b have the same sums and since $(L,P) \in A(a,b)$, we find that the polytopes $M(a^1,b^1)$ and $M(a^2,b^2)$ are non-empty.

Let $T(a^1,b^1,x^1)$ be a basis set for the vertex $x^1 = (x^1_{ij}) \in M(a^1,b^1)$ and let $T(a^2,b^2,x^2)$ be a basis set for the vertex $x^2 = (x^2_{ij}) \in M(a^2,b^2)$. It is easily seen that the matrix x with components

$$x_{ij} = \begin{cases} x^1_{ij} & \text{if} \quad (i,j) \in L \times P, \\ x^2_{ij} & \text{if} \quad (i,j) \in \overline{L} \times \overline{P}, \\ 0 & \text{otherwise} \end{cases}$$

is a vertex of $M(a,b)$. Here $\overline{L} = N_m \setminus L$, $\overline{P} = N_n \setminus P$. At the same time, because of the obvious equalities $|T(a^1,b^1,x^1)| = k+r-1$, $|T(a^2,b^2,x^2)| = m-k+n-r-1$, the number of its positive components is less than $m+n-1$. Thus we have obtained a contradiction of the assumption that the polytope $M(a,b)$ was non-degenerate.

Sufficiency. For any vertex x of $M(a,b)$, when $A(a,b)=\emptyset$, we have, by Lemma 1.3, that $x_{ij} > 0$, $(i,j) \in T(a,b,x)$. Thus, since $|T(a,b,x)| = m+n-1$ for all vertices $x \in M(a,b)$, we see that the polytope $M(a,b)$ is non-degenerate. //

The following is an important property of the transportation polytope.

Proposition 1.4 *The polytope* $M(a,b)$ *is integral if and only if the vectors* a *and* b *are integral.*

Proof Sufficiency follows from Lemma 1.3. Necessity follows by the method of contradiction. //

§2 BASES AND SPANNING TREES

2.1 The Number of Bases

The following theorem was proved by Simmonard & Hadley (1959).

Theorem 2.1 *The number of bases* $\beta(m,n)$ *of the transportation polytope of order* $m\times n$ *is given by the formula*

$$\beta(m,n) = n^{m-1}m^{n-1} . \tag{2.1}$$

Proof Consider the $((m+n-1)\times mn)$-matrix $\bar{R}$ which is obtained from the matrix R (§1) by removing the $(m+n)^{th}$ row. It was shown earlier that R had rank $m+n-1$. Thus, since the matrix R is absolutely unimodular (§4, Ch.4), we have, using Proposition 4.5 (Ch.1) that $\beta(m,n) = \det(\bar{R}\bar{R}^T)$.

Carrying out elementary determinant operations, we have the following chain of equalities :

$$\beta(m,n) = \begin{bmatrix} n & & & & 1 & 1 & \cdots & 1 \\ & n & & & 1 & 1 & \cdots & 1 \\ & & \ddots & & \vdots & \vdots & \ddots & \vdots \\ & & & n & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & m & & & \\ 1 & 1 & \cdots & 1 & & m & & \\ \vdots & \vdots & \cdots & \vdots & & & \ddots & \\ 1 & 1 & \cdots & 1 & & & & m \end{bmatrix} = \begin{bmatrix} n & -n & \cdots & -n & & & & 1 \\ & n & & & & & & 1 \\ & & \ddots & & & & & \vdots \\ & & & n & & & & 1 \\ 1 & & & & m & & & \\ 1 & & & & & m & & \\ \vdots & & & & & & \ddots & \\ 1 & & & & -m & -m & \cdots\ -m & m \end{bmatrix}$$

$$= \begin{bmatrix} n & & & & & & & m \\ & n & & & & & & 1 \\ & & \ddots & & & & & \vdots \\ & & & n & & & & 1 \\ 1 & & & & m & & & \\ 1 & & & & & m & & \\ \vdots & & & & & & \ddots & \\ 1 & & & & & & & \\ n-1 & & & & & & & m \end{bmatrix} = \begin{bmatrix} n & & & & & & & m \\ & n & & & & & & \\ & & \ddots & & & & & \\ & & & n & & & & \\ & & & & m & & & \\ & & & & & m & & \\ & & & & & & \ddots & \\ n-1 & & & & & & & m \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & & & & & \\ & n & & & & & & \\ & & \ddots & & & & & \\ & & & n & & & & \\ & & & & m & & & \\ & & & & & m & & \\ & & & & & & \ddots & \\ n-1 & & & & & & & m \end{bmatrix} = m^{n-1}n^{m-1} .$$

Here, all elements not shown are zero. //

2.2 Spanning Trees

The following theorem establishes a connection between the bases of a transportation polytope and the spanning trees of the corresponding complete bipartite graph.

Theorem 2.2 *There is a bijection between the set of spanning trees of the complete marked bipartite graph* $K_{m,n}$ *and the set of bases of the transportation polytope of order* $m \times n$.

The proof is similar to the proof of Proposition 1.2 (Ch. 5) with the difference that instead of the complete bipartite graph $K_{n,n}$ we need to consider the graph $K_{m,n}$. //

Theorems 2.1 and 2.2 have the following corollary.

Corollary 2.3 *The number of spanning trees of the complete marked bipartite graph* $K_{m,n}$ *is given by the formula*

$$\Delta(m,n) = m^{n-1}n^{m-1} . \tag{2.2}$$

There are many different proofs of formulae (2.1) and (2.2) in the literature (see, for example, Austin (1960), Scoins (1962), Szwarz & Wintgen (1965), Olah (1968), Moon (1969), Yemelichev & Kononenko (1970)) In deriving these formulae and their extensions, use has been made of Stirling numbers, Prüfer numbers, generating functions and the Kirchoff and Binet-Cauchy formulae.

Theorem 2.4 (Klee & Witzgall 1968) *The number of spanning trees* $G(U,V)$ *of the complete marked bipartite graph* $K_{m,mq+1}$, $m \geq 2$, $q \geq 1$, *for which the condition* $\deg i = q+1$, $i \in U$ *is satisfied, is given by the number* $(mq+1)!(mq+1)^{m-2}/(q!)^m$.

We precede the proof of this theorem, which will be needed in §8, by the following lemma.

Lemma 2.5 *Let the vertices of a graph be marked with the numbers* $1,2,\dots,m$, $m \geq 2$. *Let* $\ell_1,\ell_2,\dots,\ell_m$ *be non-negative whole numbers such that* $\sum_{i=1}^{m} \ell_i = m-2$. *Then, the number of spanning trees of the complete marked graph* K_m *for which* $\deg i = \ell_i+1$, $i \in N_m$, *is given by*

$$\nu(\ell_1,\ell_2,\dots,\ell_m) = \frac{(m-2)!}{\ell_1!\ell_2!\dots\ell_m!} .$$

Proof We use induction on m. For $m=2$ the lemma is trivial.

Let $\ell_1 \geq \ell_2 \geq \dots \geq \ell_m$. Let the number k be determined by $\ell_k > 0$, $\ell_{k+1} = 0$. Fix a number $r \in N_k$. Since the vertex with index m is a terminal vertex we have, by the inductive assumption, that the number of spanning trees containing the edge (m,r) is equal to $(m-3)!/\{\ell_1!\ell_2!\dots\ell_{r-1}!(\ell_r-1)!\ell_{r+1}!\dots\ell_k!\}$. Now, summing over all $r \in N_k$ we obtain the assertion of the lemma. //

Proof of Theorem 2.4. Fix a particular $v \in V$. Since $G(U,V)$ is a tree, given any vertex $i \in U$ there is a unique path connecting the vertices i and v. Let $(i,j(i))$ be the edge in this path which is incident to i. The forest obtained from the tree $G(U,V)$ by the simultaneous removal of the edges $(1,j(1)),(2,j(2)),\dots,(m,j(m))$ will be denoted by $R^v(G)$. Let

$$Q^v = \{R^v(G) : G(U,V) \quad D_{m,mq+1}\} ,$$

where $D_{m,mq+1}$ is the set of spanning trees of the graph $K_{m,mq+1}$ which satisfy the condition in the theorem. Since every connected component $R_i^v(G)$ of the forest $R^v(G)$ has one vertex $i \in U$ and q vertices in the set $V\setminus\{v\}$ which are neighbours of i, we have the equation

$$|Q^v| = \frac{(mq)!}{(q!)^m} \tag{2.3}$$

Further, for each tree $G(U,V) \in D_{m,mq+1}$ containing the graph $R^v(G)$, we construct the tree $\bar{G}$ with vertices $U \cup \{v\}$ according to the following rule: $(i,v) \in \bar{G}$ if $(i,v) \in G(U,V)$; $(i,j) \in \bar{G}$ if either $(i,k) \in G(U,V)$, $(j,k) \in R_j^v(G)$ or $(j,k) \in G(U,V)$, $(i,k) \in R_i^v(G)$. From our construction it follows that there exist non-negative whole numbers ℓ_v, ℓ_i, $i \in N_m$, such that

$$\ell_v + 1 = \deg_{\bar{G}} v = \deg_{G(U,V)} v ,$$

$$\ell_i + 1 = \deg_{\bar{G}} i = \sum_{j \in E_i} \deg j - q + 1 , \qquad i \in N_m ,$$

where $\deg_{\bar{G}} i$ is the degree of the vertex i in the graph $\bar{G}$ and $E_i = \{j \in V : (i,j) \in R_i^v(G)\}$.

On the other hand, it is easily verified that the tree $\bar{G}$ has the following properties:

1) if the edge $(i,j) \in \bar{G}$, $j \neq v$, is incident to the vertex i in the path joining vertex i to vertex v, then $(i,j) \in G(U,V)$ for any $k \in R_j^v(G)$;

2) if $(i,j) \in \bar{G}$, $j = v$, then $(i,j) \in G(U,V)$.

Since the number of vertices $k \in R_j^v(G)$, $k \in V$, is equal to q, then by properties 1) and 2) the number of such trees is equal to $q^{\sum_{i=1}^{m} \ell_i}$. By Lemma 2.5 and equation (2.3) this gives

$$|D_{m,mq+1}| = \frac{(mq)!}{(q!)^m} \sum_{\ell_1+\dots+\ell_m+\ell_v=m-1} \frac{(m-1)!}{\ell_1!\dots\ell_m!\,\ell_v!}\, q^{\sum_{i=1}^{m}\ell_i}$$

$$= \frac{(mq+1)!}{(q!)^m}(mq+1)^{m-2} . \quad //$$

§3 FACES

In this section we classify transportation polytopes according to their numbers of faces. Throughout this chapter a face of a transportation polytope of order $m \times n$, $m,n \geqq 2$ will be understood to be a face of maximal dimension (facet), that is a $(d-1)$-face where $d = (m-1)(n-1)$. Since a transportation polytope of order 2×2 has only two vertices, this case will be excluded in what follows.

It is clear that the facets of the transportation polytope $M(a,b)$ of order $m\times n$ must be point sets which belong to coordinate hyperplanes, that is, nonempty sets of the form

$$F_{sq}(a,b) = \{x = (x_{ij})_{m\times n} \in M(a,b) : x_{sq} = 0\}, \quad (s,q) \in N_m \times N_n .$$

The question arises as to what additional conditions the components of the vectors a and b should satisfy in order that the set $F_{sq}(a,b)$ should be a facet of the transportation polytope? Such a characterization is given by the following theorem due to Klee & Witzgall (1968).

Theorem 3.1 *The set* $F_{sq}(a,b)$ *is a facet of the transportation polytope* $M(a,b)$ *of order* $m\times n$, $mn > 4$, *if and only if*

$$a_s + b_q < \sum_{i=1}^{m} a_i .$$

Proof 1) Sufficiency. Since there is a matrix $x' \in M(a,b)$ with components $x'_{ij} > 0$ for all $(i,j) \neq (s,q)$, the only constraints which are satisfied as equalities on the whole set $F_{sq}(a,b)$ are conditions (1.1) and $x_{sq} = 0$. It is easy to see that the determinant of the matrix consisting of the coefficients of the unknowns $x_{1p}, x_{2p}, \ldots, x_{mp}, x_{\ell q}, x_{sq}, x_{1j}$ for all $j \neq p,q$, where $p \neq q$, $\ell \neq s$, is equal to unity. Taking into account the known fact that any equation of the system (1.1) is a consequence of the remaining $m+n-1$ equations, we find that the rank of the system of constraints satisfied as equations on the set $F_{sq}(a,b)$ is equal to $m+n$. Hence, by Proposition 4.1 (Ch.1), the set $F_{sq}(a,b)$ is a facet of the polytope $M(a,b)$.

2) Necessity. Let $x \in F_{sq}(a,b)$. Then we have the relations

$$b_q = \sum_{i=1}^{m} x_{iq} = \sum_{i\neq s} x_{iq} \leqq \sum_{i\neq s} a_i .$$

Thus, $b_q + a_s \leqq \sum_{i=1}^{m} a_i$. This inequality can not be satisfied as an equality for otherwise the set $F_{sq}(a,b)$ would only contain one element. Since $mn > 4$ this would contradict the fact that $F_{sq}(a,b)$ is a facet of the polytope $M(a,b)$. //

The following theorem gives a criterion for a transportation polytope to belong to the class of polytopes with a given number of facets (Yemelicheva & Kravtsov 1977).

Theorem 3.2 *Let* $2 \leq m \leq n$, $n \geq 3$, $0 \leq k \leq n$. *The transportation polytope* $M(a,b)$ *of order* $m \times n$ *has* $(m-1)n + k$ *facets if and only if the following conditions are satisfied :*

1) *When* $k = n-1$

$$b_2 < \sum_{i=2}^{m} a_i \leqq b_1 \quad , \quad a_2 < \sum_{j=2}^{n} b_j \leqq a_1 \ ; \tag{3.1}$$

2) *When* $0 \leqq k < n-m$

$$b_{n-k+1} < \sum_{i=2}^{m} a_i \leqq b_{n-k} \ ; \tag{3.2}$$

3) *When* $n-m \leqq k \leqq n$, $k \neq n-1$, *either* (3.2) *holds or*

$$a_{n-k+1} < \sum_{j=2}^{n} b_j \leqq a_{n-k} \ , \tag{3.3}$$

where $a_1 \geqq a_2 \geqq \ldots \geqq a_m$, $b_1 \geqq b_2 \geqq \ldots \geqq b_n$, $a_0 = b_0 = +\infty$, $a_{m+1} = b_{n+1} = 0$.

Proof Sufficiency. Let $k = n-1$. Then, putting $c = \sum_{i=1}^{m} a_i$, we have from conditions (3.1) :

$$a_i + b_j \begin{cases} \geqq c \ , & \text{if } (i,j) = (1,1), \\ < c \ , & \text{if } (i,j) \neq (1,1). \end{cases}$$

Hence, by Theorem 3.1 every set $F_{ij}(a,b)$, $(i,j) \neq (1,1)$, and only these sets are facets of the polytope $M(a,b)$. Hence, $f_{d-1}(M(a,b)) = mn - 1$.

Now let $k \neq n-1$. The following two cases can occur.

a) Condition (3.2) holds for $M(a,b)$. Then

$$a_1 + b_j \geqq c \ , \qquad j = 1,2,\ldots,n-k,$$

$$a_1 + b_j < c \,, \qquad j = n-k+1, n-k+2, \dots, n,$$

$$a_i + b_j < c \,, \qquad i = 2,3,\dots,m, \quad j = 1,2,\dots,n.$$

By Theorem 3.1 this means that all sets $F_{ij}(a,b)$, $(i,j) \notin \{(1,1),(1,2),\dots,(1,n-k)\}$ and only those sets are facets of $M(a,b)$. Hence $f_{d-1}(M(a,b)) = (m-1)n + k$.

b) Condition (3.3) holds for $M(a,b)$. This case may be dealt with in the same way as case a). This completes the proof of sufficiency.

Before proving necessity we introduce the concept of a critical pair of a polytope and establish some of its properties.

A pair $(s,q) \in N_m \times N_n$ is called a *critical pair of the polytope* $M(a,b)$ *of order* $m \times n$ if the inequality

$$a_s + b_q \geqq \sum_{i=1}^{m} a_i$$

holds. Clearly, a pair (s,q) is a critical pair of a non-degenerate polytope $M(a,b)$ if and only if for every matrix $x \in M(a,b)$ the component $x_{sq} > 0$.

<u>Lemma 3.3</u> *Let* (s,q), (r,t) *be critical pairs of a transportation polytope of order* $m \times n$, $mn > 4$, *then either* $s = r$ *or* $q = t$.

Proof Let $q \neq t$. Then the inequality $a_s + b_q \geqq \sum_{j=1}^{n} b_j$ implies that $a_s \geqq b_t$. Now suppose, in addition, that $s \neq r$, then in the same way we obtain $b_t \geqq a_s$. Since $mn > 4$ it is clear that one of the inequalities $a_s \geqq b_t$, $b_t \geqq a_s$ must be strict. This contradiction proves the lemma. //

Proof of Necessity in Theorem 3.2. Let $f_{d-1}(M(a,b)) = (m-1)n + k$. Then, by Theorem 3.1, the number of critical pairs of $M(a,b)$ is equal to $n-k$. By Lemma 3.3, in the cases $m < n-k$ these pairs are given by $(1,j)$, $j \in N_{n-k}$, and in the cases $m \geqq n-k$ the pairs are given by either $(1,j)$, $j \in N_{n-k}$ or $(i,1)$, $i \in N_{n-k}$ (see Figures 38,39). It is easily seen that all the conditions of the theorem are satisfied. //

Lemma 3.3 implies that the largest possible number of critical pairs of a transportation polytope of order $m \times n$, $2 \leqq m \leqq n$, $n \geqq 3$, is the number n. Consequently, theorem 3.1 shows that the minimal number

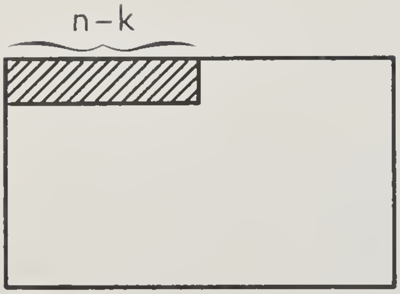


Fig. 38.

Fig. 39.

of facets in the class of transportation polytopes of order $m \times n$, $2 \leq m \leq n$, $n \geq 3$ equals $(m-1)n$. But for any transportation polytope $M(a,b)$ of order $m \times n$ we have $f_{d-1}(M(a,b)) \leq mn$. Thus, Theorem 3.2 has the following corollary.

Corollary 3.4 *For* $2 \leq m \leq n$, $n \geq 3$ *any integer of the form* $(m-1)n+k$, *where* $0 \leq k \leq n$, *and only such integers, can equal the number of faces of a transportation polytope of order* $m \times n$.

We also have the following theorem which is analagous to theorem 3.2.

Theorem 3.2' *Let* $2 \leq m \leq n$, $n \geq 3$, $0 \leq k \leq n$. *A non-degenerate transportation polytope* $M(a,b)$ *of order* $m \times n$ *has* $(m-1)n+k$ *facets if and only if the conditions of theorem* 3.2 *are satisfied and if the inequalities* (3.1)-(3.3) *are strict.*

§4 DIAMETER

The diameter is an important characteristic of a polytope because it represents a bound on the maximum number of iterations which are necessary in solving a linear programming problem on the polytope. The classification of the non-degenerate transportation polytopes in the previous section according to the number of facets enables us now to evaluate the maximum and minimum diameter in each of these classes.

4.1 Auxiliary Facts

First, let us examine the process of passing from one vertex $x = (x_{ij})_{m \times n}$ of the polytope $M(a,b)$ to a neighbouring vertex.

Let (i_1,j_1) be any index pair in the set $(N_m \times N_n) \setminus T(a,b,x)$. Let $H = T(a,b,x) \cup \{(i_1,j_1)\}$. First, we delete those columns of the matrix x which contain only one element of H. Then, in the resulting matrix we delete those rows containing only one element of H. We then resume the same process on the columns and then on the rows etc.. We continue this process until we obtain a submatrix $\bar{x}$ of x such that in every line (row or column) of $\bar{x}$ there are at least two elements of H. It is easily seen that the pair (i_1,j_1) together with other pairs corresponding to basis elements of $\bar{x}$ form a cycle $L = \{(i_1,j_1),(i_2,j_1),(i_2,j_2),\ldots,(i_s,j_s),(i_1,j_s)\}$. The uniqueness of the cycle follows from Theorem 2.2.

We obtain a new vertex x' whose components are given by the formulae

$$x'_{ij} = \begin{cases} x_{ij} - \eta, & \text{if } (i,j) \in \{(i_2,j_1),(i_3,j_2),\ldots,(i_1,j_s)\} \\ x_{ij} + \eta, & \text{if } (i,j) \in \{(i_1,j_1),(i_2,j_2),\ldots,(i_s,j_s)\} \\ x_{ij} & \text{otherwise,} \end{cases}$$

where $\eta = \min(x_{i_2j_1}, x_{i_3j_2}, \ldots, x_{i_sj_{s-1}}, x_{i_1j_s})$. The operation of passing from the vertex x to the vertex x' is called a *transposition through the cycle* L. The vertices x and x' are adjacent by definition 1.1 of Ch.2.

With each vertex $x = (x_{ij})_{m\times n} \in M(a,b)$ we associate a bipartite graph $G_x(U,V) \subset K_{m,n}$ which contains the edge (i,j) if $x_{ij} > 0$. Similarly, we define the bipartite graph $G_{T(a,b,x)}(U,V)$ whose edges correspond to the pairs $(i,j) \in T(a,b,x)$.

The following obvious lemmas will be required later.

Lemma 4.1 *Let* x *and* y *be distinct vertices of the transportation polytope* $M(a,b)$. *The following statements are equivalent:*

1) *the vertices* x *and* y *are adjacent;*
2) *the graph* $G_x(U,V) \cup G_y(U,V)$ *contains a unique cycle;*
3) *there are basis sets* $T(a,b,x)$, $T(a,b,y)$ *of the vertices* x *and* y *such that the graph* $G_{T(a,b,x)}(U,V) \cup G_{T(a,b,y)}(U,V)$ *contains a unique cycle.*

Lemma 4.2 *If there exist two vertices* x *and* y *of the nondegenerate transportation polytope* $M(a,b)$ *of order* $m\times n$ *which have* t *basis variables in common, then* $\operatorname{diam} M(a,b) \geq m+n-t-1$.

4.2 Minimum Diameter

Let $\mathcal{M}(m,n,k)$ be the set of all non-degenerate transportation polytopes of order $m\times n$, $2\leq m\leq n$ with $(m-1)n+k$ facets. Let

$$d(m,n,k) = \min_{M(a,b)\in\mathcal{M}(m,n,k)} \operatorname{diam} M(a,b) .$$

Theorem 4.3 *The minimum diameter in the class of nondegenerate transportation polytopes of order* $m\times n$, $2\leq m\leq n$, *with* $(m-1)n+k$ *facets is given by the formulae*

$$d(m,n,k) = \begin{cases} m+k-1 & \textit{for} \quad k=0,1, \\ m+1 & \textit{for} \quad k=2,3,\ldots,n. \end{cases}$$

Proof Every vertex of $M(a,b)\in\mathcal{M}(m,n,0)$ with $m\leq n$ corresponds to a matrix constructed as in Figures 40 and 41, where the case shown in Fig. 41 can only occur when $m=n$. Hatched positions correspond to positive components at the vertex, while double hatching indicates positions corresponding to the critical pairs of the polytope (§3). Note that in any line which does not contain a critical pair there is only one positive component, which can occur in any position. Figures 42-46 show the cases which can occur corresponding to vertices in the class $\mathcal{M}(m,n,1)$. The cases in Fig. 44 and Fig. 45 can only occur when $m=n$ while the case of Fig. 46 can only occur when $n=m+1$. In Fig. 42 (44) for each row (column) with index $k\geq 2$ there is a single positive component, except for one row (column) in which there are two such components.

Thus, for any polytope $M(a,b)\in\mathcal{M}(m,n,k)$ with $k=0,1$, we have from the given structure of its vertices that

$$p(a,b) = \min_{x,y\in\operatorname{vert} M(a,b)} |T(a,b,x)\cap T(a,b,y)| = n-k .$$

The structure of the vertices obtained by using the north-west and north-east corner methods (see Problem 17) shows that for any polytope of order $m\times n$ we have the inequality $p(a,b)\leq n$ $(m\leq n)$. It is not difficult to see that $p(a,b)=n$ only for polytopes of the class $\mathcal{M}(m,n,0)$ and that $p(a,b)=n-1$ only for polytopes of the class $\mathcal{M}(m,n,1)$. Consequently

$$p(a,b)\leq n-2 \quad \forall\, M(a,b)\in\mathcal{M}(m,n,k), \quad k=2,3,\ldots,n.$$

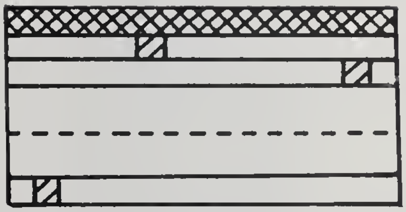


Fig. 40.

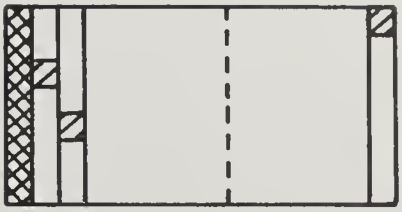


Fig. 41.

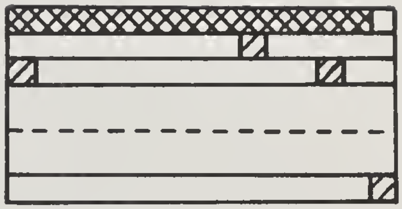


Fig. 42.

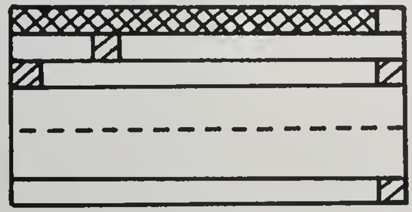


Fig. 43.

Fig. 44.

Fig. 45.

Fig. 46.

Thus, by Lemma 4.2 we have

$$d(m,n,k) \geqq \begin{cases} m+k-1 & \text{for } k=0,1, \\ m+1 & \text{for } k=2,3,\dots,n. \end{cases}$$

We now show that in each class $\mathcal{M}(m,n,k)$, $0 \leqq k \leqq n$, there is a polytope whose diameter does not exceed $m-1$ if $k=0$, or m if $k=1$ or $m+1$ if $2 \leqq k \leqq n$.

Case 1. $k=0$. Let the condition $\sum_{i=2}^{m} a_i < b_n$ be fulfilled for a polytope $M(a,b) \varepsilon \mathcal{M}(m,n,0)$, (see Fig. 40). In the other possible case for $M(a,b) \varepsilon \mathcal{M}(m,n,0)$ in which $\sum_{j=2}^{n} b_j < a_m$ the proof is symmetrically analagous. It is clear that for any two distinct vertices x and y of $M(a,b)$ we have $|T(a,b,x) \cap T(a,b,y)| = n+\ell$ where $0 \leqq \ell \leqq m-2$. Since $x=(x_{ij})_{m\times n} \neq y$, there is an index pair $(s,p) \varepsilon T(a,b,y) \setminus T(a,b,x)$, $s>1$. Then there is a pair $(s,q) \varepsilon T(a,b,x)$, $p \neq q$, and hence a vertex

$x' \in M(a,b)$ with components

$$x'_{ij} = \begin{cases} x_{ij} - \min(x_{1p}, x_{sq}) , & \text{if } (i,j) \in \{(1,p),(s,q)\}, \\ x_{ij} + \min(x_{1p}, x_{sq}) , & \text{if } (i,j) \in \{(1,q),(s,p)\}, \\ x_{ij} , & \text{otherwise.} \end{cases}$$

Clearly, the distance $r(x,x') = 1$ and $|T(a,b,x')\ T(a,b,y)| = n + \ell + 1$. Thus, after $m - \ell - 1$ similar transformations we travel from vertex x to vertex y , that is $r(x,y) \leqq m - \ell - 1$. Hence $\operatorname{diam} M(a,b) \leqq m - 1$.

Case 2. $1 \leqq k \leqq n$. Consider the polytope $M(a,b)$ of order $m \times n$ such that

$$b_{n-k-1} - b_n < \sum_{j=1}^{n-1} b_j - a_1 < \min(a_m, b_{n-k} - b_n) . \tag{4.1}$$

where $a_1 \geqq a_2 \geqq \dots \geqq a_m$, $b_1 \geqq b_2 \geqq \dots \geqq b_n$. These inequalities imply that $M(a,b) \in \mathcal{M}(m,n,k)$ (see Theorem 3.2') and that the set of vertices of $M(a,b)$ is representable in the form

$$\operatorname{vert} M(a,b) = \bigcup_{h=1}^{k+1} V_h(a,b) ,$$

where $V_h(a,b)$, $1 \leqq h \leqq k$, is the set of those vertices of $M(a,b)$ such that

$$x_{1j} > 0 , \quad j = 1,2,\dots,n-h,n-h+2,\dots,n,$$
$$x_{1,n-h+1} = 0 , \quad x_{i,n-h+1} > 0 , \quad i = 2,3,\dots,m,$$

and such that there is exactly one positive component among the remaining components which can occur at any location (i,j) , $i = 2,3,\dots,m$, $j = 1,2,\dots,n-h,n-h+2,\dots,n$ (see Fig. 47); also $V_{k+1}(a,b)$ is the set of vertices of $M(a,b)$ such that $x_{1j} > 0$, $j \in N_n$, and such that in each row with index i , $i = 2,3,\dots,m$, there is exactly one positive component which can be at any location in this row (see Fig. 48) with the exception of cases in which all the components of a column with index t , $t = n-k+1, n-k+2, \dots, n$ are positive.

Consider all possible ways in which the vertices x and y can belong to the classes $V_h(a,b)$, $h \in N_{k+1}$.

Let $x,y \in V_h(a,b)$, $1 \leqq h \leqq k$, $x \neq y$. Then, by Lemma 4.1, the vertices x and y are adjacent.

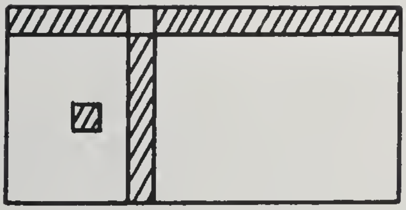


Fig. 47.

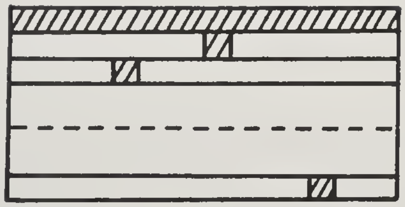


Fig. 48.

Let $x, y \in V_{k+1}(a,b)$, $x \neq y$. Then the proof of the inequality $r(x,y) \leq m-1$ is carried out as in the case $k=0$ with the difference that the index pair $(s,p) \in T(a,b,y) \setminus T(a,b,x)$ is chosen so that for some pair $(s,q) \in T(a,b,x)$ the inequality $x_{sq} < x_{1p}$ is satisfied. The conditions (4.1) ensure that such a pair always exists.

Now let $x \in V_h(a.b)$, $1 \leq h \leq k$ and $y \in V_{k+1}(a,b)$. Choose $x^0 \in V_{k+1}(a,b)$ such that $r(x,x^0) = 1$. Then $r(x^0,y) \leq m-1$ and so $r(x,y) \leq m$.

We have thus shown that when $k=1$ we have the inequality $\operatorname{diam} M(a,b) \leq m$.

Finally, let $x \in V_h(a,b)$, $y \in V_g(a,b)$, $1 \leq h < g \leq k$. Let x^0 and y^0 be those vertices in $V_{k+1}(a,b)$ for which $r(x,x^0) = 1$, $r(y,y^0) = 1$. Then, since $r(x^0,y^0) \leq m-1$ we have $r(x,y) \leq m+1$.

Thus we have shown that for $2 \leq k \leq n$ we have $\operatorname{diam} M(a,b) \leq m+1$. //

If we examine the proof of the first part of Theorem 4.3 more carefully we notice that we have, in fact, proved a stronger result : a non-degenerate transportation polytope of order $m \times n$, $2 \leq m \leq n$. has minimum diameter $m-1$ if and only if it has the minimum number of facets.

4.3 Lower Bound for Maximum Diameter

Theorem 4.4 *Let* $3 \leq m \leq n$, $n \geq 4$, $1 \leq k \leq n$. *In the class of non-degenerate transportation polytopes of order* $m \times n$ *with* $(m-1)n+k$ *facets the maximum diameter is not less than* $m+k-1$.

Proof First let $1 \leq k \leq n-1$. Consider a polytope $M(a,b) \in \mathcal{M}(m,n,k)$ such that

$$\sum_{\substack{j=1 \\ j \neq n-k}}^{n} b_j < a_1 < \sum_{j=1}^{n-k} b_j . \qquad (4.2)$$

Let x and y be the vertices of this polytope constructed by the north-west corner and the north-east corner methods respectively (see Problem 17). Then by condition (4.2) we have $|T(a,b,x) \cap T(a,b,y)| = n-k$. Hence, Lemma 4.2 gives the inequality $\operatorname{diam} M(a,b) \geq m+k-1$.

Now let $k = n$. We examine first the particular case in which $(m,n) = 1$. Let $a^* = \underbrace{(n,n,\ldots,n)}_{m}$, $b^* = \underbrace{(m,m,\ldots,m)}_{n}$. By Theorem 3.2', $M(a^*,b^*) \in \mathcal{M}(m,n,n)$.

For the case in which $m = 3$ and $n = 4$ there is a vertex x with positive components $x_{11} = x_{34} = 3$, $x_{12} = x_{33} = 1$, $x_{22} = x_{23} = 2$ and a vertex y with positive components $y_{13} = y_{32} = 3$, $y_{14} = y_{31} = 1$, $y_{24} = y_{21} = 2$ so that by Lemma 4.2 the theorem is true in this case. In the following we will assume that $m \geq 3$, $n \geq 5$.

An *elementary transformation of a matrix* is any transformation of the form : a) an interchange of any two columns; b) an interchange of any two rows. Clearly, any elementary transformation transforms a vertex of $M(a^*,b^*)$ into another vertex of the same polytope.

Let x be the vertex of $M(a^*,b^*)$ constructed by the north-west corner method. Note that every column of the matrix x contains not more than two positive components.

We show how to use a sequence of elementary transformations to pass from the vertex x to a vertex y which does not intersect it, that is, a vertex y such that $T(a^*,b^*,x) \cap T(a^*,b^*,y) = \emptyset$. To do this To do this we interchange the columns in x with indices j and $n-j+1$, for $j = 1,2,\ldots n/2$. These transformations convert the vertex x into the vertex x' which would be constructed by the north-east corner method. It is easy to see that the sets $T(a^*,b^*,x)$ and $T(a^*,b^*,x')$ intersect either in the p^{th}-row, where $p \neq 1,m$, or in the $(n+1)/2^{th}$-column (if n is odd).

In the first case we interchange the rows of x' numbered p and m. If the vertex x'' so obtained still intersects x then to find the desired vertex y it suffices to interchange successively the columns of x numbered $j-1$ and j, $j = 2,3,\ldots,n$.

In the second case (which can happen when $m \geqq 4$) we can interchange the columns of x' numbered $(n+1)/2$ and n to obtain a vertex which does not intersect x. Hence, by Lemma 4.2 diam $M(a^*,b^*) \geqq m+n-1$.

In the case $(m,n) \neq 1$, it is necessary to consider the polytope $M(a',b')$ of order $m \times n$, determined by the vectors $a' = (n+(m-1)/m, n-1/m, \ldots, n-1/m)$ and $b' = (m,m,\ldots,m)$. By Theorem 3.2' $M(a',b') \in \mathcal{M}(m,n,n)$. It is easy to see that all the reasoning given above for the case $(m,n) = 1$ is still valid, for the first row of the matrix is not affected by the elementary transformations described above. Hence diam $M(a',b') \geqq m+n-1$. //

4.4 An Upper Bound for the Diameter

The purpose of this section is to prove the following proposition.

Theorem 4.5 *The diameter of the transportation polytope of order* $m \times n$, $m,n \geqq 2$ *does not exceed* mn.

We recall that an upper bound on the maximum diameter $\Delta(s,\gamma)$ in the class of s-polytopes with γ facets was obtained in §2, Ch.2, and took the form

$$\Delta(s,\gamma) \leqq 2^{s-3}\gamma .$$

It is easily seen that for the transportation polytopes this bound is too large.

First, we prove the following analogue of theorem 2.2, Ch.2.

Lemma 4.6 *Given any degenerate transportation polytope, there exists a non-degenerate transportation polytope of the same order and with a diameter not smaller than the diameter of the first polytope.*

Proof Let $M(a,b)$ be a degenerate transportation polytope of order $m \times n$. It is clear that a sufficiently small number ε can be found such that the polytope $M(a(\varepsilon),b(\varepsilon))$ defined by the vectors $a(\varepsilon) = (a_1+\varepsilon, a_2+\varepsilon, \ldots, a_m+\varepsilon)$ and $b(\varepsilon) = (b_1, b_2, \ldots, b_{n-1}, b_n+m\ )$ will be non-degenerate (see Problem 1).

We show that each vertex $x^0 = (x^0_{ij})_{m\times n}$ of the degenerate polytope $M(a,b)$ can be associated with a vertex $x^0(\varepsilon)$ of $M(a(\varepsilon),b(\varepsilon))$ in such a way that $x^0(0) = x^0$. Indeed, let

$$c_{ij}(x^0) \begin{cases} = 0, & \text{if } x^0_{ij} > 0, \\ > 0 & \text{otherwise.} \end{cases}$$

It is obvious that the only vertex at which the minimum of of the function $F(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}(x^0)x_{ij}$ is attained is the vertex x^0.

Let $F(x^0(\varepsilon)) = \min_{x \in M(a(\varepsilon),b(\varepsilon))} F(x)$. Then, from the optimality criterion for the transportation problem (see Problem 13), it follows that $F(x^0(0)) = \min_{x\in M(a,b)} F(x)$. Hence $x^0(0) = x^0$.

Let diam $M(a,b) = r(x^0,y^0)$. To prove Lemma 4.6 it suffices to establish the inequality $r(x^0,y^0) \leqq r(x^0(\varepsilon),y^0(\varepsilon))$.

Since $x^0 \neq y^0$ it follows that $x^0(\varepsilon) \neq y^0(\varepsilon)$. Let the shortest chain connecting the vertices $x^0(\varepsilon)$ and $y^0(\varepsilon)$ contain s edges. If $x(\varepsilon)$ and $y(\varepsilon)$ are adjacent vertices of the polytope $M(a(\varepsilon),b(\varepsilon))$ then either $x(0) = y(0)$ or $x(0)$ and $y(0)$ are adjacent vertices of the polytope $M(a,b)$. Thus there is a chain between the vertices x^0 and y^0 of $M(a,b)$ whose length does not exceed s. //

Proof of Theorem 4.5. By Lemma 4.6 we can restrict our considerations to the case of non-degenerate polytopes.

We use induction on the number $k = m + n$. When $k = 4$ the theorem can be checked directly.

For a vertex $x = (x_{ij})_{m\times n}$ of $M(a,b)$ we introduce the notation

$$R(x) = \{(i,j) \in N_m \times N_n : x_{ij} = \min(a_i, b_j)\}$$

This set is clearly not empty for any vertex of $M(a,b)$.

Let $x = (x_{ij})_{m\times n}$ and $y = (y_{ij})_{m\times n}$ be arbitrary vertices of $M(a,b)$. Let $\alpha = \min\left(\min_{(i,j)\in R(x)} x_{ij}, \min_{(i,j)\in R(y)} y_{ij}\right)$. For definiteness suppose that $\alpha = y_{pq} = b_q$. If $\alpha = y_{pq} = a_p$ the proof is similar except that Case 1 is the only possible case.

Suppose first that $x_{pq} = b_q$. Let $\bar{x}$ and $\bar{y}$ denote the matrices which are obtained respectively from x and y by deleting the q^{th} column. The matrices $\overline{x}$ and $\overline{y}$ are vertices of a non-degenerate polytope $M(\overline{a},\overline{b})$ of order $m\times(n-1)$ defined by the vectors

$$\overline{a} = (a_1,a_2,\ldots,a_{p-1},a_p-b_q,a_{p+1},\ldots,a_m),$$

$$\overline{b} = (b_1,b_2,\ldots,b_{q-1},b_{q+1},\ldots,b_n).$$

It is clear that $r(x,y) \leq r(\overline{x},\overline{y})$. Hence by the induction hypothesis we have $r(x,y) \leq m(n-1)$.

Now let $x_{pq} < b_q$. Let $s_p(x)$ and $t_q(x)$ denote respectively the number of positive components in the p^{th}-row and the q^{th}-column of x , excluding the component x_{pq} which may be positive. For definiteness let $m \leq n$.

Case 1 . Suppose that $s_p(x)+t_q(x) \leq m$. There are two possibilities.

a) $x_{pq} > 0$. We construct a vertex $x' = (x'_{ij})_{m\times n}$ of $M(a,b)$ whose components are determined as follows :

$$x'_{ij} = \begin{cases} x_{ij} - \min(x_{hq},x_{p\ell}), & \text{if } (i,j) \in \{(h,q),(p,\ell)\}, \\ x_{ij} + \min(x_{hq},x_{p\ell}), & \text{if } (i,j) \in \{(p,q),(h,\ell)\}, \\ x_{ij} & \text{otherwise,} \end{cases}$$

where $x_{hq} = \max\limits_{\substack{1\leq i\leq m \\ i\neq p}} x_{iq} > 0$, $x_{p\ell} = \max\limits_{\substack{1\leq j\leq n \\ j\neq q}} x_{pj} > 0$. Clearly, the vertices x and x' are adjacent and $x'_{pq} > x_{pq}$ and one of the components x'_{hq} and $x'_{p\ell}$ is zero. Continuing this process and noting that $s_p(x)+t_q(x) \leq m$, we construct a vertex $z = (z_{ij})_{m\times n} \in M(a,b)$ for which $z_{pq} = b_q$ and $r(x,z) \leq m-1$.

b) $x_{pq} = 0$. Then the pair (p,q) together with some pairs from the set $T(a,b,x)$ forms a unique cycle. Performing a transposition through this cycle we obtain an adjacent vertex x' with $s_p(x')+t_q(x') \leq m$, (see paragraph 1). Now proceeding as in a) we obtain a vertex $z = (z_{ij})_{m\times n} \in M(a,b)$ for which $z_{pq} = b_q$ and $r(x,z) \leq m$.

Thus, in case 1 we can always construct a vertex z of $M(a,b)$ whose distance from x is not greater than m and whose distance from y , by the induction hypothesis, is not greater than $m(n-1)$.

Hence $r(x,y) \leq mn$.

Case 2. Suppose that $s_p(x) + t_q(x) \geq m+1$. Since there is an index $u \in \{1,2,\ldots,q-1,q+1,\ldots,n\}$ such that $x_{pu} = b_u$ and $b_u \geq b_q$, then $\sum_{\substack{i \neq p \\ x_{iq} > 0}} x_{iq} \leq b_u$. Hence, repeating the arguments given in case 1 , we obtain the inequality $r(x,y) \leq mn$. //

§5 POLYTOPES WITH THE MINIMAL NUMBER OF VERTICES

Although it is relatively easy to classify transportation polytopes according to the numbers of their facets, a similar classification according to numbers of vertices has not so far been obtained. In this section we describe transporation polytopes which have the minimal possible number of vertices in both the non-degenerate and degenerate cases.

Throughout this section we will assume that the components of the vectors a and b are ordered as follows:

$$a_1 \geq a_2 \geq \cdots \geq a_m \quad , \quad b_1 \geq b_2 \geq \cdots \geq b_n \ .$$

Since $f_0(M(a,b)) = f_0(M(b,a))$ we can also assume that $m \leq n$.

5.1 Nondegenerate Polytopes

The following theorem (Yemelichev & Kononenko 1971) gives a criterion for a non-degenerate transportation polytope of order $m \times n$, $2 \leq m \leq n$, to belong to the class of polytopes with the minimum number n^{m-1} of vertices. It was first obtained by Demuth (1961).

Theorem 5.1 *A non-degenerate transportation polytope* $M(a,b)$ *of order* $m \times n$, $2 \leq m \leq n$, $n \geq 3$ *has the minimal number* n^{m-1} *of vertices if and only if it has the minimal number of facets, that is, the following inequalities hold* :

1) *when* $m < n$

$$\sum_{i=2}^{m} a_i < b_n \ ; \tag{5.1}$$

2) *when* $m = n$ *either* (5.1) *holds, or*

$$\sum_{j=2}^{n} b_j < a_m \ . \tag{5.2}$$

Proof We show first that for any polytope $M(a,b)$ of order $m\times n$, $2\leq m\leq n$, $n\geq 3$ satisfying conditions (5.1) or (5.2) the number of vertices is given by

$$f_0(M(a,b)) = n^{m-1} . \tag{5.3}$$

We assume that condition (5.1) is satisfied since when (5.2) is satisfied we can transform to the first case by considering $M(b,a)$ instead of $M(a,b)$.

For any vertex $x = (x_{ij})_{m\times n}$ of $M(a,b)$ condition (5.1) implies that $x_{1j} > 0$, $j \in N_n$. Hence, since the number of positive components of the vertex x is $m+n-1$, there must be exactly one positive component in each of the rows numbered i, $i = 2,3,\ldots,m$ and this component can occur in any column (see Fig. 40). Let the nonzero element in the i^{th}-row occur in the j_i^{th}-column. Then the number of vertices of $M(a,b)$ is equal to the number of $(m-1)$-tuples $(j_2,j_3,\ldots,j_m)$ whose elements are chosen from N_n. This number is n^{m-1}.

We now show that if $M(a,b)$ is a polytope of order $m\times n$, $2\leq m\leq n$, $n\geq 3$ for which conditions (5.1) and (5.2) are not satisfied, then

$$f_0(M(a,b)) > n^{m-1} . \tag{5.4}$$

We use induction on the number $p = m+n$. When $p = 5$ the inequality (5.4) can be verified directly. Suppose it is true for $p = m+n-1$. We consider three possible cases.

a) $a_m > b_n$, $m < n$. Then by (5.3), which has just been proved, we have by induction

$$f_0(M(a,b)) \geq \sum_{i=1}^{m} f_0(M(a^i,b^n)) \geq m(n-1)^{m-1} > n^{m-1} ,$$

where $a^i = (a_1,a_2,\ldots,a_{i-1},a_i-b_n,a_{i+1},\ldots,a_m)$, $b^n = (b_1,b_2,\ldots,b_{n-1})$. The final inequality can be verified using the well-known inequality $(1+1/k)^k < 3$, $k = 1,2,\ldots$.

b) $a_m < b_n$, $m \leq n$. Then

$$f_0(M(a,b)) \geq \sum_{j=1}^{n} f_0(M(a^m,b^j)) \geq (n-1)n^{m-2} + f_0(M(a^m,b^n)) ,$$

where $a^m = (a_1,a_2,\ldots,a_{m-1})$, $b^j = (b_1,b_2,\ldots,b_{j-1},b_j-a_m,b_{j+1},\ldots,b_n)$.

Since $a_m < b_n$, we have $\sum_{i=2}^{m-1} a_i > b_n - a_m$. Thus, by the induction hypothesis we have $f_0(M(a^m, b^n)) > n^{m-2}$. Hence $f_0(M(a,b)) > n^{m-1}$.

c) $a_m > b_n$, $m = n$. This case reduces to case b) if we replace the polytope $M(a,b)$ by the polytope $M(b,a)$. We have already noted that $f_0(M(a,b)) = f_0(M(b,a))$. //

Theorems 3.2' and 5.1 have the following corollary :

Corollary 5.2 *All polytopes in the class* $\mathcal{M}(m,n,0)$ *are combinatorially equivalent.*

5.2 Degenerate Polytopes

Theorem 5.3 *A degenerate transportation polytope* $M(a,b)$ *of order* $m \times n$, $2 \leq m \leq n$ *has the minimum number* $n!/(n-m+1)!$ *of vertices if and only if the following conditions are satisfied* :

1) *when* $m = 2$, $a_m = b_n$; (5.5)

2) *when* $3 \leq m \leq n$, $a_m = b_1 = b_2 = \ldots = b_n$, $a_1 = (n-m+1)b_1$. (5.6)

Proof If $m = 2$ it is clear that $f_0(M(a,b)) = n$. Now let $3 \leq m \leq n$. Then the number of vertices of $M(a,b)$ is equal to the number of ways in which n objects can be distributed among m sets of which one contains $n - m + 1$ objects while the other sets contain one object each. Hence $f_0(M(a,b)) = n!/(n-m+1)!$.

To complete the proof it suffices to note that any degenerate transportation polytope $M(a,b)$ of order $m \times n$, $2 \leq m \leq n$ which does not satisfy conditions (5.5) or (5.6) satisfies the inequality $f_0(M(a,b)) > n!/(n-m+1)!$. This inequality is proved in the same way as inequality (5.4) and will therefore be left to the reader. //

Theorems 3.2 and 5.3 have the following corollary.

Corollary 5.4 *Every degenerate transportation polytope of order* $m \times n$, $3 \leq m \leq n$ *with a minimum number of vertices has a maximum number of faces.*

Note. Since for $3 \leq m \leq n$ we have $n!/(n-m+1)! < n^{m-1}$, every transportation polytope of order $m \times n$, $3 \leq m \leq n$ with the minimum number of vertices is degenerate.

§6 BASIC CONCEPTS

The central results of this chapter are the criterion for the maximum number of vertices of a transportation polytope and the apparatus for calculating this number. This section is devoted to explaining the concepts of equivalence, regularity and spectrum which will be needed in deriving our results.

6.1 Equivalence

With the vertex $x = (x_{ij})_{m\times n}$ of the polytope $M(a,b)$ of order $m\times n$ we associate the set

$$\mathcal{K}(a,b,x) = \{(i,j) \in N_m \times N_n : x_{ij} > 0\}.$$

When x is a non-degenerate vertex it is clear that $\mathcal{K}(a,b,x) = T(a,b,x)$. Let $M(a^0,b^0)$ and $M(a^1,b^1)$ be two polytopes of the same order.

Definition 6.1 The vertices $x^0 \in M(a^0,b^0)$ and $x^1 \in M(a^1,b^1)$ are called *equivalent vertices* if $\mathcal{K}(a^0,b^0,x^0) = \mathcal{K}(a^1,b^1,x^1)$. If to each vertex of the polytope $M(a^0,b^0)$ there corresponds an equivalent vertex of $M(a^1,b^1)$ and conversely, then we say that these polytopes are *equivalent polytopes* and write $M(a^0,b^0) \sim M(a^1,b^1)$.

