Fundamentals of MATHEMATICS

FOR JEE MAIN AND ADVANCED

FUNCTIONS AND GRAPHS

Sanjay Mishra

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Delhi • Chennai

Published by Pearson India Education Services Pvt. Ltd, CIN: U72200TN2005PTC057128, formerly known as TutorVista Global Pvt. Ltd, licensee of Pearson Education in South Asia.

ISBN 978-93-325-5658-4

eISBN 978-93-325-7632-2

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Preface

Calculus is a branch of mathematics which is considered as gateway course for almost all career streams closely associated with mathematics like Engineering, Applied Physics, Astrophysics, Computer Science, Electronics, etc. It is the most basic mathematical tool dealing with rate of changes (variations). Indeed this beautiful branch was not born in a moment of divine inspiration, rather it gradually came in to existence as a variety of apparently different ideas and procedures merged together to form a systematic coherent pattern.

Over the years, scientists, economists, and other researchers have studied the relationships existing between various quantities. For example, an engineer may need to know how the illumination from a light source on an object is related to the distance between the object and the source; a biologist may wish to investigate how the population of a bacterial colony varies with time in the presence of a toxin; an economist may wish to determine the relationship between consumer demand for a certain commodity and its market price. The mathematical study of such relationship involves the concept of a ***function***. The concept of function is an essential pre-requisite for understanding the most basic idea of calculus. The purpose of this book is to lay foundation for the same to understand Calculus.

Functions are the central objects of investigation in most fields of modern mathematics. It can be understood as a machine which for each input x returns exactly one corresponding output. $f(x)$. There are many ways to represent a function. Some functions may be defined by formula or algorithm that tells how to compute the output for a given input.

Since the input and output of a function can be expressed as an ordered pair, ordered such that the first element is the input (or tuple of inputs, if the function takes more than one input), and the second is the output. For instance, given a function $f(x) = x^2$, we have the ordered pair (1, 1), (−2, 4), etc. for satisfying it. If both input and output are real numbers, this ordered pair can be viewed as the Cartesian coordinates of a point on the graph of the function. Therefore, functions can also be represented by a picture, called its graph. Graph of function is set of all those points (input x, output y) that satisfy the relation $y = f(x)$. Graphs last longer in our mind(s) as they provide a visual impact to our memory. They also reveal information about functions which may not be readily evident from verbal or an algebraic description.

Calculus constitutes a major share in the syllabus of IIT JEE MAINS/ADVANCED and other competitive engineering examinations therefore its in-depth analysis is essential. This book has been written to serve as a basic text book as well as an exercise book for Calculus having a special focus on problem solving. I feel this will not only fulfill the needs of a pre-college students (i.e., students of Class XI and XII) but will also meet the advanced requirements of students who are preparing for various engineering entrance examinations like IIT-JEE MAINS /ADVANCED, BIT-SAT, and other state engineering entrance examinations.

Fundamentals of Mathematics: Functions and Graphs develop a deep insight of topics, such as Relations, Functions; their classifications and their exhaustive properties. Topic on Graph Theory provides details about various functions, their graphical nature and the variation observed in their graphs as they pass through different mathematical transformations. Method of sketching an unknown curve(s) using calculus shall be an essential skill to be learned which is sufficiently discussed in detail. I have experienced in my teaching career that if the core concepts of functions and their graphs is well laid upon in the mind of students from beginning then mastering the giant like calculus becomes very convenient. From a JEE aspirant's point of view, this topic is highly scoring in Mathematics as far as competitive exams are concerned. It is commonly observed among many students that they tend to develop a phobia against calculus. This is due to the non-familiarity with the detail concepts of functions and their properties. Further, there exists a lack of good books that lay down these core concepts in a lucid and student friendly manner.

This book has a well-arranged content list which will help students and teachers both to conveniently access the chapters and their sub topics. Each chapter is divided in to several topics. Each topic contains its theory with sufficient number of worked out illustrative problems. This will enable the students to develop an applicative understanding of the concepts. This is then followed by a textual exercise of both objective and subjective problems, testing the student's understanding of concepts. At the end of the theory of each chapter, a large set of solved examples of both objective and subjective type has been given. This will involve application of all the concepts learnt in the chapter so that students can develop expertise over the chapter.

The tutorial exercise given at the end contains a large number of multiple choice problems of single and multiple correct options, comprehension passage, column matching problems, numerical integer type questions to facilitate the students to do the thorough revision of entire chapter for raising their level of understanding of these topics. For teachers, this text book will be quite helpful as it will provide a set of well-graded problems, arranged in topic and sub-topic wise, that be used as a home assignment given to their students.

Any suggestions for improving the content of this book are most welcome and will be gratefully acknowledged.

Sanjay Mishra

Acknowledgements

I am really grateful to 'Pearson Education' for showing their faith in me and for providing me an opportunity to transform my yearning, my vast experience of teaching and knowledge in to the present comprehensive text book, Fundamentals of Mathematics. I would like to thank all my friends and teachers, for their valuable criticism, support and advice that was really helpful to carve out this work. Interactions with my students have aspired me to write this book. I wish to thank my parents and all my family members, for their patience and support in bringing out this book and donating their valuable share of time. I extend my special thanks to my team, including assisting teachers, managers and computer operators, for their hard work and dedication in completing this task.

Sanjay Mishra

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1 CHAPTER

Relations

INTRODUCTION

To study relation between two sets we must do nothing but to look around different relations in our day to day business, *e.g.*, friendship relation, relation between two countries, relationship between two families etc. For the existence of all the above relations there must be some method or rule (correspondence) of association by which the relation is actually established. So relation between two sets can be defined as a rule by which elements of first set actually correspond with the elements of the second set. In my view without the study of relations the actual understanding of functions and further study of calculus is not possible.

So this chapter which includes study of sets and relations between sets prepares a strong foundation for further study of calculus.

Cartesian Product of Two Sets

Cartesian product of two sets A and B is a set containing the ordered pairs (a, b) such that $a \in A$ and $b \in B$. It is denoted by

$A \times B$. That is, $A \times B = \{ (a, b): a \in A \text{ and } b \in B\}$. If set $A = \{ a_1, a_2, a_3\}$ and $B = \{b_1, b_2\}$, then

$A \times B = \{(a_1, b_1),(a_1, b_2),(a_2, b_1), (a_2, b_2), (a_3, b_1), (a_3, b_2)\}$, and

$B \times A = \{(b_1, a_1), (b_1, a_2), (b_1, a_3), (b_2, a_1), (b_2, a_2), (b_2, a_3)\}$.

Clearly $A \times B \neq B \times A$ until A and B are equal.

REMARKS:

1. *Since $A \times B$ has elements as ordered pairs therefore it can be geometrically located on X-Y plane by considering set A on X-axis and set B on Y-axis.*
2. *Cartesian product of n sets $A_1, A_2, A_3,....,A_n$ is denoted by $A_1 \times A_2 \times A_3 \times .... \times A_n$ and is the set of n ordered tuples* i.e., $A_1 \times A_2 \times A_3 \times .... \times A_n = \{(a_1, a_2, a_3, ..., a_n) : a_i \in A_i,\ i = 1, 2, 3, .., n\}$. *Cartesian product of n sets represents n dimensional space.*
3. *$A \times B \times C$ and $(A \times B) \times C$ are not same.*
   $A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}$, *whereas* $(A \times B) \times C = \{((a, b), c) : a \in A, b \in B, c \in C\}$

Number of Elements in Cartesian product $A \times B$

If number of elements in A denoted by $n(A) = m$ and number of elements in B denoted by $n(B) = n$, then number of elements in $(A \times B) = m \times n$. i.e., $n(A \times B) = n(A) \times n(B)$.

Since $A \times B$ contains all such ordered pairs of the type (a, b) such that $a \in A$ and $b \in B$, that means it includes all possibilities in which the elements of set A can be related with the elements of set B. Therefore, $A \times B$ contains $n(A) \times n(B)$ number of elements.

Properties and laws of Cartesian Product

Distributive Laws

(a) Cartesian product distributes over union and intersection of sets.

i.e., $A \times (B \cup C) = (A \times B) \cup (A \times C)$ and $A \times (B \cap C) = (A \times B) \cap (A \times C)$ for every group of sets A, B and C.

Proof: Let $(x, y) \in A \times (B \cup C)$

$\Rightarrow$ $x \in A$ and $y \in (B \cup C)$

$\Rightarrow$ $x \in A$ and $y \in B$ or $y \in C$

$\Rightarrow$ $x \in A$ and $y \in B$ or $x \in A$ and $y \in C$

$\Rightarrow$ $(x, y) \in (A \times B) \cup (A \times C)$

$A \times (B \cup C)$ is a subset of $(A \times B) \cup (A \times C)$

Similarly, we can prove that $(A \times B) \cup (A \times C)$ is subset of $A \times (B \cup C)$.

therefore, $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Similarly, we can prove $A \times (B \cap C) = (A \times B) \cap (A \times C)$ for every group of sets A, B and C.

(b) Cartesian product distributes over subtraction of sets

i.e., $A \times (B - C) = (A \times B) - (A \times C)$

Proof: Let $(x, y) \in A \times (B - C)$

$\Rightarrow$ $x \in A$ and $y \in B - C$

$\Rightarrow$ $x \in A$ and $y \in B$ but $y \notin C$

$\Rightarrow$ $x \in A, y \in B$ and $x \in A, y \notin C$

$\Rightarrow$ $(x, y) \in (A \times B)$ but $(x, y) \notin (A \times C)$

$\Rightarrow$ $(x, y) \in (A \times B) - (A \times C)$

$\Rightarrow$ $A \times (B - C) \subseteq (A \times B) - (A \times C)$

Similarly $(A \times B) - (A \times C) \subseteq A \times (B - C)$

$\Rightarrow$ $A \times (B - C) = (A \times B) - (A \times C)$.

2. **Cartesian Product is not Associative:** Cartesian product of sets is not associative in nature, i.e., $A \times (B \times C) \neq (A \times B) \times C$

As the elements of $A \times (B \times C)$ are of the type $(a, (b, c))$, whereas the elements of $(A \times B) \times C$ are of the type $((a, b), c)$; $a \in A$, $b \in B$, $c \in C$.

3. **Cartesian Product is not Commutative:** Cartesian product of sets is not commutative in nature.

i.e., $A \times B \neq B \times A$ until $A = B$.

4. **Cardinality of Cartesian Product:**

(a) If A and B are two sets, then $n(A \times B) = n(A) \times n(B)$.

(b) If A and B are sets having k number of common elements, i.e., $n(A \cap B) = k$, then the number of elements common to $A \times B$ and $B \times A = k^2$.

Proof: Let $C \subset B$ and $C \subset A$ and C is largest such set and let $n(C) = k$

Now, $C \times C \subset A \times B$ and $C \times C \subset B \times A$

$\Rightarrow$ $n[(A \times B) \cap (B \times A)] = n(C) \cdot n(C) = k \cdot k = k^2$.

5. Intersection of cross product is equal to cross product of intersection:

That is, for sets A, B, S and T : $(A \times B) \cap (S \times T) = (A \cap S) \times (B \cap T)$

6. **For subset A of B and C of D:** We have

(a) $(A \times C) \subseteq (B \times C)$ for every set C.

(b) $(A \times C) \subseteq (B \times D)$

(c) $(A \times A) \subseteq (A \times B) \cap (B \times A)$

7. For complementary sets B' and C' of sets B and C

(a) $A \times (B' \cup C')' = (A \times B) \cap (A \times C)$

(b) $A \times (B' \cap C')' = (A \times B) \cup (A \times C)$

Proof: (a) $A \times (B' \cup C')'$

$= A \times (B \cap C)$ [By Demorgan's Law]

$= (A \times B) \cap (A \times C)$

(b) $A \times (B' \cap C')'$

$= A \times (B \cup C)$ [By Demorgan's Law]

$= (A \times B) \cup (A \times C)$

8. $A \times (B \Delta C) = (A \times B) \Delta (A \times C)$

Proof: $A \times (B \Delta C) = A \times [(B - C) \cup (C - B)]$

$= [A \times (B - C)] \cup [A \times (C - B)]$

$(\because A \times (B \cup C) = (A \times B) \cup (A \times C))$

$= [(A \times B) - (A \times C)] \cup [(A \times C) - (A \times B)]$

$(\because A \times (B - C) = (A \times B) - (A \times C))$

$= (A \times B) \Delta (A \times C)$

ILLUSTRATION 1: If the number of elements in A is 2 and number of elements in B is 3, then find

(i) the number of elements in the power set of $A \times B$.

(ii) the number of elements in the power set of $(A \times B) \times (A \times B)$

SOLUTION: (i) Since $n(A) = 2$; $n(B) = 3$, then $n(A \times B) = 6$

So number of subsets of $A \times B = 2^6$ $\Rightarrow$ $n(P(A \times B)) = 2^6 = 64$

(ii) Number of subsets of $[(A \times B) \times (A \times B)] = 2^{[n(A \times B)]^2} = 2^{36}$.

$\therefore$ $n(P[(A \times B) \times (A \times B)]) = 2^{36}$.

ILLUSTRATION 2: Let A and B be two non-empty sets having n elements in common, then prove that $A \times B$ and $B \times A$ have n^2 elements in common.

SOLUTION: We have $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

On replacing C by B and D by A, we get $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$

It is given that A and B have n elements in common, so $[(A \cap B) \times (B \cap A)]$ has n^2 elements

But $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$

$\therefore$ $[(A \times B) \cap (B \times A)]$ has n^2 elements. Hence $A \times B$ and $B \times A$ have n^2 elements in common.

ILLUSTRATION 3: Given two sets A and B defined as $A = \{a : a \in \mathbb{Z} \text{ and } |a - 2| \le 3\}$ and $B = \{b : b \in \mathbb{Z} \text{ and } |b - 4| \le 2\}$.

Find $A \times B$ and $B \times A$. Represent them geometrically and also find number of elements common in $A \times B$ and $B \times A$.

SOLUTION: $A = \{a : a \in \mathbb{Z} \text{ and } |a - 2| \le 3\}$; $B = \{b : b \in \mathbb{Z} \text{ and } |b - 4| \le 2\}$.

To find: $A \times B$ and $B \times A$

Here $-3 \le a - 2 \le 3 \Rightarrow -1 \le a \le 5$ and $-2 \le b - 4 \le 2 \Rightarrow 2 \le b \le 6$

$\therefore$ $A = \{-1, 0, 1, 2, 3, 4, 5\}$; $B = \{2, 3, 4, 5, 6\}$

$(A \times B)$ can be represented geometrically by taking the set A along x-axis and set B along y-axis as shown below in Figure 1.1.

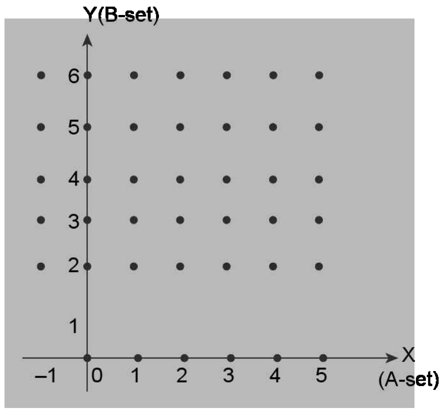


FIGURE 1.1

Now, $(B \times A)$ can be represented geometrically by taking set B along x-axis and set A along y-axis as shown below in Figure 1.2.

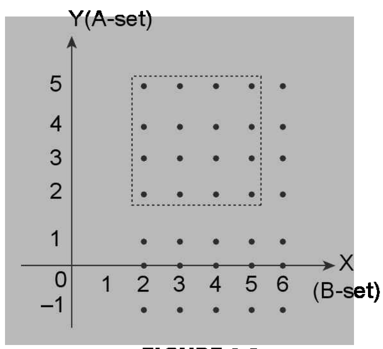


FIGURE 1.2

Clearly, $A \times B$ and $B \times A$ have 16 elements in common as represented enclosed by rectangular region in above graph. Note that A and B have 4 elements in common.

$\therefore$ $(A \times B)$ and $(B \times A)$ have $(4)^2$ elements in common

ILLUSTRATION 4: Given two sets A and B defined as $A = \{a : a \in \mathbb{R} \text{ and } |a - 2| \le 3\}$ and $B = \{b : b \in \mathbb{Z} \text{ and } |b - 2| \le 4\}$. Find $A \times B$ and $B \times A$. Represent them geometrically and also find number of elements common in $A \times B$ and $B \times A$. Fiind the answer if $b \in \mathbb{R}$.

SOLUTION: $A = \{a : a \in \mathbb{R} \text{ and } |a - 2| \le 3\}$; $B = \{b : b \in \mathbb{Z} \text{ and } |b - 2| \le 4\}$.

Here $|a - 2| \le 3 \Rightarrow -3 \le a - 2 \le 3 \Rightarrow -1 \le a \le 5 \Rightarrow A = [-1, 5]$

And $|b - 2| \le 4 \Rightarrow -4 \le b - 2 \le 4 \Rightarrow -2 \le b \le 6; b \in \mathbb{Z} \Rightarrow B = \{-2, -1, 0, 1, 2, 3, 4, 5, 6\}$

$\therefore$ $(A \times B)$ can be represented geometrically by taking set A along x-axis and set B along y-axis as shown below in Figure 1.3.

Further, $(B \times A)$ can be represented geometrically by taking set B along x-axis and set A along y-axis as shown in Figure 1.4.

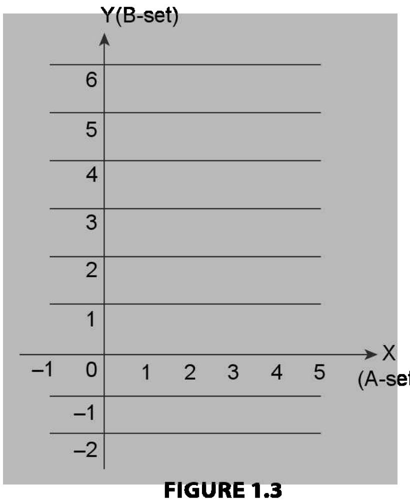


FIGURE 1.3

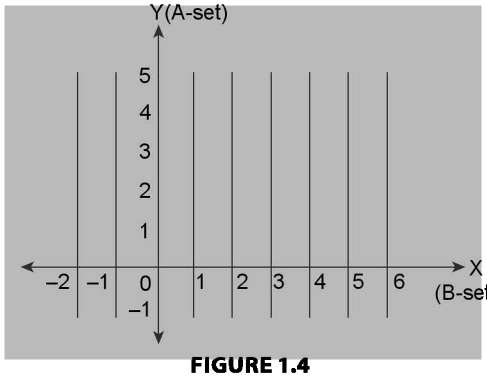


FIGURE 1.4

Since A and B have seven elements in common, i.e., $-1, 0, 1, 2, 3, 4, 5$.

$\therefore$ $(A \times B)$ and $(B \times A)$ have $(7)^2 = 49$ elements in common.

Further, if $b \in \mathbb{R}$, then $B = [-2, 6]$.

Then $(A \times B)$ and $(B \times A)$ can be represented geometrically as shown in Figure 1.5 and 1.6 respectively.

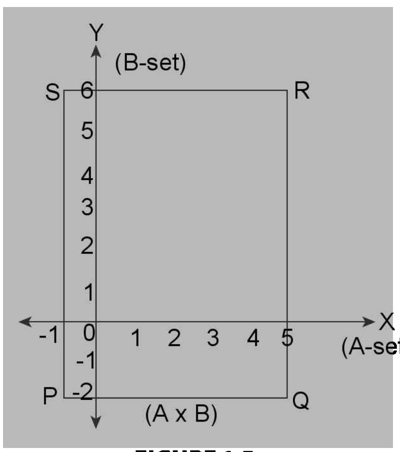


FIGURE 1.5

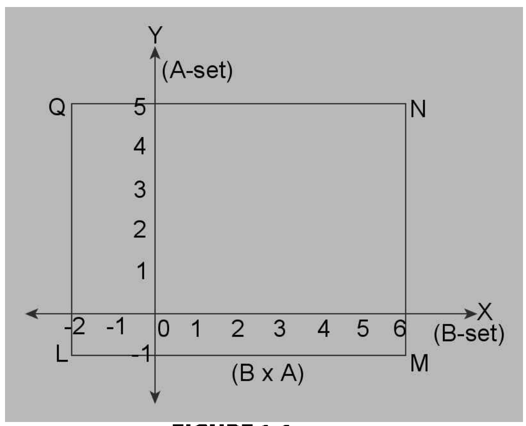


FIGURE 1.6

In above case when $A = [-1, 5]$ and $B = [-2, 6]$; $A \times B$ and $B \times A$ have infinitely many uncountable number of common elements each occurring in square bounded by straight lines $x = -1$, $x = 5$, $y = -1$ and $y = 5$. That is, $A \times B$ and $B \times A$ have 36 square units area in common.

ILLUSTRATION 5: Given two sets A and B defined as $B = \{y : y \in \mathbb{R} \text{ and } |y| \leq 4\}$ and $A = \{x : x \in \mathbb{R} \text{ and } |x| \leq 3$ and $x^2 + 3x \geq 0\}$. Find the region common to $R : A \to B$ and $A \times B$; where $R = \left\{(x, y) : \frac{x^2}{9} + \frac{y^2}{4} \leq 1 \text{ and } (x, y) \in (A \times B)\right\}$.

SOLUTION: $A = \{x : x \in \mathbb{R} \text{ and } |x| \leq 3; x^2 + 3x \geq 0\}$; $B = \{y : y \in \mathbb{R} \text{ and } |y| \leq 4\}$.

Now for $x \in A$; $x \in [-3, 3]$ and $(x)(x + 3) \geq 0$

$\Rightarrow$ $x \in [-3, 3]$ and $x \in (-\infty, -3] \cup [0, \infty)$

$\Rightarrow$ $x \in [0, 3]$

$\therefore$ $A = [0, 3]$ and for $y \in \mathbb{R}$; $|y| \leq 4$

$\Rightarrow$ $y \in [-4, 4] \Rightarrow B = [-4, 4]$

$\therefore$ $A \times B = [0, 3] \times [-4, 4]$ is the rectangular region bounded by straight lines $x = 0$; $x = 3$; $y = -4$ and $y = 4$ as shown in Figure 1.7.

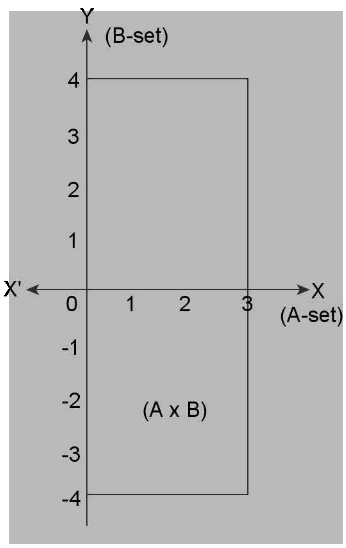


FIGURE 1.7

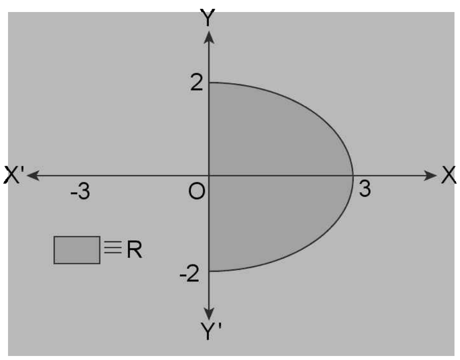


FIGURE 1.8

Further $R : A \longrightarrow B$ is defined by $R = \left\{(x, y) : \frac{x^2}{9} + \frac{y^2}{4} \leq 1\right\}$, represents the right half of the interior region of standard ellipse with length of semi major axis 3 and semi minor axis 2 as shown in Figure 1.8.

$\because$ R lies wholly inside the region $(A \times B)$.

Thus, $R \cap (A \times B) = R$

$= 1/2 \times \pi \times (3) \times (2) = 3\pi$ square units ($\because$ Area of ellipse $= \pi AB$).

TEXTUAL EXERCISE–1: (SUBJECTIVE)

1. If $A = \{x : x \in \mathbb{N}, x \leq 4\}$; $B = \{y : y \in W, y \leq 2\}$; W = set of whole numbers.
   (a) Find $A \times B$ (b) Find $n(A \times B)$

2. Plot $A \times B$ and $B \times A$ on x-y plane, if
   (a) $A = \{x : x \in \mathbb{R}, 1 \leq x \leq 4\}$; $B = \{y : y \in W, y \leq 2\}$
   (b) $A = \{y : y \in \mathbb{R}, 0 \leq y \leq 2\}$; $B = \{x : x \in \mathbb{N}, x \leq 4\}$

3. Find the area of common region represented by following two relations.
   (a) $A \times B$; where $A = \{x : x \in \mathbb{R}, -1 \leq x \leq 1\}$; $B = \{y : y \in \mathbb{R}, -1 \leq y \leq 1\}$ and $|x| + |y| \leq 1$

4. Find out $n[(A \times B) \cap (B \times A)]$ iff
   $A = \{x : x \in$ non-negative integers and $x^2 - 2x - 8 \leq 0\}$ and $B = \{x : x \in \mathbb{N}$ and $x^2 - 5x + 4 \leq 0\}$

5. What will be the number of elements in $[(A \times B) \cap (B \times A)]$ if
   (i) Elements of set A are natural numbers and that of B in above problem are real numbers
   (ii) Elements of set A and B are real numbers.
   And how $A \times B$ and $B \times A$ can be geometrically represented and is it possible to represent the same in case elements are rationals or irrational numbers?

Answer Keys

1. (a) $A \times B = \{(1,0),(1,1), (1,2), (2,0),(2,1),(2,2), (3,0),(3,1),(3,2),(4,0),(4,1),(4,2)\}$ (b) 12
3. 2 Square units
4. 16
5. (i) 16
   (ii) Infinitely many found in square of area 9 square units bounded by straight lines $x = 1$, $x = 4$, $y = 1$ and $y = 4$. If elements of A and B are rationals or irrationals, it would not be possible to represent $A \times B$ and $B \times A$ geometrically as between any two real numbers there lies infinitely many rationals and irrationals.

TEXTUAL EXERCISE–1: (OBJECTIVE)

1. Which one of the following is not true?
   (a) $A \times (B - C) = (A \times B) - (A \times C)$
   (b) $A - (B \times C) = (A - B) \times (A - C)$
   (c) $A - (B \times C) = A$; A, B, C are subsets of U
   (d) $A \times (B \times C) \neq (A \times B) \times C$; A, B, C are subsets of U

2. Which of the following is/are not true?
   (a) $(A \cap B) \cap C = A \cap (B \cap C)$
   (b) $(A \cup B) \cup C = A \cup (B \cup C)$
   (c) $(A \times B) \times C = A \times B \times C = A \times (B \times C)$
   (d) $(A - B) - C = A - (B - C)$

3. Which of the following is/are not true?
   (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   (b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (c) $A \cup (B \cup C) = (A \cup B) \cup (A \cup C)$ and $A \cap (B \cap C) = (A \cap B) \cap (A \cap C)$
   (d) $A \times (B \times C) = (A \times B) \times (A \times C)$

4. Which of the following is/are not true?
   (a) If $A, B \subseteq U$ (universal set), then $A \times B$ is also a subset of U
   (b) If $A, B, \subseteq U$ (universal set), then $A \cup B$ and $A \cap B$ are subsets of U
   (c) If $A, B \subseteq U$ (universal set), then $A - B$ and $A \Delta B$ are subsets of U
   (d) If $A, B \subseteq U$, then $A \times B$ is a subset of $U \times U$

5. Which of the following is/are true?
   (a) If $A \subseteq B \subseteq C$, then $(A \times B) \subseteq (A \times C)$
   (b) If $A \subseteq B \subseteq C$, then $(A \times B) \subseteq (B \times C)$
   (c) If $A \subseteq B \subseteq C$, then $(A \times B) \cup (B \times C) = (B \times C)$
   (d) If $A \subseteq B \subseteq C$, then $(A \times B) \cap (A \times C) = (A \times B)$

6. Which of the following is/are true?
   (a) $A \times C \subseteq B \times C \Rightarrow A \subseteq B$
   (b) $A \times C \subseteq B \times C \Rightarrow A \subseteq B \subseteq C$

(c) $A \times B \subseteq B \times C \quad \Rightarrow A \subseteq B \subseteq C$
(d) $A \times B = B \times C \quad \Rightarrow A = B = C$

7. If $A = \{2, 3, 4, 5\}$; $B = \{x : x$ is a letter of English alphabet$\}$, then n($A \times B$) is
(a) 26 (b) 52
(c) 104 (d) $(4)^{26}$

8. If $A = \{2, 3, 4, 5\}$; $B = \{a, e, i, o, u\}$, then number of subsets of $A \times B$ is
(a) $(2)^{20}$ (b) $(4)^{10}$
(c) $(16)^5$ (d) None of these

9. If $A = \{\phi\}$; $B = \{\{\{\}\}\}$, then n($A \times B$) is
(a) 0 (b) 3
(c) 1 (d) Can't be defined

10. If $(A \times B)$ is an infinite set, then what can be said about A and B ?
(a) A and B both must be infinite sets
(b) A is finite and B is infinite
(c) A is infinite and B is finite
(d) At least one of A and B is an infinite set

11. Let $A = \{(x, y): x, y \in \mathbb{R}$ and $x^2 = 4y\}$ and $B = \{(x, 4); x \in \mathbb{R}\}$, then the set $(A \cap B) \cup \{(0,0)\}$ forms
(a) vertices of an equilateral triangle
(b) vertices of an isosceles triangle
(c) vertices of an square
(d) vertices of an scalene triangle

12. Let $A = \{(x, y); x, y \in \mathbb{R}$ and $x^2 = 4y\}$ and $B = \{(x, y), x\ y \in \mathbb{R}$ and $y = |x|\}$, then the area of triangle joining the points represented by $A \cap B$ is
(a) 16 (b) 4
(c) $4\sqrt{2}$ (d) None of these

13. If $A = \{\phi, \{\{\{\}\}\}\}$, then $n[P(P(A \times A))]$ is
(a) 2^{16} (b) $(16)^4$
(c) $(256)^2$ (d) None of these

14. If $n(A) = 50$; $n(B) = 60$; $n(A \cup B) = 100$, then $n[(A \times B) \cap (B \times A)]$ is
(a) 10 (b) 100
(c) $(50x\ 60)^2$ (d) None of these

15. If $n(A \cap B) = 2$; $n(B \cap C) = 3$; $n(C \cap A) = 5$; $n(A \cap B \cap C) = 1$, then $n[(A \times B \times C) \cap (B \times C \times A)]$ is
(a) 6 (b) 8
(c) 30 (d) None of these

Answer Keys

1. (b)	**2.** (c,d)	**3.** (d)	**4.** (a)	**5.** (a,b,c,d)	**6.** (a,c,d)	**7.** (c)	**8.** (a,b,c)	**9.** (c)	**10.** (d)
11. (b)	**12.** (a)	**13.** (a,b,c)	**14.** (b)	**15.** (c)					

BASIC DEFINITIONS

1. **Quantity:** Any thing on which mathematical operation such as addition, subtraction, multiplication, division can be performed is called a Quantity.
2. **Constant:** A constant is a quantity whose value does not change under different mathematical circumstances.

   Such a quantity retains the same value through out a mathematical operation.
3. **Absolute Constant:** When a quantity represents the same elements in all circumstances, then it is known as absolute constant. Such a constant retains the same value in every mathematical operation.

   Thus, 1, 2, 3, …, $\sqrt{2}$, $\sqrt{3}$, π, e, etc, are the absolute constant.
4. **Arbitrary Constant:** A constant which retains the same value throughout in one problem but may have different values for different problems is called arbitrary constant. It is usually denoted by a, b, c, …, l, m, n etc.

   For example in the equation of a straight line $y = mx + c$; m and c are arbitrary constant as they have same values for one line and other values for the other lines.
5. **Variable:** A quantity which varies, i.e., a quantity x, which assumes different values from a given set X, is called as a variable over the set X. For example in the equation: $y = mx + c$; $ax + by + c = 0$; $x^2 + y^2 = a$, x and y are variables. There are two types of variables as given below:

   (i) **Independent Variable:** *A* quantity (say x) which represents any arbitrary member belonging to a set, is called an independent variable this nomenclature is based upon the fact that the values of x can be arbitrarily chosen.

