PHILLIP E. JOHNSON

A History of Set Theory

VOLUME SIXTEEN

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A HISTORY
OF SET THEORY

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DEDICATION

TO MY PARENTS

ACKNOWLEDGMENTS

A number of people deserve credit for whatever is good in this work. *A History of Set Theory* was originally written as a Ph.D. thesis at George Peabody College for Teachers under the direction of Dr. J. Houston Banks, major professor. Without his aid and comfort throughout the trying period of doctoral study, work on the thesis could never even have begun. Special appreciation is also due to Dr. Raymond C. Norris, minor professor, and to Dr. Kenneth S. Cooper for his help in matters pertaining to historiography. The library staffs of the Peabody College Library and the central division of the Joint University Libraries were most helpful in locating needed materials. The Latin translations are due to Msgr. Albert Seiner. Mrs. Emma Rosenberg gave valuable aid on some of the more difficult German passages. Dr. Otto Bassler of the Peabody mathematics department gave counsel on Chapter Six, which was not a part of the thesis, as well as lending pertinent materials from his personal library. Dr. Banks also gave helpful suggestions for the last chapter. Finally, grateful appreciation is expressed to Professor Howard Eves, who brought the manuscript to the attention of the publisher.

Being historical, the work naturally depends almost entirely on what other people have written. Hopefully, proper credit has been given throughout; pains were taken to assure this. Although a number of sources were consulted and valuable information derived from them, two works are of such general excellence and were relied on so heavily that they deserve special praise. Philip E. B. Jourdain's *Contributions to the Founding of the Theory of Transfinite Numbers*, a translation into English of two of Cantor's works, which also contains a valuable introduction and notes, was especially helpful for Chapter Three; A. A. Fraenkel's excellent biography of Georg Cantor (in German) was almost the sole source for Chapter Two and was also helpful for Chapter Three. The reliance on Cantor's papers is obvious; his work and what it has led to is what the whole thing is about.

The photographs of Georg Cantor, Leopold Kronecker, Karl Weierstrass, Ernst Kummer, and Bertrand Russell are here reproduced with the kind permission of the Library of Columbia University from whose David Smith Collection they were taken. The photograph of Richard Dedekind appears in the first volume of his *Gesammelte Mathematische Werke*, published in Germany by Friedrich Vieweg & Sohn GmbH.

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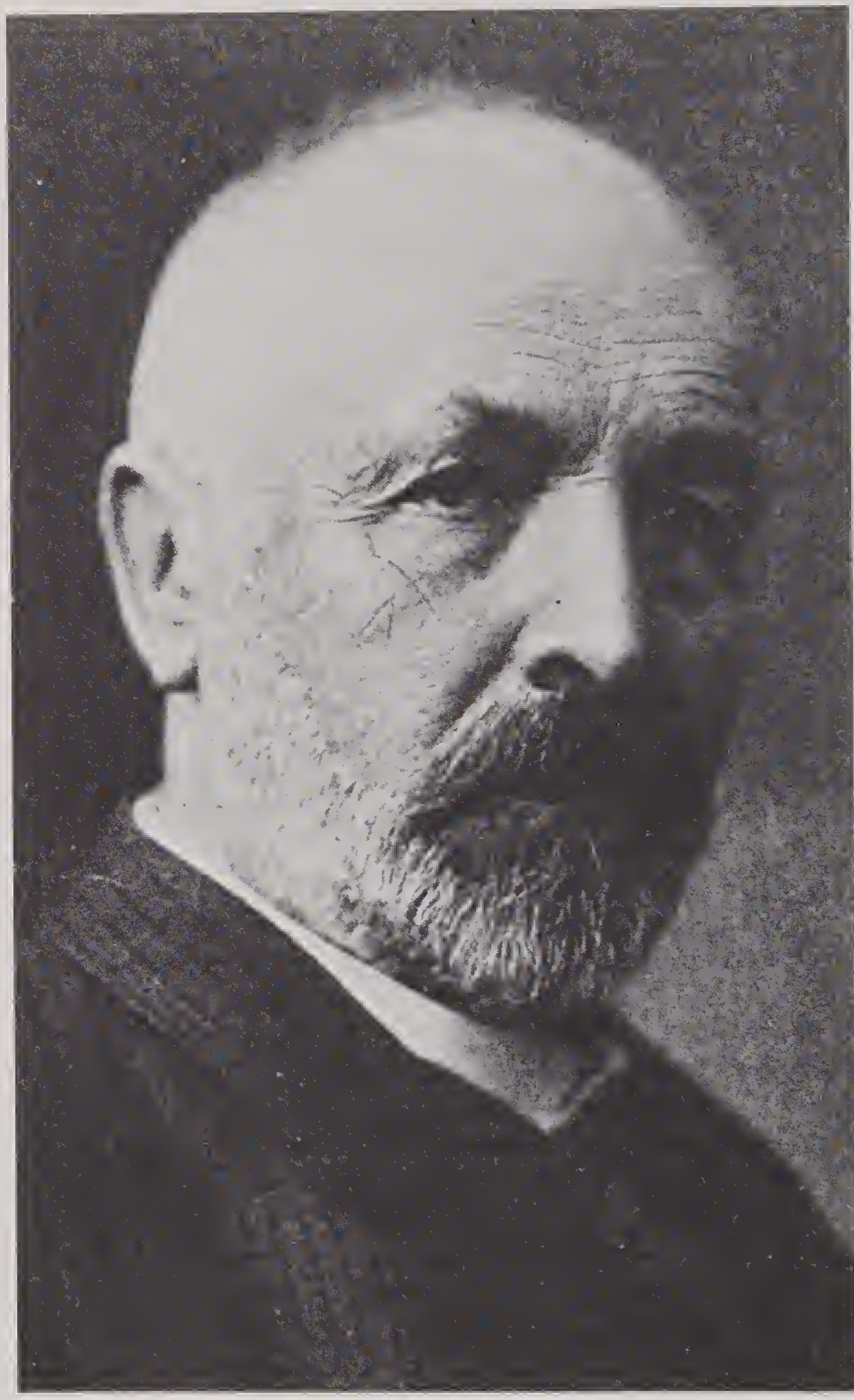


GEORG CANTOR (1845–1918)

INTRODUCTION

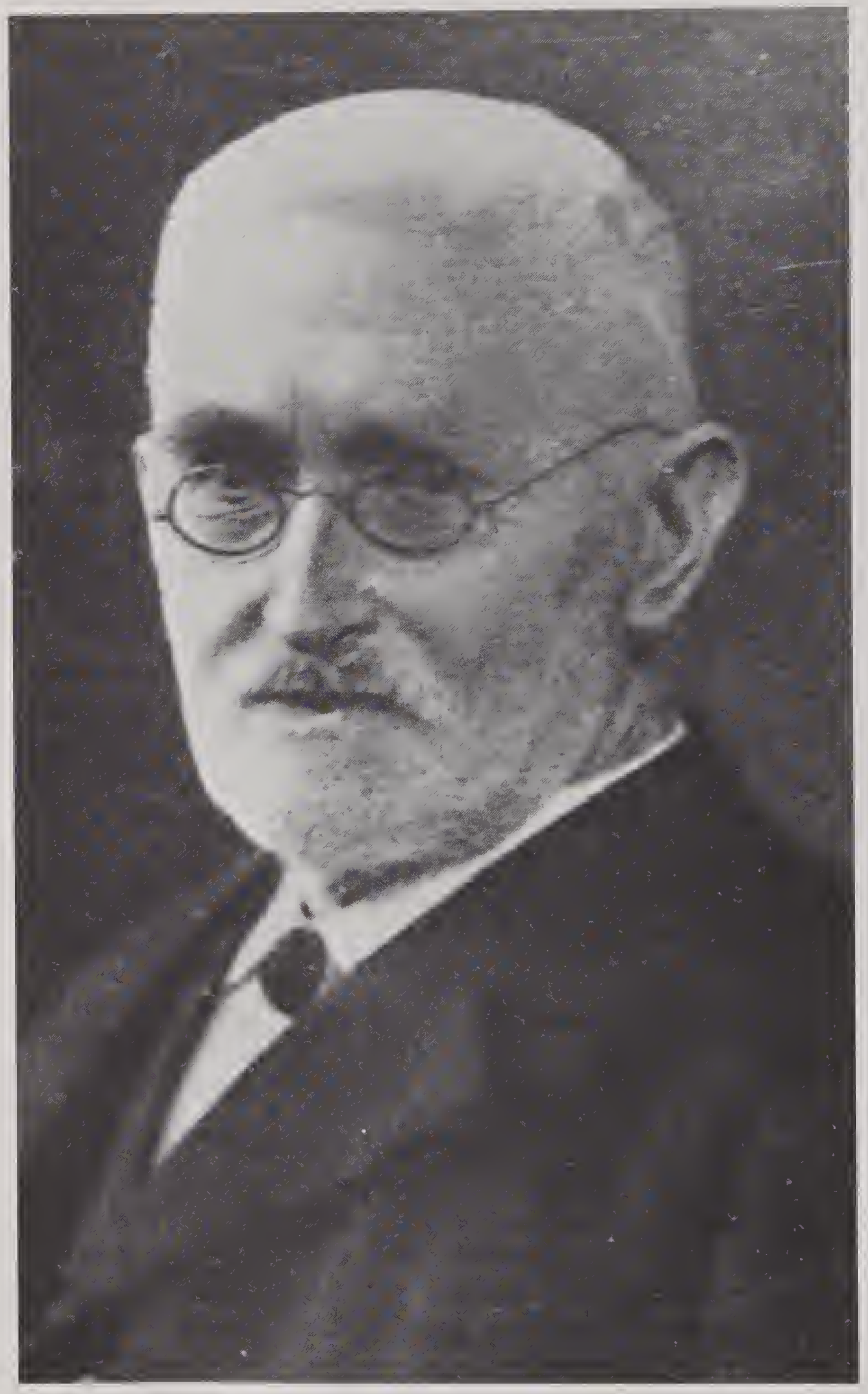
Georg Cantor's creation of the theory of sets was a development of the utmost importance for all of mathematics and for modern analysis in particular. His work appears to differ from other mathematical discoveries in that it came into being without being preceded by a long evolutionary period such as is associated with calculus or non-Euclidean geometry, for example. Often in mathematics several people make similar discoveries at about the same time; this was not the case with Cantor's discoveries. Although others made important contributions to set theory soon after Cantor started his work, the creation of the theory appears to be Cantor's alone.

The purpose of this study is to trace the history of set theory and its influence on the foundations of mathematics from its earliest beginnings up to the start of the axiomatic theories. It begins with a delineation of the work done by Cantor and follows its influence on other mathematicians' work. Conjectures made by Cantor in the course of his studies became fruitful areas of research for other workers in set theory. The paradoxes which unexpectedly arose from his work have been the cause of important study in the foundations of mathematics. In discussing this sequence of events, primary emphasis is devoted to the basic features of the general theory of sets as opposed to the theory of point sets. The main interest is in the subject matter such as that found in Kamke's *Theory of Sets.*[1] Since the study is built around Cantor's work and its consequences, however, an attempt is made to discuss in some measure all areas of set theory to which Cantor contributed.

Hopefully, a history of one area of mathematics might stimulate interest in that area and, more largely, in the history of mathematics as a whole. A history of set theory in its naive or intuitive form should prove particularly worthwhile to teachers of mathematics at all levels. Although some parts of the book will not be readily understood by one without background in the foundations of mathematics or set theory, these parts can be skimmed without loss of continuity and the reader can still profit from the overall picture of the history. Finally, the study should prove helpful to people interested in research in the foundations of mathematics.

The history is primarily concerned with set theory as developed by Cantor in its naive or intuitive form in contradistinction to the axiomatic theories which developed using Cantor's work as a base. Although some account is given of the development of the axiomatic theories, the major portion of the work is concerned with the genesis and early development of set theory as fostered by Cantor.

CHAPTER 2



J. W. RICHARD DEDEKIND (1831–1916)

GEORG CANTOR

Georg Ferdinand Ludwig Philipp Cantor was born on March 3, 1845 in St. Petersburg (now Leningrad), Russia, but should properly be ranked among the German mathematicians, having spent the greater part of his life in German universities. His parents were the prosperous merchant, Georg Waldemar Cantor and his artistic wife, Maria Boehm. The father was born in Copenhagen, Denmark, but moved to St. Petersburg while still a young man. Because of pulmonary disease, Cantor's father retired in 1856 to Frankfurt, Germany, where he died in 1863. Maria Cantor outlived her husband by thirty-three years.

The mathematician Georg Cantor was the eldest of three children born to the Georg Waldemar Cantors. All three children had artistic natures: Georg's brother, Constantin, who became a German Army officer, was a fine pianist; the sister, Sophie Nobiling, was an accomplished designer; Georg was a mathematician who was also interested in philosophy. The artistic temperaments of the children seem to have come from the mother's side of the family. Her grandfather, Ludwig Boehm, was a conductor, whose brother, Josef, in Vienna, was a teacher of the famous violinist Joachim; her brother was a musician; her sister Annette had a daughter, Olga, who was a painter and teacher at the Munich Kunstgewerbeschule.

Cantor's father was born a Jew but became a convert to Protestantism at least as early as 1845. Cantor's mother was a staunch Catholic. Cantor, brought up as a Protestant, was also deeply interested in his mother's faith. The differing religious backgrounds of his parents did not produce religious indifference in Cantor, but, on the contrary, fostered a strong religious sense. He was a devout Christian and knowledgeable theologian.

Cantor's first instruction was at home under a private tutor, followed by attendance at an elementary school in St. Petersburg. Although he showed an early burning desire to take up the study of mathematics, his father wanted him to study engineering for the purpose of earning a livelihood. The son submitted at first and went in 1860 to the Grossherzogliche Hoehere Gewerbeschule (Grand-Ducal Higher Polytechnic, later changed to Technische Hochschule) at Darmstadt. Before this he had briefly attended the Wiesbaden Gymnasium as well as private schools in Frankfurt and then in 1859 the Grossherzoglich Hessiche Provinzialrealschule in Darmstadt. When he left the Darmstadt Realschule in September, 1860, he was given the following certificates: "Mathematics: his industry and zeal exemplary; his knowledge of lower mathematics, including trigonometry, is very good" . . . "Descriptive Geometry: industry, attentiveness and achievements special." After leaving the Polytechnic in August, 1862 with a view to getting a maturity-certificate in languages and natural sciences, he devoted himself in the fall of 1862 to these subjects and secured the mark "sehr gut" (the highest mark given) in most subjects.[1]

On the occasion of Cantor's confirmation in 1860, his father sent him a long letter expressing the high hopes which his parents and "all the other family connections in Germany as well as in Russia and Denmark" had placed on him. The father stated that they "expect from you nothing less than that you become a Theodor Schaeffer and later, perhaps, if God so wills, a shining star in the engineering firmament."[2] Despite the elder Cantor's wish that his son become an engineer, however, the deep interest which the younger Cantor had in mathematics finally had its effect on the father, and he dropped his objection to the son's pursuing a career in mathematics. For his father's permission to seek a university career in mathematics, Georg was deeply thankful. In a letter to his father dated May 25, 1862 his oldest known letter, Cantor writes:

> My Dear Papa!
> How much your letter pleased me, you can imagine for yourself; the letter settles my future. The last days were for me ones of doubt and indecision; I could not come to any decision. Duty and inclination moved in continual conflict. Now I

> am happy when I see that it will not displease you if I follow my feelings in my choice. I hope you will live to find pleasure in me, dear father; for my soul, my whole being, lives in my calling; what a man wishes to do, and that to which an inner compulsion drives him, that will he accomplish.[3]

In the fall of 1862 Cantor began his higher studies at Zurich, but left after the first term in the spring of 1863 because of the death of his father. In the fall of 1863 he entered the University of Berlin, where he studied mathematics, physics, and philosophy. The three great mathematicians at Berlin, Ernst Eduard Kummer, Karl W. T. Weierstrass, and Leopold Kronecker, attracted some of the best minds to study there; and because of the comparatively small number of students, they were able to give considerable help and suggestions to their pupils. Kronecker stimulated a lively interest in number theory in Cantor, but by far the greatest influence on Cantor's scientific career was exerted by Weierstrass. With respect to the two disciplines other than mathematics which Cantor studied at this time, there is no exact knowledge available of his interest in physics, but it is known that he had an astounding familiarity with philosophical literature.[4] During his stay at Berlin, Cantor was away only during the summer term of 1866 when he attended the University of Goettingen, following the usual German custom of studying for a time at another university.

According to E. Lampe, Cantor belonged in his Berlin period, in addition to the Mathematical Association (*Mathematischer Verein*), to a smaller circle of young colleagues who met every week in Rahmel's wine house.[5] Apart from occasional guests, this circle included also Henoch (later editor of "Fortschritte"), E. Lampe, F. Mertens, Max Simon, and L. W. Thomé. Thomé developed a special attachment to Cantor. Another well-known colleague of Cantor at Berlin was Hermann Amandus Schwarz, who was two years older than Cantor. Contrary to the example of his teacher Weierstrass, Schwarz later met Cantor's ideas with great distrust.

On 14 December 1867 Cantor received the degree of doctor on the basis of a severely classical dissertation based on a study of the *Disquisitiones Arithmeticae* of Carl Friedrich Gauss as well as on the number theory of Adrien-Marie Legendre. This work was written on a difficult point which Gauss had left aside concerning the solution in integers x, y, z of the indeterminate equation

$$ax^2 + by^2 + cz^2 = 0,$$

where a, b, c are any given integers. Entitled "De aequationibus secundi gradus indeterminatis" ("On Indeterminate Equations of the Second Degree"), the thesis was dedicated to his guardians, Eduard Flersheim and Bernhard Horkheimer. In the customary disputation against opposing doctors, Cantor defended three theses against his colleagues Hennoch, Simon, and Lampe, who had already received their degrees. The three theses were:

1. In arithmetica methodi mere arithmeticae analyticis longe praestant ("In Arithmetic Merely Arithmetic Methods Far Surpass Analytical Methods").

2. Num spatii ac temporis realitas absoluta sit, propter ipsam controversiae quaestionem pluris facienda est quam solvendi ("Since It Is Disputed, the Question of the Absoluteness of Space and Time Is More Important Than Its Solution").

3. In re mathematica ars propoenedi quaestionem pluris facienda est quam solvendi ("In Mathematics the Art of Proposing a Question Must Be Held of Higher Value Than Solving It").

Cantor's doctoral dissertation, and indeed all his early works until the early 1870s, although excellent, gave no hint of the great mathematical originator which he was to become. That he had ability was obvious, but it was only around 1871 that the life of Cantor began to exceed the up-to-that-time normal development of a gifted scientist.

After earning his doctorate, it appears that Cantor taught for a short time at a girls' school in Berlin. At any rate, after passing the required state examination for teachers, he was a member of Schellbach's Seminar for Teachers of Mathematics in 1868. In the spring of 1869 he established himself as Privatdozent at the University of Halle on the Basis of the paper "De transformatione formarum ternariarum quadraticarum" ("On the Transformation of Ternary Quadratic Forms"). This paper, his fifth, was in number theory, as his first four had been. Although he was especially attracted to the beauty and purity of number theory, he rarely came back to this area later. He became Extraordinarius at Halle in 1872 and Ordinarius in 1879. In 1905 he was released from his official duties and in 1913 he resigned his post altogether. It was during these years at Halle that he published his great and immortal works on set theory.

Although he did not spend much time in preparation, Cantor's lectures were clear, orderly, lively, and stimulating according to his students, at least during times of normal health. He did the best job in the areas which were of most interest to him. Function theory, much in the background at Halle, was not one of his stronger interests, but he showed great interest in group theory. Only occasionally did he lecture about his works in set theory. The number of students attending his lectures was often quite small: not infrequently he had only one to three students in class. Although strict, Cantor was a warm and faithful friend to his students.

Cantor trained a large number of candidates for the state examination for the teacher's diploma, but comparatively few produced doctoral dissertations under his direction. Apparently Cantor immediately carried out the ideas that occurred to him and did not leave much for his students to explore. He did not produce any appreciable number of researchers at Halle. Coming mainly from Berlin, most doctoral candidates arrived at Halle with their dissertations ready and received their degrees after a brief stay.