For a non-degenerate polytope it is easy to see that this definition is equivalent to definition 1.6, Ch.3 .

As before let

$$\mu_{I,J}(a,b) = \sum_{i\in I} a_i - \sum_{j\in J} b_j \quad , \quad I \subseteq N_m,\ J \subseteq N_n ,$$

$$\mathcal{A}_{m\times n} = \{(I,J) : 1 \in I \subset N_m \ , \ \emptyset \neq J \subset N_n\}.$$

Equivalent polytopes are described by the following theorem.

Theorem 6.1 $M(a^0,b^0) \sim M(a^1,b^1)$ *if and only if*

$$\operatorname{sign} \mu_{I,J}(a^0,b^0) = \operatorname{sign} \mu_{I,J}(a^1,b^1) \qquad \forall (I,J) \in \mathcal{A}_{m\times n} . \quad (6.1)$$

Proof Sufficiency. We introduce some auxiliary concepts. A *line of a matrix* is, as before, any row or column of the matrix. A *simple line* is a line which contains a single nonzero component.

Let $x^0 = (x^0_{ij})_{m\times n}$ be any vertex of the polytope $M(a^0,b^0)$ defined by the vectors $a^0 = (a^0_1,a^0_2,\dots,a^0_m)$ and $b^0 = (b^0_1,b^0_2,\dots,b^0_n)$. Since the number of positive components of any vertex of a transportation polytope of order $m\times n$ does not exceed $m+n-1$, the matrix x^0 contains at least one simple line. This means that there is an index pair (s,k) such that $x^0_{sk} = \min(a^0_s,b^0_k)$.

For definiteness suppose that $a^0_s \leqq b^0_k$, that is, the s^{th}-row of x^0 is a simple line.

Clearly, there are vertices of $M(a^1,b^1)$ defined by the vectors $a^1 = (a^1_1,a^1_2,\dots,a^1_m)$ and $b^1 = (b^1_1,b^1_2,\dots,b^1_n)$ which have the component $x^1_{sk} = \min(a^1_s,b^1_k)$. But, because of our previous assumption and condition (6.1) we have $a^1_s \leqq b^1_k$, so there is a matrix which is a vertex of $M(a^1,b^1)$ whose s^{th}-row is a simple line.

If we now delete the s^{th}-row of the matrix $x^0 \in M(a^0,b^0)$, then among its remaining lines there must be at least one simple line. Continuing this process we construct a vertex $x^1 \in M(a^1,b^1)$ which is equivalent to the vertex x^0.

The above argument shows that to every vertex of $M(a^0,b^0)$ there is an equivalent vertex of $M(a^1,b^1)$ and conversely.

Necessity. Suppose that $M(a^0,b^0) \sim M(a^1,b^1)$ but that there exists a pair $(L,P) \in \mathcal{A}_{m\times n}$ for which (6.1) is not satisfied. Without loss of generality we may consider only the case

$$\mu_{L,P}(a^0,b^0) > 0\ , \tag{6.2}$$

$$\mu_{L,P}(a^1,b^1) \leqq 0\ . \tag{6.3}$$

The condition (6.2) implies that there is at least one vertex $x' \in M(a^0,b^0)$ with components $x'_{ij} = 0$, $(i,j) \in \bar{L}\times P$, while among the components x'_{ij}, $(i,j) \in L\times\bar{P}$ there is at least one positive component.

On the other hand condition (6.3) shows that none of the vertices of $M(a^1,b^1)$ are equivalent to x'. But this contradicts the equivalence of $M(a^0,b^0)$ and $M(a^1,b^1)$. This proves necessity. //

<u>Corollary 6.2</u> *Let* $M(a^0,b^0) \sim M(a^1,b^1)$, *then*

$$M(a^0,b^0) \sim M(\lambda a^1 + (1-\lambda)a^0\ ,\ \lambda b^1 + (1-\lambda)b^0)\quad ,\ \forall\lambda \in [0,1]\ .$$

Let $M(a^0,b^0)$ be a transportation polytope of order $m\times n$ and let ρ be a positive number, assumed small. We introduce a set of polytopes which are close, in some sense, to $M(a^0,b^0)$:

$$Q^\rho(a^0,b^0) = \{M(a,b) : \max_{1\leq i\leq m} |a_i - a_i^0| \leq \rho \ , \ \max_{1\leq j\leq n} |b_j - b_j^0| \leq \rho\} \ .$$

We prove the following property of this set.

<u>Corollary 6.3</u> *Let* $M(a^0,b^0)$ *be a non-degenerate transportation polytope of order* $m\times n$ *and let*

$$0 < \rho \leq \min_{(I,J)\in A_{m\times n}} \frac{|\mu_{I,J}(a^0,b^0)|}{m+n} \ .$$

Then, any polytope $M(a,b)\in Q^\rho(a^0,b^0)$ *is equivalent to* $M(a^0,b^0)$.

Proof We have from the given condition

$$\mu_{I,J}(a^0,b^0) - \rho(|I| + |J|) \leq \mu_{I,J}(a,b)$$

$$\leq \mu_{I,J}(a^0,b^0) + \rho(|I| + |J|) \qquad \forall (I,J)\in A_{m\times n} \ ,$$

which together with the obvious inequalities

$$|\mu_{I,J}(a^0,b^0)| > \rho(|I| + |J|) \qquad \forall (I,J)\in A_{m\times n}$$

give

$$\operatorname{sign} \mu_{I,J}(a^0,b^0) = \operatorname{sign} \mu_{I,J}(a,b) \qquad \forall (I,J)\in A_{m\times n} \ .$$

By Theorem 6.1, this establishes the corollary. //

6.2 <u>Regularity</u>

<u>Definition 6.2</u> The transportation polytope $M(a,b)$ is called a k-*degenerate polytope* if $|A(a,b)| = k$. A 1-degenerate polytope $M(a,b)$ for which $\mu_{L,P}(a,b) = 0$, is called (L,P)-*degenerate* .

Let $M(a^0,b^0)$, $M(a^1,b^1)$ be transportation polytopes of the same order. Let

$$a^{\lambda} = \lambda a^1 + (1-\lambda)a^0 \ , \ b^{\lambda} = \lambda b^1 + (1-\lambda)b^0 \quad \text{where} \quad 0 \leqq \lambda \leqq 1 \ .$$

Definition 6.3 A pair of non-degenerate transportation polytopes $M(a^0,b^0)$, $M(a^1,b^1)$ of the same order is called an (L,P)-*regular pair* if there is a number $\lambda^* \varepsilon (0,1)$ such that the polytope $M(a^{\lambda^*},b^{\lambda^*})$ is (L,P)-degenerate while $M(a^{\lambda},b^{\lambda})$ is non-degenerate for all $\lambda \varepsilon (0,1)$, $\lambda \neq \lambda^*$. The polytope $M(a^{\lambda^*},b^{\lambda^*})$ is called the *centre of the* (L,P)-*regular pair* of polytopes $M(a^0,b^0)$ and $M(a^1,b^1)$.

Let $c = (c_1,c_2,\ldots,c_t)$ and let $T \subset N_t$ be a non-empty subset, then we define the vector $c[T]$ to be made up of those components of c whose indices belong to T.

Let there exist a pair $(L,P) \varepsilon A_{m\times n}$ for which $\mu_{L,P}(a,b) = 0.$ We define the number

$$\delta_{L,P}(a,b) = f_0(M(a[L],b[P]))f_0(M(a[\bar{L}],b[\bar{P}])) \ .$$

The following theorem lays the basis for treating the problem of enumerating the vertices of a transportation polytope.

Theorem 6.4 *Let* $M(a^0,b^0)$ *and* $M(a^1,b^1)$ *be an* (L,P)-*regular pair of transportation polytopes with centre* $M(a^{\lambda^*},b^{\lambda^*})$, *then putting*

$$\gamma_{L,P}(a^1,b^1) = (n|L| - m|P|) \ \text{sign} \ \mu_{L,P}(a^1,b^1) \ ,$$

we have

$$f_0(M(a^1,b^1)) = f_0(M(a^0,b^0)) + \delta_{L,P}(a^{\lambda^*},b^{\lambda^*}) \ \gamma_{L,P}(a^1,b^1). \quad (6.4)$$

Before proving the theorem we examine two auxiliary lemmas.

Lemma 6.5 *The pair of transportation polytopes* $M(a^0,b^0)$ *and* $M(a^1,b^1)$ *is an* (L,P)-*regular pair with centre* $M(a^{\lambda^*},b^{\lambda^*})$ *if and only if*

$$\mu_{L,P}(a^{\lambda^*},b^{\lambda^*}) = 0 \quad , \quad \mu_{L,P}(a^0,b^0)\mu_{L,P}(a^1,b^1) < 0 \quad ,$$

$$\mu_{I,J}(a^0,b^0)\mu_{I,J}(a^1,b^1) > 0 \quad \textit{for all} \quad (I,J) \neq (L,P) \ .$$

The proof uses the definition 6.3 and the linearity of the functions $\mu_{L,P}(a^{\lambda},b^{\lambda})$ with respect to λ.

Lemma 6.6 *Let* $M(a,b)$ *be a transportation polytope of order* $m\times n$. *Then* $M(a,b)$ *has a vertex whose smallest positive component is*

$$t = \min_{(I,J)} |\mu_{I,J}(a,b)|$$

where the minimum is taken over all pairs (I,J), $I \subseteq N_m$, $J \subset N_n$, *for which* $\mu_{I,J}(a,b) \neq 0$.

Proof By Lemma 1.3 we have

$$t \leqq \min_{x\in \operatorname{vert} M(a,b)} \ \min_{(i,j)\in K(a,b,x)} x_{ij} . \tag{6.5}$$

Let $t = |\mu_{L,P}(a,b)|$. Choose a pair (k,r) such that

$$(k,r) \in \begin{cases} L\times\overline{P} , & \text{if } \mu_{L,P}(a,b) = t , \\ \overline{L}\times P , & \text{if } {}_{L,P}(a,b) = -t . \end{cases}$$

Consider the polytope $M(a',b')$ defined by the vectors

$$a' = (a_1,a_2,\ldots,a_{k-1},a_k - t,a_{k+1},\ldots,a_m) ,$$

$$b' = (b_1,b_2,\ldots,b_{r-1},b_r - t,b_{r+1},\ldots,b_n) .$$

Since $\mu_{L,P}(a',b') = 0$ there is at least one vertex $x' \in M(a',b')$ with components $x'_{ij} = 0$, $(i,j) \in (L\times\overline{P}) \cup (\overline{L}\times P)$. Hence, there is a vertex $x \in M(a,b)$ with component $x_{kr} = t$, that is

$$t \geqq \min_{x\in \operatorname{vert} M(a,b)} \ \min_{(i,j)\in K(a,b,x)} x_{ij} . \tag{6.6}$$

The desired equality follows from the inequalities (6.5) and (6.6). //

Proof of Theorem 6.4. Let $V_{L,P}(a^{\lambda^*},b^{\lambda^*})$ be the set of those vertices of $M(a^{\lambda^*},b^{\lambda^*})$ for which $x_{ij} = 0$, $(i,j) \in (L\times\overline{P}) \cup (\overline{L}\times P)$. Since $M(a^{\lambda^*},b^{\lambda^*})$ is (L,P)-degenerate we have

$$|V_{L,P}(a^{\lambda^*},b^{\lambda^*})| = \delta_{L,P}(a^{\lambda^*},b^{\lambda^*}) , \tag{6.7}$$

$$|K(a^{\lambda^*},b^{\lambda^*},x)| = m+n-2 \qquad \forall x \in V_{L,P}(a^{\lambda^*},b^{\lambda^*}) . \tag{6.8}$$

By the linearity of the function $\mu_{L,P}(a^{\lambda},b^{\lambda})$ with respect to λ, given $\eta > 0$ there exists $\delta > 0$ such that

$$|\mu_{L,P}(a^{\lambda},b^{\lambda})| < \eta \qquad\qquad \forall\,\lambda \in \Lambda = \{\lambda : |\lambda-\lambda^*| < \delta\}. \qquad (6.9)$$

Let

$$0 < \eta < \min_{(I,J)} |\mu_{I,J}(a^{\lambda^*},b^{\lambda^*}) \quad , \qquad (6.10)$$

where the minimum is taken over all pairs (I,J) , $I \subseteq N_m$, $J \subset N_n$, for which $\mu_{I,J}(a^{\lambda^*},b^{\lambda^*}) \neq 0$. Choose $\lambda_0,\lambda_1 \in \Lambda$ such that $0 < \lambda_0 < \lambda^* < \lambda_1 < 1$.

Clearly, the pair of polytopes $M(a^{\lambda_0},b^{\lambda_0})$ and $M(a^{\lambda_1},b^{\lambda_1})$ is an (L,P)-regular pair with centre $M(a^{\lambda^*},b^{\lambda^*})$. Hence, by Lemma 6.5,

$$\operatorname{sign} \mu_{L,P}(a^{\lambda_0},b^{\lambda_0}) = -\operatorname{sign} \mu_{L,P}(a^{\lambda_1},b^{\lambda_1}) .$$

Suppose first that $\operatorname{sign} \mu_{L,P}(a^{\lambda_1},b^{\lambda_1}) = 1$. Consider some vertex $y \in V_{L,P}(a^{\lambda^*},b^{\lambda^*})$. By conditions (6.8)-(6.10) and lemma 6.6, for any pairs $(i,j) \in L\times\overline{P}$ and $(i,j) \in \overline{L}\times P$ there exist vertices $x_y^{(i,j)} \in M(a^{\lambda_1},b^{\lambda_1})$ and $\hat{x}_y^{(i,j)} \in M(a^{\lambda_0},b^{\lambda_0})$ respectively, such that

$$K(a^{\lambda^*},b^{\lambda^*},y) \cup \{(i,j)\} = \begin{cases} K(a^{\lambda_1},b^{\lambda_1},x_y^{(i,j)}) & \text{if } (i,j) \in L\times\overline{P} , \\ K(a^{\lambda_0},b^{\lambda_0},\hat{x}_y^{(i,j)}) & \text{if } (i,j) \in \overline{L}\times P . \end{cases}$$

Since $M(a^{\lambda_0},b^{\lambda_0})$ and $M(a^{\lambda_1},b^{\lambda_1})$ are an (L,P)-regular pair with centre $M(a^{\lambda^*},b^{\lambda^*})$, we have by lemma 6.5 that for every pair $(I,J) \neq (L,P)$

$$\operatorname{sign} \mu_{I,J}(a^{\lambda_0},b^{\lambda_0}) = \operatorname{sign} \mu_{I,J}(a^{\lambda_1},b^{\lambda_1}) .$$

Repeating the arguments used to prove sufficiency in theorem 6.1, it is not difficult to show that $f_0(M(a^{\lambda_1},b^{\lambda_1})) = f_0(M(a^{\lambda_0},b^{\lambda_0})) + W_1 - W_0$, where W_1 is the number of vertices $x_y^{(i,j)}$, $(i,j) \in L\times\overline{P}$, $y \in V_{L,P}(a^{\lambda^*},b^{\lambda^*})$ and W_0 is the number of vertices $\hat{x}_y^{(i,j)}$, $(i,j) \in \overline{L}\times P$, $y \in V_{L,P}(a^{\lambda^*},b^{\lambda^*})$. Using (6.7) this gives

$$f_0(M(a^{\lambda_1},b^{\lambda_1})) = f_0(M(a^{\lambda_0},b^{\lambda_0})) + (n|L| - m|P|)\delta_{L,P}(a^{\lambda^*},b^{\lambda^*}) \qquad (6.11)$$

The case $\operatorname{sign} \mu_{L,P}(a^{\lambda_1}, b^{\lambda_1}) = -1$ may be dealt with similarly. We obtain

$$f_0(M(a^{\lambda_1}, b^{\lambda_1})) = f_0(M(a^{\lambda_0}, b^{\lambda_0})) - (n|L| - m|P|)\delta_{L,P}(a^{\lambda*}, b^{\lambda*}) \tag{6.12}$$

Combining (6.11) and (6.12) we obtain

$$f_0(M(a^{\lambda_1}, b^{\lambda_1})) = f_0(M(a^{\lambda_0}, b^{\lambda_0})) + \delta_{L,P}(a^{\lambda*}, b^{\lambda*})\gamma_{L,P}(a^{\lambda_1}, b^{\lambda_1}).$$

To establish (6.4) it remains to show that

$$f_0(M(a^{\lambda_0}, b^{\lambda_0})) = f_0(M(a^0, b^0)),$$

$$f_0(M(a^{\lambda_1}, b^{\lambda_1})) = f_0(M(a^1, b^1)).$$

For any pair $(I,J) \varepsilon A_{m\times n}$ we have by lemma 6.5

$$\operatorname{sign} \mu_{I,J}(a^{\lambda_0}, b^{\lambda_0}) = \operatorname{sign} \mu_{I,J}(a^0, b^0),$$

$$\operatorname{sign} \mu_{I,J}(a^{\lambda_1}, b^{\lambda_1}) = \operatorname{sign} \mu_{I,J}(a^1, b^1).$$

Then from theorem 6.1

$$M(a^{\lambda_0}, b^{\lambda_0}) \sim M(a^0, b^0), \quad M(a^{\lambda_1}, b^{\lambda_1}) \sim M(a^1, b^1),$$

which concludes the proof of theorem 6.4. //

Corollary 6.7 *Let* $M(a^0, b^0)$, $M(a^1, b^1)$ *be an* (L,P)*-regular pair of transportation polytopes. Then* $f_0(M(a^0, b^0)) = f_0(M(a^1, b^1))$ *if and only if* $n|L| = m|P|$.

6.3 The Spectrum

The concept of the spectrum of two transportation polytopes is a very fruitful one. This concept will enable us to obtain a number of incisive results in the following sections about both the numerical and the structural characteristics of transportation polytopes.

Definition 6.4 The *spectrum* of two transportation polytopes $M(a^0, b^0)$ and $M(a^1, b^1)$ of the same order is the set

$S(a^0,b^0,a^1,b^1)$ of all numbers $\lambda \in (0,1)$ for which $M(a^\lambda,b^\lambda)$ is a degenerate polytope.

We can formulate the criterion for two transportation polytopes to be equivalent (theorem 1.9, Ch.3) in terms of their spectrum.

<u>Theorem 6.8</u> *Two non-degenerate transportation polytopes are equivalent if and only if their spectrum is empty.*

<u>Definition 6.5</u> The spectrum is called *finite* if the number of its elements is finite, otherwise it is called *infinite* . As before we will use the notation

$$A(a,b) = \{(I,J) \in A_{m\times n} : \mu_{I,J}(a,b) = 0\} .$$

<u>Proposition 6.9</u> *The spectrum* $S(a^0,b^0,a^1,b^1)$ *is infinite if and only if* $A(a^0,b^0) \cap A(a^1,b^1) \neq \emptyset$.

Proof Sufficiency follows directly from the linearity of $\mu_{I,J}(a^\lambda,b^\lambda)$ with respect to λ . Necessity is easily shown using a proof by contradiction and noting that the set $A_{m\times n}$ is finite. //

This proposition implies the following properties of the spectrum :

a) two transportation polytopes, one of which is non-degenerate, have a finite spectrum;

b) if the spectrum $S(a^0,b^0,a^1,b^1)$ is finite, then

$$|S(a^0,b^0,a^1,b^1)| \leqq 2(2^{m-1}-1)(2^{n-1}-1) ;$$

c) if the spectrum $S(a^0,b^0,a^1,b^1)$ is infinite, it is equal to the entire interval $(0,1)$.

<u>Definition 6.6</u> The finite non-empty spectrum $S(a^0,b^0,a^1,b^1)$ is called a *simple spectrum* if every polytope $M(a^\lambda,b^\lambda)$, $\lambda \in S(a^0,b^0,a^1,b^1)$, is (I_λ,J_λ)-degenerate. An empty spectrum is also regarded as simple.

Note that when the spectrum $S(a^0,b^0,a^1,b^1)$ is not simple, there is at least one number $\lambda \in S(a^0,b^0,a^1,b^1)$ such that $|A(a^\lambda,b^\lambda)| \geqq 2$.

We state a theorem which guarantees the existence of a simple spectrum.

Theorem 6.10 *Let* $M(a^0,b^0)$, $M(a^1,b^1)$ *be transportation polytopes of order* $m\times n$. *Let* $M(a^0,b^0)$ *be a non-degenerate polytope and let* ρ *be such that*

$$0 < \rho \leqq \min_{(I,J)\varepsilon \mathcal{A}_{m\times n}} \frac{|\mu_{I,J}(a^0,b^0)|}{m+n} .$$

Then, there is a polytope $M(a,b)\,\varepsilon\, Q^\rho(a^0,b^0)$ *such that the spectrum* $S(a,b,a^1,b^1)$ *is simple.*

Proof Let $\mathcal{B}_{m\times n}$ be the set of all subsets of $\mathcal{A}_{m\times n}$ which contain not less than two elements. For each $D\,\varepsilon\,\mathcal{B}_{m\times n}$ we define the set $\Omega(D)$ of all $(m+n)$-vectors $(a,b) = (a_1,a_2,\ldots,a_m,b_1,b_2,\ldots,b_n)$ with real positive components such that

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j ,$$

$$\sum_{i\varepsilon I} a_i = \sum_{j\varepsilon J} b_j \qquad \forall (I,J)\,\varepsilon\, D ,$$

$$\sum_{i\varepsilon I} a_i \neq \sum_{j\varepsilon J} b_j \qquad \forall (I,J)\,\varepsilon\, \mathcal{A}_{m\times n}\setminus D .$$

From Proposition 4.1, Ch.1 , we deduce that $\dim \Omega(D) \leqq m+n-3$, $D\,\varepsilon\,\mathcal{B}_{m\times n}$. Consequently, the dimension of any affine set $\Delta(D)$ generated by the vector (a^1,b^1) and the set $\Omega(D)$ does not exceed $m+n-2$. Thus, setting $N = \{M(a,b) : (a,b)\,\varepsilon \bigcup_{D\varepsilon\mathcal{B}_{m\times n}} \Delta(D)\}$ and recalling that $\mathcal{B}_{m\times n}$ is a finite set and that $\dim\{(a,b) : M(a,b)\,\varepsilon\, Q^\rho(a^0,b^0)\} = m+n-1$, we find that $N^* = Q^\rho(a^0,b^0)\setminus N \neq \emptyset$.

It follows by our construction that for every polytope $M(a,b)\,\varepsilon\, N^*$ the spectrum $S(a,b,a^1,b^1)$ is simple. Indeed, suppose that for some polytope $M(a',b')\,\varepsilon\, N^*$ this is not so. Then, there is at least one number $\lambda\,\varepsilon\, S(a',b',a^1,b^1)$ such that

$$|\mathcal{A}(\lambda a^1 + (1-\lambda)a', \lambda b^1 + (1-\lambda)b')| \geqq 2 ,$$

that is $M(a',b') \notin N^*$. This contradiction completes the proof. //

§7 POLYTOPES WITH THE MAXIMUM NUMBER OF VERTICES

7.1 First Criterion

Definition 7.1 The transportation polytope of order $m \times n$ determined by the vectors $a^* = \underbrace{(n,\ldots,n)}_{m}$ and $b^* = \underbrace{(m,\ldots,m)}_{n}$, is called *central*.

Klee & Witzgall (1968) conjectured that the central transportation polytope of order $m \times n$ had the maximum possible number of vertices when m and n were coprime. In 1972 Bolker (1972) verified this conjecture. A generalization of this result is given by the following criterion for a transportation polytope of order $m \times n$ to belong to the class of polytopes with the maximum possible number, $\phi(m,n)$, of vertices, (see Yemelichev & Kravtsov, 1976iii).

Theorem 7.1 *A transportation polytope* $M(a,b)$ *of order* $m \times n$ *has the maximum possible number of vertices,* $\phi(m,n)$, *if and only if it is non-degenerate and the spectrum* $S(a,b,a^*,b^*) = \emptyset$.

We prepare for this theorem with a number of lemmas.

Lemma 7.2 *Let* $M(a,b)$ *be any* k-*degenerate* $(k \geq 1)$ *polytope of order* $m \times n$, $mn > 4$. *Then, there exists an* ℓ-*degenerate* $(0 \leq \ell \leq k-1)$ *polytope* $M(a',b')$ *of the same order, such that*

$$f_0(M(a',b')) > f_0(M(a,b)) .$$

Proof Fix a pair $(L,P) \in A(a,b)$. For definiteness we will assume that

$$\max(|L||\overline{P}|, |\overline{L}||P|) = |L||\overline{P}| . \qquad (7.1)$$

Now choose an index pair $(s,t) \in L \times \overline{P}$. Define vectors $a' = (a'_1, a'_2, \ldots, a'_m)$ and $b' = (b'_1, b'_2, \ldots, b'_n)$ by

$$a'_i = \begin{cases} a_i , & \text{if } i \neq s, \\ a_i + \eta , & \text{if } i = s, \end{cases}$$

$$b'_j = \begin{cases} b_j , & \text{if } j \neq t, \\ b_j + \eta , & \text{if } j = t, \end{cases}$$

where $0 < \eta < \min_{x \in \mathrm{vert} M(a,b)} \min_{(i,j) \in K(a,b,x)} x_{ij}$

Consider any vertex $x \in M(a,b)$ with components $x_{ij} = 0$, $(i,j) \in (L \times \overline{P}) \cup (\overline{L} \times P)$. From the way η was chosen it is clear that to every pair $(i,j) \in L \times \overline{P}$ there corresponds a vertex $x^{(i,j)} \in M(a',b')$ such that $K(a',b',x^{(i,j)}) \supseteq K(a,b,x) \cup \{(i,j)\}$. Hence the vertex x can be associated with the vertices in the set $\bigcup_{(i,j) \in L \times \overline{P}} x^{(i,j)}$, whose cardinality is not less than two (since $mn > 4$ and by assumption (7.1)).

Further, for each vertex $x \in M(a,b)$ with component $x_{st} > 0$ there is a vertex $x' \in M(a',b')$ with components

$$x'_{ij} = \begin{cases} x_{ij} + \eta, & \text{if } (i,j) = (s,t), \\ x_{ij} & \text{otherwise.} \end{cases}$$

Finally, let x be a vertex of $M(a,b)$ such that $x_{st} = 0$ and such that there is at least one positive component in the set $\{x_{ij} : (i,j) \in L \times \overline{P}\}$. For each such vertex there is a vertex $x' \in M(a',b')$ with components

$$x'_{ij} = \begin{cases} x_{ij} - \eta, & \text{if } i = s_r,\ j = t_{r+1},\ r = 0,1,\dots,k-1, \\ x_{ij} + \eta, & \text{if } i = s_r,\ j = t_r,\ r = 0,1,\dots,k-1, \\ x_{ij} & \text{otherwise,} \end{cases}$$

in the case where the pair (s,t) and some pairs from the set $K(a,b,x)$ form a cycle : $(s,t),(s,t_1),(s_1,t_1),(s_1,t_2),\dots,(s_{k-1},t_r),(s_k,t)$ where $s_0 = s$, $t_k = t$. Otherwise, we define

$$x'_{ij} = \begin{cases} \eta, & \text{if } (i,j) = (s,t), \\ x_{ij} & \text{otherwise.} \end{cases}$$

Collecting all the results obtained, we have $f_0(M(a',b')) > f_0(M(a,b))$. //

We define

$$A(a^0,b^0,a^1,b^1) = \bigcup_{\lambda \in S(a^0,b^0,a^1,b^1)} A(a^\lambda,b^\lambda).$$

From the linearity of $\mu_{I,J}(a^\lambda,b^\lambda)$ with respect to λ the following lemma is obvious.

<u>Lemma 7.3</u> *Let* $M(a^0,b^0)$, $M(a^1,b^1)$ *be two transportation polytopes of order* $m \times n$ *and let at least one of these polytopes be non-*

-degenerate. Then the pair $(L,P) \in A_{m\times n}$ *is contained in* $A(a^0,b^0,a^1,b^1)$ *if and only if*

$$\mu_{L,P}(a^0,b^0)\mu_{L,P}(a^1,b^1) < 0 .$$

Lemma 7.4 *Let* $M(a^0,b^0)$ *be a non-degenerate transportation polytope of order* $m\times n$ *and let*

$$0 < \rho \leqq \min_{(I,J)\in A_{m\times n}} \frac{|\mu_{I,J}(a^0,b^0)|}{m+n} .$$

Then for any polytope $M(a^1,b^1)$ *of the same order we have*

$$A(a^0,b^0,a^1,b^1) = A(a,b,a^1,b^1) \qquad \forall M(a,b) \in Q^\rho(a^0,b^0)$$

Proof Suppose that there exist polytopes $M(a^1,b^1)$ and $M(a,b) \in Q^\rho(a^0,b^0)$ such that $A(a^0,b^0,a^1,b^1) \neq A(a,b,a^1,b^1)$. This means that there is a pair $(L,P) \in A_{m\times n}$ for which one of the following cases occurs:

a) $(L,P) \notin A(a^0,b^0,a^1,b^1)$, $(L,P) \in A(a,b,a^1,b^1)$;

b) $(L,P) \in A(a^0,b^0,a^1,b^1)$, $(L,P) \notin A(a,b,a^1,b^1)$.

Case a). By Lemma 7.3

$$\mu_{L,P}(a,b)\mu_{L,P}(a^1,b^1) < 0 . \tag{7.2}$$

By Corollary 6.3 , $M(a,b) \sim M(a^0,b^0)$. Hence, by Theorem 6.1 we have $\mu_{L,P}(a,b)\mu_{L,P}(a^0,b^0) > 0$. Together with (7.2) this implies that $\mu_{L,P}(a^0,b^0)\mu_{L,P}(a^1,b^1) < 0$, so that, by Lemma 7.3, $(L,P) \in A(a^0,b^0,a^1,b^1)$. This contradiction proves the Lemma for case a). Case b) can be proved similarly. //

Proof of Theorem 7.1. Necessity. Note first that Lemma 7.2 shows that the polytope $M(a,b)$ is non-degenerate.

Suppose that $S(a,b,a^*,b^*) \neq \emptyset$. Consider the case where the spectrum $S(a,b,a^*,b^*)$ is simple. List the elements of this spectrum in increasing order $\lambda_1 < \lambda_2 < \dots < \lambda_T$, $T \geqq 1$. Choose a number λ_0 such that $\lambda_1 < \lambda_0 < 1$ if $T = 1$ and $\lambda_1 < \lambda_0 < \lambda_2$ if $T > 1$. Then, if $a^\lambda = \lambda a^* + (1-\lambda)a$, $b^\lambda = \lambda b^* + (1-\lambda)b$, then the pair of polytopes $M(a,b)$ and and $M(a^{\lambda_0},b^{\lambda_0})$ are $(I_{\lambda_1},J_{\lambda_1})$-regular with centre $M(a^{\lambda_1},b^{\lambda_1})$.

Applying Theorem 6.4 to this pair and using the obvious inequality

$$\mu_{I_{\lambda_1}J_{\lambda_1}}(a^{\lambda_0},b^{\lambda_0})\mu_{I_{\lambda_1}J_{\lambda_1}}(a^*,b^*) > 0 ,$$

we obtain $f_0(M(a^{\lambda_0},b^{\lambda_0})) > f_0(M(a,b)) = \phi(m,n)$ which is impossible.

Now suppose that the spectrum $S(a,b,a^*,b^*)$ is not simple. Since $M(a,b)$ is non-degenerate, Corollary 6.3 and Theorem 6.10 show that there is a non-degenerate polytope $M(a',b')$ such that $S(a',b',a^*,b^*)$ is simple and $f_0(M(a',b')) = f_0(M(a,b))$.

Repeating the argumants of the previous case we again show the existence of a polytope whose number of vertices exceed $\phi(m,n)$. Thus necessity has been proved.

Sufficiency. Let $M(a',b')$ be a polytope with the maximum possible number of vertices. Then, by the necessity part of the theorem, the spectrum $S(a',b',a^*,b^*) = \emptyset$. From this and from the fact that the spectrum $S(a,b,a^*,b^*)$ is also empty we have

$$A(a,b,a',b') = A(a^*,b^*) . \tag{7.3}$$

If $A(a^*,b^*) = \emptyset$ then the spectrum $S(a,b,a',b') = \emptyset$ and by Theorem 6.8 we have $f_0(M(a,b)) = f_0(M(a',b')) = \phi(m,n)$.

Now suppose $A(a^*,b^*) \neq \emptyset$. Consider first the case in which $S(a,b,a',b')$ is a simple spectrum. Let the elements of this spectrum be $\lambda_1 < \lambda_2 < \ldots < \lambda_T$. Choose any π_t such that $\lambda_t < \pi_t < \lambda_{t+1}$, $t=0,1,\ldots,T$, where $\lambda_0 = 0$, $\lambda_{T+1} = 1$. Let $a^\lambda = \lambda a' + (1-\lambda)a$, $b^\lambda = \lambda b' + (1-\lambda)b$. It is clear that for each $t \in N_T$ the pair of polytopes $M(a^{\pi_{t-1}},b^{\pi_{t-1}})$ and $M(a^{\pi_t},b^{\pi_t})$ is $(I_{\lambda_t},J_{\lambda_t})$-regular with centre $M(a^{\lambda_t},b^{\lambda_t})$. From (7.3) we have

$$n|I_{\lambda_t}| = m|J_{\lambda_t}| \qquad \forall t \in N_T ,$$

and hence, applying Theorem 6.4 to any such pair of polytopes we have $f_0(M(a,b)) = f_0(M(a',b')) = \phi(m,n)$.

If the spectrum $S(a,b,a',b')$ is not simple then by Corollary 6.3 and Theorem 6.10 there exists a non-degenerate polytope $M(a'',b'') \in Q^\rho(a',b')$ such that the spectrum $S(a,b,a'',b'')$ is simple and $f_0(M(a'',b'')) = f_0(M(a',b'))$. Since $M(a'',b'') \in Q^\rho(a',b')$ we have by Lemma 7.4 that

$A(a,b,a'',b'') = A(a,b,a',b')$, whence, from (7.3) we have $A(a,b,a'',b'') = A(a^*,b^*)$. The argument now proceeds as in the previous case. //

Since the central transportation polytope of order $m \times n$ is non-degenerate if and only if $(m,n) = 1$, we obtain immediately from Theorems 6.8 and 7.1 the following Corollary.

Corollary 7.5 *All transportation polytopes of order* $m \times n$ *with the maximum possible number of vertices are equivalent if and only if* $(m,n) = 1$.

7.2 Second Criterion

Using the first criterion, Kononenko & Trukhanovsky (1978, 1979) obtained conditions for the number of vertices of a transportation polytope to be maximal which were easier to check than those of Theorem 7.1.

Theorem 7.6 *Let* $2 \leqq m \leqq n$, $n \geqq 3$, $m = tp$, $n = tq$ *where* t *is the highest common factor of* m *and* n ,

$$ph - qg = 1 \quad , \quad 0 \leqq g \leqq p-1 \; , \; 1 \leqq h \leqq q \; , \tag{7.4}$$

$$a_1 \geqq a_2 \geqq \dots \geqq a_m \quad , \quad b_1 \leqq b_2 \leqq \dots \leqq b_n \; . \tag{7.5}$$

Then the non-degenerate transportation polytope $M(a,b)$ *has the maximum number of vertices if and only if the inequalities*

$$\sum_{i=1}^{g+\beta p} a_i < \sum_{j=1}^{h+\beta q} b_j \quad , \quad \beta = 0,1,2,\dots,t-2,(t-1)\mathrm{sign}(q-1), \tag{7.6}$$

are satisfied. (*If* $g = 0$, $\beta = 0$, *we put* $\sum_{i=1}^{g+\beta p} a_i = 0$.)

We remark that the numbers g and h can be calculated by the following formulae (Khinchin 1964):

$$h/g = \begin{cases} q_\ell/p_\ell \, , & \text{if } \ell \text{ is odd, } \ell > 1, \\ (q-q_\ell)/(p-p_\ell) \, , & \text{if } \ell \text{ is even,} \end{cases}$$

$$h = 1 \ , \ g = 0 \qquad\qquad \text{if} \quad \ell = 1 \ ,$$

where

$$\frac{q}{p} = [q_1;q_2,\ldots,q_\ell] \ , \quad \frac{q_\ell}{p_\ell} = [q_1;q_2,\ldots,q_{\ell-1}] \ .$$

Here $[q_1;q_2,\ldots,q_\ell]$ is a continued fraction expansion.

We begin by proving the following lemma.

Lemma 7.7 *Let the numbers* m,n,p,q,t,h,g *satisfy the conditions of theorem* 7.6. *Then we have*

$$\left[\frac{\alpha g + \beta p}{m}\right] = \left[\frac{\alpha h + \beta q}{n}\right] , \tag{7.7}$$

$\forall \alpha \in N_p$, $\beta = 0,1,2,\ldots,t-2,(t-1)\operatorname{sign}(q-1)$.

Proof If $p = 1$ the lemma is easily checked directly. Hence, assume that $p \geqq 2$. In this case we show that (7.7) holds for any $\beta = 0,1,2,\ldots$.

Suppose first that $\alpha < p$. Put

$$\alpha g + \beta p = ms_1 + r_1 \ , \qquad 0 \leqq r_1 \leqq m-1 \ ,$$

$$\alpha h + \beta q = ns_2 + r_2 \ , \qquad 0 \leqq r_2 \leqq n-1,$$

where s_1 , s_2 are whole numbers. Suppose $s_1 < s_2$. then $m(\alpha h+\beta q) - n(\alpha g+\beta p) \geqq n$. Hence, using (7.4) we obtain $\alpha t \geqq n$. But since $p \leqq q$, we have $\alpha t < n$, $\alpha \in N_{p-1}$. This contradiction shows that $s_1 \geqq s_2$. If we suppose $s_1 > s_2$ we similarly obtain the contradiction $-\alpha t \geqq m$. Hence $s_1 = s_2$.

Now let $\alpha = p$. Put $p(g+\beta) = ms+r$, $0 \leqq r \leqq m-1$. Then $g+\beta = sd+r^*$, where $r^* = r/p$ is a whole number, $0 \leqq r^* \leqq d-1$. Thus, using (7.4) we have $ph + \beta q = q(g+\beta) + 1 = sn + qr^* + 1$. This shows that $s = [(ph+\beta q)/n]$, since $qr^* + 1 \leqq q(d-1) + 1 < n$. //

Proof of Theorem 7.6 Necessity. By Theorem 7.1 we have

$$(n|I| - m|J|)\mu_{I,J}(a,b) > 0 \ , \quad n|I| \neq m|J| \ , \tag{7.8}$$

which is equivalent to

$$\mu_{I,J}(a,b) < 0 \ , \qquad |I| \leqq m-1 \ , \quad |J| = [n|I|/m] + 1 \ . \tag{7.9}$$

Since $g + \beta p \leqq m - 1$ for $\beta \leqq t - 1$ and since by (7.4) $[n|I|/m] + 1 = h + \beta q$ for $|I| = g + \beta p$, these inequalities imply that

$$\mu_{I,J}(a,b) < 0 \ ; \quad |I| = g + \beta p \ , \ |J| = h + \beta q \ ,$$

$$\beta = 0,1,2,\ldots,t-2,(t-1)\operatorname{sign}(q-1) \ ,$$

which is (7.6).

Sufficiency. From inequality (7.6) and conditions (7.5) we have

$$\begin{aligned} &\mu_{I,J}(a,b) < 0 \ , \quad |I| = g + \beta p \ , \quad |J| = h + \beta q \ , \\ &\beta = 0,1,2,\ldots,t-2,(t-1)\operatorname{sign}(q-1). \end{aligned} \tag{7.10}$$

Thus, if $\beta = 0$ we have $\mu_{I,J}(a,b) < 0$, $|I| = g$, $|J| = h$. Hence, by Lemma 7.7 we have the inequalities

$$\mu_{I,J}(a,b) < 0 \ , \ |I| = r(\alpha g/m) \ , \ |J| = r(\alpha h/n) \ , \ \alpha \in N_{p-1} \ , \tag{7.11}$$

where $r(v/w)$ is the remainder on dividing v by w .

We now establish the following equalities

$$\left[\frac{g + \beta p + r(\alpha g/m)}{m}\right] = \left[\frac{h + \beta q + r(\alpha h/n)}{n}\right] , \tag{7.12}$$

$$\alpha = 0,1,2,\ldots,p-1, \quad \beta = 0,1,2,\ldots,t-2,(t-1)\operatorname{sign}(q-1).$$

Note first that equations (7.12) follow from (7.7) in the case $\alpha = 0$.

For every $\alpha \in N_p$ and $\beta = 0,1,\ldots,t-2,(t-1)\operatorname{sign}(q-1)$ we have by (7.7)

$$\alpha g + \beta p = ms(\alpha,\beta) + r(\alpha g + \beta p/m) \ ,$$

$$\alpha h + \beta q = ns(\alpha,\beta) + r(\alpha h + \beta q/n) \ .$$

Thus, for any $\alpha \in N_{p-1}$ we have

$$\left[\frac{g + \beta p + r(\alpha g/m)}{m}\right] = \left[\frac{(\alpha+1)g + \beta p - ms(\alpha,0)}{m}\right]$$

$$= \left[\frac{(s(\alpha+1,\beta) - s(\alpha,0))m + r((\alpha+1)g + \beta p/m)}{m}\right]$$

$$= s(\alpha+1,\beta) - s(\alpha,0) .$$

and we may obtain similarly

$$\left[\frac{h + \beta q + r(\beta h/n)}{n}\right] = s(\alpha+1,\beta) - s(\alpha,0) .$$

Thus equations (7.12) have been proved. These equalities, together with inequalities (7.10) and (7.11) give

$$\mu_{I,J}(a,b) < 0 . \quad |I| = r(\alpha g+\beta p/m) , \quad |J| = r(\alpha h+\beta q/n) ,$$
$$\alpha \in N_p , \quad \beta = 0,1,2,\ldots,t-2,(t-1)\operatorname{sign}(q-1) .$$

It should also be noted that for any $\alpha_1,\alpha_2 \in N_p$ and $\beta_1,\beta_2 \in \{0,1,2,\ldots,t-2,(t-1)\operatorname{sign}(q-1)\}$ we have $r(\alpha_1 g+\beta_1 p/m) \neq r(\alpha_2 g+\beta_2 p/m)$, if $\alpha_1 \neq \alpha_2$ or $\beta_1 \neq \beta_2$. Hence from the equality

$$\left[\frac{nr(\alpha g + \beta p/m)}{m}\right] + 1 = r(\alpha h+\beta q/n) ,$$

we obtain (7.9) or equivalently (7.8). Noting that $M(a,b)$ is non-degenerate we conclude from Theorem 1.1 that the polytope contains the maximum number of vertices. //

Corollary 7.8 *Let* m *and* n *be coprime integers. Then the transportation polytope* $M(a,b)$ *of order* $m\times n$ *has the maximum number of vertices if and only if*

$$\sum_{i=1}^{g} a_i < \sum_{j=1}^{h} b_j . \tag{7.13}$$

To prove this it suffices to verify that if m and n are coprime then (7.13) implies that the polytope $M(a,b)$ is non-degenerate, since this polytope is equivalent to the central polytope $M(a^*,b^*)$.

7.3 Necessary Conditions

Later, in §10 we will need the following simple indication of the maximality of the number of vertices of a transportation polytope.

Theorem 7.9 *Let* $2 \leq m \leq n$, $a_1 \leq a_2 \leq \cdots \leq a_m$, $b_1 \leq b_2 \leq \cdots \leq b_n$. *Then if* $M(a,b)$ *is any transportation polytope of order* $m \times n$ *with the maximum number of vertices, the following inequalities hold:*

1) *if* $m = n$

$$a_m < b_1 + b_2 \quad , \quad b_n < a_1 + a_2 \ ;$$

2) *if* $n = mq$, $q > 1$

$$a_m < \sum_{j=1}^{q+1} b_j \quad , \quad \sum_{j=(m-1)q+2}^{n} b_j < a_1 \ ;$$

3) *if* $n = mq + r$, $q \geq 1$, $1 \leq r \leq m-1$

$$a_m < \sum_{j=1}^{q+1} b_j \quad , \quad \sum_{j=n-q+2}^{n} b_j < a_1 \ .$$

There are examples which show that in general these conditions are not sufficient.

Proof Each case has a similar proof. We will therefore consider only case 1). Suppose that there is a polytope $M(a,b)$ of order $m \times n$ with maximal number of vertices for which either $a_m > b_1 + b_2$ or $b_n > a_1 + a_2$.

Clearly, if $a_m > b_1 + b_2$ the spectrum $S(a,b,a^*,b^*)$ contains the number $(a_m - b_1 - b_2)/(a_m - b_1 - b_2 + m)$, while if $b_n > a_1 + a_2$ it contains the number $(b_n - a_1 - a_2)/(b_n - a_1 - a_2 + m)$. Hence the spectrum $S(a,b,a^*,b^*)$ is not empty. Thus, by Theorem 7.1 $f_0(M(a,b)) < \phi(m,n)$. But this contradicts the assumption that $f_0(M(a,b)) = \phi(m,n)$. //

The following theorem establishes a connection between polytopes with the maximum number of facets and polytopes with the maximum number of vertices.

Theorem 7.10 (Klee & Witzgall 1968) *Every transportation polytope of order* $m \times n$, $mn > 4$ *with the maximum number of vertices has the maximum number of facets.*

Proof Suppose that there is a polytope $M(a,b)$ of order $m \times n$ with the maximum number of vertices such that $f_{d-1}(M(a,b)) < mn$. Then, by Theorem 3.2' , we must have

$$\alpha = \max_{1 \leq i \leq m} a_i + \max_{1 \leq j \leq n} b_j - \sum_{j=1}^{n} b_j > 0 \ .$$

From this and from the obvious inequality $mn > m+n$ (for $mn > 4$), we obtain

$$\frac{\alpha}{\alpha+mn-m-n} \in S(a,b,a^*,b^*) .$$

Hence, by Theorem 7.1 $f_0(M(a,b)) < \phi(m,n)$ which is impossible. //

§8 CALCULATION OF $\phi(m,n)$

8.1 Enumeration Theorems

We present here two approaches to the calculation of the maximum number $\phi(m,n)$ of vertices in the class of transportation polytopes of order $m\times n$. This will enable us to reduce the calculation of $\phi(m,n)$ to the calculation of the number of vertices of several polytopes of lower order.

First Method. The basis of this method is Theorem 6.4 on the increment in the function $f_0(M(a^\lambda,b^\lambda))$ as the parameter λ passes through an element of the spectrum.

Let the spectrum $S(a,b,a^*,b^*)$ be simple. Then, for every $\lambda \in S(a,b,a^*,b^*)$ the polytope $M(a^\lambda,b^\lambda)$ determined by the vectors $a^\lambda = \lambda a + (1-\lambda)a^*$ and $b^\lambda = \lambda b + (1-\lambda)b^*$ is (I_λ,J_λ)-degenerate. Using the notation of §6 for $\lambda \in S(a,b,a^*,b^*)$ we define the size of the increment $\delta_\lambda = \delta_{I_\lambda,J_\lambda}(a^\lambda,b^\lambda)|n|I_\lambda| - m|J_\lambda||$, which by Proposition 6.9 is always positive.

Theorem 8.1 (Yemelichev & Kravtsov 1978) *Let* $M(a,b)$ *be a non-degenerate transportation polytope of order* $m\times n$ *whose spectrum* $S(a,b,a^*,b^*)$ *is simple. Then the maximum number of vertices in the class of transportation polytopes of order* $m\times n$ *is given by the formula*

$$\phi(m,n) = f_0(M(a,b)) + \sum_{\lambda\in S(a,b,a^*,b^*)} \delta_\lambda .$$

Proof If $S(a,b,a^*,b^*) = \emptyset$ then the equality $\phi(m,n) = f_0(M(a,b))$ follows from Theorem 7.1.

Now suppose $S(a,b,a^*,b^*) \neq \emptyset$. Let the elements of this spectrum be ordered: $\lambda_1 < \lambda_2 < \dots < \lambda_T$, $T \geqq 1$. Choose numbers π_t such that $\lambda_t < \pi_t < \lambda_{t+1}$. $t=0,1,2,\ldots,T$, where $\lambda_0 = 0$, $\lambda_{T+1} = 1$. Since the spectrum is simple it follows that for each $t \varepsilon N_T$ the pair of polytopes $M(a^{\pi_{t-1}}, b^{\pi_{t-1}})$ and $M(a^{\pi_t}, b^{\pi_t})$ is $(I_{\lambda_t}, J_{\lambda_t})$-regular with centre $M(a^{\lambda_t}, b^{\lambda_t})$. Applying Theorem 6.4 to each of these pairs of polytopes and recalling that

$$\operatorname{sign} \mu_{I_{\lambda_t}, J_{\lambda_t}}(a^{\pi_t}, b^{\pi_t}) = \operatorname{sign} \mu_{I_{\lambda_t}, J_{\lambda_t}}(a^*, b^*) \quad , \ t \varepsilon N_T \ ,$$

we obtain

$$f_0(M(a^{\pi_t}, b^{\pi_t})) \quad = \quad f_0(M(a^{\pi_{t-1}}, b^{\pi_{t-1}})) + \delta_{\lambda_t} \quad , \quad t \varepsilon N_T \ .$$

Whence

$$f_0(M(a^{\pi_T}, b^{\pi_T})) \quad = \quad f_0(M(a,b)) + \sum_{t=1}^{T} \delta_{\lambda_t} \ .$$

Since the polytope $M(a^{\pi_T}, b^{\pi_T})$ satisfies the condition of Theorem 7.1 it must have the maximum number $\phi(m,n)$ of vertices. //

Second Method. This is based on the enumeration of special vertices of polytopes of lower order. These special vertices are those $x = (x_{ij})$ each of whose columns contains at least two positive components.

We suppose that $2 \leq m \leq n$. Let $n = mq + r$ where r is the remainder on dividing n by m . We introduce the notation

$$K(t) = \{k \varepsilon Z_m^+ : \sum_{i=1}^{m} k_i = t\} \ , \quad t = 0,1,2,\ldots,m-r-1 \quad \text{for} \quad r > 0;$$

$$K(m-1) = \{(\underbrace{1,1,\ldots,1}_{m-1},0)\},$$

$$K(t) = \emptyset \quad , \qquad\qquad t = 0,1,2,\ldots,m-2 \quad \text{for} \quad r = 0.$$

With each vector $k = (k_1, k_2, \ldots, k_m) \varepsilon \bigcup_{t=0}^{m-r-1} K(t)$ we associate a polytope $M(a^k, b^k)$ of order $m \times (r + \sum_{i=1}^{m} k_i)$ associated with the vectors

$$a^k \quad = \quad (m^2 k_1 + rm - 1, \ldots, m^2 k_{m-1} + rm - 1),$$

$$b^k \quad = \quad (m^2, m^2, \ldots, m^2) \ .$$

A vertex $(x^0_{ij})_{m\times n}$ of $M(a,b)$ is called *special* if $x^0_{ij} < b_j$ for all $i \in N_m$ and $j \in N_n$. Let $\gamma(a,b)$ denote the number of special vertices of the polytope $M(a,b)$.