(ii) **Dependent Variable:** A quantity (say y) whose value depends upon some chosen values of an independent variable (x), is named as dependent variable.

For example, volume of sphere $V = (4/3)\,\pi r^3$ and surface area $S = 4\pi r^2$, here V and S are dependent variables and r is an independent variable.

6. **Domain of a Variable:** The set of all possible values of variable x is called the domain of the variable x.
7. **Continuous Variable:** Continuous variable is a variable that can take all the numerical values from one real number to another real number or between two given real numbers. For a continuous variable domain is an interval (open, closed or semi-open, semi-closed).
8. **Discrete Variable:** Quantities which are incapable of taking all possible values between two given numbers are called discrete or discontinuous variables.

RELATIONS

A relation R from set X to Y ($R : X \to Y$) is a correspondence between set X to set Y by which none, one or more elements of X are associated with none, one or more elements of Y. Therefore a relation (or binary relation) R, from a non-empty set X to another non-empty set Y, is a subset of $X \times Y$. i.e., $R : X \to Y$ is nothing but subset of $A \times B$. For example, consider a set X and Y as set of all males and females members respectively of a royal family of the kingdom Ayodhya.

Thus X = {Dashrath, Ram, Bharat, Laxman, Shatrughan} and Y = {Koshaliya, Kaikai, Sumitra, Sita, Mandavi, Urmila, Shrutkirti} and a relation R is defined as x was husband of y from set X to set Y.

Then R = {(Dashrath, Koshaliya), (Ram, Sita); (Bharat, Mandavi); (Laxman, Urmila); (Shatrughan, Shrutkirti); (Dashrath, Kaikai); (Dashrath, Sumitra)}

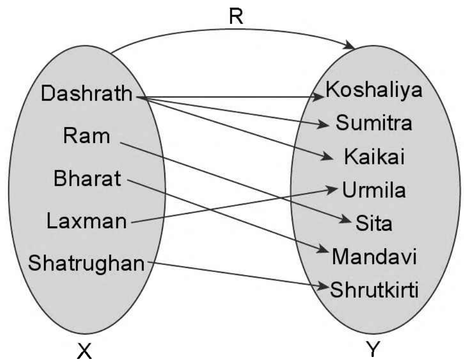


FIGURE 1.9

ILLUSTRATION 6: Are the following set of ordered pairs (correspondences) are relations from set A to set B?

(i) $R = \{(2, 4), (3, 6), (2, 6), (4, 8)\}$; $A = \{2, 3, 4\}$; $B = \{4, 6, 8\}$

(ii) If $A = \{a, b, c, d\}$ and $B = \{p, q, r, s\}$

(a) $R_1 = \{(a, p), (b, r), (c, s)\}$ (b) $R_2 = \{(q, b), (c, s), (d, r)\}$

(c) $R_3 = \{(a, p), (a, q), (d, p), (c, r), (b, r)\}$

SOLUTION: (i) R is a relation from A to B as every ordered pair of R is an element of $A \times B$.

(ii) (a) Clearly $R_1 \subseteq A \times B$. So, R_1 is a relation from A to B.

(b) Since $(q, b) \in R_2$ but $(q, b) \notin A \times B$. So $R_2 \nsubseteq A \times B$. Thus, R_2 is not a relation from A to B.

(c) Clearly $R_3 \subseteq A \times B$. So, R_3 is a relation from A to B.

Note: (i) If a is related to b, then symbolically it is written as $(a\,R\,b)$ or $(a, b) \in \mathbb{R}$; where a is pre-image and b is image.

(ii) If a is not related to b, then symbolically it is written as $a \not R\, b$ or $(a, b) \notin \mathbb{R}$.

ILLUSTRATION 7: A relation '~' from $\mathbb{C}$ to $\mathbb{R}$ is defined as $z \sim x$ iff $|z| = x$; $z \in \mathbb{C}$, $x \in \mathbb{R}$. Which of the following are correct?

(a) $(3 + 4i) \sim 5$ (b) $2 \sim -2$

(c) $1 + i \sim 2$ (d) $i \sim 1$

SOLUTION: As $|3 + 4i| = \sqrt{3^2 + 4^2} = 5$, therefore, $(3 + 4i) \sim 5$ $\Rightarrow$ (a) holds

Also, $|2| = 2 \neq -2$, therefore, $2 \sim -2$ is not true. $\Rightarrow$ (b) is incorrect.

and $|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2} \neq 2$, therefore, $1 + i \sim 2$ is not true. $\Rightarrow$ (c) is incorrect

Next $i \sim 1$ as $\sqrt{0^2 + 1^2} = 1$. $\Rightarrow$ (d) holds

DOMAIN, CO-DOMAIN AND RANGE OF RELATION

Domain: Domain of a relation R from set A to set B is the collection of elements of set A which are participating in the correspondence. That is, it is set of all pre-images under the relation R. For example, Domain of $R = \{(1, 5), (2, 10), (3, 6)\}$ is

$D_R = \{1,2,3\}$ where R is a relation from set $A = \{1,2,3,4\}$ to set $B = \{5,6,7,8,9,10\}$.

Co-Domain: Co-domain of a relation R from set A to set B is set B itself irrespective of the fact whether an element of set B is related with any element of A or not. For example, $B = \{5, 6, 7, 8, 9, 10\}$ is co-domain of above relation R.

Range: Range of a relation R from set A to set B is the set of those elements of set B which are participating in the correspondence, i.e., set of all images under the relation R. For the above relation range is given by the set $R_R = \{5,6,10\}$

ILLUSTRATION 8: Let R be the relation from the set of whole numbers W to $\mathbb{N}$ the set of natural numbers defined by

$R = \{(x, y)\} : x + 3y = 12, x \in W, y \in \mathbb{N}\}$

Find (i) R (ii) Domain of R

(iii) co-domain of R (iv) Range of R

SOLUTION: (i) We have, $x + 3y = 12$

$\Rightarrow x = 12 - 3y$

Putting $y = 1, 2, 3, 4$, we get $x = 9, 6, 3, 0$ respectively.

Also for $y > 4$, $x \notin W$

$\therefore R = \{ (9, 1), (6, 2), (3, 3), (0,4)\}$

(ii) Domain of $R = D_R = \{ 0, 3, 6, 9\}$

(iii) Co-domain of $R = \mathbb{N}$ (set of natural numbers)

(iv) Range of $R = R_R = \{1,2,3,4\}$.

ILLUSTRATION 9: If $A = \{2, 3, 4, 5\}$; $B = \{2, 3, 4, 6, 7\}$, then find the domain, co-domain and range of the relations from A to B defined below.

(a) $R_1 = \{(x, y) : y = x + 1\}$ (b) $R_2 = \{(x, y) : x + y > 6\}$

(c) $R_3 = \{(x, y) : y < x\}$ (d) $R_4 = \{(x, y) : x + y = 7\}$

SOLUTION: (a) $R_1 = \{(x, y) : y = x + 1\} = \{(2, 3), (3, 4), (5, 6)\}$

$\Rightarrow$ Domain of $R_1 = D_{R_1} = \{2,3,5\}$; Co-domain $= B$;

Range of $R_1 = R_{R_1} = \{3,4,6\}$

(b) $R_2 = \{(x, y) : x + y > 6\}$

$= \{(2, 6), (2, 7), (3, 4), (3, 6), (3, 7), (4, 3), (4, 4), (4, 6), (4, 7), (5, 2), (5, 3), (5, 4), (5, 6), (5, 7)\}$

$\Rightarrow$ Domain of $R_2 = D_{R_2} = \{2,3,4,5\} = A$; Co-domain = B;

Range of $R_2 = R_{R_2} = \{2,3,4,6,7\} = B$

(c) $R_3 = \{(x, y) : y < x\} = \{(3, 2), (4, 2), (4, 3), (5, 2), (5, 3), (5, 4)\}$

$\Rightarrow$ Domain of $R_3 = D_{R_3} = \{3,4,5\}$; Co-domain = B; Range of $R_3 = R_{R_3} = \{2,3,4\}$

(d) $R_4 = \{(x, y) : x + y = 7\} = \{(3, 4), (4, 3), (5, 2)\}$

$\Rightarrow$ Domain of $R_4 = D_{R_4} = \{3,4,5\}$; Co-domain = B; Range of $R_4 = R_{R_4} = \{2,3,4\}$

UNIVERSAL RELATION FROM SET *A* TO SET *B*

Since $A \times B$ contains all possible ordered pairs which relate each element of A to every element of B, therefore $(A \times B)$ is largest possible relation defined from set A to set B, and hence also known as Universal relation from A to B.

NUMBER OF RELATIONS FROM SET *A* TO SET *B*

Since each relation from A to B is a subset of Cartesian product $A \times B$, therefore number of relations that can be defined from set A to set B is equal to the number of subsets of $A \times B$. Thus number of relations from A to $B = 2^{n(A \times B)} = 2^{n(A) \times n(B)}$.

RELATION ON *A* SET

A relation R from a set A to itself is called relation on set A.

For example, let $A = \{1,2,3,4,9,16\}$. Define a relation from set A to itself as 'a R b, if b is square of a but $a \neq b$', then

$R = \{(2, 4), (3, 9),(4, 16)\}$. Here domain = $\{2, 3, 4\}$; co-domain = A; range = $\{4, 9, 16\}$.

ILLUSTRATION 10: Let R be the relation of $\mathbb{N}$ defined as $x\,R\,y$ iff $x + 2y = 8$. Find the domain and range of R.

SOLUTION: Domain of $R = \{ x \in \mathbb{N} : x \text{ R } y \text{ for some } y \in \mathbb{N}\}$.

Hence, $x = 8 - 2y, y \in \mathbb{N} = \begin{Bmatrix} 6 \text{ when } y = 1 \\ 4 \text{ when } y = 2 \\ 2 \text{ when } y = 3 \end{Bmatrix}$

Thus, $R = \{(2, 3), (4, 2), (6, 1)\}$.

$\Rightarrow D_R = \{2, 4, 6\}; R_R = \{1, 2, 3\}$

REPRESENTATION OF RELATION IN DIFFERENT FORMS

(i) **By representing the relation as a set of ordered pairs (Roster form):**
In this method we represent the relation by a set containing ordered pairs (a, b), where $a \in A$ and $b \in B$ such that $\boldsymbol{aRb}$ as shown below for the relation R from $A = \{1, 2, 3, 4\}$ to set $B = \{2, 3, 4, 5, 6, 7\}$ when $b \in B$ is to be related to $a \in A$ such that $b = 2a + 1$.
$R = \{(1,3), (2, 5), (3, 7)\}$.

(ii) **Analytical method or set builder from:** In this method we represent the relation as
$R = \{(a, b) : a \in A, b \in B; a\ldots.b\}$; where the dots are replaced by an equation connecting image b with its pre-image a. For example, let R be a relation from set $A = \{1, 2, 3, 4\}$ to set $B = \{2, 3, 4, 5, 6, 7\}$ given by $R = \{(1,3),(2,5),(3,7)\}$, then it can be represented by $R = \{(x, y) : x \in A, y \in \text{B}; x\,R\,y \text{ iff } y = 2x + 1\}$.

(iii) **Graphical representation or representation by lattice:** In this method we take set X along x-axis and set B along y-axis, then plot the points $(a, b) \in$

$\mathbb{R}$ in x-y plane. For example, in above illustration the relation can be represented as shown in the diagram given below.

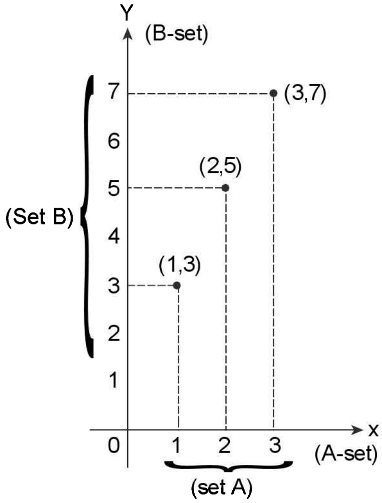


FIGURE 1.10

(iv) **By arrow Diagram:** In this method, we represent the set A and set B by two circles or by two ellipses and join the images and their pre-images by using arrows as shown below for above illustration.

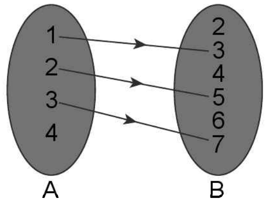


FIGURE 1.11

(v) **Tabular form:** In this form of representation of a relation R from set A to set B, elements of A and B are written in the first column and first row respectively. If $(a, b) \in \mathbb{R}$, then we write '1' in the row containing a and the column containing b and if $(a, b) \notin \mathbb{R}$, then we write '0' in the row containing a and the column containing b.

For example, for the relation $R = \{(1, 3), (2, 5), (3, 7)\}$ from set $A = \{1, 2, 3, 4\}$ to set $B = \{2, 3, 4, 5, 6, 7\}$ we have following tabular representation.

R	2	3	4	5	6	7
1	0	1	0	0	0	0
2	0	0	0	1	0	0
3	0	0	0	0	0	1
4	0	0	0	0	0	0

ILLUSTRATION 11: Given $A = \{1, 2, 3, 4\}$ and $B = \{x : x \in \mathbb{Z}$ and $0 < |x| \le 4\}$, then represent the relations R_1, R_2, R_3 defined from A to B by using ordered pairs and by arrow diagrams.

(a) $y \, R_1 \, x$ iff $y = x + 1$ (b) $y \, R_2 \, x$ iff $y = 2x$

(c) $y \, R_3 \, x$ iff $y^2 = x$

SOLUTION: $A = \{1, 2, 3, 4\}$ and $B = \{x : x \in \mathbb{Z}$ and $0 < |x| \le 4\}$

$= \{x : x \in \mathbb{Z}; -4 \le x \le 4; x \ne 0\}$

$= \{-4, -3, -2, -1, 1, 2, 3, 4\}$

(a) Ordered pair representation of $R_1 = \{(1, 2), (2, 3), (3, 4)\}$

Arrow diagram representation of R_1

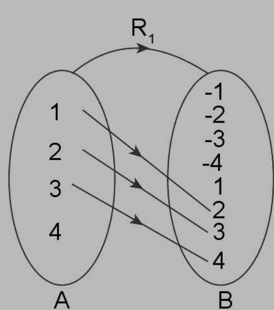


FIGURE 1.12

(b) Ordered pair representation of $R_2 = \{(1, 2), (2, 4)\}$

Arrow diagram representation of R_2

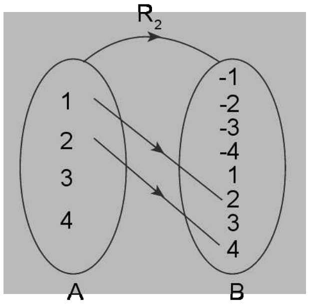


FIGURE 1.13

(c) Ordered pair representation of $R_3 = \{(1, 1), (1, -1), (4, 2), (4, -2)\}$

Arrow diagram representation of R_3

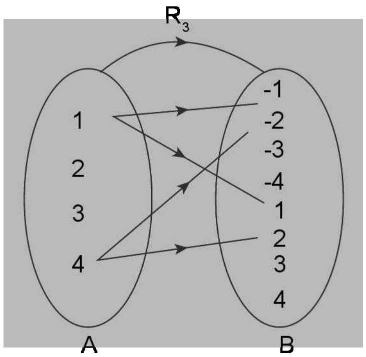


FIGURE 1.14

ILLUSTRATION 12: The relations R_1 and R_2 are defined from $\mathbb{R}$ to $\mathbb{R}$ as given below:

$R_1 = \{(x, y) : |x - 3| \le 1, |y - 3| \le 1\}$ and

$R_2 = \{(x, y) : 4x^2 + 9y^2 - 32x - 54y + 109 \le 0\}$. Show that $R_1 \subset R_2$.

SOLUTION: $|x - 3| \le 1$

$\Rightarrow -1 \le x - 3 \le 1$

$\Rightarrow 2 \le x \le 4$ and $|y - 3| \le 1$

$\Rightarrow 2 \le y \le 4$

Therefore, the relation R_1 consists of the points, on and inside the square bounded by the lines

$x = 2, x = 4, y = 2$ and $y = 4$.

Now for relation R_2.

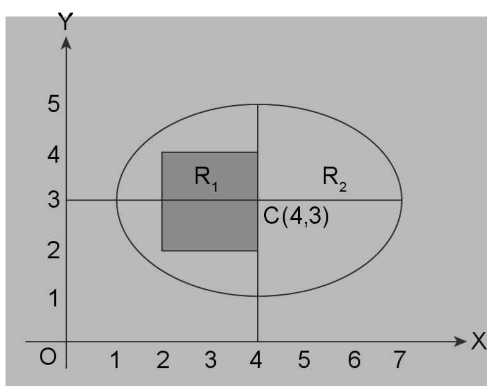


FIGURE 1.15

Also $4x^2 + 9y^2 - 32x - 54y + 109 \le 0$

$\Rightarrow 4(x^2 - 8x + 16) + 9(y^2 - 6y + 9) \le 36$

$$\Rightarrow \quad \frac{(x-4)^2}{9}+\frac{(y-3)^2}{4}\le 1$$

$\Rightarrow$ The set R_2 consists of all the points inside or on the ellipse with centre (4, 3) and semi-major and semi-miner axes of lengths three and two units respectively as shown in the above Figure 1.15. From above Figure 1.15 it is clear that $R_1 \subseteq R_2$

■ CLASSIFICATION OF RELATIONS

One–One or Injective Relation

If different elements of set X are related with different elements of set Y. That is,, no two different elements of domain are related to same element of set Y, then R is said to be one-one relation or injective relation from set X to set Y.

For example, R_1: $\{(x_1, y_1), (x_2, y_2)\}$.

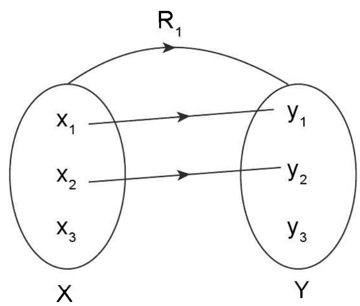


FIGURE 1.16

Many–One Relation

When there exists at least one group having more than one element of set X which are related with same element of set Y, then R is said to be many–one relation from set X to set Y.

For example, R_2 : $\{(x_1, y_1), (x_2, y_1), (x_3, y_3)\}$.

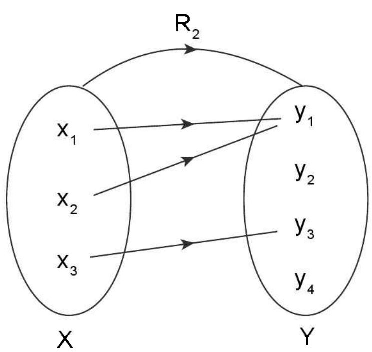


FIGURE 1.17

One–Many Relation

Relation R from set X to set Y is said to be one–many if there exists an element in set X which is related with more than one element of set Y.

For example, R_3 : $\{(x_1, y_1), (x_1, y_2), (x_2, y_3)\}$

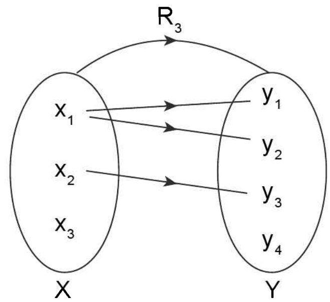


FIGURE 1.18

Many–Many Relation

Relation R from set X to set Y is said to be many–many if it is many–one as well as one–many.

For example, R_4: $\{(x_1, y_1), (x_1, y_2), (x_2, y_3), (x_3, y_3)$.

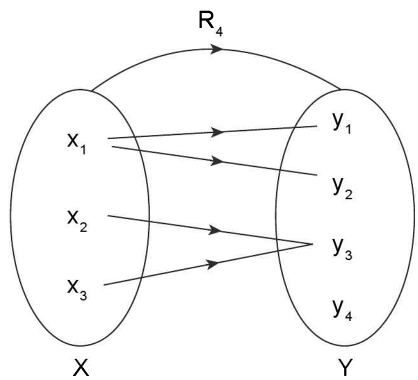


FIGURE 1.19

Onto Relation (Surjective Relation)

A relation $R : X \to Y$ is said to be onto or surjective relation if there is no such element $y \in Y$ which is not related with any $x \in X$, that is, for each $y \in Y$ there exist at least one element x in X which is related with y. In such a relation,

Range (R_R) = Co-Domain, That is, range of onto relation is nothing but the co-domain of the relation.

For example, R_5: $\{(x_1, y_1), (x_1, y_2), (x_2, y_3), (x_2, y_4)\}$

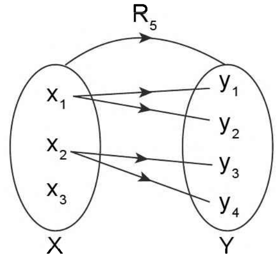


FIGURE 1.20

REMARK:

In onto relation all elements of set X may or may not participate in relation but all elements of co-domain set Y participate in relation.

INTO RELATION

A relation $R : X \to Y$ is said to be into iff there exist at least one $y \in Y$ which is not related with any $x \in X$. That is, if Range (R_R) $\subset$ Co-Domain, that is, range of relation is a proper subset of co-domain. For example,

$$R_6: \{(x_1, y_1), (x_1, y_2), (x_2, y_3)\}$$

Clearly under relation R_6, y_4 has no pre-image in X.

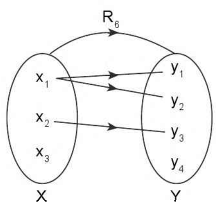


FIGURE 1.21

One–One–Onto Relation (Bijective Relation)

A relation $R : X \to Y$ is said to be bijective relation iff it is both one–one as well as onto.

For example,

R_7: $\{(x_1, y_2), (x_2, y_1), (x_3, y_3),$

where $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3)\}$

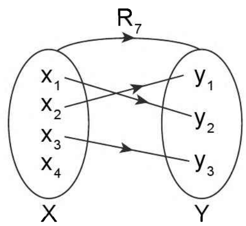


FIGURE 1.22

ILLUSTRATION 13: Find the domain and range of following relations

(a) R_i from set X to set Y ; where X = {Dashrath, Ram, Bharat, Laxman, Shatrughan} and Y = {Koshaliya, Kaikai, Sumitra, Sita, Mandavi, Urmila, Shrutkirti} and classify them as many–one/one–one/one–many/many–many as well as onto/into relations.

(i) R_1 : {(Dashrath, Koshaliya), (Dashrath, Sumitra), (Ram, Sita), (Laxman, Urmila)}

(ii) R_2: {(Ram, Sita),(Laxman, Urmila), (Bharat, Mandavi)}

(b) $R : \mathbb{R} \to \mathbb{R}$ defined as

(i) $R_1 = \{(x, y) : x^2 = y^2\}$

(ii) $R_2 = \{(x, y): y \geq 0, y \leq x, x + y \leq 1\}$

(iii) $R_3 = \{(x, y) : x^2 + y^2 \leq 1 ; x \geq 0\}$

(iv) $R_4 = \{(x, y) : |y| \leq |x| \leq 1\}$

(v) $R_5 = \{(x, y) : |y| = \sin x\}$

SOLUTION: (a) (i) For R_1: Domain $= D_{R_1}$ = {Dashrath, Ram, Laxman}; Range $= R_{R_1}$ = {Koshaliya, Sumitra, Sita, Urmila}. Clearly R_1 is one–many and into.

(ii) For R_2: Domain = D_{R_2} = {Ram, Laxman, Bharat}; Range = R_{R_2} = {Sita, Urmila, Mandavi}. Clearly R_2 is one-one and into.

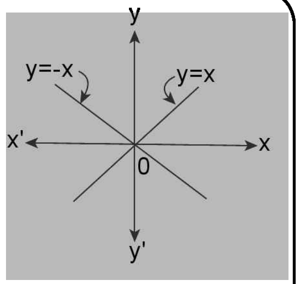


FIGURE 1.23

(b) (i) $R_1 = \{(x, y) : x^2 = y^2\}$

Now, $y^2 = x^2$

$\Rightarrow$ $y = \pm x$

$\Rightarrow$ the given relation contains a pair of straight lines given by $y = x$ and $y = -x$.

That is, $R = \{(x, y) : y = x \text{ or } y = -x\}$

Therefore, domain and range of relation R are $\mathbb{R}$ (set of real numbers) and the relation is many–many and onto.

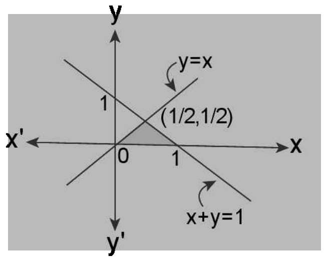


FIGURE 1.24

(ii) $R_2 = \{(x, y) : y \geq 0, y \leq x, x + y \leq 1\}$;

R_2 represents the shaded region bounded by x-axis, the straight line $y = x$ and the straight line $x + y = 1$ as shown in Figure 1.24 given.

Therefore, Domain of $R_2 = [0, 1]$; Range = $[0, 1/2]$.

Clearly R_2 is many–many and into.

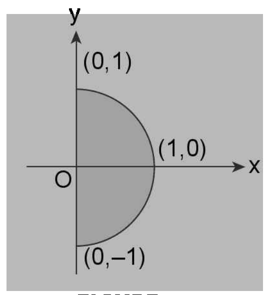


FIGURE 1.25

(iii) $R_3 = \{(x, y) : x^2 + y^2 \leq 1; x \geq 0\}$; R_3 represents the region of semi circle (interior and semi circular arc) of circle $x^2 + y^2 = 1$ on right side of y-axis as shown below.

Therefore, Domain of $R_3 = [0, 1]$; Range = $[-1, 1]$; many–many and into

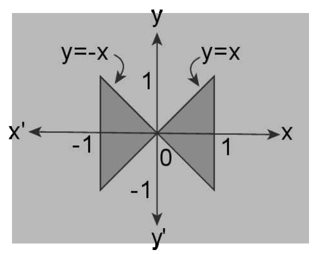


FIGURE 1.26

(iv) $R_4 = \{(x, y) : |y| \leq |x| \leq 1\}$; R_4 represents the area bounded by straight lines $y = \pm x$ and $x = \pm 1$ as shown below

Therefore, Domain $R_4 = [-1, 1]$; Range = $[-1, 1]$; many–many and into

(v) $R_5 = \{(x, y) : |y| = \sin x\}$; R_5 represents the positive portion of sine curve along with its reflection on x-axis as shown below.

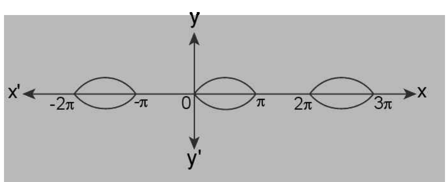


FIGURE 1.27

Therefore, Domain of $R_5 = \bigcup_{n \in \mathbb{Z}} [2n\pi, (2n+1)\pi]$; Range of $R_5 = [-1, 1]$;

The relation R_5 is many–many and into.

ILLUSTRATION 14: Test for injectivity and surjectivity the following relations

(a) $y^2 = 2x - 4$ (b) $x^2 = 2y + 1$

(c) $y = |x| + 2$ (d) $y = \sqrt{(x-2)(x-4)}$

(e) $y^2 = x^2 - 4$ (f) $xy = 4$

(g) $y = x + \frac{1}{x}$ (h) $|y| = |\tan x|$

SOLUTION: (a) $y^2 = 2x - 4 = 2(x - 2)$. The above equation represents a right handed parabola with its vertex at (2, 0) as jhshown below.

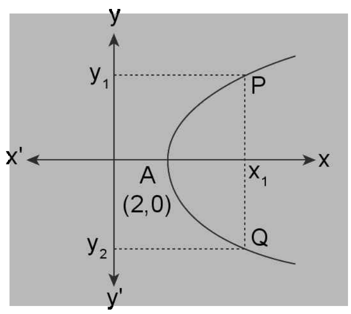


FIGURE 1.28

Clearly we have infinitely many pairs of points on the curves having same abscissa in $(2, \infty)$, one such pair P and Q having co-ordinates (x_1, y_1) and (x_1, y_2) is shown in above diagram.

Thus the given equation represents one–many relation, that is, not an injective relation (one–one). Domain of above relation is $[2, \infty)$ and range is $(-\infty, \infty)$. Thus the relation is surjective (onto) from $\mathbb{R} \to \mathbb{R}$.

(b) $x^2 = 2y + 1 = 2\left(y + \frac{1}{2}\right)$.

The above equation represents an upwards parabola with its vertex at $\left(0, -\frac{1}{2}\right)$ as shown below.

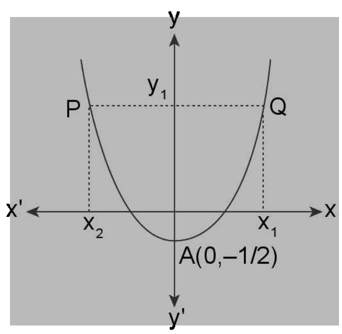


FIGURE 1.29

Clearly for every input x, output $y = \frac{1}{2}(x^2 - 1)$ is unique however we get infinitely many pairs of points on the curve having same ordinate but different abscissae one such pair of points P and Q is shown on the graph having their co-ordinates (x_1, y_1) and (x_2, y_1) respectively.

Thus the given equation represents many–one relation and hence not an injective relation. Clearly domain of relation is $\mathbb{R}$ and range of relation is $\left[-\frac{1}{2}, \infty\right)$. Thus the relation from $\mathbb{R} \to \mathbb{R}$ is not surjective (onto).

However the relation from $\mathbb{R}$ to $\left[-\frac{1}{2}, \infty\right)$ is surjective.

(c) $y = |x| + 2$

The graph representing the above relation is as shown below

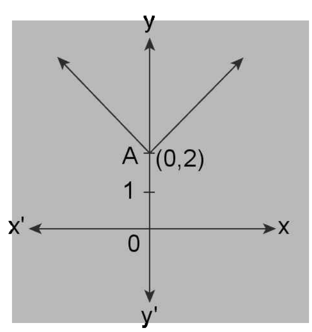


FIGURE 1.30

Clearly the relation is many–one and hence not injective. Also domain of relation is $(-\infty, \infty)$ and range $(2, \infty)$. Thus, the given relation is not surjective from $\mathbb{R}$ to $\mathbb{R}$. But it is surjective from $\mathbb{R}$ to $[2, \infty)$.

(d) $y = \sqrt{(x-2)(x-4)}$; Domain of relation = $(-\infty, 2] \cup [4, \infty)$ and range = $[0, \infty)$

Thus, the relation is not surjective from $\mathbb{R}$ to $\mathbb{R}$. But it is so from $\mathbb{R}$ to $[0, \infty)$.

Also $y = 0$ at $x = 2$ and at $x = 4$

$\Rightarrow$ the relation is many–one and hence not injective.

(e) $y^2 = x^2 - 4$

$\Rightarrow$ $x^2 - y^2 = 4$

$\Rightarrow$ $\dfrac{x^2}{4} - \dfrac{y^2}{4} = 1$

which represents a rectangular hyperbola as shown below.

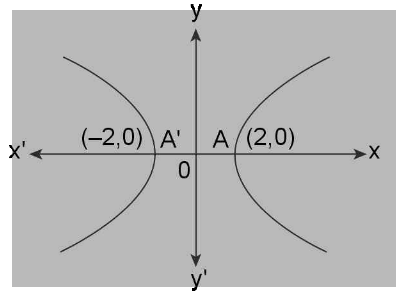


FIGURE 1.31

Clearly the relation is one–many as well as many–one with domain $(-\infty, -2] \cup [2, \infty)$ and range = $(-\infty, \infty)$.

Thus, the relation is not injective but it is surjective from $\mathbb{R} \to \mathbb{R}$.

(f) $xy = 4$

$\Rightarrow$ $y = 4/x$

The graph of above relation is as shown below:

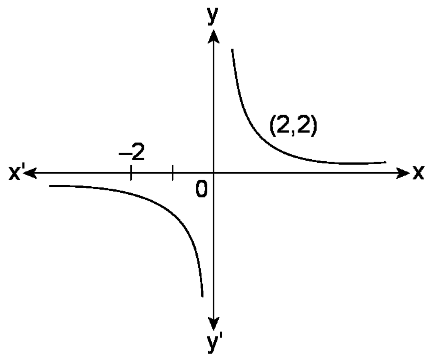


FIGURE 1.32

Clearly the given equation represents one-one relation, i.e., injective with domain $\mathbb{R} \sim \{0\}$ and range $\mathbb{R} \sim \{0\}$.