In 1872, on one of his trips to Switzerland, which apparently were not rare in his younger years, Cantor met Richard Dedekind, one of the most prominent contributors of the nineteenth century to the theory of algebraic numbers. A warm friendship developed between the two men. They frequently met in person, mostly in Harzburg, and also carried on a correspondence from which numerous letters are preserved from 1873–1879 and from 1899. The mathematical content of the letters is limited, but they give a good idea of Cantor's working habits and mood of that time as well as the contrasts in the nature of the two men. Cantor wrote in his first letter about his need to talk with Dedekind about scientific objects and to come closer to him in a personal relationship. In a letter dated 31 August 1899, Cantor mentioned the inspiration which he received from Dedekind's classic writings. To a greater extent than becomes evident from the letters, however, the differences in Cantor's early and later publications dealing with set theory show how deeply Dedekind's abstract, logical way of thinking influenced Cantor. The friendship was as important to Cantor professionally as it was personally.

Another important event in Cantor's personal life was his meeting with his future wife, Vally Guttmann, who lived in Berlin with her three brothers. They became engaged in the spring of 1874 and were married in the summer of the same year. On their wedding trip the young couple went to Interlaken, Switzerland, where they saw a lot of Dedekind. At Halle, Cantor's wife helped provide a comfortable musical atmosphere and pleasant company in their home for his own students as well as students of other disciplines. The Cantors had two sons and four daughters, none of whom had a specific mathematical gift. After the birth of the children, Cantor spent vacations with his family in the Harz, where he could engage in one of his favorite pastimes, hiking. One of the Cantors' daughters, Frau Gertrud Vahlen, was especially helpful to Fraenkel in his writing of the biography of Georg Cantor, which, as previously mentioned, is the primary authority for the present biographical sketch of Cantor.

When Cantor went to Halle, the mathematician Heine was an Ordinarius there. Of decisive importance to Cantor's professional career was Heine's imspiring his young colleague to take up the study of trigonometric series soon after his arrival in Halle. Cantor attacked the subject with zeal and met with a considerable number of successes. His papers six through ten, as well as some later papers, dealt with trigonometric series. But more important than the successes in work with trigonometric series per se was the fact that this work led him to the theory of point sets, and at the same time, to the transfinite ordinals, as is shown in his tenth paper. These papers contain both the first basic ideas of the theory of point sets and the work which, in addition to set theory, has made Cantor immortal: the theory of irrational numbers as fundamental series. The tenth paper also shows Cantor's recognition of the necessity to use in geometry an axiom which is known today under the name *Cantor's Axiom*. At the same time and independently of Cantor's work, Dedekind's "Stetigkeit und irrationale Zahlen" ("Continuity and Irrational Numbers") was published, a topic with which Cantor occupied himself again and again.

Cantor's eleventh paper proved a frequently-used theorem in algebra for which no proof was given in textbooks. His twelfth paper dealt with the history of the calculus of probability, which had already occupied him for several years, although he had never done research in any branch of applied mathematics.[6] He was directed to this subject by the intention to read a paper to the Naturforschende Gesellschaft (Society of the Friends of Nature) of Halle, the theme of which had to be understandable to all.

In 1874 an article appeared in *Crelle's Journal* that marks the hour of the birth of set theory. A subsequent work, in 1878, remained in the publishing room of *Crelle's Journal* longer than usual for that time, apparently due to Kronecker's skeptical view of Cantor's idea. Kronecker was on the editing staff of the journal then as in 1874. This delay seems to have been quite irritating to Cantor. He was tempted to withdraw the manuscript and publish it as a special article, but Dedekind persuaded him against this action. Cantor complained to Dedekind that the printing was delayed, in spite of the promises of the editors, "in a manner which is utterly unexplainable to me'', and that a work which had arrived later was given preference. Weierstrass defended the work, according to Cantor. The differences at Crelle's were solved and the work appeared on time, but after the unfortunate events surrounding the publication of Cantor's paradoxical result (considered paradoxical at that time), he never published another article in *Crelle's Journal*.

As Cantor's work became more and more advanced, Kronecker's opposition grew. Weierstrass, on the other hand, showed complete understanding of Cantor's endeavors. Cantor believed that both Weierstrass and Charles Hermite had early prejudices against him, systematically stimulated by Kronecker, but that they both completely overcame these prejudices.[7] Indeed, Weierstrass applied some of Cantor's theory to part of his own work.

According to Fraenkel, a surprising and little-known observation by Cantor seems to prove that as early as the beginning of the 1870s he had the scope of his theories, as well as the opposition to them, clearly in mind. In 1870 the idea of the transfinite numbers came to him for the first time. In 1873 he recognized the significance of the possibility of counting and the connection between it and the continuum. In the period 1879–1884, he published practically his complete theory of sets, of which some of the fundamental points had been given before. This period was Cantor's most intesive working period during which he brought forth the incredible development of his ingenious ideas; it was also the time of the most difficult crisis in his life, which affected him until his death.

The editor of the *Mathematische Annalen* performed a bold deed, but also gained immortal merit, in publishing Cantor's six-part treatise of 1879–1884. There was tremendous opposition to his work by Kronecker and other prominent mathematicians of that time. In the words of Cantor's biographer, this treatise ''belongs to the events in history where a completely new idea of epochal thought which stands in total opposition to the ideas of the past and present bursts forth and crystallizes with increasing clearness, becoming all the time clearer to its creator in its boldness and newness."[8] Fraenkel comments further that this treatise belongs, at least in its fifth part, not only to the field of mathematics and philosophy, but is of importance altogether for the history of science and human thinking.

Some passages from the fifth part of Cantor's treatise give a hint of the hard fight which took place in those years and which was greatly responsible for the decline in his health. Therein he speaks of the necessity for extending the number concept and says that without such extension it would be impossible to execute the smallest step forward in set theory. He uses this as his justification for introducing apparently strange or foreign ideas into his considerations. He acknowledges that he places himself in his enterprise in a certain contrast to widely held ideas about the infinite.

The fifth part of the treatise, a work of utmost importance, was also published separately with an added preface.[9] In the preface Cantor states that he most certainly does not believe that he can speak the last work about such a complicated, difficult, and encompassing subject as that presented by the infinite. However, he continues, since he has come to certain conclusions about this topic in his many years of research, and since these conclusions have become more and more firm during the course of his studies, he feels that he has a certain obligation to put them in form and to make them known.

In retrospect, the delay in getting his important paper of 1878 accepted and the hard struggle to gain recognition for his works of 1879–1884 appear to be major causes for the complete breakdown that Cantor suffered in the spring of 1884. The formidable array of influential colleagues against Cantor's work, chief of whom was Kronecker, was almost certain to have an adverse effect on Cantor. Kronecker enjoyed an almost undisputed respect at that time and it is natural that a substantial number of colleagues would be on his side in the contest between him and Cantor. In addition, Cantor was only moderately satisfied with his appointment as regular professor at Halle and would have preferred the wider field of work offered by Berlin. But his attempts to get transferred to the University of Berlin were frustrated by the opposition of Schwarz and, especially, of Kronecker. In the year 1884 alone, he wrote as many as fifty-two letters to G. M. Mittag-Leffler attacking Kronecker. In one of these (26 January 1884), he referred to Kronecker's writings as "miserable scribblings."[10] Later he said in another letter to Mittag-Leffler that only time would tell which ideas would prove more fruitful, his or Kronecker's. Cantor's work of 1883—against the natural numbers playing so great a role and in favor of freedom of mathematical creation—was unmistakably directed against Kronecker. Cantor's relations with Kronecker seem to have been good until about 1880.

Possibly Kronecker has been blamed too much for Cantor's breakdown; it has been seen that there were a number of contributing causes to this condition. While it is popularly held that Kronecker's animosity toward Cantor was personal, it appears that Kronecker's attacks came at least partly from his own scientific convictions and were not directed against Cantor the man as much as has often been supposed. Cantor's bitterness toward Kronecker was considered by his family and physician to be the chief cause of his psychical disease, and they persuaded Cantor to seek a reconciliation with Kronecker. The reconciliation was effected, but it was not real, as Cantor soon realized. Odd as it may seem, Cantor became diffident and developed a lack of faith in the value of his own work during this period. In this year of mental crisis, 1884, he actually applied to be allowed to lecture on philosophy instead of mathematics. But at the beginning of 1885 his mental crisis was essentially over and his confidence in his own work was reestablished. Although some of Cantor's works after this time are certainly important, Fraenkel states that the most significant and fruitful period of Cantor's life ends with the year 1884.

The successful attempts of Kronecker and other prominent mathematicians to keep him suppressed probably had much to do with Cantor's part in the founding of the Deutsche Mathematiker-Vereinigung in 1890. The association was intended to protect young researchers against the possibility of being injured by the overpowering influence of some mathematicians.

Even in the year of his death, 1891, Kronecker remained critical of Cantor's work. In a lecture to his students in summer semester of that year he very adversely criticized Cantor's mathematical papers. This act on Kronecker's part was certainly inexcusable. It is not generally considered quite cricket for one scientist to deliver a savage attack on the work of a contemporary to his students; there are journals available for objectively handling such disagreements.

The death of Kronecker, coupled with the sincere friendship of such an influential man as Mittag-Leffler and the affection which Weierstrass always felt for him, were probably helpful in making life more tolerable for Cantor. He was soon to begin receiving the recognition which he deserved.

Public recognition of Cantor and his theory before the very end of the nineteenth century was meager, even in his own country. In 1869 he received regular membership in the Naturforschende Gesellschaft of Halle. In 1878 he became a corresponding member of the Society of Sciences in Goettingen, apparently the only German academy or university outside of Halle which honored Cantor's merits publicly. In the same year he received a call to the academy in Muenster, which he declined. As already mentioned he was appointed to the Ordinariats at Halle in 1879, a newly established post there. In 1885, the year in which Cantor essentially recovered from his mental crisis, his ideas were being taken up by others, among the first of whom were A. Harnack, M. Lerch, and E. Phragmén. But it was not until 1897, at the first International Congress of Mathematicians, held at Zurich, that Cantor was generally recognized. There A. Hurwitz openly expressed his great admiration of Cantor and proclaimed him as one by whom the theory of functions has been enriched. Jacques Hadamard expressed his opinion that the notions of the theory of sets were known and indispensable instruments.

Applications of Cantor's ideas also began to find places in papers and books. The second edition of Meyer's *Elemente der Arithmetik und Algebra*, published in 1885, was influenced by Cantor's work in transfinites; the work introduced especially the number concept in a set-theoretic manner. Henri Poincaré and G. M. Mittag-Leffler made use of his ideas in some of their work. In France, L. Couturat's *L'infini mathématique* of 1896 used Cantor's notions and has a thorough appendix dedicated to the ideas of Cantor. The theory found a place in *Enzyklopaedie der Mathematischen Wissenschaften* in a separate article of 1898, in E. Borel's *Lecons sur la théorie des fonctions* of 1898, and in R. Baire's *Sur les fonctions der variables réelles* of 1899. The first text books on set theory seem to have been A. Schoenflies' *Entwickelung der Lehre von den Punktmannigfaltigkeiten*, 1900; The *Theory of Sets of Points* by Professor W. H. Young and his wife Dr. G. C. Young, 1906; and Hausdorff's *Grundzuege der Mengenlehre*. Numerous other books and papers have been written which show the influence of Cantor's set theory. His work is now accepted as a fundamental contribution to all mathematics and particularly to the foundations of analysis.

The early twentieth century saw Cantor being accorded recognition of another kind by foreign countries. He was made an Honorary Member of the London Mathematical Society in 1901 and of the Mathematical Society of Charkov, and corresponding member of the Reale Istituto Veneto di Scienze, Lettere ed Arti of Venice. The Christiana University in 1902 and the St. Andrews University in 1911 conferred honorary doctors' degrees on him. In 1904 the Royal Society of London awarded him its Sylvester Medal. In 1915 his seventieth birthday was celebrated as an event of international importance. In spite of the war, German mathematicians from near and far took part in the celebration. At that time a marble bust of Cantor was subscribed to. It was placed on the staircase at the University of Halle in 1928.

Certainly Cantor was pleased with the honors which belatedly came to be bestowed upon him. Part of a letter from him to Professor W. H. Young was translated and read by Young in the Presidential Address to the London Mathematical Society on 13 November 1924. The letter is probably indicative of the gratitude which Cantor felt for all the honors and awards bestowed upon him. In the letter he says, "I shall always remain grateful to the London Mathematical Society for the award of the Sylvester Medal, . . ." He goes on to say that he feels himself at one with the high-minded inhabitants of Great Britain, but that it is quite otherwise with the Germans, "who do not know me, although I have lived among them fifty-two years."[11]

Some comments about Cantor by a few leading mathematicians will be of interest. Writing of Germany's neglect of Cantor's theory, Schoenflies noted that, while Cantor possessed in the *Annalen* a ready outlet for his publications, before the year 1884 he had hardly any scientific influence on others.[12] Schoenflies expressed the opinion that mathematical science would ever remain indebted to Cantor for his ceaseless belief in and defense of his creations. Fraenkel said that a great pioneer of science was given to the mathematical world in Georg Cantor. He stated further that Cantor's work had opened up entirely new, rigorous, and revolutionary avenues of research for analysis. The ideas of the theory of point sets have even been of value for physical applications. David Hilbert quoted Hermann Minkowski as saying that future history would regard Cantor as one of the most deep-thinking mathematicians of those times. Minkowski also expressed regrets that the opposition by Kronecker on grounds not purely mathematical could disturb the joys of Cantor in his scientific researches.[13] Hilbert and Minkowski, along with Hurwitz, were probably the first in Germany to recognize the originality and significance of Cantor's set theory. Professor W. H. Young pointed out that at the turn of the century *Mengenlehre* was an all but unknown term, but that a complete change has come over the field of mathematics and that it would now be hard to find a writer on analysis who does not directly or indirectly utilize the concepts and even the theorems of the theory of sets of points.[14] Much more testimony could be cited to show the deep respect which has come to be accorded to the mathematical genius of Cantor and the regard for the importance of his results.

Cantor's illness recurred throughout his life; he died in the psychiatric clinic at Halle on 6 January 1918. He had lived long enough to see the beginning of the tremendous impact which his theory was destined to have on the mathematical world and to enjoy the pleasure of the belated recognition which he so much deserved.

CHAPTER 3

LEOPOLD KRONECKER (1823–1891)

CANTOR'S SET-THEORETICAL WORKS

The birth of set theory can now be recognized in Cantor's paper (13). Insofar as set theory is concerned it was preceded only by his work with trigonometric series, out of which grew the consideration of point sets. His paper (10), to be discussed later, contained, among other things, the first basic ideas of point sets.

The paper (13) immediately makes clear that it is possible to distinguish different kinds of infinity. In the naive infinite, different orders of infinity do not exist; if something is infinite, it is infinite. In this paper, however, Cantor shows that there are at least two different orders of infinity: the set of real algebraic numbers can be put in one-to-one correspondence with the set of natural numbers (positive whole numbers), whereas the same is not true of the set of real numbers. According to Fraenkel, Cantor himself had first thought that the continuum could be put in one-to-one correspondence with the set of natural numbers.

The set of all real roots of ordinary algebraic equations of the type

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0,$$

where n is a natural number, a_i an integer, $i = 0, 1, \ldots, n$, and $a_0 > 0$, are called *algebraic numbers*. To see that this set of numbers can be put into one-to-one correspondence with the set of natural numbers, define the *index* of an equation such as the above as the natural number

$$a_0 + |a_1| + |a_2| + \ldots + |a_{n-1}| + |a_n| + n.$$

Since both n and a_0 are at least 1, there exists exactly one equation of index 2, namely $x = 0$, so that the only root of equations of index 2 is the number zero. The only equations of index 3 are $2x = 0$, $x \pm 1 = 0$, $x^2 = 0$, yielding roots $0, -1, 1$. For each index there exists only a finite number of equations and hence a finite number of roots corresponding. Any non-real roots are rejected. Thus the set of real algebraic numbers can be arranged in a sequence by arranging the numbers corresponding to the index in order of magnitude, and then arranging the various indices in their order of magnitude. In arranging the numbers any number that was a root of an equation of lower index may be rejected. Since every equation has an index all algebraic numbers appear in the sequence and the proof is completed.

The proof in the paper (13) that the set of real numbers cannot be put in one-to-one correspondence with the set of natural numbers is not the familiar one using Cantor's famous diagonal process. Cantor proved the theorem for the first time using nested intervals. The following proof, simpler and perhaps now more common than his first proof, is essentially due to his later work (30).

Suppose by way of contradiction that there exists a one-to-one correspondence between the set of natural numbers and the set of real numbers. Denote the real number paired with each natural number n in this correspondence by r_n. Also let the digit in the nth decimal place of r_n be denoted by $a_n{}^n$. Let a real number $r = 0.\, a_1 a_2 a_3 \ldots a_n \ldots$ be defined such that, for each n, $a_n = 1$ if $a_n{}^n \neq 1$, and $a_n = 2$ if $a_n{}^n = 1$. Since r is a real number, it must itself be an r_n, say r_i. But r_i has $a_i{}^i$ as the ith decimal digit, whereas r has a different digit a_i. From this contradiction it must be concluded that there does not exist a one-to-one correspondence between the set of natural numbers and the set of real numbers.

Cantor remarks in the introduction to (13) that by combining these two theorems (the algebraic numbers can be put in one-to-one correspondence with the set of natural numbers whereas the reals cannot) there results a proof of the theorem first proved by Liouville that in each given interval there exist infinitely many transcendental (non-algebraic) real numbers.

The concept of one-to-one correspondence between sets is the fundamental idea in Cantor's memoir (14). Some important theorems pertaining to this kind of relation between various sets are given and suggestions are made of a classification of sets on this basis.

At the beginning of the work Cantor introduces the idea of equivalence, and based on this explains the concept of power in a more concrete manner than he does later in the systematic presentation (32). If two well-defined sets can be put into one-to-one correspondence, they have the same *power* (*Machtigkeit*) or are *equivalent*. Cantor borrowed the term *power* from Jakob Steiner, who used it in a special, but allied, sense. When a set is finite, the notion of power corresponds to that of number, for two finite sets have the same power if and only if the number of their elements is the same.

A *subset* (*Bestandteil*) of a set is defined as any other set whose elements are also elements of the original set (the present meaning of *proper subset*). A subset of a finite set always has a power less than that of the set itself. That this is not the case with infinite sets was first noticed by Bernard Bolzano several years earlier, and was used in a definition of *infinite* by Dedekind, independently of Bolzano and Cantor, in 1887.[1] For example, it is easy to see, as Cantor points out, that the set of natural numbers has the same power as that subset of it consisting of the even natural numbers. Hence, from the circumstance that an infinite set M is equivalent to a subset of N, it can only be concluded that the power of M is less than that of N if it is known that these powers are unequal.

The collection of sets having the smallest infinite power is an extraordinarily rich and extensive system. It is easy to show that the set of natural numbers has the smallest infinite power. Moreover it is also easy to show that if M is a set having this power, then any infinite subset of M has this same power. Cantor also gives here the theorem that if $M', M'', M''', \ldots$ is any finite or simply infinite sequence of sets each having the power of the natural numbers, then the set resulting from the union of these sets has this same power. Also included in this paper is Cantor's interesting and well-known proof that the set of rational numbers has the smallest infinite power. These are some of the principal theorems in the paper concerning sets having the power of the natural numbers (Cantor will later use the term *countable* to refer to such sets).

The proof that the real number continuum is not countable has already been discussed. The idea of going over from the one-dimensional to the multi-dimensional to get higher transfinite powers had previously occupied Cantor in 1874, as his correspondence with Dedekind shows. To see a problem here at all required a new attitude since it was commonly assumed that points in two-space cannot be traced back to one-space. Yet Cantor proceeded to prove in his paper (14) the independence of the power of the continuum from its number of dimensions. That this fact surprised even himself is evident in another of his letters to Dedekind prior to publication of the proof. Cantor observes that the result can be expanded to the case of a countable (later terminology) infinity of dimensions. He also conjectures that the two powers of the rational numbers and the real numbers exhaust all possibilities for infinite subsets of the continuum. Time has shown that he was overly optimistic in that it has now been shown in the realm of axiomatic set theory that the conjecture is neither provable nor disprovable.

In close connection with the work (14) just discussed stands the attempt in Cantor's paper (15) of a proof of the following theorem: If two regions G_m and G_n of the dimension numbers m and n relate to each other continuously so that to each point of G_m at least one point of G_n, to each point of G_n at most one point of G_m corresponds, then $n \geq m$. This theorem would include the general theorem of the invariance of the number of dimensions. It had been proved by Lueroth for the cases $m = 1$ and $m = 2$.[2] He had received the idea through Cantor's work (14). Cantor refers in this paper to corresponding endeavors of Thomae and Netto and also to Lueroth's work.

Evidently only Brouwer first brought an absolute proof of the invariance of the dimension numbers.[3]

The foremost line of ancestry of Cantor's theory of point sets has to be traced back to the very important sequence of essays (16). Insofar as the theory of point sets specifically is concerned, it was preceded only by the introduction of a few less important concepts in (10). His paper (10) shows clearly how the consideration of general point sets grew out of his research into trigonometric series.