The following theorem provides an apparatus for calculating the number $\phi(m,n)$.

Theorem 8.2 (Yemelichev, Kravtsov & Averbukh 1976) *The maximun number of vertices in the class of transportation polytopes of order* $m\times n$ *is given by*

$$\phi(m,n) = \frac{n!}{(q!)^m} \sum_{t=0}^{m-r-1} \sum_{k\in K(t)} \frac{\gamma(a^k,b^k)}{(r+t)!} \prod_{i=1}^{m} \prod_{p=0}^{k_i-1} (q-p). \tag{8.1}$$

(*If* $k_i = 0$ *we consider that* $\prod_{p=0}^{k_i-1}(q-p) = 1$.)

Proof It follows easily from Theorem 7.1 that for any natural numbers m and n the polytope $M(\bar{a},\bar{b})$ of order $m\times n$, defined by the vectors $\bar{a} = (mn-1,\dots,mn-1,mn+m-1)$, $\bar{b} = (m^2,m^2,\dots,m^2)$ has $\phi(m,n)$ vertices.

Let $0 \leq t \leq m-r-1$, $k = (k_1,k_2,\dots,k_m) \in K(t)$. The number of vertices of the polytope $M(\bar{a},\bar{b})$ for which the condition $|\{j\in N_n : x_{ij} = m^2\}| = q - k_i$, $i \in N_m$, holds is equal to the number $\gamma(a^k,b^k)$ multiplied by the number of ways in which n elements can be distributed among $(m+1)$ sets of which the i^{th}-set $(i \in N_m)$ contains $q - k_i$ elements and the $(m+1)^{th}$-set contains $r+t$ elements, namely

$$\gamma(a^k,b^k)\,\frac{n!}{\prod_{i=1}^{m}(q-k_i)!\,(r+t)!} \tag{8.2}$$

Summing (8.2) over all $k \in K(t)$ and $t = 0,1,2,\dots,m-r-1$ we obtain the following formula:

$$\phi(m,n) = n! \sum_{t=0}^{m-r-1} \sum_{k\in K(t)} \gamma(a^k,b^k)\left(\frac{1}{(r+t)!\prod_{i=1}^{m}(q-k_i)!}\right) \tag{8.3}$$

Now note that $\prod_{i=1}^{m}(q-k_i)! \prod_{p=0}^{k_i-1}(q-p) = (q!)^m$. This, together with (8.3) yields the desired equation. //

8.2 Bolker's Conjecture

We prove here a theorem which was proposed by Bolker(1972) in the form of a conjecture.

Theorem 8.3 *The maximum number of vertices* $\phi(m,n)$ *in the class of transportation polytopes of order* $m\times n$, $2\leq m\leq n$, *is given by the formula*

$$\phi(m,n) = \frac{n!}{(q!)^m} P(q,m,r) ,$$

where $n = mq+r$ *and* r *is the remainder on dividing* n *by* m , *and* $P(q,m,r)$ *is a polynomial in* q *with leading term* $m^{m-2}q^{m-r-1}$.

Proof The formula (8.1) is easily transformed to the form

$$\phi(m,n) = \frac{n!}{(q!)^m}\left(q^{m-r-1} \sum_{k\varepsilon K(m-r-1)} \frac{\gamma(a^k,b^k)}{(m-1)!} + R(q,m,r)\right), \quad (8.4)$$

where $R(q,m,r)$ is a polynomial in q of degree not greater than $m-r-2$.

Let $H_{m,m-1}$ be the set of spanning trees of the graph $K_{m,m-1}$ for which the condition $\deg j = 2$, $j \varepsilon V$, is satisfied. Let $\Gamma(a^k,b^k)$ denote the set of special vertices of $M(a^k,b^k)$. Since $|H_{m,m-1}| = |D_{m-1,m}|$ then, by (8.4) and Theorem 2.4, Bolker's conjecture will be verified if we establish a bijection between the sets $H_{m,m-1}$ and $\Gamma = \bigcup_{k\varepsilon K(m-r-1)} \Gamma(a^k,b^k)$ and also prove the relation $\Gamma(a^{k^1},b^{k^1})\cap\Gamma(a^{k^2},b^{k^2}) = \emptyset$ for any two distinct vectors k^1 and k^2 in $K(m-r-1)$.

It is clear that every special vertex $x = (x_{ij})_{m\times(m-1)}$ of any non-degenerate transportation polytope of order $m\times(m-1)$ is constructed so that $|\{i\varepsilon N_m : x_{ij}>0\}| = 2$, $j \varepsilon N_{m-1}$. Thus every such vertex corresponds with a graph $G_x(U,V)\,\varepsilon\, H_{m,m-1}$.

We show that every graph $G(U,V)\,\varepsilon\, H_{m,m-1}$ can be associated with a vertex x in Γ . Let $G(U^j,V^j)$ and $G(\overline{U}^j,\overline{V}^j)$ be those trees obtained from $G(U,V)$ by deleting the vertex $j \varepsilon V$ together with the edges incident at j . Assume that the vertex numbered 1 belongs to the set U^j . In order to construct the vertex we are seeking, consider the following system of linear equations determined by the tree $G(U,V)$,

$$\sum_{s=1}^{m} k_s = m - r - 1 \,,$$
$$\sum_{s \in U^j} k_s = \left[\frac{(m-r+1/m)|U^j| - 1}{m}\right] \qquad \forall j \in N_{m-1}. \tag{8.5}$$

Using induction on m we can show that the determinant of this system is ± 1. Hence (8.5) has a unique integral solution $k^0 = (k_1^0, k_2^0, \ldots, k_m^0)$. From the fact that there is a unique basis of the transportation polytope of order $m \times (m-1)$ corresponding to each tree in $H_{m,m-1}$ (Theorem 2.2), we can show, by contradiction, that the vector k^0 is non-negative. Thus $k^0 \in K(m-r-1)$. Then, by Lemma 1.3, the matrix $x = (x_{ij})_{m \times (m-1)}$ with components

$$x_{ij} = \begin{cases} 0, & \text{if } (i,j) \notin G(U,V), \\ \sum_{s \in \overline{U}^j} k_s m + (r - 1/m)|\overline{U}^j| - m|\overline{V}^j| \,, & \text{if } (i,j) \in G(U,V)\,,\ i \in \overline{U}^j \,, \\ \sum_{s \in U^j} k_s m + 1 + (r - 1/m)|U^j| - m|V^j| \,, & \text{if } (i,j) \in G(U,V)\,,\ i \in U^j \,. \end{cases}$$

is a vertex of the polytope $M(a^{k^0}, b^{k^0})$. Clearly x, the vector so constructed, lies in Γ and $G(U,V) = G_x(U,V)$.

We now show that if $x_1 \neq x_2$ then $G_{x_1}(U,V) \neq G_{x_2}(U,V)$. Let

Let $x_1 \in \Gamma(a^{k^1}, v^{k^1})$, $x_2 \in \Gamma(a^{k^2}, b^{k^2})$, $k^1, k^2 \in K(m-r-1)$. Suppose that $G_{x_1}(U,V) = G_{x_2}(U,V)$. Then, by Lemma 1.3, it is easily seen that the vectors k^1, k^2 satisfy system (8.5) which has a unique solution. Hence $k^1 = k^2$ which implies $x_1 = x_2$. This contradiction completes the proof. //

8.3 <u>Explicit Formulae</u>

Explicit formulae for calculating $\phi(m,n)$ have not yet been found in the general case. Such formulae are known only for special cases where $n = mq$, $mq \pm 1$, $mq-2$. We consider the derivation of these formulae.

<u>Proposition 8.4</u>

$$\phi(m, mq+1) = \frac{(mq+1)!}{(q!)^m} (mq+1)^{m-2} \,.$$

Proof By Theorems 2.4 and 7.1 it suffices to establish a bijection between the set of vertices of the central polytope $M(a^*,b^*)$ of order $m\times(mq+1)$ and the set of spanning trees $D_{m,mq+1}$ (see §2.2).

Let x be a vertex of $M(a^*,b^*)$. Since every component x_{ij} of this vertex does not exceed m, the number of positive components in each row of x is not less than $q+1$. Hence, since the number of positive components of x is $m(q+1)$ we see that the tree $G_x(U,V)\ \varepsilon\ D_{m,mq+1}$.

If x', x'' are two distinct vertices of $M(a^*,b^*)$ then $G_{x'}(U,V) \neq G_{x''}(U,V)$.

Let $G(U,V)\ \varepsilon\ D_{m,mq+1}$. We show that there is a vertex of $M(a^*,b^*)$ such that

$$G_x(U,V) = G(U,V)\,. \tag{8.6}$$

The deletion of any edge (i,j) splits the graph $G(U,V)$ into two trees. The graph containing the vertex labelled i will be denoted by $G(U_{ij},V_{ij})$. By Lemma 1.3 the matrix $x=(x_{ij})_{m\times(mq+1)}$ with components

$$x_{ij} = \begin{cases} 0, & \text{if } (i,j) \not\varepsilon\, G(U,V), \\ (mq+1)|U_{ij}| - m|V_{ij}|, & \text{if } (i,j)\ \varepsilon\ G(U,V), \end{cases}$$

is a vertex of $M(a^*,b^*)$. For this vertex the condition (8.6) is clearly satisfied. //

The following is a direct corollary of Theorem 8.3.

<u>Corollary 8.5</u>

$$\phi(m,mq-1) = \frac{(mq-1)!}{(q!)^m}\, m^{m-2}\,.$$

The formulae for the number of vertices of the central polytope of order $m\times n$ when $n = mq+1$ and $n = mq-1$ were first derived by Klee & Witzgall (1968). In 1972 Bolker (1972) showed that this polytope had the maximum number of vertices in these cases.

<u>Corollary 8.6</u>

$$\phi(m,mq) = \frac{(mq)!}{(q!)^m}\, m^{m-2} q^{m-1}\,.$$

Proof It follows from formula (8.1) that

$$\phi(m,mq) = \frac{(mq)!}{(q!)^m(m-1)!}\,\gamma(a',b')q^{m-1},$$

where $a' = (m - 1/m, m - 1/m, \ldots, m - 1/m, 1 - 1/m) \in E_m$ and $b' = (m,\ldots,m) \in E_{m-1}$. From Theorem 8.3 this gives $\gamma(a',b') = (m-1)!\,m^{m-2}$ which establishes the proposition. //

<u>Proposition 8.7</u> (Yemelichev, Kravtsov & Krachkovsky 1977ii)

$$\phi(m,mq-2) = \frac{(mq-2)!}{(q!)^m}\left(m^{m-2}q + \frac{\phi(m,m-2)}{(m-2)!}\right).$$

Proof Using (8.1) we have from Theorem 8.3 that

$$\phi(m,mq-2) = \frac{(mq-2)!}{(q!)^m}\left(m^{m-2}q + \frac{\gamma(a',b')}{(m-2)!}\right),$$

where $a' = (m - 1 - 1/m, m - 2 - 1/m, \ldots, m - 2 - 1/m) \in E_m$ and $b' = (m,m,\ldots,m) \in E_{m-2}$. Since every vertex of $M(a',b')$ is special we have, by Theorem 7.1, the equality $\gamma(a',b') = \phi(m,m-2)$. //

According to the previous proposition the calculation of $\phi(m,mq-2)$ reduces to the calculation of $\phi(m,m-2)$. The formula for $\phi(m,mq-2)$ takes the form (Krachkovsky 1978)

$$\phi(m,m-2) = \frac{(m-1)!\,(m-2)!}{2}\sum_{(s,t)}\frac{s^{s-1}t^{t-1}(m-s-t)^{m-s-t-1}}{s!\,t!\,(m-s-t-1)!},$$

where the sum is taken over all numbers s and t satisfying the inequalities $1 \leq s \leq [m^2/(2m+1)]$, $1 \leq t \leq [m^2/(2m+1)]$, $s+t > [m^2/(2m+1)]$. The derivation of this formula is left to the reader.

§9 <u>MINIMUM NUMBER OF VERTICES IN THE CLASS OF NON-DEGENERATE POLYTOPES WITH A GIVEN NUMBER OF FACETS</u>

Throughout this section we assume that the components of the vectors a and b are ordered as follows : $a_1 \geq a_2 \geq \ldots \geq a_m$, $b_1 \geq b_2 \geq \ldots \geq b_n$.

<u>Theorem 9.1</u> (Yemelichev, Kravtsov & Krachkovsky 1977iii and 1978i) *Let* $2 \leq m \leq n$, $n \geq 3$, $1 \leq k \leq n$. *The minimum number of vertices in the class of non-degenerate transportation polytopes of order* $m\times n$ *with* $(m-1)n + k$ *facets is equal to* $n^{m-1} + k(mn - m - n)$.

Proof Let $M(a^0,b^0)$ be a non-degenerate polytope of order $m\times n$ satisfying the conditions

$$b^0_{n-k+1} - b^0_n < \sum_{j=1}^{n-1} b^0_j - a^0_1 < \min(a^0_m , b^0_{n-k} - b^0_n) . \qquad (9.1)$$

By Theorem 3.2' this polytope has $(m-1)n+k$ faces. We show that it has $n^{m-1}+k(mn-m-n)$ vertices.

Conditions (9.1) imply that the elements in the first row of any matrix $x=(x_{ij})_{m\times n} \in M(a^0,b^0)$ are constructed in the following way:

a) $x_{1j} > 0$, $\forall j \in N_{n-k}$;

b) among the elements x_{1j} , $j=n-k+1,\dots,n$, only one can equal zero.

Hence $\operatorname{vert} M(a^0,b^0) = \bigcup_{s=1}^{k+1} V_s(a^0,b^0)$, where $V_s(a^0,b^0)$, $1 \leqq s \leqq k$, is the set of those vertices of $M(a^0,b^0)$ for which $x_{1,n-s+1}=0$ and $V_{k+1}(a^0,b^0)$ is the set of vertices of $M(a^0,b^0)$ for which $x_{1j}>0$, $\forall j \in N_n$. It may be verified directly that every set $V_s(a^0,b^0)$, $s \in N_k$, contains $(m-1)(n-1)$ elements while $V_{k+1}(a^0,b^0)$ contains $(n^{m-1}-k)$ elements. Hence $f_0(M(a^0,b^0)) = n^{m-1}+k(mn-m-n)$.

We show next that the number of vertices of any non-degenerate transportation polytope $M(a,b)$ of order $m\times n$, $2 \leqq m \leqq n$, $n \geqq 3$ with $(m-1)n+k$ facets satisfies the inequality

$$f_0(M(a,b)) \geqq n^{m-1}+k(mn-m-n) . \qquad (9.2)$$

We prove this inequality by induction on the number $p=m+n+k$. For $p \leqq 8$ the inequality (9.2) can be verified directly.

1st Case. Suppose that for $m+k<n$ the condition

$$0 < \sum_{i=2}^{m} a_i - b_{n-k+1} < \min(a_m,b_n) , \qquad (9.3)$$

is satisfied, while for $m+k \geqq n$ either (9.3) or

$$0 < \sum_{j=2}^{n} b_j - a_{n-k+1} < \min(a_m,b_n) \qquad (9.4)$$

is satisfied.

Let $\bar{a}=(\bar{a}_1,\dots,\bar{a}_m)$, $\bar{b}=(\bar{b}_1,\dots,\bar{b}_n)$ where

$\bar{a}_i = a_i/(\sum_{s=1}^{m} a_s)$, $\bar{b}_j = b_j/(\sum_{s=1}^{m} a_s)$. Also let $a^\lambda = \lambda\bar{a} + (1-\lambda)a^*$, $b^\lambda = \lambda\bar{b} + (1-\lambda)b^*$ where $0 \leq \lambda < \infty$. Then since one of the conditions (9.3),(9.4) is satisfied there are numbers $\lambda' > 1$ and $1 \leq \ell \leq k$, such that $M(a^{\lambda'}, b^{\lambda'}) \in \mathcal{M}(m,n,k-\ell)$.

There are two possibilities.

1) The spectrum $S(a^{\lambda'}, b^{\lambda'}, \bar{a}, \bar{b})$ is simple. Then, there is a number $1 < \pi \leq \lambda'$ such that the polytope $M(a^\pi, b^\pi) \in \mathcal{M}(m,n,k-1)$. Order the elements of the spectrum $S(a^\pi, b^\pi, \bar{a}, \bar{b})$ as follows : $\lambda_1 < \lambda_2 < \ldots < \lambda_T$, $T \geq 1$. Choose any numbers π_t such that $\lambda_t < \pi_t < \lambda_{t+1}$, $t = 0,1,2,\ldots,T$, where $\lambda_0 = \pi$, $a^{\lambda_{T+1}} = \bar{a}$, $b^{\lambda_{T+1}} = \bar{b}$. For every $t \in N_T$ it is clear that the polytopes $M(a^{\pi_{t-1}}, b^{\pi_{t-1}})$ and $M(a^{\pi_t}, b^{\pi_t})$ are a $(I_{\lambda_t}, J_{\lambda_t})$-regular pair with centre $M(a^{\lambda_t}, b^{\lambda_t})$. Applying Theorem 6.4 to any such pair of polytopes and noting the obvious inequalities

$$\mu_{I_{\lambda_t}, J_{\lambda_t}}(a^{\pi_t}, b^{\pi_t})\, \mu_{I_{\lambda_t}, J_{\lambda_t}}(a^*, b^*) > 0 \qquad \forall t \in N_T$$

we obtain

$$f_0(M(a^{\pi_t}, b^{\pi_t})) = f_0(M(a^{\pi_{t-1}}, b^{\pi_{t-1}})) + \delta_{I_{\lambda_t}, J_{\lambda_t}}(a^{\lambda_t}, b^{\lambda_t})\,|n|I_{\lambda_t}| - m|J_{\lambda_t}||, t \in N_T.$$

Observing the inequality

$$\sum_{t=1}^{T} \delta_{I_{\lambda_t}, J_{\lambda_t}}(a^{\lambda_t}, b^{\lambda_t})\,|n|I_{\lambda_t}| - m|J_{\lambda_t}|| \geq mn - m - n ,$$

we have

$$f_0(M(\bar{a}, \bar{b})) \geq f_0(M(a^\pi, b^\pi)) + mn - m - n .$$

Then, since $f_0(M(a,b)) = f_0(M(\bar{a}, \bar{b},))$, we have, using the induction hypothesis

$$f_0(M(a^\pi, b^\pi)) \geq n^{m-1} + (k-1)(mn - m - n) .$$

Thus inequality (9.2) is proved.

2) The spectrum $S(a^{\lambda'}, b^{\lambda'}, \bar{a}, \bar{b})$ is not simple. Then by Theorem 6.10 and Corollary 6.3, for every number ρ^* satisfying the inequality

$$0 < \rho^* \leqq \min\left(\min_{(I,J)\varepsilon A_{m\times n}} \frac{|\mu_{I,J}(\bar{a},\bar{b})|}{m+n} , \min_{(I,J)\varepsilon A_{m\times n}} \frac{|\mu_{I,J}(a^{\lambda'},b^{\lambda'})|}{m+n} \right),$$

there is a polytope $M(a',b') \varepsilon Q^{\rho^*}(a^{\lambda'},b^{\lambda'})$ such that a) $S(a',b',a^*,b^*)$ is a simple spectrum; b) $f_0(M(a',b')) = f_0(M(a^{\lambda'},b^{\lambda'}))$; c) $M(a',b') \varepsilon \mathcal{M}(m,n,k-\ell)$, where $1 \leq \ell \leq k$.

By the choice of the number ρ^* there is a number $0 < \pi < 1$ such that $M(a^\pi,b^\pi)$ defined by the vectors $a^\pi = \pi a' + (1-\pi)a^*$ and $b^\pi = \pi b' + (1-\pi)b^*$ belongs to $Q^{\rho^*}(\bar{a},\bar{b})$. By Corollary 6.3, $M(a^\pi,b^\pi) \varepsilon \mathcal{M}(m,n,k)$ and $f_0(M(a^\pi,b^\pi)) = f_0(M(\bar{a},\bar{b}))$. Repeating the argument used in the first part we deduce that

$$f_0(M(a,b)) > n^{m-1} + k(mn - m - n) .$$

2nd Case. Now suppose that conditions (9.3) and (9.4) are not satisfied for the polytope $M(a,b) \varepsilon \mathcal{M}(m,n,k)$. We examine two possibilities.

1) $a_m < b_n$. Note that in this case $m \geqq 3$. It is easily checked that every polytope $M(a^m,b^j)$, $j \varepsilon N_n$, defined by the vectors $a^m = (a_1,a_2,\ldots,a_{m-1})$ and $b^j = (b_1,b_2,\ldots,b_{j-1},b_j-a_m,b_{j+1},\ldots,b_n)$ belongs to the class $\mathcal{M}(m-1,n,k)$. Hence, by the induction hypothesis, we obtain the inequalities

$$f_0(M(a,b)) \geqq \sum_{j=1}^{n} f_0(M(a^m,b^j)) \geqq n(n^{m-2}+k(mn-2n-m+1)) > n^{m-1} + k(mn - m - n) .$$

The last inequality is true for $3 \leqq m \leqq n$, $n \geqq 4$, $1 \leqq k \leqq n$.

2) $a_m > b_n$. Every polytope $M(a^i,b^n)$, $i \varepsilon N_m$, given by the vectors $a^i = (a_1,\ldots,a_{i-1},a_i-b_n,a_{i+1},\ldots,a_m)$ and $b^n = (b_1,b_2,\ldots,b_{n-1})$ belongs either to the class $\mathcal{M}(m,n-1,k-1)$ or to the class $\mathcal{M}(m,n-1,k)$. Hence, by the induction hypothesis we have for $n \geqq 5$

$$f_0(M(a,b)) \geqq \sum_{i=1}^{m} f_0(M(a^i,b^n)) \geqq m((n-1)^{m-1}+(k-1)(mn-2m-n+1)) > n^{m-1}+k(mn-m-n).$$

The last inequality may be checked directly for $m = 2,3,4,5$ while for $m \geqq 6$ it follows from the known inequalities

$$(1 - 1/m)^{m-1} > 1/3 , \qquad m = 6,7,8,\ldots . \quad //$$

Theorems 5.1 and 9.1 have the following corollary.

Corollary 9.2 (Yemelichev & Kravtsov 1976i) *There is no non-degenerate transportation polytope* $M(a,b)$ *of order* $m \times n$, $2 \leqq m \leqq n$, $n \geqq 3$, *whose number of vertices satisfies*

$$n^{m-1} < f_0(M(a,b)) < n^{m-1} + mn - m - n \, .$$

In other words $n^{m-1} + mn - m - n$ is the next largest ('almost' minimal) number of vertices after the minimum number in the class of non-degenerate transportation polytopes of order $m \times n$, $2 \leqq m \leqq n$, $n \geqq 3$.

§10 ASYMPTOTICS

In this section we examine the asymptotic behaviour of certain classes of transportation polytopes. It is shown that as the order increases the ratio of the number of polytopes with the maximum number of facets to the total number of polytopes tends to unity, while the ratio of the number of polytopes with the minimum or maximum number of vertices to the total number tends to zero.

We consider the open regular simplex in E_k

$$U_k = \{c \in E_k : \sum_{i=1}^{k} c_i = 1 \, , \, c_i > 0 \, , \, i \in N_k\} \, .$$

Let

$$W_{m\times n} = U_m \times U_n = \{(a,b) : a \in U_m \, , \, b \in U_n\}$$

To each pair of vectors $(a,b) \in W_{m\times n}$ there corresponds a transportation polytope $M(a,b)$ of order $m \times n$. Let $W^{\xi}_{m\times n}$ denote the subset of pairs (a,b) in $W_{m\times n}$ for which the associated polytopes possess the property ξ . We say that *almost all* transportation polytopes have *property* ξ , if

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \frac{\mu(W^{\xi}_{m\times n})}{\mu(W_{m\times n})} = 1 \, .$$

If this limit tends to zero we say that *almost none* of the transportation polytopes have this property. In what follows $\mu(W)$ is the Lebesgue measure of the set W in the space

$$E_{m+n-2} = \{(a,b) : \sum_{i=1}^{m} a_i = \sum_{i=1}^{n} b_j = 1\} .$$

Theorem 10.1 *Almost all transportation polytopes have the maximum number of facets.*

Proof Let ξ represent the property of having the maximum number of facets. Then by Theorem 3.2 we have

$$W^{\xi}_{m\times n} = \{(a,b) \in W_{m\times n} : \max_{1\leq i\leq m} a_i + \max_{1\leq j\leq n} b_j \leq 1\}.$$

Let

$$U_k(x) = \{c\in U_k : \max_{1\leq i\leq k} c_i > x\} , \; \overline{U}_k(x) = \{c\in U_k : \max_{1\leq i\leq k} c_i = x\} .$$

With this notation and using the obvious inequality $\max_{1\leq i\leq k} c_i \geq 1/k$ we have

$$W_{m\times n} \setminus W^{\xi}_{m\times n} = (U_m(\tfrac{n-1}{n})\times U_n) \cup \bigcup_{1/m\leq x\leq (n-1)/n} (\overline{U}_m(x)\times U_n(1-x)) .$$

From this, using the following well known properties of measure $\mu(W\times V) = \mu(W)\times\mu(V)$; $\mu(W\cup V) = \mu(W)+\mu(V)$ if $W\cap V = \emptyset$; $\mu(W\setminus V) = \mu(W) - \mu(V)$ if $V\subseteq W$; and bearing in mind the equality

$$F_k(x) = \frac{\mu(U_k(x))}{\mu(U_k)} = \sum_{i=1}^{[1/x]} (-1)^{i-1} \binom{k}{i} (1-ix)^{k-1} ,$$

which is proved in Feller (1957,1966), we find that

$$\frac{\mu(W^{\xi}_{m\times n})}{\mu(W_{m\times n})} = 1 - \frac{m}{n^{m-1}} + \int_{1/m}^{(n-1)/n} F'_m(x)F_n(1-x)\, dx ,$$

where $F'_m(x)$ is the derivative of $F_m(x)$.

Dividing the interval of integration into the two intervals $[1/m,1/2]$ and $[1/2,(n-1)/n]$ and applying integration by parts to the first integral we obtain

$$\int_{1/m}^{(n-1)/n} F'_m(x)F_n(1-x)\, dx = \frac{mn}{2^{m+n-2}} - \frac{n}{m^{n-1}} -$$

$$-\int_{1/m}^{1/2} F_m(x)F_n'(1-x)\,dx + \int_{1/2}^{(n-1)/n} F_m'(x)F_n(1-x)\,dx .$$

Hence, noting that $F_k(x) = k(1-x)^{k-1}$ for $\frac{1}{2} \leqq x \leqq 1$, and $F_k(1-x) = kx^{k-1}$ for $1/k \leqq x \leqq \frac{1}{2}$. we find that

$$\frac{\mu(W^{\xi}_{m\times n})}{\mu(W_{m\times n})} \geqq 1 - \frac{m}{n^{m-1}} + \frac{mn}{2^{m+n-2}} - \frac{n}{m^{n-1}} - \int_{1/m}^{1/2} n(n-1)x^{n-2}\,dx -$$

$$-\int_{1/2}^{(n-1)/n} m(m-1)(1-x)^{m-2}\,dx$$

$$= \left(1-\frac{m}{2^{m-1}}\right)\left(1-\frac{n}{2^{n-1}}\right) .$$

Taking the limit as $m,n \to \infty$ and noting that $\mu(W^{\xi}_{m\times n}) \leqq \mu(W_{m\times n})$ we obtain the assertion of the theorem. //

The following is a direct corollary of Theorem 10.1 and Theorem 5.1.

<u>Corollary 10.2</u> *There are almost no non-degenerate transportation polytopes with the minimum number of vertices.*

Bolker (1972) posed the following question: is it true that almost all transportation polytopes have the maximum number of vertices? The following theorem due to Krachkovsky (1979) gives a negative answer to this question.

<u>Theorem 10.3</u> *There are almost no transportation polytopes with the maximum number of vertices.*

Proof Assume for definiteness that $m \leqq n$. Define the set $W^{\xi}_{m\times n} \subseteq W_{m\times n}$ by the condition $(a,b) \in W^{\xi}_{m\times n}$ if the components of the vectors a and b satisfy the constraints:

$$a_{i_1} + \sum_{r=1}^{n-2} b_{j_r} < 1 , \quad b_{j_1} + \sum_{r=1}^{m-2} a_{i_r} < 1 \quad \text{if } m = n ;$$

$$\sum_{r=1}^{q-1} b_{j_r} + \sum_{r=1}^{m-1} a_{i_r} < 1 , \quad a_{i_1} + \sum_{r=1}^{n-q-1} b_{j_r} < 1 \quad \text{if } n = mq , q > 1 ;$$

$\sum_{r=1}^{q} b_{j_r} + \sum_{r=1}^{m-1} a_{i_r} < 1$, $a_{i_1} + \sum_{r=1}^{n-q-1} b_{j_r} < 1$ if $n = mq + r$, $q \geqq 1$, $1 \leqq r \leqq m-1$,

where $a_{i_1} \geqq a_{i_2} \geqq \cdots \geqq a_{i_m}$, $b_{j_1} \geqq b_{j_2} \geqq \cdots \geqq b_{j_n}$.

By Theorem 7.9 every pair $(a,b) \in W_{m\times n}$ for which the polytope $M(a,b)$ has the maximum number of vertices, belongs to $W^{\xi}_{m\times n}$. Thus, to prove the theorem it suffices to show that

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \frac{\mu(W_{m\times n} \setminus W^{\xi}_{m\times n})}{\mu(W_{m\times n})} = 1 .$$

Consider the set of vectors $U^{t}_{k}(x) = \left\{ c \in U_k : \sum_{r=1}^{t} c_{i_r} > x ,\right.$ $\left. c_{i_1} \geqq c_{i_2} \geqq \cdots \geqq c_{i_k} \right\}$, $t \leq k$. Using the obvious inequalities $\sum_{r=1}^{t} c_{i_r} \geqq t/k$, $c \in U_k$, we have the inclusions

$$U^{m-2}_{m}\left(\frac{n-1}{n}\right) \times U_n \subseteq W_{m\times n} \setminus W^{\xi}_{m\times n} \qquad \text{for } m = n ,$$

$$U^{m-1}_{m}\left(\frac{n-q+1}{n}\right) \times U_n \subseteq W_{m\times n} \setminus W^{\xi}_{m\times n} \qquad \text{for } n = mq , \quad q > 1 ,$$

$$U^{m-1}_{m}\left(\frac{n-q}{n}\right) \times U_n \subseteq W_{m\times n} \setminus W^{\xi}_{m\times n} \qquad \text{for } n = mq + r , \quad r,q \geqq 1 .$$

Therefore

$$\frac{\mu(W_{m\times n} \;\; W^{\xi}_{m\times n})}{\mu(W_{m\times n})} \geqq \begin{cases} \dfrac{\mu\left(U^{m-2}_{m}\left(\frac{n-1}{n}\right)\right)}{\mu(U_m)} & \text{for } m = n , \\[2ex] \dfrac{\mu\left(U^{m-1}_{m}\left(\frac{n-q+1}{n}\right)\right)}{\mu(U_m)} & \text{for } n = mq , \quad q > 1 , \\[2ex] \dfrac{\mu\left(U^{m-1}_{m}\left(\frac{n-q}{n}\right)\right)}{\mu(U_m)} & \text{for } n = mq+r , \quad r,q \geqq 1. \end{cases}$$

Now, using the equality

$$\frac{\mu(U^{t}_{k}(x))}{\mu(U_k)} = 1 - \sum_{t/x \leqq i \leqq k} (-1)^{k-i} \frac{k!(ix-t)^{k-1}}{(k-i)!\,it!\,(i-t)!\,(i-t)^{t-1} t^{k-t-1}}$$

with $t/k \leqq x \leqq 1$, which is derived in Mauldon (1951), we find that

$$\frac{\mu(W_{m\times n}\setminus W^{\xi}_{m\times n})}{\mu(W_{m\times n})} \geqq \begin{cases} 1-\dfrac{1}{2^{m-2}}\,\dfrac{m-3}{m-2}+\dfrac{1}{(m-2)m^{m-2}} & \text{for } m=n, \\ 1-\dfrac{1}{q^{m-1}} & \text{for } n=mq\ ,\ q>1, \\ 1-\left(\dfrac{r}{mq+r}\right)^{m-1} & \text{for } n=mq+r,\ r,q\geqq 1. \end{cases}$$

Thus, if $m \leqq n$ we have

$$\lim_{\substack{m\to\infty\\ n\to\infty}} \frac{\mu(W^{\xi}_{m\times n})}{\mu(W_{m\times n})} = 0 .$$

The case $m \geqq n$ can be treated in a precisely analagous way. The statement of the theorem is now clear. //

EXERCISES

1. Let $M(a,b)$ be a degenerate transportation polytope of order $m\times n$. Show, that there exists a number $\delta > 0$, such that for any η , satisfying $0 < \eta \leqq \delta$, the polytope $M(a(\eta),b(\eta))$, defined by the vectors $a(\eta) = (a_1+\eta,a_2+\eta,\dots,a_m+\eta)$ and $b(\eta) = (b_1,b_2,\dots,b_{n-1},b_n+m\eta)$, is non-degenerate.

2. (Orden 1956). Let $M(a,b)$ be a transportation polytope of order $m\times n$ with whole number valued vectors $a = (a_1,a_2,\dots,a_m)$ and $b = (b_1,b_2,\dots,b_n)$. Then, the polytope $M(a',b')$, defined by the vectors $a' = (a_1,a_2,\dots,a_{m-1},a_m+1)$, $b' = (b_1+1/n,b_2+1/n,\dots,b_n+1/n)$ can never be degenerate.

3. If the transportation polytope $M(a,b)$ of order $m\times n$ has a unique degenerate vertex, then there exists a single pair of indices $(s,t) \in N_m\times N_n$ such that $a_s + b_t = \sum_{i=1}^{m} a_i$.

4. If the matrix $x = (x_{ij})_{m\times n} \in M(a,b)$ is a vertex, then it contains at least $\max(m,n) - \min(m,n) + 1$ simple lines (see §6,1).

5. Let $B = (R^{i_1,j_1},R^{i_2,j_2},\dots,R^{i_{m+n-1},j_{m+n-1}})$ be a basis of the transportation polytope of order $m\times n$, and let $\overline{B}$ be a $(m+n-1)\times(m+n-1)$ matrix obtained from B by crossing out any row. Show, that by means of suitable permutations of the rows and columns of $\overline{B}$, it can be transformed into a triangular matrix with components $r_{ii} = 1$, $\forall i \in N_{m+n-1}$,

and $r_{ij} = 0$ if $i > j$.

6. The following properties of the bi-partite graph $G(U,V)$, $|U| = m$, $|V| = n$, are equivalent : (1) the graph is connected and has no cycles; (2) the graph has $m+n-1$ edges and no cycles; (3) the graph is connected and has $m+n-1$ edges; (4) the graph has no cycles, but the addition of one extra edge creates a unique cycle; (5) the graph is connected but the removal of any edge makes it disconnected; (6) any pair of vertices in the graph are connected by a unique chain.

7. Show that: (1) the point x of the polytope $M(a,b)$ is a vertex of this polytope if and only if the bi-partite graph $G_x(U,V)$ has no cycles; (2) if the graph $G_x(U,V)$ is a forest, consisting of t trees, then the vertex x of the polytope $M(a,b)$ of order $m\times n$ has $m+n-t$ non-zero elements.

8. (Olah 1968). A graph is called an *n-partite graph of order* $t_1\times t_2\times\ldots\times t_n$, if its vertex set can be partitioned into n pairwise disjoint subsets $U_1 , U_2 , \ldots , U_n$ such that $|U_i| = t_i$, $\forall i \in N_n$, and such that every edge joins vertices lying in different subsets. The number of spanning trees in a complete marked (labelled) n-partite graph of order $t_1\times t_2\times\ldots\times t_n$ is given by the formula

$$\Delta(t_1,t_2,\ldots,t_n) = \left(\sum_{i=1}^{n} t_i\right)^{n-2} \prod_{j=1}^{n} \left(\sum_{i=1}^{n} t_i - t_j\right)^{t_j-1} ,$$

which generalizes (2.2).

9. (Votyakov & Frumkin 1976). Show that the number of bases $\beta(m,n)$ of a transportation polytope of order $m\times n$ satisfies the recurrence relation $(m \leq n)$:

$$\beta(m,n) = \sum_{j=1}^{m-1} \binom{n}{j} m^{n-j} \sum_{r=0}^{j-1} (-1)^r \binom{j}{r} m^r \beta(m,j-r) .$$

Hence derive the formula

$$\beta(m,n) = m^{n-1} \sum_{j=1}^{m-1} \binom{n}{j} \sum_{r=0}^{j-1} (-1)^r \binom{j}{r} (j-r)^{m-1} .$$

10. (Yemelichev & Kravtsov 1976ii). Consider the transportation problem:

minimize $F(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij}$ where $x = (x_{ij})_{m\times n} \in M(a,b)$, and c_{ij} are given real numbers. The point of the polytope $M(a,b)$ at which the minimum of $F(x)$ is attained is called the *optimal solution of the transportation problem* . Establish the following assertions:

(1). If there exists a pair $(\ell,t) \in N_m \times N_n$ such that $c_{it} - c_{\ell t} > \max_{j \neq t}(c_{ij} - c_{\ell j})$, $i = 1,2,\dots,\ell-1,\ell+1,\dots,m$, then, for any optimal solution of the transportation problem we have $x_{\ell t} = \min(a_\ell, b_t)$.

(2). Let $x = (x_{ij})_{m \times n}$ be a point of the polytope $M(a,b)$ such that $x_{ij} = 0$, $\forall (i,j) \in D \times E$, where $\emptyset \neq D \subset N_m$, $\emptyset \neq E \subset N_n$. If there exists a pair $(s,t) \in D \times E$ satisfying

$$c_{st} + \min_{(i,j) \in \bar{D} \times \bar{E}} c_{ij} > \max_{(i,j) \in D \times \bar{E}} c_{ij} + \max_{(i,j) \in \bar{D} \times E} c_{ij} ,$$

then $x_{st} = 0$ for any solution of the transportation problem.

11. Put $\delta^q_{pj} = c_{pj} - c_{qj}$, $p \neq q$, $j \in N_n$ and assume that $\delta^q_{p1} \geqq \delta^q_{p2} \geqq \dots \geqq \delta^q_{pn}$. We define: $S^q_p = \{j \in N_n : j \leqq k\}$, where the number k satisfies the inequalities $\sum_{j=1}^{k} b_j \leqq a_q \leqq \sum_{j=1}^{k+1} b_j$. The following assertions are true:

(1). (Kurtsevich & Kravtsov 1974). If $S^q_p \neq \emptyset$ for some pair $p,r \in N_m$, then there is an optimal solution of the transportation problem with components $x_{pj} = 0$, $\forall j \in S^q_p$.

(2). (Yemelichev & Kravtsov 1976ii). If, for some pair $p,q \in N_m$ the set $\underline{S}^q_p = \left\{ j \in S^q_p : \delta^q_{pj} > \delta^q_{p,|S^q_p|+1} \right\} \neq \emptyset$, for any optimal solution of the transportation problem we have $x_{pj} = 0$, $\forall j \in \underline{S}^q_p$.

(3). (Kravtsov 1973). If, for some $q \in N_m$, the set $S^q = \bigcap_{\substack{p \in N_m \\ p \neq q}} S^q_p \neq \emptyset$, then there is an optimal solution of the transportation problem for which $x_{qj} = b_j$, $\forall j \in S^q$.

(4). Let there exist non-empty sets L_1 , P_1 given by $L_1 \subset N_m$, $P_1 = \bigcup_{q \in L_1} \bigcap_{p \in L_2} S^q_p \subset N_n$ such that $\sum_{q \in L_1} a_q = \sum_{j \in P_1} b_j$. Then, the solution of the transportation problem reduces to the solution of the following subproblems T_s (s=1,2):

minimize $\sum_{t \in L_s} \sum_{j \in P_s} c_{ij} x_{ij}$ subject to $\sum_{i \in L_s} x_{ij} = b_j$, $\forall j \in P_s$,

$$\sum_{j \in P_s} x_{ij} = a_i \ , \ \forall i \in L_s \ , \quad x_{ij} \geqq 0 \ , \ \forall (i,j) \in L_s \times P_s ,$$

where $L_2 = N_m \setminus L_1$ and $P_2 = N_n \setminus P_1$.

12. Fix a number $p \in N_m$. Define the numbers x^0_{ij} by the rule:

$$x^0_{pj} = b_j - \sum_{i \neq p} x^0_{ij} \,, \quad \text{when} \quad i = p \,; \quad \text{and}$$

$$x^0_{ij} = \min \left\{ \sum_{j=1}^{n} (c_{ij} - c_{pj}) x_{ij} \; : \; \sum_{j=1}^{n} x_{ij} = a_i \,, \; 0 \leqq x_{ij} \leqq b_j \,, \; \forall j \in N_n \right\} .$$

Show, that the matrix $x^0 = (x^0_{ij})_{m \times n} \in M(a,b)$ is an optimal solution of the transportation problem if and only if $x^0_{pj} \geqq 0$, $\forall j \in N_n$.

13. The matrix $(x_{ij})_{m \times n} \in M(a,b)$ is an optimal solution of the transportation problem (see Ex. 10) if and only if there exist numbers u_i, $i \in N_m$, v_j, $j \in N_n$, such that

$$u_i + v_j = c_{ij} \,, \quad \text{if} \quad x_{ij} > 0 \,; \quad u_i + v_j \leqq c_{ij} \,, \quad \text{if} \quad x_{ij} = 0 \,.$$

This is a very well known criterion of optimality for the transportation problem.

14. Let $R \subset N_m$. The set $\left\{ x = (x_{ij})_{m \times n} : \sum_{i=1}^{m} x_{ij} = b_j \,, \forall j \in N_n \,, \right.$

$\left. \sum_{j=1}^{n} x_{ij} = a_i \,, \forall i \in R \,, \sum_{j=1}^{n} x_{ij} \leqq a_i \,, \forall i \in \overline{R} \,, \; x_{ij} \geqq 0 \,, \forall (i,j) \in N_m \times N_n \right\} \neq \emptyset$ if and only if $\sum_{i \in R} a_i \leqq \sum_{j \in N_n} b_j \leqq \sum_{i \in N_m} a_i$.

15. (Shvartin 1978). Let $\eta > 0$, $(p,q) \in N_m \times N_n$. Two matrices $(c^1_{ij})_{m \times n}$, $(c^2_{ij})_{m \times n}$ are called η-*neighbourly relative to the pair* (p,q) if $c^1_{ij} = c^2_{ij}$ for all $(i,j) \neq (p,q)$ and if $0 \leqq |c^1_{pq} - c^2_{pq}| < \eta$. Two transportation problems are called η-neighbourly relative to the pair (p,q) if their feasible sets are identical and if their cost matrices are η-neighbourly relative to the pair (p,q). Show that:

(1). Whatever the numbers $\eta > 0$, $N > 0$ and the pair $(p,q) \in N_m \times N_n$, it is always possible to construct two η-neighbourly transportaion problems relative to the pair (p,q) such that if $(x_{ij})_{m \times n}$ is any optimal solution of one of these problems and $(x'_{ij})_{m \times n}$ is any optimal solution of the other then $\sum_{i=1}^{m} \sum_{j=1}^{n} |x_{ij} - x'_{ij}| \geqq N$;

(2). For any pair $(p,q) \in N_m \times N_n$ and any $\eta > 0$ it is always possible to construct two η-neighbourly transportation problems relative to the pair (p,q) which have the same optimal solution.

16. Consider the two transportation problems T_s $(s=1,2)$: minimize $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}^{s}$ subject to $(x_{ij}^{s})_{m\times n} \in M(a^s, b^s)$, where $\sum_{i=1}^{m} a_i^1 = \sum_{i=1}^{m} a_i^2$. Let $(x_{ij}^{*s})_{m\times n}$ be an optimal solution of problem T_s , $s = 1,2$. The following assertions are true:

(1). (Ogurtsova, Skaletskaya & Skaletsky 1973). If $\sum_{i=1}^{m} |a_i^2 - a_i^1| \leqq \Delta$, $\sum_{j=1}^{n} |b_j^2 - b_j^1| \leqq \Delta$, then $|x_{ij}^{*2} - x_{ij}^{*1}| \leqq \Delta$, $\forall (i,j) \in N_m \times N_n$.

(2). (Intrator & Lev 1976). If $a_i^1 \leqq a_i^2$, $\forall i \in N_{m-1}$, $a_m^1 \geqq a_m^2$, $b_j^1 = b_j^2$, $\forall j \in N_n$, then $x_{mj}^{*} \geqq x_{mj}^{*}$, $\forall j \in N_n$.

17. Let $U = \{(i_1,j_1),(i_2,j_2),\ldots(i_{mn},j_{mn})\}$ be a listing of all the pairs in the set $N_m \times N_n$ in some order. Show that the following procedure constructs a vertex of the transportation polytope $M(a,b)$: Following the order of the pairs in the set U we define successively $x_{i_1,j_1} = \min(a_{i_1}, b_{j_1})$, $x_{i_t,j_t} = \min(a_{i_t} - \sum_{(i_t,k)} x_{i_t,k}, b_{j_t} - \sum_{(k,j_t)} x_{k,j_t})$, where the sum is taken over all pairs (i_t,k) and (k,j_t) which precede (i_t,j_t) in the list U .

When $U = \{(1,1),(1,2),\ldots.(1,n),(2,1),(2,2),\ldots,(2,n),\ldots,(m,1),(m,2),\ldots(m,n)\}$, this is usually called the *North-west Corner Method* , while, when $U = \{(m,n),(m,n-1),\ldots,(m,1),(m-1,n),(m-1,n-1),\ldots,(m-1,1),\ldots,(1,n),(1,n-1),\ldots,(1,1)\}$, it is called the *South-east Corner Method* .

18. For any polytope $M(a,b)$ in the class $\mathcal{M}(m,n,0)$, $2 \leqq m \leqq n$, we have the equations:

$$f_i(M(a,b)) = \binom{(m-1)n}{mn-m-n+1-i} , \quad i = mn-2m-n+2,\ldots,mn-m-n .$$

19. (Klingman & Russel 1974). Consider the problem of minimizing the function $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$ subject to the constraints:

$$\sum_{j=1}^{n} x_{ij} \geqq a_i , \ i = 1,2,\ldots,m_1 ; \quad \sum_{j=1}^{n} x_{ij} = a_i , \ i = m_1+1,m_1+2,\ldots,m_2 ;$$

$$\sum_{j=1}^{n} x_{ij} \leqq a_i , \ i = m_2+1,m_2+2,\ldots,m ; \quad \sum_{i=1}^{m} x_{ij} \geqq b_j , \ j = 1,2,\ldots,n_1 ;$$

$$\sum_{i=1}^{m} x_{ij} = b_j \ , \ j = n_1+1, n_1+2, \ldots, n_2 \quad ; \quad \sum_{i=1}^{m} x_{ij} \leqq b_j \ , \ j = n_2+1, n_2+2, \ldots, n \ ,$$

$$x_{ij} \geqq 0 \ , \ \forall (i,j) \in N_m \times N_n \ .$$

This problem can be reduced to the solution of the following transportation problem:

$$\text{minimize} \ \sum_{i=1}^{m+1} \sum_{j=1}^{n+1} c_{ij} x_{ij} \ , \ \text{subject to} \ \sum_{j=1}^{n+1} x_{ij} = a_i \ , \ \forall i \in N_{m+1} \ ,$$

$$\sum_{i=1}^{m+1} x_{ij} = b_j \ , \ \forall j \in N_{n+1} \ , \quad x_{ij} \geqq 0 \ , \ \forall (i,j) \in N_{m+1} \times N_{n+1} \ ,$$

where $a_{m+1} = \alpha - \sum_{i=1}^{m} a_i$, $b_{n+1} = \alpha - \sum_{j=1}^{n} b_j$, $\alpha \geqq 2 \sum_{j=1}^{n} b_j$,

$$c_{i,n+1} = \min_{1 \leqq j \leqq n_1} c_{ij} \ , \ i = 1,2,\ldots,m_2 \ ,$$

$$c_{i,n+1} = 0 \ , \ i = m_2+1, m_2+2, \ldots, m \ ,$$

$$c_{m+1,j} = \min_{1 \leqq i \leqq m_1} c_{ij} \ , \ j = 1,2,\ldots,n_2 \ ,$$

$$c_{m+1,j} = 0 \ , \ j = n_2+1, n_2+2, \ldots, n \ , \quad c_{m+1,n+1} = 0 \ .$$

The connection between the optimal solutions of these two problems is given by the following formulae

$$x^*_{ij} = \begin{cases} x^{**}_{ij} + x^{**}_{i,n+1} \ , & \text{if} \ i \in N_{m_2} \ , \ j \in \{t : c_{it} = \min_{1 \leqq j \leqq n_1} c_{ij}\} \ , \\ x^{**}_{ij} + x^{**}_{m+1,j} \ , & \text{if} \ i \in \{r : c_{rj} = \min_{1 \leqq i \leqq m_1} c_{ij}\} \ , \ j \in N_{n_2} \ , \\ x^{**}_{ij} & \text{otherwise,} \end{cases}$$

where $(x^*_{ij})_{m \times n}$ is the optimal solution of the original problem and $(x^{**}_{ij})_{(m+1)\times(n+1)}$ is the optimal solution of the second problem.

20. (Yemelichev 1969, Yemelichev & Kovalev 1972). The function $F(x)$ is called *Schur concave* on the convex set M , if it satisfies the following conditions: (a) $F(x)$ is a strictly concave function on the convex set M , i.e. $F(\lambda x_1 + (1-\lambda)x_2) > \lambda F(x_1) + (1-\lambda)F(x_2)$, $x_1, x_2 \in M$, $\lambda \in (0,1)$; (b) $F(x)$ is a symmetrical function, i.e. a function which does not change its value when the components of the vector x are permuted.