Thus the given relation is not surjective from $\mathbb{R}$ to $\mathbb{R}$, but it is so from $\mathbb{R} \sim \{0\}$ to $\mathbb{R} \sim \{0\}$.

(g) $y = x + \dfrac{1}{x}$.

The above equation represents the graph as shown below

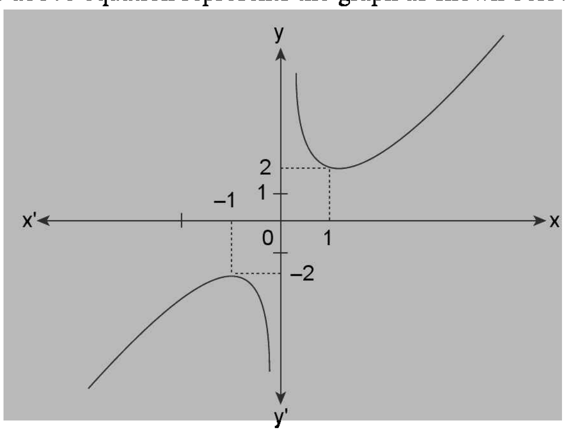


FIGURE 1.33

Clearly the given relation is many–one and hence not an injective relation with domain = $\mathbb{R} \sim \{0\}$ and range = $\mathbb{R} \sim (-2, 2)$.

Thus, the relation from $\mathbb{R}$ to $\mathbb{R}$ is not surjective.

(h) $|y| = |\tan x|$

Let $|\tan x| = g(x)$

$\Rightarrow$ $|y| = g(x)$

The graph of $g(x)$ is as shown below

Now $|y| = g(x)$

$\Rightarrow$ $y = \pm\, g(x)$

Thus, graph of $|y| = g(x)$ can be obtained by reflecting the graph of $g(x)$ on x-axis and also considering the graph of $g(x)$ along with as shown below.

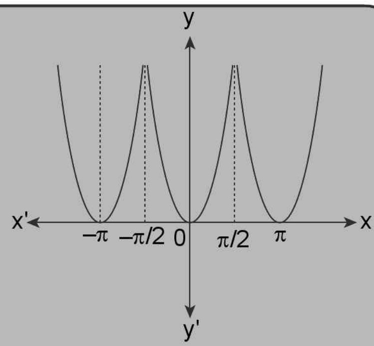


FIGURE 1.34

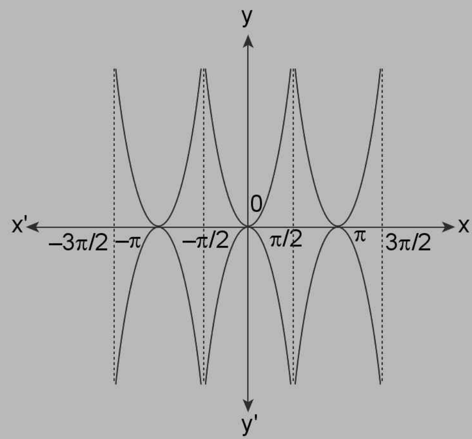


FIGURE 1.35

Graph of $|y| = |\tan x|$

Clearly the given relation represents one–many as well as many–one relation. Domain of relation is $\mathbb{R} \sim \left\{(2n+1)\dfrac{\pi}{2}; n \in \mathbb{Z}\right\}$ and range of relation is $(-\infty, \infty)$. Thus the relation is not injective but it is surjective from $\mathbb{R}$ to $\mathbb{R}$.

TEXTUAL EXERCISE–2: (SUBJECTIVE)

1. Let $A = \{1, 2, 3, 4, 6\}$. Let R is the relation on A defined by $\{(a, b): a, b \in A, b$ is divisible by $a\}$
 (i) Write R in roster form
 (ii) Find the domain of R
 (iii) Find the range of R.

2. Let $R = \{(x, y) : x, y \in \mathbb{Z}; (x + y)(y + 2004) + 1 = 0\}$.
 (i) Write R in roster form
 (ii) Find the domain of R
 (iii) Find the range of R.

3. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3, \ldots, 66, 67\}$. If R be a relation from the set A to the set B defined by
 (i) is square root of (ii) is cube root of
 Find R and also its domain and range.

4. Determine the domain and range of the following relations on $\mathbb{N}$.
 (i) $R = \{(x, x^3) : x$ is a prime number less than $10\}$
 (ii) $R = \{(4x + 3, 1 - x) : x \le 4; x \in \mathbb{N}\}$
 (iii) $R = \left\{\left(x, \dfrac{1}{x}\right) : 0 < x < 4 \text{ and } x \text{ is natural number}\right\}$

5. Let $R = \{(x, y) : x, y \in W; y = 2x - 4\}$. If $(a, -2) \in R$ and $(4, b^2) \in R$, then find the relation $R_1 = \{(a, b)\}$.

6. Find a linear relation between the components of the ordered pairs of relation R given by
$R = \{(0, 2), (-1, 5), (2, -4), ...\}$

7. Let $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3, 4, 5\}$ and R be a relation from A to B defined by $a\ R\ b$ iff $a|b$
(i) represent R by lattice
(ii) represent R in tabular form

8. Let $A = \{1, 2, 3\}$, $B = \{4, 5\}$ and R be a relation from A into B given by $R = \{(2, 4), (2, 5), (3, 5)\}$. Represent R:
(i) in tabular form (ii) by arrow diagram

9. Figure 1.36 shows a relation R from set A to set B. Write this relation in
(i) set builder form (ii) roster form

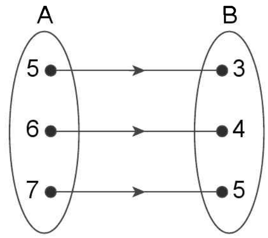


FIGURE 1.36

10. Test the given relations for injectivity and surjectivity.
(a) $y^2 = 2x - 3$ (b) $y = x^2 + 4$
(c) $y = ax^2 + bx + c$ (d) $y^3 = 5x + 4$
(e) $y = \sqrt{3x-4}$

Answer Keys

1. $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (6, 6)\}$; $D_R = R_R = A$
2. $R = \{(2002, -2003), (2006, -2005)\}$, $D_R = \{2002, 2006\}$, $R_R = \{-2003, -2005\}$.
3. (i) $R = \{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25)\}$; $D_R = A$; $R_R = \{1, 4, 9, 16, 25\}$
(ii) $R = \{(1, 1), (2, 8), (3, 27), (4, 64)\}$; $D_R = \{1, 2, 3, 4\}$; $R_R = \{1, 8, 27, 64\}$
4. (i) $D_R = \{2, 3, 5, 7\}$; $R_R = \{8, 27, 125, 343\}$ (ii) $D_R = \{\ \}$; $R_R = \{\ \}$
(iii) $D_R = \{\}$; $R_R = \{\}$
5. $R_1 = \{(1, 2), (1, -2)\}$ **6.** $3x + y = 2$
9. (i) $R = \{(x, y) : y = x - 2 \text{ and } x \in \{5, 6, 7\}\}$ (ii) $R = \{(5, 3), (6, 4), (7, 5)\}$
10. (a) injective, surjective (b) neither injective nor surjective, many–one
(c) neither injective nor surjective, many–one (d) injective, surjective
(e) injective but not surjective.

TEXTUAL EXERCISE–2: (OBJECTIVE)

1. Let $A = \{2, 3, 4, 5\}$, then how many relations can be defined on set A?
(a) $(16)^2$ (b) $(2)^{16}$
(c) $(16)^4$ (d) $(4)^{16}$

2. Let $A = \{a, e, i, o, u\}$, then how many relations can be defined on power set of A?
(a) $(2)^{10}$ (b) $(2^{10})^2$
(c) $(2)^{2^{10}}$ (d) None of these

3. Let $A = \{1, 2, 3\}$ and $B = \{a, b, c, d\}$; then the number of relations that can defined from set A to set B is
(a) $(2)^{12}$ (b) $(12)^2$
(c) $(2)^7$ (d) None of these

4. If $n(A) = m$ and $n(B) = n$ and $m > n$. Let k = number of relations defined from set A to set B and p = number of relations defined from set B to set A, then
(a) $k > p$ (b) $k < p$
(c) $k = p$ (d) Can't be predicted surely

5. Define a relation R on set $A = \{2, 3, 5, 6, 10\}$ as 'xRy if '$x < y$ and x divides y', then the domain of relation is
(a) $\{2, 3, 5, 6, 10\}$ (b) $\{2, 3, 5\}$
(c) $\{2, 3, 5, 6\}$ (d) None of these

6. Define a relation R on set of natural numbers defined as xRy if '$3x + 5y = 53$', then the range set of relation is
   (a) $\{1, 4\}$ (b) $\{1, 4, 8\}$
   (c) $\{1, 4, 7, 10\}$ (d) None of these

7. Define relations R_1 and R_2 on set $A = \{2, 3, 5, 7, 10\}$ as xR_1y if $x|(y - 1)$ and xR_2y if $x + y = 10$, then the relation R given by $R = R_1 \cap R_2$ is
   (a) $\{\ \}$ (b) $\{(3, 7)\}$
   (c) $\{(3, 7), (5, 5)\}$ (d) None of these

8. Define a relation R on set of natural numbers as $x\,Ry$ if '$x + y$ divides 10', then the relation is
   (a) One–one (b) Many–one
   (c) One–many (d) None of these

9. Define a relation R on set $A = \{2, 7, 9, 11\}$ defined as $x\,R\,y$ if 'x divides y' then the relation R is
   (a) Many–one
   (b) One–many
   (c) One–one
   (d) A function from set A to set A

10. For a relation to be a function from set A to set B, it should be
   (a) One–one
   (b) One–many
   (c) Many–one
   (d) Domain of relation should be A

11. If $A = \{x : x^2 - 3x + 2 = 0; x \in \mathbb{R}\}$, and R is a universal relation on A, then R is;
   (a) $\{(1,1),(2,2)\}$ (b) $\{(1,1)\}$
   (c) $\{\ \}$ (d) $\{(1,1),(1,2),(2,1),(2,2)\}$

12. If $A = \{1,2,3\}$ and $R_1 = \{(1,2),(3,2),(1,3)\}$; $R_2 = \{(1,3), (3,6), (2,1), (1,2)\}$, then
   (a) R_1 is a relation and R_2 is not on A
   (b) R_1 and R_2 both are relations on A
   (c) R_1 and R_2 both are not relations on A
   (d) None of these

13. Let $A = \{a, b, c, d\}$, $B = \{b, c, d, e\}$. Then $n[(A \times B) \cap (B \times A)]$ is equal to
   (a) 3 (b) 6
   (c) 9 (d) None of these

Answer Keys

1. (b,c) **2.** (c) **3.** (a) **4.** (c) **5.** (b) **6.** (c) **7.** (b) **8.** (b,c) **9.** (c,d)
10. (a,c,d) **11.** (d) **12.** (a) **13.** (c)

TYPES OF RELATIONS

Reflexive Relation

$R : X \to Y$ is said to be reflexive iff $x\,R\,x\ \forall\ x \in X$. That is, every element of X, must be related to itself.

Therefore if for each $x \in X$; $(x, x) \in R$, then relation R is called reflexive relation. For a relation to be reflexive from set X to set Y, X must be a subset of Y.

REMARK:

If $R : X \to Y$ is a reflexive relation, then its domain is X. For example, if R is a relation on set of integers ($\mathbb{Z}$) defined by 'xRy iff x divides y', then it is reflexive and hence its domain set is $\mathbb{Z}$.

ILLUSTRATION 15: If $X = \{x_1, x_2, x_3\}$ and $Y = \{x_1, x_2, x_3, x_4, x_5\}$, then find which of the following is/are reflexive relation.

(a) R_1: $\{(x_1, x_1), (x_2, x_2)\}$
(b) R_2: $\{(x_1, x_1), (x_2, x_2), (x_3, x_3)\}$
(c) R_3: $\{(x_1, x_1), (x_2, x_2), (x_3, x_3), (x_1, x_3), (x_2, x_4)\}$
(d) R_4: $\{(x_1, x_1), (x_2, x_2), (x_3, x_4)\}$

SOLUTION: (a) Not reflexive as (x3, x3) $\notin$ R1 (b) Reflexive
(c) Reflexive (d) Non-reflexive because $(x_3, x_3) \notin R_4$.

Identity Relation

A relation $R : X \to Y$ is said to be an identity relation if each element of X is related to it self only. For example, if $X = \{x_1, x_2, x_3\}$ and $Y = \{x_1, x_2, x_3, x_4\}$, then the relation $R = \{(x_1, x_1), (x_2, x_2), (x_3, x_3)\}$ is an identity relation from set X to set Y.

REMARKS

1. *Every identity relation from set X to set Y is reflexive relation from set X to set Y, but converse is not true. that is, every reflexive relation need not be identity. For example, $R : X \to Y$; where $X = \{x_1, x_2, x_3\}$ and $Y = \{x_1, x_2, x_3, x_4\}$, then the relation $R = \{(x_1, x_1), (x_2, x_2), (x_3, x_3), (x_1, x_2)\}$ is reflexive but not identity relation from set X to set Y because $x_1 R x_1$ as well as $x_1 R x_2$.*
2. *If R is a relation from set X to itself, then the relation is called relation on set X.*
   (a) *R is said to be reflexive on set X if $xRx \ \forall \ x \in X$*
   (b) *R is said to be identity relation on set X if $xRx \ \forall \ x \in X$ and x is not related to any other element and it is denoted by I_x.*
3. ***Symmetric Relation:*** *$R : X \to Y$ is said to be symmetric iff $(x, y) \in R \Rightarrow (y, x) \in R$*
   That is, *$x R y \Rightarrow y R x$. For example, perpendicularity of lines in a plane is symmetric relation.*

ILLUSTRATION 16: If $X = \{a, b, c\}$ and $Y = \{a, b, c, d, e, f\}$; then find which of the following relations is/are symmetric relation (s).
(i) $R_1 = \{ \}$ that is, void relation (ii) $R_2 = \{(a, b)\}$
(iii) $R_3 = \{(a, b), (b, a), (a, c), (c, a), (a, a)\}$

SOLUTION: (i) R_1 is symmetric relation by default i.e., we have no counter example such that $(a, b) \in R_1$ but $(b, a) \notin R_1$.
(ii) R_2 is not symmetric because $(a, b) \in R_2$ but $(b, a) \notin R_2$ and
(iii) R_3 is symmetric as $(a, b), (a, c), (a, a) \in R$ and $(b, a), (c, a), (a, a) \in R$

Transitive Relation

$R : X \to Y$ is said to be transitive iff $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$
i.e., $x R y$ and $yR z$
$\Rightarrow x R z$.

For example, the relation 'being sister of' among the members of a family is always transitive.

ILLUSTRATION 17: If $X = \{a, b, c\}$ and $Y = \{a, b, c, d, e\}$; then which of the following is/are transitive relation(s).
(a) $R_1 = \{ \}$ (b) $R_2 = \{(a, a)\}$
(c) $R_3 = \{(a, a), (c, d)\}$ (d) $R_4 = \{(a, b), (b, a), (a, c), (a, a), (c, a)\}$

SOLUTION: (a) R_1 is transitive relation because it is null relation, i.e., we have no counter example to disprove its non-transitivity.
(b) R_2 is transitive relation because all singleton relations are transitive.

(c) R_3 is transitive relation as we have no counter example to prove its non-transitivity, that is, R_3 does not contain ordered pairs of the type (x, y) and (y, z) for which (x, z) is not contained in R_3.

(d) R_4 is not transitive relation as (b, a); $(a, c) \in R_4$ but $(b, c) \notin R_4$.

NOTE:

(*i*) *Every null relation is a transitive relation.*

(*ii*) *Every singleton relation is a transitive relation.*

(*iii*) *Universal and identity relations are reflexive as well as transitive.*

Anti-symmetric Relation

A relation R from set X to set Y is said to be an anti-symmetric relation iff $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ i.e., for two different elements $x \in X$ and $y \in Y$, the relation R does not contain the ordered pairs (x, y) and (y, x) simultaneously.

e.g., Relations 'being subset of'; 'is greater than or equal to' and 'identity relation' are anti-symmetric relations.

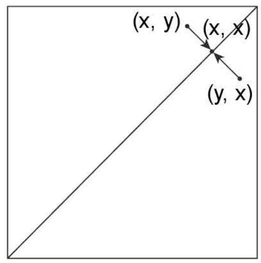


FIGURE 1.37

REMARK:

A relation R from set X to set Y may be both symmetric as well anti-symmetric, any one or not both e.g. let X = {1, 2, 3, 4} and Y = {1, 2, 3, 4, 5, 6}

R:AA, where n(A) = m

Number of reflexive relation $= 2^{m^2-m}$

Number of symmetric relation $= 2^{\frac{m}{2}+m} = 2^{\frac{m^2+m}{2}}$

From m × m matrix choose all diagonal element and rest element have two choices each. Each elements on diagonal and above have two choices, then their symmetric pair is always to be taken.

Consider the relations:

(i) $R_1 = \{(1,1), (2,2)\}$

(ii) $R_2 = \{(1, 2), (2, 1), (2, 3), (3, 2)\}$

(iii) $R_3 = \{(1, 1), (2, 2), (3, 4)\}$

(iv) $R_4 = \{(1, 2), (2, 1), (3, 4)\}$

(i) R_1 is symmetric as whenever ordered pair $(x, y) \in R_1$ $\Rightarrow (y, x) \in R_1$.
Also R_1 is anti-symmetric as for no two different elements x, y the ordered pairs (x, y) and (y, x) occur in R_1.

(ii) R_2 is symmetric but not anti-symmetric as $(1, 2)$, $(2, 3) \in R_2$
$\Rightarrow$ $(2, 1), (3, 2) \in R_2$ but $1 \neq 2$ and $2 \neq 3$.

(iii) R_3 is anti-symmetric but not symmetric as $(3, 4) \in R_3$ but $(4, 3) \notin R_3$.

(iv) R_4 is neither symmetric nor anti-symmetric as $(3, 4) \in R_4$ but $(4, 3) \notin R_4$ and $(1, 2), (2, 1)$ both are in R_4 but $1 \neq 2$.

ILLUSTRATION 18: Let R be a relation on the set $\mathbb{N}$ of natural numbers defined by $xRy \Leftrightarrow$ 'x divides y' for all $x, y \in \mathbb{N}$.

SOLUTION: This relation is an anti-symmetric relation on set $\mathbb{N}$.

Since for any two numbers $a, b \in \mathbb{N}$, $.a \mid b$ and $b \mid a \Rightarrow a = b$

i.e., $a R b$ and $b R a$

$\Rightarrow$ $a = b$.

It should be noted that this relation is not anti-symmetric on the set $\mathbb{Z}$ of integers, because we find that for any non-zero integer a, $aR(-a)$ and $(-a) R a$. but $a \neq -a$.

ILLUSTRATION 19: Let a relation R_1 on the set $\mathbb{R}$ of real numbers be defined as $(a, b) \in R_1 \Leftrightarrow 1 + ab > 0$ $\forall\, a, b \in \mathbb{R}$.

Show that R_1 is reflexive and symmetric but not transitive.

SOLUTION: We observe the following properties:

Reflexivity: Let a be an arbitrary element of $\mathbb{R}$. Now $1 + a \,.\, a = 1 + a^2 > 0$

$\Rightarrow$ $(a, a) \in R_1$ Thus $(a, a) \in R_1$ for all $a \in \mathbb{R}$. So R_1 is reflexive on $\mathbb{R}$

Symmetry: Let $(a, b) \in R_1$ $\Rightarrow$ $1 + ab > 0$

$\Rightarrow$ $1 + ba > 0$ $\Rightarrow$ $(b, a) \in R_1$.

$\therefore$ R_1 is symmetric relation on $\mathbb{R}$

Transitivity: Let $a = 4$, $b = -1/5$, $c = -1$, then $1 + ab = 1 + 4(-1/5) = 1/5$

$\Rightarrow$ $1 + ab > 0$ $\Rightarrow$ $(a, b) \in R_1$.

Also $1 + bc = 1 + (-1/5)(-1) = 6/5$

$\Rightarrow$ $1 + bc > 0$ $\Rightarrow$ $(b, c) \in R_1$

Now $1 + ac = 1 + 4(-1) = -3 < 0$

$\Rightarrow$ $(a, c) \notin R_1$.

$\therefore$ R_1 is not transitive.

Equivalence Relation

A relation R from a set X to set Y (R: $X \to Y$) is said to be an equivalence relation iff it is reflexive, symmetric as well as transitive. The equivalence relation is denoted by $\sim$. *e.g* Relation 'is equal to' (Equality). Similarity and congruency of triangles, parallelism of lines are equivalence relations.

ILLUSTRATION 20: Let $A = \{1, 2, 3\}$. Which of the following is not an equivalence relation on A?

(a) $\{(1, 1), (2, 2), (3, 3)\}$

(b) $\{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$

(c) $\{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$

(d) None of these

SOLUTION: All the subsets of $A \times A$, given in (a), (b) and (c) are equivalence relation on A, as they are reflexive, symmetric and transitive.

ILLUSTRATION 21: Prove that the relation R on the set $\mathbb{Z}$ of all integer numbers defined by $(x, y) \in R$ $\Leftrightarrow x - y$ is divisible by n is an equivalence relation on $\mathbb{Z}$, where n is a fixed integer.

SOLUTION: We observe the following properties

Reflexivity: For any $a \in \mathbb{Z}$, we have $a - a = 0 \times n \Rightarrow a - a$ is divisible by $n \Rightarrow (a, a) \in R$

Thus, $(a, a) \in R$ for all $a \in \mathbb{Z}$. So, R is reflexive on $\mathbb{Z}$.

Symmetricity: Let $(a, b) \in R$. $\Rightarrow (a - b)$ is divisible by n

$\Rightarrow (a - b) = np$ for some $p \in \mathbb{Z}$ $\Rightarrow$ $b - a = n(-p)$

$\Rightarrow$ $b - a$ is divisible by n $\Rightarrow$ $(b, a) \in R$

Thus $(a, b) \in R$ $\Rightarrow$ $(b, a) \in R$ for all $a, b \in R$. So R is symmetric on $\mathbb{Z}$.

Transitivity: Let $a, b, c \in \mathbb{Z}$, such that $(a, b) \in R$ and $(b, c) \in R$. Then $(a, b) \in R$

$\Rightarrow$ $(a - b)$ is divisible by n $\Rightarrow$ $a - b = np$ for some $p \in \mathbb{Z}$, $(b, c) \in R$

$\Rightarrow$ $(b - c)$ is divisible by n $\Rightarrow$ $b - c = nq$ for some $q \in \mathbb{Z}$

$\therefore$ $(a, b) \in R$ and $(b, c) \in R$ $\Rightarrow$ $a - b = np$ and $b - c = nq$

$\Rightarrow$ $(a - b) + (b - c) = np + nq$ $\Rightarrow$ $a - c = n(p + q)$

$\Rightarrow$ $a - c$ is divisible by n $\Rightarrow$ $(a, c) \in R$

Thus, $(a, b) \in R$ and $(b, c) \in R$ $\Rightarrow$ $(a, c) \in R$ for all $a, b, c \in \mathbb{Z}$

So, R is transitive relation on $\mathbb{Z}$.

Thus, R being reflexive, symmetric and transitive is an equivalence relation on $\mathbb{Z}$

ILLUSTRATION 22: Given the relation $R = \{(1, 2), (2, 4)\}$ on set of natural numbers $\mathbb{N}$, add a minimum number of ordered pairs so that the enlarged relation is symmetric, transitive and reflexive, i.e., equivalence.

SOLUTION: The enlarged relatio$n$ = $\{(1, 2), (2, 1), (2, 4), (4, 2), (1, 4), (4, 1), (1, 1), (2, 2), (4, 4)\}$ is reflexive, symmetric and transitive i.e., equivalence.

ILLUSTRATION 23: A relation R on the set of complex numbers $\mathbb{C}$ is defined by z_1Rz_2 iff $\dfrac{z_1 - z_2}{z_1 + z_2}$ is real. Show that R is an equivalence relation.

SOLUTION: Given $z_1\, R\, z_2$ iff $\dfrac{z_1 - z_2}{z_1 + z_2}$ is real.

Reflexivity: Now z_1Rz_1 if $\dfrac{z_1 - z_1}{z_1 + z_1}$ is real. i.e., iff 0 is real, which is true.

$\Rightarrow$ The relation R is reflexive

For Symmetry: Also $z_1\, R\, z_2$ $\Rightarrow$ $\dfrac{z_1 - z_2}{z_1 + z_2}$ is real

$\Rightarrow$ $\dfrac{z_2 - z_1}{z_1 + z_2}$ is real $\Rightarrow$ $z_2\, R\, z_1$.

$\Rightarrow$ the relation R is symmetric.

For Transivity: Let $z_1 = a + ib$, $z_2 = c + id$, $z_3 = e + if$ be any three complex numbers, where a, b, c, d, e, f are reals.

Then $z_1\, R\, z_2$ iff $\dfrac{z_1 - z_2}{z_1 + z_2}$ is real, i.e., iff $\dfrac{(a-c)+i(b-d)}{(a+c)+i(b+d)}$ is real.

That is, iff $\dfrac{(a-c)+i(b-d)}{(a+c)+i(b+d)} \times \dfrac{(a+c)-i(b+d)}{(a+c)-i(b+d)}$ is real

That is, iff $\dfrac{(a^2-c^2+b^2-d^2)+i[(a+c)(b-d)-(a-c)(b+d)]}{(a+c)^2+(b+d)^2}$ is real.

i.e., iff $(a + c)(b - d) - (a - c)(b + d) = 0$
i.e., iff $(ab - ad + bc - cd - ab - ad + bc + cd) = 0$
i.e., iff $2bc - 2ad = 0$ i.e., $ad = bc$.

Now $z_1 R z_2$ and $z_2 R z_3$ $\Rightarrow ad = bc$ and $cf = ed$
$\Rightarrow adcf = bced$ $\Rightarrow af = be$ when $cd \neq 0$
$\Rightarrow z_1 R z_3$ and when $cd = 0$, then transitivity is obvious
$\Rightarrow$ the relation R is transitive.

Hence, the given relation R is an equivalence relation.

ILLUSTRATION 24: If R_1 and R_2 are two equivalence relations on a non-empty set A, then show that $R_1 \cup R_2$ need not be an equivalence relation.

SOLUTION: When R_1 and R_2 are equivalence relations, then both R_1 and R_2 are reflexive as well as symmetric.

Also $R_1 \cup R_2$ will be reflexive and symmetric, but $R_1 \cup R_2$ need not be transitive.

For example, on set $A = \{1, 2, 3\}$

Both $R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$ and $R_2 = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$ are equivalence relations but $R_1 \cup R_2$ is not an equivalence relation as $(1, 2) \in R_1 \cup R_2$ and $(2, 3) \in R_1 \cup R_2$ but $(1, 3) \notin R_1 \cup R_2$, thus showing that $R_1 \cup R_2$ is not transitive.

ILLUSTRATION 25: Which of the following is not an equivalence relation on $\mathbb{Z}$, the set of integers?

(a) $xRy \Leftrightarrow x + y$ is an even integer. (b) $xRy \Leftrightarrow x < y$.
(c) $xRy \Leftrightarrow x - y$ is an even integer. (d) $xRy \Leftrightarrow x = y$

SOLUTION: (a) **Reflexivity:** Let xRx iff $x + x$ is an even integer
iff $2x$ is an even integer, which is true.

Symmetry: Let $xRy \Rightarrow x + y$ is even integer $\Rightarrow y + x$ is even integer
$\Rightarrow yRx$

Transitivity: Let xRy and $yRz \Rightarrow x + y$ is even integer and $y + z$ is even integer.
$\Rightarrow (x + y) + (y + z)$ is even integer
$\Rightarrow x + 2y + z$ is even integer
$\Rightarrow x + z$ is even integer as the sum of two even integers is an even integer.
$\Rightarrow xRz$.

Thus, the given relation is an equivalence relation

(b) **Reflexivity:** Let xRx iff $x < x$ which is false. Thus R is non-reflexive.

Symmetry: Let $xRy \Rightarrow x < y \not\Rightarrow y < x$ i.e., $y \not R x$. Thus R is non-symmetric.

Transitivity: Let xRy and $yRz \Rightarrow x < y$ and $y < z \Rightarrow x < z$. Thus R is transitive.

Thus, the given relation is not an equivalence relation.

(c) Reflexivity: Let xRx iff $x - x$ is an even integer.
Iff zero is an even integer, which is true.

Symmetry: Let $xRy \Rightarrow x - y$ is even integer $\Rightarrow y - x$ is even integer.
$\Rightarrow yRx$

Transitivity: Let xRy and $yRz \Rightarrow x - y$ is even integer and $y - z$ is even integer.
$\Rightarrow (x - y) + (y - z)$ is even integer

$\Rightarrow$ $x - z$ is even integer

$\Rightarrow$ xRz $\Rightarrow$ R is transitive

Thus the given relation is an equivalence relation

(d) **Reflexivity:** Let xRx iff $x = x$ which is true. Thus, R is reflexive.

Symmetry: Let $xRy \Rightarrow x = y \Rightarrow y = x$. Thus, R is symmetric.

Transitivity: Let xRy and $yRz \Rightarrow x = y$ and $y = z$

$\Rightarrow$ $x = z$. Thus, R is transitive.

Thus, the given relation is an equivalence relation.

ILLUSTRATION 26: Let $A = \{a, b, c\}$ and $R = \{(a, a), (b, b), (c, c), (b, c)\}$ be a relation on A. Show that R is

(a) reflexive (b) non-symmetric

(c) Anti-symmetric (d) transitive

SOLUTION: (a) **Reflexivity:** Since x R x for all $x \in A$, therefore, R is reflexive.

(b) **Non-symmetry:** As bRc but $c \not R b$ i.e., $(b, c) \in \mathbb{R}$ but $(c, b) \notin \mathbb{R}$, therefore, R is not symmetric.

(c) **Anti-symmetry:** As for any two distinct $x, y \in \mathbb{R}$, both (x, y) and (y, x) do not belong to R.

Therefore, R is anti-symmetric.

(d) **Transitivity:** Further R is transitive as whenever xRy and yRz, then xRz.

(Here, bRb and $bRc \Rightarrow bRc$).

Equivalence class of an element: If R is an equivalence relation on a set A i.e., $R : A \rightarrow A$ is an equivalence relation, then for an element $x \in A$, equivalence class of element x is denoted by $[x]$ and is defined as $[x] = \{a \in A : (a, x) \in R\}$.

Further various equivalence classes of elements of A are either identical or disjoint and their union covers the set A.

If $A_1, A_2, A_3, \dots, A_n$ are equivalence classes on set A, then either $A_i = A_j$ or $A_i \cap A_j = \phi$ $\forall$ i, j and $\bigcup_{i=1}^{n} A_i = A$.

The sequence $A_1, A_2, A_3, \dots, A_n$ called a partition of A if each A_i represents a different equivalence class,

i.e., $Ai \cap Aj = \phi$ and $\bigcup_{i=1}^{n} Ai = A$.

ILLUSTRATION 27: In the set W of whole numbers an equivalence relation R is defined as follows. '$a R b$ iff both a and b leave same remainder when divided by 5'.

SOLUTION: Equivalence class of $1 = [1] = \{a \in W; a R 1\} = \{a \in W; a = 5p + r; 1 = 5q + r$, for some $p, q \in W)\}$

$= \{a \in W; a = 5p + 1\}$ as $1 = 5q + r$

$\Rightarrow$ $q = 0, r = 1$

$\therefore$ $[1] = \{1, 6, 11, 16, 21 \dots\}$

COMPOSITION OF RELATIONS

Let R and S be two relations from set A to B and B to C respectively. Then we can define a relation SoR from A to C such that $(a, c) \in So\,R \Leftrightarrow \exists\, b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.

This relation is called the composition of R and S. Diagrammatically it is as shown bellow.