It has already been mentioned in Chapter 2 that Cantor's theory of irrational numbers stems from (10). At first these numbers were not a purpose in themselves, but an instrument; he comes back to them more thoroughly in his fifth essay of (16). The first introduction of limit points and derivatives of point sets is also contained in (10). For a set of points P in a finite interval, a *limit point* (*Grenzpunkt*) of P is any point of the straight line such that in any interval within which this point is contained there are an infinity of points of P. Cantor called every point of P which is not a limit point of P an *isolated point*. Every point of P either is or is not a limit point of P. The set of all limit points of P is called the *first derived system* (*erste Ableitung*) P'. If P' is not finite, a second derived system P'' can be deduced by the same process, and so on. These notions will be used in (16).

In the first essay of the sequence (16), Cantor submits linear sets to a close analysis by classifying them according to certain principles. He stresses that the idea of derived sets can be extended to several dimensions. Sets are divided according to three points of view:

a. According to their behavior when continuing the derivation process. In this connection, a point set P_γ is said to be of the *first kind and γth species* if $P^{(\gamma)}$ consists of merely a finite set of points. It is of the *second kind* if the sequence $P', P'', P''', \ldots, P^{(\gamma)}, \ldots$ is infinite.

b. According to their behavior in a given interval. The sets which are dense everywhere in the interval are different from those which are not dense everywhere. Paul du Bois Reymond independently used the same idea.[4] A point set P is defined to be *everywhere dense* in a closed interval if every subinterval of the interval contains points of P.

c. According to power. Cantor puts in the background for the time being the question whether the countable and the continuous exhaust all infinite point sets, and gives a somewhat simpler proof than in (13) that the linear continuum is not of the first (infinite) power.

In his third essay in the sequence (16) Cantor extends the concepts previously developed for linear sets to sets situated in continua of n dimensions, especially the concepts of limit points, of derivatives, and of density. There is a passage at the beginning of the essay which deals with the linear point sets as being given. The general character of the power concept is stressed. There are some reflections as to under what circumstances an infinite set is "well-defined." He explains that a set of elements belonging to any sphere of thought is *well-defined* when, in consequence of its definition and of the logical principle of the excluded middle, (i) it must be considered as intrinsically determined whether any object belonging to this sphere belongs to the set or not, and (ii) whether two objects belonging to the set are equal or not, in spite of formal differences in the manner in which they are expressed. Cantor says that the concept of power may be considered an attribute of every well-defined set.

Also in the third essay of (16) is the first use of the word *countable* (*denumerable, enumerable*) to describe a set which can be put in a one-to-one correspondence with the set of positive integers and is consequently of the first infinite power. Cantor mentioned in a letter to Jourdain that he first formed the idea of countability in 1873.[5] Also in this essay is the theorem that in an n-dimensional space an infinite set of n-dimensional continuous subregions, separated from each other and meeting at most at their boundaries, is countable (with every subregion the points of its boundary are considered as belonging to it). Finally Cantor makes the interesting remark that if from an n-dimensional continuum any countable and everywhere dense set is removed, the remainder U, if $n \geq 2$, does not cease to be continuously connected, in the sense that any two points N, N' of U can be connected by a continuous line composed of circular arcs all of whose points belong to U.

Cantor used the concept of countability in his paper (20). He adapted some of the most elementary set-theoretical understandings to a problem of analysis which was suggested by Weierstrass. The problem dealt with producing, from a function having in a certain place some singularity, functions having the same kind of singularity at a countable and everywhere dense set in a given real interval.

In the fourth essay of the sequence (16) Cantor introduces the concept of the isolated (n-dimensional) point set and states a few facts about it. If a set Q (in a continuum of n dimensions) is such that none of its points is a limit point, it is *isolated*. He proceeds to prove six theorems on countable point sets.

As was mentioned in Chapter 2, the fifth essay of (16) was also published separately, with extra notes, as *Grundlagen einer allgemeinen Mannichfaltigkeitslehre*. In a note to the *Grundlagen*, Cantor remarked that he meant by the term *Mannichfaltigkeitslehre* a doctrine embracing very much which before he had attempted to develop only in the special form of an arithmetical or geometrical theory of sets (*Mengenlehre*). By a manifold or set he understood generally any multiplicity which can be thought of as one, that is, any totality of definite elements which can be bound up into a whole by means of a law. Cantor repeatedly emphasized this character of unity.

The *Grundlagen* begins by drawing a distinction between two types of infinity which appear in mathematics. The term *improper infinite* (*Uneigentlich-Unendliches*) is used for a magnitude which either increases above all limits or decreases to an arbitrary smallness, but always remains finite; so that it may be called a variable finite. The *proper infinite* (*Eigentlich-Unendliches*) is typified by the infinite real integers, and, to emphasize this, the old symbol "∞," which was and is used also for the improper infinite, was here replaced by "ω." Cantor points out that, although it would be contradictory to speak of a greatest number of the set of positive integers, there is nothing objectionable in imagining a new number, ω, which is to express that the entire set of positive integers is given by its law in its natural order of succession. He goes on to develop larger numbers of this type by what he calls "generation and limitation principles." These numbers are, of course, recognized now as ordinal numbers; he returns to a discussion of them later and so names them. Cantor observes here that in the case of finite sets this conception of number coincides with power, but these two concepts diverge in the case of infinite sets. This new concept of number serves to make precise the concept of power which Cantor had used often already. Cantor is able in this context to define his new numbers independently of the theory of derivatives. Without this extension of the concept of number, Cantor says that it would hardly be possible for him to make without constraint the least step forward in the theory of sets. He says that he was logically forced to these new numbers almost against his will. He stresses the important idea that the fact that these numbers do not have all the qualities of the finite numbers or have certain qualities which cannot go with the finite numbers cannot be reason enough to reject them. Each enlargement of a basic principle, he says, brings the loss of certain characteristics; he refers to the example of the complex numbers as an enlargement of the reals. The development of Cantor's new numbers and the distinction between this concept and the concept of power is of significance for the theory of all (finite and infinite) arithmetic.

Any well-defined set whose elements have a given definite succession such that there is a *first* element, a definite element that follows every other (if it is not the last), and to any finite or infinite set a definite element belongs which is the *next* following element in the succession to them all (unless there are no following elements in the succession) is called by Cantor a *well-ordered* set. Two well-ordered sets are *similar* (using Cantor's later terminology) if a one-to-one correspondence is possible between them such that, if a and b are any two different elements of the one, and a′ and b' are the two corresponding elements of the other, if a precedes or follows b, then a′ respectively precedes or follows b'.

There is an interesting discussion in the *Grundlagen* of the conditions under which the introduction into mathematics of a new concept, such as ω, is to be regarded as justified. Cantor says that mathematics is, in its development, quite free; that it is only subject to the self-evident condition that its concepts be both free from contradiction in themselves and stand in fixed relations, arranged by definitions, to previously formed and tested concepts. In the introduction of new numbers, in particular, it is only obligatory to give such definitions of them as will afford them such a definiteness, and under certain circumstances, such a relation to the older numbers, as permits them to be distinguished from one another in given cases. He says that a number, as soon as it satisfies all these conditions, can and must be considered as existent and real in mathematics.

The *Grundlagen* gives a good account of the slow and sure way in which the transfinite numbers forced themselves on the mind of Cantor and also shows to a considerable degree Cantor's philosophical and mathematical traditions. Both here and in Cantor's later works are discussions of opinions on infinity held by mathematicians and philosophers of all times. Such names as Aristotle, René Descartes, Baruch (or Benedict) Spinoza, Thomas Hobbes, George Berkeley, John Locke, Wilhelm Leibniz, Bernard Bolzano, and many others are found in his works. There is evidence of deep erudition and painstaking search after new views on infinity to analyze. Many pages have been devoted by Cantor to the schoolmen and the fathers of the church.

There is some lack of attention in details to be noticed in Cantor's fifth essay of (16), possibly due in some measure to the turmoil through which his life was going at this time. Some of the mathematics found here is later treated far more completely and is drawn up with far more attention to logical form than was the *Grundlagen.* Of course, much of this increased attention to details later may be due to the seasoning of the ideas in Cantor's mind over the years.

In the fifth essay, as in the third, Cantor rejects the actually infinitely small. He also opposes the finitistic concept of Kronecker. Cantor recognized certain methodical advantages by Kronecker, but he thought some of Kronecker's ideas were not only basically unproductive with reference to the progress of science but were even wrong.

Cantor defines addition and multiplication of transfinite numbers and mentions some of the basic properties. He will return to a fuller investigation of transfinite arithmetic in later papers.

Finally, in the rather lengthy *Grundlagen*, Cantor discusses and defines the concept of "continuum." He explains that he intends to give a strictly mathematical analysis without going into the metaphysical side. He briefly refers to the discussions of the continuum concept of Leucippus, Democritus, Aristotle, Epicurus, Lucretius, and Thomas Acquinas. He emphasizes that he will not draw upon the concepts of time or space for the understanding of the continuum since one needs instead, in the opposite direction, for the understanding of space and time, a mathematically exact definition of the continuum.

He starts, supported by the arithmetical concept of the real number, from the arithmetical space G_n: $(x_1, x_2, x_3, \ldots, x_n)$, where each x can assume any real value from $-\infty$ to $+\infty$ independently of the others. Every such system is an "arithmetical point" of G_n, and the "distance" of two such points is defined by

$$+\sqrt{(x_1' - x_1)^2 + (x_2' - x_2)^2 + \ldots + (x_n' - x_n)^2}.$$

By an "arithmetical point set" P contained in G_n is meant any set of points G_n selected out of it by a law.

A point set P is *closed* if every limit point of P is a point of P. The set P is *perfect* if P is closed and if every point of P is a limit point of P. (This language differs from Cantor's; the word *closed* was used later.)

Perfect sets are not always everywhere dense. Cantor gave an example of a bounded perfect point set which is everywhere dense in no interval. The example, in another form, had been published already in 1875 by H. J. S. Smith but obviously was not known to Cantor.[6] Since perfect sets are not always everywhere dense, they are not fitted for the complete definition of a continuum, although the continuum certainly must be perfect. Cantor defined a point set P to be *connected* if, for t and t' any two points of P and ϵ a given arbitrarily small positive number, a finite number of points $t_1, t_2, t_3, \ldots, t_n$ of P exist such that the distances $tt_1, t_1t_2, t_2t_3, \ldots, t_nt'$ are all less than ϵ. Cantor comments that all the geometric point-continua known to us are connected. He says that he believes the two properties *perfect* and *connected* are the necessary and sufficient characteristics of a *point-continuum*. In a note Cantor stresses the independence of his definition of the continuum from the dimension. He observes that the set of all continuous functions has the power of the continuum (and probably also the set of all integrable functions). It should be mentioned that Cantor was critical of Bolzano's treatment of the continuum. It also seemed to him that Dedekind only emphasized a property of a continuum, namely that which it has in common with all other perfect sets.[7]

The last paragraphs of the fifth essay give, on the basis of Cantor's generation and limitation principles, a sketch of the theory of the numbers of the "second number class." The first number class has the smallest infinite power and Cantor proceeded to show that the totality of the numbers of his second number class has the next higher power. From these researches into ordinal numbers and powers Cantor derived the idea that any set whatsoever can be well-ordered, and this he stated with a promise to return to the subject later. Cantor also gives a proof of the equivalence theorem for sets of the power of the second number class: If M is any well-defined set of the second power, M' is a subset of M and M'' is a subset of M', and M'' has the same power as M, then M' has the same power as M and hence as M''. Cantor remarked that this theorem has general validity and promised to return to it later; obviously the proof is still lacking in the total presentation (32).

Cantor's sixth essay in the sequence (16) is especially rich in mathematical contents. He begins with a general explanation of the interval nesting method. He observes that the interval nesting method developed by him is, according to its basic ideas, very old and is used in newer times by Joseph-Louis Lagrange, Adrien-Marie Legendre, P. G. Lejeune Dirichlet, and Augustin-Louis Cauchy, and, in some treatises, by Bolzano and Weierstrass. Certain objections (Kronecker's) against this method of proof are compared by Cantor with the false conclusions of Zeno.

Cantor gives proofs in the sixth essay of a few assertions of the preceding essay. Several theorems concerning countable sets are stated and proved. Among these is the theorem that a perfect set is not of the first (infinite) power. The treatise (22), previously published, also gives a proof of this theorem as well as proofs of two other theorems contained in the sixth essay.

The concept *closed* was introduced in the sixth essay (using this term). Cantor showed that each closed set can be presented as the derivative of a set. Cantor's paper (25) contains a generalization of sets which are not closed. The concept *dense in itself* was also introduced in the sixth essay. A nonempty point set M is *dense in itself* if M is contained in its derived set M'.

Some of Cantor's work in the sixth essay of (16) deals with content theory.[8] Also, a small part of his paper (23) deals with content theory. In the latter paper Cantor observes that he needs his content theory for investigations in the dimensions of continuous sets. A dimension theory on this basis was later developed by Hausdorff.[9] At the same time that Cantor's paper came out Stolz published a work wherein there is a content definition (for linear sets) which agrees with that of Cantor.[10] Cantor used an integral to give a definition of content for an n-dimensional point set P. In the sixth essay, he proved essentially only the theorem that the content of the point set P is the same as the content of the derived set P'. He drew attention to the existence of perfect sets of content 0, which are nowhere dense, while a perfect set which is nowhere dense can very well have a positive content. Cantor's definitions were intended to apply to the case of closed sets. According to his opinion closed sets should be sufficient for general observation. His beginnings of the continuum problem are essentially conditioned by the over-estimation of the closed set. When F. Bernstein's dissertation reduced the significance of these sets in a decisive manner, Cantor was very surprised and shortly after interrupted certain power investigations that he had been busy with at that time. A development of content theory, based on the work of Cantor and others, may be found in Young's *The Theory of Sets of Points.*[11] Such mathematicians as Otto Stolz, A. Harnack, Camille Jordan, and Émile Borel, among others, also investigated the content of sets.

Part of the sixth essay of (16) deals with Cantor's researches on the power of perfect sets. The beginning of the paper (23) is also dedicated to examining the power of perfect sets. The construction procedure which Cantor uses in the sixth essay of the linear sets that are nowhere dense has special significance for the theory of real functions of a variable.[12] Cantor himself makes an important use of this in (23) by constructing a continuous monotone function whose derivative disappears everywhere outside the nowhere dense, perfect set P. Especially interesting is the case that P possesses Cantor's content 0 (Lebesque's measure 0).

At the end of the sixth essay Cantor comes back again to the continuum problem. After his researches into the power of perfect sets, he feels that the power of the continuum as the second transfinite can be determined by making use also of some former theorems. As is known from a letter to Mittag-Leffler, Cantor considered, as starting point of the proof, a closed set of the power $\aleph_1$. This construction he did not succeed in establishing, however.

Cantor used the term *adherent* to denote any isolated point of a set and the term *coherent* to denote any limiting point that is also a point of the set. The set of all the adherents he called the *adherence* and that of all the coherents the *coherence*. Some of the work which Cantor did on his theory of adherences and coherences may be found in Young's *The Theory of Sets of Points*. Early applications of the theory of point sets to the theory of functions were made by C. Jordan, T. Brodén, W. F. Osgood, R. Baire, C. Arzelà, A. Schoenflies, and many others. The next endeavor here will be to trace Cantor's development, during the years 1883 to 1895, of the theory of the transfinite cardinal and ordinal numbers.

Cantor's papers (26), (27), and (28), which speak especially to a philosophical circle of readers, give an account of the development that the theory of transfinite numbers underwent in his mind from 1883 to 1890. A large part of the discussion is concerned with philosophers' denials of the possibility of infinite numbers as well as extracts from letters to and from philosophers and theologians. Cantor contends that all so-called proofs denying the possibility of actually infinite numbers are false in that they begin by attributing to the numbers in question all the properties of finite numbers, whereas the infinite numbers must constitute quite a new kind of number.

In 1883 Cantor had begun to lecture on his view of cardinal numbers and types of order as general concepts related to sets and that arise from these sets by abstractions from the nature of the elements. He said that every set of distinct things can be regarded as a unitary thing in which the things first mentioned are constitutive elements. By abstracting both from the nature of the elements and from the order in which they are given, the *cardinal number*, or *power* of the set is obtained. The cardinal number of the set is a general concept in which the elements, as so-called units, have so grown organically into one another as to make a unitary whole in which no one of them ranks above the others. Two different sets *have the same cardinal number* when and only when they are equivalent to one another. Cantor further said that there is no contradiction when, as often happens with infinite sets, two sets of which one is a subset of the other have the same cardinal number. He regarded the non-recognition of this fact as the principal obstacle to the introduction of infinite numbers. In dealing with ordered sets, if the act of abstraction referred to is only performed with respect to the nature of the elements, so that the ordinal rank in which these elements stand to one another is kept in the general concept, the organic whole arising is what Cantor called *order* (*ordinal*) *type*, or in the special case of well-ordered sets an *ordinal number*. This ordinal number is the same concept that Cantor had referred to earlier, under another name, in his *Grundlagen*. Two ordered sets *have the same order type* if they are similar to each other. Cantor said that these are the roots from which develops with logical necessity the organism of transfinite theory of types and, in particular, the transfinite ordinal numbers. He proclaimed that he hoped soon to publish the theory in a systematic form.[13]

In a letter of 1884 Cantor pointed out that the *cardinal number* of a set M is the general concept under which fall all sets equivalent to M. He said that one of the most important problems of the theory of sets consists of determining the various powers of the sets in the whole of nature, insofar as they can be know. He thought he had solved this problem as to its principal part in his *Grundlagen* by the development of the general concept of ordinal number. (Another term was used in the *Grundlagen*).[14]

In this same letter, which was writtern to Kurd Lasswitz and whose contents had been given in a lecture in Freiburg, Cantor departed from the custom followed in the *Grundlagen* for writing the product of two ordinal numbers. He now wrote the multiplier on the right and the multiplicand on the left.

The importance of this alteration is that now $\alpha^{\beta} \cdot \alpha^{\gamma} = \alpha^{\beta + \gamma}$ whereas the notation of the *Grundlagen* would yield $\alpha^{\beta} \cdot \alpha^{\gamma} = \alpha^{\gamma + \beta}$ (for ordinal numbers, $\beta + \gamma \neq \gamma + \beta$).[15]

Both at the end of this letter and in a letter of 1886 Cantor discussed the sense in which ω may be regarded as the limit to which the variable finite whole number *n* tends. He noted that ω is the least transfinite ordinal number that is greater than all finite numbers in the same way that $\sqrt{2}$ is the limit of certain variable, increasing, rational numbers, except for the fact that the difference between $\sqrt{2}$ and these approximating fractions becomes as small as desired, whereas $\omega - n$ is always equal to ω. This difference between ω and $\sqrt{2}$ does not in any way, however, alter the fact that ω is to be considered as definite and completed as is $\sqrt{2}$, and that ω has no more trace of the numbers *n* which tend to it than $\sqrt{2}$ has of the approximating fractions. Cantor said that the transfinite numbers are in a sense new irrationalities. He felt the best method of defining finite irrational numbers to be the same in principle as his method of introducing transfinite numbers. He said that the transfinite numbers stand or fall with finite irrational numbers, that in their inmost being both are alike, both being definitely marked off modifications of the actually infinite. He pointed out that while ω may be considered the limit of the increasing, finite, whole numbers *n*, ω is *not* a maximum of the finite numbers, for there is no such thing. Indeed, any number *n*, however great, is quite as far from ω as the least finite number.[16]

In two letters of 1886 Cantor returned in great detail to the distinction between the "potential" and "actual" infinite of which he had made a great point under the names "improper" and "proper" infinite, respectively, in his *Grundlagen*. It will be recalled that the improper (here "potential") infinite is a variable finite; in order that such a variable may be completely known, the domain of variability must be determinable, and this domain can only be, in general, an actually infinite set of values. Thus every potential infinite presupposes an actual infinite. The "domains of variability" which are studied in the theory of sets are the foundations of arithmetic and analysis.

Further, besides actual infinite sets, natural abstractions from these sets must be considered in mathematics; these form the material of the theory of transfinite numbers.[17]

By 1885, Cantor had developed to a large extent his theory of cardinal numbers and order types. In his paper (28) he laid particular stress on the theory of order types and entered into details which he had not published before concerning the definition of order type in general, of which ordinal number is a particular case. Here he introduced the notation $\overline{\overline{M}}$ to denote the cardinal number of a set M, indicating by the double bar that a double act of abstraction is to be performed. The ordinal number of M he denoted by $\overline{M}$, thus denoting that a single act of abstraction is to be performed.