The vertex x^0 of the polytope $M(a,b)$ is called a *local minimum point* of the function $F(x)$, if $F(x^0) \leqq F(x)$ for all vertices x which are adjacent to x^0.

The vertex $x \in M(a,b)$ is called *ideal* if, given any triple of nonzero components $x^0_{kp}, x^0_{kq}, x^0_{tq}$, one of the following inequalities holds:

$$x^0_{kp} \geqq x^0_{kq} + x^0_{tq} \quad , \quad x^0_{tq} \geqq x^0_{kp} + x^0_{kq} .$$

Show, that a vertex of a transportation polytope is a local minimum point of a Schur concave function if and only if it is ideal. It follows from this that the location of local minimum points of Schur concave functions on a transportation polytope does not depend on the behaviour of the function but depends only on the geometry of the polytope.

21. (Yemelichev 1965). The algorithm of Ex. 17 for constructing a vertex of a transportation polytope is called the *greatest element method* if the sequence U is defined in the following way : i_1 is the index of the largest component of the vector a, and j_1 is the index of the largest component of the vector b and so on recursively.

Show that a vertex of a transportation polytope, constructed by the greatest element method, is a local minimum point of a Schur concave function (see previous exercise).

22. (Yemelichev & Kononenko 1971, Yemelicheva & Kravtsov 1977). Show that a vertex of a transportation polytope constructed by the greatest element method is a global minimum point of a Schur concave function on any polytope having the minimum number of facets.

23. (Kravtsov 1979). Let $(s,q), (r,t) \in N_m \times N_n$, $(s,q) \neq (r,t)$. The set $\{x = (x_{ij})_{m\times n} \in M(a,b) : x_{sq} = x_{rt} = 0\}$ is a $(d-2)$-face of the polytope $M(a,b)$ of order $m\times n$, $\min(m,n) \geqq 2$, $mn > 4$, if and only if the following conditions are satisfied: $a_s < \sum_{j\neq q} b_j$, $a_r < \sum_{j\neq t} b_j$ when $s\neq r$, $q\neq t$; $a_s < \sum_{j\neq q,t} b_j$ when $s=r$, $q\neq t$; $b_q < \sum_{j\neq s,r} a_i$ when $s\neq r$, $q=t$.

Show that:

(1) the non-degenerate transportation polytope of order $m\times n$, $3 \leqq m \leqq n$, has the minimum number, $\frac{(m-1)n}{2}$, of $(d-2)$-faces if and only if it has the minimum number of vertices.

(2) the transportation polytope of order $m\times n$, $m,n \geqq 4$, has the maximum number, $\frac{mn}{2}$, of $(d-2)$-faces if and only if

$$a_1 < \sum_{j=3}^{n} b_j \; , \; b_1 < \sum_{i=3}^{m} a_i \; , \text{ where } \; a_1 \geqq a_2 \geqq \ldots \geqq a_m \; , \; b_1 \geqq b_2 \geqq \ldots \geqq b_n .$$

(3) every transportation polytope of order $m\times n$, $m,n \geqq 4$, having the minimum or the maximum number of vertices has the maximum

has the maximum number of $(d-2)$-faces. The converse is not generally true;

(4) the number of $(d-2)$-faces of any non-degenerate transportation polytope $M(a,b)$ of order $m\times n$, $3 \leqq m \leqq n$, $mn > 9$, with $(m-1)n + k$ facets $(0\leq k \leq n)$ satisfies the inequalities

$$\binom{(m-1)n}{2} + k(m-1)(n-1) \leqq f_{d-2}(M(a,b)) \leqq \binom{(m-1)n+k}{2},$$

where the lower and the upper bounds are attainable.

24. Let $(s_1,q_1), (s_2,q_2), (s_3,q_3) \in N_m \times N_n$, $(s_1,q_1) \neq (s_2,q_2)$, $(s_1,q_1) \neq (s_3,q_3)$, $(s_2,q_2) \neq (s_3,q_3)$. By analogy with Ex. 23, find conditions for the set $\{x = (x_{ij})_{m\times n} \in M(a,b) : x_{s_1q_1} = x_{s_2q_2} = x_{s_3q_3} = 0\}$ to be a $(d-3)$-face of the transportation polytope $M(a,b)$ of order $m\times n$ and show that:

(1) the transportation polytope $M(a,b)$ of order $m\times n$, $m, n \geqq 5$, has the maximum number, $\binom{mn}{3}$, of $(d-3)$-faces if and only if

$$a_1 < \sum_{j=4}^{n} b_j, \quad b_1 < \sum_{i=4}^{m} a_i, \quad \text{where} \quad a_1 \geqq a_2 \geqq \ldots \geqq a_m, \quad b_1 \geqq b_2 \geqq \ldots \geqq b_n;$$

(2) the maximum number of $(d-3)$-faces in the class $\mathcal{M}(m,n,k)$ where $4 \leqq m \leqq n$, $n \geqq 5$, is given by $\binom{(m-1)n+k}{3}$.

25. Establish the following properties of equivalence for transportation polytopes:

(1) $M(a,b) \sim M(\alpha a, \alpha b)$ for any $\alpha > 0$;

(2) if $M(a^0,b^0) \sim M(a^1,b^1)$, then $M(a^0,b^0) \sim M(a^0+a^1, b^0+b^1)$. (The converse is not true).

26. The following assertions are true:

(1) the number of equivalence classes in the set $\mathcal{M}(m,n,1)$, $2 \leqq m < n$, is equal to $mn(2^{m-1}-1)$ when $m < n-1$ and is equal to $n(n-1)(2^{n-2}-1)$ when $m = n-1$;

(2) Let $\sigma(m,n)$ be the number of equivalence classes in the set of all transportation polytopes of order $m\times n$ with the maximum number of vertices and with $(m,n) \neq 1$. Then $\sigma(m,n)$ satisfies

$$2|\zeta(a^*,b^*)| \leq \sigma(m,n) \leq 2^{|\zeta(a^*,b^*)|},$$

where $|\zeta(a^*,b^*)| = \frac{1}{2}\sum_{(x,y)} \binom{m}{x}\binom{n}{y}$. Here, the sum is taken over all integral solutions of the system $my = nx$, $1 \leqq x \leqq m-1$, $1 \leqq y \leqq n-1$.

(3) The number of equivalence classes in the set of all degenerate transportation polytopes of order $m \times n$ having the minimum number of facets is given by $m(2^n-1)$ when $m < n$, and by $2m(2^m-1)$ when $m = n$. For the case of non-degenerate transportation polytopes with the minimum number of facets the number of equivalence classes is given by m if $m < n$, and by $2m$ if $m = n$.

27. The 1-degenerate transportation polytope of order $m \times n$ has the minimum number of vertices $(\max(m,n))^{\min(m,n)-1}$ if and only if $M(a,b)$ is defined by vectors a and b such that:

(1) $\sum_{i=1}^{m} a_i - \max_{1 \leqq i \leqq m} a_i = \min_{1 \leqq j \leqq n} b_j$ if $m \leqq n$,

$\sum_{j=1}^{n} b_j - \max_{1 \leqq j \leqq n} b_j = \min_{1 \leqq i \leqq m} a_i$ if $m \geqq n$;

(2) $M(a,b)$ has the minimum number of facets.

28. The diameter of a non-degenerate transportation polytope of order $m \times n$, $2 \leqq m \leqq n$, with $(m-1)n+1$ facets does not exceed $m+1$. For non-degenerate transportation polytopes of order $2 \times n$, $n \geqq 3$, with $n+k$ facets, $0 \leqq k \leqq n$, the diameter does not exceed $k+1$, if $k = 0,1,\ldots,n-1$, and does not exceed n if $k = n$. In particular, this proves the maximum diameter conjecture for the case of non-degenerate transportation polytopes of order $2 \times n$.

29. (Balinski 1974). $\operatorname{diam} M(a,b) = m+n-1$, if $3 \leqq m \leqq n$, and $a = (q_1m+r, q_2m+r, \ldots, q_mm+r)$, $b = (m,m,\ldots,m) \varepsilon E_n$, where $q_i \geqq 1$ is a whole number, $n = \sum_{i=1}^{m} q_i + r$, $r = 1$ or $r = m-1$.

30. (Yemelichev, Kravtsov & Krachkovski 1979). Every integer from $m-1$ to $m+n-1$ can occur as the diameter of a transportation polytope of order $m \times n$, $3 \leqq m \leqq n$.

31. (Yemelichev, Kravtsov & Krachkovski 1978ii). Let the numbers m and n be relatively prime and let $m,n \geqq 3$. Then the diameter of any transportation polytope of order $m \times n$ having the maximum number of vertices is not less than $m+n-1$.

32. If all the vertices of the transportation polytope $M(a,b)$ of order $2 \times n$, $n \geqq 3$, are degenerate, then

(1) $b_1 = b_2 = \ldots = b_n$,

(2) the polytope has the maximum or the minimum number of facets.

33. A polytope all of whose facets have the same number of vertices, and all of whose vertices are incident to the same number of facets, is called *special*. The following transportation polytopes are special:

(1) central polytopes ;

(2) a non-degenerate polytope with the minimum number of vertices ;

(3) a polytope of order $m \times n$ with the maximum number of vertices, where m and n are co-prime.

34. Let $M(a,b)$ be a degenerate transportation polytope of order $m \times n$, $3 \leq m \leq n$, with the minimum number of vertices. The number of vertices belonging to any facet $F_{ij}(a,b)$ is given by the formula:

$$f_0(F_{ij}(a,b)) = \begin{cases} \dfrac{(n-1)!(n-1)}{(n-m+1)!}, & \text{if } a_i \neq \max\limits_{1 \leq s \leq m} a_s , \\ \\ \dfrac{(n-1)!(m+1)}{(n-m+1)!}, & \text{if } a_i = \max\limits_{1 \leq s \leq m} a_s \end{cases}$$

35. Every non-degenerate transportation polytope of order $m \times n$, $2 \leq m \leq n$, has at least $(m-1)n$ facets, each of which contains not less than n^{m-2} vertices.

36. (Yemelichev & Kravtsov 1978). Let $M(a^0,b^0)$, $M(a^1,b^1)$ be non-degenerate transportation polytopes of order $m \times n$ with the minimum and the maximum number of vertices respectively. If $2 \leq m < n$, then these polytopes do not have any equivalent vertices. If $m = n$, then $M(a^0,b^0)$ and $M(a^1,b^1)$ can have up to $m!$ equivalent vertices

37. Show that the condition in Theorem 7.10 is sufficient only when $m + n \leq 6$.

38. Let $\phi_v(m,n)$ be the maximum number of vertices in the class of degenerate transportation polytopes of order $m \times n$. Then the following relations are true (Kravtsov 1976i & 1976ii) :

$$\phi_v(m,n) \geq (mn-m-n+2)\phi(m-1,n-1) , \quad m,n > 1 ;$$

$$\phi_v(m,n) \leq \phi(m,n) - mn + m + n , \quad 3 \leq m \leq n , \quad mn > 9 ;$$

$$\phi_v(2,n) = \left(n - \left[\frac{n}{2}\right]\right)\left(\binom{n}{[n/2]} - 1\right) + 1 .$$

When m and n are coprime we have (Kononenko & Trukhanovsky 1978) :

$$\phi_v(m,n) \geq \phi(m,n) - (mq-qp-1)\phi(p,q)\phi(m-p,n-q) ,$$

where $mq - np = 1$, $0 < p < m$, $0 < q < n$.

39. An analysis of the proof of Theorem 6.4 leads directly to the following result. Let $M(a^0,b^0)$ and $M(a^1,b^1)$ be an (L,P)-regular pair of polytopes with centre $M(a^{\lambda^*},b^{\lambda^*})$, then putting $\alpha = \max(|P|(m-|L|), |L|(n-|P|)) - 1$, $\beta = \min(|P|(m-|L|), |L|(n-|P|)) - 1$, we have

$$f_0(M(a^{\lambda^*},b^{\lambda^*})) = \max(f_0(M(a^0,b^0)), f_0(M(a^1,b^1))) -$$
$$- \alpha\delta_{L,P}(a^{\lambda^*},b^{\lambda^*}) = \min(f_0(M(a^0,b^0)), f_0(M(a^1,b^1))) - \beta\delta_{L,P}(a^{\lambda^*},b^{\lambda^*}) .$$

40. Let $M(a^0,b^0)$ and $M(a^1,b^1)$ be transportation polytopes of the same order, and let at least one of them be non-degenerate. Then $|S(a^0,b^0,a^1,b^1)| = |A(a^0,b^0,a^1,b^1)|$ if and only if the spectrum $S(a^0,b^0,a^1,b^1)$ is simple. Here $A(a^0,b^0,a^1,b^1) = \bigcup_{\lambda\in S(a^0,b^0,a^1,b^1)} A(a^\lambda,b^\lambda)$

41. Let $\lambda_1 < \lambda_2 < \ldots < \lambda_T$ be the numbers in the simple spectrum $S(a^0,b^0,a^1,b^1)$, where $a^1 = a^*$, $b^1 = b^*$, $M(a^0,b^0)$ is a non-degenerate transportation polytope of order $m\times n$, $m \leq n$, with the minimum number of vertices. Further, let $\lambda_t < \pi_t < \lambda_{t+1}$, $t = 0,1,\ldots,T$, $\lambda_0 = 0$, $\lambda_{T+1} = 1$. The following assertions are true:

(1) $f_{d-1}(M(a^{\pi_t},b^{\pi_t})) = f_{d-1}(M(a^{\pi_{t-1}},b^{\pi_{t-1}})) + 1$ if and only if there exists $k \in N_n$, such that
$$\lambda_t = \frac{a_1^0 - \sum_{j=1}^{n} b_j^0 + b_{n-k+1}^0}{a_1^0 - \sum_{j=1}^{n} b_j^0 + b_{n-k+1}^0 + mn - m - n}$$
where $a_1^0 \geqq a_2^0 \geqq \ldots \geqq a_m^0$, $b_1^0 \geqq b_2^0 \geqq \ldots \geqq b_n^0$;

(2) if the number k in (1) exists, then
$$f_0(M(a^{\pi_t},b^{\pi_t})) = f_0(M(a^{\pi_{t-1}},b^{\pi_{t-1}})) + mn - m - n .$$

42. (Yemelichev & Trukhanovski 1977). Let $2 \leqq m \leqq n$, $n \geqq 3$, and let $k > mn$ be an integer. Define the vectors a and b by:

$a = (m(n-1)+1+(10^k-1)^{-1}(1-10^{-kn})-10^{-kn}(10^k-1)^{-1}(1-10^{-k(m-1)}), 1+10^{-k(n+1)},$
$1+10^{-k(n+2)},\ldots,1+10^{-k(n+m-1)})$,

$b = (m+10^{-k}, m+10^{-2k}, \ldots, m+10^{-nk})$. Then, the spectrum $S(a,b,a^*.b^*)$ is simple and the polytope $M(a,b)$ is non-degenerate and has the minimum number of vertices.

43. (Yemelichev & Trukhanovski 1977). Let $M(a,b)$ be a non-degenerate polytope of order $m\times n$, $2 \leqq m \leqq n$, $n \geqq 3$, with the minimum number of vertices such that the spectrum $S(a,b,a^*,b^*)$ is simple. Then

$$|S(a,b,a^*,b^*)| = \sum_{k=1}^{m-1} \binom{m-1}{k-1} \sum_{t=[kn/m]+1}^{n-1} \binom{n}{t} .$$

Hence, by Theorem 6.4 , the number $\tau(m,n)$ of distinct values of the function $f_0(M(a,b))$ on the class of non-degenerate transportation polytopes of order $m \times n$, $2 \leqq m \leqq n$, satisfies the inequality

$$\tau(m,n) \geqq \sum_{k=1}^{m-1} \binom{m-1}{k-1} \sum_{t=[kn/m]+1}^{n-1} \binom{n}{t} + 1 .$$

44. (Yemelichev, Kravtsov & Krachkovski 1979). Let $2 \leqq m < n$ and let $1 \leqq k < n - m$ or $k = n - 1$. The polytope $M(a,b) \in \mathcal{M}(m,n,k)$ has the maximum number of vertices if and only if the spectrum $S(a,b,a^k,b^k) = \emptyset$, where

$$a^k = (mk+((m-1)n+1)(n-k-1) , n , n , \ldots , n) \in E_m ,$$

$$b^k = ((m-1)n+1,\ldots,(m-1)n+1,(m-1)n,\underbrace{m,m,\ldots,m}_{k}) \in E_n .$$

We note that in the remaining cases, where $m = n$ or $m < n$, $k \geqq n - m$, $k \neq n - 1 , n$, the set $\mathcal{M}(m,n,k)$ can be partitioned into two subsets in each of which the class of polytopes with the maximum number of vertices can be characterized in a similar way.

45. Let $2 \leqq m \leqq n$, $0 \leqq k \leqq n$. The number of vertices in any polytope in the class $\mathcal{M}(m,n,k)$ does not exceed the number

$$n^{m-2}m^{k-1}(mn - km + k) .$$

46. (Yemelichev, Kravtsov & Krachkovski 1979). Let $2 \leqq m \leqq n$, $1 \leqq k \leqq n-2$. Then, the maximum number of vertices in the class $\mathcal{M}(m,n,k)$ is equal to $m^k(n-k)^{m-1}$ if $k \leqq n/m$ and is greater than $m^k(n-k)^{m-1}$ if $k > n/m$.

47. (Kravtsov 1976i). Let the sequences of non-negative numbers $\alpha_1,\alpha_2,\ldots,\alpha_m$, $\beta_1,\beta_2,\ldots,\beta_n$ satisfy the constraints:

$$\sum_{i=1}^{m} \alpha_i = \sum_{j=1}^{n} \beta_j < 1 ,$$

$$(m,n) \neq 1 \Rightarrow \sum_{i \in I} \alpha_i \neq \sum_{j \in J} \beta_j \quad \forall I \subset N_m , J \subset N_n .$$

Then the polytope $M(a,b)$, defined by the vectors $a = (n+\alpha_1,n+\alpha_2,\ldots,n+\alpha_m)$ and $b = (m+\beta_1,m+\beta_2,\ldots,m+\beta_n)$, has the maximum number of vertices.

48. (Yemelichev, Kravtsov & Krachkovski 1978ii). Show that the necessary conditions on Theorem 7.9 are also sufficient in the case $n = mq + 1$.

49. (Klee & Witzgall 1968). Show that

$$\phi(m,n) \geqq n\phi(m-1,n) \quad , \quad m,n>1 \; ,$$

$$\lim_{\substack{m\to\infty\\ n\to\infty}} \frac{\log \phi(m,n)}{\log \beta(m,n)} = 1 \; ,$$

where $\beta(m,n)$ is the number of bases of a transportation polytope of order $m\times n$.

50. Using Proposition 8.4 and Corollary 8.5 , show that

$$\frac{(mq+1)!}{(q!)^m}(mq+1)^{m-2}m^{r-1} \leq \phi(m,mq+r) \leq \frac{(mq+m-1)!}{(q!)^m}\, m^{r-1} \quad , \quad \forall r \in N_{m-1} \; .$$

The upper bound improves the estimate given in Likhachev & Yemelichev 1974 and in Likhachev 1975.

51. For every number $t \in \{0,1,\ldots,m-r-1\}$ we define $K(t) = \{k \in Z_m^+ : \sum_{i=1}^m k_i = t\}$. With each vector $k \in \bigcup_{t=0}^{m-r-1} K(t)$ we associate a polytope $M(a^k,b^k)$ of order $m\times(r+\sum_{i=1}^m k_i)$, defined by the vectors $a^k = (k_1m+r, k_2m+r,\ldots,k_mm+r)$ and $b^k = (m,m,\ldots,m)$. Prove the following assertion, which is a generalization of Theorem 8.2 : the number of vertices of the transportation polytope $M(a,b)$ of order $m\times n$, $2 \leq m \leq n$, defined by the vectors $a = (q_1m+r, q_2m+r,\ldots,q_mm+r)$ and $b = (m,m,\ldots,m)$, where $q_i \geqq 0$ is an integer and $n = \sum_{i=1}^m q_i + r$, $0 \leq r \leq m-1$, is given by

$$\frac{n!}{\prod_{i=1}^m q_i!} \sum_{t=0}^{m-r-1} \; \sum_{k\in K(t)} \frac{\gamma(a^k,b^k)}{(r+t)!} \prod_{i=1}^m \prod_{\rho=0}^{k_i-1} (q_i-\rho) \; .$$

When $n = \sum_{i=1}^m q_i + m - 1$ this gives the formula of Balinski (1974).

$$f_0(M(a,b)) = \frac{n!\, m^{m-2}}{\prod_{i=1}^m q_i!} \; .$$

52. (Balinski 1974). The number of vertices of the transportation polytope $M(a,b)$ of order $m\times n$, $2 \leq m \leq n$, defined by the vectors $a = (mq_1+1, mq_2+1,\ldots,mq_m+1)$ and $b = (m,m,\ldots,m)$, where $q_i \geqq 0$ is an

integer and $n = \sum_{i=1}^{m} q_i + 1$,is given by $(n!\,n^{m-2}) / \left(\prod_{i=1}^{m} q_i! \right)$.

53. There is a transportation polytope of order $m\times n$, $m,n \geqq 2$, which has a facet containing at least $(m-1)!\times(n-1)!$ vertices.

54. (Kravtsov 1976i). We say that the vertex x of the transportation polytope $M(a,b)$ of order $m\times n$, $2 \leqq m \leqq n$, has *degeneracy of degree* k , $0 \leqq k \leqq m-1$, if $|K(a,b,x)| = m+n-k-1$. There exists a transportation polytope of order $m\times n$, $2 \leqq m \leqq n$, among whose vertices there is a vertex of any degree of degeneracy from 0 to $m-1$. Given any k , $1 \leqq k \leqq m-2$, there does not exist a transportation polytope of order $m\times n$, $3 \leqq m \leqq n$, for which all of its vertices have degeneracy of degree k .

55. (Yemelichev, Kravtsov & Krachkovski 1978ii). Let $2 \leqq m \leqq n$, $1 \leqq k \leqq n$. The polytope $M(a,b) \in \mathcal{M}(m,n,k)$ has the minimum number , $n^{m-1} + k(mn-m-n)$, of vertices if and only if:

(1) when $m = 2$, $b_{n-k+1} < a_2 < \min(b_{n-k}, b_{n-1}-b_n)$;

(2) when $3 \leqq m < n$

$$b_{n-k+1} - b_n < \sum_{j=1}^{n-1} b_j - a_1 < \min(a_m , b_{n-k}-b_n) \; ; \qquad (*)$$

(3) when $m = n \geqq 5$ either (*) holds, or

$$a_{m-k+1} - a_m < \sum_{i=1}^{m-1} a_i - b_1 < \min(b_n , a_{m-k}-a_m) \; ,$$

where $a_1 \geqq a_2 \geqq \ldots \geqq a_m$, $b_1 \geqq b_2 \geqq \ldots \geqq b_n$, $b_0 = a_0 = +\infty$.

56. (Kravtsov & Krachkovski 1978). For any k , $1 \leqq k \leqq [2n/3]$ there does not exist any non-degenerate transportation polytope $M(a,b)$ of order $m\times n$, $2 \leqq m \leqq n$, $n \geqq 5$, whose number of vertices satisfies:

$$n^{m-1} + (k-1)(mn-m-n) < f_0(M(a,b)) < n^{m-1} + k(mn-m-n) \; .$$

57. Let $M(a,b) \in \mathcal{M}(m,n,k)$, $2 < m \leqq n$, $0 \leqq k \leqq n$. Show that among its facets there are no more than k $(d-1)$-simplexes, and, in the case in which this polytope has the minimum number of vertices, the number of $(d-1)$-simplexes is exactly k .

58. (Kravtsov & Krachkovski 1978). For the problem of minimizing a Schur concave function (see Ex. 20) on any transportation polytope of order $m\times n$, $2 \leqq m \leqq n$, with the minimum number of vertices and with $(m-1)n + k$ facets , $0 \leqq k \leqq n$, it is possible to designate $k+1$ vertices one of which always provides the global minimum.

59. The graph of a polytope $M(a,b) \in \mathfrak{M}(m,n,k)$, $2 \leq m \leq n, n \geq 5$, having the minimum number of vertices is Hamiltonian.

60. The minimum radius of a non-degenerate transportation polytope of order $m \times n$, $2 \leq m \leq n$, $n \geq 3$, is equal to $m - 1$.

61. For any $s > 1$ the number of $(s-1)$-faces of an s-face of a transportation polytope does not exceed $6(s-1)$ (Gibson 1976). Using this result, it is shown in Krachkovski & Yemelichev (1979) that the diameter of a transportation polytope of order $m \times n$, $2 \leq m \leq n$, is no greater than $2^{2m-4}(n - m + 1)(20m - 17)$.

62. (Yemelichev, Kravtsov & Krachkovsky 1977i). The constant term in the polynomial $P(q,m,r)$ in Theorem 8.3 is $\phi(m,r)/r!$. The remaining terms of this polynomial, other than the highest term, are not known.

63. (Kononenko & Trukhanovsky 1978). Let m and n be co-prime integers. Then, there does not exist a non-degenerate transportation polytope of order $m \times n$, $2 \leq m \leq n$, whose number of vertices satisfies the inequality $\sigma(m,n) < f_0(M(a,b)) < \phi(m,n)$, where $\sigma(m,n) = \phi(m,n) - \phi(p,q)\phi(m-p,n-q)$, and the numbers p and q satisfy the relations

$$mq - np = 1 \quad , \quad 0 < p < m \quad , \quad 0 < q < n \quad . \qquad (**)$$

Hence, the number $\sigma(m,n)$ is the nearest possible vertex number to $\phi(m,n)$, i.e. it is the 'almost' maximum number of vertices in class of non-degenerate transportation polytopes of order $m \times n$, $(m,n) = 1$.

Show that the non-degenerate polytope $M(a,b)$ of order $m \times n$, $(m,n) = 1$, $2 \leq m \leq n$, has the 'almost' maximum number of vertices if and only if

$$\sum_{i=1}^{p} a_i < \sum_{j=1}^{q} b_j \quad , \quad \sum_{i=1}^{p} a_i < b_{q+1} + \sum_{j=1}^{q-1} b_j \quad ,$$

where $a_1 \geq a_2 \geq \dots \geq a_m$, $b_1 \leq b_2 \leq \dots \leq b_n$, and the numbers p and q satisfy the conditions (**).

64. Show that the Lebesgue measure in the space E_{m+n-1} of the set of vector pairs $(a,b) = (a_1,a_2,\dots,a_m,b_1,b_2,\dots,b_n)$, which define degenerate transportation polytopes, is equal to zero. In particular, it follows from this that there are almost no transportation polytopes with the minimum number of vertices, since every transportation polytope of order $m \times n$, $m,n \geq 3$, with the minimum number of vertices is degenerate.

65*. Find necessary and sufficient conditions for two classical transportation polytopes of the same order to have the same number of vertices.

66*. Is it true that the maximum number of vertices in the class of classical transportation polytopes of order $m \times n$, $2 \le m \le n$, is not less than the number $n!\,m^{m-2} q^{m-r-1}/(q!)^m$, where $n = mq + r$, and r is the remainder on dividing n by m? This assertion is true for the cases $r = 0, 1, m-1, m-2$ (see §8).

67*. Let $2 \le m \le n$, $n \ge 5$, $1 \le k \le n$, $d = (m-1)(n-1)$. Is it true that a non-degenerate transportation polytope of order $m \times n$ with $(m-1)n + k$ facets has the minimum number of vertices if and only if there are exactly k $(d-1)$-simplexes among its facets (see Ex. 57)?

68. Theorem 7.1 can be reformulated as follows: the non-degenerate transportation polytope $M(a,b)$ of order $m \times n$ has the maximum number of vertices if and only if

$$\sum_{i \in I} a_i < \sum_{j \in J} b_j \text{ , if } n|I| < m|J| \text{ ; } \sum_{i \in I} a_i > \sum_{j \in J} b_j \text{ , if } n|I| > m|J|,$$

where $I \subset N_m$, $J \subset N_n$.

In this chapter we study the feasible sets of transportation problems with prohibitions and with bounded communication flows as well as generalized and symmetric transportation problems. For these polytopes we examine the possibility of representation as a product of polytopes of lower order. Existence theorems are formulated and limits on the number of facets are obtained. Polytopes of maximum dimension are determined as well as those which are simplexes.

§1 TRUNCATED TRANSPORTATION POLYTOPES

Transportation problems with bounded flows are widely known. This section is concerned with the feasible sets of such problems.

Definition 1.1 A *truncated transportation polytope* of order $m\times n$, $m,n \geqq 1$, is the feasible set of a transportation problem with bounded flows, that is, a set of the form

$$M(a,b,D) = \{x = (x_{ij})_{m\times n} : \sum_{j=1}^{n} x_{ij} = a_i \ , \ i \in N_m \ ,$$
$$\sum_{i=1}^{m} x_{ij} = b_j \ , \ j \in N_n \ , \ 0 \leqq x_{ij} \leqq d_{ij} \ , \ (i,j) \in N_m \times N_n \ ,$$

where the components of the vectors $a = (a_1, a_2, \ldots, a_m)$ and $b = (b_1, b_2, \ldots, b_n)$ and the elements of the matrix $D = (d_{ij})_{m\times n}$ are positive real numbers, and $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

Note that if $d_{ij} \geqq \min(a_i, b_j)$, $(i,j) \in N_m \times N_n$ then this polytope coincides with a classical transportation polytope.

From Gale's theorem (Corollary 4.12, Ch.4) we have the following criterion.

Proposition 1.1 *The truncated transportation polytope $M(a,b,D)$ of order $m\times n$ is not empty if and only if*

$$\sum_{i=1}^{m} \min(a_i, \sum_{j\in J} d_{ij}) \geqq \sum_{j\in J} b_j \qquad J\subseteq N_n . \qquad (1.1)$$

1.1 Polytopes of Maximum Dimension

In contrast to the classical transportation polytope of order $m\times n$ whose dimension is always $(m-1)(n-1)$, the truncated transportation polytope of order $m\times n$, $m,n \geqq 2$, may have a dimension less than this.

Theorem 1.2 (Kravtsov & Korzinkov 1979) *The truncated transportation polytope $M(a,b,D)$ of order $m\times n$, $m,n\geqq 2$, has maximum dimension $(m-1)(n-1)$ if and only if the following conditions are satisfied:*

$$\sum_{j=1}^{n} d_{ij} > a_i \qquad i\in N_m , \qquad (1.2)$$

$$\sum_{i=1}^{m} \min(a_i, \sum_{j\in J} d_{ij}) > \sum_{j\in J} b_j \qquad J\subset N_n . \qquad (1.3)$$

Proof Necessity. The necessity of conditions (1.2) is clear.

We now establish the necessity of conditions (1.3). Since $M(a,b,D)$ is non-empty, Proposition 1.1 ensures that the inequalities (1.1) are satisfied. Suppose there exists a subset $J\subset N_n$, $J\neq N_n$, such that

$$\sum_{i=1}^{m} \min(a_i, \sum_{j\in J} d_{ij}) = \sum_{j\in J} b_j . \qquad (1.4)$$

This means that there is an index $i_0\in N_m$ such that $a_{i_0} > \sum_{j\in J} d_{i_0 j}$, and hence that for any matrix $(\eta_{ij})_{m\times n}$ with positive elements we have

$$\sum_{i=1}^{m} \min(a_i , \sum_{j\in J} (d_{ij} - \eta_{ij})) < \sum_{j\in J} b_j :$$

On the other hand, since there is a matrix $x\in M(a,n,D)$ such that

$$0 < x_{ij} < d_{ij} \qquad \forall (i,j)\in N_m\times N_n , \qquad (1.5)$$

there is a matrix $(\eta_{ij})_{m\times n}$ with positive elements such that

$$\sum_{i=1}^{m} \min(a_i, \sum_{j\in J}(d_{ij}-\eta_{ij})) \geq \sum_{j\in J} b_j . \qquad (1.6)$$

This contradiction proves the necessity of condition (1.3).

Sufficiency. From conditions (1.3) we have

$$\eta = \min_J (\sum_{i=1}^{m} \min(a_i, \sum_{j\in J} d_{ij}) - \sum_{j\in J} b_j) > 0 ,$$

where the minimum is taken over all proper subsets $J \subset N_n$. Let $0 < \eta_{ij} \leqq \eta/(mn)$, $(i,j) \in N_m \times N_n$. Then the elements of the matrix $D' = (d_{ij} - \eta_{ij})_{m\times n}$ satisfy the inequality (1.6) for any $J \subset N_n$ and also for $J = N_n$ by condition (1.2). Hence, by Proposition 1.1, $M(a,b,D') \neq \emptyset$. Hence there is a matrix $x \in M(a,b,D)$ satisfying (1.5). Therefore, by Proposition 4.1, Ch.1, the dimension of $M(a,b,D)$ is $(m-1)(n-1)$. //

1.2 A Representation Theorem

The truncated transportation polytope $M(a,b,D)$ of order $m\times n$, $m,n \geqq 2$, is called *regularly truncated* if its dimension is maximum, that is $\dim M(a,b,D) = (m-1)(n-1)$. When at least one of the numbers m or n equals unity the polytope $M(a,b,D)$ is called regularly truncated if $d_{ij} > \min(a_i, b_j)$, $(i,j) \in N_m \times N_n$.

When $M(a,b,D)$ is a non-regularly truncated transportation polytope of order $m\times n$ we introduce the following sets :

$$P = \{(i,j) \in N_m \times N_n : x_{ij} = d_{ij} \quad \forall x = (x_{ij})_{m\times n} \in M(a,b,D)\},$$

$$Q = \{(i,j) \in N_m \times N_n : x_{ij} = 0 \quad \forall x = (x_{ij})_{m\times n} \in M(a,b,D)\}$$

Theorem 1.2 shows that $P \neq \emptyset$ in this case.

Theorem 1.3 *Every non-regularly truncated, non-empty transportation polytope of order* $m\times n$, $m,n \geqq 2$, *which does not degenerate to a single point, can be represented uniquely as the product of a set of regularly truncated transportation polytopes and a point, that is* $M(a,b,D) = M(a^1,b^1,D^1) \otimes M(a^2,b^2,D^2) \otimes \ldots \otimes M(a^k,b^k,D^k) \otimes R(P,Q)$, *where* $R(P,Q)$ *is a point with coordinates*

$$x_{ij} = \begin{cases} 0, & \text{if } (i,j) \in Q, \\ d_{ij}, & \text{if } (i,j) \in P. \end{cases}$$

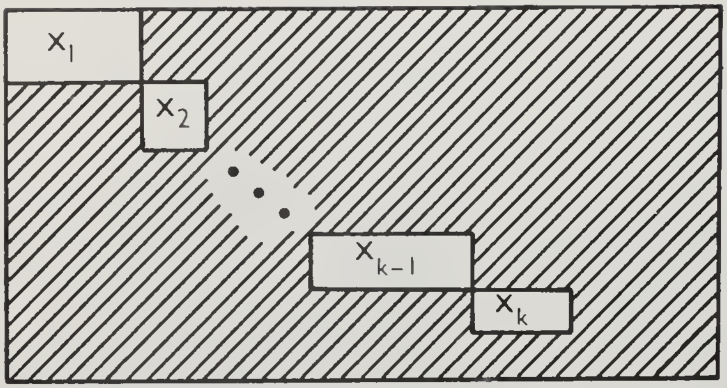


Fig. 49.

This means that after suitable permutations of rows and columns every matrix of such a polytope $M(a,b,D)$ can be represented in the form depicted in Figure 49. Here the shaded region is the set $P \cup Q$ of constant components, and $x^p = (x^p_{ij})_{i\varepsilon I_p, j\varepsilon J_p}$ is a matrix of the polytope $M(a^p,b^p,D^p)$, $p \varepsilon N_k$.

The importance of this theorem resides partly in the fact that the problem of extremizing a function $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij}$ on a non-regularly truncated transportation polytope which does not reduce to a point can be broken down into the solution of k problems of lower order:

$$\operatorname{extr}\{ \sum_{i\varepsilon I_p} \sum_{j\varepsilon J_p} c_{ij}x^p_{ij} : (x^p_{ij}) \varepsilon M(a^p,b^p,D^p)\} \quad , \quad p \varepsilon N_k .$$

Proof of Theorem 1.3. Let $M(a,b,D)$ be a truncated transportation polytope of order $m\times n$ such that $0 < \dim M(a,b,D) < (m-1)(n-1)$. Then, by Theorem 1.2 either there is an i N_m such that $\sum_j d_{ij} = a_i$ or there is a subset $J \subset N_n$, for which equality (1.4) holds.

We define the sets $I_1, J_1, \tilde{I}_1, \tilde{J}_1, J_1'$ relative to $M(a,b,D)$ as follows.

If $I_1 = \{i \varepsilon N_m : \sum_{j=1}^{n} d_{ij} = a_i\} \neq \emptyset$, then $\tilde{I}_1 = N_m \setminus I_1$, $J_1' = N_n$, $\tilde{J}_1 = \{j \varepsilon J_1' : \sum_{i\varepsilon I_1} d_{ij} < b_j\}$.

If $\sum_{j=1}^{n} d_{ij} > a_i$, $\forall i \varepsilon N_m$, then among all the subsets $J \subset N_n$

satisfying (1.4) we choose a set J_1' of maximum cardinality. It is easy to see that such a set is unique. Indeed, if there are two such sets $J_1'' \neq J_1'$, $|J_1''| = |J_1'|$ satisfying (1.4) then the set $J_1' \cup J_1''$ will also satisfy (1.4). This contradicts the manner in which J_1' was chosen.

Further, we put $J_1 = N_n \setminus J_1'$, $I_1 = \{i \in N_m : \sum_{j \in J_1'} d_{ij} < a_i\}$, $\tilde{I}_1 = N_m \setminus I_1$, $\tilde{J}_1 = \{j \in J_1' : \sum_{i \in I_1} d_{ij} < b_j\}$.

For the sets $I_1, J_1, \tilde{I}_1, \tilde{J}_1, J_1'$ we define the following transportation polytopes

$$M(a^1,b^1,D^1) = \{(x_{ij})_{i \in I_1, j \in J_1} : \sum_{j \in J_1} x_{ij} = a_i - \sum_{j \in J_1'} d_{ij},\ i \in I_1,$$

$$\sum_{i \in I_1} x_{ij} = b_j,\ j \in J_1,\ 0 \leqq x_{ij} \leqq d_{ij},\ (i,j) \in I_1 \times J_1\},$$

$$M(\tilde{a}^1,\tilde{b}^1,\tilde{D}^1) = (x_{ij})_{i \in \tilde{I}_1, j \in \tilde{J}_1} : \sum_{j \in \tilde{J}_1} x_{ij} = a_i,\ i \in \tilde{I}_1,$$

$$\sum_{i \in \tilde{I}_1} x_{ij} = b_j - \sum_{i \in I_1} d_{ij},\ j \in \tilde{J}_1,\ 0 \leqq x_{ij} \leqq d_{ij},\ (i,j) \in \tilde{I}_1 \times \tilde{J}_1\}.$$

By definition we consider that $M(a^1,b^1,D^1) = \emptyset$ if one of the sets I_1 or J_1 is empty. Similarly $M(\tilde{a}^1,\tilde{b}^1,\tilde{D}^1) = \emptyset$ if $\tilde{I}_1 = \emptyset$ or $\tilde{J}_1 = \emptyset$.

Clearly, $M(a^1,b^1,D^1)$ is a regularly truncated transportation polytope if $J_1 \neq \emptyset$.

Now consider the polytope $M(\tilde{a}^1,\tilde{b}^1,\tilde{D}^1)$. If it is empty then the theorem is proved (k=1). If it is a regularly truncated transportation polytope then we put $M(a^2,b^2,D^2) = M(\tilde{a}^1,\tilde{b}^1,\tilde{D}^1)$, $P = I_1 \times J_1'$, $Q = \tilde{I}_1 \times ((J_1 \cup J_1') \setminus \tilde{J}_1)$ and the theorem is proved. If $M(\tilde{a}^1,\tilde{b}^1,\tilde{D}^1)$ is not regularly truncated then in a similar way we define the sets $I_2, J_2, \tilde{I}_2, \tilde{J}_2, J_2'$ and the corresponding polytopes $M(a^2,b^2,D^2)$, $M(\tilde{a}^2,\tilde{b}^2,\tilde{D}^2)$. Clearly, after a finite number of steps we arrive at a polytope $M(\tilde{a}^s,\tilde{b}^s,\tilde{D}^s)$ which is either regularly truncated or empty. Since $\dim M(a,b,D) > 0$, there are k, $k \geq 1$, non-empty polytopes among the polytopes $M(a^p,b^p,D^p)$, $p \in N_s$. The sets P and Q are respectively

$$\bigcup_{p=1}^{s} (I_p \times J_p') \quad \text{and} \quad \bigcup_{p=1}^{s} (\tilde{I}_p \times ((J_p \cup J_p') \setminus \tilde{J}_p)).$$

The polytope $M(a,b,D)$ is clearly uniquely expressible as a product of regularly truncated transportation polytopes. This completes the proof. //

Corollary 1.4

$$f_0(M(a,b,D)) = \prod_{p=1}^{k} f_o(M(a^p,b^p,D^p)) ,$$

$$f_{d-1}(M(a,b,D)) = \sum_{p=1}^{k} f_{d_p-1}(M(a^p,b^p,D^p)) ,$$

$$\operatorname{diam} M(a,b,D) = \sum_{p=1}^{k} \operatorname{diam} M(a^p,b^p,D^p) ,$$

$$\dim M(a,b,D) = \sum_{p=1}^{k} \dim M(a^p,b^p,D^p) .$$

Corollary 1.5 *The polytope* $M(a,b,D)$ *is simple if and only if the polytopes* $M(a^p,b^p,D^p)$, $p \in N_k$, *are simple.*

1.3 Facets of a Regularly Truncated Polytope

The facets of a regularly truncated polytope of order $m \times n$ are non-empty sets of the form:

$$F^-_{ij}(a,b,D) = \{x \in M(a,b,D) : x_{ij}=0\} , \quad (i,j) \in N_m \times N_n ,$$

$$F^+_{ij}(a,b,D) = \{x \in M(a,b,D) : x_{ij}=d_{ij}<\min(a_i,b_j)\} , \quad (i,j) \in N_m \times N_n .$$

The following lemma is obvious.

Lemma 1.6 *The set* $F^-_{st}(a,b,D)$ $(F^+_{st}(a,b,D))$, $(s,t) \in N_m \times N_n$, *is a facet of the regularly truncated transportation polytope* $M(a,b,D)$ *of order* $m \times n$ *if and only if there is a matrix* $x \in F^-_{st}(a,b,D)$ $(x \in F^+_{st}(a,b,D))$ *such that*

$$0 < x_{ij} < \min\,(a_i,b_j,d_{ij}) \qquad \textit{for all} \quad (i,j) \neq (s,t) .$$

Theorem 1.7 *For* $m,n \geq 3$, *every integer of the form* $(m-1)(n-1)+k$, *where* $1 \leq k \leq mn+m+n-1$ *and only these integers can be the number of facets of a regularly truncated transportation polytope of order* $m \times n$.

Proof The number of facets of any regularly truncated transportation polytope $M(a,b,D)$ of order $m\times n$ clearly satisfies the constraints

$$(m-1)(n-1) < f_{d-1}(M(a,b,D)) \leqq 2mn .$$

The proof that any integer in this range equals the number of faces of some regularly truncated transportation polytope of order $m\times n$ will be carried out separately for the cases: $k=1$, $2 \leqq k \leqq (m-1)(n-1)+2$, $(m-1)(n-1)+3 \leqq k \leqq mn+m+n-1$.

Case 1. Let $k=1$. Consider the regularly truncated transportation polytope $M(a',b',D')$ of order $m\times n$, $m,n \geqq 2$, defined by the vectors $a' = ((n-1)(3m-2)+1,3,3,\dots,3)$, $b' = (3m-2,\dots,3m-2)$ and the matrix D' with elements

$$d'_{ij} = \begin{cases} 3m-2, & \text{if } i=1 , j \in N_{n-1} \\ 2, & \text{if } i=1 , j=n, \\ 3, & \text{if } i=2,3,\dots,m , j \in N_n . \end{cases}$$

By Lemma 1.6 we find that the facets of $M(a',b',D')$ are given by the sets $F^-_{ij}(a',b',D')$, $(i,j) \in \{2,3,\dots,m\}\times N_{n-1}$, $F^+_{1n}(a',b',D')$ and only these. Consequently

$$f_{d-1}(M(a',b',D')) = (m-1)(n-1)+1 .$$

Case 2. Let $2 \leqq k \leqq (m-1)(n-1)+2$. Consider the regularly truncated transportation polytope $M(a'',b'',D'')$ of order $m\times n$, $m,n \geqq 2$, $mn > 4$, defined by the vectors $a'' = (4m(n-1)-1,4,4,\dots,4)$, $b'' = (4m,4m,\dots,4m,4m-5)$ and by the matrix D'' with elements

$$d''_{ij} = \begin{cases} 4m, & \text{if } i=1 , j \in N_{n-1} , \\ 2, & \text{if } (i,j) \in H_1 \cup \{(1,n)\} , \\ 4 & \text{otherwise,} \end{cases}$$

where H_1 is some subset of pairs (i,j) taken from the set $\{2,3,\dots,m\}\times N_{n-1}$ whose cardinality equals $k-2$. As in the first case, by Lemma 1.6 we find that the faces of $M(a'',b'',D'')$ are precisely the sets

$F^-_{1n}(a'',b'',D'')$, $F^-_{ij}(a'',b'',D'')$, $(i,j)\in\{2,3,\dots,m\}\times N_{n-1}$, $F^+_{ij}(a'',b'',D'')$, $(i,j)\in H_1\cup\{(1,n)\}$. Hence

$$f_{d-1}(M(a'',b'',D'')) = (m-1)(n-1)+k .$$

Case 3. Let $(m-1)(n-1)+3\le k\le mn+m+n-1$. In this case it suffices to consider the regularly truncated transportation polytope $M(a^3,b^3,D^3)$ of order $m\times n$, $m,n\ge 3$ given by the vectors $a^3=(n,n,\dots n)$, $b^3=(m,m,\dots,m)$ and the matrix D^3 with elements

$$d^3_{ij} = \begin{cases} \max\left(\frac{m+1}{m-1}, \frac{n+1}{n-1}\right) & \text{for } (i,j)\in H_2 , \\ \min(m,n) & \text{for } (i,j)\notin H_2 , \end{cases}$$

where H_2 is some subset of pairs $(i,j)\in N_m\times N_n$ of cardinality $k-m-n+1$ and to verify that

$$f_{d-1}(M(a^3,b^3,D^3)) = (m-1)(n-1)+k . \quad //$$

Corollary 1.8 *Among the regularly truncated transportation polytopes of order* $m\times n$, $m,n\ge 3$, *there are* $(m-1)(n-1)$-*simplexes.*

§2 (k,t)-TRUNCATED TRANSPORTATION POLYTOPES

In this section we study the feasible set of a transportation problem with prohibitions; namely, we study polytopes $M_{k,t}(a,b) = \{(x_{ij})_{m\times n}\in M(a,b) : x_{ij}=0 \text{ if } n-t-1<j-i<k-m+1\}$, where $a=(a_1,\dots,a_m)$ and $b=(b_1,b_2,\dots,b_n)$ are vectors with real positive components, $0\le k,t\le \min(m,n)-1$ and k , t are integers.

The polytope $M_{k,t}(a,b)$ is called a (k,t)-*truncated transportation polytope of order* $m\times n$.

Figure 50 shows schematically an arbitrary matrix belonging to $M_{k,t}(a,b)$ (the shaded locations are those which can contain non-zero components).

The polytope $M_{k,t}(a,b)$ reduces to a classical transportation polytope if $k=t=0$.

2.1 An Existence Criterion

Recall that the equality $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ is a necessary and

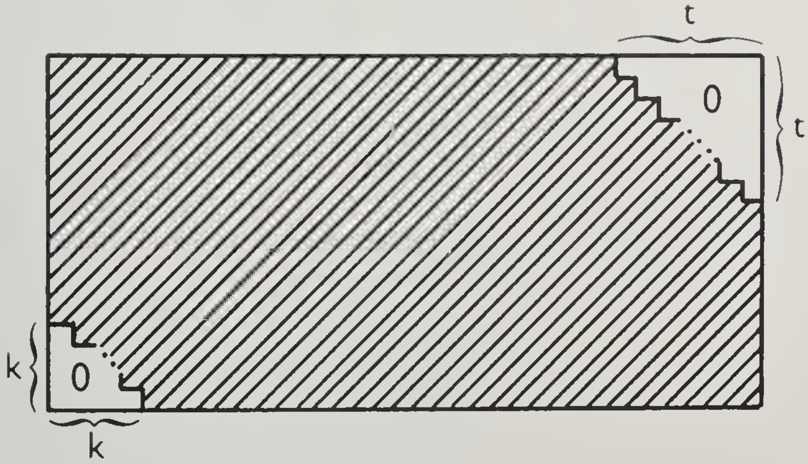


Fig. 50.

sufficient condition for the existence of a solution to the classical transportation problem. But this equation does not suffice to guarantee that a (k,t)-truncated transportation polytope should be non-empty.

Theorem 2.1 *Let* $m,n \geqq 2$, $k+t \geqq 1$. *The* (k,t)-*truncated transportation polytope* $M_{k,t}(a,b)$ *of order* $m\times n$ *is non-empty if and only if the following conditions are satisfied:*

1) *if* $kt > 0$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \,, \tag{2.1}$$

$$\sum_{i=1}^{s} a_i \leqq \sum_{j=1}^{s+n-t-1} b_j \qquad \forall s \in N_t \,, \tag{2.2}$$

$$\sum_{j=1}^{s} b_j \leqq \sum_{i=1}^{s+m-k-1} a_i \qquad \forall s \in N_k \,; \tag{2.3}$$

2) *if* $k=0$, $t>0$ (2.1) *and* (2.2)

3) *if* $t=0$, $k>0$ (2.1) *and* (2.3)

Proof Necessity. The necessity of (2.1) is obvious. Let $t>0$, $x^0 = (x^0_{ij})_{m\times n} \in M_{k,t}(a,b)$. Then from the way the matrix x^0 is constructed we have

$$\sum_{i=1}^{s} a_i = \sum_{i=1}^{s} \sum_{j=1}^{n} x^0_{ij} \leqq \sum_{j=1}^{s+n-t-1} \sum_{i=1}^{m} x^0_{ij} = \sum_{j=1}^{s+n-t-1} b_j \,, \ \forall s \in N_t \,,$$

Thus, constraint (2.2) is satisfied. The constraint (2.3) may be checked similarly.