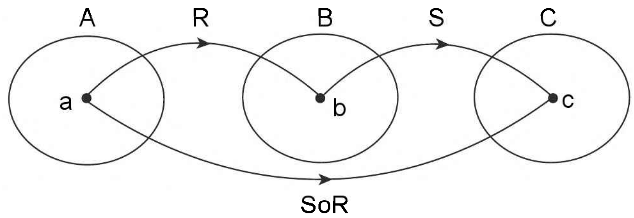


FIGURE 1.38

ILLUSTRATION 28: If $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$, $C = \{p, q, r, s\}$ be three sets such that $R = \{(1, a), (2, c), (1, c), (2, d)\}$ is a relation from A to B and $S = \{(a, s), (b, q), (c, r)\}$ is relation from B to C. Then find SoR and RoS.

SOLUTION: $SoR = \{(1, s), (2, r), (1, r)\}$. In this case RoS does not exist as S is a relation from B to C and R is a relation from A to B. RoS exists if S is a relation from B to C, than R should be from C to other set A. In general $RoS \neq SoR$.

INVERSE OF A RELATION

Let A, B be two sets and let R be a relation from a set A to B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by $R^{-1} = \{(b, a): (a, b) \in R\}$. Clearly, $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$.

Also, Dom (R) = Range (R^{-1}) and range (R) = Dom (R^{-1}).

e.g. Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 3, 4, 5\}$.

Define a relation R from A to B as xRy iff $y = x + 1$, then $R = \{(1, 2), (2, 3), (3, 4), (4, 5)\}$.

$\Rightarrow \quad R^{-1} = \{(2, 1), (3, 2), (4, 3), (5, 4)\}$

Thus, we can define R^{-1} a relation (say R_1) from B to A as xR_1y iff $y = x - 1$. The arrow diagram given below represents the relations R and R^{-1}.

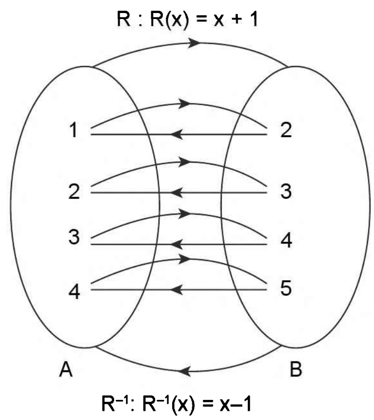


FIGURE 1.39

REMARK:

$(SoR)^{-1} = R^{-1}oS^{-1}$; where R is a relation from A to B and S is a relation from B to C.

Proof: *Let $(x, y) \in (SoR)^{-1}$*

$\Leftrightarrow$ $(y, x) \in (SoR) \Leftrightarrow (SoR)(y) = x$

$\Leftrightarrow$ $S[R(y)] = x \Leftrightarrow (R(y), x) \in S \Leftrightarrow (x, R(y)) \in S^{-1}$

$\Leftrightarrow$ $S^{-1}(x) = R(y) \Leftrightarrow R(y) = S^{-1}(x) \Leftrightarrow R^{-1}(S^{-1}(x)) = y$

$\Leftrightarrow$ $R^{-1}oS^{-1}(x) = y \Leftrightarrow (x, y) \in R^{-1}oS^{-1}$

$\Leftrightarrow$ $SoR^{-1} = R^{-1}oS^{-1}$.

ILLUSTRATION 29: Let $A = \{1, 2, 3, 4\}$; $B = \{5, 6, 7, 8\}$; $C = \{9, 10, 11, 12\}$ and let $R = \{(1, 5), (2, 6), (3, 7)\}$ and $S = \{(5, 9), (6, 11), (8, 12)\}$, be the relations from A to B and B to C respectively. Then prove that $(SoR)^{-1} = R^{-1}oS^{-1}$.

SOLUTION: Here $(SoR) = \{(1, 9), (2, 11)\}$ $\Rightarrow (SoR)^{-1} = \{(9, 1), (11, 2)\}$

Also $S^{-1} = \{(9, 5), (11, 6), (12, 8)\}$ and $R^{-1} = \{(5, 1), (6, 2), (7, 3)\}$

$\Rightarrow R^{-1}oS^{-1} = \{(9, 1), (11, 2)\}$ $\therefore (SoR)^{-1} = R^{-1}oS^{-1}$.

ILLUSTRATION 30: Let A be the set of first ten natural numbers and let R be a relation on A defined by $(x, y) \in R \Leftrightarrow x + 2y = 10$. i.e., $R = \{(x, y): x \in A, y \in A \text{ and } x + 2y = 10\}$.

Express R and R^{-1} as sets of ordered pairs. Determine also

(i) Domains of R and R^{-1} (ii) Range of R and R^{-1}.

SOLUTION: We have $(x, y) \in R \Leftrightarrow x + 2y = 10 \Leftrightarrow y = \dfrac{10-x}{2}$, $x, y \in A$;

where $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Now, for $x = 1$, $y = \dfrac{10-1}{2} = \dfrac{9}{2} \notin A$

This shows that 1 is not related to any element in A.

Similarly, we can observe that 3, 5, 7, 9 and 10 are not related to any element of A under the defined relation. Further, we find that for $x = 2$, $y = \dfrac{10-2}{2} = 4 \in A$

$\therefore (2, 4) \in R$. $\Rightarrow$ For $x = 4$, $y = \dfrac{10-4}{2} = 3 \in A$

$\Rightarrow (4, 3) \in R$ $\Rightarrow$ For $x = 6$, $y = \dfrac{10-6}{2} = 2 \in A$

$\Rightarrow (6, 2) \in R$ $\Rightarrow$ For $x = 8$, $y = \dfrac{10-8}{2} = 1 \in A$

$\Rightarrow (8,1) \in R$

Thus $R = \{(2, 4), (4, 3), (6, 2), (8, 1)\}$ $\Rightarrow R^{-1} = \{ (4, 2), (3, 4), (2, 6), (1, 8)\}$

Clearly, Domain $(R) = \{2, 4, 6, 8\}$ = Range (R^{-1}) and range $(R) = \{ 4, 3, 2, 1\}$ = Domain (R^{-1}).

■ IMPORTANT REMARKS AT A GLANCE

- ❑ If number of elements in A: $n(A) = m$ and $n(B) = n$, then number of elements in $(A \times B) = m \times n$.
- ❑ Since $A \times B$ contains all such ordered pairs of the type (a, b) such that $a \in A$ and $b \in B$, that means it includes all possibilities in which the elements of set A can be related with the elements of set B. Therefore, $A \times B$ is termed as largest possible relation defined from set A to set B, also known as universal relation from A to B.
- ❑ If $A \subseteq B$, then $(A \times B) \cap (B \times A) = A^2 = A \times A$
- ❑ If A has m elements and B has n elements, then number of relations that can be defined from A to $B = 2^{m \times n}$.
- ❑ If A is a set containing n elements, then the number of relations that can be defined on set $A = (2)^{n^2}$.
- ❑ If A and B are two non-empty sets having n elements in common, then $A \times B$ and $B \times A$ have n^2 elements in common.
- ❑ If A is related to B, then symbolically it is written as (aRb), where a is pre-image and b is image.
- ❑ If A is not related to B, then symbolically it is written as $a \not R\ b$.
- ❑ Every relation from $A \to B$ is a subset of $A \times B$.
- ❑ Every function is a relation but every relation cannot be a function.
- ❑ Relations can be represented in following 5 ways
  1. **As a set of ordered pair:** (Roster form)
  2. Graphically (by lattice) $\to$ Plotting the points on $x - y$ plane
  3. Diagrammatically (by using arrow diagrams)

4. **Analytically** (set builder form) → Representing relation as an equation in x and y. e.g., $y = \sqrt{x}$
5. **Tabular form:** Arranging the elements of set A along first column and the elements of set B along first row and putting '1' in the row containing a and column containing b if $(a, b) \in R$ and 0 if $(a, b) \notin R$

- ❑ All identity relations are reflexive but all reflexive relations are not identity.
- ❑ If a relation comes out to be neither One–many nor Many–one, then it is classified as One–one relation.
- ❑ ϕ or { } is called null or void relation. i.e., relation having no element. e.g. If $A = \{2, 3, 5\}$, $B = \{7, 11\}$, then the relation R from A to B defined by $(a, b) \in R$, if 'a divides b' is a null or void relation, because there is no element in A which divides any element of B. i.e., $R = \{\ \}$ or ϕ.
- ❑ Every null relation is a transitive relation.
- ❑ Every singleton relation is a transitive relation.
- ❑ Universal and identity relations are reflexive, as well as transitive.
- ❑ Identity relation is symmetric as well as anti-symmetric both.
- ❑ Union of two reflexive (or symmetric) relations on a set A also reflexive (or symmetric) on set A.
- ❑ Union of two transitive relations need not be transitive on set A.
- ❑ Union of two equivalence relations need not be equivalence.

TEXTUAL EXERCISE–3: (SUBJECTIVE)

1. Let R be a relation from $\mathbb{N}$ to $\mathbb{N}$ defined by $R = \{(a, b): a^2 + b^2 = ab\}$. Are the following true?
   (i) $(a, a) \in R\ \forall\ a \in \mathbb{N}$
   (ii) $(a, b) \in R \Rightarrow (b, a) \in R$
   (iii) $(a, b) \in R; (b, c) \in R \Rightarrow (a, c) \in R$
2. Let R be a relation from $\mathbb{Q}$ to $\mathbb{Q}$ defined by $R = \{(a, b): a, b \in \mathbb{Q}$ and $a - b \in \mathbb{Z}\}$. Show that
   (i) $(a, a) \in R$ for all $a \in \mathbb{Q}$
   (ii) $(a, b) \in R \Rightarrow (b, a) \in R$
3. Let $A = \{2, 3, 6\}$. Which of the following relations on A are reflexive?
   (i) $R_1 = \{(2, 2), (3, 3), (6, 6)\}$
   (ii) $R_2 = \{(2, 2), (3, 3), (3, 6), (6, 3)\}$
   (iii) $R_3 = \{(2, 2), (3, 6), (2, 6)\}$
   (iv) $R_4 = \{(2, 2), (3, 3), (3, 6), (6, 6) (6, 3)\}$
4. Let $A = \{1, 2, 3, 5\}$. Which of the following relations on A are symmetric?
   (i) $R_1 = \{(1, 2), (2, 3)\}$
   (ii) $R_2 = \{(1, 2), (2, 3), (3, 5), (3, 2), (5, 3), (2, 1)\}$
   (iii) $R_3 = \{(1, 1), (2, 2), (3, 3), (5, 5), (3, 5)\}$
   (iv) $R_4 = A \times A$
5. Let $A = \{2, 3, 4, 5, 6\}$. Which of the following relations on A are transitive?
   (i) $R_1 = \{(2, 3), (3, 6)\}$
   (ii) $R_2 = \{(2, 2), (3, 4), (4, 4), (4, 5)\}$
   (iii) $R_3 = \{(2, 4), (4, 5), (2, 5)\}$
   (iv) $R_4 = \{(2, 3), (3, 2), (4, 5), (5, 4)\}$
6. Let $A=\{1, 2, 3, 4\}$.Which of the following relations on A are reflexive, symmetric and transitive
   (i) $R_1 = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 3), (3, 4)\}$
   (ii) $R_2 = \{(2, 4), (4, 2)\}$
   (iii) $R_3 = \{(4, 3), (2, 4)\}$
   (iv) $R_4 = A \times A$
7. Let R be the relation on the set R of real numbers defined by $(a, b) \in R$ if $1 - ab > 0$. Show that the relation R is symmetric, but not reflexive and transitive.
8. Test whether the following relations are reflexive, symmetric and transitive.
   (i) R_1 on $\mathbb{Z}$ defined by $(a, b) \in R_1$ iff $|a - b| \le 7$
   (ii) R_2 on $\mathbb{Q}$ defined by $(a, b) \in R_2$ iff $ab = 4$.
   (iii) R_3 on $\mathbb{R}$ defined by $(a, b) \in R_3$ iff $a^2 - 4ab + 3b^2 = 0$.
9. Classify the following relations in terms of Reflexive (A) Symmetric (B) and Transitive (C) relations also state which of them are equivalence relation.
   R_1: $\mathbb{Z} \to \mathbb{Z}$, $\{a R_1 b$ iff a is divisible by b;$\}$ where $\mathbb{Z}$ is set of all integers.
   R_2: $\Delta \to \Delta$ where Δ is set of all triangles defined as$\{\Delta_1 R_2 \Delta_2$ iff Δ_1 and Δ_2 are similar$\}$ similarity of triangles.
   R_3: $L \to L$, where L is set of all lines on a given plane defined as $\{L_1 R_3 L_2$ iff L_1 is $\parallel L_2\}$ parallelism of lines
   R_4: $L \to L$, where L is set of all lines on a given plane defined as $\{L_1 R_4 L_2$ if $L_1 \perp L_2\}$ perpendicularity of lines

R_5: $H \to H$, where H is set of all members of a family and relation is being sister of. i.e., $\{a\,R_5\,b$ iff a is sister of $b\}$

R_6: $\mathbb{N} \to \mathbb{N}$, where $\mathbb{N}$ is a natural numbers defined as $\{a\,R_6\,b$ iff $a - b \in \mathbb{N}\}$

R_7: $\mathbb{R} \to \mathbb{R}$; where $\mathbb{R}$ is set of real numbers defined as $\{a\,R_7\,b$ iff $a \geq b\}$

10. Let $\mathbb{N}$ denote the set of all natural numbers and R a relation on $\mathbb{N} \times \mathbb{N}$. Which of the following is an equivalence relation?

(a) $(a, b)\,R\,(c, d)$ iff $ad\,(b + c) = bc\,(a + d)$
(b) $(a, b)\,R\,(c, d)$ iff $a + d = b + c$
(c) $(a, b)\,R\,(c, d)$ iff $ad = bc$
(d) None of these

11. For real numbers x and y, we write $x\,R\,y \Leftrightarrow x^2 - y^2 + \sqrt{3}$ is an irrational number. Then the relation R is

(a) reflexive (b) symmetric
(c) transitive (d) None of these

Answer Keys

1. (i) False (ii) True (iii) True
3. (i) and (iv) **4.** (ii) and (iv) **5.** (iii)
6. (i) only reflexive (ii) only symmetric (iii) Nothing
(iv) Reflexive, symmetric as well as transitive i.e., equivalence.
7. (i) Reflexive and symmetric but not transitive. (ii) only symmetric. (iii) only transitive.
9. $R_1 \to C$; $R_2 \to A, B, C$; $R_3 \to A, B, C$; $R_4 \to B$; $R_5 \to C$; $R_6 \to C$; $R_7 \to A, C$
10. a, b, c **11.** a

TEXTUAL EXERCISE–3: (OBJECTIVE)

1. Define a relation R on set $A = \{2, 3, 4, 9, 16\}$ as xRy if '$x^2 = y$', then ROR^{-1} is

(a) Identity relation on set A
(b) Identity relation on Domain set of R^{-1}
(c) Reflexive relation on A
(d) Reflexive and symmetric relation on Domain of R^{-1}

2. Define two relations R_1 and R_2 on set $\{10, 20, 30, 40, 50\}$ as xR_1y if '$y - x = 10$' and xR_2y if x divides $(x + y)$; then the relation $(R_1\,o\,R_2)^{-1}$ is

(a) $\{(20, 10), (30, 10), (30, 20), (40, 10), (40, 30), (50, 10), (50, 20), (50, 40)\}$
(b) $\{(20, 10), (30, 20), (40, 30), (50, 20)\}$
(c) $\{(20, 10), (30, 30), (40, 40), (50, 20)\}$
(d) None of these

3. The relation on set of natural numbers defined as xRy if '$x + y + 4xy$' is an even number, then R is

(a) reflexive (b) symmetric
(c) transitive (d) equivalence

4. Let $A = \{2, 3, 4, 5\}$ and let $R = \{(2, 2), (3, 3), (4, 4)$ $(5, 5), (2, 3), (3, 2), (3, 5), (5, 3)\}$ be a relation in A. Then R is

(a) reflexive and transitive
(b) reflexive and symmetric
(c) reflexive and anti-symmetric
(d) None of the above

5. If R is a relation on A such that $R = R^{-1}$, then R is

(a) reflexive (b) symmetric
(c) transitive (d) None of these

6. Let R and S be two relations on a set A. Then:

(a) R and S are transitive, then $R \cup S$ is also transitive
(b) R and S are reflexive, then $R \cup S$ is equivalence
(c) R and S are symmetric, then $R \cup S$ is equivalence
(d) R and S are reflexive, then $R \cup S$ is reflexive

7. A set of points in a plane is denoted by P. Let $P_1, P_2 \in P$, then $P_1 R\,P_2 \Leftrightarrow P_1$ and P_2 are equidistant from origin; then which statement(s) is/are correct?

(a) R is reflexive but not symmetric.
(b) R is reflexive, symmetric but not transitive
(c) R is reflexive, symmetric and transitive
(d) R is not an equivalence relation.

8. $\mathbb{N}$ is the set of natural numbers. The relation R is defined on $\mathbb{N}$ as follows $(a, b)\,R(c, d) \Leftrightarrow a + d = b + c$ is

(a) reflexive (b) symmetric
(c) transitive (d) All of these

9. The relation R defined on the set $\mathbb{N}$ of natural numbers by $xRy \Leftrightarrow 2x^2 - 3xy + y^2 = 0$ is
(a) symmetric but not reflexive
(b) only symmetric
(c) not symmetric but reflexive
(d) None of these

10. The minimum number of elements that must be included to the relation $R = \{(1,2), (2,3)\}$ on the set $\{1,2,3\}$ so that it is equivalence is
(a) 4 (b) 7
(c) 6 (d) 5

11. If R is a relation from a set A to a set B and S is a relation from set B to a set C, then the relation SoR:
(a) is from A to C (b) is from C to A
(c) Does not exist (d) None of these

12. If $R \subseteq A \times B$ and $S \subseteq B \times C$ be two relations, then $(SoR)^{-1}$ is equal to
(a) $S^{-1}oR^{-1}$ (b) $R^{-1}oS^{-1}$
(c) SoR (d) RoS

13. Consider the following statements:
(1) Identify relation on a finite set A is the greatest relation on A
(2) The universal relation on a set containing at least two elements is not anti-symmetric.
(3) The union and intersection of two symmetric relations are also symmetric relations.
(a) only 1 (b) only 2 and 3
(c) only 3 and 1 (d) All of these

14. If R and R' are symmetric relations (not disjoint) on a set A, then the relation $R \cap R'$ is
(a) reflexive (b) symmetric
(c) transitive (d) None of these

15. Let m = number of elements in a reflexive relation defined on set A and n = number of elements in an identity relation defined on set A, then
(a) $m \geq n$ (b) $m \leq n$
(c) $m = n$ (d) None of these

16. The set $S = \{1, 2, 3, \ldots, 12\}$ is to be partitioned into three sets A, B, C of equal size. Thus, $A \cup B \cup C = S$, $A \cap B = B \cap C = A \cap C = \phi$. The number of ways to partition S is
(a) $\dfrac{12!}{3!(3!)^4}$ (b) $\dfrac{12!}{(4!)^3}$
(c) $\dfrac{12!}{(3!)^4}$ (d) $\dfrac{12!}{3!(4!)^3}$

Answer Keys

1. (b,d) **2.** (a) **3.** (a,b,c,d) **4.** (b) **5.** (b) **6.** (d) **7.** (c) **8.** (d) **9.** (c) **10.** (b)
11. (a) **12.** (b) **13.** (b) **14.** (b) **15.** (a) **16.** (b)

MULTIPLE-CHOICE QUESTIONS

SECTION–I

OBJECTIVE-TYPE SOLVED EXAMPLE

1. If $A = \{1, 2, 5, 6\}$ and $B = \{1, 2, 3\}$, then $(A \times B) \cap (B \times A)$ equals.

(a) $\{(1, 1), (2, 1), (6, 1), (3, 2)\}$
(b) $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$
(c) $\{(1, 1), (2, 2)\}$
(d) $(1, 1), (1, 2), (2, 5), (2, 6)\}$

Solution: (b) $A = \{1, 2, 5, 6\}$ and $B = \{1, 2, 3\}$
Now $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$
$[\because (A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)]$
$= (A \cap B) \times (A \cap B) = \{1, 2\} \times \{1, 2\}$
$= \{(1, 1), (1, 2), (2, 1), (2, 2)\}$.

2. If A, B, C are non-empty sets such $A \cap C = \phi$, then what is $(A \times B) \cap (C \times B)$?

(a) $A \times C$ (b) $A \times B$
(c) $B \times C$ (d) ϕ

Solution: **(d)** $(A \times B) \cap (C \times B)$
$= (A \cap C) \times (B \cap B)$
$= \phi \times B = \phi$

3. If $A = \{a, b\}, B = \{c, d\}, C = \{d, e\}$, then $\{(a, c), (a, d), (a, e), (b, c), (b, d), (b, e)\}$ is equal to

(a) $A \cap (B \cup C)$ (b) $A \cup (B \cap C)$
(c) $A \times (B \cup C)$ (d) $A \times (B \cap C)$

Solution: (c) Clearly $A \cap (B \cup C)$ and $A \cup (B \cap C)$ do not contain ordered pairs.
So the answer can't be (*a*) and (*b*)
Now $(B \cap C) = \{d\}$ and $A \times (B \cap C)$
$= \{a, b\} \times \{d\} = \{(a, d), (b, d)\}$
So (d) is incorrect.
Next $(B \cup C) = \{c, d, e\}$
$\Rightarrow A \times (B \cup C) = \{a, b\} \times \{c, d, e\} = \{(a, c), (a, d), (a, e), (b, c), (b, d), (b, e)\}$
$\therefore$ (c) is correct.

4. If $S_1 = \{1, 2, 3, ..., 20\}, S_2 = \{a, b, c, d\}, S_3 = \{b, d, e, f\}$. The number of elements of $(S_1 \times S_2) \cup (S_1 \times S_3)$ is

(a) 100 (b) 120
(c) 140 (d) 40

Solution: (b) $S_1 \times S_2$ has $20 \times 4 = 80$ elements
$S_1 \times S_3$ has $20 \times 4 = 80$ elements
Number of common elements $= 20 \times 2 = 40$
$[\because b$ and d are common elements in S_2 and $S_3]$
$\therefore$ Number of elements of $(S_1 \times S_2) \cup (S_1 \times S_3)$
$= 80 + 80 - 40 = 160 - 40 = 120$.

5. Let $A = \{(x, y) : y = e^x, x \in \mathbb{R}\}, B = \{(x, y) : y = e^{-x}, x \in \mathbb{R}\}$. Then:

(a) $A \cap B = \phi$ (b) $A \cap B \neq \phi$
(c) $A \cup B = \mathbb{R}$ (d) None of these

Solution: (b) $y = e^x$ and $y = e^{-x}$
$\Rightarrow e^x = e^{-x}$ $\Rightarrow e^{2x} = 1$
$\Rightarrow e^{2x} = e^0$ $\Rightarrow 2x = 0$
$\Rightarrow x = 0$
$\therefore y = e^0 = 1$
$\therefore$ A and B meet at (0, 1)
Hence, $A \cap B \neq \phi$

6. Let $A = \{(x, y) : y = e^x, x \in \mathbb{R}\}, B = \{(x, y) : y = x, x \in \mathbb{R}\}$. Then:

(a) $B \subseteq A$ (b) $A \subseteq B$
(c) $A \cap B = \phi$ (d) $A \cup B = A$

Solution: (c) $y = e^x$ and $y = x$
$\Rightarrow e^x = x$
$\Rightarrow$ No $x \in \mathbb{R}$ as e^x and $y = x$ do not intersect each other.
Hence $A \cap B = \phi$.

7. Consider the following with regard to a relation R on a set of real numbers defined by xRy if and only if $3x + 4y = 5$. Consider the following three statements:

1. $0R1$ **2.** $1R\frac{1}{2}$

3. $\frac{2}{3}R\frac{3}{4}$

Which of the above are correct?

(a) 1 and 2 only (b) 1 and 3 only
(c) 2 and 3 only (d) 1, 2 and 3

Solution: (c) $3x + 4y = 5$
$\Rightarrow y = \frac{5 - 3x}{4}$

we are to find y for $x = 0, 1, \frac{2}{3}$ (By given options)

$\Rightarrow$ $y = \frac{5}{4}$ for $x = 0$; $y = \frac{1}{2}$ for $x = 1$ and

$y = \frac{3}{4}$ for $x = \frac{2}{3}$

$\Rightarrow$ $1R\frac{1}{2}$ and $\frac{2}{3}R\frac{3}{4}$

$\Rightarrow$ (2) and (3) are true

$\Rightarrow$ (c) is correct option.

8. If the set A and B are defined as: $A = \left\{(x, y) : y = \frac{1}{x}, 0 \neq x \in \mathbb{R}\right\}$, $B = \{(x, y) : y = -x, x \in \mathbb{R}\}$, then

(a) $A \cap B = A$ (b) $A \cap B = B$
(c) $A \cap B = \phi$ (d) None of these

Solution: (c) $y = \frac{1}{x}$ and $y = -x$

$\Rightarrow$ $\frac{1}{x} = -x$

$\Rightarrow$ $x^2 = -1$

$\Rightarrow$ $x \notin \mathbb{R}$.

$\therefore$ $A \cap B = \phi$.

9. What is the set of points (x, y) satisfying the equation $x^2 + y^2 = 4$ and $x + y = 2$?

(a) $\{(2, 0), (-2, 0), (0, 2)\}$
(b) $\{(0, 2), (0, -2)\}$
(c) $\{(0, 2), (2, 0)\}$
(d) $\{(2, 0), (-2, 0), (0, 2), (0, -2)\}$

Solution: (c) $x^2 + y^2 = 4$; $x + y = 2$

$\Rightarrow$ $x^2 + (2 - x)^2 = 4$

$\Rightarrow$ $2x^2 - 4x = 0$

$\Rightarrow$ $x = 0$ or $x = 2$

$\therefore$ for $x = 0, y = 2$ and for $x = 2, y = 0$

$\therefore$ $\{(x, y) : x^2 + y^2 = 4; x + y = 2\} = \{(0, 2), (2, 0)\}$

10. Let $R = \{(1, 3), (4, 2), (2, 4), (2, 3), (3, 1)\}$ be a relation on the set $A = \{1, 2, 3, 4\}$. The relation R is

(a) a function (b) reflexive
(c) not symmetric (d) transitive

Solution: (c) R is not a function as element 2 has two images 4 and 3.

Also R is not reflexive as no element of A is related with itself.

Further $(2, 3) \in R$ but $(3, 2) \notin R$

Hence, R is not symmetric.

Also R is not transitive as $(2, 4), (4, 2) \in R$ but $(2, 2) \notin R$.

11. Let $R = \{(3, 3), (6, 6), (9, 9), (12, 12), (6, 12), (3, 9), (3, 12), (3, 6)\}$ be a relation on the set $A = \{3, 6, 9, 12\}$.

The relation is

(a) reflexive and transitive only
(b) reflexive only
(c) an equivalence relation
(d) reflexive and symmetric only

Solution: (a) R is reflexive and transitive only.

Since $(3, 3), (6, 6), (9, 9), (12, 12) \in R$

$\Rightarrow$ R is Reflexive

and $(3, 6), (6, 12), (3, 12) \in R$

$\Rightarrow$ R is transitive. Further R is not symmetric as $(3, 9), (6, 12), (3, 6), (3, 12) \in R$ but their inverses $(9, 3), (12, 6), (6, 3), (12, 3) \notin R$.

12. Let W denotes the set of words in the English dictionary. Define the relation R by: $R = \{(x, y) \in W \times W \mid$ the words x and y have at least one letter in common$\}$. Then R is

(a) not reflexive, symmetric and transitive
(b) reflexive, symmetric and not transitive
(c) reflexive, symmetric and transitive
(d) reflexive, not symmetric and transitive

Solution: (b) **Reflexivity:** Clearly $(x, x) \in R\ \forall\ x \in W$ as every word has all letters common to itself. So, R is reflexive.

Symmetricity: Let $(x, y) \in R$, then $(y, x) \in R$ as x and y have at least one letter in common.

So, R is symmetric.

Transitivity: But R is not transitive for example let x = DELHI, y = CHANDIGARH and z = AMBALA, then $(x, y) \in R$ and $(y, z) \in R$ but $(x, z) \notin R$.

13. Let X be the set of all graduates in India. Elements x and y in X are said to be related if they are graduates of the same university. Which one of the following statements is correct?

(a) Relation is symmetric and transitive only
(b) Relation is reflexive and transitive only
(c) Relation is reflexive and symmetric only
(d) Relation is reflexive, symmetric and transitive.

Solution: (d) **Reflexivity:** xRx iff x and x are graduates of same university, which is true.

Symmetricity: $xRy \Rightarrow x$ and y are graduates from same university

$\Rightarrow$ y and x are graduates from same university

$\Rightarrow$ yRx

Transitivity: xRy and yRz

$\Rightarrow$ x and y are graduates from same university as well as y and z are graduates from same university.

$\Rightarrow$ x, y and z are graduates from same university

$\Rightarrow$ x and z are graduates from same university

$\Rightarrow$ xRz.

14. Let M be set of men and R is a relation 'is son of' defined on M. Then R is

(a) an equivalence relation
(b) a symmetric relation only
(c) a transitive relation only
(d) None of the above

Solution: (d) **Reflexivity:** As nobody can be son of himself so the relation R can't be reflexive

Symmetric: xRy

$\Rightarrow$ x is son of y

$\Rightarrow$ y is father of x

$\Rightarrow$ y can't be son of x

$\Rightarrow$ $y \not{R}\, x$.

Transitivity: Let xRy and yRz

$\Rightarrow$ x is a son of y and y is a son of z

$\Rightarrow$ x is a grand son of z

$\Rightarrow$ $x \not{R}\, z$.

$\therefore$ The given relation is neither reflexive nor symmetric, transitive

15. Let $\mathbb{R}$ be the real line. Consider the following subsets of the plane $\mathbb{R} \times \mathbb{R}$.

$S = \{(x, y) : y = x + 1 \text{ and } 0 < x < 2\}$, $T = \{(x, y): x - y \text{ is an integer}\}$. Which one of the following is true?

(a) neither S nor T is an equivalence relation on $\mathbb{R}$
(b) both S and T are equivalence relations on $\mathbb{R}$
(c) S is an equivalence relation on $\mathbb{R}$ but T is not
(d) T is an equivalence relation on $\mathbb{R}$ but S is not

Solution: (d) **Reflexivity:** $T = \{(x, y): x - y \in \mathbb{Z}\}$

xTx iff $x - x = 0 \in \mathbb{Z}$; which is true

$\Rightarrow$ T is a reflexive relation

Symmetricity: Let xTy

$\Rightarrow x - y \in \mathbb{Z}$ $\quad\Rightarrow y - x \in \mathbb{Z}$

$\Rightarrow yTx$ $\quad\therefore$ T is symmetrical also

Transitivity: Let xTy, yTz

$\Rightarrow x - y \in \mathbb{Z}$ and $y - z \in \mathbb{Z}$

$\Rightarrow (x - y) + (y - z) \in \mathbb{Z}$

$\Rightarrow x - z \in \mathbb{Z}$ $\quad\Rightarrow xTz$

$\Rightarrow$ T is also transitive.

Hence, T is an equivalence relation.

Clearly, $x \neq x + 1 \Rightarrow (x, x) \notin S$

$\therefore$ S is not reflexive

$\Rightarrow$ S can't be an equivalence relation.

16. Consider the following relations. $R = \{(x, y) \mid x, y$ are real numbers and $x = wy$ for some rational number $w\}$

$S = \{\left(\frac{m}{n}, \frac{p}{q}\right) \mid m, n, p$ and q are integers such that, $q \neq 0$, $n \neq 0$ and $qm = pn\}$. Then

(a) Neither R nor S is an equivalence relation
(b) S is an equivalence relation but R is not an equivalence relation
(c) R and S both are equivalence relations
(d) R is an equivalence relation but S is not an equivalence relation.

Solution: (b) **Reflexivity:** xRx as $x = 1.x$ and 1 is a rational number.

$\frac{m}{n} S \frac{m}{n}$ iff $m.n = m.n$, which is true.

Thus, R and S are reflexive relations.

Symmetricity: Let xRy

$\Rightarrow x = wy$ for some rational number w

$\Rightarrow y = \frac{1}{w}x$

$\Rightarrow$ yRx provided $w \neq 0$, but if $x = 0$ and y is any non-zero integer, then $x = 0y$ and there exist no rational number w for which $y = wx$.