In dealing with the theory of cardinal numbers, Cantor defined the addition and multiplication of two cardinal numbers and proved the funadamental laws about them. It is characteristic of Cantor's views that he distinguished very sharply between a set and a cardinal number that belongs to it: "Does not the first-mentioned [the set] stand *opposite* as the object while the last-mentioned [the belonging cardinal number] is an abstract image of it in *our* mind?"[18] The addition and multiplication of order types and the fundamental laws are also dealt with in (28). The addition and multiplication of cardinal numbers and order types and the fundamental laws about them are treated much the same as in his memoir of 1895. Fortunately, the memoir of 1895 is available in translation.[19]

The idea of power may be explained on the basis of the notion of equivalence, but in the context of (28) the fact that equivalent sets have the same power becomes a provable theorem. Also, Cantor again affirms comparability, again without proof.

The notion of the n-ple ordered set (an ordered set of any finite number of dimensions) is introduced in (28). On this concept he bases the idea of n-ple order types. In an n-ple ordered set, Cantor says, "If we make abstraction of the nature of the elements, while we retain their rank in all the n different directions, an intellectual picture, a general concept, is generated in us, and I call this the n-ple order type."[20] Cantor had previously defined similarity (although he did not use this term) of two well-ordered sets. Here he defines two n-ply ordered sets M and N to be *similar* if it is possible to make their elements correspond uniquely and completely so that, if E and E' are any two elements of M, and F and F' are the two corresponding elements of N, then for $\gamma = 1, 2, \ldots, n$ the relation of rank of E and E' in the γth direction inside the set M is exactly the same as the relation of rank of F to F' in the γth direction inside the set N.

A part of (28) deals with finite sets and their cardinal numbers. Complete induction proves that a finite set cannot be equivalent to any subset.

Cantor gave some interesting criticisms of the number concept as advanced by H. von Helmholtz and Kronecker. Cantor had arrived at a very clear notion that the most essential part of the number concept lay in the unitary idea. In essays of 1886 Helmholtz and Kronecker started with the last and most unessential feature in the treatment of ordinal numbers: the words or other signs which are used to represent these numbers.

In one of the letters in the essay (28), Cantor elaborates on a short remark made in the *Grundlagen* about the infinitely small. He gives a detailed (but incomplete) proof of the nonexistence of actually infinitely small magnitudes, which Fraenkel says is hardly absolutely valid.

Hermann Schwarz's "Ein Beitrag zur Theorie der Ordnungstypen" of 1888, stimulated by an 1887 lecture of Cantor's, simplifies Cantor's respective ideas and continues them in a more general manner. In this respect a representation of the finite order types by G. Vivanti is also worth noting.

The immediately preceding papers of Cantor's which have been discussed dealt with philosophical questions to a considerable extent. The conclussion of the treatises by Cantor to set theory are formed by two purely mathematical works.

Cantor's paper (30) deals with an important question in the theory of transfinite numbers. Cantor stresses that by means of generalization of the diagonal procedure used to prove the non-countability of the reals, it is possible to form from one set another set having greater power. This proof, already accomplished in the *Grundlagen* by means of number classes, of the existence of infinitely many powers is accompanied by a much simpler deduction, which avoids the detour over the ordinal numbers. Cantor takes as an example for the proof of the general theorem that the set of all single-valued real functions defined on the closed interval 0 to 1 has a greater power than the continuum. In the proof Cantor uses comparability of powers. He apparently was not quite so conscious of the difficulties connected with comparability as in his next publication (32). In the terminology introduced in (32), it can be said that the memoir (30) contains a proof that 2, when exponentiated by a transfinite cardinal number, gives rise to a cardinal number which is greater than the cardinal number first mentioned. In (30) can be seen the origins of the concept of "covering" (*Belegung*) which is defined in (32). The introduction of this concept is probably the most striking advance in the principles of the theory of transfinite numbers from 1885 to 1895.

The great double treatise (32) is the last work of his own which Cantor published. Free of the philosophical and critical encumbrances of (28) it was destined for a mathematical public. The two memoirs were published in 1895 and 1897. Some of the early principal advances in set theory since 1897 will be given in a later chapter.

Fraenkel points particularly to (32) along with (16) as the "two quite great and immortal works of Cantor." He mentions that of the "classical" theorems of abstract sets, there is lacking in (32) only the equivalence theorem. He also notes that from (16) to (32) there is a considerable movement from the observation of the sets to that of the numbers. There is also progress to be noted in the direction of clarification and systematization which makes (32) didactically very usable even today.

Cantor begins the paper with the well-known set definition: A *set* (*Menge*) is any collection into a whole M of definite and separate objects m of our intuition or thought. The objects are called the *elements* of M.

Cantor follows the definition by noting what is meant by the union of several disjoint sets into a single set.[21] The definition which he gives of "subset" (A *subset* of a set M is any other set M_1 whose elements are also elements of M) is what is today called a *proper subset*—the term *subset* is used in this chapter in the Cantorian sense. He follows this with the obvious statement that if M_2 is a subset of M_1, and M_1 is a subset of M, then M_2 is a subset of M.

The notation $\overline{\overline{M}}$ to denote the cardinal number, or power, of M is retained in this paper and is still in use today. The equivalence of sets is also discussed again.

There is the express remark here by Cantor that in this build-up comparability is neither self-understood nor provable. It will be recalled that earlier he had used comparability. He says that while, for cardinal numbers $\mathfrak{a}$ and $\mathfrak{b}$, of the three relations

$$\mathfrak{a} = \mathfrak{b}, \mathfrak{a} < \mathfrak{b}, \mathfrak{b} < \mathfrak{a},$$

each one of them excludes the others, the theorem that one of these three relations must necessarily be realized is by no means self-evident. He stated that the theorem that if $\mathfrak{a}$ and $\mathfrak{b}$ are any two cardinal numbers, then *either* $\mathfrak{a}=\mathfrak{b}$, $\mathfrak{a}>\mathfrak{b}$, or $\mathfrak{a}<\mathfrak{b}$ will be capable of proof only after we have gained a survey over the ascending sequence of the transfinite numbers and an insight into their connection.[22] Some theorems concerning equivalence are mentioned as being easily derived once this theorem is known. Among them is the Bernstein equivalence theorem: If A and B are sets such that A is equivalent to a subset B_1 of B and B is equivalent to a subset

A_1 of A, then A and B are equivalent. F. Bernstein proved the theorem in Cantor's seminar in 1898; it was also proved independently by E. Schröder.[23]

A section on the addition and multiplication of powers is followed by a section on the exponentiation of powers. The section on exponentiation of powers contains the important definition of covering. "By a covering of the set N with elements of the set M we understand a law by which with every element n of N a definite element of M is bound up, where one and the same element of M can come repeatedly into application." Cantor is able to introduce the idea of function in a quite general form.

After giving the formal laws of operation which hold for exponentiated powers, Cantor points out that by means of $2^{\aleph_0} = c$, which he proved, and by quite short purely algebraic calculations, the whole contents of his paper (14) become deducible. In this connection it is noted that the facts $c^{\gamma} = c$, where γ is any finite number, and $c^{\aleph_0} = c$ mean that both the γ – dimensional and the $\aleph_0$ – dimensional continuum have the power of the one-dimensional continuum. Cantor proved both these equalities.

A treatment in the sense of (28) of the finite cardinal numbers is followed by a theory of the countable sets. Among the theorems proved in the section dealing with finite cardinal numbers is the one that states that in every set of different finite cardinal numbers there is a smallest. The first proof given in section six of the paper, dealing with countable sets and entitled "The Smallest Transfinite Cardinal Number Aleph-zero," implicitly uses the Axiom of Choice. (Given any collection of disjoint, non-empty sets, there exists a set having exactly one element from each of the given sets.) This axiom was commonly used in mathematics without question and without an explicit statement that such a principle was being used. The first explicit statement of the axiom was given by Ernst Zermelo in 1904 for the purpose of proving the famous and still controversial well-ordering theorem.[24] Cantor's implicit use of the axiom is obvious in the proof that every transfinite set contains a countably infinite subset. Immediately following this theorem is the theorem that any transfinite subset of a set *S* having cardinal number $\aleph_0$ also has cardinal number $\aleph_0$.

The essential difference between finite and transfinite sets, to which Cantor had referred in (14) and to which he also points here, is brought out most clearly in two other theorems of the section on countable sets:

1. Every finite set is such that it is equivalent to none of its subsets.

2. Every transfinite set has subsets which are equivalent to it.

For every transfinite cardinal number, there exists a next one greater, and also to every unlimitedly ascending well-ordered set of transfinite cardinal numbers there is a next one greater. Cantor points to 1882 as the date of his discovery of this fact and mentions that it was exposed in his *Grundlagen* as well as in volume 21 of the *Annalen*. As a firm foundation for showing that cardinal numbers have these properties, he sets forth his theory of order types.

Fraenkel points out that the obvious link to (30) and the application to finite sets are left out of the theory of power. He contends that this in the only way that Cantor could have overlooked the purposefulness and necessity of the introduction of the empty set. The empty set had been used before in the algebra of logic, but it only later found its place in set theory (by means of Zermelo's work).

Cantor's observations here of order types is limited to simply ordered sets. A set *M* is *simply ordered* if a definite order of precedence rules over its elements *m*, so that, of every two elements m_1 and m_2, one takes the "lower" and the other the "higher" rank. Of three elements m_1, m_2, and m_3, for example, if m_1 is of lower rank than m_2, and m_2 is of lower rank than m_3, if then m_1 is of lower rank than m_3. The order type of a set *M* is denoted by $\overline{M}$. Two ordered sets M and N are defined as being *similar* if they can be put into a biunivocal correspondence with one another in such a manner that, if m_1 and m_2 are any two elements of M, and n_1 and n_2 are the corresponding elements of N, then the relative rank of m_1 to m_2 in M is the same as that of n_1 to n_2 in N. If an ordered set with order number α has all the relations of precedence of its elements inverted, the order number of the newly obtained set is denoted by Cantor as $^*\alpha$, a notation still in use. The order types ω and $^*\omega + \omega$ are also discussed.

To the basic concepts which had already been introduced in (28) Cantor adds the idea of a type class. New also in comparison with (28), and mostly also in comparison with (16), are the sections 9–11. Section 9 discusses the order type η of the set of all rational numbers which are greater than 0 and less than 1, in their natural order of precedence. It is shown that the order type η is established by the qualities countably infinite, with no element in the set being either lowest or highest in rank, and the set's being everywhere dense (between every two elements of the set there are others).

As preparation for the next task of the paper, Cantor introduces "fundamental series of the first order." For any simply ordered transfinite set M those subsets which have the types ω (typified by the natural numbers in their usual order) and $^*\omega$ (typified by the negatives of the natural numbers in their usual order) are called *fundamental series of the first order* contained in M. With their help, he defines the concepts "dense in itself," "closed," and "perfect." If there exists in M an element m_0 which has such a position with respect to the ascending fundamental series $\{a_\nu\}$ that

a. for every ν, a_ν precedes m_0,

b. for every element m of M that precedes m_0 there exists a certain number ν_0 such that a_ν succeeds m, for $\nu \geqq \nu_0$, than m_0 is a *limiting element* of $\{a_\nu\}$ in M.

If a set M consists of limiting elements, it is *dense in itself*. If to every fundamental series in M there is a limiting element in M, then M is a *closed* set. Any set that is both dense in itself and closed is a *perfect* set. Using these concepts enables Cantor, at the end of (32I), to characterize very simply the order type θ of the linear continuum of all real numbers between 0 and 1 inclusive: An ordered set M which is perfect and in which a countable subset is densely situated (between any two elements of M there are elements of the subset) has the order type θ.

The continuation (32II), devoted to the well-ordered sets, gives in new systematic garb much that was contained in the *Grundlagen*. Only the beginning is occupied with general theory (sections 12–14). Cantor begins with a definition of well-ordered set which, apart from the wording, is identical with that which was introduced in (16V). Several elementary theorems are then stated and proved, among which are the following:

1. Every subset of a well-ordered set has a first element.
2. If a simply ordered set is such that both it and every one of its subsets have a first element, then the set itself is well-ordered.
3. Every subset of a well-ordered set is also a well-ordered set.
4. Every set which is similar to a well-ordered set is also a well-ordered set.

The introduction of the concept of segment of a well-ordered set is followed by a chain of theorems about the similarity of well-ordered sets and their segments. If f is any element of the well-ordered set F that is different from the initial element f_1, then the set of all elements of F which precede f is defined as a *segment* (*Abschnitt*) of F. That a well-ordered set is similar to no subset of any one of its segments is among the theorems proven. Perhaps in this sequence of theorems the theorem should be stressed that states that two well-ordered sets are similar if for each segment of the one there is a similar segment of the other, and vice versa. Comparability of ordinal numbers (order types of well-ordered sets) follows readily by way of disjunction from the following theorem:

If *F* and *G* are any two well-ordered sets, then either

a. *F* and *G* are similar to each other, or

b. there is a definite segment B_1 of *G* to which *F* is similar, or

c. there is a definite segment A_1 of *F* to which *G* is similar; and each of these three cases excludes the two others.

After citing several equalities and inequalities for working with ordinal numbers, Cantor goes into the addition of infinitely many ordinal numbers. He concludes this part with a proof for an earlier-mentioned statement that all simply ordered sets of given finite cardinal number have one and the same order type.

Beginning with section 15, the remainder of the work is devoted to Cantor's second number class. (It will be recalled that he dealt with this topic in the *Grundlagen*.) The second number class is the designation for the totality $\{\alpha\}$ of order types α of well-ordered sets of the cardinal number $\aleph_0$. By the first number class is understood the totality $\{\nu\}$ of finite ordinal numbers. The second number class is proven to have a least number which is designated by ω. The power of the second number class is proven to be the second greatest transfinite cardinal number, $\aleph_1$.

While Cantor's publications were here finished, his occupation with set theory was not. Notable is his continuing correspondence with Dedekind, especially concerning his endeavors with the continuum problem. The continuum problem had a deep impression on Cantor and even produced doubts in his mind whether set theory in its present form could be maintained as a scientific build-up. However, the positive influence which Cantor's work has had, and continues to have, on all of mathematics is undeniable.

CHAPTER 4

KARL W. T. WEIERSTRASS (1815–1897)

THE PARADOXES

Cantor's initial development of set theory was not built up explicitly on the basis of axioms. Analysis of his proofs, however, indicates that almost all of the theorems proved by him can be derived from three axioms: (1) the axiom of extensionality for sets, (2) the axiom of abstraction, and (3) the axiom of choice.[1] The axiom of extensionality states that two sets are identical if and only if they have the same members. The axiom of abstraction says that, given any property, there exists a set whose members are just those entities having that property. The axiom of choice states that, given a collection of mutually disjoint, nonempty sets, there exists a set which has as its elements exactly one element from each set in the given collection of sets. The paradoxes, or antinomies, which arose on the heels of Cantor's set theory were important in motivating the development of new, restricted axioms for set theory.[2]

In the late nineteenth century, paradoxes began to be discovered in the fringes of Cantor's general theory of sets. The discovery of these paradoxes was one of the most profoundly disturbing crises in the foundations of mathematics and has still not been resolved to the satisfaction of all concerned.

A study of mathematics from Greek antiquity to the present reveals that the foundations of mathematics has undergone two other crises which have been resolved. Neither of them was easily nor quickly settled, however.[3]

The first crisis arose in the fifth century B.C., precipitated by the unexpected discovery that not all geometrical magnitudes of the same kind are commensurable with one another. The discovery that like magnitudes may be incommensurable proved to be highly devastating, since the Pythagorean development of magnitudes was built upon the firm intuitive belief that all like magnitudes are commensurable. The entire Pythagorean theory of proportion with all its consequences had to be scrapped as unsound. This crisis was resolved in about 270 B.C. By Eudoxus's revised theory of magnitude and proportion. Eudoxus's treatment of incommensurables coincides essentially with the modern exposition of irrational numbers that was given by Richard Dedekind in 1872. Many of Eudoxus's theorems are contained in Euclid's *Elements*; Eudoxus's treatment of incommensurables may be found in the fifth book of the *Elements*.

The second crisis followed the discovery of calculus by Isaac Newton and Gottfried Wilhelm Leibniz in the late seventeenth century. It was easy to be carried away by the power and applicability of the new tool: the successors of Newton and Leibniz failed to consider sufficiently the solidity of the base upon which the subject was founded. With the passage of time paradoxes arose in increasing numbers, revealing a serious crisis in the foundations of mathematics. That the edifice of analysis was being built upon sand became more and more apparent. Finally, in the early nineteenth century, Augustin-Louis Cauchy took the first steps toward resolving the crisis by replacing the hazy method of infinitesimals with the precise method of limits. With the subsequent so-called arithmetization of analysis by Karl Weierstrass and his followers, it was felt that the second crisis in the foundations had been overcome.

The third crisis in the foundations of mathematics materialized with shocking suddenness in the late nineteenth century. The discovery of paradoxes in set theory naturally cast the validity of the whole foundational structure of mathematics in doubt, since so much of mathematics is permeated with set concepts to the extent that it can be said to have set theory as a basic foundation.

The first of the modern paradoxes was published by the Italian mathematician Cesare Burali-Forti in the year 1897.[4] His formulation of the paradox was not altogether satisfactory, as he had confused Cantor's well-ordered sets with his own "perfectly order sets." Burali-Forti defined a set u to be perfectly order if: u has a first element; every element of u (provided it is not the last) has an immediate successor; where x is any element of u, either x has no immediate predecessor, or there is an element *y* of *u* such that *y* precedes *x*, *y* has no immediate predecessor, and only a finite number of elements of *u* lie between *y* and *x*. Although Burali-Forti's definition of perfectly ordered set was not the equivalent of Cantor's definition of well-ordered set, the paradox could still be established on the basis of Cantor's definition of well-ordered set. Burali-Forti himself soon realized his error and published a note admitting the mistake and pointing out that the contradiction still holds with the correct definition.

In the theory of transfinite ordinal numbers, it is shown that every well-ordered set has a unique ordinal number; every segment of ordinals has an ordinal number which is greater than any other ordinal in the segment; and the set of all ordinal numbers in natural order is well ordered. The Burali-Forti paradox may be stated as follows: consider the ordinal number of the set of all ordinals. Since this ordinal number must itself be an element of that set, it must be the order type of the segment of all ordinals less than itself. Therefore the set of all ordinal numbers is similar to one of its own segments, which is impossible.[5]

Fraenkel says that Cantor himself had come across the paradox in 1895 at the latest—two years before Burali-Forti published his result—and had communicated it to Hilbert in 1896. Although Copi points out the uncertainty surrounding this claim of priority, he yet accepts it as being probably true. Assuming that Cantor did discover the paradox first, there is the question of why he nevertheless attacked Burali-Forti's article. Supposition as to the reason for Cantor's attack on the article if he was already familiar with the contradiction is at best guesswork.[6] What is certain, however, is that whoever first discovered the paradox, Burali-Forti does have the distinction of having been the first to publish a modern instance of the logical paradoxes.

When the Burali-Forti paradox first appeared, it did not create the sensation that might be expected. This fact is probably largely due to Burali-Forti's mistaken definition of well-ordered set in the original presentation, which would naturally serve to render the whole matter suspect. Certainly contradictions can be easily obtained if mistakes are allowed in the derivations.

Copi points out several other reasons why Burali-Forti's derivation of the contradiction, even after the required corrections are made, did not bring forth more response. Burali-Forti's formulation of the paradox was not in the most convincing form possible. Also the result was not presented as a contradiction demanding resolution, but, rather, was presented as part of an attempted proof that ordinal numbers are not necessarily comparable. Cantor had already proved the comparability of the ordinals, however, and his proof was convincing.

The division alluded to in Chapter 2 between Cantor and Kronecker may be still another reason why Burali-Forti's contradiction did not get more attention. It appears that those mathematicians who at that time were at all concerned with matters pertaining to the foundations of mathematics were pretty much either on the side of Cantor and his followers—accepting and working with the multiple infinities involved in the theory of point sets and transfinite cardinals and ordinals—or on the side of Kronecker and his followers—rejecting the infinities of Cantor and the nonconstructive methods freely used by the Cantorians. The Burali-Forti paradox appeared as a minor skirmish in the larger war of what constitutes admissible methods in mathematics. It is not overly surprising, in the mathematical environment of that time, if both sides may have experienced some lapse of objectivity.