Sufficiency. Let $kt > 0$. Suppose that for the polytope $M_{k,t}(a,b)$ the conditions (2.1)-(2.3) are satisfied. We will construct a matrix $(x^0_{ij})_{m\times n}$ which is a point of the polytope $M_{k,t}(a,b)$. For definiteness suppose $a_m \leqq b_n$. If $b_n < a_m$ then consider the polytope $M_{t,k}(b,a)$ instead of $M_{k,t}(a,b)$.

From the conditions of the theorem there is a number r such that $0 \leqq r \leqq m-t-1$, and

$$\sum_{i=m-r}^{m} a_i \leqq b_n < \sum_{i=m-r-1}^{m} a_i .$$

Let $x^0_{mj} = 0$, $j = k+1, k+2, \ldots, n-1$. Also let $x^0_{in} = a_i$, $i = m-r, m-r+1, \ldots, m$, if $r = m-t-1$, and

$$x^0_{in} = \begin{cases} a_i & \text{for } i = m-r, m-r+1, \ldots, m, \\ b_n - \sum\limits_{i=m-r}^{m} a_i & \text{for } i = m-r-1, \\ 0 & \text{for } i < m-r-1 \end{cases}$$

if $r < m-t-1$.

Consider the (k_1,t_1)-truncated transportation polytope $M_{k_1,t_1}(a',b')$ of order $(m-r-1)(n-1)$ defined by the numbers $t_1 = t-1$,

$$k_1 = \begin{cases} k-r-1, & \text{if } k-r-1 > 0, \\ 0, & \text{if } k-r-1 \leqq 0, \end{cases}$$

and by the vectors $a' = (a'_1, a'_2, \ldots, a'_{m-r-1})$, $b' = (b'_1, b'_2, \ldots, b'_{n-1})$, where $a'_i = a_i$, $i \in N_{m-r-1}$, if $\sum\limits_{i=m-r}^{m} a_i = b_n$, and

$$a'_i = \begin{cases} a_i & \text{for } i \in N_{m-r-2}, \ r < m-2 \\ \sum\limits_{i=m-r-1}^{m} a_i - b_n & \text{for } i = m-r-1, \end{cases}$$

if $\sum\limits_{i=m-r}^{m} a_i < b_n$; $b'_j = b_j$, $j \in N_{n-1}$.

If $k_1 = t_1 = 0$ this polytope is a classical transportation polytope, and if $k_1 + t_1 \geqq 1$ then it is easily checked that all the conditions of the theorem are satisfied. Thus continuing the process described we obtain a matrix $(x^0_{ij})_{m\times n} \varepsilon M_{k,t}(a,b)$.

When $t=0$ or $k=0$ the proof of sufficiency is similar except that the number r is given by the inequalities

$$\sum_{i=m-r}^{m} a_i \leqq b_n < \sum_{i=1}^{m} a_i \ , \ 0 \leqq r \leqq m-1 \ , \text{ or } \sum_{j=n-r}^{n} b_j \leqq a_m < \sum_{j=1}^{n} b_j \ , \ 0 \leqq r \leqq n-1.$$

This completes the proof. //

We remark that in the case $m=n$, $k=t=n-2$, this theorem reduces to Theorem 1 in Lev (1972).

2.2 Polytopes of Maximum Dimension

Every vertex $x=(x_{ij})_{m\times n}$ of $M_{k,t}(a,b)$ is constructed so that $x_{ij}=0$ if $j-i>n-t-1$ or if $i-j>m-k-1$. From the equalities

$$|\{(i,j)\ \varepsilon\ N_m\times N_n\ :\ i-j>m-k-1\}| = k(k+1)/2 \ ,$$

$$|\{(i,j)\ \varepsilon\ N_m\times N_n\ :\ j-i>n-t-1\}| = t(t+1)/2 \ ,$$

we conclude that the maximum dimension of a (k,t)-truncated transportation polytope of order $m\times n$ cannot be greater than $d=(m-1)(n-1)-\{k(k+1)+t(t+1)\}/2$. On the other hand for $m,n>2$, $m+n-k-t>3$, a (k,t)-truncated transportation polytope whose dimension is d is given by $M_{k,t}(a,b)$ where

$$a_i = \sum_{j=1}^{n} x^0_{ij} \ , \ i\ \varepsilon\ N_m \quad , \quad b_j = \sum_{i=1}^{m} x^0_{ij} \ , \ j\ \varepsilon\ N_n \ .$$

Here

$$x^0_{ij} = \begin{cases} 0, & \text{if } k-m+1>j-i>n-t-1, \\ 1 & \text{otherwise.} \end{cases}$$

Note that when $m+n-k-t\leqq 3$, a non-empty (k,t)-truncated polytope of order $m\times n$ always degenerates to a point.

Thus the maximum dimension of a (k,t)-truncated transportation polytope is given by the number d .

<u>Theorem 2.2</u> *Let* $m,n\geqq 2$, $k+t\geqq 1$, $m+n-k-t>3$. *The following conditions are necessary and sufficient for a non-empty* (k,t)-*truncated transportation polytope* $M_{k,t}(a,b)$ *of order* $m\times n$ *to have maximum*

dimension:

1) *if* $kt > 0$,

$$\sum_{i=1}^{s} a_i < \sum_{j=1}^{s+n-t-1} b_j \qquad \forall s \in N_n , \qquad (2.4)$$

$$\sum_{j=1}^{s} b_j < \sum_{i=1}^{s+m-k-1} a_i \qquad \forall s \in N_k ; \qquad (2.5)$$

2) *if* $t = 0$, $k > 0$ *then* (2.5);

3) *if* $k = 0$, $t > 0$ *then* (2.4).

Proof Necessity. Case 1). Suppose that one of the conditions (2.4) or (2.5) is not satisfied. Then, by Theorem 2.1, there is a number $r \in N_{\max(k,t)}$, for which one of the equalities $\sum_{i=1}^{r} a_i = \sum_{j=1}^{r+n-t-1} b_j$ or $\sum_{j=1}^{r} b_j = \sum_{i=1}^{r+m-k-1} a_i$ is true. For definiteness suppose that the first of these is true. This means that for any point of $M_{k,t}(a,b)$ we have $x_{ij} = 0$, $i = r+1, r+2, \ldots, m$, $j = 1,2,\ldots,r+n-t-1$. Hence $\dim M_{k,t}(a,b) < d$. This contradiction shows the necessity of conditions (2.4),(2.5).

The proof for cases 2) and 3) is similar.

Sufficiency. Consider the case $kt > 0$. The other cases have a similar proof. Since $M_{k,t}(a,b)$ satisfies conditions (2.4) and (2.5), for every pair $(p,q) \in Q = \{(i,j) \in N_m \times N_n : i-j \leqq m-k-1 , j-i \leqq n-t-1\}$ there is a point $x^{(p,q)} = (x_{ij})_{m\times n}$ in $M_{k,t}(a,b)$ for which $x_{pq} > 0$. Therefore the point $x^0 = (x^0_{ij}) = \sum_{(p,q)\in Q} \alpha_{pq} x^{(p,q)}$ of $M_{k,t}(a,b)$, where $\sum_{(p,q)\in Q} \alpha_{pq} = 1$, $0 < \alpha_{pq} < 1$, satisfies the conditions $x^0_{ij} > 0$, $(i,j) \in Q$. Hence the polytope $M_{k,t}(a,b)$ has maximum dimension. //

2.3 A Representation Theorem

Let $m+n-k-t \geqq 3$. Then the (k,t)-truncated transportation polytope $M_{k,t}(a,b)$ of order $m\times n$ is called *regular* if its dimension is maximum, that is

$$\dim M_{k,t}(a,b) = d = (m-1)(n-1) - \tfrac{1}{2}\{k(k+1) + t(t+1)\} .$$

In particular, every classical transportation polytope is regular.

Clearly, any (k,t)-truncated transportation polytope of order $m\times n$ with $\min(m,n)=2$ is either regular or degenerates to a point.

For a (k,t)-truncated transportation polytope $M_{k,t}(a,b)$ of order $m\times n$ we define the set $Q = \{(i,j)\varepsilon N_m\times N_n : x_{ij}=0,\ x\ \varepsilon\ (x_{ij})\ \varepsilon\ M_{k,t}(a,b)\}$

The following theorem is analagous to Theorem 1.3.

Theorem 2.3 *Every* (k,t)*-truncated transportation polytope of order* $m\times n$, $m,n>2$, *which is not regular and which does not degenerate to a point may be represented uniquely in the form of a product of regular polytopes, that is*

$$M_{k,t}(a,b) = M_{k_1,t_1}(a^1,b^1) \otimes M_{k_2,t_2}(a^2,b^2) \otimes \dots \otimes M_{k_p,t_p}(a^p,b^p) \otimes R(Q) \ ,$$

where $R(Q)$ *is a point with coordinates* $x_{ij}=0$, $\forall(i,j)\ \varepsilon\ Q$.

It is easily seen that any matrix of a (k,t)-truncated transportation polytope $M_{k,t}(a,b)$ which is not regular and which does not degenerate to a point takes the form given in Figure 51. Here the unshaded region represents the set of zero components and $x^\ell = (x^\ell_{ij})_{i\varepsilon I_\ell, j\varepsilon J_\ell}$ is some matrix of the polytope $M_{k_\ell,t_\ell}(a^\ell,b^\ell)$, $\ell\ \varepsilon\ N_p$.

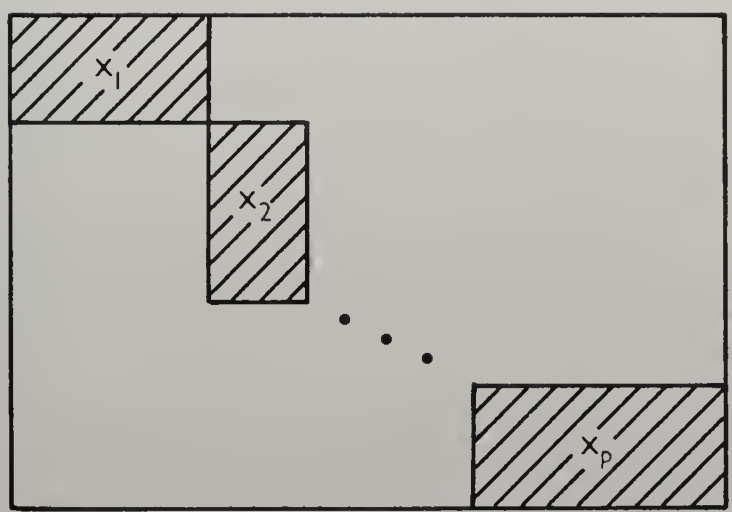


Fig. 51.

Corollary 2.4

$$f_0(M_{k,t}(a,b)) = \prod_{\ell=1}^{p} f_0(M_{k_\ell,t_\ell}(a^\ell,b^\ell)) ,$$

$$\operatorname{diam} M_{k,t}(a,b) = \sum_{\ell=1}^{p} \operatorname{diam} M_{k_\ell,t_\ell}(a^\ell,b^\ell) ,$$

$$\dim M_{k,t}(a,b) = \sum_{\ell=1}^{p} \dim M_{k_\ell,t_\ell}(a^\ell,b^\ell) ,$$

$$f_{d-1}(M_{k,t}(a,b)) = \sum_{\ell=1}^{p} f_{d_\ell-1}(M_{k_\ell,t_\ell}(a^\ell,b^\ell)) .$$

Here d_ℓ *is the dimension of the polytope* $M_{k_\ell,t_\ell}(a^\ell,b^\ell))$.

2.4 Simplexes

Theorem 2.5

Let $k+t \geqq 1$, $d = (m-1)(n-1) - \frac{1}{2}\{k(k+1) + t(t+1)\}$. *Among the regular* (k,t)*-truncated transportation polytopes of order* $m\times n$, $m,n \geqq 2$, *there are* d*-simplexes only in the cases:*

1) $m+n-k-t = 3$;
2) $(k-1)(t-1) = 0$;
3) $\min(m,n) = 3$, $\max(m,n) \geqq 3$, $\max(k,t) = 2$;
4) $m = n = 4$, $\min(k,t) = 0$, $\max(k,t) = 3$.

Note that in the case of the classical transportation polytopes of order $m\times n$ there are $\max(m,n)$-simplexes only in the case $\min(m,n) = 2$ (see §5, Ch.6).

Proof Clearly, a regular (k,t)-truncated transportation polytope $M_{k,t}(a,b)$ of order $m\times n$ is a d-simplex if and only if it has $d+1$ vertices. In case 1) every regular $(k,1)$-truncated transportation polytope is a 0-simplex.

Case 2). Let, for example, $t=1$. Consider the regular $(k,1)$-truncated transportation polytope $M_{k,1}(a^0,b^0)$ of order $m\times n$ where $a^0 = (3n-4,3,3,\dots,3)$, $b^0 = (3,3,\dots,3m-4)$. For any vertex $x = (x_{ij})_{m\times n}$ of $M_{k,1}(a^0,b^0)$ the inequalities $x_{1j} > 0$, $j \in N_{n-1}$, $x_{in} > 0$, $i = 2,3,\dots,m$ are satisfied. Among the remaining components of the vertex x there is only one positive component, which can occur at any of the places : $(i,j) \in \{2,3,\dots,m\}\times N_{n-1}$, $i-j \leqq m-k-1$. Consequently, $f_0(M_{k,1}(a^0,b^0)) = d+1$.

Case 3). For definiteness, let $m=3$, $t=2$. Consider the regular $(k,2)$-truncated transportation polytope $M_{k,2}(a^1,b^1)$ of order $3\times n$, $n \geq 3$, defined by the vectors $a^1 = (4n-9,3,6)$, $b^1 = (4,4,\dots,4)$. It is easily seen that the elements of any matrix $x = (x_{ij})_{3\times n} \in M_{k,2}(a^1,b^1)$ are constructed in the following way : $x_{1j} > 0$, $j \in N_{n-2}$, $x_{2,n-1} > 0$, $x_{3,n-1} > 0$, $x_{3n} > 0$. Thus $f_0(M_{k,2}(a^1,b^1)) = d+1$.

Case 4). For definiteness, let $t=0$. It is clear that $f_0(M_{0,3}(a^2,b^2)) = d+1$ when $a^2 = (3.6.4.6)$, $b^2 = (6,4,6,3)$.

Since any two vertices of a d-simplex are adjacent, to complete the proof it remains to show that in all remaining cases there are non-adjacent vertices in any regular (k,t)-truncated transportation polytope $M_{k,t}(a,b)$ of order $m\times n$.

We suppose that $a_1 \geqq b_1$. If $a_1 < b_1$ then consider $M_{t,k}(b,a)$ instead of $M_{k,t}(a,b)$.

Let the integer p satisfy the inequalities

$$2 \leq p \leqq n-t \quad , \quad \sum_{j=1}^{p-1} b_j \leqq a_1 \leqq \sum_{j=1}^{p} b_j .$$

Now let

$$(i_1,j_1) = \begin{cases} (1,p+1), & \text{if } p=2 \text{ , } k=m-1 \text{ , } m \leqq n \text{ ,} \\ (2,p-1) & \text{otherwise,} \end{cases}$$

$$(i_1',j_1') = \begin{cases} (m-1,\ell), & \text{if } x_{ms} > 0 \text{ , } s=q,q+1,\dots,n \text{ ,} \\ (m,h) & \text{otherwise,} \end{cases}$$

where

$$q = \max(p+1,k+1) \quad , \quad \ell = \begin{cases} n-1 \text{ ,} & \text{if } t=m-1 \text{ , } m \leqq n \text{ ,} \\ n & \text{otherwise,} \end{cases}$$

$$h = \max\{s : q \leqq s \leqq n-1 \text{ , } x_{ms} = 0\}.$$

Since the parameters m,n,k,t do not satisfy conditions 1)-4), there is a vector $y = (y_{ij})_{m\times n}$ of $M_{k,t}(a,b)$ with components $y_{i_1,j_1} > 0$, $y_{i'_1,j'_1} > 0$. Together with the obvious equalities $x_{i_1,j_1} = 0$, $x_{i'_1,j'_1} = 0$, this implies that the vertices x and y are not adjacent.

§3 THE DISTRIBUTION POLYTOPE

Let $a = (a_1,a_2,\ldots,a_n)$ be a vector with real non-negative components.

Definition 3.1 The set

$$M(a) = \{x = (x_{ij})_{n\times n} : \sum_{j=1}^{n} x_{ij} = 1 , \ i \in N_n ,$$
$$\sum_{i=1}^{n} a_i x_{ij} = a_j , \ j \in N_n , \ x_{ij} \geqq 0 , \ (i,j) \in N_n \times N_n\} ,$$

is called a *distribution polytope* of order $n\times n$.

This polytope is clearly always non-empty.

If $a_1 = a_2 = \ldots = a_n > 0$ the distribution polytope $M(a)$ reduces to the assignment problem polytope. Thus, in this case the number of vertices of $M(a)$ is $n!$.

Reverz (1961) conjectured that the number of vertices of a distribution polytope $M(a)$ of order $n\times n$ defined by a positive vector a could not exceed $n!$. However, in 1964, Perfect & Mirsky (1964) proved the following theorem.

Theorem 3.1 *If all components of the vector* a *are positive and are not all equal, then the number of vertices of the distribution polytope* $M(a)$ *of order* $n\times n$, $n \geqq 3$ *is greater than* $n!$.

Proof Without loss of generality we can assume that $0 \leqq a_1 \leqq a_2 \leqq \ldots \leqq a_n = 1$. Then the non-singular affine map $y_{ij} = a_i x_{ij}$, $i \neq j$, $y_{jj} = a_j(x_{jj} - 1) + 1$ maps the polytope $M(a)$ into

$$R(a) = \{x \in M_n : x_{ii} \geqq 1 - a_i , \ i \in N_n\} .$$

Thus to prove the theorem it suffices to show that the number of vertices of the polytope $R(a)$ is greater than the number of vertices of the polytope M_n associated with the assignment problem. To do this we set up a bijection between the set of all vertices of M_n and a proper subset of the vertices of $R(a)$.

Let $y = (y_{ij})_{n \times n} \in \text{vert } M_n$ and let the permutation

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{pmatrix},$$

be such that

$$y_{ij} = \begin{cases} 1 & \text{if } j = \pi_i , \\ 0 & \text{if } j \neq \pi_i . \end{cases}$$

We describe a method of constructing a vertex $y^* = (y^*_{ij})_{n \times n} \in R(a)$ corresponding to the vertex y. To do this we express the permutation π as a product of independent cycles $\pi = \delta_1 \delta_2 \ldots \delta_r$. For each cycle $\delta = \langle b,c,d,\ldots,j,k \rangle$ whose length is greater than 1 we define the quantity $\mu_\delta = \max(1 - a_b, 1 - a_c, 1 - a_d, \ldots, 1 - a_j, 1 - a_k)$. Clearly $0 \leqq \mu_\delta < 1$. Let

$$y^*_{bb} = y^*_{cc} = y^*_{dd} = \cdots = y^*_{jj} = y^*_{kk} = \mu_\delta ,$$

$$y^*_{bc} = y^*_{cd} = y^*_{de} = \cdots = y^*_{jk} = y^*_{kb} = 1 - \mu_\delta ,$$

and for every cycle $\delta(m)$ of unit length we let $y^*_{mm} = 1$. All the remaining components are put equal to zero.

We show that the matrix y^* constructed in this way is a vertex of $R(a)$. Suppose the opposite. Then there exist two distinct matrices $z^1 = (z^1_{ij})_{n \times n}$, $z^2 = (z^2_{ij})_{n \times n} \in R(a)$ not equal to y^* such that $y^* = (z^1 + z^2)/2$. This implies that if $y^*_{ij} = 0$ or 1 then $z^1_{ij} = z^2_{ij} = y^*_{ij}$. It remains to determine the positive elements in the rows and columns with indices $b,c,d,\ldots,j,k$.

Suppose, for definiteness that $\mu_\delta = 1 - a_b$. Then, since for any point of $R(a)$ it is true that $z_{bb} \geqq 1 - a_b$, we have that $y^*_{bb} = z^1_{bb} = z^2_{bb} = 1 - a_b$. Examining in succession the components at the locations $(b,c),(c,c),(c,d),\ldots,(j,j),(j,k),(k,k)$ and repeating this argument for each cycle δ whose length exceeds unity, we obtain $y^* = z^1 = z^2$. This contradiction proves that $y^* \in \text{vert } R(a)$.

It is easily seen that the matrices y and y^* have positive off-diagonal elements in exactly the same positions. Hence $y_1 \neq y_2 \Rightarrow y^*_1 \neq y^*_2$. Thus, we have found $n!$ distinct vertices of the polytope $R(a)$.

We now show that there is at least one vertex z of $R(a)$ which is distinct from those already listed. Since the numbers $a_1, a_2, \ldots, a_n$ are not all equal there is an $a_s < 1$, where $1 \leqq s \leqq n-1$. We examine two possible cases.

Case 1. Let $s = 1$. Then

$$
z = \left[\begin{array}{ccc|cccc}
1-a_1 & 0 & a_1 & & & & \\
a_1 & 0 & 1-a_1 & & & & \\
0 & 1 & 0 & & & & \\
\hline
 & & & 1 & & & \\
 & & & & 1 & & \\
 & & & & & \ddots & \\
 & & & & & & 1
\end{array}\right].
$$

Case 2. Let $s > 1$. Then if $a_{s-1} + a_s \geqq 1$

$$
\begin{array}{c}
\left[\begin{array}{cccc|ccc|cccc}
1 & & & & & & & & & & \\
 & 1 & & & & & & & & & \\
 & & \ddots & & & & & & & & \\
 & & & 1 & & & & & & & \\
\hline
 & & & & 1-a_{s-1} & 0 & a_{s-1} & & & & \\
 & & & & a_{s-1}+a_s-1 & 1-a_s & 1-a_{s-1} & & & & \\
 & & & & 1-a_s & a_s & 0 & & & & \\
\hline
 & & & & & & & 1 & & & \\
 & & & & & & & & 1 & & \\
 & & & & & & & & & \ddots & \\
 & & & & & & & & & & 1
\end{array}\right] = z \\
\underbrace{\qquad}_{s-2} \qquad\qquad\qquad\qquad\qquad\qquad \underbrace{\qquad}_{n-s-1}
\end{array}
$$

and if $a_{s-1} + a_s < 1$

$$
\begin{array}{c}
\left[\begin{array}{ccc|ccc|ccc}
1 & & & & & & & & \\
 & \ddots & & & & & & & \\
 & & 1 & & & & & & \\
\hline
 & & & 1-a_{s-1} & 0 & a_{s-1} & & & \\
 & & & 0 & 1-a_s & a_s & & & \\
 & & & a_{s-1} & a_s & 1-a_s-a_{s-1} & & & \\
\hline
 & & & & & & 1 & & \\
 & & & & & & & \ddots & \\
 & & & & & & & & 1
\end{array}\right] = z. \\
\underbrace{\qquad}_{s-2} \qquad\qquad\qquad\qquad\qquad\qquad \underbrace{\qquad}_{n-s-1}
\end{array}
$$

Here all elements not indicated are zero. It is easily verified that in each case z is a vertex of $R(a)$. Also, z is distinct from each of the vertices y^* constructed above, since $u(z)+v(z)\geqq n+1$ and $u(y^*)+v(y^*)=n$, where $u(x)$ is the number of diagonal elements of x equal to unity and $v(x)$ is the number of positive off-diagonal elements of x. //

Other results on the number of vertices of a distribution polytope have been obtained by Perfect & Mirsky (1964) and by Dubois (1973). These results are included in the exercises to this chapter.

EXERCISES

1. The regularly truncated transportation polytope $M(a,b,D)$ of order $m\times n$ is degenerate if and only if there exist partitions $N_m = I_1\cup I_2$, $N_n = J_1\cup J_2$, and also subsets (possibly empty) $I_t'\subseteq\{j:(i,j)\in I_t\times\overline{J}_t,\ d_{ij}<\min(a_i,b_j)\}$, $J_t'\subseteq\{i:(i,j)\in\overline{I}_t\times J_t,\ d_{ij}<\min(a_i,b_j)\}$, $t=1,2$, such that

$$\{x=(x_{ij})_{i\in I_t,j\in J_t}:\ \sum_{j\in J_t}x_{ij}=a_i-\sum_{j\in J_t'}d_{ij},\ \forall i\in I_t,$$

$$\sum_{i\in I_t}x_{ij}=b_j-\sum_{i\in I_t'}d_{ij},\forall j\in J_t,\ 0\leqq x_{ij}\leqq d_{ij},\forall(i,j)\in I_t\times J_t\}\neq\emptyset,\ t=1,2.$$

2. Is it true that for every degenerate truncated transportation polytope of order $m\times n$ there exists a non-degenerate truncated transportation polytope of the same order whose number of vertices is not smaller?

3. (Kravtsov & Korzinkov 1979). For every non-degenerate classical transportation polytope $M(a,b)$ of order $m\times n$, $2\leqq m\leqq n$, there exists a pair of indices $(s,t)\in N_m\times N_n$ and a matrix $D=(d_{ij})_{m\times n}$ such that $f_0(M(a,b,D)\geqq f_0(M(a,b))+(m-2)f_0(M(a^s,b^k))$, where $a^s=(a_1,a_2,\ldots,a_{s-1},a_s-b_k,a_{s+1},\ldots,a_m)$, $b^k=(b_1,b_2,\ldots,b_{k-1},b_{k+1},\ldots,b_n)$. Using this fact show that, for $2\leqq m\leqq n-1$ the following inequality holds:

$$\psi(m,n)\geqq\phi(m,n)+(m-2)((n-1)^{m-1}+(n-1)(mn-2m-n+1)),$$

where $\psi(m,n)$ is the maximum number of vertices in the class of regularly truncated transportation polytopes of order $m\times n$.

4. Let $M(a,b,D)=M(a^1,b^1,D^1)\otimes M(a^2,b^2,D^2)\otimes\ldots\otimes M(a^p,b^p,D^p)\otimes R(P,Q)$ and $m,n\geqq 2$. A generalization of the first two assertions of Corollary 1.4 are given by the following formulae:

$$f_k(M(a,b,D)) = \sum_{\substack{i_1+i_2+..+i_p=k \\ i_1..i_p \geqq 0}} \prod_{s=1}^{p} f_{i_s}(M(a^s,b^s,D^s)) ,$$

for $k = 0,1,2,...,mn-m-n$. Here, by definition

$$f_i(M(a',b',D')) = \begin{cases} 0, & \text{if } i > (m'-1)(n'-1), \\ 1, & \text{if } i = (m'-1)(n'-1), \end{cases}$$

where the polytope $M(a',b',D')$ has order $m' \times n'$.

5. (Kravtsov & Korzinkov 1979). Let $m,n \geqq 2$, q_i , h_i , $i \in N_m$ be natural numbers such that $\sum_{i=1}^{k} q_i \leqq m$, $\sum_{i=1}^{k} h_i \leqq n$. Every number of the form $\rho = \sum_{i=1}^{k} (q_i-1)(h_i-1)$, and only these numbers, can equal the dimension of a truncated transportation polytope of order $m \times n$.

6. (Kravtsov & Korzinkov 1979). Let $n \geqq 3$. Every number of the form $n+k$, $k = 0,1,2,...,n$, and only these numbers can equal the number of facets of a regularly truncated transportation polytope of order $2 \times n$.

7. (Kravtsov & Korzinkov 1979). Let $m,n \geqq 2$, $mn > 4$, $1 \leqq k \leqq (m-1)(n-1)+2$. Then, the minimum number of vertices in the class non-degenerate regularly truncated transportation polytopes of order $m \times n$ with $(m-1)(n-1)+k$ facets is given by $k(m-1)(n-1) - k + 2$.

8. There is no non-degenerate regularly truncated transportation polytope $M(a,b,D)$ of order $m \times n$, $m,n \geqq 3$, whose number of vertices satisfies the inequalities

$$(m-1)(n-1)+1 < f_0(M(a,b,D)) < 2(m-1)(n-1) .$$

9. Can any number $t \in N_n$ equal the number of vertices of a truncated transportation polytope of order $m \times n$, $2 \leqq m \leqq n$?

10. Formulate conditions on the components of the vectors a and b and the elements of the matrix D which guarantee that a truncated transportation polytope reduces to a point.

11. The number of bases of a (k,t)-truncated transportation polytope of order $m \times n$ is given by

$$c \cdot \frac{(m-1)!\,(n-1)!}{(m-k-1)!\,(n-k-1)!} m^{n-k-1} n^{m-k-1} , \quad \text{where} \quad 0 < c < 1 .$$

12. Necessary conditions for a truncated transportation polytope $M(a,b,D)$ of order $m \times n$ to be non-empty are given by:

$$\sum_{j=1}^{n} d_{ij} \geqq a_i \ , \ \forall i \in N_m \quad ; \quad \sum_{i=1}^{m} d_{ij} \geqq b_j \ , \ \forall j \in N_n \ .$$

Show that these conditions are not sufficient when $m,n > 2$.

13. The feasible set for a transportation problem with prohibitions is a transportation polytope $M(a,b,D)$ of order $m \times n$ with matrix D whose elements satisfy the conditions

$$d_{ij} \begin{cases} = 0 \ , & \text{if } (i,j) \in \alpha \ , \\ \geqq \min(a_i, b_j) \ , & \text{if } (i,j) \notin \alpha \end{cases}$$

where α is the set of prohibited locations, that is, some non-empty subset of the set $N_m \times N_n$. Such a polytope is called α-*truncated* . Using similar arguments to those used in the proof of Theorem 2.1, Ch. 6, obtain the following formula for the number of bases $\beta'(m,n)$ of a non-empty α-truncated transportation polytope of order $m \times n$, $m,n \geqq 2$ (it is assumed that no column of the constraint matrix (1.1), Ch. 6, corresponding to an element in the set α enters into the basis)

$$\beta'(m,n) = \prod_{j=2}^{n} \sum_{i=1}^{m} \lambda_{ij} \begin{vmatrix} \sum_{j=1}^{n} \lambda_{1j} + c_{11} & c_{12} & \cdots & c_{1m} \\ c_{21} & \sum_{j=1}^{n} \lambda_{2j} + c_{22} & \cdots & c_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ c_{m1} & c_{m2} & \cdots & \sum_{j=1}^{n} \lambda_{mj} + c_{mm} \end{vmatrix}$$

where

$$c_{ij} = - \sum_{j=2}^{n} \frac{\lambda_{ik}\lambda_{jk}}{\sum_{i=1}^{m} \lambda_{ik}} \ , \quad \lambda_{ij} = \begin{cases} 0, & \text{if } (i,j) \in \alpha \ , \\ 1, & \text{if } (i,j) \notin \alpha \ . \end{cases}$$

It is clear that this formula gives the number of spanning trees of a marked (labelled) bipartite graph for which the numbers

$n_i = \sum_{j=1}^{n} \lambda_{ij}$, $\forall i \in N_m$, $m_j = \sum_{i=1}^{m} \lambda_{ij}$, $\forall j \in N_n$, represent the degrees of the vertices and (λ_{ij}) is the incidence matrix of the graph.

For the case in which $n = n_1 \geqq n_2 \geqq \cdots \geqq n_m \geqq 1$, $m = m_1 \geqq m_2 \geqq \cdots \geqq 1$, $\lambda_{ij} = 0 \Rightarrow \lambda_{ik} = 0$ for all $k > j$, $\forall i \in N_m$, a formula for the number of spanning trees is given by (Yemelichev, Kononenko & Likhachev 1972, Yemelichev & Kononenko 1974):

$$\beta'(m,n) = \prod_{j=2}^{n} m_j \prod_{i=2}^{m} n_i \ .$$

14. (Yemelichev & Kravtsov 1975). Consider the α-truncated polytope (see Exercise 13) for the case in which α is given by

$$\alpha = \bigcup_{r=h}^{t} \Big(\{m_{r-1}+1, m_{r-1}+2, \ldots, m_r\} \times \{n_{s-r+h-1}+1, n_{s-r+h-1}+2, \ldots, n_s\} \Big) \ , \ a_i > 0 \ ,$$

$b_j > 0$, $2 \leqq h \leqq t \leqq s$, $0 = m_0 < m_1 < \ldots < m_t = m$, $0 = n_0 < n_1 < \ldots < n_s = n$.

A necessary and sufficient condition for this polytope to be non-empty is given by:

$$\sum_{i=1}^{m_r} a_i \geqq \sum_{j=n_{s-r+h-2}+1}^{n_s} b_j \quad , \quad r = h-1, h, \ldots, t-1.$$

15. Consider the transportation problem T with constraints on partial sums of the variables:

minimize $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$; subject to $\sum_{j=1}^{n} x_{ij} \leqq a_i$, $\forall i \in N_m$;

$\sum_{i=1}^{m} x_{ij} = b_j$, $\forall j \in N_n$; $\sum_{i \in P_s} x_{ij} \geqq d_{sj}$, $\forall (s,j) \in N_k \times N_n$; $x_{ij} \geqq 0$, $\forall (i,j) \in N_m \times N_n$; where $c_{ij} \geqq 0$, $a_i > 0$, $b_j > 0$, $d_{sj} \geqq 0$, $\bigcup_{s=1}^{k} P_s = N_m$, $P_s \cap P_t = \emptyset$, for all $s \neq t$. Necessary and sufficient conditions for this problem to have a solution are given by:

$$\sum_{i=1}^{m} a_i \geqq \sum_{j=1}^{n} b_j \ ; \ \sum_{i \in P_s} a_i \geqq \sum_{j=1}^{n} d_{sj} \ , \ \forall s \in N_k \ ; \ b_j \geqq \sum_{s=1}^{k} d_{sj} \ , \ \forall j \in N_n \ .$$

Show, that the solution of problem T reduces to the solution of a classical transportation problem of order $m \times ((k+1)(n+1))$.

16. Let $R \subseteq N_m$, $\bigcup_{k=1}^{s} P_k = N_m$, $\bigcup_{r=1}^{t} Q_r = N_n$, $P_k \cap P_r = Q_k \cap Q_r = \emptyset$ for $k \neq r$. Consider the problem:

minimize $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$; subject to $\sum_{i=1}^{m} x_{ij} = b_j$, $\forall j \in N_n$;

$\sum_{j=1}^{n} x_{ij} = a_i$, $\forall i \in R$; $\sum_{j=1}^{n} x_{ij} \leqq a_i$, $\forall i \in \overline{R}$; $d_{kr}^{-} \leqq \sum_{i \in P_k} \sum_{j \in Q_r} x_{ij} \leqq d_{kr}^{+}$, $\forall (k,r) \in N_s \times N_t$, $x_{ij} \geqq 0$, $\forall (i,j) \in N_m \times N_n$.

Necessary and sufficient conditions for this problem to have a solution are given by:

$$\sum_{i\varepsilon R} a_i \leq \sum_{j=1}^{n} b_j \leq \sum_{i=1}^{m} a_i \ ; \ \sum_{k=1}^{s} \min(\alpha_k,\gamma_k) \geq \sum_{r\varepsilon L} \beta_r \ ; \ \alpha_k \geq 0 \ , \ \beta_r \geq 0 \ ,$$

$\forall k \varepsilon N_s$, $\forall r \varepsilon N_t$, $\forall L \subseteq N_t$, where

$$\alpha_k = \sum_{i\varepsilon P_k} a_i - \sum_{r=1}^{t} d^-_{kr} \ , \ \beta_r = \sum_{j\varepsilon Q_r} b_j - \sum_{k=1}^{s} d^-_{kr} \ , \ \gamma_k = \sum_{r\varepsilon L} (d^+_{kr} - d^-_{kr}) \ .$$

17. Let $N_m = \bigcup_{k=1}^{s} P_k$, $P_k \cap P_t = \emptyset$ for $k \neq t$. Consider the problem:

minimize $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij}$; subject to $(x_{ij})_{m\times n} \varepsilon M(a,b)$, and

$$d^-_{kj} \leq \sum_{i\varepsilon P_k} x_{ij} \leq d^+_{kj} \ , \ \forall (k,j) \varepsilon N_s \times N_n \ .$$

The existence of an index r , $1 \leq r \leq s$, for which $\sum_{i\varepsilon P_r} a_i = \sum_{j=1}^{n} d^-_{rj}$, is a necessary and sufficient condition for this problem to reduce to the solution of the following sub-problems:

1). $$\min \left\{ \sum_{i\varepsilon P_r} \sum_{j=1}^{n} c_{ij}x_{ij} : \sum_{i\varepsilon P_r} x_{ij} = d^-_{rj} \ , \forall j \varepsilon N_n \ , \ \sum_{j=1}^{n} x_{ij} = a_i \ , \ \forall i \varepsilon P_r \ , \ x_{ij} \geq 0 \ , \ \forall (i,j) \varepsilon P_r \times N_n \right\} ,$$

2). $$\min \left\{ \sum_{i\varepsilon \overline{P}_r} \sum_{j=1}^{n} c_{ij}x_{ij} : \sum_{i\varepsilon \overline{P}_r} x_{ij} = b_j - d^-_{rj} \ , \forall j \varepsilon N_n \ , \ \sum_{j=1}^{n} x_{ij} = a_i \ , \ \forall i \varepsilon \overline{P}_r \ , \ d^-_{kj} \leq \sum_{i\varepsilon P_k} x_{ij} \leq d^+_{kj} \ , \ \forall (k,j) \varepsilon (N_s \setminus \{r\}) \times N_n \ , \ x_{ij} \geq 0 \ , \ \forall (i,j) \varepsilon \overline{P}_r \times N_n \right\} ,$$ where $\overline{P}_r = N_m \setminus P_r$.

18. Formulate necessary and sufficient conditions on the components of the vectors a and b for the (k,t)-truncated transportation polytope to reduce to a point.

19. The number of bases of a (k,k)-truncated transportation polytope of order $n\times n$ is given by $\{(2+\sqrt{3})^n - (2-\sqrt{3})^n\}/2\sqrt{3}$, when $k=n-2$ (Yemelicheva 1974), and by $n^{2n-5}(n-2)(n^2-2n+2)$, when $k = 1$.

20. (Yemelicheva 1974). A necessary and sufficient condition for the (n-2,n-2)- truncated transportation polytope of order $n\times n$ to be degenerate is the existence of an index t, $1 \leq t \leq n-1$, for which at least one of the following conditions is satisfied:

$$1).\ \sum_{i=1}^{t} a_i = \sum_{j=1}^{t-1} b_j ,\ b_0 = 0 ,\quad 2).\ \sum_{i=1}^{t} a_i = \sum_{j=1}^{t} b_j ,\quad 3)\ \sum_{i=1}^{t} a_i = \sum_{j=1}^{t+1} b_j ,$$

$$4).\ \sum_{i=1}^{t} a_i = \sum_{j=1}^{t-1} b_j + b_{t+1} ,\quad 5)\ \sum_{j=1}^{t} b_j = \sum_{i=1}^{t-1} a_i + a_{t+1} ,\quad a_o = 0 .$$

21. (Yemelicheva 1974). The minimum number of vertices in the class of non-degenerate (n-2,n-2)-truncated transportation polytopes of order $n\times n$ is given by $(n - 2[n/2] + 2)3^{[n/2] - 1}$.

22. Every whole number from 0 to $n-1$, with the exception of $n-2$, and only these numbers, can equal the dimension of an (n-2,n-2)-truncated transportation polytope of order $n\times n$, $n > 2$.

23. (Yemelicheva 1974). Every whole number γ such that $n + [n/2] - 1 \leq \gamma \leq 2n - 1$, and only these numbers, can equal the number of (n-2)-faces of a non-degenerate (n-2,n-2)-truncated transportation polytope of order $n\times n$.

24. (Yemelicheva 1974). Let the components of the vectors $a = (a_1,a_2,\ldots,a_n)$ and $b = (b_1,b_2,\ldots,b_n)$ satisfy the conditions $a_i = b_i$ for all $i \in N_n$. Derive the following formulae:

$$\begin{aligned} f_0(M_{n-2,n-2}(a,b)) &= 2^{n-1} , \quad \text{if } a_1 < a_2 < \ldots < a_n ; \\ &= u_{n+1} , \quad \text{if } a_1=a_2=\ldots=a_n \text{ or if } a_1>a_2=a_3=\ldots=a_n=a ,\ a_1\geq 2a; \\ &= u_r \sum_{i=1}^{n-r+1} u_i + u_{n-r+1}\sum_{i=1}^{r} u_i - u_r u_{n-r+1} , \\ &\quad \text{if } a_r<a_1=a_2=\ldots=a_{r-1}=a_{r+1}=\ldots=a_n ,\quad 1\leq r\leq n . \end{aligned}$$

Here u_i is the i^{th}-Fibonacci number. In particular, it follows from this that the number of vertices in the (n-2,n-2)-truncated polytope for the assignment problem of order $n\times n$ ($a_i=b_i=1$) equals the $(n+1)^{th}$-Fibonacci number.

25. Let $m,n\geq 3$, $0\leq k,t\leq \min(m,n) - 3$. The maximum number of facets in the class of regularly (k,t)-truncated transportation polytopes of order $m\times n$ is given by the number

$$mn - \frac{k(k+1) + t(t+1)}{2} .$$

26. Consider the problem of minimizing $F_n(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$, where $x = (x_{ij})_{m\times n} \in M_{n-2,n-2}(a,b)$, and $a_i = b_i = 1$, $\forall i \in N_n$. For a fixed t , $1 \leq t \leq n$, let x^t be the optimal solution of the problem of minimizing the function $F_t(x)$ on the polytope $M_{t-2,t-2}(a,b)$. Derive the following recurrence relation:

$$F_t(x^t) = \min(F_{t-1}(x^{t-1}) + c_{tt} , F_{t-2}(x^{t-2}) + c_{t-1,t} + c_{t,t-1}) ,$$

for $3 \leq t \leq n$.

27. (Dubois 1973). When $a = b$ we call the classical transportation polytope $M(a,b)$ *symmetrical* . It is easy to see that such a polytope is a special case of a distribution polytope (when $a_i > 0$, $\forall i \in N_n$) .

The following statements are true:

1) if $0 < a_1 < a_2 = a_3 = \dots = a_n$, then

$$f_0(M(a,b)) = ((n-1)!)^2 \sum_{k=0}^{n-1} 1/k! ;$$

2) if $0 < a_1 = a_2 = \dots = a_{n-1} < a_n < 2a_1$, then

$$f_0(M(a,b)) = (n-1)! \sum_{k=0}^{n-1} 1/k! ;$$

3) if the symmetric transportation polytope $M(a,b)$ has the maximum number of vertices, then the numbers $a_1, a_2, \dots, a_n$ are all distinct.

4) the maximum number of vertices in the class of symmetric transportation polytopes of order $n\times n$ is not less than $\prod_{k=0}^{n-1} (k^2+1)$.

28. (Perfect & Mirsky 1964). Derive the following relations for the number of vertices of the distribution polytope $M(a)$ of order $n\times n$:

$f_0(M(a)) \geqq n!$, if $a_1=0$, $a_i>0$, $i=2,3,\dots,n$;

$f_0(M(a)) \geqq n^k (n-k)!$, if $a_1=a_2=\dots=a_k=0$, $a_i>0$, for $i=k+1,k+2,\dots,n$, $2\leqq k\leqq n-1$;

$f_0(M(a)) \geqq 2(n-1)(n-1)!$, if $a_1 = \frac{1}{2}$, $a_i = 1$, $i > 1$;

$f_0(M(a)) = n!$, if $a_1 = 0$, $a_2 = a_3 = \dots = a_n$.

29. The minimum number of vertices in the class of distribution polytopes of order $n\times n$ is equal to $n!$. Formulate necessary and sufficient conditions for a distribution polytope to have the minimum number of vertices.

30. A *generalized transportation polytope* of order $m \times n$ is a set of matrices $x = (x_{ij})_{m\times n}$ satisfying $\sum_{i=1}^{m} \alpha_{ij} x_{ij} = 1$, $\forall j \in N_n$, $\sum_{j=1}^{n} \beta_{ij} x_{ij} = 1$, $\forall i \in N_m$, $x_{ij} \geqq 0$, $\forall (i,j) \in N_m \times N_n$, where $(\alpha_{ij})_{m\times n}$ and $(\beta_{ij})_{m\times n}$ are matrices with real positive elements. Prove the following assertions:

1) any whole number from $(m-1)(n-1)$ to mn and only these numbers can equal the number of facets of a non-degenerate generalized transportation polytope of order $m\times n$, $m,n \geq 3$;

2) every whole number from 1 to $m+n-4$ can occur as the diameter of a non-degenerate generalized transportation polytope of order $m\times n$, $2 \leq m \leq n$, $n \geq 3$.

31. The solution of the transportation problem with bounded flows:

minimize $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$ subject to $(x_{ij})_{m\times n} \in M(a,b,D)$,

can be reduced to the following transportation problem with prohibitions:

minimize $\sum_{i=1}^{m(n+1)} \sum_{j=1}^{n(m+1)} c'_{ij} x_{ij}$ subject to $\sum_{i=1}^{m(n+1)} x_{ij} = b'_j$, $\forall j \in N_{n(m+1)}$,

$\sum_{j=1}^{n(m+1)} x_{ij} = a'_i$, $\forall i \in N_{m(n+1)}$, $x_{ij} \geqq 0$, $\forall (i,j) \in \alpha$, $x_{ij} = 0$, $\forall (i,j) \notin \alpha$,

where

$$a'_i = \begin{cases} a_i & \text{if } i \in N_m , \\ d_{kt} & \text{if } i = m+(k-1)n+t ,\ k \in N_m ,\ t \in N_n , \end{cases}$$

$$b'_j = \begin{cases} b_j & \text{if } j \in N_n , \\ d_{kt} & \text{if } j = kn+t ,\ k \in N_m ,\ t \in N_n , \end{cases}$$

$$c'_{ij} = \begin{cases} c_{kj} & \text{if } i = m+(k-1)n+j ,\ k \in N_m ,\ j \in N_n , \\ 0 & \text{otherwise,} \end{cases}$$

$\alpha = \{(m+(k-1)n+t,\ t) : k\in N_m,\ t\in N_n\} \cup \{(k, kn+t) : k\in N_m,\ t\in N_n\} \cup \{(m+k, n+k) : k\in N_{mn}\}$

The optimal solution of the original problem is given by the formula $x^*_{kj} = x^{**}_{ij}$, where $i = m+(k-1)n+j$, $k\in N_m$, $j\in N_n$, and (x^{**}_{ij}) is the optimal solution of the new problem having dimension $(mn+m)\times(mn+n)$.

32*. Prove or disprove the conjecture $\lim_{m,n\to\infty} \psi(m,n)/\beta(m,n) = 1$, where $\psi(m,n)$ is the maximum number of vertices in the class of regularly truncated transportation polytopes of order $m\times n$, and $\beta(m,n)$ is the number of bases of a regularly truncated transportation polytope of order $m\times n$ (see Ex. 49, Ch. 6).

33*. Is it true that almost all truncated transportation polytopes have the maximum number of facets?

34*. Let $k+t\geqq 1$. Is it true that the number of vertices of any regular (k,t)-truncated transportation polytope of order $m\times n$ does not exceed the number $2\phi(m,n)/\{k(k+1)+t(t+1)\}$?

35. Extend the results of §2 to the case of a transportation polytope with prohibitions of the following type:
$x_{ij}=0$, if $t_q-n+1>i-j>m-k_r-1$, $1\leqq r\leqq s$, $1\leqq q\leqq p$, where $1\leqq s$, $p\leqq \min(m,n)-1$, $0\leqq t_1$, $k_1\leqq \min(m,n)-1$, $k_1>k_2>\ldots>k_s$ and $t_1>t_2>\ldots>t_p$, k_r,t_q are whole numbers.

36. Consider the feasible set of a transportation problem with prohibitions which are organized in a special way:
$M^s_{m\times n}(a,b)=\{x=(x_{ij})_{m\times n}\,;\,x\varepsilon M(a,b)\,,\ x_{ij}=0\,,\ \forall(i,j)\varepsilon G\}$, where $G=(N_m\times N_{n_s})\setminus\bigcup_{p=1}^{s}(R_p\times Q_p)$. Here, $s\geqq 2$ is a natural number, $R_p=\{\overline{m_{p-1}+1,m_p}\}$, $Q_p=\{\overline{n_{p-1}+1,n_p}\}$, and m_p, n_p are numbers such that $0=n_0<n_1<n_2<\ldots<n_s<n_{s+1}=n$, $0=m_0<m_1<m_2<\ldots<m_s=m$. Every such polytope is called a *transportation polytope of order* $m\times n$ *with* s *blocks*. Prove the following assertions:

1) the polytope $M^s_{m\times n}(a,b)$ is non-empty if and only if the inequalities

$$\sum_{i\varepsilon R_p} a_i \geqq \sum_{j\varepsilon Q_p} b_j \quad ,\ \forall p\,\varepsilon\, N_s\ , \tag{*}$$

are satisfied, and at least one of them is satisfied strictly;

2) the polytope $M^s_{m\ n}(a,b)$ has maximum dimension d, where

$$d=m(n_{s+1}-n_s)-m\ \ -n+\sum_{p=1}^{s}(m_p-m_{p-1})(n_p-n_{p-1})\ ,$$

if and only if all of the inequalities (*) are strict.

A natural generalization of the classical transportation problem is the following p-indexed m-fold transportation problem:

minimize the expression

$$\sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_p=1}^{n_p} c_{i_1 i_2 \ldots i_p} x_{i_1 i_2 \ldots i_p}$$

subject to the constraints

$$\sum_{i_{k_1}=1}^{n_{k_1}} \sum_{i_{k_2}=1}^{n_{k_2}} \cdots \sum_{i_{k_m}=1}^{n_{k_m}} x_{i_1 i_2 \ldots i_p}$$

$$= \quad b_{i_1 \ldots i_{k_1-1} * i_{k_1+1} \ldots i_{k_2-1} * i_{k_2+1} \ldots \ldots i_{k_m-1} * i_{k_m+1} \ldots i_p} ,$$

$$\forall i_s \in N_{n_s} \ , \ s \in N_p \ , \ s \neq k_1, k_2, \ldots, k_m \ , \ 1 \leqq k_1 \leqq k_2 \leqq \ldots \leqq k_m \leqq p \ ,$$

$x_{i_1 i_2 \ldots i_p} \geqq 0$ for all sets $(i_1, i_2, \ldots, i_p)$, where

$b_{i_1 \ldots i_{k_1-1} * i_{k_1+1} \ldots i_{k_2-1} * i_{k_2+1} \ldots \ldots i_{k_m-1} * i_{k_m+1} \ldots i_p}$ and

$c_{i_1 i_2 \ldots i_p}$ are given real numbers, m is a given fixed number,

$1 \leqq m \leqq p-1$, $p \geqq 2$, $n_s > 1$, $\forall s \in N_p$. When $m = 1$ the problem is called *planar* . When $m = p-1$ it is called a *p-indexed transportation problem with axial sums.*

Many practically important problems in various branches of science, technology and production reduce to planar transportation problems.