$\Rightarrow$ $y \not{R}\, x$ thus R is not symmetric.

Next let $\frac{m}{n} S \frac{p}{q}$

$\Rightarrow mq = pn$ $\quad\Rightarrow pn = mq$

$\Rightarrow \frac{p}{q} S \frac{m}{n}$

$\Rightarrow$ S is a symmetric relation.

Transitivity: Let xRy and yRz

$\Rightarrow x = w_1 y$ and $y = w_2 z$ for some $w_1, w_2 \in \mathbb{Q}$

$\Rightarrow x = w_1(w_2 z) = (w_1.w_2)z$

$\Rightarrow xRz$ as $w_1.w_2 \in \mathbb{Q}$

Now let $\frac{m}{n} S \frac{p}{q}$ and $\frac{p}{q} S \frac{r}{t}$; where $n, q, t \neq 0$

$\Rightarrow$ $mq = np$ and $pt = qr$

$\Rightarrow$ $mqpt = npqr$

$\Rightarrow$ $mt = nr$, as $q \neq 0$, $p \neq 0$ (as if $p = 0$, then $mq = 0$

$\Rightarrow$ $m = 0 \Rightarrow \frac{m}{n} = \frac{p}{q} = \frac{r}{t} = 0$)

$\Rightarrow$ $\frac{m}{n} S \frac{r}{t}$

$\Rightarrow$ S is transitive.

$\Rightarrow$ S is an equivalence relation but R is not an equivalence relation

17. Consider the following relation R on the set of real square matrices of order 3.

$\{(A, B) | A = P^{-1} BP$ for some invertible matrix $P\}$

A: R is an equivalence relation.

R: For any two invertible 3×3 matrices M and N, $(MN)^{-1} = N^{-1}M^{-1}$. Now answer

(a) If both assertion and reason are correct, and reason is the correct explanation of the assertion.

(b) If both assertion and reason are correct, but reason is not correct explanation of the assertion.

(c) If assertion is correct, but reason is incorrect

(d) If assertion is incorrect, but reason is correct

Solution: (a) **Reflexivity:** As $A = P^{-1}AP$ when $P = I_3$, the identity matrix of order 3,

we get $(A, A) \in R$ $\forall$ real square matrix A of order 3.

$\therefore$ R is reflexive.

Symmetricity: Let us assume $(A, B) \in R$

$\Rightarrow$ There exists a non-singular matrix P such that $A = P^{-1} BP$

$\Rightarrow$ $B = PAP^{-1} = (P^{-1})^{-1} AP^{-1}$

$$\left(\because |P^{-1}| = \frac{1}{|P|} \neq 0 \Rightarrow P^{-1} \text{ is invertible}\right)$$

Thus, $(B, A) \in R$

$\therefore$ R is symmetric

Next, assume that $(A, B) \in R$ and $(B, C) \in R$.

$\Rightarrow$ $A = P^{-1} BP$ and $B = Q^{-1} CQ$ for some invertible matrices P and Q

$\therefore$ $A = P^{-1} (Q^{-1} C Q)P = (P^{-1} Q^{-1}) C (QP)$

$= (QP)^{-1} C (QP)$ [using Reason]

$\Rightarrow$ $(A, C) \in R$. [$\because |QP| = |Q|.|P| \neq 0$ as $|P| \neq 0$, $|Q| \neq 0$

$\therefore$ Assertion is true and Reason is true and is correct explanation for Assertion.

SECTION–II

SUBJECTIVE-TYPE SOLVED EXAMPLE

1. Let $A = \{1, 2\}$ and $B = \{3, 4\}$. Write all subsets of $A \times B$.

Solution: Given, $A = \{1, 2\}$ and $B = \{3, 4\}$

$\therefore$ $A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$

Subsets of $A \times B$ are ϕ, $\{(1, 3)\}$, $\{(1, 4)\}$, $\{(2, 3)\}$, $\{(2, 4)\}$, $\{(1, 3), (1, 4)\}$, $\{(1, 3), (2, 3)\}$, $\{(1, 3), (2, 4)\}$, $\{(1, 4), (2, 3)\}$, $\{(1, 4), (2, 4)\}$, $\{(2, 3), (2, 4)\}$, $\{(1, 3), (1, 4), (2, 3)\}$, $\{(1, 3), (1, 4), (2, 4)\}$, $\{(1, 4), (2, 3), (2, 4)\}$, $\{(1, 3), (2, 3), (2, 4)\}$, $A \times B$

2. If $A = \{2, 3\}$, $B = \{4, 5\}$ and $C = \{5, 6\}$, then find

(i) $A \times (B \cup C)$ (ii) $A \times (B \cap C)$

(iii) $(A \times B) \cup (B \times C)$

Solution: (i) $B \cup C = \{4, 5, 6\}$

$A \times (B \cap C) = \{2, 3\} \times \{4, 5, 6\} = \{(2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)\}$

(ii) $B \cap C = \{5\}$

$A \times (B \cap C) = \{2, 3\} \times \{5\} = \{(2, 5), (3, 5)\}$

(iii) $A \times B = \{2, 3\} \times \{4, 5\} = \{(2, 4), (2, 5), (3, 4), (3, 5)\}$

$B \times C = \{4, 5)\} \times \{5, 6\} = \{(4, 5), (4, 6), (5, 5), (5, 6)\}$

$\therefore$ $(A \times B) \cup (B \times C) = \{(2, 4), (2, 5), (3, 4), (3, 5), (4, 5), (4, 6), (5, 5), (5, 6)\}$

3. Let $A = \{1, 2, 3, 4\}$ and $S = \{(a, b) : a \in A, b \in A, a$ divides $b\}$. Write S explicitly

Solution: Since 1 divides 2, 3, 4 and 2 divides 4

3 does not divide any of 1, 2 and 4

4 does not divide any of 1, 2 and 3

Also 1 divides 1, 2 divides 2, 3 divides 3 and 4 divides 4.

Hence, $S = \{(a, b) : a \in A, b \in A, a$ divides $b\}$

$= \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$

4. If $\mathbb{R}$ is the set of all real numbers, what do the Cartesian products $\mathbb{R} \times \mathbb{R}$ and $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ represent?

Solution: $\mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\}$
Hence $\mathbb{R} \times \mathbb{R}$ represents the coordinates of all the points in x-y plane
Again $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z)\} : x, y, z \in \mathbb{R}\}$
Hence $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ represents the coordinates of all the points in three dimensional space.

5. If $A \times B = \{(a, x), (a, y), (b, x), (b, y)\}$. Find A and B

Solution: Given $A \times B = \{(a, x), (a, y), (b, x), (b, y)\}$
$\therefore$ A = set of first components of all ordered pairs in $A \times B = \{a, b\}$ and B = set of second components of all ordered pairs in $A \times B = \{x, y\}$

6. If A and B are two sets such that $A \times B$ consists of 6 elements and if three elements of $A \times B$ are (3, 3), (1, 3) and (2,5), then what are its remaining elements?

Solution: Since $(1, 3), (2, 5), (3, 3) \in A \times B$. So clearly $1, 2, 3 \in A$ and $3, 5 \in B$
Given $n(A \times B) = 6 \quad \Rightarrow n(A).\ n(B) = 6$
But $1, 2, 3 \in A$ and $3, 5 \in B$
Hence, $A = \{1, 2, 3\}$ and $B = \{3, 5\}$
$\therefore$ $A \times B = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 3), (3, 5)\}$
$\therefore$ Remaining elements of $A \times B$ are (3, 5), (1, 5) and (2, 3).

7. Let A and B be two sets such that $n(A) = 5$ and $n(B) = 2$. If a, b, c, d, e are distinct elements and $(a, 2), (b, 3), (c, 2), (d, 3), (e, 2)$ are in $A \times B$, then find the sets A and B.

Solution: Since a, b, c, d, e are distinct and $(a, 2), (b, 3), (c, 2), (d, 3), (e, 2)$ are elements of $A \times B$.
Therefore, a, b, c, d, e are in A.
But $n(A) = 5$
$\therefore$ $A = \{a, b, c, d, e\}$
Again $n(B) = 2$ and $(a, 2), (b, 3) \in A \times B$
$\therefore$ $B = \{2, 3\}$.

8. The Cartesian product $A \times A$ has 9 elements among which two are $(-1, 0)$ and $(0, 1)$. Find the set A and the remaining elements of $A \times A$.

Solution: Let $n(A) = m$
Given, $n(A \times A) = 9 \quad \Rightarrow n(A).\ n(A) = 9$
$\Rightarrow m\ .\ m = 9 \quad \Rightarrow m = 3$
$\therefore$ $n(A) = 3$.
Now $(-1, 0) \in A \times A$
$\Rightarrow -1 \in A$ and $0 \in$ A
Again $(0, 1) \in A \times A$
$\Rightarrow 0 \in A$ and $1 \in$ A
Thus, $-1 \in A$, $0 \in A$ and $1 \in A$
But A has exactly three elements, therefore $A = \{-1, 0, 1\}$
Remaining elements of $A \times A$ are $(-1, -1), (-1, 1), (0, -1), (0, 0), (1, -1), (1, 0), (1, 1)$.

9. Prove that $A \times \phi = \phi$ for any set A.

Solution: $(x, y) \in A \times \phi \Rightarrow y \in \phi$ (contradiction)
Thus there is no element (x, y) in $A \times \phi$
$\therefore$ $A \times \phi = \phi$

10. Prove that $A \times B = \phi \Leftrightarrow A = \phi$ or $B = \phi$

Solution: Let $A \times B = \phi$ .Then it is to prove that either $A = \phi$ or $B = \phi$
Suppose, if possible, neither $A = \phi$ nor $B = \phi$.
Then there are elements $x \in A$ and $y \in B$ so that $(x, y) \in A \times B = \phi$ i.e., $(x, y) \in \phi$
This is a contradiction
Hence, our supposition is wrong. So either $A = \phi$ or $B = \phi$
Thus, $A \times B = \phi \Rightarrow A = \phi$ or $B = \phi$ ...(1)
Conversely, let either $A = \phi$ or $B = \phi$, then it is to prove that $A \times B = \phi$.
Suppose, if possible $A \times B \neq \phi$, then there is an element $(x, y) \in A \times B$ so that $x \in A$ and $y \in B$, i.e., $A \neq \phi$ and $B \neq \phi$
This is contradictory to our supposition
$\therefore$ $A \times B \neq \phi$ is impossible
$\therefore$ $A \times B = \phi$
Thus $A = \phi$ or $B = \phi$
$\Rightarrow$ $A \times B = \phi$ ....(2)
From (1) and (2), we have $A \times B = \phi \Leftrightarrow A = \phi$ or $B = \phi$.

11. If A and B are two sets, then prove $A \times B$ and $B \times A$ have an element in common iff A and B have an element in common.

Solution: $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
$\therefore$ $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$
$= (A \cap B) \times (A \cap B) = C \times C$, where $C = A \cap B$
$\therefore$ $(A \times B) \cap (B \times A) \neq \phi \Leftrightarrow C \times C \neq \phi \Leftrightarrow C \neq \phi \Leftrightarrow A \cap B \neq \phi$

12. If A and B are any two non-empty sets, then prove that $A \times B = B \times A \Leftrightarrow A = B$.

Solution: If part: Let $A = B$ ...(1)
To prove $A \times B = B \times A$
Now, $A \times B = A \times A \quad [\because B = A]$

$= B \times A$ [$\because A = B$]

Only if part: Let $A \times B = B \times A$ .....(2)

To prove $A = B$. Let $a \in A$ and $b \in B$

$\Rightarrow (a, b) \in A \times B$

$\Rightarrow (a, b) \in B \times A$ [From (2)]

$\Rightarrow a \in B$ and $b \in A$

Thus $a \in A \Rightarrow a \in B$

$\Rightarrow A \subseteq B$ and $b \in B \Rightarrow b \in A \Rightarrow B \subseteq A$ .....(3)

Thus $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$

13. Let A and B be non-empty sets such that $A \times B = A \times C$. Show that $B = C$

Solution: Given, $A \times B = A \times C$ ....(1)

A and B are non-empty sets. We are to prove $B = C$.

Let b be an arbitrary element of B. A is non-empty

$\Rightarrow$ there exists $a \in A$

Now, $a \in A$ and $b \in B$

$\Rightarrow (a, b) \in A \times B$

$\Rightarrow (a, b) \in A \times C$ [$\because A \times B = A \times C$]

$\Rightarrow b \in C$

Thus, $b \in B \Rightarrow b \in C$

$\Rightarrow B \subseteq C$ .... (2)

Again, let c be an arbitrary element of C.

A is non-empty

$\Rightarrow$ there exist $a \in A$.

Now, $a \in A$ and $c \in C$

$\Rightarrow (a, c) \in A \times C$

$\Rightarrow (a, c) \in A \times B$ [From (1)]

$\Rightarrow c \in B$

Thus $c \in C \Rightarrow c \in B$

$\Rightarrow C \subseteq B$ ... (3)

From (2) and (3), $B = C$.

16. Find the linear relation between the components of the ordered pairs of the relation R, where $R = \{(2, 1), (4, 7), (1, -2), \ldots\}$

Solution: Given $R = \{(2, 1), (4, 7), (1, -2)\ldots\}$

Let $y = ax + b$ be the linear relation between the components of R.

Since $(2, 1) \in R$

$\Rightarrow y = ax + b \Rightarrow 1 = 2a + b$ ...(1)

Also $(4, 7) \in R$

$\Rightarrow y = ax + b \Rightarrow 7 = 4a + b$ ....(2)

Subtracting (1) from (2), we get $2a = 6 \Rightarrow a = 3$

Substituting $a = 3$ in (1), we get $b = -5$

Substituting these values of a and b in $y = ax + b$, we get $y = 3x - 5$, which is the required linear relation between the components of the given relation.

17. Let $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$. Let R be a relation from A into B defined by

$R = \{(1, x), (1, z), (3, x), (4, y)\}$

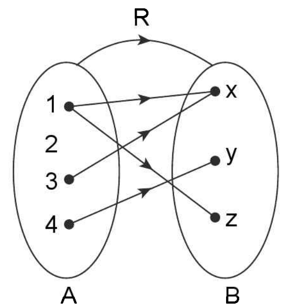
(i) Draw arrow diagram of relation R.

(ii) Represent R in the tabular form

Solution: (i) Given $R = \{(1, x), (1, z), (3, x), (4, y)\}$

As $1Rx$, $1Rz$, $3Rx$ and $4Ry$

Therefore, the given figure below is the arrow diagram of relation R.



(ii) Given $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$, $R = \{(1, x), (1, z), (3, x), (4, y)\}$

Here $1Rx$, $1Rz$, $3Rx$ and $4Ry$

While representing a relation R from A to B in tabular form, we write elements of A in first column and elements of B in first row, keeping the position corresponding to first row and first column to represent relation R.

If $(a, b) \in R$, we write 1 in the row containing a and column containing b and if $(a, b) \notin R$, then we write 0 in the row containing a and the column containing b. The tabular form of relation R has been shown in the table given below.

R	x	y	z
1	1	0	1
2	0	0	0
3	1	0	0
4	0	1	0

Solved Column Matching

1. **Column I**

(i) $R = \{(x, y) : (x. y)$ is a perfect square of a natural number and $x, y \in \mathbb{N}\}$ is

(ii) $R = \{(x, y) : y = (x)^{1/n}$ for some $n \in \mathbb{N}$; $x, y \in \mathbb{N}\}$ is

(iii) $R = \{(x, y) : (x, y)$ is a point on circle of radius $2\sqrt{2}$ units having centre at origin and $x^2 - y^2 = 0\}$ is

(iv) $R = \{(x, y) : |x - y| \leq 4; x, y \in \mathbb{R}\}$ is

Column II

(a) Reflexive
(b) Symmetric
(c) Transitive
(d) One–many
(e) Many–one

Ans. (i) → (a), (b), (c), (d), (e);
(ii) → (a), (c), (d), (e);
(iii) → (a), (b), (c), (d), (e);
(iv) → (a), (b), (c), (d), (e)

Solution: (i) $R = \{(x. y) : x . y$ is a perfect square of a natural number$\}$

xRx as x^2 is a perfect square of natural number x

Now xRy ⇒ $x.y$ is a perfect square

⇒ $y.x$ is a perfect square ⇒ yRx

Also xRy, yRz

⇒ $x{\cdot}y = k_1^2, y{\cdot}z = k_2^2$ for some $k_1, k_2 \in \mathbb{N}$

Now $(x{\cdot}y)\,(y{\cdot}z) = k_1^2 \,.\, k_2^2$

$$\Rightarrow \quad x.z = \frac{k_1^2 k_2^2}{y^2} = \left(\frac{k_1 k_2}{y}\right)^2,$$

which must be square of a natural number as left hand side is also a natural number.

⇒ xRz ⇒ R is transitive.

Since $2.2 = 4 = (2)^2 \Rightarrow 2R2$

Also $2.8 = 16 = (4)^2 \Rightarrow 2R(16)$

Thus the relation is one–many. Also the relation is symmetric

⇒ The relation is many–one

∴ (i) → (a), (b), (c), (d), (e)

(ii) $R = \{(x, y) : y = (x)^{1/n}$ for some $n \in \mathbb{N}; x, y \in \mathbb{N}\}$

Since $x = (x)^{1/1}\ \forall\ x \in \mathbb{N}$

⇒ xRx

⇒ R is reflexive.

Now if we consider $3 = (27)^{1/3}$

⇒ $27 = (3)^3$

⇒ $27 \not{R} 3$ as the power of 3 is not of the form $\frac{1}{n}$; $n \in \mathbb{N}$.

⇒ R is not symmetric.

Now xRy, yRz

⇒ $x = (y)^{1/n}$ for some $n \in \mathbb{N}$,

and $y = (z)^{1/m}$ for some $m \in \mathbb{N}$,

⇒ $x = (z)^{1/nm}$ ⇒ R is transitive.

Also $2 = (8)^{1/3}$ and $2 = (64)^{1/6}$

⇒ $2R8$ and $2R(64)$

⇒ R is one–many

Also, $8 = (64)^{1/2}$ and $4 = (64)^{1/3}$

⇒ $8R(64)$ and $4R(64)$

⇒ R is many–one.

⇒ (ii) → (a), (c), (d), (e)

(iii) $R = \{(x, y) : (x, y)$ is a point on circle of radius $2\sqrt{2}$ units having centre at origin and $x^2 - y^2 = 0\}$

$= \{(x, y) : x^2 + y^2 = 8, x^2 - y^2 = 0\}$

$= \{(x, y) : x^2 + y^2 = 8, y = \pm x\} = \{(x, \pm x) : x^2 = 4\}$

$= \{(x, \pm x) : x = \pm 2\} = \{(2, -2), (2, 2), (-2, 2), (-2, -2)\}$

∵ $(2, 2), (-2, -2) \in \mathbb{R}$

⇒ R is reflexive.

Also aRb

⇒ $bRa\ \forall\ a, b \in \{2, -2\}$

⇒ R is symmetric

Also aRb and bRc

⇒ $aRc\ \forall\ a, b, c \in \{2, -2\}$

⇒ R is transitive

Also $2R(-2)$ and $2R2$

⇒ R is One–many

∴ R is symmetric

⇒ R is Many–one

⇒ (iii) → (a), (b), (c), (d), (e)

(iv) $R = \{(x, y) : |x - y| \leq 4; x, y \in \mathbb{R}\}$

Since $|x - x| = 0 \leq 4$

⇒ $xRx\ \forall\ x \in \mathbb{R}$ ⇒ R is reflexive.

Further xRy ⇒ $|x - y| \leq 4$

⇒ $|y - x| \leq 4$ ⇒ R is symmetric.

Now let xRy and yRz

⇒ $|x - y| \leq 4$ and $|y - z| \leq 4$

Now $|x - z| = |(x - y) + (y - z)| \leq |x - y| + |y - z| \leq 4 + 4 = 8$ (∵ $|a + b| \leq |a| + |b|$)

∴ xRz

⇒ R is transitive

Now $2R2$ and $2R4$ as $|2 - 2| \leq 4$ and $|2 - 4| \leq 4$

⇒ R is one–many

∵ R is symmetric

⇒ R is many–one

⇒ (iv) → (a), (b), (c), (d), (e)

TUTORIAL EXERCISE

SECTION–III

OBJECTIVE-TYPE QUESTIONS (ONLY ONE CORRECT ANSWER)

1. If a set A has n distinct elements, the number of all relations on A is
(a) 2^{n^2} (b) n^2
(c) 2^n (d) None of these

2. Let $A = \{2, 3, 4, 5\}$ and let $R = \{(2, 2), (3, 3), (4, 4)$ $(5, 5), (2, 3), (3, 2), (3, 5), (5, 3)\}$ be a relation on A. Then R is
(a) reflexive and transitive
(b) reflexive and symmetric but not transitive
(c) reflexive and anti-symmetric
(d) none of the above

3. If R is a relation on A such that $R = R^{-1}$, then R is
(a) reflexive (b) symmetric
(c) transitive (d) None of these

4. If $A =\{x : x^2 - 3x + 2 = 0\}$, and R is a universal relation on A, then R is
(a) $\{(1, 1), (2, 2)$ (b) $\{1, 1)\}$
(c) $\{\phi\}$ (d) $\{(1,1), (1,2), (2,1), (2,2)\}$

5. An integer m is said to be related to another integer n if m is a multiple of n. Then the relation $\mathbb{R} : \mathbb{Z} \sim \{0\} \to \mathbb{Z}$ is
(a) reflexive and symmetric
(b) reflexive and transitive
(c) symmetric and transitive
(d) equivalence relation

6. If $A = \{5, 6, 7\}$ and $B = \{1, 2, 3, 4\}$, then number of elements in set $(A \times B) \times B$ is equal to
(a) 36 (b) 48
(c) 16 (d) None of these

7. If R be a relation from $A = \{2, 3, 4\}$ to $B = \{3, 5\}$ such that $a < b$ $((a, b) \in R)$, then RoR^{-1} does not contain
(a) (3, 3) and (5, 5) (b) (3, 3)
(c) (5, 5) (d) None of these

8. If R and R' are symmetric relations (not disjoint) on a set A, then the relation $R \cap R'$ is
(a) reflexive (b) symmetric
(c) transitive (d) None of these

9. A and B are two sets having 3 and 4 elements respectively and having 2 elements in common. The number of non-empty relations which can be defined from A to B is equal to
(a) 2^5 (b) $2^{10} - 1$
(c) $2^{12} - 1$ (d) None of these

10. If the relation $R : A \to B$, where $A = \{1, 2, 3, 4\}$ and $B = \{ 1, 3, 5\}$ is defined by $R = \{ (x, y); x < y, x \in A, y \in B\}$, then R^{-1} o R is equal to
(a) $\{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$
(b) $\{(3, 1), (5, 1), (5, 2), (5, 3), (5, 4)\}$
(c) $\{(3, 3), (3, 5), (5, 3), (5, 5)\}$
(d) None of these

11. The relation 'less than' in the set of natural numbers is
(a) only symmetric
(b) only transitive
(c) only reflexive
(d) equivalence relation

12. Let $P = \{(x, y)| x^2 + y^2 = 1, x, y \in \mathbb{R}\}$. Then P is
(a) reflexive (b) symmetric
(c) transitive (d) anti-symmetric

13. Let R be an equivalence relation on a finite set A having n elements. Then the number of ordered pairs in R is
(a) less than n
(b) greater than or equal to n
(c) less than or equal to n
(d) None of these

14. For real numbers x and y, we write $xRy \Leftrightarrow x - y + \sqrt{5}$ is an irrational number. Then the relation R is
(a) reflexive (b) symmetric
(c) transitive (d) None of these

15. Let X be a family of sets and R be a relation on X defined by 'A is disjoint from B'. Then R is
(a) reflexive (b) symmetric
(c) anti-symmetric (d) transitive

16. The relation R defined on the set $A = \{1, 2, 3, 4, 5\}$ by $R = \{(x, y) : |x^2 - y^2| < 16\}$ is given by
(a) $\{(1, 1), (2, 1), (3, 1), (4, 1), (2, 3)\}$
(b) $\{(2, 2), (3, 2), (4, 2), (2, 4)\}$
(c) $\{(3, 3), (3, 4), (5, 4), (4, 3), (3, 1)\}$
(d) None of these

17. A relation R is defined from $\{2, 3, 4, 5\}$ to $\{3, 6, 7, 10\}$ by $xRy \Leftrightarrow$ x is relatively prime to y. Then domain of R is
(a) $\{2, 3, 5\}$ (b) $\{3, 5\}$
(c) $\{2, 3, 4\}$ (d) $\{2, 3, 4, 5\}$

18. If $R = \{(x, y) \mid x, y \in \mathbb{Z}, x^2 + y^2 \leq 4\}$ is a relation on $\mathbb{Z}$, then domain of R is
(a) $\{0, 1, 2\}$ (b) $\{0, -1, -2\}$
(c) $\{-2, -1, 0, 1, 2\}$ (d) None of these

19. Let R be a reflexive relation on a set A and I be the identity relation on A. Then
(a) $R \subset I$ (b) $I \subset R$
(c) $R = I$ (d) None of these

20. The relation 'is subset of' on the power set $P(A)$ of a set A is
(a) symmetric
(b) anti-symmetric
(c) equivalence relation
(d) None of these

21. The relation R defined on a set A is anti-symmetric if $(a, b) \in R \Rightarrow (b, a) \in R$ for
(a) every $(a, b) \in R$
(b) no $(a, b) \in R$
(c) no $(a, b) \in R, a \neq b$
(d) None of these

22. The void relation on a set A is
(a) reflexive
(b) symmetric and transitive
(c) reflexive and symmetric
(d) reflexive and transitive

23. Which one of the following relations on $\mathbb{R}$ is an equivalence?
(a) $aR_1b \Leftrightarrow |a| = |b|$
(b) $aR_2b \Leftrightarrow a \geq b$
(c) $aR_3b \Leftrightarrow a$ divides b
(d) $aR_4b \Leftrightarrow a < b$

24. If R an equivalence relation on a set A, then R^{-1} is
(a) reflexive only
(b) symmetric but not transitive
(c) equivalence
(d) None of these

25. The relation 'congruence modulo m' is
(a) reflexive only
(b) transitive only
(c) symmetric only
(d) an equivalence relation

26. Let R and S be two equivalence relations on a set A. Then
(a) $R \cup S$ is an equivalence relation on A
(b) $R \cap S$ is an equivalence relation on A
(c) $R - S$ is an equivalence relation on A
(d) None of these

27. Let $R = \{(1, 3), (2, 2), (3, 2)\}$ and $S = \{(2, 1), (3, 2), (2, 3)\}$ be two relations on set $A = \{1, 2, 3\}$. Then RoS =
(a) $\{(1, 3), (2, 2), (3, 2), (2, 1), (2, 3)\}$
(b) $\{(3, 2), (1, 3)\}$
(c) $\{(2, 3), (3, 2), (2, 2)\}$
(d) $\{(2, 3), (3, 2)\}$

28. $x^2 = xy$ is a relation which is
(a) symmetric (b) reflexive
(c) transitive (d) None of these

29. The number of reflexive relations of a set with four elements is equal to
(a) 2^{16} (b) 2^{12}
(c) 2^8 (d) 2^4

30. If A is the set of even natural numbers less than 8 and B is the set of prime numbers less than 7, then the number of relations from A to B is
(a) 2^9 (b) 9^2
(c) 3^2 (d) $2^9 - 1$

31. If $A = \{(x, y) : x^2 + y^2 = 25\}$ and $B = \{(x, y) : x^2 + 9y^2 = 144\}$, then $A \cap B$ contains
(a) one point (b) three points
(c) two points (d) four points

32. Let R be the relation on the set R of all real numbers defined by aRb iff $|a - b| \leq 1$. Then R is
(a) reflexive and symmetric
(b) symmetric only
(c) transitive only
(d) anti-symmetric only

SECTION–IV

OBJECTIVE-TYPE QUESTIONS (MORE THAN ONE CORRECT ANSWER)

1. Let $X = \{1, 2, 3, 4, 5\}$ and $Y = \{1, 3, 5, 7, 9\}$. Which of the following is/are relations from X to Y?
(a) $R_1 = \{(x, y) | y = 2 + x, x \in X, y \in Y\}$
(b) $R_2 = \{(1, 1), (2, 1), (3, 3), (4, 3), (5, 5)\}$
(c) $R_3 = \{(1, 1), (1, 3), (3, 5), (3, 7), (5, 7)\}$
(d) $R_4 = \{(1, 3), (2, 5), (2, 4), (7, 9)\}$

2. $\mathbb{N}$ is the set of natural numbers. The relation R is defined on $\mathbb{N}$ as follows $(a, b)\ R(c, d) \Leftrightarrow a + d = b + c$ is
(a) reflexive
(b) symmetric
(c) transitive
(d) anti-symmetric

3. If $A = \{(x, y): y = e^{2x}, x \in \mathbb{R}\}$ and $B = \{(x, y): y = e^{-2x}, x \in \mathbb{R}\}$, then:
(a) $(A \cap B) = \phi$
(b) $(A \cap B) \neq \phi$
(c) $A \cap B$ is a singleton set
(d) None of these

4. If $A = \{(x, y): x^2 + y^2 \leq 1; x, y \in \mathbb{R}\}$ and $B = \{(x, y): x^2 + y^2 \geq 4; x, y \in \mathbb{R}\}$, then:
(a) $A - B \neq \phi$ (b) $B - A = \phi$
(c) $(A \cap B) \neq \phi$ (d) $(A \cap B) = \phi$

5. A relation $R : \mathbb{R} \to \mathbb{R}$ such that aRb iff $a \leq b^2$ is.
(a) reflexive (b) not symmetric
(c) non-transitive (d) not equivalence

6. A relation R on set $A = \{1, 2, 3, 4, 5, 6\}$ given by $R = \{(x, y) : y = x + 1\}$ is.
(a) not reflexive
(b) symmetric
(c) non-transitive
(d) not an equivalence relations.

7. Let R and be two relations on a set A. Then
(a) R and S are transitive, then $R \cup S$ is also transitive
(b) R and S are transitive, then $R \cap S$ is also transitive
(c) R and S are reflexive, then $R \cap S$ is also reflexive
(d) R and S are symmetric, then $R \cup S$ is also symmetric

8. Let n be a fixed positive integer. Define a relation R on the set $\mathbb{Z}$ of integers by $aRb \Leftrightarrow n\ |(a - b)$. Then R is
(a) reflexive
(b) symmetric
(c) transitive
(d) an equivalence relation

9. Let A be the non-void set of the children in a family. The relation 'x is a brother of y' on A is
(a) reflexive (b) symmetric
(c) transitive (d) None of these

SECTION–V

ASSERTION AND REASON-TYPE QUESTION

1. A: A relation $R : \mathbb{Z} \to \mathbb{Z}$ defined such that aRb iff $a|b$ (a divides b), then R is reflexive relation.
R: A relation $R : A \to B$ is called reflexive iff $aRa\ \forall\ a \in A$.

2. A: If $n(A) = m$ and $n(B) = k$, then total number of relations that can be defined from set A to set $B = 2^{km}$.
R: Every relation $R : A \to B$ is a subset of $A \times B$.

3. A: A relation $R : \mathbb{C} \to \mathbb{C}$ such that $(a, b) \in R$ iff $|a - b| \leq 1$, then R is an equivalence relation.
R: A relation is called equivalence iff it is reflexive, symmetric as well as transitive.

4. A: A relation $A \times B$ is universal relation (largest relation) defined from set A to set B.
R: Any relation $R : A \to B$ is always a subset of $A \times B$.

5. A: A relation $R : \mathbb{R} \to \mathbb{R}$ such that aRb iff $2|(a - b)$ is not an equivalence relation.
R: For a relation to be an equivalence it has to be reflexive, symmetric as well as transitive.