Another reason which Copi mentions for Burali-Forti's result receiving no more attention than it did was the tendency toward narrow specialization among mathematicians during the nineteenth century. There was a tremendous proliferation of mathematics during the century, and no one person could be expected to master the whole of the field. To mathematicians tied down to their own specialties the Burali-Forti article, written in the logical symbolism of G. Peano, was quite unlikely to be very much noticed. The language itself was new and strange, and not intelligible to a large number of mathematicians.

In view of the circumstances just described, it seems likely that Burali-Forti's contradiction would have received little serious attention if it had been the only one to appear. After a short while, however, other paradoxes began to appear in abundance. The importance of these paradoxes in motivating the development of new, restricted axioms for set theory will be dealt with in the next chapter.

As early as 1899 Cantor discovered that a paradox arises by considering the set of all powers (cardinal numbers) or the set of all that is conceivable. The result was not published until 1932 (posthumously).

Cantor's paradox, very similar to the Burali-Forti paradox, may be stated as follows: Consider the cardinal number of the set of all sets. It is clear that this is the greatest possible cardinal number. But by a standard theorem of intuitive set theory, the set of all subsets of a set has a greater cardinal number than the set itself. Therefore, the cardinal number of the set of all subsets of the set of all sets is greater than the greatest possible cardinal number, an obvious contradiction.

Whereas Burali-Forti's and Cantor's paradoxes involve results of set theory, in 1902 Bertrand Russell discovered a paradox based on just the concept of set itself. Russell's antinomy was also discovered independently by Ernst Zermelo.[7] The paradox comes about by considering the set of all sets which have the property of not being members of themselves. For example, the set of all men is not a man, whereas the set of all sets is a set.

Suppose the set of all sets which are members of themselves is denoted by M, and the set of all sets which are not members of themselves is denoted by N. Note that the axiom of abstraction would guarantee such a set. Does N belong to M? If N is a member of itself, then N is a member of M and not of N, and

N is not a member of itself. Contrariwise, if *N* is not a member of itself, then *N* is a member of *N* and not of *M*, and *N* is a member of itself. Since either case leads to a contradiction, a paradox is apparent.

Russell also showed how to recast his paradox in purely logical instead of set-theoretic terminology. A property is "predicable" if it applies to itself, "impredicable" if it does not apply to itself. For example, the property "abstract" applies to itself and hence is predicable. On the other hand the property "concrete" is also abstract and not concrete. The paradoxical consequence follows that "impredicable" is impredicable if and only if "impredicable" is not impredicable.

Russell communicated his paradox to Gottlob Frege just after the latter had completed the last volume of his great two-volume treatise on the foundations of arithmetic. Frege's work, which stemmed from the need of a sounder basis for mathematics, used the concepts of set and set of all sets. His consternation can be judged from his acknowledgment of Russell's communication at the end of his treatise:

> A scientist can hardly meet with anything more undesirable than to have the foundation give way just as the work is finished. In this position I was put by a letter from Mr. Bertrand Russell as the work was nearly through the press.[8]

There are many popularized forms of the Russell paradox appearing in the literature. One of the best known of these forms was given by Russell himself in 1919. A certain village barber shaves everyone in the village who does not shave himself. The question is, does the barber shave himself? If he shaves himself, then he should not according to the given principle; if he does not shave himself, then he should according to the principle.

Another well-known popularization of the Russell paradox is the catalogue paradox. Suppose the Librarian of Congress compiles, for inclusion in the Library of Congress, a bibliographic catalogue of all those bibliographic catalogues in the Library of Congress which do not list themselves. Does the catalogue list itself? This popularization is due to Ferdinand Gonseth in 1932.[9]

The two popularizations can be dispensed with. In the first case the village barber just could not do what the principle says, and in the second case the library could not make a catalogue satisfying the stated requirements. But these explanations do not apply to the Russell paradox; in terms of logic as it was known in the nineteenth century, the situation is inexplicable.

Except for the popularizations given of Russell's paradox, the paradoxes considered so far are logical or mathematical paradoxes. The popularizations would be referred to as semantical (sometimes "epistemological") paradoxes. The English logician F. P. Ramsey proposed this distinction in 1925. Curry holds that Ramsey was not quite correct in his view that mathematics does not have to take account of the semantical paradoxes, in that some of the most significant results of modern logical have come from a deeper study of them.[10] Roughly speaking, logical paradoxes arise from purely mathematical constructions, whereas semantical paradoxes arise from direct consideration of the language used to talk about mathematics or logic.

Quite a number of other paradoxes arose in the early years following Burali-Forti's result. Some of these modern paradoxes, falling more or less within the context of set theory, can be seen to be related to several ancient paradoxes of logic. For example, the Cretan philosopher Epimenides (sixth century B.C.) is supposed to have made the statement, "Cretans are always liars."[11] This statement, if true, makes the speaker a liar for telling the truth. The Epimenides paradox, known also as "the liar," appears in stark form in the statement attributed to Eubulides (fourth century B.C.): "This statement I am now making is false." The quoted statement can neither be true nor false without entailing a contradiction.[12]

Another of the semantical antinomies, significant since it is a sort of caricature of Cantor's diagonal method, was the Richard paradox of 1905, due to Jules Richard.[13] An essentially instructive and ingenious simplification of Richard's paradox is due to G. G. Berry in 1906. Consider the expression, "the least natural number not nameable in fewer than twenty-two syllables." This expression names in twenty-one syllables a natural number which by definition cannot be named in fewer than twenty-two syllables. Various modifications exist.

In 1908 K. Grelling and L. Nelson called attention to a paradox which they regarded as only a variant of Russell's paradox. The Grelling paradox can be stated quite simply. Among English adjectives there are some that have the property that they denote, such as "short," "polysyllabic," and "English." Let adjectives which have this property be called autological and all others be called heterological. In the latter class would be such adjectives as "long," "monosyllabic," "blue," and "hot." Paradoxically, the adjective "heterological" is heterological if and only if it is autological.

One of the more interesting paradoxes, discovered in 1924 by two distinguished Polish mathematicians, S. Banach and A. Tarski, depends on the axiom of choice for its derivation. The choice axiom, inadvertently used by Cantor and others, appears to have been explicitly alluded to first in a paper of G. Peano in 1890 concerning an existence proof for a system of ordinary differential equations. He writes:

> However since one cannot apply infinitely many times an arbitrary law by which one assigns to a class an individual of that class, we have formed here a definite law by which, under suitable assumptions, one assigns to every class of a certain system an individual of that class.[14]

In 1902 Beppo Levi alluded to such a principle while dealing with the statement that the union of a disjoint set t of non-empty sets has a cardinal number greater than or equal to the cardinal number of t. The axiom of choice was introduced formally by Ernst Zermelo in 1904 for the purpose of proving the famous and still controversial well-ordering theorem. This theorem will be given in the next chapter.

It is conventional to assign to any configuration a number 0, 1, 2, or 3 to denote its dimensionality. The problem of deciding whether an object has 0, 1, 2, or 3 dimensions appears to be a simple and obvious one that can be solved intuitively without mathematical analysis. A remarkable paradox was uncovered, however, which shows that the intuitive ideas about dimensionality, as well as area, are lacking in precision and are often wholly misleading.

A number, called a *measure*, can be uniquely assigned to every figure in the plane so that the following three conditions will be satisfied: (a) the word "congruent" being used in the elementary geometry sense, two congruent figures are to have the same measure; (b) if a figure is divided into two parts, the sum of the measures assigned to each of the two parts is to be exactly equal to the measure assigned to the orignal figure; (c) as a model for determining the method of assigning a measure to each figure in the plane, the measure 1 is assigned to the square whose side has a length of one unit. The measure can be assigned analytically (by means of point sets) without using the traditional concepts of classical geometry.

Now this same problem of assigning a measure to surfaces was found to be unsolvable and even led to paradoxes. The same methods which had been quite fruitful in investigations in the plane, when applied to the surface of a sphere proved inadequate to determine a unique measure.

It seems sensible to set up the following conditions in attempting to assign a measure to a surface: (a) the same measure shall be assigned to congruent surfaces; (b) the sum of the measures assigned to each of two component parts of a surface shall be equal to the measure assigned to the original surface; (c) if S denotes the entire surface of a sphere of radius r, the measure assigned to S shall be $4\pi r^2$.

The German mathematician Felix Hausdorff showed that a measure cannot be uniquely assigned analytically to the portions of the surface of a sphere so that the listed conditions will be satisfied. He showed that the surface of a sphere can be divided into three separate and distinct parts, *A*, *B*, and *C*, so that *A* is congruent to *B* and *B* is congruent to *C*, and not only is *A* congruent to *C*, but also *A* is congruent to $B + C$. Thus, if a measure is assigned to *A*, the same measure must be assigned to *B* and to *C* and also to $B + C$, which implies that the measures assigned to *A*, *B*, and *C* are all equal to 0. But by the third condition listed above, the sum of the measures assigned to the parts of the surface of a sphere of radius *r* must be equal to $4\pi r^2$.

Banach and Tarski have extended the implications of Hausdorff's paradoxical theorem to three-dimensional space. Their conclusions are generally conceded to be rigorous and unimpeachable if the choice axiom is granted.

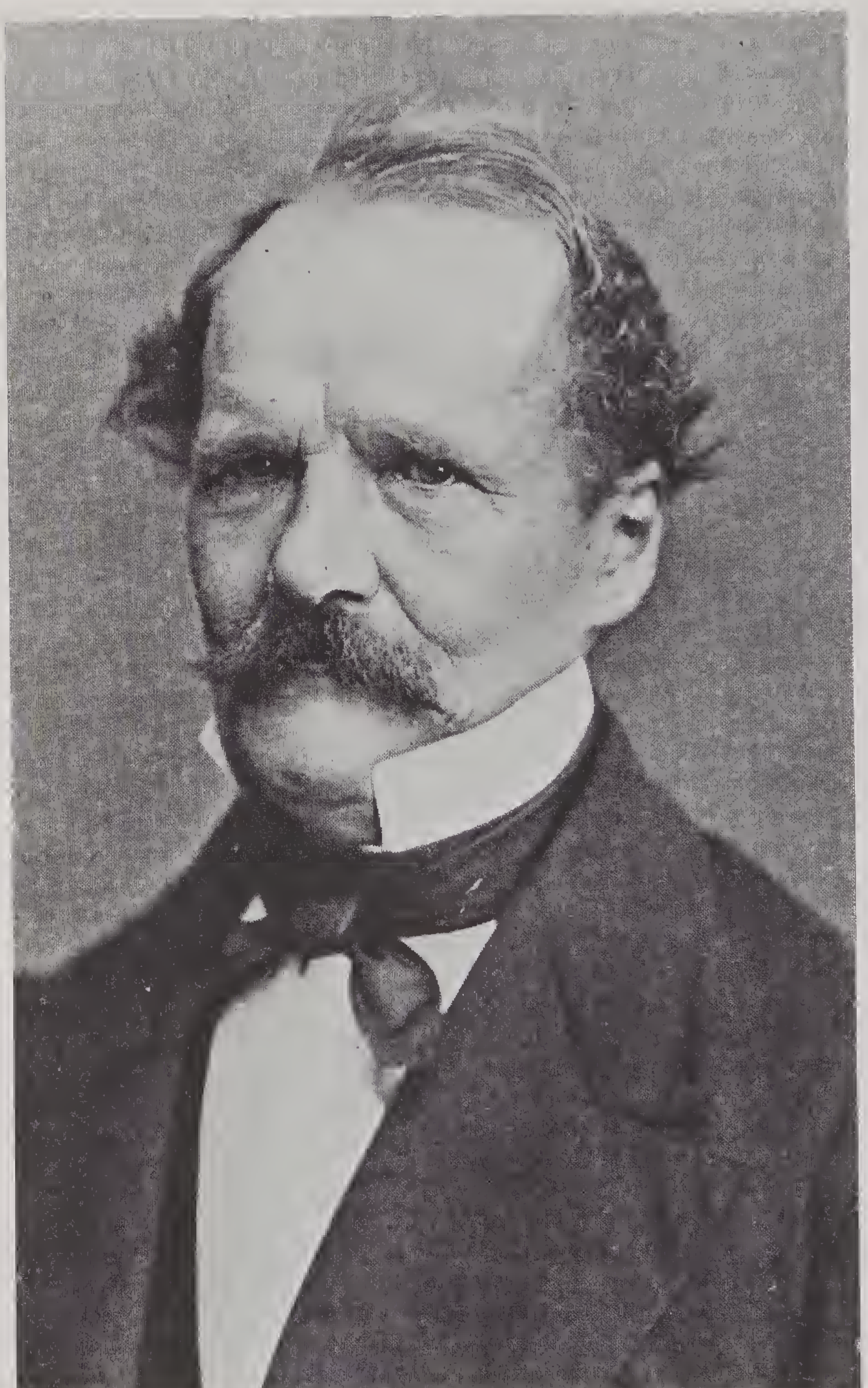
Imagine two spheres in three-dimensional space, one very large (like the sun, for example) and the other very small (like a pea). Denote the large sphere by *S* and the small sphere by *P*. The entire solid spheres of both *S* and *P* are being referred to, not just the surfaces of the two spherical objects. The theorem of Banach and Tarski holds that the following operations can theoretically be carried out.

Divide *S* into a great many small parts. Each part is to be separate and distinct and the totality of the parts is to be finite in number. Designate these parts by $s_1, s_2, s_3, \ldots, s_n$. Together these small parts will make up the entire sphere *S*. Similarly *P* may be divided into an equal number of mutually exclusive parts, $p_1, p_2, p_3, \ldots, p_n$, which together make up *P*. The proposition goes on to say that if *S* and *P* have been cut up in a suitable manner, so that the little portion s_1 of *S* is congruent to the little portion p_1 of *P*, s_2 congruent to p_2, s_3 congruent to p_3, up to s_n congruent to p_n, this process will exhaust not only all the little portions of *P*, but all the tiny portions of *S* as well.[15]

In other words, *S* and *P* may both be divided into a finite number of disjoint parts so that every single part of one is congruent to a unique part of the other, and so that after each small portion of *P* has been matched with a small portion of *S*, no portion of *S* will be left over. The paradox here lies not in the simple one-to-one correspondence between the elements of the two sets, but in the fact that each element is matched with one which is completely congruent to it. To state the result more generally, Banach and Tarski showed that in a Euclidean space of dimension greater than two, two arbitrary bounded sets with interior points are equivalent by finite decomposition, that is, the two sets can be decomposed into the same finite number of disjoint parts with a one-to-one correspondence of congruence between their respective parts.[16] The Hausdorff result of 1914 and the Banach-Tarski paradox of 1924 induced many mathematicians to reject the axiom of choice, by means of which the statements were proved.

The paradoxes cited, and others, have had a profound influence on the foundations of mathematics. The influence exerted by the paradoxes on the foundations of set theory will be investigated in the next chapter.

CHAPTER 5



ERNST E. KUMMER (1810–1893)

SOME EARLY TWENTIETH CENTURY DEVELOPMENTS

In 1904 Ernst Zermelo published his monumental proof that there exists a well ordering for any set whatsoever, a theorem which had been conjectured by Cantor.[1] The axiom of choice, on which the proof of the well-ordering theorem hinges, was explicitly stated by Zermelo at the suggestion of Erhard Schmidt. Émile Borel pointed out that not only does Zermelo's proof depend on the axiom of choice, but indeed the theorem is actually equivalent to the axiom.[2]

The paper by Zermelo touched off a controversy about the axiom of choice which still continues today. Cantor, and others, had used the axiom implicitly without being aware that a principle was being used that might be suspect. Use of the axiom is clear in proofs which say to select (without giving a rule for the selection) an element x_1 from an infinite set S, then x_2 from $S—X_1$, and so on. It is easy to see how the axiom can be used implicitly. Today, most careful writers state when they are making use of the axiom of choice.

A popular example—the original version of which was given by Bertrand Russell—serves to clarify the kind of circumstances wherein the axiom of choice is needed. In an infinite collection of pairs of shoes, the axiom of choice is not needed to establish the existence of a set containing exactly one element from each of the pairs, for a rule can be given to select the left shoes, for example. But in the case of an infinite collection of pairs of socks all alike as to size, color, and so on, no such rule is available; appeal must be made to the axiom of choice if the assertion is made that there exists a set containing exactly one sock from each pair.

The proof of Zermelo's theorem is a proof of pure existence and gives no way for effectively carrying out the well ordering. To appreciate the result, one need only consider the set of real numbers between 0 and 1. This set is not well ordered using the relation "is less than" in the ordinary sense. No one has been able to effect a well ordering of the set of real numbers or any other set having cardinal number greater than the reals.

Attitudes toward the axiom of choice run the gamut from total rejection, through various degrees of skepticism, to complete acceptance. In topology, the axiom is used unhesitatingly: apparently little of this subject can be derived without its use. Although a large portion of analysis can be established without use of the choice axiom, measure theory and those parts of modern analysis which rely extensively on topological ideas also make use of the axiom. However, algebraists are inclined to proceed as far as possible without making use of the axiom. Proofs avoiding use of the axiom of choice are zealously sought for a number of famous theorems, particularly in analysis and in set theory.

Since the axiomatization of set theory (to be discussed later), some particularly significant work has been done concerning the axiom of choice and also Cantor's continuum hypothesis. The developments concerning these two well-known statements are explained by analogy with non-Euclidean geometry in an article by Cohen and Hersh.[3]

Gödel proved in about 1940 that the axiom of choice is relatively consistent with a well-known postulate set of set theory; that is, if the other axioms of set theory are consistent, the addition of this axiom will not lead to a contradiction. Gödel also achieved some progress on the continuum hypothesis, with which Cantor's efforts of the 1880s and Hilbert's of the 1920s had been unsuccessful. He showed that the (generalized) continuum hypothesis is relatively consistent with the other axioms of set theory. It should be noted that Gödel's result does not constitute a proof of the continuum hypothesis, only a proof that it cannot be disproved.

In 1963 Paul Cohen made a major breakthrough by constructing an example in which the axiom of choice does not hold but the other axioms of set theory do. Thus, to Gödel's proof that the axiom of choice cannot be disproved is added the result that it cannot be proved. Cohen also added the fact that the continuum hypothesis cannot be proved to Gödel's discovery that it cannot be disproved. Thus two of the vexing questions of the foundations have been solved within the realm of axiomatic set theory.

In the twentieth century a considerable amount of literature concerning the paradoxes has appeared, and numerous attempts at a solution have been offered. Despite the vast literature devoted to the paradoxes and the variety of explanations offered for them, however, there is at present no one explanation which is universally accepted.

If the paradoxes discussed in the last chapter are examined carefully, it will be seen that most of them involve a set S and a member m of S whose definition depends upon S. Such a definition is called *impredicative*. Russell's paradox is a good example of an impredicative procedure, and his popularized barber paradox points this procedure out in a very lucid way. If S denotes the set of all members of the village and m denotes the barber, then m is defined impredicatively as "that member of S who shaves all those members and only those members of S who do not shave themselves." The definition of the barber involves the members of the village and the barber himself is a member of the village.

Henri Poincaré, whose views will be discussed later in connection with the intuitionist school of mathematics, considered the cause of the paradoxes to lie in impredicative definitions. Russell enunciated the same explanation in his vicious circle principle: no set can contain members definable only in terms of this set, or members involving or presupposing this set. Outlawing impredicative definitions would rid set theory of the paradoxes dependent on such definition. Parts of mathematics which mathematicians would be extremely reluctant to discard, however, also contain impredicative definitions. For example, the least upper bound of a given set of real numbers is defined as the smallest member of the set of all upper bounds of the given set.

Hermann Weyl, in his 1918 book *Das Kontinuum*, undertook to find out how much of analysis could be constructed without the use of impredicative definitions. He was able to obtain a fair part of analysis, but not the important theorem that every nonempty set of real numbers having an upper bound has a least upper bound.

Set theory can be rid of the known paradoxes by constructing the theory on a sufficiently restrictive axiomatic basis. Such a procedure suffers the natural criticism that it merely avoids the paradoxes instead of explaining them and that other kinds of paradoxes may still occur even when the known ones have been eliminated. Nevertheless, the development of axiomatic theories must be viewed as an important contribution to the further study of the problem.

The historically first axiomatization of set theory was given by Zermelo in 1908. The troublesome axiom of abstraction was modified so as to avoid the paradoxes arising therefrom in the *axiom schema of separation*. Zermelo did not go so far as to reject impredicative definitions entirely. He calls a statement *definite* if it can be decided in a non-arbitrary way whether or not any object satisfies the statement. Slightly paraphrased, Zermelo's formulation of the axiom schema is: if a statement $\varphi(x)$ is definite for all elements of a set M, then there is always a subset M_φ of M which contains exactly those elements x of M for which $\varphi(x)$ is true. The axiom schema permits the separating off of the elements of a given set that satisfy some property and forming a set consisting of just these elements. For example, if the set of animals is known to exist, the axiom schema of separation can be used to assert the existence of the set of animals that has the property of being human.[4]

One alternative closely connected to Zermelo-Fraenkel set theory is von Neumann-Bernays-Gödel set theory. J. von Neumann's axiom system was simplified by R. M. Robinson.