Particularly well known are problems arising in planning the transportation of different types of goods.

Classical transportation problems arise in finding the optimal transportation scheme for carrying a homogeneous product from points of production to points of consumption. A triply-indexed planar transportation problem can arise in the transportation of an inhomogeneous product. A quadruply-indexed planar transportation problem occurs in considering models of transporting an inhomogeneous product using several means of transport. In general, if p factors have to be considered in solving a transportation problem, this gives rise to a p-indexed planar transportation problem.

A p-indexed transportation problem with axial sums arises from a transportation problem in which the points of production produce a semi-finished product which requires finishing before delivery to the points of consumption. The finishing of the product is carried out at intermediate points which can be classified into $p-2$ groups.

This chapter deals with the feasible sets of planar and axial transportation problems as well as the multi-indexed selection problem. The axial transportation polytopes are studied in §1 where we obtain degeneracy conditions, a criterion for a polytope to belong to the class of non-degenerate polytopes with the minimum number of vertices and a formula for the minimal number of integral vertices. Planar transportation polytopes are studied in §2 and well-known existence conditions are obtained. Bounds on the dimension are obtained together with conditions for the existence of simplexes. Section 3 is devoted to calculating the number of options in a multi-indexed selection problem.

§1 AXIAL TRANSPORTATION POLYTOPES

1.1 Definitions and Basic Properties

Definition 1.1 An *axial transportation polytope* of order $n_1 \times n_2 \times \ldots \times n_p$, $p \geq 2$, is a set $M(a^1, a^2, \ldots, a^p)$ of 'matrices' $x = (x_{i_1 i_2 \ldots i_p})$ whose elements satisfy the following constraints:

$$\sum_{i_1=1}^{n_1} \cdots \sum_{i_{s-1}=1}^{n_{s-1}} \sum_{i_{s+1}=1}^{n_{s+1}} \cdots \sum_{i_p=1}^{n_p} x_{i_1 i_2 \ldots i_p} = a^s_{i_s} \qquad (1.1)$$

$$\forall i_s \in N_{n_s}, \quad \forall s \in N_p,$$

$$x_{i_1 i_2 \ldots i_p} \geqq 0 \quad \text{for all p-tuples} \quad (i_1, i_2, \ldots, i_p) \;, \tag{1.2}$$

where $a^s = (a_1^s, a_2^s, \ldots, a_{n_s}^s)$ is a vector with real positive components.

When $p = 2$ this is clearly a classical transportation polytope.

Let

$$\sum_{i_s=1}^{n_s} a_{i_s}^s = K \qquad , \quad \forall s \in N_p \;. \tag{1.3}$$

Since the matrix x with elements

$$x_{i_1 i_2 \ldots i_p} = \prod_{s=1}^{p} a_{i_s}^s / K^{p-1} \;, \quad i_s \in N_{n_s} \;, \; s \in N_p$$

satisfies the constraints (1.1) and (1.2) it is clear that the axial transportation polytope $M(a^1, a^2, \ldots, a^p)$ is non-empty if and only if the condition (1.3) holds.

Proposition 1.1 *The dimension of the axial transportation polytope of order* $n_1 \times n_2 \times \ldots \times n_p$ *is equal to*

$$\prod_{s=1}^{p} n_s - \sum_{s=1}^{p} n_s + p - 1 \;.$$

We omit the proof. It can be constructed exactly on the lines of the proof of Proposition 1.1, Ch. 6.

1.2 A Criterion for Degeneracy

It was shown in §1, Ch. 6 that a necessary and sufficient condition for a classical transportation polytope to be degenerate is the existence of at least one pair of non-empty subsets $I \subset N_m$, $J \subset N_n$, such that $\sum_{i \in I} a_i = \sum_{j \in J} b_j$. In this section we generalize these conditions to the case of a multi-indexed axial transportation polytope and we show that they are sufficient for such a polytope to be degenerate.

We need first to define some new concepts.

Let there be given a system of vectors:

$b^s = (b_1^s, b_2^s, \ldots, b_{n_s}^s)$, $s \in N_p$, such that $b_{i_s}^s \geqq 0$, $i_s \in N_{n_s}$, $\forall s \in N_p$, $\sum_{i_s=1}^{n_s} b_{i_s}^s = K$, $\forall s \in N_p$, and a vector $r = (r_1, r_2, \ldots, r_p)$, $r_s \in N_{n_s}$ such that $b_{r_s}^s > 0$, $\forall s \in N_p$. We transform the system of vectors $b^1, b^2, \ldots, b^p$ into the system $c^s = (c_1^s, c_2^s, \ldots, c_{n_s}^s)$, $s \in N_p$, by the rule:

$$c_{i_s}^s = \begin{cases} b_{i_s}^s & \text{if } i_s \neq r_s , \\ b_{i_s}^s - \alpha & \text{if } i_s = r_s , \end{cases}$$

where $0 < \alpha \leqq \min_{1 \leqq s \leqq p} b_{r_s}^s$.

We say that the system of vectors $b^1, b^2, \ldots, b^p$ is transformed into the system $c^1, c^2, \ldots, c^p$ by means of an *elementary transformation* .

Definition 1.2 A system of vectors $a^1, a^2, \ldots, a^p$ which defines a non-empty polytope $M(a^1, a^2, \ldots, a^p)$, i.e. which satisfies the conditions (1.3) , is called a *normal system* .

Let $t(a)$ denote the number of zero components of the vector a .

Definition 1.3 A normal system of vectors $a^1, a^2, \ldots, a^p$ is called k-*reducible* ($k \geqq 1$) if there is a sequence of k elementary transformations which transform the system $a^1, a^2, \ldots, a^p$ into the system $\bar{a}^1, \bar{a}^2, \ldots, \bar{a}^p$ for which $\sum_{i=1}^{p} t(\bar{a}^i) > k$.

When $\alpha = \min_{1 \leqq s \leqq p} b_{r_s}^s$ the elementary transformation is called a *special transformation* . Clearly a sequence of special transformations which transforms a system of vectors $a^1, a^2, \ldots, a^p$ into a system of null vectors is at the same time a means of constructing a vertex of the polytope $M(a^1, a^2, \ldots, a^p)$. Hence, a normal system of vectors $a^1, a^2, \ldots, a^p$ is always L-reducible, where

$$L = \sum_{s=1}^{p} n_s - p + 1$$

If a normal system of vectors $a^1, a^2, \ldots, a^p$ is k-reducible for some $k < L$ then we will call it a *reducible system*.

Since the rank of system (1.1) is equal to L, every vertex of an axial transportation polytope of order $n_1 \times n_2 \times \ldots \times n_p$ contains no more than L nonzero components. Thus a degenerate point of an axial transportation polytope of order $n_1 \times n_2 \times \ldots \times n_p$ has less than L positive components.

Lemma 1.2 *An axial transportation polytope* $M(a^1, a^2, \ldots, a^p)$ *is degenerate if and only if the system of vectors* $a^1, a^2, \ldots, a^p$ *is reducible.*

The proof of the Lemma follows from the fact that the existence of a degenerate point of a polytope implies the existence of a degenerate vertex.

Definition 1.4 A normal system of vectors is called *divisible* if there are subsets $S_1, S_2 \subset N_p$, $S_1 \cap S_2 = \emptyset$, $S_1 \cup S_2 \neq \emptyset$ and non-empty subsets $J_s \subset N_{n_s}$, $s \in S_1 \cup S_2$ such that

$$\sum_{s \in S_1} \sum_{j \in J_s} a_j^s = \sum_{s \in S_2} \sum_{j \in J_s} a_j^s .$$

Theorem 1.3 *If the system of vectors* $a^1, a^2, \ldots, a^p$ *is divisible then the axial transportation polytope* $M(a^1, a^2, \ldots, a^p)$ *is degenerate.*

Proof By Lemma 1.2 it suffices to show that every divisible system of vectors $a^1, a^2, \ldots, a^p$ is reducible. We prove this by induction on the number $t = |S_1| + |S_2|$ (see Definition 1.4). For the case $t = 2$ the assertion is trivial.

Let

$$\sum_{j \in J_s} a_j^s = a_s , \quad \sum_{j \in \bar{J}_s} a_j^s = b_s , \quad s \in S_1 \cup S_2 ,$$

where $\bar{J}_s = N_{n_s} \setminus J_s$, and let $\gamma = \min\left(\min_{s \in S_1 \cup S_2} a_s , \min_{s \in S_1 \cup S_2} b_s \right)$.

First suppose that $\gamma = a_{s_0}$, $s_0 \in S_1$. Then there is a sequence of special transformations which transforms the system of vectors

$a^1, a^2, \ldots, a^p$ into a system $\bar{a}^1, \bar{a}^2, \ldots, \bar{a}^p$ such that

$$\sum_{s \in S_1 \setminus \{s_0\}} \sum_{j \in J_s} \bar{a}_j^s = \sum_{s \in S_2} \sum_{j \in J_s} \bar{a}_j^s .$$

By the induction hypothesis, the system so obtained is reducible. When $s_0 \in S_2$ the proof is similar.

Now suppose that $\gamma = b_{s_0}$. Assume, for definiteness, that $|S_1| > |S_2|$. Choose a subset $S \subset S_1 \setminus \{s_0\}$ whose cardinality is $|S_1| - |S_2|$. Then

$$\sum_{s \in S_1'} \sum_{j \in J_s'} a_j^s = \sum_{s \in S_2'} \sum_{j \in J_s'} a_j^s ,$$

where

$$S_1' = S_1 \setminus S , \quad S_2' = S_2 \cup S ,$$

$$J_s' = \begin{cases} J_s , & \text{if } s \in S , \\ \bar{J}_s , & \text{if } s \in (S_1 \setminus S) \cup S_2 . \end{cases}$$

Consequently, as in the previous case, there is a sequence of special transformations which transforms our system into a reducible system. //

1.3 Polytopes with the Minimum Number of Vertices

In this section we describe non-degenerate axial transportation polytopes with the minimum number of vertices and we also derive a formula for determining this number. Corresponding results for classical transportation polytopes were obtained in §5, Ch. 6. In proving the results of this section certain technical complications arise. We therefore note only the main lines of the argument and do not dwell on the details.

For the remainder of this section we assume that $n_1 \geqq n_2 \geqq \cdots \geqq n_p \geqq 1$, $a_1^s \geqq a_2^s \geqq \cdots \geqq a_{n_s}^s$, $\forall s \in N_p$, $a_1^s < a_1^{s+1}$ if $n_s = n_{s+1}$. (The case $a_1^s = a_1^{s+1}$ cannot occur since the polytope $M(a^1, a^2, \ldots, a^p)$ is assumed to be non-degenerate (see §1.2)).

We present some definitions which are necessary for this section.

Definition 1.5 The axial transportation polytope $M(a^1.a^2,\dots,a^p)$ of order $n_1\times n_2\times\dots\times n_p$ is called *regular* if

$$a^s_{n_s} + \sum_{k=s+1}^{p} a^k_1 > (p-s)K , \qquad \forall s \in N_{p-1} .$$

We recall that an axial transportation polytope of order $n_1\times n_2\times\dots\times n_p$ is non-degenerate if each of its vertices contains exactly $\sum_{s=1}^{p} n_s - p + 1$ positive components.

Lemma 1.4 *Every regular axial transportation polytope of order* $n_1\times n_2\times\dots\times n_p$ *is non-degenerate.*

This lemma can be proved using induction on p. Note that the lemma is obvious when $p=2$.

Let $\mathcal{M}_p$ denote the class of all regular axial transportation polytopes of order $n_1\times n_2\times\dots\times n_p$.

Lemma 1.5 *If* $M(a^1,a^2,\dots,a^p) \in \mathcal{M}_p$ *, then the number of vertices of this polytope is equal to* $\prod_{s=2}^{p}\left(\prod_{i=1}^{s-1} n_i\right)^{n_s-1}$.

In proving this lemma we use the fact that every vertex $x^0 = (x^0_{i_1 i_2 \dots i_p})$ of $M(a^1,a^2,\dots,a^p) \in \mathcal{M}_p$ has the following form:

1) $x^0_{i_1 1 1 \dots 1} > 0$, $\qquad \forall i_1 \in N_{n_1}$;

2) for any $s \in \{2,3,\dots,p\}$ and every $i_s \in \{2,3,\dots,n_s\}$ there is a vector $(i^0_1,i^0_2,\dots,i^0_{s-1})$, $i^0_k \in N_{n_k}$, $k \in N_{s-1}$ such that

$x^0_{i^0_1 i^0_2 \dots i^0_{s-1} i^0_s 1 1 \dots 1} > 0$, $x^0_{i_1 i_2 \dots i_{s-1} i_s 1 1 \dots 1} = 0$ for all $(i_1,i_2,\dots,i_{s-1}) \neq (i^0_1,i^0_2,\dots,i^0_{s-1})$. Hence the number of vertices of any polytope $M(a^1,a^2,\dots,a^p) \in \mathcal{M}_p$ is equal to the product of the number of choices of $(n_s - 1)$ numbers, with repetitions, from the sets $\{1,2,\dots,\prod_{i=1}^{s-1} n_i\}$, $s = 2,3,\dots,p$, that is $\prod_{s=2}^{p}\left(\prod_{i=1}^{s-1} n_i\right)^{n_s-1}$.

Lemma 1.6 *If the non-degenerate axial transportation polytope* $M(a^1,a^2,\dots,a^p) \notin \mathcal{M}_p$ *, then the number of its vertices is greater than* $\prod_{s=2}^{p}\left(\prod_{i=1}^{s-1} n_i\right)^{n_s-1}$.

The proof of this lemma can be carried out along the lines of the proof of the analagous theorem for the classical transportation polytopes (see §5, Ch. 6).

Lemmas 1.5 and 1.6 yield immediately the following theorems.

Theorem 1.7 (Yemelichev, Kononenko & Likhachev 1972) *The minimum number of vertices in the class of non-degenerate axial transportation polytopes of order* $n_1 \times n_2 \times \dots \times n_p$ *is equal to* $\prod_{s=2}^{p}\left(\prod_{i=1}^{s-1} n_i\right)^{n_s-1}$.

Theorem 1.8 (Yemelichev, Kononenko & Likhachev 1972) *A non-degenerate axial transportation polytope has the minimum number of vertices if and only if it is regular.*

1.4 The Minimum Number of Integral Vertices

In this section we assume that the vectors $a^1,a^2,\dots,a^p$ defining an axial transportation polytope are integral.

As we have mentioned earlier (§1.2) a sequence of special transformations which transform the system of vectors $a^1,a^2,\dots,a^p$ into a system of zero vectors gives, at the same time, a method of constructing a vertex of $M(a^1,a^2,\dots,a^p)$. Thus, in any polytope $M(a^1,a^2,\dots,a^p)$ defined by integral vectors, there exist integral vertices. On the other hand, since the constraint matrix in (1.1) is not unimodular for $p > 2$, we have by Theorem 2.1, Ch. 4, that there are polytopes with non-integral vertices. An example of such a polytope is the polytope $M(a^1,a^2,a^3)$ of order 2×2×2 defined by the vectors $a^1 = a^2 = a^3 = (1.1)$. Indeed, the matrix $(x_{i_1 i_2 i_3})_{2\times 2\times 2}$ with elements

$$x_{i_1 i_2 i_3} = \begin{cases} 1/2 & \text{if } (i_1,i_2,i_3) \in \{(1,1,1),(2,1,2), \\ & \qquad (1,2,2),(2,2,1)\}, \\ 0 & \text{otherwise,} \end{cases}$$

is a vertex of this polytope.

We solve here the problem of finding the minimum number of integral vertices in the class of axial transportation polytopes of order $n_1 \times n_2 \times \ldots \times n_p$.

Let $f_0^z(M)$ be the number of integral vertices of the polytope M. Later we will need the following lemma.

Lemma 1.9 *Let* $p \geq 3$, $1 \leq r \neq k \leq p$. *Then the number of integral vertices of any axial transportation polytope* $M(a^1, a^2, \ldots, a^p)$ *of order* $n_1 \times n_2 \times \ldots \times n_p$ *satisfies the inequality*

$$f_0^z(M(a^1, a^2, \ldots, a^p)) \geq f_0^z(M(a^1, a^2, \ldots, a^{r-1}, a^{r+1}, \ldots, a^p)) . f_0^z(M(a^r, a^k)).$$

Proof Let $x = (x_{i_r i_k})$ be a vertex of the polytope $M(a^r, a^k)$ and let $y = (y_{i_1 i_2 \ldots i_{r-1} i_{r+1} \ldots i_p})$ be an integral vertex of the polytope $M(a^1, a^2, \ldots, a^{r-1}, a^{r+1}, \ldots, a^p)$. We consider an algorithm for constructing the nonzero components of a vertex $z = (z_{i_1 i_2 \ldots i_p})$ of $M(a^1 a^2, \ldots, a^p)$. The t-th step $(1 \leq t \leq n_k)$ in the algorithm is as follows. We define

$$x_{i_r^0 t} = \min x_{i_r t},$$

$$y_{i_1^0 i_2^0 \ldots i_{r-1}^0 i_{r+1}^0 \ldots i_{k-1}^0 t i_{k+1}^0 \ldots i_p^0} = \min y_{i_1 i_2 \ldots i_{r-1} i_{r+1} \ldots i_{k-1} t i_{k+1} \ldots i_p},$$

where the first minimum is taken over all indices i_r, $1 \leq i_r \leq n_r$, for which $x_{i_r t} > 0$, and the second minimum is taken over all vectors $(i_1, i_2, \ldots, i_{r-1}, i_{r+1}, \ldots, i_{k-1}, i_{k+1}, \ldots, i_p)$ for which

$$y_{i_1 i_2 \ldots i_{r-1} i_{r+1} \ldots i_{k-1} t i_{k+1} \ldots i_p} > 0.$$

Let $z_{i_1^0 i_2^0 \ldots i_r^0 \ldots i_{k-1}^0 t i_{k+1}^0 \ldots i_p^0} = \Delta t$, where $\Delta t = \min(x_{i_r^0 t}, y_{i_1^0 i_2^0 \ldots i_{r-1}^0 i_{r+1}^0 \ldots i_{k-1}^0 t i_{k+1}^0 \ldots i_p^0})$. We transform the matrices x, y and the vectors $a^1, a^2, \ldots, a^p$ according to the formulae

$$x'_{i_r t} = \begin{cases} x_{i_r t}, & \text{if } i_r \neq i_r^0, \\ x_{i_r t} - \Delta t, & \text{if } i_r = i_r^0, \end{cases}$$

$$y'_{i_1\ldots i_{r-1}i_{r+1}\ldots i_{k-1}t i_{k+1}\ldots i_p}$$

$$= y_{i_1\ldots i_{r-1}i_{r+1}\ldots i_{k-1}t i_{k+1}\ldots i_p} - \Delta t\ , \qquad \text{if}$$

$$(i_1,\ldots,i_{r-1},i_{r+1},\ldots,i_{k-1},i_{k+1},\ldots,i_p) = (i_1^0,\ldots,i_{r-1}^0,i_{r+1}^0,\ldots,i_{k-1}^0,i_{k+1}^0,\ldots,i_p^0)$$

$$\text{or} \quad = y_{i_1\ldots i_{r-1}i_{r+1}\ldots i_{k-1}t i_{k+1}\ldots i_p} \qquad \text{otherwise,}$$

$$\bar{a}^s_{i_s} = \begin{cases} a^s_{i_s}\ , & \text{if } i_s \neq i_s^0\ ,\ s \in N_p\ , \\ a^s_{i_s} - \Delta t\ , & \text{if } i_s = i_s^0\ ,\ s \in N_p\ , \end{cases}$$

where $i_k^0 = t$.

If $\bar{a}^k_t > 0$ we repeat the t^{th} step of the algorithm. If $\bar{a}^k_t = 0$ and $t < n_k$ we proceed to the $(t+1)^{th}$ step of the algorithm. In the case $\bar{a}^k_t = 0$ and $t = n_k$ the algorithm terminates.

Thus from any integral vertices $x \in M(a^r,a^k)$ and $y \in M(a^1,a^2,\ldots,a^{r-1},a^{r+1},\ldots,a^p)$ it is always possible to construct an integral vertex $z \in M(a^1,a^2,\ldots,a^p)$. It is clear that distinct vertex pairs $(x_1,y_1),(x_2,y_2)$, where $x_1,x_2 \in M(a^r,a^k)$, $y_1,y_2 \in M(a^1,\ldots,a^{r-1},a^{r+1},\ldots,a^p)$ correspond to distinct vertices of $M(a^1,a^2,\ldots,a^p)$. This completes the proof. //

The following theorem is due to Yemelicheva & Kononenko (1974).

Theorem 1.10 *The minimum number of integral vertices in the class of axial transportation polytopes of order* $n_1\times n_2\times\ldots\times n_p$, $p \geq 3$, *is equal to*

$$\frac{\left(\left(\max\limits_{1\leq i\leq p} n_i\right)!\right)^{p-1}}{\prod\limits_{s=1}^{p}\left(\max\limits_{1\leq i\leq p} n_i - n_s + 1\right)!}$$

Proof Let $n_1 = \max\limits_{1\leq s\leq p} n_s$. By Lemma 1.9 we have the inequality

$$f_0^z(M(a^1,a^2,\ldots,a^p)) \geq \prod_{s=2}^{p} f_0^z(M(a^1,a^s)) .$$

This, and Theorems 5.1 and 5.3, Ch. 6, imply that

$$f_0^z(M(a^1,a^2,\dots,a^p)) \geqq \frac{(n_1!)^{p-1}}{\prod_{s=1}^{p}(n_1 - n_s + 1)!}$$

For definiteness, let $n_2 \geqq n_3 \geqq \dots \geqq n_p \geqq 1$. Consider the axial transportation polytope $M(a^1,a^2,\dots,a^p)$ of order $n_1 \times n_2 \times \dots \times n_p$, $p \geqq 3$ defined by the vectors $a^s = (n_1 - n_s + 1, 1, 1, \dots, 1) \in E_{n_s}$, $s \in N_p$.

By Theorem 1.3, this polytope is degenerate. It is easily seen that every integral vertex $x = (x_{i_1 i_2 \dots i_p}) \in M(a^1,a^2,\dots,a^p)$ is constructed as follows:

$$x_{1 i_2^1 \dots i_p^1} = 1,$$

$$x_{2 i_2^2 \dots i_p^2} = 1,$$

$$\dots\dots\dots$$

$$x_{n_1 i_2^{n_1} \dots i_p^{n_1}} = 1,$$

and all other components are zero. Also, for any $s \in \{2,3,\dots,p\}$ the choice of the numbers $i_s^1, i_s^2, \dots, i_s^{n_1}$ from the numbers $1,2,\dots,n_s$, is made so that each of the numbers $2,3,\dots,n_s$ occurs exactly once. The converse is also true. Consequently

$$f_0^z(M(a^1,a^2,\dots,a^p) = \prod_{s=2}^{p} \prod_{i=1}^{n_s-2} (n_1 - i) = \frac{(n_1!)^{p-1}}{\prod_{s=1}^{p}(n_1 - n_s + 1)!}$$

§2 PLANAR TRANSPORTATION POLYTOPES

In this section, for the sake of simplicity, we will only consider triply-indexed planar transportation polytopes. The results obtained here can be extended to general multi-indexed polytopes. Some of these generalizations can be found in the problems for this chapter.

Definition 2.1 A *triply-indexed transportation polytope* of order $m \times n \times k$ is the feasible set of a planar transportation problem, that is, a set $M(A,B,C)$ of matrices $x = (x_{ijt})_{m \times n \times k}$, whose elements satisfy the following conditions:

$$\sum_{j=1}^{n} x_{ijt} = a_{it} \qquad \forall (i,t) \in N_m \times N_k , \tag{2.1}$$

$$\sum_{i=1}^{m} x_{ijt} = b_{jt} \qquad \forall (j,t) \in N_n \times N_k , \tag{2.2}$$

$$\sum_{t=1}^{k} x_{ijt} = c_{ij} \qquad \forall (i,j) \in N_m \times N_n , \tag{2.3}$$

$$x_{ijt} \geqq 0 \qquad \forall (i,j,t) \in N_m \times N_n \times N_t , \tag{2.4}$$

where $A = (a_{it})_{m\times k}$, $B = (b_{jt})_{n\times k}$, $C = (c_{ij})_{m\times n}$ are matrices with real non-negative elements.

2.1 Necessary Conditions for a Non-empty Polytope

Consistency conditions for a system of linear equations and inequalities are given by a well known generalization of the Kronecker-Capelli Theorem. However, these conditions are difficult to verify so that for a concrete system, such as, for example, (2.1)-(2.4) which determines the planar transportation polytope $M(A,B,C)$, the problem arises of obtaining simple, easily verified conditions for consitency, or, as we shall say, conditions for a non-empty polytope.

At the present time only necessary conditions have been obtained for the planar transportation polytope $M(A,B,C)$ to be non-empty. We present some of them here.

The following conditions are obviously necessary for $M(A,B,C)$ to be non-empty and we will assume from now on that they are satisfied:

$$\sum_{i=1}^{m} a_{it} = \sum_{j=1}^{n} b_{jt} \qquad \forall\, t \in N_k ,$$

$$\sum_{t=1}^{k} a_{it} = \sum_{j=1}^{n} c_{ij} \qquad \forall\, i \in N_m ,$$

$$\sum_{t=1}^{k} b_{jt} = \sum_{i=1}^{m} c_{ij} \qquad \forall\, j \in N_n .$$

Schell's Conditions (Schell, 1955)

Let

$$\Gamma_{ijt} = \min(a_{it}, b_{jt}, c_{ij}), \qquad (i,j,t) \in N_m \times N_n \times N_k .$$

For the triply-indexed planar transportation polytope $M(A,B,C)$ of order $m \times n \times k$ to be non-empty it is necessary that the following conditions are satisfied:

$$\sum_{j=1}^{n} \Gamma_{ijt} \geqq a_{it} \qquad \forall (i,t) \in N_m \times N_k ,$$

$$\sum_{i=1}^{m} \Gamma_{ijt} \geqq b_{jt} \qquad \forall (j,t) \in N_n \times N_k ,$$

$$\sum_{t=1}^{k} \Gamma_{ijt} \geqq c_{ij} \qquad \forall (i,j) \in N_m \times N_n .$$

Indeed, if $(x^0_{ijt})_{m \times n \times k} \in M(A,B,C)$ then we must have $x^0_{ijt} \leqq \Gamma_{ijt}$, $(i,j,t) \in N_m \times N_n \times N_k$ so that Schell's conditions are satisfied.

The following example shows that Schell's conditions are not, in general, sufficient. Let

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 3 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & 3 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

It can be verified directly that Schell's conditions are satisfied.

However, since $a_{32} = a_{33} = 1$, $b_{22} = b_{23} = b_{32} = b_{33} = 0$, we must have for any point $(x_{ijt})_{3\times3\times3}$ of $M(A,B,C)$ that $x_{312} = x_{313} = 1$. On the other hand, by (2.3) we have $x_{311} + x_{312} + x_{313} = c_{31} = 1$. Hence $x_{311} = -1$ which contradicts (2.4). Thus $M(A,B,C)$ is empty.

Let $\Gamma^1_{ijt} = \min(a_{it}, b_{jt}, c_{ij})$, $(i,j,t) \in N_m \times N_n \times N_k$, and for $r = 1,2,3,\ldots$ define the numbers

$$\gamma^r_{ijt} = \max(0, (a_{it} - \sum_{p \neq j} \Gamma^r_{ipt}), (b_{jt} - \sum_{\ell \neq i} \Gamma^r_{\ell jt}), (c_{ij} - \sum_{q \neq t} \Gamma^r_{ijq})) ,$$

$$\Gamma^{r+1}_{ijt} = \min(\Gamma^r_{ijt}, (a_{it} - \sum_{p \neq j} \gamma^r_{ipt}), (b_{jt} - \sum_{\ell \neq i} \gamma^r_{\ell jt}), (c_{ij} - \sum_{q \neq t} \gamma^r_{ijq})) .$$

It is not difficult to see that if there is a number Γ^r_{ijk} which is negative than $M(A,B,C)$ is empty.

The following conditions are strcnger than Schell's conditions.

Haley's conditions (Haley 1963)

For a triply-indexed planar transportation polytope $M(A,B,C)$ of order $m\times n\times k$ to be non-empty it is necessary that there exist an index $h \geqq 1$ for which the following conditions are satisfied:

$$\Gamma^h_{ijt} = \Gamma^{h+1}_{ijt} \qquad \forall (i,j,t) \in N_m\times N_n\times N_k ,$$

$$\gamma^h_{ijt} = \gamma^{h+1}_{ijt} \qquad \forall (i,j,t) \in N_m\times N_n\times N_k ,$$

$$\sum_{i=1}^{m} \gamma^h_{ijt} \leqq b_{jt} \leqq \sum_{i=1}^{m} \Gamma^h_{ijt} \qquad \forall (j,t) \in N_n\times N_k ,$$

$$\sum_{j=1}^{n} \gamma^h_{ijt} \leqq a_{it} \leqq \sum_{j=1}^{n} \Gamma^h_{ijt} \qquad \forall (i,t) \in N_m\times N_k ,$$

$$\sum_{t=1}^{k} \gamma^h_{ijt} \leqq c_{ij} \leqq \sum_{t=1}^{k} \Gamma^h_{ijt} \qquad \forall (i,j) \in N_m\times N_n .$$

The proof follows from the obvious fact that if $M(A,B,C) \neq \emptyset$ for any number $r = 1,2,3,\ldots$ we have the inequalities $\gamma^r_{ijt} \leqq x_{ijt} \leqq \Gamma^r_{ijt}$, $(i,j,t) \in N_m\times N_n\times N_k$. Moreover, as r increases the lower bound γ^r_{ijt} is non-decreasing and the upper bound Γ^r_{ijt} is non-increasing.

Before giving further necessary conditions for $M(A,B,C)$ to be non-empty we introduce some notation.

Let $I \subset N_m$, $J \subset N_n$. We denote the sum of the elements of the matrix $(z_{ij})_{m\times n}$ for which the index i takes all values in I and the index j takes all values in J by $z(I,J)$. Similarly, for $I\subset N_m$, $J \subset N_n$, $T \subset N_k$ we define the number $z(I,J,T) = \sum_{i\in I} \sum_{j\in J} \sum_{t\in T} z_{ijt}$ with respect to a given matrix $(z_{ijt})_{m\times n\times k}$.

The Moravek-Vlach Conditions (Moravek & Vlach 1967)

For the triply-indexed planar transportation polytope $M(A,B,C)$ of order $m\times n\times k$ to be non-empty it is necessary that the following inequalities be satisfied:

$$a(I,T) - b(J,T) + c(\overline{I},T) \geqq 0 , \quad I \subset N_m , J \subset N_n , T \subset N_k .$$

To prove this it suffices to note that for any subsets $I \subset N_m$, $J \subset N_n$, $T \subset N_k$ and for any matrix $x \in M(A,B,C)$ we have the relations

$$a(I,T) = x(I,J,T) + x(I,\bar{J},T) ,$$

$$b(J,T) = x(I,J,T) + x(\bar{I},J,T) ,$$

$$c(I,J) = x(I,J,T) + x(I,J,\bar{T}) .$$

We present an example which shows that Haley's conditions are not sufficient in general.

Example. Let the planar transportation polytope $M(A,B,C)$ of order $5\times 8\times 2$ be defined by the matrices

$$A = \begin{bmatrix} 1 & 7 \\ 2 & 6 \\ 7 & 1 \\ 6 & 2 \\ 6 & 2 \end{bmatrix} , \quad B = \begin{bmatrix} 1 & 4 \\ 1 & 4 \\ 1 & 4 \\ 1 & 4 \\ 4 & 1 \\ 4 & 1 \\ 4 & 1 \\ 3 & 2 \end{bmatrix} , \quad C = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

It may be verified that Haley's conditions are satisfied for $M(A,B,C)$:

$$\Gamma^1_{ijt} = \Gamma^2_{ijt} = 1 \qquad \forall (i,j,t) \in N_5\times N_8\times N_2 ,$$

$$\gamma^1_{ijt} = \gamma^2_{ijt} = 0 \qquad \forall (i,j,t) \in N_5\times N_8\times N_2 ,$$

$$0 \leqq b_{jt} \leqq 5 \qquad \forall (j,t) \in N_8\times N_2 ,$$

$$0 \leqq a_{it} \leqq 8 \qquad \forall (i,t) \in N_5\times N_2 ,$$

$$0 \leqq c_{ij} \leqq 2 \qquad \forall (i,j) \in N_5\times N_8 .$$

On the other hand, the Moravek-Vlach conditions are not satisfied for the subsets $I = \{3,4,5\}$, $J = \{1,2,3\}$, $T = \{2\}$. Thus $M(A,B,C)$ is empty.

The Haley and the Moravek-Vlach conditions have the following generalization.

Smith's Condition (Smith 1973)

A necessary condition for a triply-indexed planar transportation polytope $M(A,B,C)$ of order $m\times n\times k$ to be non-empty is the existence of an index $h \geqq 1$ for which the following conditions hold:

$$\Gamma^h_{ijt} = \Gamma^{h+1}_{ijt} = \Gamma_{ijt} \qquad \forall (i,j,t) \in N_m\times N_n\times N_k \ ,$$

$$\gamma^h_{ijt} = \gamma^{h+1}_{ijt} = \gamma_{ijt} \qquad \forall (i,j,t) \in N_m\times N_n\times N_k \ ,$$

$$\gamma(I,\bar{J},T)+\gamma(\bar{I},J,\bar{T}) \leqq a(I,T) - b(J,T) + c(\bar{I},J)$$

$$\leqq \Gamma(I,\bar{J},T)+\Gamma(\bar{I},J,\bar{T}) \qquad \forall\ T\subset N_k\ ,\ I\subset N_m\ ,\ J\subset N_n\ . \tag{2.5}$$

By Haley's conditions it suffices to establish the necessity of the inequalities (2.5). Let $I\subset N_m$, $J\subset N_n$, $T\subset N_k$. Then, by Haley's conditions we have the inequalities

$$\gamma(I,J,T)+\gamma(I,\bar{J},T) \leqq a(I,T) \leqq \Gamma(I,J,T)+\Gamma(I,\bar{J},T)\ ,$$

$$\gamma(I,J,T)+\gamma(\bar{I},J,T) \leqq b(J,T) \leqq \Gamma(I,J,T)+\Gamma(\bar{I},J,T)\ ,$$

$$\gamma(\bar{I},J,T)+\gamma(\bar{I},J,\bar{T}) \leqq c(\bar{I},J) \leqq \Gamma(\bar{I},J,T)+\Gamma(\bar{I},J,\bar{T})\ .$$

Now, using the Moravek-Vlach conditions we obtain the desired inequalities.

2.2 The Polytope Dimension

Let us note, first of all, that it is possible for a triply-indexed planar transportation polytope to degenerate to a point. To show this, consider the polytope $M(A,B,C)$ of order $m\times n\times k$ defined by the matrices

$$A = \begin{bmatrix} n & 1 & \dots & 1 \\ n & 1 & \dots & 1 \\ . & . & \dots & . \\ n & 1 & \dots & 1 \\ n-1 & n & \dots & n \end{bmatrix},\quad B = \begin{bmatrix} m-1 & m & \dots & m \\ m & 1 & \dots & 1 \\ . & . & \dots & . \\ m & 1 & \dots & 1 \\ m & 1 & \dots & 1 \end{bmatrix},\quad C = \begin{bmatrix} k & 1 & \dots & 1 \\ k & 1 & \dots & 1 \\ . & . & \dots & . \\ k & 1 & \dots & 1 \\ k-1 & k & \dots & k \end{bmatrix}.$$

It is clear that the only point of this polytope is the point $(x_{ijt})_{m\times n\times k}$ with elements: $x_{m11} = 0$,

$$x_{ijt} = \begin{cases} 0 , & \text{if } (i,j,t) \in \{1,2,..,m-1\}\times\{2,3,..,n\}\times\{2,3,..,k\} \\ 1 & \text{otherwise.} \end{cases}$$

Let us now examine the question of determining the maximum dimension of a triply-indexed planar transportation polytope. The following theorem answers the question.

<u>Theorem 2.1</u> *The maximum dimension of the triply-indexed planar transportation polytope of order* $m\times n\times k$ *equals* $(m-1)(n-1)(k-1)$.

Proof It is easily seen that the rank of the system of linear equations

$$\sum_{j=1}^{n} x_{ijt} = a_{it} , \qquad \forall i \in N_m \setminus \{1\} , \; t \in N_k ,$$
$$\sum_{i=1}^{m} x_{ijt} = b_{jt} \qquad \forall (j,t) \in N_n \times N_k ,$$
$$\sum_{t=1}^{k} x_{ijt} = c_{ij} \qquad \forall (i,j) \in N_m \times N_n ,$$

is equal to $\beta = mk + nk + mn - m - n - k + 1$. On the other hand, since any $m + n + k - 1$ equations of the system (2.1)-(2.3) are a consequence of all the remaining equations, the rank of the constraint matrix cannot exceed β . Thus the rank of the system of linear equations (2.1)-(2.3) is equal to β . Hence, by Proposition 4.1, Ch. 1, the dimension of any triply-indexed planar transportation polytope $M(A,B,C)$ of order $m\times n\times k$ satisfies $\dim M(A,B,C) \leq (m-1)(n-1)(k-1)$.

At the same time, a triply-indexed planar transportation polytope whose dimension is $(m-1)(n-1)(k-1)$ is given by the polytope $M(A^*,B^*,C^*)$ of order $m\times n\times k$, given by the matrices

$$A^* = \begin{pmatrix} n & n & \dots & n \\ n & n & \dots & n \\ \cdot & \cdot & \dots & \cdot \\ n & n & \dots & n \end{pmatrix}, \; B^* = \begin{pmatrix} m & m & \dots & m \\ m & m & \dots & m \\ \cdot & \cdot & \dots & \cdot \\ m & m & \dots & m \end{pmatrix}, \; C^* = \begin{pmatrix} k & k & \dots & k \\ k & k & \dots & k \\ \cdot & \cdot & \dots & \cdot \\ k & k & \dots & k \end{pmatrix} ,$$

since it contains a point $(x_{ijt})_{m\times n\times k}$ satisfying $x_{ijt} > 0$, $\forall(i,j,t) \in N_m \times N_n \times N_k$. //

2.3 Simplexes

For classical transportation polytopes of order $m\times n$ there exist $\max(m,n)$-simplexes only when $\min(m,n) = 2$ (§5, Ch.6).

The following theorem is due to Fedenya (1977).

Theorem 2.2 *Among the triply-indexed planar transportation polytopes of order* $m\times n\times k$, $m,n,k \geq 2$, *there exist* $(m-1)(n-1)(k-1)$-*simplexes.*

Proof Consider the triply-indexed planar transportation polytope $M(A_0,B_0,C_0)$ of order $m\times n\times k$, given by the matrices

$$A_0 = \begin{pmatrix} n & 3(n-1)+1 & \cdots & 3(n-1)+1 \\ 3(n-1)+1 & 3 & \cdots & 3 \\ \cdots & \cdots & \cdots & \cdots \\ 3(n-1)+1 & 3 & \cdots & 3 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} m & 3(m-1)+1 & \cdots & 3(m-1)+1 \\ 3(m-1)+1 & 3 & \cdots & 3 \\ \cdots & \cdots & \cdots & \cdots \\ 3(m-1)+1 & 3 & \cdots & 3 \end{pmatrix},$$

$$C_0 = \begin{pmatrix} k & 3(k-1)+1 & \cdots & 3(k-1)+1 \\ 3(k-1)+1 & 3 & \cdots & 3 \\ \cdots & \cdots & \cdots & \cdots \\ 3(k-1)+1 & 3 & \cdots & 3 \end{pmatrix}.$$

It is clear that a vertex of $M(A_0,B_0,C_0)$ is given by the matrix $(x^*_{ijt})_{m\times n\times k}$ with elements

$$x^*_{11t} = 1 \ , \ t \in N_k \ , \ x^*_{1j1} = 1 \ , \ j \in N_n \ , \ x^*_{i11} = 1 \ , \ i \in N_m \ ,$$

$$x^*_{ij1} = 3 \ , \ i \in N_m\setminus\{1\} \ , \ j \in N_n\setminus\{1\},$$

$$x^*_{i1t} = 3 \ , \ i \in N_m\setminus\{1\} \ , \ t \in N_k\setminus\{1\},$$

$$x^*_{1jt} = 3 \ , \ j \in N_n \setminus \{1\} \ , \ t \in N_k \setminus \{1\} \ , \quad x^*_{ijt} = 0 \quad \text{otherwise.}$$

It is not difficult to see that every other vertex $(x_{ijt})_{m\times n\times k}$ of $M(A_0,B_0,C_0)$ has the form

$$x_{ijt} = \begin{cases} x^*_{ijt} - 1 \ , & \text{if } (i,j,t) \in K^- \ , \\ x^*_{ijt} + 1 \ , & \text{if } (i,j,t) \in K^+ \ , \\ x^*_{ijt} & \text{otherwise} \ , \end{cases}$$

where

$$K^- = \{(i_0,j_0,1),(i_0,1,t_0),(1,j_0,t_0),(1,1,1)\} \ ,$$

$$K^+ = \{(i_0,j_0,t_0),(i_0,1,1),(1,j_0,1),(1,1,t_0)\} \ .$$

Here (i_0,j_0,t_0) is some fixed triple of indices taken from the set $\{2,3,\ldots,m\}\times\{2,3,\ldots,n\}\times\{2,3,\ldots,k\}$. Consequently, $M(A_0,B_0,C_0)$ is a non-degenerate polytope and $f_0(M(A_0,B_0,C_0)) = (m-1)(n-1)(k-1)+1$. Thus, taking into account the fact that the dimension of any non-degenerate triply-indexed transportation polytope of order $m\times n\times k$ is equal to $(m-1)(n-1)(k-1)$ we conclude that $M(A_0,B_0,C_0)$ is a $(m-1)(n-1)(k-1)$-simplex. //

The following is an immediate consequence of Theorem 2.2.

Corollary 2.3 *The minimum number of vertices in the class of non-degenerate triply-indexed planar transportation polytopes of order* $m\times n\times k$ *is equal to* $(m-1)(n-1)(k-1)+1$.

§3 PLANS FOR A MULTI-INDEXED SELECTION PROBLEM

Just as the multi-indexed transportation problems are a natural generalization of the classical transportation problem, so the multi-indexed selection problem is a generalization of the well-known assignment problem.

In this section we establish the connection between plans for a multi-indexed selection problem and orthogonal systems of multi-dimensional cubes and, in particular cases, with finite projective planes (Yemelichev & Kononenko 1974).

3.1 Orthogonal Systems of Cubes

A *p-cube* of order n is a set of n^p elements each corresponding to a point in p-dimensional space whose coordinates are given by $i_1, i_2, \ldots, i_p$ where $i_s \in N_n$, $s \in N_p$. A 2-cube is naturally called a *square*. We will denote a p-cube by $A = (a_{i_1 i_2 \ldots i_p})_n$ and we will suppose that $a_{i_1 i_2 \ldots i_p} \in N_n$, $n \geqq 2$, $p \geqq 1$.

Note that with any n^p-vector with elements in N_n we can associate a p-cube of order n by ordering the coordinates of the elements of the cube lexicographically. This correspondence is clearly a bijection. We will therefore formulate the following definitions and propositions both in terms of cubes and in terms of n^p-column-vectors.

Let us fix the values of the indices $i_1 = i_1^0$, $i_2 = i_2^0$, ..., $i_{s-1} = i_{s-1}^0, i_{s+1} = i_{s+1}^0, \ldots, i_p = i_p^0$. The sequence of n points with coordinates $i_1^0, i_2^0, \ldots, i_{s-1}^0, i_s, i_{s+1}^0, \ldots, i_p^0$, where $i_s \in N_n$ is called a *line* of the p-cube A of order n. If every line of a cube A contains precisely the numbers $1, 2, \ldots, n$ in some order, then it is called a *Latin p-cube* of order n. The Latin cubes thus defined are a natural generalization of the Latin squares of order n. An example of a Latin cube is the p-cube with elements

$$a_{i_1 i_2 \ldots i_p} = 1 + r\left(\sum_{s=1}^{p} v_s i_s / n\right), \quad i_s \in N_n, \quad s \in N_p,$$

where the numbers v_s are coprime with n and $r(u/v)$ is the remainder on dividing u by v.

For example, the 3-cubes of order 3 shown in Figure 52 are Latin cubes.

<u>Definition 3.1</u> A system of p-cubes of order n

$$A^1, A^2, \ldots, A^p, \quad \text{where} \quad A^s = (a^s_{i_1 i_2 \ldots i_p})_n, \tag{3.1}$$

is called *orthogonal* if all n^p possible p-tuples $(a^1_{i_1 i_2 \ldots i_p}, a^2_{i_1 i_2 \ldots j_p}, \ldots, a^p_{i_1 i_2 \ldots i_p})$ are distinct.

If we apply the same permutation of the n^p elements of each cube simultaneously in (3.1) the property of orthogonality is preserved.

Let $E^1 = (e^1_{i_1 i_2 \ldots i_p})$, $E^2 = (e^2_{i_1 i_2 \ldots i_p})$, ..., $E^p = (e^p_{i_1 i_2 \ldots i_p})$ be p-cubes of order n such that $e^s_{i_1 i_2 \ldots i_p} = i_s$,

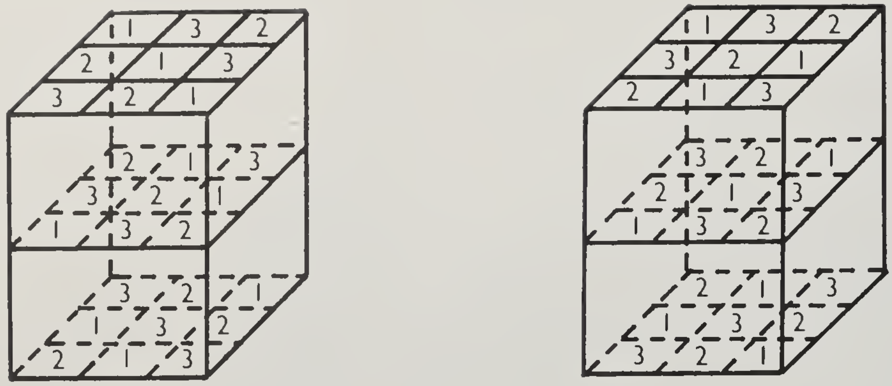


Fig. 52.

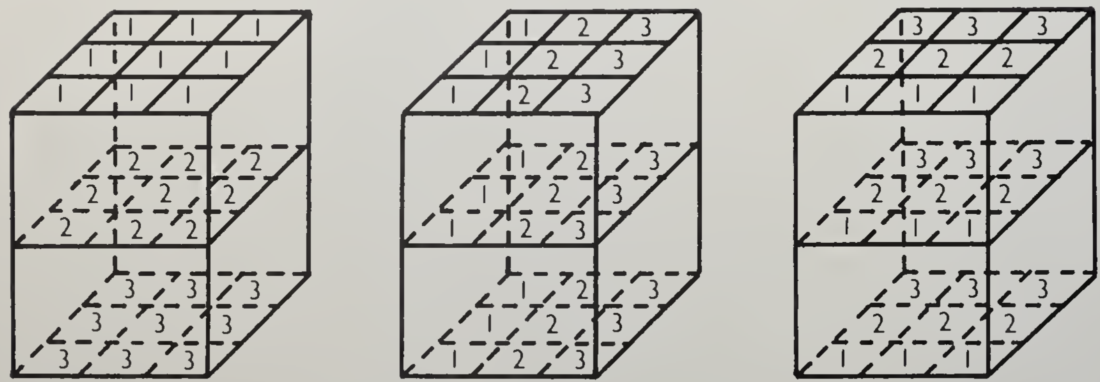


Fig. 53.

for all $i_s \in N_n$ and for all $(i_1, i_2, \ldots, i_{s-1}, i_{s+1}, \ldots, i_p)$. Such a system of cubes is called *normal* .

For example, the two squares of order 3 given by

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix} , \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

form a normal system, Again, the three cubes of order 3 shown in Fig. 53 also form a normal system.

It is easy to see that a normal system of cubes is orthogonal.

Definition 3.2 A system of p-cubes of order n

$$A^1, A^2, \dots, A^t \qquad (p \leq t) \tag{3.2}$$

is called (t,n,p)-*orthogonal* if any set of p cubes in (3.2) form an orthogonal system.

If $p = 2$, the concept of a (t,n,p)-orthogonal system coincides with the usual concept of a set of t pairwise orthogonal square matrices of order n, whose existence is equivalent to the existence of $t - 2$ pairwise orthogonal Latin squares (Ryser 1963, Hall 1967, Rybnikov 1972).

Obviously, the concepts of orthogonality and (t,n,p)-orthogonality can be applied equally to n^p-column-vectors. In particular, the array consisting of p columns such that its rows constitute all possible p-tuples whose elements are in N_n forms an orthogonal system of column vectors.

We next establish the connection between the property of orthogonality and the property of being a Latin cube.

Consider the (t,n,p)-orthogonal system (3.2). Given any fixed coordinate position with coordinates $i_1^0, i_2^0, \dots, i_p^0$, there exists, by the orthogonality of the system A^j, $j \in N_p$, another coordinate position $i_1', i_2', \dots, i_p'$ such that $a^s_{i_1' i_2' \dots i_p'} = i_s^0$, $s \in N_p$. We interchange the elements in each of these two positions in each of the cubes of the system (3.2). In doing this, as already noted, we do not violate the orthogonality of the system.

As a result of a sequence of such transformations it is clear that the (t,n,p)-orthogonal system (3.2) can be reduced to the following canonical form:

$$E^1, E^2, \dots, E^p, D^{p+1}, \dots, D^t, \tag{3.3}$$

where $E^1, E^2, \dots, E^p$ is a normal system of cubes.

We show further that every cube D^i, $i = p+1, p+2, \dots, t$ of system (3.3) is Latin. To do this we establish that the elements in any line of D^i are precisely the numbers $1, 2, \dots, n$ in some order.