SECTION–VI

LINKED COMPREHENSION-TYPE QUESTIONS

A: A relation R on a set 'B' is a subset of $B \times B$, i.e., $R = \{(x, y) : x, y \in B\} \subseteq B \times B$. If $(x, y) \in R$, then we write this fact as xRy (read as 'x is related to y'). Now R is called Reflexive, if $x R x \ \forall\ x \in B$, symmetric if xRy

$\Rightarrow yRx$ and transitive if

xRy and $yRz \Rightarrow xRz$. If R is reflexive, symmetric and transitive, then it is called an equivalence relation. Let R_1 and R_2 be two relations on the set ($\mathbb{N} \times \mathbb{N}$ where $\mathbb{N}$ is set) of natural numbers defined as $R_1 = \{((a, b)\ (c, d)) : ad = bc\}$ and $R_2 = \{((a, b), (c, d)) : a + d = b + c\ \}$. Now, choose the most appropriate alternative in the following

1. (a) R_1 is relfexive but R_2 is not reflexive
(b) R_1 is not reflexive but R_2 is reflexive
(c) R_1 and R_2 both are reflexive
(d) R_1 and R_2 both are not reflexive

2. Which of the following is true?
(a) R_1 is symmetric but R_2 is not symmetric
(b) R_1 and R_2 both are symmetric
(c) R_2 is symmetric but R_1 is not symmetric
(d) R_1 and R_2 both are not symmetric

3. Which of the following is correct?
(a) R_1 is transive but R_2 is not transitive
(b) R_1 and R_2 both transitive
(c) R_1 is not transitive but R_2 is transitive
(d) R_1 and R_2 both are not transitive

4. Which of the following is true?
(a) R_1 and R_2 both are equivalence relations
(b) R_1 is equivalence but R_2 is not
(c) R_1 is not equivalence but R_2 is equivalence
(d) R_1 and R_2 both are not equivalence.

SECTION–VII

COLUMN MATCHING-TYPE QUESTION

1. **Column A**

(i) $R : A \to A$ where $A = \{1, 2, \ldots.., 14\}$ defined by $R = \{(x, y) : 3x - y > 0\}$ is
(ii) $R : \mathbb{N} \to \mathbb{N}$ defined by $R = \{(x, y) : y = x + 4 \text{ and } x < 4\}$ is
(iii) $R : \mathbb{Z} \to \mathbb{Z}$ defined by $R = \{(x, y): x - y \in \mathbb{Z}\}$ is
(iv) $R : A \to A$, where A is set of humans in a town defined by $R = \{(a, b)$:a and b work in same place$\}$ is

Column B

(a) Reflexive relation
(b) Symmetric relation
(c) Transitive relation
(d) Equivalence relation
(e) Anti-symmetric relation

2. **Column A**

(i) R: $\mathbb{R} \to \mathbb{R}$: defined as aRb iff $a \le b$
(ii) $R : \mathbb{R} \to \mathbb{R}$: defined as aRb iff $a \le b^3$ is
(iii) $R : A \to A$ defined as $R = \{(a, b):|a - b| \text{ is even}\}; A = \{1, 2, 3, 4, 5\}$ is
(iv) $R : H \to H$ where H is set of Human beings defined as $R = \{(x, y) : x \text{ and } y \text{ have same age}\}$ is

Column B

(a) Reflexive
(b) Symmetric
(c) Transitive
(d) Equivalence
(e) Anti-symmetric
(f) None of these

3. **Column A**

(i) $R = \{(X, Y) : X, Y \in P(A); X \cap Y \ne \phi\}$, where $P(A)$ denotes the power set of A, is
(ii) $R : \{(x, y) : x, y \in \mathbb{N} \text{ and L.C.M. } (x, y) = 12\}$ is
(iii) $R = \{(x, y) : x, y \in \mathbb{C}$ and y is complex conjugate of $x\}$; $\mathbb{C}$ is the set of complex numbers.
(iv) $R = \{(x, y) : 2x^2 + 3y^2 - 5xy = 0; x, y \in \mathbb{R}\}$ is

Column B

(a) Reflexive
(b) Symmetric
(c) Transitive
(d) Surjective
(e) Injective.

Answer Keys

SECTION–III

1. (a)	**2.** (b)	**3.** (b)	**4.** (d)	**5.** (b)	**6.** (b)	**7.** (d)	**8.** (b)	**9.** (c)	**10.** (d)
11. (b)	**12.** (b)	**13.** (b)	**14.** (a)	**15.** (b)	**16.** (d)	**17.** (d)	**18.** (c)	**19.** (b)	**20.** (b)
21. (c)	**22.** (b)	**23.** (a)	**24.** (c)	**25.** (d)	**26.** (b)	**27.** (c)	**28.** (b)	**29.** (b)	**30.** (a)
31. (d)	**32.** (a)								

SECTION–IV

1. (a,b,c)	**2.** (a,b,c)	**3.** (b,c)	**4.** (a,d)	**5.** (b,c,d)	**6.** (a,c,d)	**7.** (b,c,d)	**8.** (a,b,c,d)	**9.** (c)

SECTION–V

1. (d)	**2.** (a)	**3.** (d)	**4.** (a)	**5.** (d)

SECTION–VI

1. (c)	**2.** (b)	**3.** (b)	**4.** (a)

SECTION–VII

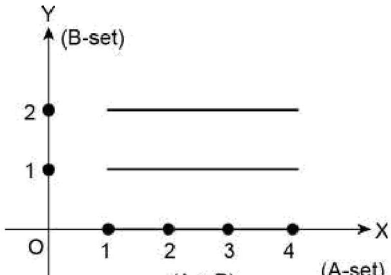
1. (i) → (a)	(ii) → (c),(e)	(iii) → (a,b,c,d)	(iv) → (a,b,c,d)
2. (i) → (a,c,e)	(ii) → (f)	(iii) → (a,b,c,d)	(iv) → (a,b,c,d)
3. (i) → (b)	(ii) → (b)	(iii) → (b,d,e)	(iv) → (a,d)

HINTS AND SOLUTIONS

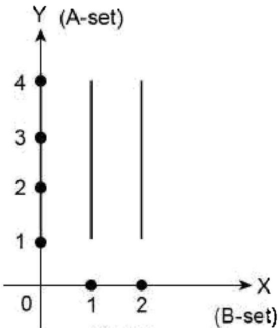
TEXTUAL EXERCISE–1: (SUBJECTIVE)

1. $A = \{x: x \in \mathbb{N}, x \leq 4\} = \{1, 2, 3, 4\}$; $B = \{y: y \in W, y \leq 2\} = \{0, 1, 2\}$.
   (a) $A \times B = \{(1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2), (3, 0), (3, 1), (3, 2), (4, 0), (4, 1), (4, 2)\}$
   (b) $n(A \times B) = n(A) \times n(B) = 4 \times 3 = 12$

2. **(a)** $A = \{x : x \in \mathbb{R};\ 1 \leq x \leq 4\}$; $B = \{y : y \in W, y \leq 2\}$
   $\Rightarrow$ $A = [1, 4]$ and $B = \{0, 1, 2\}$
   $\therefore$ $(A \times B)$ will be plotted in shown below.

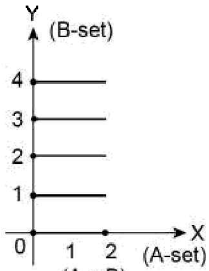


(A x B)

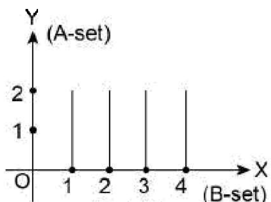


(B x A)

   (b) $A = \{y: y \in \mathbb{R};\ 0 \leq y \leq 2\}$; $B = \{x: x \in \mathbb{N}, x \leq 4\}$
   $\Rightarrow$ $A = [0, 2]$ and $B = \{0, 1, 2, 3, 4\}$
   $\therefore$ $(A \times B)$ and $(B \times A)$ will be plotted in shown below.

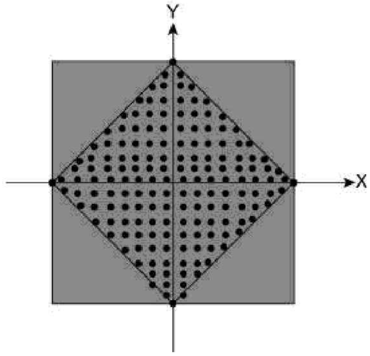


(A x B)



(B x A)

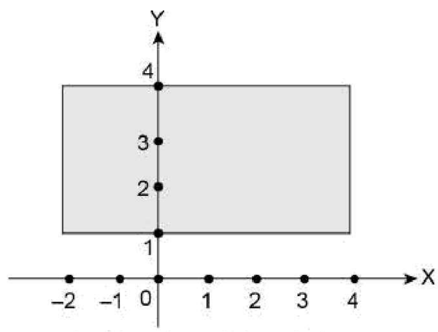
3. $A = \{x: x \in \mathbb{R}, -1 \leq x \leq 1\}$; $B = \{y: y \in \mathbb{R}, -1 \leq y \leq 1\}$
   $\Rightarrow$ $x \in A = [-1, 1]$ and $y \in B = [-1, 1]$ and $|x| + |y| \leq 1$.
   The common region is shown by shaded portion and equals $4\left(\frac{1}{2}(1)(2)\right) = 2$ sq. units.



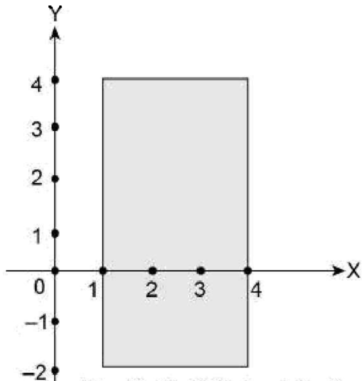
Here ■ $= (A \times B)$; ■ $= \{(x, y): |x| + |y| \leq 1\}$ = common Area.

4. A = $\{x: x \in$ non-negative integers and $x^2 - 2x - 8 \leq 0\}$ and $B = \{x: x \in \mathbb{N}$ and $x^2 - 5x + 4 \leq 0\}$
   $\Rightarrow$ A $= \{x: x \geq 0, x \in \mathbb{Z}$ and $(x - 4)(x + 2) \leq 0\} = \{0, 1, 2, 3, 4\}$ and $B = \{x: x\ \mathbb{N}$ and $(x - 1)(x - 4) \leq 0\} = \{1, 2, 3, 4\}$
   $\Rightarrow$ $A \cap B = \{1, 2, 3, 4\}$
   $\Rightarrow$ We know that $[(A \times B) \cap (B \times A)] = [(A \cap B) \times (B \cap A)]$
   $\Rightarrow$ $n[(A \times B) \cap (B \times A)] = n[(A \cap B) \times (B \cap A)] = [n(A \cap B)]^2 = (4)^2 = 16$

5. **(i)** If elements of set A are natural, and that of B are real in above problem, then
   A $= \{1, 2, 3, 4\}$ and $B = [1, 4]$; then $(A \cap B) = \{1, 2, 3, 4\}$
   $\Rightarrow$ $n[(A \times B) \cap (B \times A)] = n[(A \cap B) \times (B \cap A)] = [n(A \cap B)]^2 = (4)^2 = 16$
   (ii) If elements of the set A and B are real numbers, then A $= [-2, 4]$; B[1, 4]
   $\Rightarrow$ $(A \cap B) = [1, 4]$ having infinitely many uncountable numbers of elements
   $\Rightarrow$ $n[(A \times B) \cap (B \times A)] = [n(A \cap B)]^2$ = infinitely many found in area of 9 square units bounded by straight lies $x = 1$,
   $x = 4, y = 1, y = 4$.
   $\therefore$ $(A \times B)$ and $(B \times A)$ will be plotted as shown below.



(A x B); A=[–2,4]; B = [1, 4]



(B x A); B=[1,4]; A = [–2, 4]

If elements of A and B are rational or irrational, then it is impossible to represent $A \times B$ and $B \times A$. Geometrical as between any two real numbers, there lie infinitely many rationals and irrationals .

TEXTUAL EXERCISE–1: (OBJECTIVE)

1. **(b)** $\because A-(B \times C)$ is a subset of A; where as $(A-B) \times (A-C)$ is a subset $A \times A$
Thus the equality is invalid.

2. **(c, d)** $\because (A \times B) \times C = \{((x, y), z): x \in A, y \in B, z \in C\}$, similarly $A \times (B \times C)$
Where as $A \times B \times C = \{(x, y, z): x \in A, y \in B, z \in C\}$,
$\therefore$ $(A \times B) \times C \neq A \times B \times C \neq A \times (B \times C)$
$\Rightarrow$ (c) is not true.
Also (d) is not true.
See the Venn–Diagrams
Clearly, $(A - B) - C \neq A - (B - C)$

3. **(d)** $\therefore A \times (B \times C) = \{(x, (y, z)): x \in A, y \in B, z \in C\}$, where as $(A \times B) \times (A \times C) = \{((x, y), (z, w)); x, z \in A, y \in B, w \in C\}$
Clearly, then elements of two sets are of different forms, thus equality for two sets is invalid.

4. **(a)** $\therefore A \times B$ is a subset of $U \times U$
$\Rightarrow$ (a) is not true.

5. **(a, b, c, d)** Each of the options can be verified by using the results
That (i) If $A \subseteq B$ and $C \subseteq D$, then $(A \times C) \subseteq (B \times D)$
(ii) If $A \subseteq B$, then $A \cap B =$ A and $A \cup B = B$.

6. **(a), (c), (d)** It is standard result that if $(A \times B) \subseteq (C \times D)$, then $A \subseteq C$ and $B \subseteq D$,
$\Rightarrow$ (a), and (c) are true, but (b) is not true.
Also, $A \times B = C \times D \Leftrightarrow A = C$ and $B = D$
$\Rightarrow$ (d) is also true.

7. **(c)** $n(A \times B) = n(A) \times n(B) = 4 \times 26 = 104$

8. **(a), (b), (c)** $n(P(A \times B) = (2)^{n(A \times B)} = (2)^{n(A) \times n(B)} = (2)^{4 \times 5} = (2)^{20} = (4)^{10} = (16)^5$

9. **(c)** $A = \{\phi\}, B = \{\{\{\}\}\}, n(A \times B) = n(A) \times n(B) = 1 \times 1 = 1$

10. **(d)** If none of A and B is infinite, $A \times B$ can't have infinitely many elements. Thus at least one of A and B must be infinite set for $A \times B$ to be infinite.

11. **(b)** $A = \{(x, y): x, y \in \mathbb{R}$ and $x^2 = 4y\}$ and $B = \{(x, 4); x \in \mathbb{R}\}$
A represents an upwards parabola and B represents a straight line $y = 4$.
$\therefore$ $A \cap B$ represents two points of intersection of parabola and straight line which are $(-4, 4)$ and $(4, 4)$.
$\therefore$ $(A \cap B) \cup \{(0, 0)\} = \{(-4, 4), (4, 4), (0, 0)\}$ forms an isosceles triangle which is not equilateral.

12. **(a)** $A = \{(x, y): x, y \in \mathbb{R}$ and $x^2 = 4y\}$
$B = \{(x, y): x, y \in \mathbb{R}$ and $y = |x|\}$
A represents upwards parabola $x^2 = 4y$, where as B represents curve $y = |x|$
$\Rightarrow$ $A \cap B$ is the set of their points of intersection.
$\Rightarrow$ $\dfrac{x^2}{4} = \pm x$ $\qquad \Rightarrow$ $x(x \pm 4) = 0$
$\Rightarrow$ $x = 0$ or $x = \pm 4$
$\Rightarrow$ $\{(0, 0), (4, 4)$ and $(-4, 4)\}$ is the set $(A \cap B)$.
$\Rightarrow$ Area of Δ formed by these points of intersection $= \dfrac{1}{2}(8)(4) = 16$ square units.

13. **(a), (b), (c)** $A = \{\phi, \{\{\{\}\}\}\}$
$n(A \times A) = n(A) \times n(A) = 4$
$\Rightarrow$ $n\{P(P(A \times A))\} = (2)^4 = 16$
$\Rightarrow$ $n\{P(P(A \times A))\} = (2)^{16} = (16)^4 = (256)^2$

14. **(b)** $n(A) = 50, n(B) = 60, n(A \cup B) = 100$
$\Rightarrow$ $n(A \cap B) = n(A) + n(B) - n(A \cup B) = 50 + 60 - 100 = 10$
$\Rightarrow$ $n[(A \times B) \cap (B \times A)] = [n(A \cap B)]^2 = (10)^2 = 100$

15. **(c)** $n[(A \times B \times C) \cap (B \times C \times A)] = n(A \cap B) \times n(B \cap C) \times n(C \cap A) = 2 \times 3 \times 5 = 30$

TEXTUAL EXERCISE–2: (SUBJECTIVE)

1. $A = \{1, 2, 3, 4, 6\}$ and $R = \{(a, b): a, b \in A, b$ is divisible by $a\}$
(i) $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (2, 2), (2, 4) (2, 6), (3, 3), (3, 6), (4, 4), (6, 6)\}$
(ii) Domain of $R = D_R = \{1, 2, 3, 4, 6\} =$ A
(iii) Range of $R = P_R = \{1, 2, 3, 4, 5, 6\} =$ A

2. $R = \{(x, y): x, y \in \mathbb{Z}; (x + y)(y + 2004) + 1 = 0\}$
$\therefore$ $(x + y)(y + 2004) = -1$ and $x, y \in \mathbb{Z}$
$\Rightarrow$ $x + y = -1, y + 2004 = 1$ or $x + y = 1, y + 2004 = -1$
$\Rightarrow$ $x = 2002, y = -2003$ or $x = 2006, y = -2005$
(i) $R = \{(2002, -2003), (2006, -2005)\}$
(ii) Domain $R = D_R = \{2002, 2006\}$
(iii) Range of $R = R_R = \{-2003, -2005\}$

3. $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3\ldots 66, 67\}$.
(i) $a\,R\,b \Leftrightarrow$ a is square root of b and $R \subseteq (A \times B)$
$\Rightarrow$ $R = \{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25)\}$
$\Rightarrow$ $D_R = \{1, 2, 3, 4, 5\} = A$ and $R_R = \{1, 4, 9, 16, 25\}$

(ii) $a\,R\,b \Leftrightarrow$ a is cube root of b and $R \subseteq (A \times B)$

$\Rightarrow$ $R = \{(1, 1), (2, 8), (3, 27), (4, 64)\}$

$\Rightarrow$ $D_R = \{1, 2, 3, 4\}$ and $R_R = \{1, 8, 27, 64\}$

4. **(i)** $R = \{(x, x^3) : x$ is a prime number less than $10\}$

$= \{(2, 8), (3, 27), (7, 343)\}$

$\Rightarrow$ $D_R = \{2, 3, 5, 7\}$: $R_R = \{8, 2, 7, 125, 343\}$

(ii) $R = \{(4x + 3, 1 - x); x \le 4; x \in \mathbb{N}\} \subseteq \mathbb{N} \times \mathbb{N}$ (given)

$= \{\ \}$ as $1 - x \in \mathbb{N}$ $\Rightarrow$ $1 - x \ge 1$

$\Rightarrow$ $x \le 0$

Which is impossible as $x \in \mathbb{N}$.

$\Rightarrow$ $D_R = \{\}, R_R = \{\}$

(iii) $R = \{(x, \frac{1}{x}); 0 < x < 4$ and $x \in \mathbb{N}\} \subseteq \mathbb{N} \times \mathbb{N}$ (given)

$\Rightarrow$ $x, \frac{1}{x} \in \mathbb{N}$, which is impossible.

$\Rightarrow$ $R = \{\ \}$ $\Rightarrow$ $D_R = \{\}$ and $R_R = \{\}$.

5. $R = \{(x, y); x, y \in W; y = 2x - 4\}$; $(a, -2) \in \mathbb{R}$ and $(4, b^2) \in \mathbb{R}$

$\Rightarrow$ $-2 = 2a - 4$ and $b^2 = 8 - 4$

$\Rightarrow$ $2a = 2$ and $b^2 = 4$ $\Rightarrow$ $a = 1$ and $b = \pm 2$

$\therefore$ $R_1 = \{(a, b)\} = \{(1, 2), (1, -2)\}$

6. Let $R = \{(x, y)\} = \{(0, 2), (-1, 5), (2, -4)\}$ and for linear relation between x and y.

Let $y = ax + b$ $\Rightarrow$ $2 = b$ and $5 = -a + b$

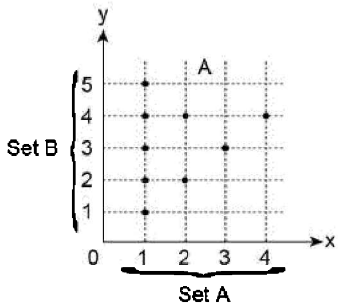
$\Rightarrow$ $a = b - 5 = 2 - 5 = -3$

$\Rightarrow$ $y = -3x + 2$ $\Rightarrow$ $3x + y = 2$

7. $A = \{1, 2, 3, 4\}$; $B = \{1, 2, 3, 4, 5\}$ and $R \subseteq A \times B$; where $a\,R\,b \Leftrightarrow a / b$

$\Rightarrow$ $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 2), (2, 4), (3, 3), (4, 4)\}$

(i) (Representation of R by Lattice diagram)



(ii) (Representation of R by tabular form)

$B \rightarrow$

R	1	2	3	4	5
1	1	1	1	1	1
2	0	1	0	1	0
3	0	0	1	0	0
4	0	0	0	1	0

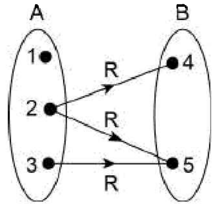
$A \uparrow$

8. $A = \{1, 2, 3\}$; $B = \{4, 5\}$; $R \subseteq A \times B$ given by $R = \{(2, 4), (2, 5), (3, 5)\}$

(i)

R	4	5
1	0	0
2	1	1
3	0	1

(ii)



9. **(i)** $R = \{(x, y) : y = x - 2$ and $x \in \{5, 6, 7\}\}$

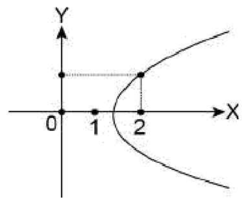
(ii) $R = \{(5, 3), (6, 4), (7, 5)\}$

10. **(a)** $y^2 = 2x - 3$

$\because$ $(2, -1)$ and $(2, 1) \in \mathbb{R}$

$\Rightarrow$ R is one–many.

From graph of $y^2 = 2x - 3$



Clearly by horizontal line test, the relation is injective but range of $R = (-\infty, \infty) \Rightarrow R$ is subjective with co-domain $\mathbb{R}$.

(b) $y = x^2 + 4$

$\because$ $(-1, 5)$ and $(1, 5) \in \mathbb{R}$

$\Rightarrow$ R is many–one $\Rightarrow$ R is not injective

Also $y \ge 4$

$\Rightarrow$ R is not surjective with co-domain $\mathbb{R}$.

(c) $y = ax^2 + bx + c$

$\because$ $\left(\frac{-b}{2a} - k, y\right)$ and $\left(\frac{-b}{2a} + k, y\right) \in \mathbb{R}\ \forall \in \mathbb{R}$ as graph is symmetric about the line $x = \frac{-b}{2a}$.

$\Rightarrow$ Relation R is many–one, i.e., not injective.

Also $y \ge -\frac{(b^2 - 4ac)}{4a}$ for $a > o$ and $y \le -\frac{(b^2 - 4ac)}{4a}$ for $a < 0$

$\Rightarrow$ Range $\ne \mathbb{R}$ R is not subjective with co-domain $\mathbb{R}$.

(d) $y^3 = 5x + 4$

$\Rightarrow$ $y = 3\sqrt{5x + 4} \in \mathbb{R}$ and range of relation $= \mathbb{R}$.

$\Rightarrow$ R is subjective.

Now, $x_2 > x_1$ $\Rightarrow$ $5x_2 > 5x_1$

$\Rightarrow$ $5x_2 + 4 > 5x_1 + 4$

$\Rightarrow$ $3\sqrt{5x_2 + 4} > 3\sqrt{5x_1 + 4}$

$\Rightarrow$ $y_2 > y_1$ where (x_1, y_1) and $(x_2, y_2) \in \mathbb{R}$

$\Rightarrow$ For $x_1 \neq x_2$ and $y_1 \neq y_2$

$\Rightarrow$ Relation is injective.

(e) $y = \sqrt{3x-4}$

Clearly range of relation $= [0, \infty) \neq \mathbb{R}$

$\Rightarrow$ Relation is not subjective,

Also for $x_2 > x_1$, $\sqrt{3x_2 - 4} > \sqrt{3x_1 - 4}$

$\Rightarrow$ $y_2 > y_1$ for $(x_1, y_1), (x_2, y_2) \in \mathbb{R}$

$\Rightarrow$ For $x_1 \neq x_2, y_1 \neq y_2$

$\Rightarrow$ Relation R is injective.

TEXTUAL EXERCISE–2: (OBJECTIVE)

1. **(b), (c)** No. of relations from A to A; where $A = \{2, 3, 4, 5\}$ = number of subset $A \times A$
$= (2)^{n(A \times A)} = (2)^{(4)^2} = (2)^{16} = (16)^4$

2. **(c)** $A = \{a, e, i, o, u\}$,
Number of relations on power set of $A = (2)^{n(P(A) \times P(A))}$
$= (2)^{[n(P(A))]^2}$
$= (2)^{\left[2^{n(A)}\right]^2} = (2)^{\left[2^5\right]^2} = (2)^{2^{10}}$

3. **(a)** $A = \{1, 2, 3\}$ and $B = \{a, b, c, d\}$;
Number of relations from set of set A to set B
$= 2^{n(A \times B)} = 2^{3 \times 4} = (2)^{12}$

4. **(c)** $n(A) = m$, $n(B) = n$ and $m > n$.
$k = n[P(A \times B)] = (2)^{n(A \times B)} = (2)^{mn}$ and $P = n[P(B \times A)]$
$= (2)^{n(B \times A)} = (2)^{nm}$
$\Rightarrow$ $k = p$

5. **(c)** $A = \{2, 3, 5, 6, 10\}$ and '$x R y$ if $x < y$ and x divides y'; $R \subseteq (A \times A)$
$\Rightarrow$ $R = \{(2, 3), (2, 5), (2, 6), (2, 10), (3, 5), (3, 6), (3, 10), (5, 6), (5, 10), (6, 10)\}$
$\Rightarrow$ Domain of $R = \{2, 3, 5, 6\}$

6. **(c)** $R \subseteq \mathbb{N} \times \mathbb{N}$; $x R y$ if '$3x + 5y = 53$'
For $x = 1, y = 10$; For $x = 6, y = 7$; For $x = 11, y = 4$; For $x = 16, y = 1$
$\Rightarrow$ Range of relation $= \{1, 4, 7, 10\}$

7. **(b)** $A = \{2, 3, 5, 7, 10\}$
$R_1 = \{(x, y): x/(y-1)\}$; $R_1 \subseteq A \times A$ and $R_2 = \{(x, y): x + y = 10\}$; $R_2 \subseteq A \times A$
$\Rightarrow$ $R_1 = \{(2, 3), (2, 5), (2, 7), (3, 7), (3, 10)\}$ and $R_2 = \{(3, 7), (5, 5), (7, 3)\}$
$\Rightarrow$ $R = R_1 \cap R_2 = \{(3, 7)\}$

8. **(b), (c)** $R \subseteq \mathbb{N} \times \mathbb{N}$ and $x R y$ if '$x + y$ divides 10'
$\Rightarrow$ $R = \{(1, 1), (1, 4), (2, 3), (3, 2), (4, 1), (1, 9), (2, 8), (3, 7), (4, 6), (5, 5), (6, 4), (7, 3), (8, 2), (9, 1)$
Clearly the relation is one–many as well as many–one.

9. **(c), (d)** $^R\subseteq A \times A$; where $A = \{2, 7, 9, 11\}$ and $x R y$ if 'x divides y'
$\Rightarrow$ $R = \{(2, 2), (7, 7), (9, 9), (11, 11)\}$
Clearly R is one–one.
Also each element of set A has a unique image in A.
Thus R is also a function from set A to itself.

10. **(a), (c), (d)** For a relation to be a function from set A to set B, it must be one–one or many–one and must have its domain A.

11. **(d)** $A = \{x : x^2 - 3x + 2 = 0; x \in \mathbb{R}\}$
$\Rightarrow$ $A = \{1, 2\}$ and $R = A \times A$
$\Rightarrow$ $R = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

12. **(a)** $A = \{1, 2, 3\}$
$R_1 = \{(1, 2), (3, 2), (1, 3)\}$
$R_2 = \{(1, 3), (3, 6), (2, 1), (1, 2)\}$
Clearly R_1 is one–many relation as well as many–one but R_2 is one–many as $(1, 3), (1, 2) \in R_2$
But $6 \notin A$, implies R_2 is not a relation on A.

13. **(c)** $A = \{a, b, c, d\}$; $B = \{b, c, d, e\}$
$n[(A \times B) \cap (B \times A)] = n[(A \cap B) \times (B \cap A)]$
$= [n(A \cap B)]^2 = (3)^2 = 9$

TEXTUAL EXERCISE–3: (SUBJECTIVE)

1. $R \subseteq \mathbb{N} \times \mathbb{N}$ such that $R = \{(a, b); a^2 + b^2 = ab\}$
(i) $(a, a) \in \mathbb{R}\ \forall\ a \in \mathbb{N}$
$\Rightarrow$ $a^2 + a^2 = a^2$ $\Rightarrow$ $a^2 = 0$, which is impossible
$\Rightarrow$ False
(ii) $(a, b) \in \mathbb{R}$ $\Rightarrow$ $a^2 + b^2 = ab$
$\Rightarrow$ $b^2 + a^2 = ba$ $\Rightarrow$ $(b, a) \in \mathbb{R}$
$\Rightarrow$ True
(iii) $(a, b) \in \mathbb{R}$ $\Rightarrow$ $a^2 + b^2 = ab$
$\Rightarrow$ $a^2 - ab + b^2 = 0$
It is a quadratic in 'a' (or b) with discriminate $= b^2 - 4b^2 = -3b^2 < 0$
$\Rightarrow$ No real values of a and b satisfy $a^2 + b^2 = ab$
$\Rightarrow$ $R = \{\ \}$
Thus, by default R is transitive \
$\Rightarrow$ True

2. **(i)** For every $a \in \theta$, $a - a = 0 \in \mathbb{Z}$
$\Rightarrow$ $(a, a) \in \mathbb{R}\ \forall\ a \in \theta$
(ii) For $a, b \in \theta$, $a - b \in \mathbb{Z}$
$\Rightarrow$ $(b - a) \in \mathbb{Z}$
$\therefore$ $(a, b) \in \mathbb{R}$ $\Rightarrow$ $(b, a) \in \mathbb{R}\ \forall\ a, b \in \theta$

3. $\therefore$ $(2, 2), (3, 3), (6, 6) \in R_1, R_4$ and $A = \{2, 3, 6\}$
$\Rightarrow$ R_1 and R_4 represented in (i) and (iv) respectively reflexive.

4. $A = \{1, 2, 3, 5\}$
Clearly, whenever $(a, b) \in R_2$
$\Rightarrow$ $(b, a) \in R_2$
$\Rightarrow$ R_2 is symmetric
Also $R_4 = A \times A$ being universal relation on A is symmetric.

5. $\therefore$ $(2, 6) \notin R_1 \Rightarrow R_1$ is not transitive. Similarly $(3, 5) \notin R_2$
$\Rightarrow$ R_2 is not transitive.
Also $(2, 2), (3, 3), (5, 5) \notin R_4$
$\Rightarrow$ R_4 is not transitive.
$\therefore$ R_3 is the only transitive relation.

6. **(i)** Only reflexive
(ii) Only symmetric

(iii) Only transitive
As there are no ordered pairs (a, b) and $(b, c) \in \mathbb{R}$.
(iv) $R_4 = A \times A$, being universal relation is reflexive, Symmetric as well as transitive, i.e., equivalence.

7. **(i)** Let a = 2
$\Rightarrow$ $1 - a^2 = 1 - 4 = -3 \not> 0$
$\Rightarrow$ (a, a), $(2, 2) \notin \mathbb{R}$
$\Rightarrow$ R is not reflexive on $\mathbb{R}$.
(ii) Clearly $(a, b) \in \mathbb{R}$
$\Rightarrow$ $1 - ab > 0$
$\Rightarrow$ $1 - ba > 0$ $\Rightarrow$ $(b, a) \in \mathbb{R}$
$\Rightarrow$ R is symmetric
(iii) Let $a = \frac{1}{2}, b = \frac{1}{5}, c = 4$
$\Rightarrow$ $1 - ab = 1 - \frac{1}{10} = \frac{9}{10} > 0$
$\Rightarrow$ $(a, b) \in \mathbb{R}$
Also $1 - bc = 1 - \frac{4}{5} = \frac{1}{5} > 0$
$\Rightarrow$ $(b, c) \in \mathbb{R}$
Now, $1 - ac = 1 - 2 = -1 < 0$
$\Rightarrow$ $(a, c) \notin \mathbb{R}$
$\Rightarrow$ R is not transitive.