Considerable improvements and additions, together with a fundamental simplification, are due to P. Bernays. Gödel modified Bernay's system chiefly for the purpose of proving the consistency of the generalized continuum hypothesis.[5]

In the von Neumann-Bernays-Gödel theory, no axiom schema of construction like that of separation is required. Instead a finite number of specific set and class constructions suffices. In this theory there is a technical distinction between sets and classes; every set is a class, but not conversely. Those classes that are not sets are called *proper classes*, and their distinguishing characteristic is that they are not members of any other class. The class of all sets exists but is a proper class. Thus, the Cantor paradox cannot be constructed. Similarly, the class of all ordinal numbers exists but is a proper class, and the Burali-Forti paradox cannot be constructed. Similar remarks apply to the Russell paradox.[6]

There are usually considered to be three principal present-day schools of mathematics: the intuitionist, formalist, and logistic schools.[7] Naturally, each school must somehow come to grips with the paradoxes of set theory. The chief interest here will be in how the schools propose to deal with the paradoxes.[8]

This division into schools is not intended to convey the idea that all mathematicians are members of one and only one of the three main schools or one of the lesser schools. Rather, it is an attempt to classify thought tendencies. It is highly likely that a given mathematician would not accept completely all of the philosophy of any given school, even though he might be considered a member of a particular school.

The intuitionist school (as a school) originated about 1908 with the Dutch mathematician L. E. J. Brouwer. Some of the intuitionist ideas had been voiced earlier by Kronecker, in the 1880s, and Poincaré, from 1902 to 1906, among others. This school has exerted considerable influence on the thinking concerning the foundations of mathematics and contains some eminent present-day adherents.[9]

The intuitionist thesis is that mathematics is to be built solely by finite constructive methods on the intuitively given sequence of natural numbers. Intuitionism is a self-generating philosophy, not relying on other philosophies or logic. According to the intuitionist view, at the very base of mathematics lies a primitive intuition, allied to our temporal sense of before and after, that allows the conception of a single object, then one more, then one more, and so on endlessly. In this way unending sequences are obtained, the best known of which is the sequence of natural numbers. From this intuitive base of the sequence of natural numbers, any other mathematical object must be built in a purely constructive manner, employing a finite number of steps or operations.

The intuitionists' insistence on constructive methods leads to a conception of mathematical existence not shared by a large number of mathematicians. For the intuitionists, an entity whose existence is to be proved must be shown to be constructible in a finite number of steps. Showing that the assumption of the entity's non-existence leads to a contradiction is not enough.

The intuitionists' insistence upon constructive procedures does away with the paradoxes of set theory. For the intuitionists a set must be considered as a law by means of which the elements of the set can be constructed in a step by step fashion. Thus a set cannot be thought of as a ready-made collection and such a concept as the "set of all sets" does not arise.

The intuitionists hold that the law of excluded middle (either a statement is true or its denial is true) should not be employed when dealing with infinite sets. A given proposition can be said to be true only when a proof of it has been constructed in a finite number of steps; false only when a proof of this situation has been constructed in a finite number of steps. The proposition is neither true nor false until one or the other of these proofs is constructed, and the law of excluded middle is inapplicable. This rejection of the law of excluded middle is one of the most spectacular features in

Brouwer's intuitionism. An example due to Heyting will make clear why the intuitionists deny the law of excluded middle for constructions involving an infinite totality.

> Consider the following two definitions: (i) k is the greatest prime such that $k - 1$ is also a prime, or $k = 1$ if such a number does not exist; (ii) m is the greatest prime such that $m - 2$ is also a prime, or $m = 1$ if such a number does not exist.
>
> Here (i) clearly defines a unique number, $k = 3$. On the other hand, there is no method at present for calculating m, since it is not known whether the sequence of pairs of twin primes is finite or not. Intuitionists therefore reject (ii) as a definition of an integer; an integer is considered to be well-defined only if a method for calculating it is given.[10]

It will be recalled that Kronecker, one of the early forerunners of intuitionism, was a severe critic of Cantor's ideas. The appearance of contradictions in set theory revived some of his objections. Poincaré's writings, in particular, reflect Kronecker's outlook. Of Poincaré's views the following are of particular significance here: every mathematical concept should be capable of explicit (finite) definition; existence of a mathematical entity should be verifiable by a finite procedure; impredicative definitions should not be employed; mathematics cannot be based on logic. Poincaré felt that most of the concepts and conclusions of Cantor's theory of sets should be excluded from mathematics. He rejected completely Zermelo's theorem, basing his rejection on the lack of definition of representative elements involved in the use of the axiom of choice.[11]

While Brouwer and some other eminent mathematicians have been reluctant to accept results proved on the basis of the axiom of choice, there is a considerable body of mathematics that up to the present day cannot be derived without its use. Although most mathematicians would be hesitant to reject this as a tool, from the intuitionistic viewpoint, theorems have not been proved when their proofs depend on the choice axiom.

It is generally conceded that the intuitionist methods do not lead to contradictions. Up to now, however, much of mathematics that most mathematicians feel is valid has not been constructed with the intuitionistic approach, and some of what is obtained is so changed as to be almost unrecognizable. Intuitionist mathematics has turned out to be less powerful and in many ways harder to develop than classical mathematics.

The thesis of the formalist school of mathematics holds that mathematics is concerned with formal symbolic systems. Mathematics is regarded by the formalists as a collection of abstract developments in which the terms are mere symbols and the statements are formulas involving these symbols. The ultimate base of mathematics is not considered to lie in logic but only in a collection of prelogical marks or symbols and in a set of operations using these marks. From this point of view, mathematics is devoid of concrete content and contains only ideal symbolic elements. Naturally, the establishment of the consistency of the various branches of mathematics becomes an important and necessary part of the formalist program. In the formalist thesis the axiomatic development of mathematics is pushed to its extreme.

The formalist school was founded by David Hilbert after completing his postulational study of geometry. Later it was developed to meet the crisis caused by the paradoxes of set theory and the challenge to classical mathematics caused by intuitionistic criticism. Hilbert talked in formalistic terms as early as 1904, but it was not until after 1920 that he and his collaborators, Paul Bernays, W. Ackerman, J. von Neumann, and others, seriously started work on what is now known as the formalist program.

Freedom from contradiction is guaranteed only by consistency proofs. Hilbert hoped to develop a direct test for consistency in his "proof theory" instead of the older consistency proofs based upon interpretations and models which merely shift the question of consistency from one domain of mathematics to another. It was thought that perhaps by analyzing mathematical concepts and processes, both logical and otherwise, and representing them by an appropriate symbolism, as in a symbolic logic, one might be able to demonstrate that the formula for a contradiction can never be obtained from the fundamental formulas and the rules laid down for manipulating the symbols. Thus consistency would be assured.

The development of the above-mentioned program was attempted by Hilbert and Bernays in their two volume *Grundlagen der Mathematik* (1934, 1939). Unforeseen difficulties arose, however, and it was not possible to complete the proof theory and exhibit proofs of consistency for all classical mathematics as Hilbert would have liked. In fact, Gödel showed, even before the publication of *Grundlagen*, that it is impossible for a sufficiently rich formalized deductive system, such as Hilbert's system for all classical mathematics, to prove consistency of the system by methods belonging to the system.

Insofar as the axiom of choice is concerned, the formalist philosophy would perhaps allow the formulation of systems in which the axiom holds, the formulation of systems in which the axiom is not assumed, and systems in which the axiom fails. As to which of these is the most useful system, the formalist would probably leave that question open. The important point would be that all of the systems are mathematics and should be considered as such.

The thesis of the logistic (also called "logicistic") school is that mathematics is a branch of logic. In this view logic becomes the progenitor of mathematics rather than just being a tool of mathematics. Mathematical concepts are to be formulated in terms of logical concepts, and all theorems of mathematics are to be developed as theorems of logic.

The notion of logic as a science containing the principles and ideas underlying all other sciences dates back quite a few years in history, at least as early as 1666 with Leibniz. Such names as Dedekind, Frege, and Peano might be considered the forerunners of the logistic school. Dedekind (1888) and Frege (1884–1903), engaged in the actual reduction of mathematical concepts to logical concepts. Peano (1889–1908) undertook the statement of mathematical theorems by means of logical symbolism. The logistic school received its definitive expression in the monumental *Principia Mathematica* of Alfred North Whitehead and Bertrand Russell (three volumes, 1910–1913). This complex work purports to be a detailed reduction of the whole of mathematics to logic. Among others, Ludwig Wittgenstein, L. Chwistek, F. P. Ramsey, C. H. Langford, R. Carnap, and W. V. Quine are some of the important names connected with supplying subsequent modifications and refinements of the program.

Corresponding to the "undefined terms" and "postulates" of a formal abstract development, the *Principia Mathematica* starts with "primitive ideas" and "primitive propositions." These primitive ideas and propositions are to be regarded as, or at least accepted as, plausible descriptions and hypotheses concerning the real world. Thus, a concrete rather than an abstract point of view prevails, and consequently no attempt is made to prove the consistency of the primitive propositions. *Principia Mathematica* aims to develop mathematical concepts and theorems from these primitive ideas and propositions, starting with a calculus of propositions, proceeding up through the theory of classes and relations to the establishment of the natural number system, and thence to all mathematics derivable from the natural number system.

Principia Mathematica employs a "theory of types" to avoid the paradoxes of set theory. Somewhat oversimply described, a theory of types sets up a hierarchy of levels of elements. The primary elements constitute those of type 0; classes of elements of type 0 constitute those of type 1; classes of elements of type 1 constitute those of type 2; and so on. In applying the theory of types, the rule is followed that all the elements must be of the same type. Adherence to this rule precludes impredicative definitions and thus avoids the paradoxes which arise from such definitions.

In order to obtain the impredicative definitions needed to establish analysis, a non-primitive and arbitrary axiom had to be introduced. The axiom drew forth severe criticism; much of the subsequent refinement of the logistic program lies in attempts to devise some method of avoiding this axiom.

Probably most mathematicians who do not concern themselves explicitly with foundational questions feel that essentially all the notions of number and set have a real existence apart from our knowledge of them, and that classical mathematics, though it needs a more secure foundation, is not actually unsound. This is the position also of Frege and Russell, the pioneers in mathematical logic, and is defended today by some of the ablest logicians.[12]

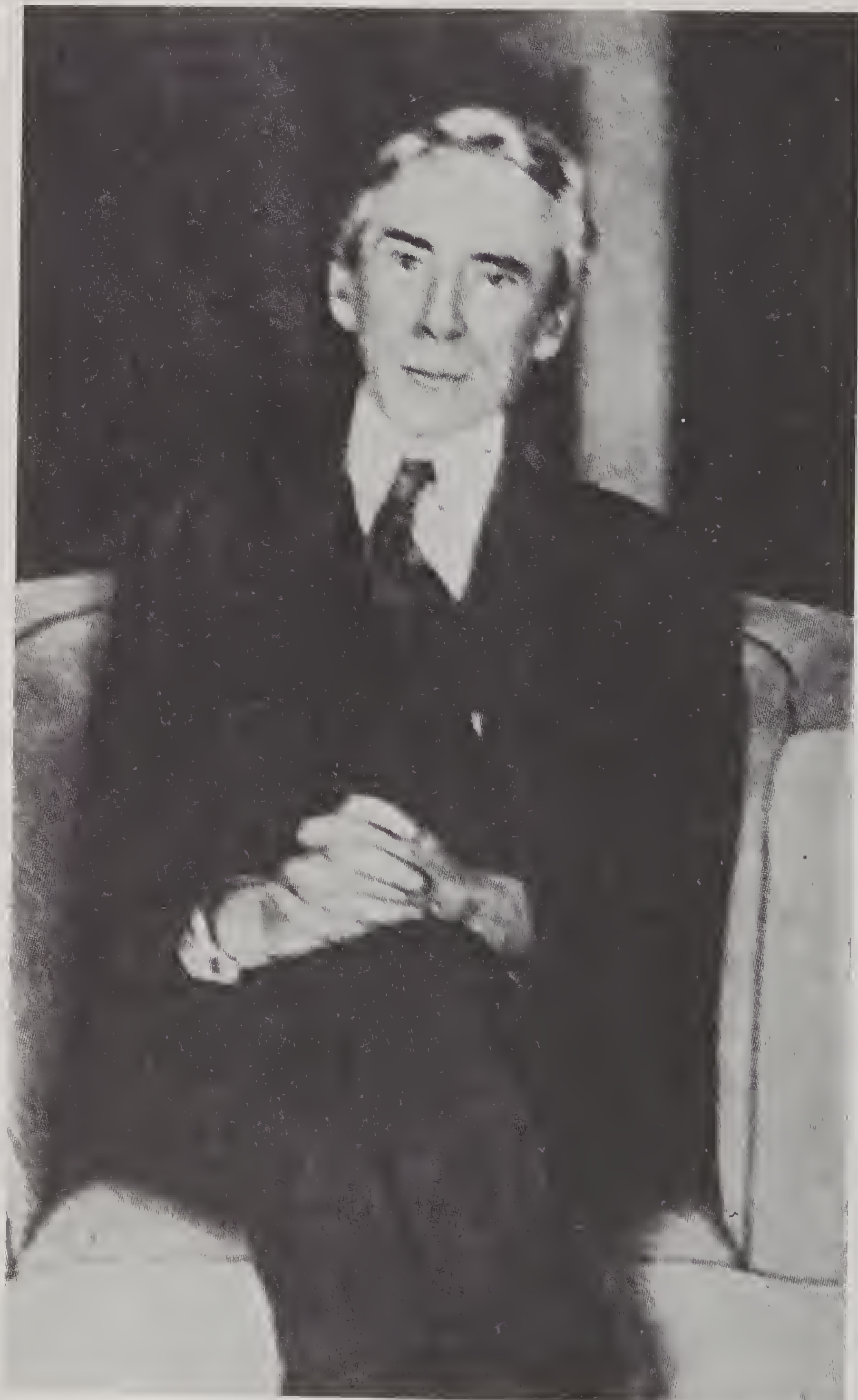
While there is no one explanation that is universally accepted as resolving the paradoxes of set theory, remarkable strides toward their resolution have been made since they first began to appear. Considering the amount of time required to put the first two crises of mathematics to rest, it must be considered remarkable that so much progress has been made on the third crisis in just a little more than half a century.

By way of summary, it has been noted that Cantor's conjecture of the well-ordering theorem was followed not only by Zermelo's proof of the theorem in 1904, but also by a precise statement of a principle often used implicitly, the axiom of choice, which was used in the proof. The axiom of choice has itself been a source of careful scrutiny, and also of controversy. The work of Gödel and Cohen on both the axiom of choice and the continuum hypothesis is particularly noteworthy. While Cantor's efforts on the continuum hypothesis had been unsuccessful, Gödel and Cohen have solved the question in the realm of axiomatic set theory.

Finally, it has been seen that the paradoxes which arose from Cantor's work have been an important stimulus to the study of questions in the foundations of mathematics. The various schools of thought have contributed worthwhile dialogue and diverse viewpoints on possible solutions to the paradoxes and on the question of what constitutes admissible methods in mathematics. The axiomatic theories which arose primarily because of the paradoxes are now important disciplines of study in their own right.

The numerous developments following Cantor's work and related to it serve to show what a stimulus his research was for further research. The continuing influence of his work is its own best monument to Georg Cantor.

CHAPTER 6



BERTRAND RUSSELL (1872–1970)

SET THEORY IN THE SCHOOL MATHEMATICS PROGRAM

During the last two decades, mathematics education at the lower levels has undergone changes of revolutionary proportions. The ten-year period 1955–1965 was an especially fruitful time for influential work by a number of groups interested in curricular reform. The groups were generally willing to sacrifice somewhat the overemphasis on rote and manipulations in an attempt to bring about a gain in reasoning ability.

Most programs developed by the groups working in the 1950s and 1960s have in common the use of set terminology and concepts as a means of clarifying and unifying the subject matter. Set theory per se has not been a subject of study at the lower levels, but concepts from the theory have generally been used freely when they serve a useful purpose. Certainly, situations should not be contrived just in order to use sets for the sake of being "modern." Some of the early integrated college algebra and trigonometry texts were guilty of having a first chapter devoted to sets and then never using the material in the rest of the book.

Over the years, a number of groups have worked on the school mathematics curriculum and have made recommendations concerning it. One of the early groups making specific recommendations as to what mathematics should be in the school program was the 1893 Subcommittee on Mathematics of the Committee of Ten on Secondary School Subjects. The Committee on College Entrance Requirements in 1899 recommended specific mathematics courses for grades 7–12. The reports of these two committees seem to carry the first evidence of any concentrated thought devoted to the consideration of decompartmentalizing the organization of the mathematics curriculum.

The International Commission on the Teaching of Mathematics had reports published by the United States Bureau of Education between the years 1911 and 1918. This commission generally surveyed the field of secondary mathematics and noted two main needs: better preparation of teachers and reduction, if not elimination, of the waste of effort involved in independent and often inadequate treatment of fundamental and broad questions by separate schools, colleges, or local systems. The Commission also made a study of elementary mathematics in the college.

The National Committee of Fifteen on Geometry Syllabus studied the problem of a syllabus for geometry. The Committee reported its recommendations in 1911.

The National Committee on Mathematical Requirements was organized in 1916 under the auspices of the Mathematical Association of America (MAA). The Committee was instructed to study the whole problem of mathematical education on the secondary and collegiate levels. The Committee's final report came out in 1923. The Committee proposed a general outline by topics rather than a detailed syllabus for the junior high school, and recommended a body of elective material to meet the aims of the senior high school.

A commission appointed in 1921 by the College Entrance Examination Board (CEEB) published reports in 1923 which made notable changes in requirements. In 1935 the Commission on Examinations in Mathematics of the CEEB completely revised the type of examinations in mathematics. This notably increased objectivity in scoring and reduced emphasis upon the traditional compartmentalized treatment of mathematics. The CEEB has to continually strive to keep its tests appropriate for the times.

The Joint Commission of the MAA and the National Council of Teachers of Mathematics (NCTM) to Study the Place of Mathematics in Secondary Education published its final report in 1940 in the *Fifteenth Yearbook* of the NCTM. In its report, the Joint Commission undertook to define the place of mathematics in the modern educational program and then to organize a mathematical curriculum for grades 7–14.

The Commission on Secondary School Curriculum was established in 1932 by the Progressive Education Association. Several committees were appointed by the Commission in the various subject fields, and among these was the Committee on the Function of Mathematics in General Education. This committee's report was published in final form in 1940.

The American Association for the Advancement of Science (AAAS) Co-operative Committee on the Teaching of Science and Mathematics, composed of representatives from several national societies, has been concerned with the problem of the preparation of teachers of science and mathematics. Several specific proposals were made in the 1946 report on "The Preparation of High School Science and Mathematics Teachers." The Committee also undertook to study the effectiveness of the teaching of science and mathematics as all levels. This effort served as the basis for a report in which appraisals were made of the science and mathematics programs from grades 1 through 12, and certain recommendations were made for improving the program. Included in the report were many pertinent recommendations for recruiting and training elementary and secondary school teachers of science and mathematics. A second report on the preparation of science and mathematics teachers, concerned with problems of the new curricula in secondary schools, was published in 1959. Typical curricula were outlined and offered as desirable basic training programs for high school mathematics and science teachers. Other activities of the AAAS were its Science Teaching Improvement Program, inaugurated in 1955, and a study of certification requirements for secondary school science and mathematics teachers, begun in 1959.

Several committees worked during the World War II years primarily to organize and direct manpower for most efficient and effective service to the war effort. Some of the reports of these committees had significant import for mathematics at the elementary and secondary levels. In 1944, the Commission on Post-War Plans was created by the Board of Directors of the NCTM to plan for effective programs in secondary mathematics in the postwar period. One of the reports of the Commission contained recommendations for improving mathematics in grades 1–14.[1]

The University of Illinois Committee on School Mathematics (UICSM), under the capable direction of Max Beberman, was one of the early influential groups of the 1950s to work towards developing materials and training teachers for a new secondary mathematics curriculum which included set concepts. An outstanding feature of the Committee has been its freedom to experiment since its beginning work in 1952. The Committee has had a broad outlook due to its joint membership from the Colleges of Education, Engineering, and Liberal Arts and Sciences. Important financial support for the UICSM came from the Carnegie Corporation of New York.