Consider the line ℓ whose points have coordinates $i_1^0, i_2^0, \dots, i_{r-1}^0, i_r, i_{r+1}^0, \dots, i_p^0$, $i_r \in N_n$, $1 \leq r \leq p$

By the definition of a normal system we have

$$e^{s}_{i_1^0 i_2^0 \ldots i_{r-1}^0 i_r i_{r+1}^0 \ldots i_p^0} = i_s^0 \ , \quad s = 1,2,\ldots,r-1,r+1,\ldots,p, \quad i_r \in N_n \ .$$

Thus all (p-1)-tuples

$$\left(e^{1}_{i_1^0 i_2^0 \ldots i_{r-1}^0 i_r i_{r+1}^0 \ldots i_p^0}, \ \ldots , \ e^{r-1}_{i_1^0 i_2^0 \ldots i_{r-1}^0 i_r i_{r+1}^0 \ldots i_p^0}, \right.$$

$$\left. e^{r+1}_{i_1^0 i_2^0 \ldots i_{r-1}^0 i_r i_{r+1}^0 \ldots i_p^0}, \ \ldots , \ e^{p}_{i_1^0 i_2^0 \ldots i_{r-1}^0 i_r i_{r+1}^0 \ldots i_p^0} \right) , \quad i_r \in N_n$$

are identical. On the other hand, the system of cubes $E^1, E^2, \ldots, E^{r-1}, D^i, E^{r+1}, \ldots, E^p$ is orthogonal for any $p+1 \leq i \leq t$.

Consequently, the line ℓ must contain all the numbers 1,2, ...,n in some order.

We have proved the following theorem.

Theorem 3.1 *For a system of cubes* (3.2) *to be* (t,n,p)-*orthogonal, it is necessary that every cube* D^i , $i = p+1, p+2, \ldots, t$ *of its canonical form* (3.3) *should be Latin.*

Theorem 3.2 *The conditions of Theorem* 3.1 *are sufficient only in the following two cases* : 1) $t = p+1$, 2) $p = 1$.

Proof In the two cases indicated the (t,n,p)-orthogonality of the system (3.3) follows from the definitions of a Latin cube and of a normal system.

Let $t \geq p+2$, $p \geq 2$ and let D be a Latin p-cube of order n . It is then easy to see that the system $E^1, E^2, \ldots E^{p-2}, D, D$ is not orthogonal so that the system $E^1, E^2, \ldots, E^p, \underbrace{D, \ldots, D}_{t-k}$ is not (t,n,p)-orthogonal. //

Two (t,n,p)-orthogonal systems are called *distinct* if their canonical forms are distinct. Let $G(t,n,p)$ denote the number of distinct (t,n,p)-orthogonal systems and let $L(n,p)$ denote the number of Latin p-cubes of order n .

The following useful corollary follows directly from Theorems 3.1 and 3.2.

Corollary 3.3 $G(t,n,p) \leq (L(n,p))^{t-p}$. *Also* $G(t,n,p) = (L(n,p))^{t-p}$ *only in the two cases* : $t = p+1$, 2) $p = 1$.

3.2 The Selection Problem and Orthogonal Systems

Let $1 \leq m < p$, $n \geq 2$. The *selection problem* $A(p,n,m)$ (compare with the p-indexed m-fold transportation problem) consists in determining the extremum of some linear function subject to the constraints

$$x_{i_1 i_2 \ldots i_p} = 1 \text{ or } 0 , \qquad \forall\, i_s \in N_n , \; s \in N_p , \tag{3.4}$$

$$\sum_{i_{k_1}=1}^{n} \sum_{i_{k_2}=1}^{n} \cdots \sum_{i_{k_m}=1}^{n} x_{i_1 i_2 \ldots i_p}$$

$$= a_{i_1 i_2 \ldots i_{k_1-1} * i_{k_1+1} \ldots i_{k_m-1} * i_{k_m+1} \ldots i_p} = 1 \tag{3.5}$$

for all $i_s \in N_n$, $s \neq k_1, k_2, \ldots, k_m$ and for all m-tuples $(k_1, k_2, \ldots k_m)$ such that

$$1 \leq k_1 < k_2 < \ldots < k_m \leq p . \tag{3.6}$$

The set of feasible selections for the problem $A(p,n,m)$, that is, the set of cubes $x = (x_{i_1 i_2 \ldots i_p})_n$ satisfying conditions (3.4), (3.5) will be denoted by $T(p,n,m)$.

It follows directly from conditions (3.4) and (3.5) that any p-cube of order n from the non-empty set $T(p,n,m)$ contains n^{p-m} non-zero elements.

Let $x^0 = (x^0_{i_1 i_2 \ldots i_p})_n \in T(p,n,m)$ and let

$$\begin{array}{l} i_1^{(1)} , i_2^{(1)} , \ldots , i_p^{(1)} , \\ i_1^{(2)} , i_2^{(2)} , \ldots , i_p^{(2)} , \\ \ldots\ldots\ldots\ldots\ldots , \\ i_1^{(n^{p-m})}, i_2^{(n^{p-m})}, \ldots , i_p^{(n^{p-m})} , \end{array} \tag{3.7}$$

be the p-tuples for which $x^0_{i_1^{(j)} i_2^{(j)} \ldots i_p^{(j)}} = 1$, $j = 1,2,\ldots,n^{p-m}$.

<u>Theorem 3.4</u> $x^0 \in T(p,n,m)$ *if and only if the system of column-vectors in table* (3.7) *is* (p,n,p-m)-*orthogonal.*

Proof Note that there are n^{p-m} distinct equations of type (3.5) with fixed $k_1, k_2, \ldots, k_m$ satisfying conditions (3.6). Thus, for the existence of a solution of such a system, satisfying conditions (3.4), it is necessary and sufficient that for any (p-m)-tuple $i_1^0, i_2^0, \ldots, i_{k_1-1}^0, i_{k_1+1}^0, \ldots, i_{k_m-1}^0, i_{k_m+1}^0, \ldots, i_p^0$ there should exist an m-tuple $i_{k_1}^0, i_{k_2}^0, \ldots, i_{k_m}^0$ such that

$$x_{i_1^0 i_2^0 \ldots i_p^0} = 1 \quad , \quad \text{and}$$

$$x_{i_1^0 i_2^0 \ldots i_{k_1-1}^0 i_{k_1} i_{k_1+1}^0 \ldots i_{k_m-1}^0 i_{k_m} i_{k_m+1}^0 \ldots i_p^0} = 0$$

for all $(i_{k_1}, i_{k_2}, \ldots, i_{k_m}) \neq (i_{k_1}^0, i_{k_2}^0, \ldots, i_{k_m}^0)$. The statement of Theorem 3.4 now follows directly. //

<u>Corollary 3.5</u> *The number of distinct selections for the problem* $A(p,n,m)$ *is equal to the number of* (p,n,p-m)-*orthogonal systems of cubes.*

The selection problems $A(p,n,p-1)$ and $A(p,n,1)$ are called *axial* and *planar* p-*indexed selection problems of order* n respectively. For these problems, Corollary 3.5 can be made more specific using Corollary 3.3.

<u>Corollary 3.6</u> *The* p-*indexed axial selection problem of order* n *has* $(n!)^{p-1}$ *distinct solutions.*

<u>Corollary 3.7</u> *The number of distinct solutions of the* p-*indexed planar selection problem of order* n *is equal to the number of* (p-1)-*dimensional Latin cubes of order* n .

For the case $p = 3$ these results were first obtained by Leue (1972).

3.3 <u>The Selection Problem and Finite Projective Planes</u>

A *finite projective plane* is a system consisting of a finite number of 'points' and 'lines' which are connected by incidence relations ('a point lies on a line' and 'a line passes through a point') which satisfy the axioms:

1) two distinct points lie on one and only one line;

2) two distinct lines pass through one and only one point;

3) there exist four distinct points, no three of which lie on the same line.

The concept of the order of a finite projective plane is introduced as follows. First, it is shown that if some line of the finite projective plane contains $n+1$ points, then every line contains $n+1$ points. The number n is called the *order of the finite projective plane.*

Finite projective planes play a leading rôle in combinatorial theory. They are intimately connected with orthogonal Latin squares (Ryser 1963, Hall 1967, Rybnikov 1972).

<u>Theorem 3.8</u> *A finite projective plane of order* $n \geq 3$ *exists if and only if there exists a* $(n+1,n,2)$*-orthogonal system of squares of order* n .

This, together with Corollary 3.5, yields the following theorem.

<u>Theorem 3.9</u> *A finite projective plane of order* $n \geq 3$ *exists if and only if the set of selections for the problem* $A(n+1,n,n-1)$ *is non-empty.*

Since there is no projective plane of order 6 it follows, in particular, that the selection problem $A(7,6,5)$ is insoluble, that is, $T(7,6,5) = \emptyset$.

It is known that a finite projective plane exists if its order n is a power of a prime number $(n \geq 3)$. The Theorem 3.9 implies the following.

<u>Corollary 3.10</u> *Let* n *be the power of a prime. Then, if* $n \geq 3$ *there is a selection which solves the problem* $A(n+1,n,n-1)$.

EXERCISES

1. (Yemelichev & Kononenko 1972). The inequality $K \geqq \sum_{s=1}^{p} n_s$ (see (1.3)) is a necessary condition for the axial transportation polytope of order $n_1 \times n_2 \times \ldots \times n_p$ to be non-degenerate. Show that the converse of Theorem 1.3 is true only in the cases: 1) p=2, 2) p=3, $\sum_{s=1}^{3} n_s = 6$.

2. Show that the following statement is false: a sufficient condition for the axial transportation polytope $M(a^1, a^2, \ldots, a^p)$ of order $n_1 \times n_2 \times \ldots \times n_p$, $p > 2$, to be non-degenerate is that for any $1 \leq k < \ell \leq p$ the classical transportation polytopes $M(a^k, a^\ell)$ are non-degenerate.

3. (Yemelichev & Kononenko 1972). Let $n_1 = \max_{1 \leq s \leq p} n_s$, $\max_{2 \leq s \leq p} n_s \geqq 2$. The number, γ, of facets of an axial transportation polytope of order $n_1 \times n_2 \times \ldots \times n_p$ must be an integer satisfying
$$\prod_{s=1}^{p} n_s - n_1 \leqq \gamma \leqq \prod_{s=1}^{p} n_s .$$

4. The set of indices $R = (r_1, r_2, \ldots, r_p)$, $r_s \in N_{n_s}$, $s \in N_p$, of the polytope $M(a^1, a^2, \ldots, a^p)$ is called *complete* if $\sum_{s=1}^{p} a^s_{r_s} \geqq (p-1)K$. Otherwise the set is *incomplete*. The s^{th}-coordinate of a complete set R is called *r-complete* if 1) $r = r_s \leqq n_s$; 2) the sets $(r_1, r_2, \ldots, r_{s-1}, j, r_{s+1}, \ldots, r_p)$ $\forall j \in N_r$ are complete but are incomplete for $j = r+1, \ldots, n_s$.

Show, that an axial transportation polytope of order $n_1 \times n_2 \times \ldots \times n_p$, $n_1 \geqq n_2 \geqq \ldots \geqq n_p$, $n_2 \geqq 2$, has γ facets, where $\prod_{s=1}^{p} n_s - n_1 \leqq \gamma \leqq \prod_{s=1}^{p} n_s$, if and only if there exists a set of indices with an r-complete coordinate, where $r = \prod_{s=1}^{p} n_s - \gamma$. When $r = 0$, this condition is interpreted to mean the absence of any complete sets.

5. (Yemelichev & Kononenko 1972). Establish the following properties: 1) Every non-degenerate axial transportation polytope with the minimum number of vertices also has the minimum number of facets.

2) For a non-degenerate polytope $M(a^1, a^2, \ldots, a^p)$ to have the minimum number of vertices, it is necessary and sufficient for any polytope $M(a^{p-k+1}, a^{p-k+2}, \ldots, a^p)$, $k = 2, 3, \ldots, p$, to have the minimum number of facets.

6. In the class of non-degenerate axial transportation polytopes of order $n_1 \times n_2 \times \ldots \times n_p$, $n_1 \geqq n_2 \geqq \cdots \geqq n_p \geqq 2$, $n_1 \geqq 3$, $p \geqq 2$, the minimum diameter is given by the number $\sum_{s=2}^{p} (n_s - 1)$. A non-degenerate regular transportation polytope has the minimum diameter (see Definition 1.5). These results were obtained by V.M.Likhachev.

7. Consider the following problem:

$$\text{Minimize} \quad \sum_{i_1=1}^{n_1} \cdots \sum_{i_p=1}^{n_p} c_{i_1 \ldots i_p} x_{i_1 \ldots i_p} \text{ , subject to}$$

$$\sum_{i_1=1}^{n_1} \cdots \sum_{i_{s-1}=1}^{n_{s-1}} \sum_{i_{s+1}=1}^{n_{s+1}} \cdots \sum_{i_p=1}^{n_p} x_{i_1 \ldots i_p} = a^s_{i_s} \quad , \forall i_s \in N_{n_s} \text{ , } s \in N_p \text{ ,}$$

$$d^s_{i_s i_p} \leqq \sum_{i_1=1}^{n_1} \cdots \sum_{i_{s-1}=1}^{n_{s-1}} \sum_{i_{s+1}=1}^{n_{s+1}} \cdots \sum_{i_{p-1}=1}^{n_{p-1}} x_{i_1 \ldots i_p} \leqq b^s_{i_s i_p} \text{ , } i_s \in N_{n_s} \text{ , } s \in N_{p-1}, i_p \in N_{n_p}$$

$x_{i_1 \ldots i_p} \geqq 0$ for all sets $(i_1, i_2, \ldots, i_p)$, where $a^s_{i_s}, d^s_{i_s i_p}, b^s_{i_s i_p}, c_{i_1 \ldots i_p}$ are given real numbers and $a^s_{i_s} > 0$, $d^s_{i_s i_p} \geqq 0$, $b^s_{i_s i_p} \geqq 0$, $c_{i_1 \ldots i_p} \geqq 0$, $\sum_{i_s=1}^{n_s} a^s_{i_s} = K$, $s \in N_p$. This problem has a solution if and only if the following inequalities are satisfied for all $s \in N_{p-1}$: (Kravtsov & Kashinski 1977)

$$a^s_{i_s} \geqq \sum_{i_p=p}^{n_p} d^s_{i_s i_p} \qquad \forall i_s \in N_{n_s} \text{ ,}$$

$$a^p_{i_p} \geqq \sum_{i_s=1}^{n_s} d^s_{i_s i_p} \qquad \forall i_p \in N_{n_p} \text{ ,}$$

$$\sum_{i_s=1}^{n_s} \min\Big(a^s_{i_s} - \sum_{i_p=1}^{n_p} d^s_{i_s i_p} \text{ , } \sum_{i_p \in I} (b^s_{i_s i_p} - d^s_{i_s i_p})\Big) \geqq \sum_{i_p \in I} \Big(a^p_{i_p} - \sum_{i_s=1}^{n_s} d^s_{i_s i_p}\Big) \text{ ,} \forall I \subseteq N_{n_p} \text{ .}$$

Conditions for the solvability of other special cases of multi-indexed transportation problems can be found in Kravtsov & Kashinski (1977) and Talanov & Shevchenko (1972).

8. The solution of the triply-indexed transportation problem with axial sums :

$$\text{minimize} \quad \sum_{i=1}^{m} \sum_{t=1}^{k} \sum_{j=1}^{n} (d_{it} + h_{tj}) x_{itj} \text{ ,}$$

$$\sum_{t=1}^{k} \sum_{j=1}^{n} x_{itj} = a_i \ , \ \forall i \in N_m \ , \ \sum_{i=1}^{m} \sum_{t=1}^{k} x_{itj} = b_j \ , \ \forall j \in N_n \ ,$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{itj} = c_t \ , \ \forall t \in N_k \ , \ \ x_{itj} \geqq 0 \ , \ \forall (i,t,j) \in N_m \times N_k \times N_n$$

can be reduced to the following pair of classical transportation problems

$$1)\ \min\left\{ \sum_{i=1}^{m} \sum_{t=1}^{k} d_{it} y_{it} : (y_{it})_{m\times k} \varepsilon M(a,c) \right\} ; 2)\ \min\left\{ \sum_{t=1}^{k} \sum_{j=1}^{n} h_{tj} z_{tj} : (z_{tj})_{k\times n} \varepsilon M(c,b) \right\} ,$$

that is, a correspondence between the optimal solution of the first problem and the optimal solutions of problems 1) and 2) is given by the formulae

$$y_{it} = \sum_{j=1}^{n} x_{itj} \ , \ \forall (i,t) \in N_m \times N_k \ ; \ z_{tj} = \sum_{i=1}^{m} x_{itj} \ , \ \forall (t,j) \in N_k \times N_n \ .$$

9. (Bolker 1976). The number of vertices of a planar transportation polytope of order $m\times n\times \underbrace{2\times 2\times \ldots \times 2}_{p-2}$, $m,n \geqq 2$, does not exceed the number $m^{n-1} n^{m-1} 2^{(p-2)(m-1)(n-1)}$.

10. Formulate sufficient conditions on the components of the vectors a^s , $s \in N_p$, such that the axial transportation polytope $M(a^1,a^2,\ldots,a^p)$ of order $n_1\times n_2\times \ldots \times n_p$ given by the integer-valued vectors $a^1,a^2,\ldots,a^p$ will have the minimum number of integral vertices.

11. (Kravtsov 1979). The maximum number of integral vertices in the class of axial transportation polytopes $M(a^1,a^2,\ldots,a^p)$ of order $n_1\times n_2\times \ldots \times n_p$, $n_1 = \max\limits_{1\leqq s\leqq p} n_s$, defined by the integral vectors $a^1,a^2,\ldots,a^p$, is not less than the number $\prod\limits_{s=2}^{p} \phi(n_1,n_s)$, where $\phi(n_1,n_s)$ is the maximum number of vertices in the class of classical transportation polytopes of order $n_1\times n_s$.

12. (Kononenko, Mikulski & Trukhanovski 1976). Let $n \geq m \geq k$. Show that Schell's conditions are sufficient only in the cases $k = 2$, $m = 2,3$.

13. The Moravek-Vlach conditions are necessary and sufficient when $\min(m,n,k) = 2$.

14. Exhibit a triply-indexed planar transportation polytope which satisfies Haley's condition but which does not satisfy Smith's condition.

15. The feasible set of a p-indexed, 1-fold transportation problem of order $n_1 \times n_2 \times \dots \times n_p$ is called a *p-indexed planar transportation polytope of order* $n_1 \times n_2 \times \dots \times n_p$. Formulate and prove the analogues of the Schell, Haley, Moravek-Vlach and Smith conditions for a p-indexed planar transportation polytope of order $n_1 \times n_2 \times \dots \times n_p$.

16. The triply-indexed planar transportation polytope $M(A,B,C)$ of order $m \times n \times 2$ is non-empty if and only if the following conditions are satisfied (Haley 1967):

$$\sum_{i=1}^{m} \min\left(a_{i1}, \sum_{j \in J} c_{ij}\right) \geq \sum_{j \in J} b_{j1}, \qquad \forall J \subseteq N_n .$$

Trukhanovski (1979) generalized these results to the case of a p-indexed planar transportation polytope of order $m \times n \times 2 \times 2 \times \dots \times 2$.

17. For the triply-indexed planar transportation polytope $M(A,B,C)$ of order $m \times n \times k$ to be non-empty it is sufficient that at least one of the following conditions is satisfied:

$$1)\quad a_{it} = \sum_{j=1}^{n} \frac{b_{jt} c_{ij}}{\sum_{r=1}^{m} c_{rj}}, \quad \forall (i,t) \in N_m \times N_k ; \quad 2)\quad b_{jt} = \sum_{i=1}^{m} \frac{a_{it} c_{ij}}{\sum_{s=1}^{n} c_{is}}, \forall (j,t) \in N_n \times N_k ;$$

$$3)\quad c_{ij} = \sum_{t=1}^{k} \frac{a_{it} b_{jt}}{\sum_{r=1}^{m} a_{rt}}, \quad \forall (i,j) \in N_m \times N_n .$$

18. Consider the triply-indexed planar transportation problem:

$$\min \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{t=1}^{k} d_{ijt} x_{ijt} : x = (x_{ijt})_{m \times n \times k} \in M(A,B,C) \right\},$$

where d_{ijt} are given real constants. Show that the point $(x_{ijt})_{m \times n \times k} \in M(A,B,C)$ is the optimal solution to this problem if and only if there exist numbers u_{it}, $(i,t) \in N_m \times N_k$, v_{jt}, $(j,t) \in N_n \times N_k$, w_{ij}, $(i,j) \in N_m \times N_n$ such that

$$u_{it} + v_{jt} + w_{ij} = d_{ijt}, \text{ if } x_{ijt} > 0 ; \quad u_{it} + v_{jt} + w_{ij} \leq d_{ijt}, \text{ if } x_{ijt} = 0 .$$

19. (Haley 1963). The solution of the axial transportation problem:

$$\text{minimize} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{t=1}^{k} d_{ijt} x_{ijt} ,$$

subject to the conditions

$$\sum_{i=1}^{m}\sum_{t=1}^{k} x_{ijt} = b_j , \quad j \in N_n , \quad \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijt} = c_t , \quad t \in N_k ,$$

$$\sum_{j=1}^{n}\sum_{t=1}^{k} x_{ijt} = a_i , \quad i \in N_m , \quad x_{ijt} \geqq 0 , \quad (i,j,t) \in N_m \times N_n \times N_k$$

can be reduced to the solution of the following triply-indexed planar transportation problem:

$$\text{minimize} \quad \sum_{i=1}^{m+1}\sum_{j=1}^{n+1}\sum_{t=1}^{k+1} d_{ijt}x_{ijt} \quad , \text{ subject to}$$

$$\sum_{i=1}^{m+1} x_{ijt} = b_{jt} , \forall (j,t) \in N_{n+1}\times N_{k+1} , \quad \sum_{j=1}^{n+1} x_{ijt} = a_{it} , \forall (i,t) \in N_{m+1}\times N_{k+1} ,$$

$$\sum_{t=1}^{k+1} x_{ijt} = c_{ij} , \forall (i,j) \in N_{m+1}\times N_{n+1} , \quad x_{ijt} \geqq 0 , \ \forall (i,j,t) \in N_{m+1}\times N_{n+1}\times N_{k+1} .$$

20. (Yemelicheva & Kononenko 1974ii). Every integer γ satisfying the relation $\prod_{s=1}^{p} n_s - \prod_{s=2}^{p} (n_s-1) \leqq \gamma \leqq \prod_{s=1}^{p} n_s$, can equal the number of facets of a p-indexed non-degenerate planar transportation polytope of order $n_1 \times n_2 \times \ldots \times n_p$, $n_1 = \max_{1 \leqq s \leqq p} n_s$.

21. (Kononenko 1973). The number of bases of a p-indexed planar transportation polytope of order $m \times n \times 2 \times 2 \times \ldots \times 2$ is

$$m^{n-1} n^{m-1} 2^{(p-2)(m-1)(n-1)} \quad (m,n \geqq 2).$$

22. (Motzkin 1952). Every p-indexed planar transportation polytope of order $m \times n \times 2 \times 2 \times \ldots \times 2$, defined by integral matrices is integral.

23. (Fedenya 1977). Theorem 2.2 generalizes in the following way to the multi-indexed case: among the p-indexed planar transportation polytopes of order $n_1 \times n_2 \times \ldots \times n_p$ there are $\left(\prod_{s=1}^{p} (n_s-1)\right)$-simplexes in two cases: 1) $n_s \geqq 2$, $s \in N_p$, p - odd; 2) $\min_{1 \leqq s \leqq p} n_s = 2$, p - even.

24. Let σ be the minimum number of vertices in the class of non-degenerate p-indexed planar transportation polytopes of order $n_1 \times n_2 \times \ldots \times n_p$, where p is even. Then σ satisfies:

$$\prod_{s=1}^{p}(n_s-1)+1 \leqq \sigma \leqq \left(\frac{\prod_{s=1}^{p}(n_s-1)}{\min\limits_{1\leq s\leq p} n_s - 1} + 1\right)^{\min\limits_{1\leq s\leq p} n_s - 1} .$$

25. (De Werra 1978). Consider the system:

$$\sum_{t=1}^{k} x_{ijt} = a_{ij} , \forall (i,j) \in N_m \times N_n , \quad \sum_{j=1}^{n} x_{ijt} = \alpha_i , \forall i \in N_m, t \in N_k$$

$$\sum_{i=1}^{m} x_{ijt} = \beta_j , \forall j \in N_n , t \in N_k ,$$

where a_{ij}, α_i, β_j are given non-negative integers. A necessary and sufficient condition for this system to have an integral solution is:

$$\sum_{i=1}^{m} a_{ij} \leqq k\beta_j , \forall j \in N_n , \quad \sum_{j=1}^{n} a_{ij} \leqq k\alpha_i , \forall i \in N_m .$$

26. Let $R \subseteq N_m$, $\bigcup_{p=1}^{s} P_p = N_m$, $\bigcup_{q=1}^{\ell} Q_q = N_n$, $P_p \cap P_q = Q_p \cap Q_q = \emptyset$ when $p \neq q$. Also, let a_i, b_j, a_{ik}, b_{jk}, d^-_{pq}, d^+_{pq} be given non-negative numbers. Consider the system:

$$\sum_{i=1}^{m}\sum_{k=1}^{t} x_{ijk} \leqq \sum_{k=1}^{t} b_{jk} , \forall t \in N_{r-1} , j \in N_n ; \quad \sum_{j=1}^{n}\sum_{k=1}^{t} x_{ijk} \leqq \sum_{k=1}^{t} a_{ik} , \forall t \in N_{r-1} , i \in N_m ;$$

$$\sum_{i=1}^{m}\sum_{k=1}^{r} x_{ijk} = b_j , \forall j \in N_n \quad ; \quad \sum_{j=1}^{n}\sum_{k=1}^{r} x_{ijk} = a_i , \forall i \in R ;$$

$$\sum_{j=1}^{n}\sum_{k=1}^{r} x_{ijk} \leqq a_i , \forall i \in \bar{R} \quad ; \quad x_{ijk} \geqq 0 , \forall (i,j,k) \in N_m \times N_n \times N_r ;$$

$$d^-_{pq} \leqq \sum_{i\in P_p}\sum_{j\in Q_q}\sum_{k=1}^{r} x_{ijk} \leqq d^+_{pq} , \forall (p,q) \in N_s \times N_\ell .$$

Necessary and sufficient conditions for this system to be consistent are given by:

$$\sum_{i\in R} a_i \leqq \sum_{j=1}^{n} b_j \leqq \sum_{i=1}^{m} a_i ,$$

$$\sum_{k=1}^{r} a_{ik} = a_i , \forall i \in R ; \quad \sum_{k=1}^{r} a_{ik} \leqq a_i , \forall i \in \bar{R} ;$$

$$\alpha_p \geqq 0 , \beta_q \geqq 0 , \forall p \in N_s , q \in N_\ell ,$$

$$\sum_{p=1}^{s} \min(\alpha_p, \gamma_p) \geqq \sum_{q\in L} \beta_q , \quad \forall L \subseteq N_\ell , \quad \text{where}$$

$$\alpha_p = \sum_{i\varepsilon P_p} a_i - \sum_{q=1}^{\ell} d^-_{pq} \;, \quad \beta_q = \sum_{j\varepsilon Q_q} b_j - \sum_{p=1}^{s} d^-_{pq} \;, \quad \gamma_p = \sum_{q\varepsilon L} (d^+_{pq} - d^-_{pq}) \;.$$

27. (Talanov & Ilichev 1979). Let a_{ij}, b_i, d_j, g be given non-negative integers. Consider the system:

$$\sum_{k=1}^{\ell} x_{ijk} = a_{ij} \;, \quad \forall i \,\varepsilon\, N_m \,, j \,\varepsilon\, N_n \;; \quad \sum_{j=1}^{n} x_{ijk} \leqq b_i \;, \forall i \,\varepsilon\, N_m \,, k \,\varepsilon\, N_\ell \;;$$

$$\sum_{i=1}^{m} x_{ijk} \leqq d_j \;, \forall j \,\varepsilon\, N_n \,, k \,\varepsilon\, N_\ell \;; \quad \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} \leqq g \;, \forall k \,\varepsilon\, N_\ell \;;$$

$$x_{ijk} \geqq 0 \;, \quad \forall (i,j,k) \,\varepsilon\, N_m \times N_n \times N_\ell \;.$$

Necessary and sufficient conditions for this system to have a solution in integers are given by:

$$\sum_{j=1}^{n} a_{ij} \leqq \ell b_i \;, \forall i \,\varepsilon\, N_m \;, \quad \sum_{i=1}^{m} a_{ij} \leqq \ell d_j \;, \forall j \,\varepsilon\, N_n \;, \quad \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \leqq \ell g \;.$$

28. (Talanov & Ilichev 1979). Let h, p, t_{ij}, a_i, b_j be given non-negative integers. Consider the system:

$$\sum_{k=1}^{\ell} x_{ijk} = t_{ij} \;, \quad \forall (i,j) \,\varepsilon\, N_m \times N_n \;,$$

$$\sum_{i=1}^{m} x_{ijk} \leqq 1 \;, \quad \forall j \,\varepsilon\, N_n \;, \; k = 1,2,\ldots,\ell h,$$

$$\sum_{j=1}^{n} x_{ijk} \leqq 1 \;, \quad \forall i \,\varepsilon\, N_m \;, \; k = 1,2,\ldots,\ell h,$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} \leqq p \;, \quad k = 1,2,\ldots,\ell h,$$

$$\sum_{j=1}^{n} \sum_{k=(s-1)h+1}^{sh} x_{ijk} \leqq a_i \;, \forall i \,\varepsilon\, N_m \;, \; s \,\varepsilon\, N_\ell \;,$$

$$\sum_{i=1}^{m} \sum_{k=(s-1)h+1}^{sh} x_{ijk} \leqq b_j \quad \forall j \,\varepsilon\, N_n \;, \; s \,\varepsilon\, N_\ell \;,$$

$$x_{ijk} \,\varepsilon\, \{0,1\} \;, \quad i \,\varepsilon\, N_m \;, \; j \,\varepsilon\, N_n \;, \; k = 1,2,\ldots,\ell h \;.$$

Necessary and sufficient conditions for this system to be consistent are

$$\sum_{j=1}^{n} t_{ij} \leqq \ell a_i \;, \forall i \,\varepsilon\, N_m \;; \quad \sum_{i=1}^{m} t_{ij} \leqq \ell b_j \;, \forall j \,\varepsilon\, N_n \;; \quad \sum_{i=1}^{m} \sum_{j=1}^{n} t_{ij} \leqq \ell ph \;.$$

29. (Yemelichev & Kononenko 1974). In the theory of Latin squares the following results are well known: 1) the number of pairwise orthogonal Latin squares of order n can not exceed $n-1$; 2) pairs of orthogonal Latin squares of order 6 and of order 2 do not exist.

Using these results, show that:

1) for a (t,n,p)-orthogonal system of cubes to exist it is necessary that $t-p < n-1$;

2) $G(t,n,p) = 0$, if $1 < p < t-1$, $n = 2,6$;

3) $G(t,n,p) = 0$, if $t \geq n+p$, $p > 1$.

30. Show, by means of an example, that when $p > 2$ the p-indexed transportation polytope defined by equations (3.5) and the conditions $x_{i_1 i_2 \ldots i_p} \geqq 0$, $\forall i_s \in N_{n_s}$, $s \in N_p$, can have non-integral vertices.

31. Let the vertices of the hypergraph G be given by the marked (labelled) points in the set $B = \bigcup_{1 \leqq k_1 < k_2 < \ldots < k_m \leqq p} B_{k_1 k_2 \ldots k_m}$, where

$B_{k_1 k_2 \ldots k_m} = \{b_{i_1 i_2 \ldots i_{k_1 -1} * i_{k_1 +1} \ldots i_{k_m -1} * i_{k_m +1} \ldots i_p}\}$, $i_s \in N_{n_s}$, $s \in N_p$.

and where each edge consists of $\binom{p}{m}$ points of the form $b_{i_1^0 i_2^0 \ldots i_{k_1 -1}^0 * i_{k_1 +1}^0 \ldots i_{k_m -1}^0 * i_{k_m +1}^0 \ldots i_p^0}$, $1 \leq k_1 < k_2 < \ldots < k_m \leqq p$.

If no two edges of G are incident and if every vertex is incident to some edge, then the hypergraph G is called a *complete* $\binom{p}{m}$*-combination*. The number of such hypergraphs is denoted by $\Gamma(p,n,m)$. When $p=2$, $m=1$, we obtain a graph which is called a *complete pairing* . Establish the following formulae:

$$\Gamma(p,n,m) = |T(p,n,m)| = G(p,n,p-m) ,$$

$$\Gamma(p,n,p-1) = (n!)^{p-1} , \quad \Gamma(p,n,1) = L(n,p-1) .$$

32. (Yemelichev & Kononenko 1974). The existence of a $(t,n,2)$-orthogonal system of Latin squares implies the existence of a $(t+2,n,2)$-orthogonal system of square matrices. However, an analagous assertion is not true even for 3-cubes. Show, by means of an example, that the existence of a $(4,3,3)$-orthogonal system of Latin cubes D_1, D_2, D_3, D_4 does not imply the existence of a $(7,3,3)$-orthogonal system $E_1, E_2, E_3, D_1, D_2, D_3, D_4$.

33. Algorithms are known which solve the assignment problem of order $n \times n$ in $O(n^3)$ operations. (Dinitz & Kronrod 1969, Kravtsov, Sherman & Averbukh 1975). However, even for the triply-indexed axial selection problem effective algorithms are not known.

34*. Is it true that every integer from 0 to $(m-1)(n-1)(k-1)$, and only these integers, can equal the dimension of a triply-indexed planar transportation polytope of order $m\times n\times k$?

35*. Prove or disprove the following statement: there is no non-degenerate axial transportation polytope $M(a^1,a^2,\ldots,a^p)$ of order $n_1\times n_2\times\ldots\times n_p$, $n_1\geqq n_2\geqq\ldots\geqq n_p\geqq 2$, $n_1\geqq 3$, whose number of vertices satisfies

$$\prod_{s=2}^{p}\left(\prod_{i=1}^{s-1} n_i\right)^{n_s-1} < f_0(M(a^1,a^2,\ldots,a^p)) < \prod_{s=2}^{p}\left(\prod_{i=1}^{s-1} n_i\right)^{n_s-1} + \prod_{s=1}^{p} n_s - \sum_{s=1}^{p} n_s + p - 2 .$$

36*. Is it true that every integer of the form $(m-1)(n-1)(k-1)+t$, where $1\leq t\leq mk+nk+mn-k-m-n$, and only these integers, can equal the number of facets of a non-degenerate planar triply-indexed transportation polytope of order $m\times n\times k$, $m,n,k\geqq 2$?

37*. Is the following statement true? There is no non-degenerate planar triply-indexed transportation polytope $M(A,B,C)$ of order $m\times n\times k$, whose number of vertices satisfies the inequalities

$$(m-1)(n-1)(k-1)+1 < f_0(M(A,B,C)) < 2(m-1)(n-1)(k-1).$$

38*. Is it true that every integer from $\sum_{s=2}^{p}(n_s-1)$ to $\sum_{s=1}^{p} n_s - p + 1$, and only these integers can equal the diameter of a non-degenerate axial transportation polytope of order $n_1\times n_2\times\ldots\times n_p$, $n_1\geqq n_2\geqq\ldots\geqq n_p\geqq 2$, $n_1\geqq 3$, $p\geqq 2$?

39. (Smith 1973). For a planar transportation polytope $M(A,B,C)$ of order $m\times n\times k$ to be non-empty it is necessary for the following inequalities to be satisfied:

$$b(J,K_1)+c(K_2,\bar{J})-a(K)\geqq 0,\quad \forall J\subseteq N_n,\ K\subseteq N_m\times N_k;$$
$$a(I,S_1)+c(\bar{I},S_2)-b(S)\geqq 0,\quad \forall I\subseteq N_m,\ S\subseteq N_n\times N_k;$$
$$a(P_2,T)+b(P_1,\bar{T})-c(P)\geqq 0,\quad \forall T\subseteq N_k,\ P\subseteq N_m\times N_n,$$

where $K_1=\{t:(i,t)\varepsilon K\}$, $K_2=\{i:(i,t)\varepsilon K\}$, $P_1=\{j:(i,j)\varepsilon P\}$, $P_2=\{i:(i,j)\varepsilon P\}$, $S_1=\{t:(j,t)\varepsilon S\}$, $S_2=\{j:(j,t)\varepsilon S\}$, and the symbol $z(R)$ denotes the sum of those elements of the matrix $(z_{ij})_{m\times n}$ for which the index pair (i,j) takes all values in the set R, $R\subseteq N_m\times N_n$.

40. Let $M(A_0,B_0,C_0)$ and $M(A_1,B_1,C_1)$ be non-empty planar transportation polytopes of the same order, then every polytope $M(A_\lambda,B_\lambda,C_\lambda)$, $0<\lambda<1$, defined by the matrices $A_\lambda=\lambda A_1+(1-\lambda)A_0$, $B_\lambda=\lambda B_1+(1-\lambda)B_0$, $C_\lambda=\lambda C_1+(1-\lambda)C_0$, is also non-empty.

1. Obtain necessary and sufficient conditions for the existence of a 3-polytope each of whose vertices has a specified number of adjacent vertices. In other words, describe those sequences which can be realized as the degrees of the vertices of a 3-polyhedral graph. Such sequences can be called *polyhedral sequences* . (Sainte Marie C. , Question 505 , Intermed. Math., 1895, 2).

2. Characterize those polyhedral sequences which correspond to a unique combinatorial type of polytope.

3. (*Walsh's conjecture*). The sequence $a_1, \dots, a_m$ is called *unimodal* if there does not exist $i < j < k < \ell$ such that $a_i < a_j > a_k < a_\ell$. Is it true that the sequence $\{a_k\}$ is unimodal if a_k is: (1) the number of k-faces of a polytope; (2) the number of non-isomorphic matroids of rank k on n elements? (Combinatorics. Proceedings Confer. Combin. Math., Oxford, 1972). (This conjecture was shown to be false by Bjorner A. *The unimodality conjecture for convex polytopes.* Bull. Amer. Math. Soc., 1981, 4, No.2)

4. When $d \geq 4$, do there exist d-polyhedral graphs which uniquely determine a polytope up to combinatorial equivalence?

5. Let the graph G be d'-polyhedral and also d"-polyhedral. Does this imply that G is d-polyhedral for all $d' \leq d \leq d''$?

6. Let M be a 3-polytope, C a simple cycle containing n vertices in the graph of M, and F a convex n-gon in the plane H . Then, there exists a polytope M' which is combinatorially equivalent to M , and an orthogonal projection $\pi : E_3 \to H$ such that $\pi(M') = F$ and $\pi^{-1}(\mathrm{bd}F)$ coincides with the vertices and edges of M' corresponding to the cycle C .

7. For any integer d there exists a finite extension $\overline{Q}$ of the rational field Q such that every combinatorial type of d-polytope is realizable in $\overline{Q}_d$.

8. (*Maximum diameter conjecture*). $\Delta(d,n) \leqq n-d$. (see §2, Ch.2)

9. Construct simplex algorithms which will traverse the vertices of a non-returning chain. Will such algorithms be polynomial?

10. (*Padberg's conjecture*). Let A be a Boolean matrix. Then the following assertions are equivalent:

(1) the relaxed polytope $M^{\leq}(A,e)$ is integral;

(2) A does not contain an $(m \times k)$-submatrix A' which has the property $\pi_{\beta,k}$ where $3 < k < n$ and $\beta = 2$, $[k/2]$, $k-1$ (compare with Theorem 5.5, Ch.4). (see Padberg 1974, Ch.4).

11. Find necessary and sufficient conditions for the polytope $M(A,b)$ to be quasi-integral for any integral vector b .

12. Find necessary and sufficient conditions for the polymatroids $M(\rho_1)$ and $M(\rho_2)$ to be combinatorially equivalent, where ρ_1 and ρ_2 are submodualr functions.

13. Obtain an analytical specification for the standardization problem polytope $H_m(Q)$ for arbitrary (not Boolean) matrix Q .

14. Which of the following problems are NP-complete: check whether a given inequality $ax \leqq b$ defines a facet of the polytope which is the convex hull of the characteristic vectors of (1) the hamiltonian cycles of a given graph; (2) the vertex packings of a given graph; (3) the vertex coverings of a given graph? (See, Karp R.M. & Papadimitriou C.H., *On linear characterizations of combinatorial optimization problems*, MIT Lab. Comput. Sci. Techn. Mem., 1980, No. 154).

15. Obtain an analytical representation of the polytope $\operatorname{conv}\{a_\pi : \pi \in H\}$ for the case in which: (1) H is the set of cyclical permutations; (2) H is the set of derangements; (3) H is the dihedral subgroup. In each case give a criterion for two vertices to be adjacent.

16. Characterize the class of polytopes for which the diameter is equal to the radius.

17. (*Siegel's conjecture*). If the maximum diameter conjecture is true for a given polytope then it is true for its intersection with a cube.

18. (*Grünbaum's conjecture for the minimum number of faces*).

$$\mu_k(d,n) = \binom{d+1}{k+1} + \binom{d}{k+1} - \binom{2d+1-n}{k+1}$$

for all $k \in N_{d-1}$, $d+1 \leqq n \leqq 2d$.

19. (*Two conjectures of Barnette*). (1) The graph of any simple 4-polytope is hamiltonian. (2) If every facet of a simple 3-polytope has an even number of edges, then the graph of this polytope is hamiltonian.

20. The diameter of any non-degenerate classical transportation polytope of order $m \times n$, $3 \leq m \leq n$, $n \geq 4$, with $(m-1)n + k$ facets, $0 \leq k \leq n$, and having the maximum number of vertices is equal to $m + k - 1$ (see Yemelichev, Kravtsov & Krachkovsky 1979, Ch.6).

21. Is it true that every whole number of the form $m + t$, where $0 \leqslant t \leqslant k-1$, and only such numbers, are realized as the diameter of a non-degenerate classical transportation polytope of order $m \times n$, $2 \leq m \leq n$, with $(m-1)n + k$ facets for each $k \in N_n$?

22. Is it true that the graph of any classical transportation polytope is hamiltonian but is not pan-cyclic.

23. The degenerate classical transportation polytope $M(a,b)$ of order $m \times n$, where m and n are coprime, has the maximum number of vertices if and only if it is 1-degenerate and the spectrum $S(a,b,a^*,b^*) = \emptyset$.

24. Almost all classical transportation polytopes have the maximum number of edges.

25. Let $\phi_1(m,n), \phi_2(m,n), \ldots, \phi_t(m,n)$ be a sequence of all possible values of the number of vertices of a classical transportation polytope of order $m \times n$.

(1) Find the number t.

(2) Is it true that $\lim\limits_{m,n\to\infty} \dfrac{\max\limits_{1\leq s\leq t} \mu(W^s_{m\times n})}{\mu(W_{m\times n})} = 0$? Here, $\mu(W)$ is the Lebesgue measure of the set W in the space E_{m+n-1}, $W^s_{m\times n} = \{(a,b) \in W_{m\times n} : f_0(M(a,b)) = \phi_s(m,n)\}$, and $W_{m\times n}$ has the same meaning as in §10, Ch.6.

26. Let the set of all classical transportation polytopes of order $m \times n$ with the maximum number of vertices be partitioned into the classes of equivalent polytopes. When m and n are coprime there is only one such class, (see Cor. 7.5, Ch.6). Determine the number of equivalence classes when $(m,n) \neq 1$ (see Reverz 1961, Ch.7).

27. The maximum number of vertices in the class of symmetric transportation polytopes of order $n \times n$ (see Ex.27, Ch.6) is not less than $\sum\limits_{j=0}^{n-1} \prod\limits_{k=0}^{j} (k^2 + 1)$ (see Dubois 1973, Ch.7).

28. Let $m,n \geq 3$, $1 \leq k \leq mn+m+n-1$. The minimum number of vertices in the class of non-degenerate regularly truncated transportation polytopes

of order $m \times n$ with $(m-1)(n-1)+k$ facets is equal to $k(m-1)(n-1)-k+2$.

This conjecture is true for $1 \leq k \leq (m-1)(n-1)+2$ (see Ex.7 Ch.7).

29. Every whole number from 1 to $m+n-1$, and only these numbers, are realized as the diameter of a transportation polytope of order $m \times n$.

30. The axial transportation polytope $M(a^1, a^2, \ldots, a^p)$ of order $n_1 \times n_2 \times \ldots \times n_p$, $n_1 = \max_{1 \leq s \leq p} n_s$, $p > 2$, defined by the integer vectors $a^1, a^2, \ldots, a^p$, has the maximum number of integral vertices if and only if every classical transportation polytope $M(a^1, a^i)$, $i = 2, \ldots, p$ has the maximum number of vertices.

31. The graph of any non-degenerate axial transportation polytope of order $n_1 \times n_2 \times \ldots \times n_p$ with the minimum number of vertices is hamiltonian.

This conjecture is true when $p = 2$ (see Ex.59,Ch.6).

32. The planar triply-indexed non-degenerate transportation polytope $M(A^*, B^*, C^*)$ of order $m \times n \times k$ defined by the matrices

$$A^* = \begin{pmatrix} n & n & .. & n \\ n & n & .. & n \\ . & . & .. & . \\ n & n & .. & n \end{pmatrix}, \quad B^* = \begin{pmatrix} m & m & .. & m \\ m & m & .. & m \\ . & . & .. & . \\ m & m & .. & m \end{pmatrix}, \quad C^* = \begin{pmatrix} k & k & .. & k \\ k & k & .. & k \\ . & . & .. & . \\ k & k & .. & k \end{pmatrix},$$

has the maximum number of vertices.

33. Is it true that every whole number from 1 to $mk + nk + mn - m - n - k + 1$, and only these numbers, are realized as the diameter of a planar triply-indexed transportation polytope of order $m \times n \times k$?

34. The maximum radius in the class of classical transportation polytopes of order $m \times n$, $m, n \geq 3$, coincides with the maximum diameter in the same class and is equal to $m + n - 1$.

35. Is there a simplex algorithm for which the number of iterations does not exceed the diameter of the polytope?

REFERENCES

Introduction

Aleksandrov, A.D. (1950) *Convex Polytopes.* (Russian). Moscow & Leningrad. Gostekhizdat.

Bartels, H. (1973) *A priori Informationen zur linearen Programmierung. - Über Ecken und Hyperflachen auf Polyedern.* Meisenheim.

Boltyanski, V.G. & Soltan, P.S. (1978) *Combinatorial geometry of different classes of convex sets.* (Russian). Kishinev: Shtiintsa.

Brückner, M. (1900) *Vielecke und Vielfläche.* Leipzig. Teubner.

Charin, V.S. (1978) *Linear mappings and convex sets.* (Russian). Kiev. Vishcha Shkola.

Chernikov, S.N. (1968) *Linear inequalities.* (Russian). Moscow. Nauka.

Dantzig, G.B. (1963) *Linear programming and extensions.* Princeton. Princeton Un. Pr.

Grünbaum, B. (1967) *Convex polytopes.* New York. Interscience.
(1970) *Polytopes, graphs and complexes.* Bull.Amer.Math. Soc. 1970, 76.

Hadwiger, H. & Debrunner, H. (1964) *Combinatorial geometry in the plane.* New York. Holt.

Kantorovich, L.V. (1939) *Mathematical methods for the organization and planning of production.* (Russian). Leningrad.

Khachian, L.G. (1980) *Polynomial algorithms in linear programming.* (Russian). Zhur.Vych.Mat. i Mat.Fiz. 1980. No.1.

Klee, V. (1965) *A class of linear programming problems requiring a large number of iterations.* Numer.Math. 1965. 7.
(1974) *Convex polyhedra and mathematical programming.* Proc.Inter.Congr.Math. Vancouver. 1974.

Kirkman, T. (1855) *On the representation and enumeration of polyhedra.* Memoirs of the Literary and Philosophical Society of Manchester. 1855. 2. No.12.

Minkowski, H. (1910) *Geometrie der Zahlen.* Leipzig.
(1911) *Theorie der konvexen Körper.* Gesamm.Abhandlungen. Leipzig. 1911. 2.

McMullen, P. (1974) *Metrical and combinatorial properties of convex polytopes.* Proc.Inter.Congr.Math. Vancouver. 1974.

McMullen, P & Shephard, G. (1971) *Convex polytopes and the upper bound conjecture.* Cambridge. 1971.

Pogorelov, A.V. (1969) *Extrinsic geometry of convex surfaces.* (Russian). Moscow. Nauka. 1969. (English translation - Am.Math.Soc. Transl. 35. Providence. 1973).

Soltan, P.S. (1976) *Extremal problems on convex sets.* (Russian). Kishinev. Shtiintsa.

Steiner, J. (1881) *Problème de situation.* Gesammelte Werke. Berlin. 1881. 1.

Todd, M. (1976) *A combinatorial generalization of polytopes.* J.Combinat.Theory. 1976. 20. No.3.

Tutte, W. (1973) *The enumerative theory of planar maps.* A Survey of Combinatorial Theory. North Holland.

Voronoj,G. (1908) *Sur quelques propriétés des formes quadratiques positives parfaites.* J.reine und angew.Math. 1908. 133.
Voronoj,G. (1908/9) *Recherches sur les paralleloedres primitifs.* J.reine und angew.Math. 1908. 134; 1909. 136.
Weyl,H. (1935) *Elementare Theorie der konvexen Polyeder.* Comment. Math.Helvetici. 1935. 7.
Yablonski,S.V. (1979) *Introduction to discrete mathematics.* (Russian). Moscow. Nauka.
Yablonski,S.V. & Lupanova,O.B. (Eds.) (1974) *Discrete mathematics and mathematical questions in cybernetics.* (Russian). Moscow. Nauka. 1974.
Yeremin,I.I. & Astafiev,N.N. (1976) *Introduction to the theory of linear and convex programming.* (Russian). Moscow. Nauka.
Yeremin,I.I. & Mazurov,V.D. (1979) *Non-stationary processes in mathematical programming.* (Russian). Moscow. Nauka.