8. R_1 and $\mathbb{Z}$ and $(a, b) \in R_1 \Leftrightarrow |a - b| \le 7$
$(a, a) \in R_1 \Leftrightarrow |a - b| \le 7 \Leftrightarrow 0 \le 7$, which is true.
$\Rightarrow$ R_1 is reflexive.
Now, $(a, b) \in R_1 \Leftrightarrow |a - b| \le R_1$
$\Leftrightarrow |b - a| \le 7 \Leftrightarrow (b, a) \in R_1$
$\Rightarrow$ R_1 is symmetric.
Further, let $a = 2$, $b = 8$, $c = 12$, then $|a - b| = 6 \le 7$
$\Rightarrow$ $(a, b) \in R_1$
And $|b - c| = 4 \le 7$
$\Rightarrow$ $(b, c) R_1$
But $|a - c| = 10 \not< 7$ $\Rightarrow$ $(a, c) R_1$
$\Rightarrow$ R_1 is not transitive.

10. R_1: $a/a\ \forall\ a \in \mathbb{Z}$ and a $\ne 0$, but 0 does not divide 0
$\Rightarrow$ R_1 is not reflexive
But $a/b \not\Rightarrow b/a$
$\Rightarrow$ $(b, a) \in R_1 \not\Rightarrow (a, b) \in R_1$
$\Rightarrow$ R_1 is not symmetric.
Clearly, a/b and b/c
$\Rightarrow$ a/c
$\Rightarrow$ transitivity of R_1
$\Rightarrow$ $R_1 \rightarrow$ (c)
R_2: Since each triangle is similar to itself
$\Rightarrow$ Reflexivity holds
Also $\Delta_1 \sim \Delta_2$
$\Rightarrow$ $\Delta_2 \sim \Delta_1$ $\Rightarrow$ Symmetricity
Also $\Delta_1 \sim \Delta_2$ and $\Delta_2 \sim \Delta_3$
$\Rightarrow$ $\Delta_1 \sim \Delta_3$ $\Rightarrow$ Transitivity
$\Rightarrow$ R_2 is equivalence relation
$\Rightarrow$ $R_2 \rightarrow A, B, C$
R_3: $L_1 \parallel L_1, L_1 \parallel L_2$ $\Rightarrow$ $L_2 \parallel L_1$
And $L_1 \parallel L_2 \parallel L_3$ $\Rightarrow$ $L_1 \parallel L_3$
$\Rightarrow$ R_3 is equivalence relation
$\Rightarrow$ $R_3 \rightarrow A, B, C$
R_4: No line is $\perp$r to itself
$\Rightarrow$ R_4 is not reflexive. Also $L_1 \perp r\ L_2$
$\Rightarrow$ $L_2 \perp r\ L_1$
$\Rightarrow$ $L_1 \parallel L_3$ $\Rightarrow$ R_4 is not transitive.
$\therefore$ $R_4 \rightarrow (B)$
R_5: No girl can be sister of herself
R_5 is not reflexive
Also if ‘a is sister of b’, then it is not necessary that ‘b is sister of a’ as, ‘b may be brother of a’.
Thus, $a\ R_5\ b$
$\Rightarrow$ $b\ R_5\ a$
$\Rightarrow$ R_5 is not symmetric.
Further $(a, b), (b, c) \in R_5$
$\Rightarrow$ ‘a is sister b’ and ‘b is sister of c’
$\Rightarrow$ ‘a is also sister of c’. Thus, R_5 is transitive.
$\therefore$ $R_5 \rightarrow$ (c)
$\Rightarrow$ R_6: $a - a = 0\ \forall\ a \in \mathbb{N}$
$\Rightarrow$ $(a, a) \notin R_6$ for any $a \in \mathbb{N}$
$\Rightarrow$ R_6: is not reflexive
Now $(a - b) \in \mathbb{N}$ $\Rightarrow$ $-(a - b) \notin \mathbb{N}$
$\Rightarrow$ $(b - a) \notin \mathbb{N}$ $\Rightarrow$ R_6 is not symmetric.
Also $a - b \in \mathbb{N}$ $\Rightarrow$ $a \ge b + 1$ and $b - c \in \mathbb{N}$
$\Rightarrow$ $b \ge c + 1$ $\Rightarrow$ $a \ge (c + 1) + 1$
$\Rightarrow$ $(a - c) \ge 2$ and $a, c \in \mathbb{N}$
$\Rightarrow$ $(a, c) \in \mathbb{N}$ $\Rightarrow$ R_6 is transitive
Thus $R_6 \rightarrow$(c)
R_7: $\because a = a\ \forall\ a \in \mathbb{R}$
$\Rightarrow$ R_7 is reflexive.
Also $a \ge b \not\Rightarrow b\ a$ e.g., $7 > 2 \not\Rightarrow 2 > 7$
$\Rightarrow$ R_7 is not symmetric.
Now $a \ge b$ and $b \ge c$
$\Rightarrow$ $a \ge c$ $\Rightarrow$ R_7 is transitive.
$\therefore$ $R_7 \rightarrow (A), (C)$

11. **(a)** $(a, b)\ R\ (c, d) \Leftrightarrow ad\ (b + c) = bc\ (a + d)$
Reflexivity: $(a, b)\ R\ (a, b)$
$\Leftrightarrow$ $ab\ (b + a) = ba\ (a + b)$, which is true
$\Rightarrow$ R is reflexive.
Symmstricity: $(a, b)\ R\ (c, d)$
$\Leftrightarrow$ $ad\ (b + c) = bc\ (a + d)$
$\Leftrightarrow$ $bc\ (a + d) = ad\ (b + c)$
$\Leftrightarrow$ $cb\ (d + a) = da\ (c + b)$
$\Leftrightarrow$ $(c, d)\ R\ (a, b)$
$\Rightarrow$ R is symmetric
Transitivity: Let $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$
$\Rightarrow$ $ad\ (b + c) = bc\ (a + d)$
$\Rightarrow$ $\left(\frac{1}{c} + \frac{1}{b}\right) = \left(\frac{1}{d} + \frac{1}{a}\right)$ (dividing by $abcd$)
And $cf\ (d + f) = de\ (c + f)$
$\Rightarrow$ $\left(\frac{1}{e} + \frac{1}{d}\right) = \left(\frac{1}{f} + \frac{1}{c}\right)$ (dividing by $cdef$)

$\Rightarrow \quad \frac{1}{c}-\frac{1}{d}=\frac{1}{a}-\frac{1}{b}=\frac{1}{e}-\frac{1}{f} \Rightarrow \frac{1}{a}-\frac{1}{b}=\frac{1}{e}-\frac{1}{f}$

$\Rightarrow \quad \frac{1}{a}+\frac{1}{f}=\frac{1}{b}+\frac{1}{e} \qquad \Rightarrow$ be $(a+f) = af(b+e)$

$\Rightarrow$ $(a, b)\, R\, (e, f)$ $\qquad \Rightarrow$ R is transitive.

(b) $(a, b)\, R\, (c, d)$ iff $a + d = b + c$

Reflexivity: $(a, b)\, R\, (a, b) \Leftrightarrow a + b = b + a$, which is true

Symmetricity: $(a, b)\, R\, (c, d) \Leftrightarrow a + d = b + c$

$\Leftrightarrow$ $b + c = a + d$

$\Leftrightarrow$ $(c + b) = (d + a)$ $\qquad \Leftrightarrow$ $(c, d)\, R\, (a, b)$

$\Leftrightarrow$ R is symmetric.

Transitivity: $(a, b)\, R\, (c, d)$ and $(c, d)\, R\, (e, f)$

$\Leftrightarrow$ $a + d = b + c$ and $c + f = d + e$

$\Leftrightarrow$ $a - b = c - d = e - f$

$\Leftrightarrow$ $a - b = e - f$ $\qquad \Leftrightarrow$ $a + f = b + e$

$\Leftrightarrow$ $(a, b)\, R\, (e, f)$

$\Leftrightarrow$ R is transitive

Thus (a), (b), (c), all represent equivalence relations.

12. $x R y \Leftrightarrow x^2 - y^2 + \sqrt{3}$ is an irrational number

Reflexivity: $x R x$

$\Leftrightarrow$ $x^2 - x^2 + \sqrt{3}$ i.e., $\sqrt{3}$ is irrational, which is true.

Symmetricity: $x R y$

$\Leftrightarrow$ $x^2 - y^2 + \sqrt{3}$ is irrational, but $\not\Rightarrow$ $y^2 - x^2 + \sqrt{3}$ is also irrational.

e.g., let $x = \sqrt{2}.4\sqrt{3}; y = 4\sqrt{3}$

$\Rightarrow$ $x^2 - y^2 + \sqrt{3} = 2\sqrt{3} - \sqrt{3} + \sqrt{3} = 2\sqrt{3}$ which is irrational .

$\Rightarrow$ $x R y$, but $y^2 - x^2 + \sqrt{3} = \sqrt{3} - 2\sqrt{3} + \sqrt{3} = 0$ which is rational.

$\Rightarrow$ $(y, x) \notin R$ $\qquad \Rightarrow$ R is not symmetric.

Transitivity: Let $x R y$ and $y R Z$. We claim that $x R z$ is nodal ways true.

Consider, $x = \sqrt[4]{3}, y = 1, z = \sqrt{2}\sqrt[4]{3}$,

then $x^2 - y^2 + \sqrt{3} = \sqrt{3} - 1 + \sqrt{3} = 2\sqrt{3} - 1 \in \overline{\mathbb{Q}} \Rightarrow xRy$

Now, $y^2 - z^2 + \sqrt{3} = 1 - 2\sqrt{3} + \sqrt{3} = 1 - \sqrt{3} \in \overline{\mathbb{Q}}$

$\Rightarrow$ $y R z$

Further, $x^2 - z^2 + \sqrt{3} = \sqrt{3} - 2\sqrt{3} + \sqrt{3} = 0 \in \mathbb{Q}$

$\not\Rightarrow$ $x R z$

$\Rightarrow$ R is not transitive.

TEXTUAL EXERCISE–3: (OBJECTIVE)

1. **(b, d)** $R = \{(2, 4), (3, 9), (4, 16)\}$

$\Rightarrow$ $R^{-1} = \{(4, 2), (9, 3), (16, 4)\}$

$\Rightarrow$ $R_0 R^{-1} = \{(4, 4), (9, 9), (16, 16)\}$

Which is identify relation on domain set of R^{-1}.

Also it is reflexive and symmetric on domain of R^{-1}

2. **(a)** $R_1 = \{(10, 20), (20, 30), (30, 40), (40, 50)\}$

$R_2 = \{(10, 10), (10, 20), (10, 30), (10, 40), (10, 50), (20, 20), (20, 40), (30, 30), (40, 40), (50, 50)\}$

$\Rightarrow$ $(R_1\, 0\, R_2)^{-1} = R_2^{-1}\, 0\, R_1^{-1} = \{(20, 10), (30, 10), (30, 20), (40, 10), (40, 30), (50, 10), (50, 20), (50, 40)\}$

3. **(a), (b), (c), (d)**

$x R y \quad \Leftrightarrow x + y + 4xy$ is even number

Reflexivity: $x R x \Leftrightarrow x + x + 4x^2 = 2x + 4x^2$ is even, which is true.

Symmetricity: $x R$ y

$\Leftrightarrow$ $x + y + 4xy = 2k$, for some $k \in \mathbb{N}$.

$\Leftrightarrow$ $y + x + 4xy = 2k$

$\Leftrightarrow$ $y R x$

Transitivity: $x R y$

$\Leftrightarrow$ $x + y + 4xy$ is even and $y R z$

$\Leftrightarrow$ $y + z + 4xy$ is even

$\Leftrightarrow$ $x + y = 2k_1$ and $y + z = 2k_2$ for some $k_1, k_2 \in \mathbb{N}$

Case (i): If y is odd, then x, z are odd

$\Rightarrow$ $(x + z)$ is even

$\Rightarrow$ $x + z + 4xz$ is even

$\Rightarrow$ $x R z$

Case (ii): If y is even, then x, z are even

$\Rightarrow$ $(x + z)$ is even

$\Rightarrow$ $x + z + 4xz$ is even

$\Rightarrow$ $x R z$

4. **(b)** $A = \{2, 3, 4, 5\}$

$R = \{(2, 2), (3, 3), (4, 4), (5, 5), (2, 3), (3, 2), (3, 5), (5, 3)\}$

Clearly R is reflexive and symmetric, but $(2, 3), (3, 5) \in \mathbb{R}$ and $(2, 5) \notin \mathbb{R}$

$\Rightarrow$ R is not transitive.

Also $(3, 2), (2, 3)$ $\mathbb{R}$ and $2 \neq 3$

$\Rightarrow$ R is not anti-symmetric

5. **(b)** $R = R^{-1}$

$\Rightarrow$ $(x, y) \in R$

$\Leftrightarrow$ $(x, y) \in R^{-1}$

$\Rightarrow$ $(x, y) \in R$ and $(y, x) \in R$

$\Rightarrow$ R is symmetric.

6. **(d)** (b) and (c) are wrong as if

$A = \{2, 3, 5\}$ and R, S are relation on set A defined by $R = \{(2, 2), (3, 3), (5, 5)\}$ and $S = \{(2, 2), (3, 3), (5, 5), (2, 5)\}$, then $(R \cup S)$ is not equivalence, also union of two transitive.

$\Rightarrow$ (a) is false .

7. **(c)** Clearly $OP_i = OP_i$ for each $P_i \in P$

$\Rightarrow$ $P_i R P_i\ \forall P_i \in P$

$\Rightarrow$ R is reflexive.

Also $OP_1 = OP_2$ $\qquad OP_2 = OP_1$

$\therefore$ $P_1 R P_2$ $\qquad P_2 R P_1$

$\Rightarrow$ R is symmetric.

Also $OP_1 = OP_2$ and $OP_2 = OP_3$

$\Rightarrow$ $OP_1 = OP_3$ $\qquad \Rightarrow$ R is transitive.

8. **(d)** $(a, b)\, R\, (c, d)$ $\qquad \Leftrightarrow$ $a + d = b + c$

$(a, b)\, R\, (a, b)$ $\qquad \Leftrightarrow$ $a + b = b + a$

Which is true $\qquad \Rightarrow$ R is reflexive.

$(a, b)\, R\, (c, d)$ $\qquad \Leftrightarrow$ $a + d = b + c$

$\Leftrightarrow$ $b + c = a + d$ $\qquad \Leftrightarrow$ $c + b = d + a$

$\Leftrightarrow$ $(c, d)\, R\, (a, d)$ $\qquad \Rightarrow$ R is symmetric

Also, $(a, b)\, R\, (c, d)$ and $(c, d)\, R\, (e, f)$

$\Leftrightarrow$ $a + d = b + c$ and $c + f = d + e$
$\Leftrightarrow$ $a - b = c - d$ and $c - d = e - f$
$\Leftrightarrow$ $a - b = e - f$ $\Leftrightarrow$ $a + f = b + e$
$\Leftrightarrow$ $(a, b)\, R\, (e, f)$ $\Rightarrow$ R is transitive.

9. (c) $2x^2 - 3xy + y^2 = 0$
$\Leftrightarrow$ $2x^2 - 2xy - xy + y^2 = 0$
$\Leftrightarrow$ $2x(x - y) - y\,(x - y) = 0$
$\Leftrightarrow$ $(2x - y)\,(x - y) = 0$
$\Leftrightarrow$ $2x = y$ or $x = y$
R is reflexive as $x = x\ \forall\ x \in \mathbb{N}$
$\therefore$ $(2, 4) \in \mathbb{R}$ as $2x = y$ holds, but $(4, 2) \notin \mathbb{R}$
$\Rightarrow$ R is not symmetric.
Also $(2, 4)$ and $(4, 8) \in \mathbb{R}$ as $2x = y$ in each pair, but $(2, 8) \notin \mathbb{R}$
$\Rightarrow$ R is not transitive.

10. (b) $R = \{(1, 2), (2, 3)\}$; $A = \{1, 2, 3\}$ and $R \subseteq A \times A$.
To make R reflexive: $(1, 1), (2, 2), (3, 3)$ must be included.
To make R symmetric: $(2, 1), (3, 2)$ must be included.
To make R transitive: $(1, 3), (3, 1)$ must be included.
Thus total 7 elements are required to be included in R.

11. (a) $R : A \to B$ and $S : BC$;
$(S\ O\ R)\ (x) = S\ (R\ (x))$
$\Rightarrow$ $x \in A$
Also $R\ (x) \in B$ $\Rightarrow$ $S(R\ (x) \in \mathrm{C}$
$\therefore$ $S\ O\ R: A \to C$

12. (b) $R \subseteq A \times B$ and $S \subseteq B \times C$
$(SOR)^{-1} = R^{-1}\ \mathrm{O}\ S^{-1}$ (standard result of composition of two relations)

13. (b) Universal relation on set A, i.e., $A \times A$ is the greatest on set A
$\Rightarrow$ 1 is incorrect
Let $A = \{a_1, a_2, \ldots, a_n\}$; $n \geq 2$
$\Rightarrow$ $A \times A = \{(a_1, a_1), (a_1, a_2), (a_1, a_3), \ldots, (a_1, a_n), (a_2, a_1), (a_2, a_2), (a_2, a_3), \ldots, (a_2, a_n), \ldots, (a_n, a_1), (a_n, a_2), \ldots, (a_n, a_n)\}$
$\therefore$ $(a_1, a_2), (a_2, a_1) \in R$, but $a_1 \neq a_2$
$\Rightarrow$ R is not anti-symmetric
$\Rightarrow$ (2) is correct
We know that union and intersection of two is also symmetric
$\Rightarrow$ (3) is correct
$\therefore$ If $(a, b) \in R \cup S$
$\Rightarrow$ $(a, b) \in R$ or S
$\Rightarrow$ $(b, a) \in R$ or S ($\therefore$ R, S are symmetric)
$\Rightarrow$ $(b, a) \in R \cup S$ $\Rightarrow$ $R \cup S$ is symmetric
Parallely if $(a, b) \in R \cap S$
$\Rightarrow$ $(a, b) \in R$ and S
$\Rightarrow$ $(b, a) \in R$ and S
$\Rightarrow$ $(b, a) \in R \cap S$
$\Rightarrow$ $R \cap S$ is symmetric

14. (b) Intersection of two symmetric relations is symmetric.

15. (a) In an identity relation a R a and $a\ R\ b$ for $a \neq b$, however in a reflexive relation a R a and a may be related to any other element different from a
$\Rightarrow$ n (reflexive relation) $\geq n$ (identify relation)
$\Rightarrow$ $m \geq n$

16. (b) $S = \{1, 2, 3, \ldots, 12\}$
Let a_1 = number of elements in set A
a_2 = number of elements in set B
a_3 = number of elements in set C
$\therefore$ A, B, C are of equal size
$\Rightarrow$ $a_1 = a_2 = a_3 = 4$
Thus, we are to divide 12 elements equally among 3 naming groups.
By the knowledge of permutations, number of ways = $\dfrac{12!}{4!4!4!} = \dfrac{12!}{(4!)^3}$

SECTION–3: (ONLY ONE CORRECT ANSWER)

1. (a) Relation from A to A is a subset of $A \times A$
Number of relation on A = Number of subsets of $A \times A$
$= (12)^{n(A) \times n(A)} = (2)^{n^2}$

2. (b) $A = \{2, 3, 4, 5\}$
$\because$ $(2, 2), (3, 3), (4, 4), (5, 5), \in R$
$\Rightarrow$ R is reflexive.
$\because$ $(2, 3), (3, 2), (3, 5)$ and $(5, 3) \in R$
$\Rightarrow$ R is symmetric.
$\because$ $(2, 3), (3, 5) \in \mathrm{R}$, but $(2, 5) \notin R$
$\Rightarrow$ R is not transitive.

3. (b) $R \subseteq A \times A$ and $R = R^{-1}$
$\Rightarrow$ $(a, b) \in R, R^{-1}$ $\Rightarrow$ $(b, a) \in R, R^{-1}$
$\Rightarrow$ R must be symmetric

4. (d) $A = \{x : x^2 - 3x + 2 = 0\}$ $\Rightarrow$ $A = \{1, 2\}$
$\Rightarrow$ $R = A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

5. (b) $\mathrm{R} = \{(m, n) : m = k_n; k \in \mathbb{Z}; k \neq 0\}$
Reflexivity:
$\because$ $m = 1.m\ \forall\ m \in \mathbb{Z} - \{0\}$
$\Rightarrow$ $(m, m) \in \forall\ m \in \mathbb{Z} - \{0\}$
$\Rightarrow$ R is reflexive
Symmetric:
$\because$ $(4, 2) \in R$ as $4 = 2(2)$, but $2 \neq k(4)$ for any $k \in \mathbb{Z}$
$\Rightarrow$ $(2, 4) \notin R$ $\Rightarrow$ R is not symmetric
Transitive:
Let $(m, n) \in R$ and $(n, k) \in R$
$\Rightarrow$ $m = k_1 n$ and $n = k_2 k$, $k_1\ k_2 \in \mathbb{Z}$
$\Rightarrow$ $(m, k) \in$
$\Rightarrow$ R is transitive.

6. (b) $A = \{5, 6, 7\}$ and $B = \{1, 2, 3, 4\}$;
$n\ [(A \times B) \times B] = n\ (A \times B) \times n\ (B)$
$= n\ (\mathrm{A}) \times n\ (B) \times n\ (B)$
$= 3 \times 4 \times 4 = 48$

7. **(d)** $A = \{2, 3, 4\}$ and $B = \{3, 5\}$
$R = \{(a, b): (a < b)\} \subseteq A \times B$
$\Rightarrow$ $R = \{(2, 3), (2, 5), (3, 5), (4, 5)\}$
$\Rightarrow$ $R^{-1} = \{(3, 2), (5, 2), (5, 3), (5, 4)\}$
$\Rightarrow$ $R \text{ o } R^{-1} = \{(3, 3), (3, 5), (5, 3), (5, 5)\}$

8. **(b)** R and R' are symmetric
Let $(R \cap R')$ contains (a, b)
$\Rightarrow$ $(a, b) \in R$ and R′
$\Rightarrow$ $(b, a) \in R$ and R′
$\Rightarrow$ $(b, a) \in R \cap R'$
$\Rightarrow$ $(R \cap R')$ is symmetric.

9. **(c)** n (Relations form A to B) $= 2^{n(A \times B)} = (2)^{3 \times 4} = (2)^{12}$
$\Rightarrow$ Number of non-empty relations $2^{12} - 1$

10. **(d)** $R = \{(x, y); x < y\}$ $A \times B$
$\Rightarrow$ $R = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$
$\Rightarrow$ $R^{-1} = \{(3, 1), (5, 1), (3, 2), (5, 2), (5, 3), (5, 4)\}$
$\Rightarrow$ $R^{-1}OR = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4,4)\}$
$\Rightarrow$ None of these

11. **(b)** $\because$ $a \nless a$, $a < b \nRightarrow b < a$ and $a < b$; $b < c \Rightarrow a < c$
$\Rightarrow$ R is only transitive.

12. **(b)** $P = \{(x, y) : x^2 + y^2 = 1, x, y \in \mathbb{R}\}$, then clearly $x R x$ only for $x = \pm\frac{1}{\sqrt{2}}$
$\Rightarrow$ R is not reflexive.
$\therefore$ $x^2 + y^2 = 1$ $\Rightarrow$ $y^2 + x^2 = 1$
$\Rightarrow$ R is symmetric
If $x = 1, y = 0$
$\Rightarrow$ $x R y$ and $y = 0, z = 1$
$\Rightarrow$ $y R z$
But $1 \not R 1$, i.e., $x \not R z$ $\Rightarrow$ R is not transitive.

13. **(b)** R is any Equivalence relation on A and $n(A) = n$
$\Rightarrow$ $a R a \ \forall a \in A$ $\Rightarrow$ $n(R) \geq n$

14. **(a)** $x R y \Leftrightarrow x - y + \sqrt{5}$ is irrational .
Reflexivity: $x R x$
$\Leftrightarrow$ $x - x + \sqrt{5}$ is irrational, i.e., $\sqrt{5}$ is irrational, which is true,
Symmetricity: Let $x = 2\sqrt{5}$, $y = \sqrt{5}$, then $x - y + \sqrt{5} = 2\sqrt{5} - \sqrt{5} + \sqrt{5} = 2\sqrt{5}$ which is irrational
$\Rightarrow$ $x R y$.
But $y - x + \sqrt{5} = \sqrt{5} - 2\sqrt{5} + \sqrt{5} = 0$ which is rational
$\Rightarrow$ $(y, x) \notin R$
$\Rightarrow$ R is not symmetric.
Let $x = 2\sqrt{5}$, $y = \sqrt{5}$, $z = 3\sqrt{5}$, then $x - y + \sqrt{5} = 2\sqrt{5}$
$\Rightarrow$ $(x, y) \in \mathbb{R}$
Now, $x - z + \sqrt{5} = 0$
$\Rightarrow$ $(x, z) \notin \mathbb{R}$ $\Rightarrow$ R is not transitive.

15. **(b)** $A, B\ X$ and $A R B \Leftrightarrow A \cap B = \phi$
Reflexivity: $A \cap A = A \neq \phi \ \forall A\ X$
$\Rightarrow$ A is not reflexive.
Symmtricity: $A \cap B = \phi$
$\Rightarrow$ $B \cap A = \phi$ $\Rightarrow$ R is symmetric.
Transitivity: Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, $C = \{2, 7, 8\}$
Clearly $A \cap B = \phi$; $B \cap C = \phi$; $A \cap C \neq \phi$
$\Rightarrow$ R is not transitive.

16. **(d)** $A = \{1, 2, 3, 4, 5\}$
$R = \{(x, y): |x^2 - y^2| < 16\}$
$R = (x, y): -16 < x^2 - y^2 < 16\}$
$R = \{(x, y): x^2 - y^2 < 16$ and $x^2 - y^2 > -16$
$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (5, 4), (5, 5)\}$

17. **(d)** $R = \{(x, y) : x$ is relatively prime to $y\}$
$= \{(2, 3), (2, 7), (3, 7), (3, 10), (4, 3), (4, 7), (5, 3), (5, 7)\}$
$\Rightarrow$ Domain of $R = \{2, 3, 4, 5\}$

18. **(c)** $x^2 + y^2 \leq 4$ $\Rightarrow$ $x \in [-2, 2]$
But $x \in \mathbb{Z}$ $\Rightarrow$ $x \in \{-2, -1, 0, 1, 2\}$

19. **(b)** Identify relation is always not true
$\Rightarrow$ $I \subset R$

20. **(b)** $A\ A, A\ B$ $\nRightarrow$ $B\ A$
And $A\ B, B\ C$ $\Rightarrow$ $A\ C$
$\Rightarrow$ R is reflexive and transitive.
Also $A\ B$ and $B\ A$
$\Rightarrow$ $A = B$
$\Rightarrow$ R is anti-symmetric.

21. **(c)** $\because$ R is anti-symmetric iff (a, b) and $(b, a) \in \mathbb{R}$
$\Rightarrow$ $a = b$

22. **(b)** $\because$ ϕ contains no elements
$\Rightarrow$ $(a, a) \notin R$ for any $a \in A$
Also by default three is no pair $(a, b) \in R$ for which $(b, a) \notin R$
$\Rightarrow$ R is symmetric. Similarly by default R is transitive.

23. **(a)** Clearly R_1 is equivalence as $|a| = |a|$, $|a| = |b|$
$\Rightarrow$ $|b| = |a|$
And $|a| = |b|$, $|b| = |c|$
$\Rightarrow$ $|a| = |c|$
R_2 is not equivalence as $a \leq b$
$\nRightarrow$ $b \leq a$
R_3 is not equivalence as a/b
$\nRightarrow$ b/a
R_4 is not equivalence as $a < b$
$\nRightarrow$ $b < a$

24. **(c)** Since R^{-1} of equivalence relation R is also equivalence.

25. **(d)** $\because$ $a \equiv a \pmod{m}$ as $m/(a - a)$
Also $a \equiv b \pmod{m}$ as $m/(a - b)$
$\Rightarrow$ $m/(b - a)$ $\Rightarrow$ $b \equiv a \pmod{m}$
Further $m/(a - b)$ and $m/(b - c)$
$\Rightarrow$ $m/(a - b) + (b - c)$ $\Rightarrow$ $m/(a - c)$
$\Rightarrow$ $a \equiv c \pmod{m}$

26. **(b)** $\because$ Union of two equivalence relations need not be equivalence, but intersection of two equivalence relation is equivalence

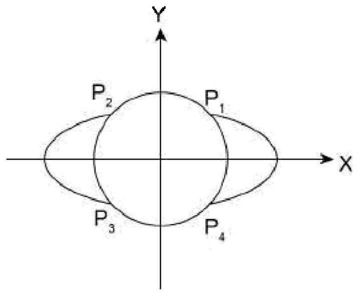
27. **(c)** $R = \{(1, 3), (2, 2), (3, 2)\}$ and $S = \{(2, 1), (3, 2), (2, 3)\}$
$\Rightarrow$ Ros = $\{(2, 3), (2, 2), (3, 2)\}$

28. **(b)** $\because x^2 = xy \quad \Rightarrow x(x - y) = 0$
$\Rightarrow x = 0$ or $x = y$
$\because x = x \ \forall x \in A \quad \Rightarrow R$ is reflexive.
Also $x R y \quad \Rightarrow x = 0$ or $x = y$
When $x = 0$, then x^2 = xy; $y \neq 0$, then $y^2 \neq yx$ as L.H.S. $\neq 0$, R.H.S = 0
$\Rightarrow R$ is not symmetric.
Let $x R y$ and $y R z$
$\Rightarrow$ Either $x = 0$ or $x = y$ and $y = 0$ or $y = z$
Case (i): $x = 0, y = 0$ and $z \neq 0$, then $x^2 = xz$
Case (ii): $x = 0, y = z \neq = 0$, then $x^2 = xz$
$\Rightarrow x R z$
Case (iii): $x = y \neq 0; y = z$
$\Rightarrow x = z \quad \Rightarrow x R z$
$\Rightarrow R$ is transitive.

29. **(b)** Let $A = \{a_1, a_2, a_3, a_4\}$, then reflexive relation.
R must contain atleast $(a_1, a_1), (a_2, a_2), (a_3, a_3)$ and (a_4, a_4)
$A \times A$ has 16 elements, out of these 16 elements, 4 elements of type (a_i, a_i) are include in reflexive relation.
Now out of remaining 12 elements 0, 1, 2, 3, …., 12 elements can be include in reflexive relation.
Thus number of ways = ${}^{12}C_0 + {}^{12}C_1 + {}^{12}C_2 + {}^{12}C_3 + \ldots + {}^{12}C_{12} = (2)^{12}$

30. **(a)** $A = \{2, 4, 6\}$ and $B = \{2, 3, 5\}$
$\Rightarrow$ Number of relation from A to $B = n[P(A \times B)] = 2^{n(A \times B)}$
$= 2^{(3)\times(3)} = (2)^9$.

31. **(d)** $A = \{(x, y) : x^2 + y^2 = 25\}$ and $B = \{(x, y) : x^2 + 9y^2 = 144\}$
A represents of circle of radius 5 and with centre at origin. B represents an ellipse with centre at origin with major axis length 12 and minor axis length 4.



$\Rightarrow A$ and B have exactly 4 points common.

32. **(a)** $a R b \quad \Leftrightarrow \quad |a - b| \leq 1$
$\therefore a R b$ as $|a - a| = 0 \leq 1$
$\Rightarrow$ R is reflexive.
Let a = 2, $b = 2.5$, $c = 3.4$, then $|a - b| = 0.5 < 1$ and $|b - c| = 0.9 < 1$
But $|a - c| = |2 - 3.4| = 1.4 \nleq 1$
$\Rightarrow (a, c) \notin R$
$\Rightarrow$ R is not transitive.
Clearly $|a - b| \leq 1$
$\Rightarrow |b - a| \leq 1$
$\Rightarrow$ R is symmetric.