Although recently the UICSM has worked at the elementary school level, its early work was concerned with grades 9–12. The notation and some of the concepts of the algebra of sets were an important part of the course content. While topics from contemporary mathematics were used, more concern was given to consistency than just an attempt to appear modern. It was found that high school teachers could teach the newer concepts when given help.

One of the important differences between the work of the older groups and the groups of the 1950s and 1960s was that some of the newer groups actually prepared and published textual materials whereas the older groups largely just made recommendations. The earlier groups were not fortunate in being heavily funded as were the groups of the 1950s and 1960s. The UICSM textbooks emphasize consistency, precision of language, structure of mathematics, and understanding of basic principles through pupil discovery. The Committee believed that high school students are greatly interested in ideas and that the understanding of basic principles and the acquiring of manipulative skill are complementary activities.

The College Entrance Examination Board (CEEB) exerts a strong influence upon the mathematics curriculum through its examinations and their influence upon teachers and authors of textbooks. The Commission on Mathematics of the CEEB was formally established and began its work in August 1955 under the chairmanship of Albert W. Tucker. The Commission issued a two-part report in the spring of 1959 on the secondary mathematics curriculum for college-capable students.[2]

In its report, the Commission cited a number of factors entering into the urgent need for curriculum revision. A large body of new mathematics has been created just in this century alone, giving rise to specialized journals in areas that were little known to mathematicians a generation ago. Also, there has been reorganization of some of the older mathematics; for example, algebra is now thought of as the study of mathematical structure, or "pattern." A first-year graduate school course in algebra today bears little resemblance to that of forty years ago. The nature of contemporary mathematics is considerably different from the older mathematical point of view. Another important factor mentioned in the report is that there are a vast number of new applications of both the older and the newer mathematics. Besides serving the obvious areas of science and engineering, mathematics is being used increasingly in social science and business. Psychologists and economists make use especially of statistics as well as other advanced mathematical techniques. Statistics is also an important tool to industry, especially in the design and analysis of industrial research experiments and in statistical quality control and sampling theory. A final compelling reason cited by the Commission for an improved college preparatory mathematics curriculum is the fast-growing need for people skilled in various branches of mathematics. All of these factors serve to point to the fact that the secondary school mathematics curriculum as it was at the time of the

Commission's report had lagged behind the growth and uses of mathematics.

The nation was faced with low levels of mathematical competence at the time of the Commission's work. Public awareness of this fact was spurred by the Russian Sputnik (October 1957), but awareness of the problem by a number of people involved in mathematics education and the beginning of the Commission's study antedate Sputnik; so, not all the curricular and instructional improvement in mathematics of the last decade should be viewed as a by-product of the post-Sputnik panic.

While the Commission's work was concerned with grades 9–12, it recognized the need for careful study of the mathematics program of grades 7 and 8 and lower. Although recommending certain subject matter for grades 7 and 8 as a proper foundation for its programs of grades 9–12, the Commission relied on other groups (such as the School Mathematics Study Group, the University of Maryland Mathematics Project, and the Curriculum Committees of the National Council of Teachers of Mathematics) to meet the needs of the earlier grades.

The Commission outlined its major proposals as follows:

1. Strong preparation, both in concepts and in skills, for college mathematics at the level of calculus and analytic geometry

2. Understanding of the nature and role of deductive-reasoning—in algebra, as well as in geometry

3. Appreciation of mathematical structure ("patterns")—for example, properties of natural, rational, real, and complex numbers

4. Judicious use of unifying ideas—sets, variables, functions, and relations

5. Treatment of inequalities along with equations

6. Incorporation with plane geometry of some coordinate geometry, and essentials of solid geometry and space perception

7. Introduction in grade 11 of fundamental trigonometry—centered on coordinates, vectors, and complex numbers

8. Emphasis in grade 12 on elementary functions (polynomial, exponential, circular)

9. Recommendation of additional alternative units for grade 12: either introductory probability with statistical applications or an introduction to modern algebra

The overriding objective of the Commission was to produce a curriculum suitable for students and oriented to the needs of mathematics, natural science, social science, business, technology, and industry in the second half of the twentieth century. While new topics were to be introduced, the primary change was in the point of view used in presenting the usual topics. Some usual topics were suggested for elimination (for example, deductive solid geometry as a course in itself) and some change in emphasis was recommended (for example, deemphasize triangle solving in trigonometry and emphasize its analytical aspects). The recommendations of the Commission were taken as a guide for a gradual change in the CEEB examinations.

Stating its belief that there is no ideal sequence of topics applicable to every school situation, the Commission gave suggested detailed outlines of its recommended courses. Some of the topics were expanded in the Commission's *Appendices* and some were prepared as classroom units by the Commission. The Commission recognized the vital role of teacher education in implementing its program, and went into detailed recommendations on what should be done to adequately prepare teachers to carry out the program.

The Commission's mathematics program for grade 9 starts with the notion of a set, and this is the first topic treated in appendix 1 (An introduction to algebra). The recommendations of the Commission do not envisage changes in the mechanics or formal manipulations of algebra. The subject matter is intended to be much the same, with differences in concept, in terminology, in some symbolism, in graphs on a line, and in the inclusion of new work dealing with inequalities. While the development of adequate skills must continue to be an important objective of the high school algebra course, there is to be a shift in emphasis from mechanical manipulations to the development of concepts, which is equally important. The object of the introductory appendix is to show how the first few weeks of instruction in algebra can be clarified and simplified by using the simple notion of a set.

After the introduction to sets and set notation in appendix 1, appendix 2 (Sets, relations, and functions) provides an introduction to the use of sets in dealing with some old topics: equations, inequalities, graphs, loci, and functions. This material is related to the work of grades 9 through 12, inclusive. Such concepts as subset, proper subset, intersection, union, universal set, empty set, and complement are introduced. In addition to the notion of set being a clarifying and simplifying concept as mentioned in appendix 1, the notion is mentioned here of its being a unifying concept. The concept of a set gives unity to the study of equations, inequalities, relations, and functions.

Appendix 9 gives a more demanding exposition of the mathematics of collections of objects than is contained in appendix 2. Part of this material is related to the Commission's "Mathematics for grade 12" and part is needed for those studying probability.

The University of Maryland Mathematics Project (UMMaP) textbooks use the notion of set in a natural way without at all belaboring the concept. When this project got underway in 1957, it was recognized that mathematics at the junior high level was perhaps the most unsatisfactory of all the mathematics courses offered. The UMMaP, under the directorship of John R. Mayor and with support from the Carnegie Corporation of New York, set out to remedy this situation.[3]

The traditional mathematics program at grade levels 7 and 8 placed heavy emphasis on the development of skills and the so-called social applications of mathematics. The subject matter was largely repetitive of what had been presented in the earlier grades. High and low achievers alike were not inspired by two more years of the same old stuff that they had been exposed to previously.

Students using the Maryland experimental materials were found to do as well on traditional tests as students in the traditional courses while (rather naturally) doing a great deal better on tests covering the new material of the project. Teachers and students alike seemed to find the Maryland program more interesting than the traditional program.

The sample courses of the Maryland program were prepared jointly by mathematicians at the University of Maryland and teachers in schools in neighboring areas of the University. A number of revisions were made in the courses after tryouts in the schools. Hardback texts were published in 1961. The books have been used in a large number of school systems and have had considerable impact on commercial texts. The UMMaP serves as a good example of the kind of success which can be obtained in curriculum improvement when mathematicians and mathematics educators work together. The work of the School Mathematics Study Group (SMSG), to be discussed next, serves as another example of the excellent results which can be obtained from this kind of joint effort.

The group having the greatest impact on modernizing the mathematics curriculum has been the SMSG. This impact has been due to a number of factors: financing by the National Science Foundation, national scope, interest in improving the teaching of mathematics in the schools at all levels, Advisory

Committee composed of representatives from diverse areas, writing teams composed of experienced high school and college teachers of mathematics, publication of experimental texts which has had great influence on the publishers of commercial texts, making available (at cost) the texts in noncommercial form when they were deemed satisfactory for classroom use, and the publication of materials for mathematics teachers.

The work of the SMSG started in 1958 at Yale University under the direction of E. G. Begle. The work of the early years dealt with the secondary (9–12) and junior high (7–8) schools. For some reason, most groups which have worked on the curriculum for all grades have preferred to work from the top down.

The task facing the SMSG writers of sample textbooks was to exhibit a total curriculum that would be recognizably superior to the one in existence. The textbooks would have to be good enough to stimulate publication of commercial texts of a similar nature. SMSG did not wish or intend to supply textbooks except on experimental and short-term bases. The program would need to be one which would be approved as a satisfactory replacement of the traditional program. With these constraints along with trying to get textbooks with the best and most appropriate mathematics available, the SMSG was faced with a formidable task.

Although not bound by what had been done previously, the writing subgroups for the high school had recourse to the considerable amount of work done by the UICSM and the recommendations of the Commission on Mathematics of the CEEB as a point of departure. While this prior work was helpful, it was still the job of each writing group to decide what should be done at its grade level.

The ninth grade subgroup decided that its subject matter should be algebra. Neither the Commission on Mathematics nor the UICSM had seen any reason for suggesting a departure from what has traditionally been a part of the ninth grade curriculum. The difficult questions of scope, sequence, and point of view remained to be worked out and would take some time to resolve.

The tenth grade subgroup decided on geometry as the appropriate content for its grade level. It was decided that the subject matter would be synthetic Euclidean geometry, but with the addition of metric postulates. The textbook was to contain a short treatment of analytic geometry near the end of the book.

The eleventh grade subgroup proposed some algebra and some trigonometry for this grade level. They wanted a book that would give some insight into the nature of mathematical thought and at the same time prepare the student to perform certain manipulations with facility.

The twelfth grade subgroup was divided into teams with each team taking responsibility for one topic. By the end of the first writing session, the twelfth grade subgroup had produced an elementary treatment of sets, relations, and functions; a full outline of circular functions; and a full outline for a course in modern algebra with typical examples and exercises.

The subgroup concerned with the seventh and eighth grades had a responsibility somewhat different from that of the high school subgroups. Instead of writing a syllabus for a course, they were to write a series of experimental units on single topics. The seventh and eighth grade subgroup was fortunate in having available the work of the UMMaP which had been under way for about a year when the SMSG started its work. The SMSG departed from what the UMMaP had done by treating algebra in its program. They decided to concentrate on providing students with experiences emphasizing logical patterns, mathematical vocabulary, and informal deduction and induction. They also decided to associate suitable applications with the mathematical ideas developed, but to keep applications secondary to an understanding of basic concepts. An informal study of geometry was ruled appropriate. A survey of the notions of measurement and approximation, elementary work with statistics and probability, and the mathematics of the lever were other topics felt to be appropriate for the seventh and eighth grades. In some areas, the SMSG subgroup borrowed heavily from the UMMaP. The director of the UMMaP was also chairman of the subgroup.

One fact of great importance that had been established by the end of the first writing session was that high school teachers and research mathematicians can work together to produce some really significant curricular changes. This excellent partnership was to continue and is one of the most important reasons for the success which the SMSG has had. For a number of years, research mathematicians had pretty much abdicated their responsibilities to the high schools in favor of their own research interests. The reblossoming of the more or less fruitful relationship between college and high school mathematics teachers that had existed many years earlier was at least as important as the preliminary materials prepared during the 1958 summer writing session. A fruitful dialogue between mathematicians and mathematics educators has been an important feature of the work of a number of the recent groups. Today the relationship among high school and college mathematics teachers and research mathematicians is far healthier than it was for a number of years before the work of the UICSM, the Commission, the UMMaP and the SMSG. In a number of universities, it is no longer chic for members of the mathematics department to look down their noses at their colleagues in the mathematics education department. Indeed, the two departments often have extremely good working relationships with each other and mutual respect for the important work of each department. The state of affairs as it exists now between mathematics and mathematics education is quite good, and it is impossible to overstress the importance of such a relationship in continually striving to improve the curriculum and teaching of mathematics.

Following the considerable preliminary work done in the first writing session, the writers began in earnest to prepare actual sample textbooks during the second writing session which got under way in the summer of 1959. Some work had been done during the academic year on the project, but most of the participants had been too busy with regular academic duties to do very much work on the project. An additional group of mathematicians and high school teachers had been recruited during the year to help in writing the textbooks.

The second writing session produced sample textbooks for grades 7, 9, 10, 11, and 12. The eighth grade volume was considered a series of experimental units rather than a textbook. The twelfth grade writing group prepared two textbooks. They scrapped the work that had been done earlier on a modern algebra course in favor of a second semester course on matrix algebra. The first-semester textbook for twelfth grade, *Elementary Functions*, started with a chapter which developed the function concept from the standpoint of sets. It was felt that students using the text at this time would be doing so without having had previous experience with sets or set concepts. Students coming along later through the SMSG program would be familiar with the set notion as the ninth grade sample textbook used set concepts and set notation wherever they would help clarify or make more precise a topic. Set notions also helped in relating numbers and geometric points. Despite vast differences in point of view, the content of the ninth grade algebra was not new. When the SMSG later on got involved in elementary work, set concepts and notation were introduced at the very earliest levels of elementary work.

After tryouts and revisions of the sample textbooks, they were made generally available. Intended for the benefit of college-capable students, this phase of the work was not terminated with the publication of sample textbooks. A great amount of work has continued to be done to aid these students as well as those not going on to college. It should also be mentioned that the SMSG has published monographs by prominent mathematicians and in-service teacher training materials.

The books prepared by the SMSG for the elementary grades differed from the traditional books even more radically than the high school books had. Following the plan of working from the top down in attempting to improve mathematics in grades K-12, the SMSG prepared textbooks for grades 4-6 and then for the lower grades. The concept of sets is an integral part of the program. Book 1 introduces such concepts as the empty set, equivalence of sets, subsets, and uses sets to aid in understanding such notions as "more than" and "fewer than." A number of set concepts are used in the books in a natural way without contriving situations to use sets just for the sake of using sets. But there is no hesitance to use set concepts when they will aid in understanding. Materials for the early grades attempt to relate to the physcial world and the environment of which children are aware. The texts move from the concrete to the abstract, from the specific to the general. That the SMSG sample texts have influenced those who prepare commercial elementary school texts to use set concepts is evident to anyone who has browsed through books at any fairly recent NCTM convention.[4]

The Committee on the Undergraduate Program in Mathematics (CUPM) of the Mathematical Association of America is another group which has had profound influence on the school mathematics program. This group was appointed in 1953 and reconstituted and organized into four panels in 1959. The Panel on Teacher Training has made recommendations and prepared course guides for the training of teachers of mathematics. That the work of this panel has had impact is evident from the number of books for pre-service and in-service mathematics teachers which reflect the panel's recommendations. Advertisements and prefaces for such books are careful to point out that SMSG and CUPM recommendations have been considered in writing the texts.

The CUPM pamphlet *Course Guides for the Training of Teachers of Elementary School Mathematics* recommends that the notions of sets, with or without notation, be part of a course in the structure of the number system for teachers of mathematics in grades K-6. The introduction of set theory notation in one unmotivated section at the beginning of the course is considered obviously undesirable. The pamphlet *Course Guides for the Training of Teachers of Junior High and High School Mathematics* recommends that a unit on logic and sets be included as a part of one of the courses recommended for teachers at these two levels.[5] Such ideas as set equality, union, intersection, complement, universal set, null set, and subset should be taught along with some of the fundamental properties which hold for sets. Chapter 3 of the *Twenty-Third Yearbook* of the NCTM is an excellent exposition on set notions with which secondary school mathematics teachers should be familiar.[6] Naturally, set notions are also used in other chapters of the book.

A large number of groups besides the ones already mentioned started work of significance for the school mathematics program during the 1950s or 1960s. A few of the better known of these are the University of Illinois Arithmetic Project; various projects by Professors Patrick Suppes and Newton S. Hawley, including *Set Theory in First Grade*, Stanford University by Professor Suppes; the Madison Project; the Greater Cleveland Mathematics Project (differing from most in working from the bottom up in developing a K-12 mathematics curriculum); the Boston College Mathematics Institute; the Ontario Mathematics Commission; Developmental Project in Secondary Mathematics at Southern Illinois University; the Secondary School Curriculum Committee of the NCTM; the Ball State Teachers College Experimental Program; and the Cambridge Conference on School Mathematics.[7]

All of the groups that have been referred to used set notions in their work. The use of sets now seems firmly entrenched in the school mathematics program. With so many groups experimenting with curricular reform, it might be expected that a number of diverse programs would originate. Most of the programs have, however, in addition to the use of sets, much of the following in common: emphasis upon structure, a logical and sequential development, earlier introduction of topics and elimination of unimportant topics, use of recent developments in mathematics, greater abstraction, refinement of nomenclature and symbolism, use of the methodology of logic, and expanded content.[8]

With the increased emphasis of the last two decades on curricular changes and increased teacher competency, prospects for the school mathematics program appear bright. To maintain an excellent mathematics program will require continual scrutiny of the curriculum and teacher training programs and continual efforts towards improving both.

NOTES

CHAPTER 1

INTRODUCTION

1. Erich Kamke, *Theory of Sets*, trans by F. Bagemihl (New York: Dover Publications, Inc., 1950).

CHAPTER 2

GEORG CANTOR

1. Abraham Adolf Fraenkel, "Georg Cantor," *Jahresbericht der Deutschen Mathematiker Vereinigung* (Hereinafter referred to as *Jahresbericht*) 39 (1930), 192–93. Unless specifically cited otherwise, Fraenkel's excellent biography of Cantor is the source for practically all of the present chapter.

2. *Ibid.*, p. 191.

3. *Ibid.*, p. 193.

4. Cantor's knowledge of philosophical writings is attested to both in his early and later works. He did not feel that mathematics and philosophy are two unrelated areas.

5. F. Engel, "Wilhelm Thomé," *Jahresbericht* 20 (1911), 265.

6. That Cantor had given some previous thought to the calculus of probability is shown in a letter to Dedekind dated 11 September 1873.

7. Cantor mentioned this in a letter of 1908 to Professor W. H. Young; cited in "The Progress of Mathematical Analysis in the Twentieth Century," *Proceedings of the London Mathematical Society* 24, Ser. 2 (1926), 423.

8. Fraenkel, "George Cantor," p. 200.

9. Under the title, *Grundlagen einer allgemeinen Mannichfaltig-*

keitslehre. Ein mathematisch-philosophischer Versuch in der Lehre des Unendlichen.

10. Arthur Schoenflies, "Die Krisis in Cantor's mathematischem Schaffen," *Acta Mathematica* 50 (1927), 5–6. The term used is *Machwerke*; in context, "miserable scribblings" seems an appropriate translation.

11. Young, "The Progress of Mathematical Analysis in the Twentieth Century," pp. 422–23.

12. Schoenflies, "Die Krisis in Cantor's mathematischem Schaffen," p. 2.

13. Ganesh Prasad, *Some Great Mathematicians of the Nineteenth Century: Their Lives and Their Works,* II (Benares City, India: Mahamandal Press, 1934), 204–7.

14. Young, "The Progress of Mathematical Analysis in the Twentieth Century," pp. 422–24.

CHAPTER 3

CANTOR'S SET-THEORETICAL WORK

1. Cantor, *Contributions to the Founding of the Theory of Transfinite Numbers* (Hereinafter referred to as *Contributions to Transfinite Numbers*), translation of (32) and provided with an introduction and notes by Philip E. B. Jourdian (New York: Dover Publications, Inc., n.d.), p. 41.

2. J. Lüroth, "Ueber Abbildung von Mannigfaltigkeiten," *Mathematische Annalen* 63 (1907), 222–38.

3. L. E. J. Brouwer, "Beweis der Invarianz der Dimensionenzahl," *Mathematische Annalen* 70 (1911), 161–65.

4. Paul du Bois-Reymond, "Erlaeuterungen zu den Anfangsgruenden der Variationsrechnung," *Mathematische Annalen* 15 (1879), 287.

5. Cantor, *Contributions to Transfinite Numbers*, p. 32.

6. Henry J. Stephen Smith, "On the Integration of Discontinuous Functions," *Proceedings of the London Mathematical Society* 6 (1874–75), 140–53.

7. Cantor, *Contributions to Transfinite Numbers*, p. 73.

8. Cantor, "Ueber unendliche, lineare Punktmannichfaltigkeiten," 23, 473–79. The notion of content is closely connected with length, area, and volume.

9. Felix Hausdorff, "Dimension und aeusseres Mass," *Mathematische Annalen* 79 (1919), 157–79.

10. Otto Stolz, "Ueber einen zu einer unendlichen Punktmenge gehoerigen Grenzwerth," *Mathematische Annalen* 23 (1884), 152–56.