Chapter 1

Ashkinuze,V.G. (1963) *Polygons and polytopes.* (Russian). Encyclopedia of elementary mathematics, 4.
Bartels,H. (1973) *A priori Informationen zur linearen Programmierung. - Über Ecken and Hyperflächen auf Polyedern.* Meisenheim.
Beinecke,L. (1974) *The maximum number of strongly connected sub-tournaments.* (Russian). In book *Graph Theory.* Moscow. Mir. 1974.
Carathéodory,C. (1907) *Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen.* Math. Ann. 1907. 64.
(1911) *Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen.* Rend.Circ. Mat.Palermo. 1911, 32.
Dehn.M. (1905) *Die Eulersche Formel in Zusammenhang mit dem Inhalt in der nicht-Euklidischen Geometrie.* Math.Ann. 1905. 61.
Euler.L. (1752) *Demonstratio nonnullarum insignium proprietatum, quibus solida hedris planis inclussa sunt praedita.* Novi. Comm.Acad.Sci.Imp.Petropol. 1752. 4. 53.
Farkas,J. (1902) *Über die Theorie der einfachen Ungleichungen.* J. reine und angew.Math. 1902. 124.
Fenchel,W. (1953) *Convex cones, sets and functions.* Princeton. 1953.
Fulkerson,D. (1971) *Blocking and anti-blocking pairs of polyhedra.* Math. Program. 1971. 1.
Gale,D. (1956) *Neighbouring vertices on a convex polyhedron.* In: *Linear inequalities and related systems.* Princeton.
(1963) *Neighborly and cyclic polytopes.* Proc.Symp.Pure Math. 1963. 7 (Convexity).
Grünbaum,B. (1967) *Convex polytopes.* New York. Interscience.
(1970) *On combinatorial spheres.* Combinatorial Structures and their Applications.
Hilbert,D. & Cohn-Vossen,S. (1932) *Geometry and the imagination.* English translation by P.Nemenyi. New York. Chelsea. 1952.
Karmanov,V.G.(1975/80) *Mathematical programming.* (Russian). Moscow. Nauka. 1975,1980.
Klee,V. (1964) *The number of vertices of a convex polytope.* Can.J. Math. 1964. 16.
Kovalev,M.M. , Milanov,P. & Isachenko,A.N. (1978) *On linear programming problems with the maximum number of bases.* Izv.Akad.Na.BSSR, ser.fiz.-mat.nauk. 1978. No.3.

Kovalev,M.M. & Isachenko,A.N. (1978) *On polytopes defined by the constraints of linear programming.* (Russian). Dokl.Akad.Na.BSSR, ser.fiz.-mat.nauk. 1978. 22, No.3.

Kowaljow,M. & Milanow,P. (1976) *Kombinatorische Eigenschaften der Lösungspolyeder der linearen Optimierung.* Intern.Wiss.Koll. TH Ilmenau. 1976. 21.

(1979) *Matroids of rank 2 and 3.* Intern.Wiss.Koll.TH Ilmenau. 1979. 24.

Kutateladze,S.S & Rubinov,A.M. (1976) *Minkowski duality and its applications.* Novosibirsk. Nauka.

Maltsev,A.I. (1970) *Foundations of linear algebra.* (Russian). Moscow. Nauka.

McMullen,P. (1977) *Valuations and Euler type relations on certain classes of convex polytopes.* Proc.London Math.Soc. 1977, 35.

McMullen,P. & Shephard,G. (1971) *Convex polytopes and the upper bound conjecture.* Cambridge.

Minkowski,H. (1897) *Allgemeine Lehrsätze über die konvexe Polyeder.* Nachr.Ges.Wiss.Göttingen. 1897.

Motzkin,T.S. (1933) *Beiträge zum Theorie der linearen Ungleichungen.* Ph.D.Thesis. Basel.

Poincaré,H. (1893) *Sur la généralisation d'un théorème d'Euler relatif aux polyedres.* Compt.Rend.Acad.Sci.Paris. 1893. 117.

Pshenichni,V.N. (1980) *Convex analysis and extremum problems.* (Russian) Moscow. Nauka.

Rockafellar,R.T. (1970) *Convex Analysis.* Princeton University Press.

Shashkin,I.A.(1963) *A note on adjacent vertices of convex polytopes.* (Russian). Uspekhi Matem.Nauk. 1963. 18. No.5.

Sommerville,D. (1927) *The relations connecting the angle-sums and volume of a polytope in space of* n *dimensions.* Proc.Roy. Soc.London. Ser.A. 1927. 115.

Stanley,R.P. (1980) *The number of faces of a simplicial convex polytope.* Adv. in Math. 1980. 35. No.3.

Steinitz,E. (1922) *Polyeder und Raumeinteilungen.* In: Enzykl.Math. Wiss. 1922. 3.

Stoer,J. & Witzgall,C. (1970) *Convexity and optimization in finite dimensions.* Berlin. Springer.

Weyl,H. (1935) *Elementare Theorie der konvexen Polyeder.* Comment. Math.Helvetici. 1935. 7.

Whitney,H. (1933) *A set of topological invariants for graphs.* Amer.J. Math. 1933. 55.

Chapter 2

Adler,J. (1974) *Lower bounds for maximum diameters of polytopes.* Math.Program.Study 1. 1974.

Balinski,M. (1961) *On the graph structure of convex polyhedra in* n-*space.* Pacific J.Math. 1961. 11.

Barnette,D. (1974) *An upper bound for the diameter of a polytope.* Discr.Math. 1974. 10.

Barnette,D. & Grünbaum,B. (1969) *On Steinitz's theorem concerning convex 3-polytopes and on some properties of 3-connected graphs.* Lecture Notes in Math. Berlin. 1969. 110.

Eberhard,V. (1891) *Zum Morphologie der Polyeder.* Leipzig. Teubner.

Eggleston,H.,Grünbaum,B. & Klee,V. (1964) *Some semicontinuity theorems for convex polytopes and cell-complexes.* Comm.Math.Helv. 1964. 39.

Grünbaum,B. (1975) *Polytopal graphs.* Studies in graph theory. Part II. Studies in Math. 1975. 12.
Harary,F. (1969) *Graph Theory.* Reading. Addison-Wesley.
Klee,V. (1974) *Polytope pairs and their relationship to linear programming.* Acta Math. 1974. 133.
Klee,V. & Minty,G. (1972) *How good is the simplex algorithm?* Inequalities III. Acad.Press.
Klee,V. & Walkup,D. (1967) *The* d-*step conjecture for polyhedra of dimension* $d < 6$. Acta.Math. 1967. 117.
Liusternik,L.(1956) *Convex figures and polytopes.* (Russian). Moscow.
Medyanik,A.I.(1972) *Some combinatorial properties of convex polytopes.* (Russian). Ukrainsk.Geom.Sbor. 1972. 12.
Nash-Williams,C.St.J.A. (1982) *A glance at graph theory: Parts* I & II. Bull.London Math.Soc. 1982. 14. pp177-212 & pp294-328.
Remesh,E.Ya. & Shteinberg,A.C. (1967) *A theorem on convex polytopes connected with the problem of finding all solutions of a system of linear inequalities.* Ukrainsk.Matem.Zhurn. 1967. 19. No.2. (Russian).
Steinitz,E. (1922) *Polyeder und Raumeinteilungen.* Enzykl.Math.Wiss. 1922. 3.

Chapter 3

Adler,J. & Dantzig,G. (1974) *Maximum diameter of abstract polytopes.* Math.Prog.Study 1.
Adler,J., Dantzig,G. & Murty,K. (1974) *Existence of* A-*avoiding paths in abstract polytopes.* Math.Program.Study 1.
Alexandroff,P.S. (1947) *Combinatorial Topology.* (Russian). Moscow. OGIZ. English translation - Rochester. New York. Graylock. 1956.
Altschuler,A. & McMullen,P. (1973) *The number of simplicial neighbourly* d-*polytopes with* $d+3$ *vertices.* Mathematika. 1973. 20.
Barnette,D.A.(1973) *Proof of the lower bound conjecture for convex polytopes.* Pacific J.Math. 1973. 46. No.2.
Bruggesser,H. & Mani,P. (1971) *Shellable decompositions of cells and spheres.* Mathem.Scand. 1971. 29.
Cayley,A. (1862) *On the* Δ-*faced polyacrons in reference to the problem of the enumeration of polyhedra.* Mem.His.Phil.Soc.Manchester. 1862. 1.
Gale,D. (1964) *On the number of faces of a convex polytope.* Can.J. Math. 1964. 16.
Grünbaum,B. (1967) *Convex Polytopes.* New York. Wiley.
Klee,V. (1974) *Polytope pairs and their relationship to linear programming.* Acta.Math. 1974. 133.
Kovalev,M.M. (1979) *Polyhedral semi-matroids.* (Russian). Izv.Akad.Na. BSSR. ser.fiz.-matem.nauk. 1979. No.3.
Kowaljow,M. & Isatchenko,A. (1979) *Semimatroide.* Vortragsthesen 24. Intern.Wiss.Koll. TH Ilmenau. 1979.
Lawrence,J. (1978) *Abstract polytopes and the Hirsch conjecture.* Math.Prog. 1978. 15. No.1.
Lloyd,K. (1970) *The number of* d-*polytopes with* $d+3$ *vertices.* Mathematika. 1970. 17.
McMullen,P. (1974) *The number of neighbourly* d-*polytopes with* $d+3$ *vertices.* Mathematika. 1974. 21.
McMullen,P. & Shephard,G. (1971) *Convex polytopes and the upper bound conjecture.* Cambridge.

Motzkin,T. (1957) *Comonotone curves and polyhedra.* Bull.Am.Math.Soc. 1957. 63.
Pontryagin,L.C. (1976) *Foundations of combinatorial topology.* (Russian) Moscow. Nauka.
Schlegel,V. (1891) *Über die verschiedenen Formen von Gruppen, welcher beliebige Punkte im* n-*dimensionalen Raum bilden können.* Arch.Math.Phys. 1891. 10.
Stanley,R. (1975) *The upper bound conjecture and Cohen-Macaulay rings.* Studies in Appl.Math. 1975. 54. No.2.
Steiner,J. (1882) *Von dem Krümmungsschwerpunkte ebener Curven.* Ges. Werke. Berlin. 1882. 2.
Walkup,D. (1978) *The Hirsch conjecture fails for triangulated 27-spheres.* Math.Oper.Res. 1978. 3. No.2.

Chapter 4

Anastasian,Yu.G. (1975) *On a class of integer programming problems.* (Russian). Kibernetika. 1975. No.3.
Balas,E. & Padberg,M. (1972/5) *On the set covering problem.* Oper.Res. 1972. 20. No.6 ; 1975. 23. No.1.
Beged-Dov,A. (1972) *Lower and upper bounds for the number of lattice points in a simplex.* J.Appl.Math. 1972. 22.
Belousov,E.G.(1977) *Introduction to convex analysis and integer programming.* (Russian). Moscow. Mos.State Univ.
Berge,C. (1970) *Graphes et hypergraphes.* Paris. Dunod.
(1972) *Balanced matrices.* Math.Program. 1972. 2. No.1.
Brown,D. (1977) *Compound and unimodular matrices.* Discr.Math. 1977. 19. No.1.
Camion,P. (1965) *Characterization of totally unimodular matrices.* Proc.Amer.Math.Soc. 1965. 16.
Cassels,J.W.S. (1959) *An introduction to the geometry of numbers.* Berlin. Springer.
Chandrasekaran,R. (1969) *Total unimodularity of matrices.* SIAM J. Appl.Math. 1969. 17. No.6.
Christofides.N. (1975) *Graph theory: an algorithmic approach.* New York. Academic Press.
Chvatal,V. (1975) *On certain polytopes associated with graphs.* J.Combin.Theory. 1975. 18.No.2.
Cunningham,W.(1977) *An unbounded matroid intersection polyhedron.* Linear Algebra and Appl. 1977. 16. No.3.
Ehrhart,E. (1977) *Polynomes arithmétiques et méthode des polyhedres en combinatoire.* Basel-Stuttgart.
Edmonds,J. (1965) *Maximum matching and a polyhedron with* 0,1-*vertices.* J.Res.Nat.Bureau Standards. 1965. 69.
(1970) *Submodular functions, matroids and certain polyhedra.* In: Comb.Structures and their Appl. Gordon and Breach.
Fiorot,J. (1972) *Generation of all integer points for given sets of linear inequalities.* Math.Prog.. 1972. 3.
Ford,L.R. & Fulkerson,D.R. (1962) *Flows in networks.* Princeton Un.Press.
Fulkerson,D.R. (1971) *Blocking and anti-blocking pairs of polyhedra.* Math.Prog. 1971. 1.
Fulkerson,D.R.,Hoffman,A. & Oppenheim,R. (1974) *On balanced matrices.* Math.Program.Study. 1.
Ghouilla-Houri,A. (1962) *Caractérisation des matrices totalement unimodulaires.* Compt.Rend.Acad.Sci. 1962. 254. No.7.

Girlich,E. & Kowaljow,M. (1981) *Nichtlineare discrete Optimierung.* Berlin. Akademie.
Gomory,R. (1969) *Some polyhedra related to combinatorial problems.* Linear Algebra and Appl. 1969. 2.
Heller,J. (1957) *On linear systems with integral valued solutions.* Pacific J.Math. 1957, 7. No.3.
(1963) *On unimodular sets of vectors.* Recent Advances in Math.Program.
Heller,J & Tompkins,C. (1958) *An extension of a theorem of Dantzig's.* In: Linear inequalities and related systems.
Hilbert,D. (1890) *Über die Theorie der algebraischen Formen.* Math.Ann. 1890. 36.
Hoffman,A. (1979) *The rôle of unimodularity in applying linear inequalities to combinatorial theorems.* Annals of Discr.Math. 1979. 4.
Hoffman,A. & Kruskal,J. (1958) *Integral boundary points of convex polyhedra.* In: Linear inequalities and related systems.
Ivanov,N.N. (1974) *The reduction of a class of integer programming problems to the covering problem.* (Russian). Izv.Akad.Na. BSSR. ser.fiz.-matem.nauk. 1974. No.4.
(1975) *A method of aggregation for an integer programming problem.* (Russian). Izv.Akad.Na.BSSR. ser.fiz.-matem.nauk. 1975, No.2.
Ivanov,N.N. & Shevchenko,V.N. (1975) *The construction of a finitely-generated semi-net.* Dokl.Akad.Na.BSSR. 1975. 19. No.9.(Rus).
Jeroslow,R. (1978) *Some basis theorems for integral monoids.* Math.Oper.Res. 1978. 2. No.2.
(1979) *An introduction to the theory of cutting planes.* Discr.Math. 1979. 5.
Kovalev,M.M. (1977) *Discrete optimization.* (Russian). Minsk. Bel.St.Univ.
(1980) *The method of partial orders.* Dokl.Akad.Na.BSSR.1980 24. No.2.
Kovalev,M.M. & Yemelicheva,V.S. (1975) *Extremal properties of partially ordered sets of planar partitions.* (Russian). In: Voprosy Kibernetiki. Moscow. Sovietskoye Radio.
Kovalev,M.M.,Isachenko,A.N. & Nguen Ngia. (1978) *Linearization of combinatorial optimization problems.* (Russian). Dokl.Akad.Na. BSSR. 1978. 22. No.10.
Kowaljow,M.M. & Girlich,E. (1978) *Minimisierung konvexer Funktionen über unbeschränkten Polymatroiden.* Internationale Tagung 'Mathematische Optimierungstheorie'
Kowaljow,M.M.,Nguen Ngia & Kühn,E. (1977) *Eine Klasse ganzzahligen Polyeder.* Humboldt.Universität zu Berlin, Tagung Math. Optimierung.
Lawler,E. (1976) *Combinatorial optimization. Networks and matroids.* New York.
Lovász,L. (1972) *Normal hypergraphs and the perfect graph conjecture.* Discr.Math. 1972. 2. No.3.
(1979) *Graph theory and integer programming.* Annals of Discr.Math. 1979. 4.
Mathews,G. (1897) *On the partition of numbers.* Proc.London Math.Soc. 1897. 28.
Meyer,R. (1974) *On the existence of optimal solutions to integer and mixed-integer programming problems.* Math.Progr. 1974.7.No.2.
Minkowski,H. (1910) *Geometrie der Zahlen.* Leipzig.
Mirsky.L. (1971) *Transversal theory.* New York. Academic Press.

Padberg,M. (1973) *On the facial structure of set packing polyhedra.* Math.Program. 1973. 5.
(1974) *Perfect zero-one matrices.* Math.Program. 1974. 6.
(1976) *A note on the total unimodularity of matrices.* Discr.Math. 1976. 14. No.3.
(1979) *Covering, packing and knapsack problems.* Annals of Discr.Math. 1979. 4.

Petrova,G.L. (1976) *A condition for finite-generation in a class of semigroups.* (Russian). Izv.Akad.Na.BSSR. ser.fiz.-matem. nauk. 1976. No.2.
(1978) *Conditions for the non-reducibility of a generating set of the semi-group of integral points of a polyhedral cone.* (Russian). Izv.Akad.Na.BSSR. ser.fiz-matem.nauk. 1978. No.2.

Poincaré,H. Proc.Lond.Math.Soc. 1901. 32.

Presburger,M.(1930) *Über die Vollständigkeit eines gewissen System der Arithmetik ganzer Zahlen, in welchem die Addition als einzige Operation hervortritt.* Sprawozdanie z I Kongresy Matematyków Krajów Słowianskich. Warszawa.

Sachkov,V.N. (1977) *Combinatorial methods of discrete mathematics.* (Russian). Moscow. Nauka.

Saigal,R. (1969) *A proof of the Hirsch conjecture on the polyhedron of the shortest route problem.* SIAM J.Appl.Math. 1969. 17.No6.

Sakarovitch,M. & MacAllister,J. (1977) *Classification de certaines matrices* 0-1. Discr.Math. 1977. 20. No.2.

Sapozhenko,A.A.,Asratian,A.S. & Kuziurin,N.N. (1977) *A survey of some results in the covering problem.* In: *Methods of discrete analysis in the solution of combinatorial problems.* Novosibirsk. 30.

Shevchenko,V.N. (1970) *On the intersection of a convex polyhedral cone with the integral lattice.* Izv.Vyssh.Ucheb.Zaved. ser. Radiofizika. 1970. 83. No.8. (Russian)
(1976) *A discrete analogue of Farkas' theorem and the aggregation problem for a system of integral linear equations.* (Russian). Kibernetika. 1976. No.2.
(1979) *Convex polyhedral cones, systems of comparison and right sections in integer programming.* (Russian). In: *Combinatoric-algebraic methods in applied mathematics.* Gorky. Gorky State Univ.

Shevchenko,V.N. & Ivanov,N.N. (1976) *On the representation of a semigroup by means of a semigroup generated by a finite set of vectors.* (Russian). Izv.Akad.Na.BSSR. ser.fiz.-matem.nauk. 1976. No.2.

Smith,H. (1861) *On systems of linear indeterminate equations and congruences.* Philosophical Trans. 1861. 151.

Stanley,R. (1974) *Combinatorial reciprocity theorems.* Advances in Mathematics. 1974. 14. No.2.

Trubin.V.A. (1969) *On a method of solution of integer linear programming problems of a special kind.* (Russian). Dokl.Akad.Nauk. SSSR. 1969. 189. No.5. (English translation: Soviet Math. Dokl. 10 (1969). No.6. pp.1544-1546).

Tucker,A. (1973) *The strong perfect graph conjecture for planar graphs.* Can.J.Math. 1973. 25. No.1.

Tutte,W. (1971) *Introduction to the theory of matroids.* Elsevier.

Veinott,A. & Dantzig,G. (1968) *Integral extreme points.* SIAM Rev. 1968. 10. No.3.

Veselov,C.I. & Shevchenko.V.N. (1978) *On the exponential growth of the coefficients in the aggregation equation.* (Russian). Kibernetika. 1978. No.4.
Votyakov , A.A. & Frumkin,M.A. (1976) *An algorithm for finding a simultaneous integral solution to a system of linear equations.* In: *Studies in discrete mathematics.* Moscow. Nauka. (Russian).
Welsh,D. (1976) *Matroid theory.* New York. Academic Press.
Zykov,A.A. (1974) *Hypergraphs.* (Russian). Uspekhi Matem.Nauk. 1974. 29. No.6.

Chapter 5

Balinski,M. & Russakoff,A. (1972) *Some properties of the assignment polytope.* Math.Progr. 1972. 3.
Beresnev,V.L.,Gimadi,E.Kh. & Dementiev,V.P. (1978) *Extremal problems of standardization.* Novosibirsk. Nauka. (Russian).
Birkhoff,G. (1946) *Trés observaciones sobre el algebra linear.* Rev. Univ.Nac.Tucuman.Revista. 1946. A5.
Brualdi,R. & Gibson,P. (1976/7) *Convex polyhedra of doubly stochastic matrices,* I,II,III. J.Combinat.Theory. 1977.A22.No.2 ; 1977.B22 ; 1977.A22.No.3 ; IV Linear Algebra and Appl. 1976. 15.No.2.
Bowman,V. (1972) *Permutation polyhedra.* SIAM J.Appl.Math. 1972. 22. No.4.
Cruse,A. (1975) *A note on doubly-stochastic matrices.* Discr.Math. 1975. No.2.
Gaihar,P. & Gupta,S. (1977) *Adjacent vertices on a permutahedron.* SIAM J.Appl.Math. 1977. 32. No.2.
Girlich,E. & Kovalev,M.M. (1974) *A class of polytopes for the standardization problem with the maximum number of vertices.* (Russian). Izv.Akad.Na.BSSR. ser.fiz.-matem.nauk. 1974. No.6.
Grötschel,M. & Padberg,M. (1977) *Lineare Characterisierungen von Travelling Salesman Problemen.* Z.Oper.Res.A. 1977. 21. No.1.
(1979) *On the symmetric travelling salesman problem.* I:*Inequalities.* Math.Program.1979.16.No.2. ; II:*Lifting theorems and facets.* Math.Prog.1979.16.No.2.
Hardy,G.H.,Littlewood,J.E. & Pólya,G. (1952) *Inequalities.* Cambridge.
Heller,J. (1955) *On the travelling salesman problem.* Proc. 2 Sympos. Linear Program. Washington. 1955. 2.
(1956) *Neighbour relations on the convex hull of cyclic permutations.* Pacific J.Math. 1956. 6. No.3.
Koontz,W. (1978) *Convex sets of some doubly stochastic matrices.* J.Combin.Theory. 1978. A24. No.1.
Kovalev,M.M. & Girlikh,E. (1978) *On the exact number of vertices of the polytope defined by the conditions of the standardization problem.* (Russian). Izv.Akad.Na.BSSR. ser.fiz.-matem.nauk. 1978. No.2.
(1980) *The structure of the feasible set of the standardization problem.* (Russian). Izv.Akad.Na.BSSR. ser.fiz.-matem. nauk. 1980. No.4.
Kovalev,M.M. & Gorunovich,S.A. (1980) *Permutation semimatroids.* (Russian). Izv.Akad.Na.BSSR. ser.fiz.-matem nauk. 1980.No.5.
Kovalev,M.M.,Isachenko,A.N. & Nguen Ngia. (1978) *Linearization of combinatorial optimization problems.* (Russian). Dokl.Akad. Na.BSSR. 1978. No.10.

Kovalev,M.M. & Kotov,V.V. (1981) *An analysis of the gradient algorithm for the solution of the travelling salesman problem.* (Russian). Zh.Vychisl.Mat. i Mat.Fiz. 1981. No.1.
Kowaljow,M.M. & Girlich,E. (1977) *Zum Problem der optimalen Standartisierung.* Mathem.Operationsforschung. ser.optimization. 1977. 8. No.1.
Kowaljow,M.M. & Isatschenko,A. (1978) *Linearisierung kombinatorische Optimierungsprobleme.* Vortragsthesen zum 'Math.Optimierung'. Universität Berlin.
Kuhn,H. (1955) *On certain convex polyhedra.* Bull.Amer.Math.Soc. 1955. 61. No.6.
Leontiev,V.K.(1975) *The stability of the travelling salesman problem.* (Russian). Zh.Vychisl.Mat. i Mat.Fiz. 1975. 15. No.5.
(1979) *Discrete extremal problems.* In: *Itogi nauki i tekhniki.* ser.Th.Prob..Math.Stat..Th.Cyber. Moscow. VINITI. 1979. (Russian).
Maurras,J. (1975) *Some results on the convex hull of the Hamiltonian cycles of symmetric complete graphs.* Combinat.Program: Meth. and Appl. Dordrecht.Boston. 1975.
Mikulski,V.E.(1974) *On the extremum of a sum of linear forms on the set of substitutions.* (Russian). Dokl.Akad.Na.BSSR.1974.18.No.10.
Norman,R. (1955) *On the convex polyhedra of the symmetric travelling salesman problem.* Bull.Amer.Math.Soc. 1955. 59. No.6.
Padberg,M. & Rao,M. (1974) *The travelling salesman problem and a class of polyhedra of diameter two.* Math.Program. 1974. 7. No.1.
Papadimitriou,C. (1978) *The adjacency relation on the travelling salesman polytope is NP-complete.* Math.Progr.1978.14.No.3.
Rado,R. (1952) *An inequality.* J.London Math.Soc. 1952. 27.
Sachkov,V.N. (1975) *On extremal points of the set of symmetric stochastic matrices.* (Russian). Matem.Sbornik. 1975.96.No.3.
(1977) *Combinatorial methods of discrete mathematics.* (Russian). Moscow. Nauka.
Sarvanov,V.I.(1977) *On polytopes connected with optimization on substitutions.* (Russian). Preprint Inst.Mat. Akad.Na.BSSR.
Savage,S.,Weiner,P. & Bagchi,A. (1976) *Neighbourhood search algorithms for guaranteeing optimal travelling salesman tours must be inefficient.* J.Comput. and Syst.Sci. 1976. 12. No.1.
Suprunenko,D.A. & Metelski,N.N. (1973) *The assignment problem and the minimization of the sum of linear forms on the symmetric group.* (Russian). Kibernetika. 1973. No.3.
Tanaev,V.A. & Shkurba,V.V (1975) *Introduction to the theory of timetables.* (Russian). Moscow. Nauka.
Yemelichev,V.A. & Kovalev,M.M. (1970/2) *The solution of certain concave programming problems by the construction of a sequence of plans.* I,II Izv.Akad.Na.BSSR. ser.fiz.-matem.nauk. 1970.No.6; and 1972.No.1. (Russian).

Chapter 6

Austin,T.L. (1960) *The enumeration of point labelled chromatic graphs and trees.* Canadian J. of Math. 1960. 12. No.4.
Balinski,M. (1974) *On two special classes of transportation polytopes.* Math.Program.Study. 11.
Bolker,E. (1972) *Transportation polytopes.* J.Combin.Th. 1972.13B.
Feller,W. (1957/66) *An introduction to probability theory and its applications.* New York. Wiley. Vol.I(1957);Vol.II(1966).

Demuth,O. (1961) *A remark on the transportation problem.* Časopis propestovati matematiky. Praha. 1961. 86.
Gibson,P. (1976) *Facets of faces of transportation polytopes.* Proc. of the seventh Conf. on Comb.Graph.Th.Comp. Louisiana.
Intrator,J. & Lev,B. (1976) *Methods for identification of vanishing variables in transportation problems and its possible applications.* Comp. and Oper.Res. 1976. 3. No.4.
Jemelitschev,V. & Krawzow,M. (1976) *Transportpolyeder.* 2ist IWK der TH Ilmenau.
Khintchine,A.Y. (1961) *Continued Fractions.* Chicago. Univ.of Ch.Press. 1964. (Translation of 3rd Russian edition of 1961).
Klee,V. & Witzgall,C. (1968) *Facets and vertices of transportation polytopes.* Math.of Decision Sciences. Part I. Am.Math.Soc. Providence.
Klingman,D. & Russell,R. (1974) *The transportation problem with mixed constraints.* Operations Research Quarterly. 1974.25.No.3.
Kononenko,A.M. & Trukhanovski,N.N. (1978) *On transportation polytopes with the maximum number of vertices.* (Russian) Izv.Akad. Na.BSSR. ser.fiz.-matem.nauk. 1978. No.5.
(1979) *A class of transportation polytopes with the maximum number of vertices.* (Russian). Izv.Akad.Na.BSSR. ser.fiz.-matem.nauk. 1979. No.5.
Krachkovski,A.P. (1978) *On a formula for the maximum number of vertices of a transportation polytope.* (Russian). Izv.Akad.Na.BSSR. ser.fiz.-matem.nauk. 1978. No.1.
(1979) *On an asymptotic problem of Bolker.* (Russian). Izv. Akad.Na.BSSR. ser.fiz.-matem.nauk. 1979. No.2.
Krachkovski,A.P. & Yemelichev,V.A. (1979) *On an upper bound for the diameter of a transportation polytope.* (Russian). Dokl.Akad. Na.BSSR. 1979. 23. No.4.
Kravtsov,M.K.(1973) *On the problem of reducing the dimension of the transportation problem.* (Russian). Izv.Akad.Na.BSSR. ser. fiz.-matem.nauk. 1973. No.2.
(1976i) *Enumerative problems on degenerate transportation polytopes.* (Russian). Izv.Akad.Na.BSSR. ser.fiz.-matem.nauk. 1976. No.6.
(1976ii) *An upper bound for the number of vertices of a degenerate transportation polytope.* (Russian). Izv.Akad.Na. BSSR. ser.fiz.-matem.nauk. 1976. No.6.
(1979) *On* (d-2)-*dimensional faces of a* d-*dimensional transportation polytope.* (Russian). Izv.Akad.Na.BSSR. ser.fiz.-matem.nauk. 1979. No.3.
Kravtsov,M.K. & Krachkovski,A.P. (1978) *On transportation polytopes with a given number of facets and the minimum number of vertices.* (Russian). Izv.Akad.Na.BSSR. ser.fiz.-matem.nauk.1978.No.5.
Kravtsov,M.K. & Yemelichev,V.A. (1976) *On transportation polytopes with a given number of facets.* Izv.Akad.Na. ser.fiz.-matem.nauk. 1979. No.3. (Russian).
Kurtsevich,K.A. & Kravtsov,M.K. (1974) *The method of dimension reduction in the transportation problem of linear programming.* (Russian). In: *Mathematical methods for the solution of economic problems.* Moscow. Nauka.
Likhachev,V.M. (1975) *On an estimate of the number of vertices of a transportation polytope.* (Russian). Izv.Akad.Na.BSSR. ser. fiz.-matem.nauk. 1975. No.2.

Likhachev,V.M. & Yemelichev,V.A. (1974) *An upper bound for the number of vertices of a transportation polytope.* (Russian). Izv.Akad. Na.BSSR. ser.fiz.-matem.nauk. 1974. No.3.
Mauldon,J. (1951) *Random division of an interval.* Proc.of the Camb. Philosophical Soc. 1951. 47. part 2.
Moon,J. (1969) *The number of labelled k-trees.* J.of Comb.Th.1969.6.
Ogurtsova,L.N.,Skaletskaya,E.I. & Skaletski,V.N. (1973) *Stability of solutions to the transportation problem.* In: *Control and Information.* (Russian). Vladivostok. 1973. 8.
Olah,G. (1968) *The problem of counting the number of certain trees.* (Russian). Studia Scientiarum Mathematicarum Hungarica. 1968. 3.
Orden,A. (1956) *The transshipment problem.* Management Science. 1956. 2. No.3.
Scoins,H. (1962) *The number of trees with nodes of alternate parity.* Proc.of the Camb.Philosophical Soc. 1962. 58.
Shvartin,C.M.(1978) *A study of the stability of transportation problems.* (Russian). Zh.Vychisl.Mat. i Mat.Fiz. 1978. 18. No.1.
Simmonard,M. & Hadley,G. (1959) *The maximum number of iterations in the transportation problem.* Naval Research Quarterly.1959.6.No.2.
Szwarz,W. & Wintgen,G. (1965) *The number of transportation bases.* Math.(Cluj). 1965. 7. No.1.
Yemelichev,V.A. (1965) *On a certain concave programming problem.* (Russian). Izv.Akad.Na.BSSR. ser.fiz.-matem.nauk. 1965.No.3.
(1969) *On the local minima in a certain multi-extremum problem.* (Russian). Izv.Akad.Na.BSSR. ser.fiz.-matem.nauk. 1969. No.1 and No.4.
Yemelichev,V.A. & Kononenko,A.M. (1970) *On a certain combinatorial enumerative problem.* (Russian). Izv.Akad.Na.BSSR. ser. fiz.-matem.nauk. 1970. No.2.
(1971) *On a certain class of transportation polytopes.* (Russian). Izv.Akad.Na.BSSR. ser.fiz.-matem.nauk. 1971.No.3.
Yemelichev,V.A. & Kovalev,M.M. (1972) *On local minima in the transportation problem with Schur-concave cost functions.* (Russian). Zh.Vychisl.Mat. i Mat.Fiz. 1972. 12. No.5.
Yemelichev,V.A. & Krachkovski,A.P. (1978) *Asymptotics of transportation polytopes.* (Russian). Izv.Akad.Na.BSSR. ser.fiz.-matem. nauk. 1978. No.4.
Yemelichev,V.A. & Kravtsov,M.K. (1976i) *On enumerative problems on transportation polytopes.* (Russian). Izv.Akad.Na.BSSR. ser.fiz.-matem.nauk. 1976. No.1.
(1976ii) *Some problems of dimension reduction for the transportation problem.* (Russian). Izv.Akad.Na.BSSR. ser. fiz.-matem.nauk. 1976. No.2.
(1976iii) *On some properties of transportation polytopes with the minimum number of vertices.* (Russian). Dokl.Akad. Nauk.SSSR. 1976. 228. No.5. (English translation: Soviet Math.Dok. 1976. 17. No.3. pp.857-860).
(1978) *On transportation polytopes with the maximum number of vertices.* (Russian). In: Voprosy Kibernetiki. Moscow. Izdat.Akad.Nauk.SSSR. 1978. 26.
Yemelichev,V.A.,Kravtsov,M.K. & Averbukh,N.D. (1976) *On the maximum number of vertices of a transportation polytope.* (Russian). Dokl.Akad.Na.BSSR. 1976. 20. No.6.

Yemelichev,V.A.,Kravtsov,M.K. & Krachkovski,A.P. (1977i) *A proof of Bolker's conjecture.* (Russian). Dokl.Akad.Nauk.BSSR. 1977. 21. No.9.
(1977ii) *On a conjecture concerning the maximum number of vertices of a transportation polytope.* (Russian). Izv. Akad.Na.BSSR. ser.fiz.-matem.nauk. 1977. No.6.
(1977iii) *Estimates of the number of vertices of transportation polytopes.* (Russian). Theses of reports to the fourth All-union conference on problems of theoretical cybernetics. Novosibirsk.
(1978i) *The minimum number of vertices of a non-degenerate transportation polytope with a specified number of facets.* (Russian). Dokl.Akad.Na.BSSR. 1978. 22. No.1.
(1978ii) *On some classes of transportation polytopes.* (Russian). Dokl.Akad.Na.SSSR. 1978. 241. No.3. (English translation: Soviet Math.Dokl. 1978. 19. No.4. pp.889-893).
(1979) *Transportation polytopes with a specified number of facets and the maximum number of vertices.* (Russian). In: *Questions of mathematical security in automated systems.* Minsk. NIIEMP in Gosplan of BSSR.
Yemelichev,B.A. & Trukhanovski,N.N. (1977) *On the spectrum of a transportation polytope.* (Russian). Izv.Akad.Na.BSSR. ser.fiz.-matem. nauk. 1977. No.5.
Yemelicheva,V.S. & Kravtsov,M.K. (1977) *On the facets of a degenerate transportation polytope.* (Russian). Izv.Akad.Na.BSSR. ser.fiz.-matem.nauk. 1977. No.5.

Chapter 7

Dubois,J. (1973) *Polytopes de transport symétriques.* Discr.Math. 1973. 4. No.1.
Kononenko,A.M. & Trukhanovski,N.N. (1978) *Enumeration of maximal trees in a certain class of bipartite graphs.* (Russian). Izv.Akad. Na.BSSR. ser.fiz.-matem.nauk. 1978. No.6.
Kononenko,A.M. & Yemelicheva,E.V. (1979) *On the number of spanning trees of a bipartite graph.* (Russian). In: *Questions of mathematical security in automated systems.* Minsk. NIIEMP in Gosplan of BSSR.
Kravtsov,M.K. & Korzinkov,A.D. (1979) *Enumerative problems on truncated transportation polytopes.* (Russian). Izv.Akad.Na.BSSR. ser.fiz.-matem.nauk. 1979. No.4.
Lev,B. (1972) *A non-iterative algorithm for tridiagonal transportation problem and its generalization.* Operations Research. 1972. 20. No.1.
Perfect,H. & Mirsky,L. *Extremal points of certain convex polyhedra.* Monatsheft Math. 1964. 68.
Reverz,P. (1961) *Seminar on Random Ergodic Theory.* 6. University of Aahus.
Yemelichev,V.A. & Kravtsov,M.K. (1975) *Solution of a large-scale transportation problem with prohibitions.* In: *Theses of reports to the second All-union seminar on mathematical reliability.* (Russian). ASUP. Moscow-Gorky.
Yemelicheva,V.S. (1974) *Enumeration problems on a certain class of transportation problems.* (Russian). Dokl.Akad.Na.BSSR. 1974. 18. No.6.

Chapter 8

Bolker,E. (1976) *Simplicial geometry and transportation polytopes.* Trans. Amer.Math.Soc. 1976. 217

De Werra,D. (1978) *Some comments on a note about timetabling.* Infor. 1978. 16. No.1.

Dinic,E.A. & Kronrod,M.A. (1969) *An algorithm for the solution of the assignment problem.* (Russian). Dokl.Akad.Nauk.SSSR. 1969. 189. No.1. (English translation: Sov.Math.Dokl. 1969. 10. No.6. pp.1324-1326).

Fedenya,O.A. (1977) *On the polytope of the multi-indexed planar transportation problem with the minimum number of vertices.* I , II. (Russian). Izv.Akad.Na.BSSR. ser.fiz.-matem.nauk. 1977. No.5 and No.6.

Haley,K. (1963) *The multi-index problem.* Oper.Res. 1963. 11. No.3. (1967) *Note on the letter by Moravek and Vlach.* Oper.Res. 1967. 15. No.3.

Hall,M. (1967) *Combinatorial theory.* Waltham. Blaisdell.

Kononenko,A.M. (1973) *On a class of hypergraphs.* (Russian). Dokl. Akad.Nauk.BSSR. 1973. 17. No.9.

Kononenko,A.M.,Mikulski,V.E. & Trukhanovski,N.N. (1976) *Feasibility conditions for triply-indexed planar transportation problems.* (Russian). Izv.Akad.Na.BSSR. ser.fiz.-matem.nauk. 1976.No.6.

Kravtsov,M.K.(1979) *On the maximum number of integral vertices of an axial transportation polytope.* (Russian). In: *Questions of mathematical security in automated systems.* Minsk. NIIEMP in Gosplan of BSSR.

Kravtsov,M.K. & Kashinski,Yu.I. (1977) *Feasibility conditions for multi-indexed transportation problems with special constraints.* (Russian). In: *Automated systems of planning calculations in the republic's planning organizations.* Minsk. NIIEMP in Gosplan of BSSR. 1977. 9.

Kravtsov,M.K.,Sherman,A.Kh. & Averbukh,N.D. (1975) *On an algorithm for the solution of the assignment problem.* (Russian). Izv. Akad.Na.BSSR. ser.fiz.-matem.nauk. 1975. No.6.

Leue,O. (1972) *Methoden zur Lösung dreidimensionaler Zuordnungsproblem.* Angewandte Informatik. 1972. No.4.

Moravek,J. & Vlach,M. (1967) *On the necessary conditions for the existence of the solution of the multi-index transportation problem.* Oper.Res. 1967. 15. No.3.

Motzkin,T. (1952) *Multi-index transport problem.* Bull.Amer.Math. Soc. 1952. 58. No.4.

Rybnikov,K.A.(1972) *Introduction to combinatorial analysis.* (Russian). Moscow. Mos.Gos.Univ.

Ryser,H.J. (1963) *Combinatorial mathematics.* Carus Math.Monograph. No.14. New York. Wiley.

Schell,E. (1955) *Distribution of a product by several properties.* Proc.2nd Sympos.Linear Progr. Washington. 1955. 2.

Smith,G. (1973) *Further necessary conditions for the existence of a solution to the multi-index problems.* Oper.Res.1973.11.No.3.

Talanov,V.A. & Ilichev,A.P. (1979) *Triply-indexed transportation problems and the composition of teaching timetables.* (Russian). Economics and mathematical methods. 1979. 15.No.4.

Talanov,V.A. & Shevchenko,V.N. (1972) *On a certain problem in a dynamical transportation net.* (Russian). Izv.Vys.Ucheb.Zav. Radiofizika. 1972. 15. No.7.

Trukhanovski,N.N. (1979) *Feasibility conditions for a class of multi-indexed planar transportation problems.* (Russian). Izv. Akad.Na.BSSR. ser.fiz.-matem.nauk. 1979. No.3.

Yemelichev,V.A. & Kononenko,A.M. (1972) *A degeneracy condition for a multi-indexed transportation polytope.* (Russian). Dokl. Akad.Na.BSSR. 1972. 16. No.6.

(1974) *On the number of plans in a multi-index selection problem.* (Russian). Dokl.Akad.Na.BSSR. 1974. 18. No.8.

Yemelichev,V.A.,Kononenko_A.M. & Likhachev,V.M. (1972) *On the polytopes of the multi-indexed transportation problem.* (Russian). Dokl.Akad.Na.BSSR. 1972. 16. No.5.

Yemelicheva,V.S. & Kononenko.A.M. (1974i) *On the minimum number of integral vertices for the multi-indexed transportation polytope.* (Russian). Izv.Akad.Na.BSSR. ser.fiz.-matem. nauk. 1974. No.3.

(1974ii) *On the minimum number of vertices and facets for the planar transportation polytope.* (Russian). Dokl.Akad. Nauk.BSSR. 1974. 18. No.9.

LIST OF SYMBOLS

A_I^J - submatrix of the matrix A formed from the rows with indices in the set I and the columns with indices in the set J .

A_G - incidence matrix of the graph G .

d - dimension of a polytope.

E_n^+ - non-negative orthant in $\mathbf{E}_n$.

$\mathbf{E}_n$ - n-dimensionsional Euclidean space.

$E_{m,n}$ - the set of $(m\times n)$-matrices with real elements.

$\bar{I} = N_s \setminus I$, where $I \subset N_s$.

K_m - complete graph on m vertices.

$K_{m,n}$ - complete bipartite graph.

$N_s = \{1,2,\ldots,s\}$.

$|S|$ - number of elements in the finite set S .

S_n - the symmetric group of permutations on n symbols.

$$\operatorname{sign} a = \begin{cases} 0, & \text{if } a = 0, \\ 1, & \text{if } a > 0, \\ -1, & \text{if } a < 0. \end{cases}$$

$Z^+ = \{0,1,2,\ldots\}$.

Z_n (Z_n^+) - the lattice of integral points in E_n (E_n^+) .

e - a vector all of whose components are equal to unity.

rank A - the rank of the matrix A .

det A - the determinant of the matrix A .

$\mathcal{B}(A)$ - semigroup, generated by the columns of the matrix A .

$\binom{n}{m}$ - the number of combinations of n objects taken n at a time.

$[\alpha]$ - the largest integer which does not exceed α .

$]\alpha[$ - the smallest integer which is not less than α .

$\{\alpha\}$ - the fractional part of α .

δ_{ij} - Kronecker's symbol.

$\emptyset$ - the empty set.

2^I - the set of all subsets of the set I .

$A_{\pm 1}$ - a (± 1)-matrix. (149)

aff A - affine hull of the column vectors of the matrix A . (11)

$\operatorname{aff}(a^1,\ldots,a^n)$ - affine hull of the points $a^1,\ldots,a^n$. (11)

$\mathcal{A}_{m\times n}$ - (265)

$\mathcal{A}(a,b)$ - (265)
bd M - boundary of the set M . (15)
$\mathcal{B}(q^1,\ldots,q^t)$ - semigroup generated by integral vectors. (134)
$C_{m,n}$ - class of $(m\times n)$-matrices with components 0 , $+1$, -1 . (150)
$C(d,n)$ - cyclic d-polytope with n facets. (24)
con A - conical hull of the columns of the matrix A . (16)
con$(a^1,\ldots,a^n)$ - conical hull of the points $a^1,\ldots,a^n$. (16)
conv$(a^1,\ldots,a^n)$ - convex hull of the points $a^1,\ldots,a^n$. (15)
$D_{m,mq+1}$ - (270)
deg v - degree of the vertex v in a graph. (64)
diam M - diameter of the polytope M . (71)
dim M - dimension of the set M . (11)
$f_k(M)$ - number of k-faces of the polytope M . (44)
$\mathcal{F}(M)$ - face-complex of the polytope M . (93)
G_A - intersection graph of a Boolean matrix A . (171)
$G(M)$ - graph of the polytope M . (62)
$G(S)$ - subgraph of the graph G , generated by the vertex set S. (63)
$G(t,n,p)$ - number of distinct (t,n,p)-orthogonal systems. (384)
$\underline{G}(x)$ - (214)
$\overline{G}$ - complement of the graph G . (172)
$G_x(U,V)$ - (275)
$G_{T(a,b,x)}(U,V)$ - (275)
$H_m(Q)$ - the standardization polytope. (245)
H^+ (H^-) - half-spaces determined by the hyperplane H . (12)
int M - interior of M . (14)
$\mathcal{K}(a,b,x)$ - (287)
$L(n,p)$ - number of Latin p-cubes of order n . (384)
M_n - bistochastic polytope. (212)
M_n^* - symmetric permutation polytope. (217)
M_n^{**} - symmetric bistochastic polytope. (255)
M_n^s - hamiltonian cycle polytope. (219)
M_n^{as} - hamiltonian tour polytope. (220)
M_x - section at vertex x of polytope M . (96)
$M_n(a)$ - permutation polytope. (227)
$M_n^m(a)$ - arrangement polytope. (237)
$M_n^+(a)$ - even permutation polytope. (235)
$M^=(A,e)$ - relaxed partition polytope. (191)
$M^{\leq}(A,e)$ - relaxed packing polytope. (170)
$M^{\geq}(A,e)$ - relaxed covering polytope. (170)
$M(A,b)$ - $\{x\in E_n^+ : Ax=b\}$ = canonical form of a polytope. (37)
$M(A^*,b^*)$ - simplest location polytope. (193)
$M(a)$ - distribution polytope. (350)

$M(a,a)$ - symmetric transportation polytope. (359)
$M(a,b)$ - classical transportation polytope. (264)
$M(a^*,b^*)$ - central transportation polytope. (262)
$M(a,b,D)$ - truncated transportation polytope. (335)
$M(a^1,\ldots,a^p)$ - p-indexed axial transportation polytope. (363)
$M(A,B,C)$ - triply indexed planar transportation polytope. (372)
$M(G)$ - matching polytope of a graph G. (175)
$M(k,n)$ - graph median polytope. (196)
$M(a^\lambda,b^\lambda)$ - (290)
$M(A,b^1,b^2,d^1,d^2)$ - (147)
$M(\rho)$ - polymatroid given by the submodular function ρ. (179)
$M_{k,t}(a,b)$ - (k,t)-truncated transportation polytope. (342)
$M_1 + M_2$ - sum of the polytopes M_1 and M_2. (26)
$M_1 \oplus M_2$ - join of the polytopes M_1 and M_2. (82)
$M_1 \otimes M_2$ - product of the polytopes M_1 and M_2. (26)
$M_1 \sim M_2$ - equivalence of the polytopes M_1 and M_2. (100)
$M_1 \cong M_2$ - combinatorial equivalence of polytopes M_1 and M_2. (94)
(M,x,y) - Dantzig figure. (77)
$\mathcal{M}(J,\mathcal{B})$ - matroid. (38)
$\mathcal{M}(m,n)$ - class of $(n-m)$-polytopes with n facets, given in canonical form. (37)
$\mathcal{M}(m,n,k)$ - set of all non-degenerate classical transportation polytopes of order $(m\times n)$ with $(m-1)n+k$ facets. (276)
$\mathcal{M}_p$ - class of regular axial transportation polytopes. (368)
$\mathcal{P} = (\mathcal{F},\mathcal{V})$ - semi-matroid. (99)
$\mathcal{P}(M)$ - semi-matroid of the polytope M. (100)
$Q(\rho)$ - unbounded polymatroid. (181)
$Q^\rho(a,b)$ - (289)
relbd M - relative boundary of the set M. (15)
relint M - relative interior of the polytope M. (14)
$r(u,v)$ - distance between the vertices u and v of a graph. (71)
$r(u/v)$ - remainder on division of u by v. (302)
$T(p,n,m)$ - set of plans for a multi-indexed selection problem. (385)
R - incidence matrix of a complete bipartite graph. (162)
$S(a^1,b^1,a^2,b^2)$ - spectrum of two classical transportation polytopes $M(a^1,b^1)$ and $M(a^2,b^2)$. (294)
$T(a,b,x)$ - basis set of vertex x of polytope $M(a,b)$. (265)
$T(p,n,m)$ - set of plans for a multi-indexed selection problem. (385)
T_d - simplex. (23)
vert M - set of vertices of a polytope M. (16)
W^* - polar of the set W. (27)
W_Z - set of integral points of the set W. (134)
$\alpha(G)$ - inner stability number. (170)
$\beta(\mathcal{M})$ - number of bases in the matroid $\mathcal{M}$. (38)
$\beta(A,b)$ - number of bases of the polytope $M(a,b)$. (37)
$\beta^*(A,b)$ - number of feasible bases of the polytope $M(a,b)$. (41)
$\gamma_{L,P}(a,b)$ - (290)
$\Delta(d,n)$ - maximum diameter in the class of d-polytopes with n facets. (6)

$\Delta_\nu(A)$ - highest common factor of all ν^{th}-order minors of matrix A. (141)

$\delta_{L,P}(a,b)$ - (290)

$\eta(M)$ - height of the polytope M. (6)

$\theta(G)$ - clique number of the graph G. (173)

$\mu_k(d,n)$ - lower bound for number of k-faces of a d-polytope with n vertices. (121)

$\mu_k^s(d,n)$ - lower bound for number of k-faces of a simplicial d-polytope with n vertices. (121)

$\mu_{I,J}(a,b)$ - (265)

$\nu(G)$ - matching number of the graph G (170)

$\rho(G)$ - edge covering number of the graph G. (170)

$\tau(G)$ - vertex covering number of the graph G. (170)

$\phi(m,n)$ - maximum number of vertices in the class of classical transportation polytopes. (296)

$\phi_k(d,n)$ - upper bound for number of k-faces of a d-polytope with n vertices. (111)

$\chi(G)$ - chromatic number of the graph G. (161)

$\omega(G)$ - plumpness of the graph G. (172)

$\langle i_1,\ldots,i_n \rangle$ - cycle. (220)

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