11. W. H. Young and Grace Chisholm Young, *The Theory of Sets of Points* (Cambridge: University Press, 1906).

12. See, for example, Arnaud Denjoy, "Mémoire sur les nombres dérivés des fonctions continues," *Journal de Mathématiques* 1, series 7 (1915), 105–240.

13. Cantor, *Contributions to Transfinite Numbers*, pp. 74–75.

14. *Ibid.*, pp. 75–76.

15. *Ibid.*, pp. 76–77.

16. *Ibid.*, pp. 77–78.

17. *Ibid.*, p. 79.

18. Cantor, "*Mitteilungen zur Lehre vom Transfiniten*," 91, 123.

19. Cantor, *Contributions to Transfinite Numbers*, pp. 85–136.

20. *Ibid.*, p. 80.

21. "We denote the uniting of many sets *M*, *N*, *P*, . . . , which have

no common elements, into a single set by (*M*, *N*, *P*, . . .). The elements of this set are, therefore, the elements of *M*, of *N*, of *P*, . . . , taken together."

22. Cantor, "Beitraege zur Begruendung der transfiniten Mengenlehre," 46, 484.

23. Raymond L. Wilder, *Introduction to the Foundations of Mathematics* (New York: John Wiley & Sons, Inc., 1952), p. 101. The theorem is also found in the literature as the Cantor-Bernstein theorem, the Schröder-Bernstein theorem, and the Cantor-Bernstein-Schröder theorem.

24. Ernst Zermelo, "Beweis dass jede Menge wohlgeordnet werden kann," *Mathematische Annalen* 59 (1904), 514–16.

CHAPTER 4

THE PARADOXES

1. Patrick Suppes, *Axiomatic Set Theory* (Princeton, New Jersey: D. Van Nostrand Company, Inc., 1960), p. 5.

2. While some persons have attempted a hair-splitting distinction between "paradox" and "antinomy," the terms are here considered equivalent in meaning. Curry, in his *Foundations of Mathematical Logic*, p. 4, considers the cause of the distinction between the terms to be due to false etymology.

3. Howard Eves and Carroll V. Newsom, *An Introduction to the Foundations and Fundamental Concepts of Mathematics* (New York: Rinehart and Company, Inc., 1958), pp. 281–82.

4. Cesare Burali-Forti, "Una questione sui numeri transfiniti," *Rendiconti del Circolo Matematico di Palermo* 11 (1897), 157–64.

5. For Burali-Forti's corrected formulation, see Irving M. Copi, "The Burali-Forti Paradox," *Philosophy of Science* 25 (1958), 281. Another formulation different from the one given above is on p. 284. Copi's article (from his Ph.D. dissertation at the

University of Michigan) is the primary source for the present history of the paradox.

6. In this connection see Copi, "The Burali-Forti Paradox," pp. 283–84.

7. Abraham Adolf Fraenkel and Yehoshua Bar-Hillel, *Foundations of Set Theory* (Amsterdam: North-Holland Publishing Company, 1958), p. 6.

8. Eves and Newsom, *An Introduction to the Foundations and Fundamental Concepts of Mathematics*, p. 283. Russell Published his paradox in 1903 in his *Principles of Mathematics.*

9. The catalogue paradox was published in 1933 in the article "La vérité `mathématique et la réalité," *L'Enseignement mathématique* 31 (1933), 96–114.

10. Haskell B. Curry, *Foundations of Mathematical Logic* (New York: McGraw-Hill Book Company, Inc., 1963), p. 5.

11. Paul quoted the statement in Epistle to Titus, I, 12, as by a Cretian prophet.

12. Stephen Cole Kleene, Introduction to *Metamathematics* (New York: D. Van Nostrand Company, Inc., 1952), p. 39.

13. Jules Richard, "Les principes des mathématiques et le problème des ensembles," *Revue générale des sciences pures et appliquées* 16 (1905), 541–43.

14. Fraenkel and Bar-Hillel, *Foundations of Set Theory*, p. 47.

15. This popularized form of the paradoxical theorem of Banach and Tarski is taken from Edward Kasner and James Newman, *Mathematics and the Imagination* (New York: Simon and Schuster, 1940), pp. 201–7.

16. This is one of the principal results in Stefan Banach's and Alfred Tarski's article "Sur la décomposition des ensembles de points en parties respectivement congruentes," *Fundamenta Mathematicae* 6 (1924), 244–77.

CHAPTER 5

SOME EARLY TWENTIETH CENTURY DEVELOPMENTS

1. Zermelo, "Beweiss, dass jede Menge wohlgeordnet werden kann," pp. 514–16. Zermelo gave a second proof in 1908 in volume 65 of *Mathematische Annalen*. Both proofs depend on the choice axiom. The proof as given in 1904 is available in English in E. Kamke's *Theory of Sets*, trans. by Frederick Bagemihl (New York: Dover Publications, Inc., 1950), pp. 112–15.

2. Eves and Newsom, *An Introduction to the Foundations and Fundamental Concepts of Mathematics*, p. 308. Suppose that every set possesses a well-ordering. Then for any collection of disjoint sets, each of them possesses a first element. The axiom of choice follows by choosing the set consisting of all these first elements. Thus Zermelo's theorem implies the axiom of choice, and since Zermelo proved that the axiom of choice implies his theorem, the two are equivalent.

3. Paul J. Cohen and Reuben Hersh, "Non Cantorian Set Theory," *Scientific American* 217 (Dec. 1967), 104–6 and 111–16.

4. Patrick Suppes, *Axiomatic Set Theory*, pp. 7–8. The system of axiomatic set theory usually called Zermelo-Fraenkel set theory (historically more appropriate, perhaps, Zermelo-Fraenkel-Skolem) is developed in this book.

5. Paul Bernays, *Axiomatic Set Theory*, with a Historical Introduction by Abraham A. Fraenkel (Amsterdam: North-Holland Publishing Company, 1958), p. 32.

6. Suppes, *Axiomatic Set Theory*, p. 12.

7. Haskell B. Curry gives a somewhat different division, *Foundations of Mathematical Logic*, pp. 8–11. In particular, he claims that logicism is not a unified view as to the nature of mathematics, but a special thesis as to the relation of logic to mathematics.

8. In the sequel, the discussion of the schools is adapted primarily

from Eves and Newsom, *An Introduction to the Foundations and Fundamental Concepts of Mathematics*, pp. 285–91, except where otherwise noted.

9. For an exposition of intuitionist mathematics, the authoritative recent work is A. Heyting's *Intuitionism: An Introduction* (Amsterdam: North-Holland Publishing Company, 1956).

10. *Ibid*., pp. 1–2.

11. Raymond L. Wilder, *Introduction to the Foundations of Mathematics*, pp. 201–2.

12. Curry, *Foundations of Mathematical Logic*, pp. 8–9.

CHAPTER 6

SET THEORY IN THE SCHOOL MATHEMATICS PROGRAM

1. For elaboration on the thumbnail sketch of the work of the committees given above, see chapter one of Charles H. Butler, F. Lynwood Wren, and J. Houston Banks, *The Teaching of Secondary Mathematics* (5th Edition; New York: McGraw-Hill Book Company, 1970).

2. *Program for College Preparatory Mathematics and Appendices* (New York: College Entrance Examination Board, 1959).

3. For more information than is given here on the work of the UMMaP, see John R. Mayor, "Research in the Learning of Mathematics at the University of Maryland," *Journal of Research in Science Teaching* 2 (1964), 304–8.

4. An excellent account of the early years of the SMSG, from which most of the information here was obtained, is William Wooton, *SMSG: The Making of a Curriculum* (New Haven and London: Yale University Press, 1965).

5. The CUPM pamphlets are available from the CUPM.

6. E. J. McShane, "Operating with Sets," *Insights into Modern Mathematics*, Twenty-Third Yearbook of the NCTM (Washington, D. C.: The National Council of Teachers of Mathematics, 1957), pp. 36–64. The Twenty-Fourth Yearbook, *The Growth of Mathematical Ideas Grades K-12*, presents mathematical understandings which are considered basic to the mathematics curriculum, and so it naturally also contains set notions.

7. For information about all of these (and a number of others) except for the Cambridge Conference, see the two pamphlets (together) *Studies in Mathematics Education: A Brief Survey of Improvement Programs for School Mathematics* (Chicago: Scott, Foresman and Company, 1960) and *An Analysis of New Mathematics Programs* (Washington, D.C.: National Council of Teachers of Mathematics, 1963).

8. Lloyd Scott, *Trends in Elementary School Mathematics* (Chicago: Rand McNally and Company, 1966), pp. 28–34.

SELECTED BIBLIOGRAPHY

CANTOR'S WORKS

1. "De aequationibus secundi gradus indeterminatis." Ph.D. dissertation, Berlin, 1867.

2. "Zwei Saetze aus der Theorie der binaeren und quadratischen Formen." *Zeitschrift fuer Mathematik und Physik* 13 (1868), 259–61.

3. "Ueber die einfachen Zahlensysteme." *Ibid.* 14 (1869), 121–28.

4. "Zwei Saetze ueber eine gewisse Zerlegung der Zahlen in unendliche Produkte." *Ibid.*, pp. 152–58.

5. "De transformatione formarum ternariarum quadraticarum." Habilitationsschrift, Halle, 1869.

6. "Ueber einen die trigonometrischen Reihen betreffenden Lehrsatz." *Journal fuer die reine und anqewandte Mathematik* (*Crelle's Journal*) 72 (1870), 130–38.

7. "Beweis, dass eine fuer jeden reellen Wert von *x* durch eine trigonometrische Reihe gegebene Funktion f (x) sich nur auf eine einzige Weise in dieser Form darstellen laesst." *Ibid.*, pp. 139–42.

8. "Notiz zu dem Aufsatze: Beweis, dass eine fuer jeden reellen Werth von *x* durch eine trigonometrische Reihe gegebene Function f(x) sich nur auf eine einzige Weise in dieser Form darstellen laesst. Bd. 72, Seite 139 dieses Journals." *Ibid.* 73 (1871), 294–96.

9. "Ueber trigonometrische Reihen." *Mathematische Annalen* 4 (1871), 139–43.

10. "Ueber die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen." *Ibid.* 5 (1872), 123–32.

11. "Algebraische Notiz." *Ibid.*, pp. 133–34.

12. "Historische Notizen ueber die Wahrscheinlichkeitsrechnung." *Bericht ueber die Sitzungen der Naturforschenden Gesellschaft zu Halle im Jahre 1873.*

13. "Ueber eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen." *Crelle's Journal* 77 (1874), 258–62.

14. "Ein Beitrag zur Mannigfaltigkeitslehre." *Ibid.* 84 (1878), 242–58.

15. "Ueber einen Satz aus der Theorie der stetigen Mannigfaltigkeiten." *Nachrichten von der Kongliga Gesellschaft der Wissenschaften und der Georg-Augusto-Universitaet zu Goettingen aus dem Jahre 1879,* ". 127–35.

16. "Ueber unendliche, lineare Punktmannichfaltigkeiten." I. *Mathematische Annalen* 15 (1879), 1–7; II. *Ibid.* 17 (1880), 355–58; III. *Ibid.* 20 (1882), 113–21; IV. *Ibid.* 21 (1883), 51–58; V. *Ibid.*, pp. 545–91; VI. *Ibid.* 23 (1884), 453–88.

17. "Bemerkung ueber trigonometrische Reihen." *Mathematische Annalen* 16 (1880), 113–14.

18. "Fernere Bemerkung ueber trigonometrische Reihen." *Ibid.*, pp. 267–69.

19. "Zur Theorie der zahlentheoretischen Funktionen." *Nachrichten von der Kongliga Gesellschaft der Wissenschaften und der Georg-Augusto-Universitaet zu Goettingen aus dem Jahre 1880*, pp. 161–69.

20. "Ueber ein neues und allgemeines Kondensationsprinzip der Singularitaeten von Funktionen." *Mathematische Annalen* 19 (1882), 588–94.

21. *Grundlagen einer allgemeinen Mannichfaltigkeitslehre: Ein mathematisch-philosophischer Versuch in der Lehre des Unendlichen.* (A separately appearing edition of 16V, with forewood.) Leipzig, 1883.

22. "Sur divers théorèmes de la théorie des ensembles de points situés dans un espace continuà *n* dimensions." *Acta Mathematica* 2 (1883), 409–14.

23. "De la puissance des ensembles parfaits de points." *Ibid.* 4 (1884), 381–92.

24. "Ludwig Scheeffer (1859–1885)." *Bibliotheca Mathematica* (1885), pp. 197–99.

25. "Ueber verschiedene Theoreme aus der Theorie der Punktmengen in einem *n*-fach ausgedehnten stetigen Raume G_n." *Acta Mathematica* 7 (1885), 105–24.

26. "Ueber die verschiedenen Standpunkte in Bezug auf das aktuale Unendliche." *Zeitschrift fuer Philosophie und philosophische Kritik* 88, N. F. (1886), 224–33.

27. "Ueber die verschiedenen Ansichten in Bezug auf die aktualunendlichen Zahlen." *Bihang till Kongliga Svenska Vetenskaps-Akademiens Handlingar* 11, Nr. 19 (1887), 1–10.

28. "Mitteilungen zur Lehre vom Transfiniten." I. *Zeitschrift fuer Philosophie und philosophische Kritik* 91, N. F. (1887), 81–125 und 252–70; II. *Ibid.* 92 (1888), 240–65.

29. "Bemerkung mit Bezug auf den Aufsatz: Zur Weierstrass-Cantor' schen Theorie der Irrationalzahlen in Math. Annalen Bd. XXXIII, p. 154." *Mathematische Annalen* 33 (1889), 476.

30. "Ueber eine elementare Frage der Mannigfaltigkeitslehre." *Jahresbericht der Deutschen Mathematiker Vereinigung* 1 (1892), 75–78.

31. "Verification jusqu'à 1000 du théorème empirique de Goldbach." *Association Francaise pour l'Avancement des Sciences, Compte Rendu de la 23me Session (Caen 1894)*, Seconde Partie, pp. 117–34.

32. "Beitraege zur Begruendung der transfiniten Mengenlehre." I. *Mathematische Annalen* 46 (1895), 481–512; II. *Ibid.* 49 (1897), 207–46.

Contributions to the Founding of the Theory of Transfinite Numbers. Paper 32 translated, and provided with an introduction and notes, by Philip E. B. Jourdain. New York: Dover Publications, Inc., n. d.

33. "Sui numeri transfiniti." *Revista di Matematica* 5 (1895), 104–9.

RELATED WORKS

Banach, Stefan, and Tarski, Alfred. "Sur la décomposition des ensembles de points en parties respectivement congruentes." *Fundamenta Mathematicae* 6 (1924), 244–77.

Bell, Eric Temple. *Men of Mathematics.* New York: Simon and Schuster, 1937.

Bernays, Paul. *Axiomatic Set Theory.* Historical Introduction by Abraham A. Fraenkel. Amsterdam: North-Holland Publishing Company, 1958.

Beth, Evert Willem. *Les fondements logiques des mathématiques.* 2nd ed. Paris: Gauthier-Villars, 1955.

———. *The Foundations of Mathematics.* Amsterdam: North-Holland Publishing Company, 1959.

Black, Max. *The Nature of Mathematics: A Critical Survey.* New York: Harcourt, Brace and Company, 1935.

Bolzano, Bernard. Paradoxes of the Infinite. Translated by Fr. Přihonský. Historical Introduction by Donald A. Steele. New Haven: Yale University Press, 1950.

Brouwer, L. E. J. "Beweis der Invarianz der Dimensionenzahl." *Mathematische Annalen* 70 (1911), 161–65.

Burali-Forti, Cesare. "Una questione sui numeri transfiniti." Rendi-*conti del Circolo Matematico di Palermo* 11 (1897), 157–64.

Cajori, Florian. *A History of Mathematics.* 2nd ed., rev. New York: The Macmillan Company, 1919.

Cohen, Paul J., and Hersh, Reuben. "Non Cantorian Set Theory." *Scientific American* 217 (December, 1967), 104–6 and 111–16.

Copi, Irving M. "The Burali-Forti Paradox." *Philosophy of Science* 25 (1958), 281–86.

Curry, Haskell B. *Foundations of Mathematical Logic.* New York: McGraw-Hill Book Company, Inc., 1963.

Eves, Howard. *An Introduction to the History of Mathematics.* 3rd ed. New York: Holt, Rinehart and Winston, 1969.

———, and Newsom, Carroll V. *An Introduction to the Foundations and Fundamental Concepts of Mathematics.* New York: Rinehart & Company, Inc., 1958.

Fraenkel, Abraham Adolf. *Abstract Set Theory.* Amsterdam: North-Holland Publishing Company, 1953.

——, *Einleitung in die Mengenlehre.* 2nd. ed. Berlin: Julius Springer, 1923.

———. "Georg Cantor." *Jahresbericht der Deutschen Mathematiker Vereinigung* 39 (1930), 189–266.

———, and Bar-Hillel, Yehoshua. *Foundations of Set Theory.* Amsterdam: North-Holland Publishing Company, 1958.

Frege, Gottlob. *The Foundations of Arithmetic.* Translated by J. L. Austin. 2nd rev. ed. Oxford: Basil Blackwell, 1953.

Gödel, Kurt. *The Consistency of the Generalized Continuum-Hypothesis with the Axioms of Set Theory.* Annals of Mathematics studies, No. 3. Princeton: Princeton University Press, 1940.

Gonseth, Ferdinand. "La vérité mathématique et la réalité." *L'Enseignement mathématique* 31 (1933), 96–114.

Grelling, K., and Nelson, L. "Bemerkungen zu den Paradoxien von Russell und Burali-Forti." *Abhandlungen der Friesschen Schule* 2, N.S. (1908), 301–24.

Hausdorff, Felix. "Dimension und aeusseres Mass." *Mathematische Annalen* 79 (1919), 157–79.

———. *Grundzuege der Mengenlehre,* Leipzig: W. de Gruyter and Company, 1914.

———. *Set Theory.* Translated by John R. Aumann, et *al.* 2nd. ed. New York: Chelsea Publishing Company, 1962.

Heyting, Arend. *Intuitionism: An Introduction.* Amsterdam: North-Holland Publishing Company, 1956.

Huntington, Edward V. *The Continuum and Other Types of Serial Order*. 2nd. ed. Cambridge, Mass.: Harvard University Press, 1917.

Kamke, Erich. *Theory of Sets*. Translated by Frederick Bagemihl. New York: Dover Publications, Inc., 1950.

Kasner, Edward, and Newman, James. *Mathematics and the Imagination*. New York: Simon and Schuster, 1940.

Kleene, Stephen Cole. *Introduction to Metamathematics*. New York: D. Van Nostrand Company, Inc., 1952.

Lüroth, J. "Ueber Abbildung von Mannigfaltigkeiten." *Mathematische Annalen* 63 (1907), 222–38.

Noether, Emmy, and Cavaillès, Jean, ed. *Briefwechsel Cantor-Dedekind.* Paris: Hermann & Cie., 1937.

Prasad, Ganesh. *Some Great Mathematicians of the Nineteenth Century: Their Lives and Their Works*. Vol. I. Benares City, India: Mahamandal Press, 1934.

Richard, Jules. "Les principes de mathématiques et le problème des ensembles." *Revue générale des sciences pures et appliquées* 16, (1905), 541–43.

Schoenflies, Arthur. "Die Krisis in Cantor's mathematischem Schaffen." *Acta Mathematica* 50 (1927), 1–23.

———. "Georg Cantor." *Mitteldeutsche Lebensbilder* 3 (1928), 548–63.

———."Zur Erinnerung an Georg Cantor." *Jahresbericht der Deutschen Mathematiker Vereinigung* 31 (1922), 97–106.

Stolz, Otto. "Ueber einen zu einer unendlichen Punktmenge gehoerigen Grenzwerth." *Mathematische Annalen* 23 (1884), 152–56.

Suppes, Patrick. *Anxiomatic Set Theory*. Princeton, New Jersey: D. Van Nostrand Company, Inc., 1960.

Wilder, Raymond L. *Introduction to the Foundations of Mathematics*. New York: John Wiley & Sons, Inc., 1952.

Young, W. H. "The Progress of Mathematical Analysis in the Twentieth Century." *Proceedings of the London Mathematical Society* 24, 2nd ser. (1926), 421–34.

———, and Young, Grace Chisholm. *The Theory of Sets of Points.* Cambridge, England: Cambridge University Press, 1906.

Zermelo, Ernst. "Beweis dass jede Menge wohlgeordnet werden kann." *Mathematische Annalen* 59 (1904), 514–16.

———. "Untersuchungen ueber die Grundlagen der Mengenlehre I." *Mathematische Annalen* 65 (1908), 261–81.

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