SECTION–IV: (MORE THAN ONE ARE CORRECT)

1. **(a), (b), (c)** $X = \{1, 2, 3, 4, 5,\}$; $Y = \{1, 3, 5, 7, 9\}$
(a) $R_1 = \{(1, 3), (3, 5), (5, 7)\}$ is a relation for X to Y.
(b) $R_2 X \times Y \Rightarrow R_2$ is a relation from X to Y.
(c) $R_3 X \times Y \Rightarrow R_3$ is also a relation from X to Y.
(d) $R_4 \not\subseteq X \times Y$ as $7 \notin X$ and $4 \notin Y$.
$\Rightarrow R_4$ is not a relation from X to Y.

2. **(a), (b), (c)**
$\because (a, b) R (a, b) \quad \Leftrightarrow \quad a + b = b + a$
$\Rightarrow R$ is reflexive.
Also $(a, b) R (c, d)$
$\Rightarrow (a + d) = (b + c)$
$\Rightarrow (b + c) = (a + d) \quad \Rightarrow (c + b) = (d + a)$
$\Rightarrow (c, d) R (a, b) \quad \Rightarrow$ R is symmetric
$(a, b) R (c, d)$ and $(c, d) R (e, f)$
$\Rightarrow a + d = b + c$ and $c + f = d + e$
$\Rightarrow a - b = c - d$ and $c - d = e - f$
$\Rightarrow a - b = e - f \quad \Rightarrow a + f = b + e$
$\Rightarrow (a, b) R (e, f) \quad \Rightarrow R$ is transitive.

3. **(b), (c)** $A = \{(x, y) : y = e^{2x}, x \in \mathbb{R}\}$ and $B = \{(x, y): y = e^{-2x}, x \in \mathbb{R}\}$
Clearly $(0, 1) \in A \cap B \quad \Rightarrow A \cap B \neq \phi$
Also $e^{2x} = e^{-2x} \quad \Leftrightarrow e^{4x} = 1$
$\Leftrightarrow x = 0$
$\Rightarrow (A \cap B)$ is singleton set.

4. **(a), (d)** $A = \{(x, y) : x^2 + y^2 \leq 1, x, y \in \mathbb{R}\}$ and $B = \{(x, y) : x^2 + y^2 \geq 4, x, y \in \mathbb{R}\}$
$\Rightarrow A$ is interior of circle $x^2 + y^2 = 1$ and B is exterior of circle $x^2 + y^2 = 4$
$\Rightarrow A \cap B = \phi$
$\Rightarrow A - B = A$ and $B - A = B$
$\Rightarrow B - A \neq \phi$ and $A - B \neq \phi$

5. **(b), (c), (d)**
$a R a \quad \Leftrightarrow a^2 \leq a^2$
$\Leftrightarrow a = 0$ or $a = 1$
$\Rightarrow R$ is not reflexive.
$a R b \quad \Leftrightarrow \quad a \leq b^2$
$\not\Rightarrow b \leq a^2 \quad \Leftrightarrow \quad (b, a) \notin R$
$\Rightarrow R$ is not symmetric
$\therefore a \leq b^2$ and $b \leq c^2 \quad \not\Rightarrow \quad a \leq c^2$
Let $a = 2$, $b = -3$, $c = 1$, then $a = 2$, $b^2 = 9$, $c^2 = 1$, $b = -3$
$\Rightarrow a < b^2$ and $b < c^2$ but $a > c^2$
$\Rightarrow$ Thus, $(a, b) \in R$ and (b) $\in R$
$\not\Rightarrow (a, c) \in R$
$\Rightarrow R$ is not transitive and equivalence

6. **(a), (c),(d)** $R = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$
$\Rightarrow R$ is not reflexive, not symmetric and not transitive.

7. **(b), (c), (d)** If R, S are reflexive, then $R \cup S$ and $R \cap S$ are also reflexive.

Same is true when R and S are symmetric. If R and S are both transitive, then $R \cap S$ is also transitive but not $R \cup S$.

8. (a), (b), (c), (d)

$a\,R\,b \quad \Leftrightarrow \quad n \mid (a-b)$

$\because \quad n/(a-a)$ i.e., $n|0 \quad \Rightarrow \quad R$ is reflexive.

$\Rightarrow \quad a\,R\,b \quad \Leftrightarrow \quad n \mid (a-b)$

$\Rightarrow \quad n \mid (b-a) \quad \Rightarrow \quad b\,R\,a$

$\Rightarrow \quad R$ is symmetric.

$a\,R\,b$ and $b\,R\,c$

$\Rightarrow \quad n \mid (a-b) \quad \Rightarrow \quad n \mid (a-c)$

$\Rightarrow \quad a\,R\,c \quad \Rightarrow \quad R$ is transitive.

$\Rightarrow \quad R$ is equivalence.

9. (c) $A \neq$

$\Rightarrow$ $(x, x) \notin R$; $(x, y)\,R$

$\Rightarrow$ $(y, x)\,R$ as x is brother of y may imply that y is sister of x

$\Rightarrow$ $(x, y)\,R$, $(y, z)\,R$

$\Rightarrow$ $(x, z)\,R$ as x is brother and y is brother of Z

$\Rightarrow$ x is brother of z

$\Rightarrow$ R is transitive.

SECTION–V: (ASSERTION AND REASON TYPE)

1. (d) Clearly R is correct.

Further $a \mid a \;\forall\; a\; \mathbb{Z} - \{0\}$, but 0 does not divide itself.

So R is not reflexive

$\Rightarrow$ Assertion is incorrect.

2. (a) Reason is true

$\Rightarrow$ Number of relation A to B

$= n[P(A \times B)] = (2)^{n(A\times B)} = 2^{n(A)\times n(B)} = (2)^{\text{km}}$

$\Rightarrow$ Assertion is true.

3. (d) Clearly reason is true

$|a - a| \le 1 \quad \Rightarrow \quad R$ is reflexive.

$|a - b| \le 1 \quad \Rightarrow \quad |b - a| \le 1$

$\Rightarrow$ R is symmetric.

If $a = 1$, $b = 1.5$, $c = 2.4$

$\Rightarrow$ $|a - b| < 1$, $|b - c| < 1$, but $|a - c| \not< 1$

$\Rightarrow$ R is not transitive and hence not equivalence.

Assertion is incorrect.

4. (a) $\because A \times B = \{(a, b) : a\,A$ and $b\,B\}$ and $R : A \to B$ is a relation having some ordered pairs (x, y), such that $x\,A$ and $y\,B$ satisfying some given condition.

$\Rightarrow$ Every relation is a subset of $A \times B$

$\Rightarrow$ $A \times B$ is the largest relation from A to B, i.e., universal relation.

$\Rightarrow$ Both assertion and reason are correct and reason correctly explains assertion.

5. (d) Clearly the reason is correct

$\because \quad 2/(a-a)$ and $2/(a-b)$

$\Rightarrow \quad 2/-(a-b) \quad \Rightarrow \quad 2/(b-a)$

$\Rightarrow$ R is reflexive and symmetric, also $2/(a-b)$ and $2/(b-c)$

$\Rightarrow$ $a\,R\,c$

$\Rightarrow$ R is transitive and hence equivalence. Thus assertion is incorrect.

SECTION–VI: (COMPREHENSION)

A:

1. **(c)**
2. **(b)**
3. **(b)**
4. **(a)**

Reflexivity: $(a, b)\,R_1\,(a, b)$

$\Leftrightarrow$ $ab = ba$, which is true

$\Rightarrow$ R_1 is reflexive.

Again $(a, b)\,R_2\,(a, b)$

$\Rightarrow$ $a + b = b + a$, which is true

$\Rightarrow$ R_2 is reflexive.

Symmetric: $(a, b)\,R_1\,(c, d)$

$\Rightarrow \quad ad = bc \quad \Rightarrow \quad bc = ad$

$\Rightarrow \quad cb = da \quad \Rightarrow \quad (c, d)\,R_1\,(a, b)$

$\Rightarrow$ R_1 is symmetric

Again $(a, b)\,R_2\,(c, d)$

$\Rightarrow \quad a + d = b + c \quad \Rightarrow \quad b + c = a + d$

$\Rightarrow \quad c + d = d + a \quad \Rightarrow \quad (c, d)\,R_2\,(a, b)$

$\Rightarrow$ R_2 is symmetric.

Transitivity: Let $(a, b)\,R_1\,(c, d)$ and $(c, d)\,R_1\,(e, f)$

$\Rightarrow \quad ad = bc$ and $cf = de \quad \Rightarrow \quad \dfrac{a}{b} = \dfrac{c}{d}$ and $\dfrac{c}{d} = \dfrac{e}{f}$

$\Rightarrow \quad \dfrac{a}{b} = \dfrac{e}{f} \Rightarrow af = be$

$\Rightarrow \quad (a, b)\,R_1\,(e, f) \quad \Rightarrow \quad R$ is transitive.

Again $(a, b)\,R_2\,(c, d)$ and $(c, d)\,R_2\,(e, f)$

$\Rightarrow$ $a + d = b + c$ and $c + f = d + e$

$\Rightarrow$ $a - b = c - d$ and $c - d = e - f$

$\Rightarrow$ $(a - b) = (e - f)$

$\Rightarrow$ $a + f = b + e$

$\Rightarrow \quad (a, b)\,R_2\,(e, f) \quad R_2$ is also transitive.

$\Rightarrow$ R_1 and R_2 both are equivalence relation.

SECTION-VII: (COLUMN–MATCHING TYPE I)

1. (i) → (a); (ii) → (c), (e); (iii) → (a), (b) , (c), (d); (iv) → (a), (b), (c), (d)

(i) $R: A \to A; A = \{1, 2, 3, \ldots, -14\}$

$R = \{(x, y): 3x - y > 0\}$

Reflexivity: $3x - x > 0 \quad \Leftrightarrow 2x > 0$

$\Leftrightarrow \quad x > 0$, which is true for $x\,A$

$\Rightarrow$ R is reflexive.

Symmeticity: Let $y = 1 \quad \Rightarrow \quad 3y = 3$ and $x = 4$

$\Rightarrow$ $3x = 12$, then $3x - y = 12 - 1 = 11 > 0$

$\Rightarrow$ $x\,R\,y$

And $3y - x = 3 - 4 = -1 < 0$

$\Rightarrow$ $(y, x) \notin R$

Transitivity: Let $x = 1$

$\Rightarrow$ $3x = 3$ and $y = 2$

$\Rightarrow$ $3y = 6$ and let $z = 5$, then $x\,R\,y$ as $3x - y = 3 - 2 = 1 > 0$, $y\,R\,z$ as $3y - z = 6 - 5 = 1 > 0$, but $(x, z) \notin R$ as $3x - z = 3 - 5 = -2 < 0$

$\Rightarrow$ R is not transitive.

$\Rightarrow$ R is not Equivalence, also (1, 2) R as 3 (1) – 2 > 0 and (2, 1) R as 3 (2) – 1 > 0

However $1 \neq 2$ $\Rightarrow$ R is not anti-symmetric

(ii) R: $\mathbb{N} \rightarrow \mathbb{N}$; $R = \{(x, y): y = x + 4 \text{ and } x < 4\}$

$\Rightarrow$ $R = \{(1, 5), (2, 6), (3, 7)\}$

$\Rightarrow$ R is not reflexive, not symmetric, but transitive and anti-symmetric by default.

(iii) R: $\mathbb{Z} \rightarrow \mathbb{Z}$; $R = \{(x, y); x - z \in \mathbb{Z}\}$

$\because$ $x - x = 0 \in \mathbb{Z}$ $\Rightarrow$ R is reflexive and $x - y \in \mathbb{Z}$

$\Rightarrow$ $y - x \in \mathbb{Z}$ $\Rightarrow$ R is symmetric

Also $x - y, y - z \in \mathbb{Z}$ $\Rightarrow$ $(x - y) + (y - z) \in \mathbb{Z}$

$\Rightarrow$ $(x - z) \in \mathbb{Z}$ $\Rightarrow$ R is transitive.

$\because$ (2, 4), (4, 2) $\in \mathbb{Z}$, but $2 \neq 4$

$\Rightarrow$ R is not anti-symmetric.

(iv) ($a R a$) as same person work with himself at same place.

Also $a R b$ $\Rightarrow$ $b R$ a and a R b is $b R c$

$\Rightarrow$ $a R c$

$\Rightarrow$ (R) is reflexive, symmetric, transitive and equivalence. However $(a, b), (b, a) \in R$,but $a \neq b$ is possible for two different person a and b.

$\Rightarrow$ R is not anti-symmetric.

2. (i) → (a), (c), (e); (ii) →(f); (iii) →(a), (b), (c), (d); (iv) → (a), (b), (c), (d)

(i) $R = \{(a, b); a \leq b; a, b \in \mathbb{R}\}$

$\because$ $a R$ a as $a = a$ $\Rightarrow$ R is reflexive

However, $a \leq b$ $\nRightarrow$ $b \leq a$, e.g., $2 < 3$

$\nRightarrow$ $3 < 2$

$\Rightarrow$ R is not symmetric, also $a \leq b$ and $b \leq c$

$\Rightarrow$ $a \leq c$

$\Rightarrow$ R is transitive, also $2 < 3$ and $3 < 2$ can't hold simultaneously.

$\Rightarrow$ R is anti-symmetric by default.

(ii) $R = \{(a, b): a \leq b^3\}$

$\because$ $a \leq a^3$ $\Rightarrow$ $a^3 - a\ 0$

$\Rightarrow$ $a(a^2 - 1)\ 0$ $\Rightarrow$ $a(a + 1)(a - 1)\ 0$

$\Rightarrow$ $a \in [-1, 0] \cup [1, \infty)$

$\Rightarrow$ R is not reflexive.

Also for $a = 4$, $b = 65$

$a < b^3$ as $4 < (65)^3$, but $b \nless a^3$ as $65 \nless 64$

$\Rightarrow$ R is not symmetric.

Further for $b = 4$, $b^3 = 64$, $c = 2$

$\Rightarrow$ $c^3 = 8$ and a = 10, we have $a < b^3$ as $10 < 64$

$\Rightarrow$ $(a, b) R$

$\Rightarrow$ $b < c^3$ as $4 < 8$ $\Rightarrow$ $(b, c) \in R$ but $a \nless c^3$ as $10 \nless 8$

$\Rightarrow$ $(a, c) \notin R$

R is not transitive.

Also (2, 3) and (3, 2) $\in R$ as $2 < (3)^3$ and $3 < (2)^3$

$\Rightarrow$ R is not anti-symmetric.

(iii) R: A → A; A = {1, 2, 3, 4, 5}

$R = \{(a, b): |a - b| \text{ is even}\}$

$\Rightarrow$ $R = \{(1, 3), (3, 1), (1, 5), (5, 1), (2, 4), (4, 2), (3, 5) (5, 3), (1, 1), (2, 2), (3, 3), (4, 4)\}$

$\Rightarrow$ R is reflexive, symmetric,

Also $|a - b|$ = even

$\Rightarrow$ $a - b = 2k_1$ and $|b - c|$ = even

$\Rightarrow$ $b - c = 2k_2$

$\Rightarrow$ $a - c = 2(k_1 + k_2)$ $\Rightarrow$ $|a - c|$ = even

$\Rightarrow$ $a R c$ $\Rightarrow$ R is transitive.

Also (2, 4), (4, 2) R, but $4 \neq 2$

$\Rightarrow$ R is not anti-symmetric.

(iv) $R = \{(x, y): x$ and have same age$\}$

Clearly, R will be reflexive, symmetric and transitive but $a R b$ and $b R$ a and $a \neq b$ is possible

$\Rightarrow$ R is not anti-symmetric.

3. (i) → (b); (ii)→ (b), (iii)→ (b), (d), (e); (iv) →(a), (d)

(i) $R = \{(X, Y): X, Y \in P\text{ (A)}; X \cap Y \neq \phi\}$

Reflexivity: $\because$ For $X \neq \phi$, $X \cap X = X = \phi$, but $\phi \cap \phi = \phi$

$\Rightarrow$ $\phi R \phi$

Subjectivity: $\Rightarrow$ R is not reflexive, also $X \cap \phi = \phi\ \forall\ X \in P\ (A)$

$\Rightarrow$ $\phi \notin$ range of Relation.

$\Rightarrow$ R is not subjective.

Symmetricity: $X R Y$

$\Rightarrow$ $X \cap Y \neq \phi$

$\Rightarrow$ $Y \cap X = \phi$ $\Rightarrow$ $Y R X$

$\Rightarrow$ R is symmetric.

Transitivity: $X RY$ and $Y R Z$

$\Rightarrow$ $X \cap Y \neq \phi$ and $Y \cap Z \neq \phi$, but $(X \cap Z)$ may be empty.

e.g., if $X = \{1, 2, 3\}$; $Y = \{2, 5, 6\}$; $Z = \{5, 8\}$, then $X \cap Y = \{2\}$; $Y \cap Z = \{5\}$, but $X \cap Z = \{\} = \phi$

$\Rightarrow$ R is not transitive

Injectivity: If X, Y, Z such that $X \cap Z \neq \phi$, $X \cap Z \neq \phi$

$\Rightarrow$ $(X, Z)\ (Y, Z)$

(ii) $R = \{(x, y): x, y \in \mathbb{N}$ and R is not injective. L.C.M $(x, y) = 12\}$

Reflexivity: $\because$ L.C.M $(x, x) = 12$

$\Leftrightarrow$ x = 12 $\Rightarrow$ R is not reflexive

Symmetric: If L.C.M $(x, y) = 12$, then L.C.M $(y, x) = 12$

$\Rightarrow$ R is symmetric

Transitive: If L.C.M $(x, y) = 12$, then $(x, y) \in R$

$\Rightarrow$ R = {(1, 12), (3, 4), (4, 3), (4, 6), (6, 4), (2, 12), (12, 2), (12, 1)}

$\therefore$ (3, 4), (4, 6) $\in R$ but (3, 6) $\notin R$

$\Rightarrow$ R is not transitive.

Also (3, 4), (4, 3) $\in R$ but $3 \neq 4$

$\Rightarrow$ R is not anti-symmetric.

$\because$ Range $\neq \mathbb{N}$

$\Rightarrow$ R is not surjective, also (3, 4), (6, 4) $\in R$

$\Rightarrow$ R is not injective

(iii) $R = \{(x, y): x, y \in \mathbb{C}$ and $y = \bar{x}\}$; $\mathbb{C}$ is the set of complex numbers.

Reflexivity: $x R x$ $\Rightarrow$ $x = \bar{x}$

$\Rightarrow$ $x \in \mathbb{R}$ i.e., $(x, x) \in R$ holds only when x is purely real

$\Rightarrow$ R is not reflexive.

Symmetricity: If $x R$ y, then $y = \bar{x}$

$\Rightarrow$ $\bar{y} = x$ $\Rightarrow$ $(y, x) \in R$

$\Rightarrow$ R is symmetric.

Transitive: $x R y$ $\Rightarrow$ $y = \bar{x}$ and $y R z$

$\Rightarrow$ $z = \bar{y}$

$\Rightarrow$ $x = z$ $\nRightarrow$ $z = \bar{x}$

$\Rightarrow$ R is not transitive.

Anti-symmetric: $x R y$ and $y R x$ for $x\ y$ e.g., $(2 + 3i) R (2 - 3i)$, but $2 + 3i \neq 2 - 3i$

$\Rightarrow$ R is not anti-symmetric.

Surjectivity: Let $a + ib \in \not\subset$ then, $(a - ib) \in \not\subset$ such that $(a - ib) R (a + ib)$

$\Rightarrow$ R is subjective.

Injectivity: Let $(z_1, z_2) \in R$ and $(z_3, z_2) \in R$ i.e., $z_2 = \overline{z_1}$ and $z_2 = \overline{z_3}$

$\Rightarrow$ $\overline{z_1} = \overline{z_3}$ $\Rightarrow$ $z_1 = z_3$

$\Rightarrow$ R is injective.

(iv) $R = \{(x, y): 2x^2 + 3y^2 - 5xy = 0; x, y \in \mathbb{R}\}$

Reflexivity: $x R x$ $\Leftrightarrow$ $2x^2 + 3x^2 - 5x^2 = 0$, which is true

$\Rightarrow$ R is reflexive.

Symmetricity: $2x^2 - 5xy + 3y^2 = 0$

$\Rightarrow$ $2x^2 - 2xy - 3xy + 3y^2 = 0$

$\Rightarrow$ $2x(x - y) - 3y(x - y) = 0$

$\Rightarrow$ $(2x - 3y)(x - y) = 0$

$\Rightarrow$ $2x = 3y$ or $x = y$

Let $x = 3, y = 2$, then $2(x) = 6$ and $3(y) = 6$

$\therefore$ $2x = 3y$

$\Rightarrow$ $x R y$, but $y \not R x$ as $2y = 4$ and $3x = 9$

$\therefore$ R is not symmetric.

Transitivity: Let $x = 3, y = 2, z = \frac{4}{3}$, then $2x = 3y$

$\Rightarrow$ $x R y$

And $2y = 3z$ $\Rightarrow$ $y R z$; but $x\ z$ and $2x\ 3z$

$\Rightarrow$ $(x, z) \notin R$

$\Rightarrow$ R is not transitive.

Surjective: Let $y \in \mathbb{R}$; then $x = \frac{3}{2}y \in \mathbb{R}$ and $2x = 3y$

$\Rightarrow$ $(x, y) \in \mathbb{R}$ $\Rightarrow$ R is surjective

Injective: $\because$ $(x, y) \in \mathbb{R}$ and $\left(\frac{3}{2}y, y\right) \in \mathbb{R}$

$\Rightarrow$ R is not injective.

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Functions

CHAPTER 2

INTRODUCTION

The term function was first used by Leibnitz to denote the dependence of one quantity on other quantities.

Examples

1. The area of a circle depends on its radius r through the equation $A = \pi r^2$. Thus, we say that A is a function of r and to denote this we write $A = f(r)$.
2. If you are traveling on a motor bike, the distance covered by you will depend upon how long you have been driving. So, we say that distance covered is a function of time. Further, the distance covered will also depend upon the speed with which you are driving. Here, we are expressing the distance covered to be a function of speed and time, because if the speed or the time increases the distance covered will increase and vice versa. The symbol f does not have a special meaning, it simply indicates dependence of one physical quantity over the other.

So, the concept of function is useful in defining the dependence of one thing (called dependent variable) on other things (called independent variables). In certain cases the dependent variables may not always increase with the increase in the independent variables. Also, the independent variable for one function may be a dependent variable for other function, which is studied under monotonicity of functions and composition of functions.

We need to understand that any relationship between two phenomenons is not necessarily a function. Actually the functions are specific type of relations. In this chapter we will also learn how to graphically represent the function and its applications to find out the solution of various problems.

DEFINITION OF FUNCTION

Let X and Y be two non-empty sets. Then a function f from set X to set Y is denoted as $f : X \to Y$ or $y = f(x)$; $x \in X$ and $y \in Y$. A function $f(x)$ from X (domain) to Y (co-domain) is defined as a relation f from set X to set Y such that each and every element of X is related with exactly one element of set Y.

Therefore, mathematically the function $f : X \to Y$ is infact a specific relation satisfying the following three conditions.

(i) $f \subset X \times Y$

(ii) for each $x \in X \Rightarrow (x, f(x)) \in f$

(iii) $(x, y_1) \in f$ and $(x, y_2) \in f \Rightarrow y_1 = y_2$

Pictorially

FIGURE 2.1

e.g.: Let $X = \{0, 1, 2, 3\}$ and $Y = \{1, 3, 5, 7, 9, 11\}$; define a function $f : X \to Y$ by $f(x) = 2x + 3$; then

$f = \{(0, 3), (1, 5), (2, 7), (3, 9)\}$. Clearly f is a function. But above is not a function when:

Set $Y = \{y : y$ is prime and $y \leq 15\}$, because 9 does not belong to Y in that case.

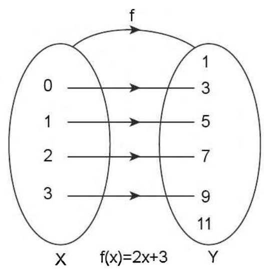


FIGURE 2.2

Machine Model of Function: $(x \in X) \to f \to (y \in Y)$ is called as function of x and abbreviated as 'y' is f of 'x'.

If a pre-image is denoted by x and image is denoted by y, then we can write:

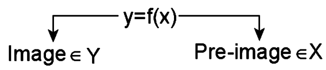


FIGURE 2.3

A function works as a machine which gets an input and produces a corresponding output as illustrated in Figure 2.4.

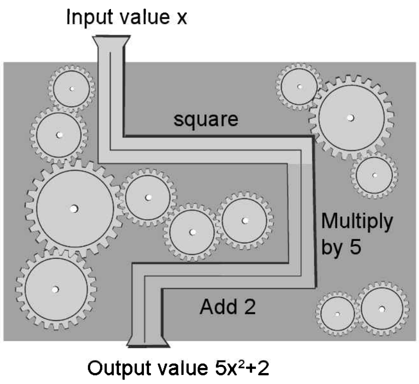


FIGURE 2.4

Notations:

$f : X \to Y$ or $X \xrightarrow{f} Y$, which reads as 'f is a function from X to Y' or 'f maps X to Y'.

Image and Pre-image: Let f be a function from set X to set Y, i.e., $f : X \to Y$ and let an element x of set X be associated to the element y of set Y through the rule 'f', then $(x, y) \in f$, i.e., $f(x) = y$, then y is called 'image of x under f' and x is called 'pre-image of y under f'.

e.g., Let $X = \{0, 1, 2, 3\}$ and $Y = \{1, 3, 5, 7, 9, 11\}$; define a function $f : X \to Y$:

$f(x) = 2x + 3$; then $f = \{(0, 3), (1, 5), (2, 7), (3, 9)\}$.

Now, $(0, 3) \in f \Rightarrow 0$ is pre-image and 3 is image,

$(1, 5) \in f \Rightarrow 1$ is pre-image and 5 is image.

$(2, 7) \in f \Rightarrow 2$ is pre-image and 7 is image.

$(3, 9) \in f \Rightarrow 3$ is pre-image and 9 is image.

ILLUSTRATION 1: If $f(x) = \dfrac{x-1}{x+1}$, then show that $\dfrac{f\left(\frac{1}{x^2}\right).\left[1+(f(x))^2\right]}{f(x)} = -2$.

SOLUTION: $f(x) = \dfrac{x-1}{x+1} \Rightarrow f\left(\dfrac{1}{x^2}\right) = \dfrac{\frac{1}{x^2}-1}{\frac{1}{x^2}+1} = \dfrac{1-x^2}{1+x^2} \quad \Rightarrow \quad \dfrac{f\left(\frac{1}{x^2}\right)[1+(f(x))^2]}{f(x)} = \dfrac{\left(\frac{1-x^2}{1+x^2}\right)\left[1+\left(\frac{x-1}{x+1}\right)^2\right]}{\left(\frac{x-1}{x+1}\right)}$

$= \dfrac{(1-x^2)}{(1+x^2)} \times \dfrac{2(x^2+1)}{(x+1)^2} \times \dfrac{(x+1)}{(x-1)} = \dfrac{2(1-x)(1+x)^2}{(x+1)^2(x-1)} = -2$. Hence, proved.

ILLUSTRATION 2: Let $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying $f(3) = 2$ and $f(x + 3) = f(3).f(x)$, then find $(f(-3))$.

SOLUTION: Since we are interested in finding $f(-3)$, we first put $x = -3$ in the relation, obtaining $f(0) = f(3)\ ((f(-3))$.

Thus, we must also know $f(0)$ in order to find $f(-3)$.

Letting $x = 0$ in the relation, $f(3) = f(3)\,f(0)$

$\Rightarrow f(0) = 1$ as $f(3) = 2$; Thus, $f(0) = f(3)\ .\ f(-3)$

$\Rightarrow 1 = (2)\ .\ f(-3) \qquad \Rightarrow f(-3) = \dfrac{1}{2}$.

ILLUSTRATION 3: Let $f : \mathbb{R} \to \mathbb{R}$, be a function defined as $f(x) = \dfrac{x-2}{x^2+2}$, find whether $f(0) + f(1) + f(2)$ and $f(0 + 1 + 2)$ are equal? Is there a real solution to the equation $f(x) = \dfrac{1}{x}$? Is there a real solution to the equation $f(x) = x$?

[redacted]

DOMAIN, CO-DOMAIN AND RANGE OF FUNCTION

If a relation f (not one–many) maps independent variables (inputs) $\in X$ to dependent variables (outputs) $\in Y$, then the subset of X containing elements x for which $f(x)$ is defined (real and finite) and $f(x) \in Y$ is called domain of f denoted by D_f

i.e., $D_f = \{x \in X : f(x)$ is finite and real $: f(x) \in Y\}$, and hence, $f : D_f \rightarrow Y$ becomes a function.

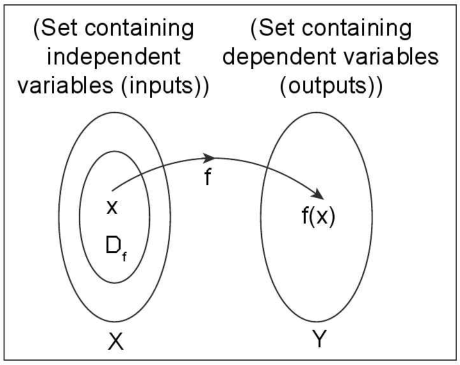


FIGURE 2.5

Natural Domain: The natural domain of a function is the largest set of real number inputs that give real number outputs of the function.

Co-domain: Set Y is called co-domain of function f.

Range of Function: If $f: D_f(\subseteq X) \rightarrow Y$ is a function with domain D_f, then the set of images y (output $\in Y$) generated corresponding to input $x \in D_f$ is called range of function, and it is denoted by R_f

i.e., $R_f = \{f(x): x \in D_f\} \subseteq Y$.

e.g., if f is a function from set $X = \{1, 2, 3, 4\}$ to set $Y = \{1, 4, 9, 16, 25, 36\}$ defined by $f(x) = x^2$; then its range set $= R_f = \{f(1), f(2), f(3), f(4)\} = \{1, 4, 9, 16\} \subseteq Y$.

Here, the elements 25, 36 are in the co-domain but not in the range.

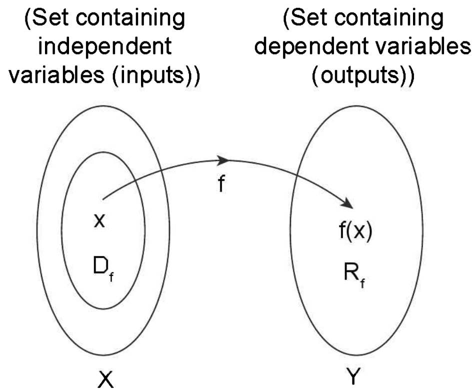


FIGURE 2.6

REMARKS

1. *To find domain of function we need to know, when does a function become undefined and when it is defined,* i.e., *we need to find those values of x where f(x) is finite and real and those values of x where f(x) is either infinite or imaginary.*
2. *When its value tends to infinity* (∞).

   e.g., $y = \dfrac{1}{x^2 - 1}$ *at* $x = \pm 1$; *f(x) is not defined at* $x = \pm 1$ *and defined* $\forall x \in \mathbb{R}$ *except for* ± 1; *therefore, domain of* $f(x) = \mathbb{R} \sim \{1, -1\}$.
3. *When it takes imaginary value.* e.g., $y = \sqrt{x - 1}$ $\forall x \in (-\infty, 1)$; *f(x) is not defined on* $(-\infty, 1)$ *and defined on* $[1, \infty)$; *therefore, domain of* $f(x) = [1, \infty)$.
4. *When it takes indeterminate form,* i.e., *becomes of the form* $\dfrac{0}{0}, \dfrac{\infty}{\infty}, 1^{\infty}, \infty^0, 0^0, \infty - \infty$ *etc*.

Example:

(i) $y = \dfrac{x+1}{x^2 - 1}$ takes $\dfrac{0}{0}$ form at $x = -1$, and hence, domain would not contain $x = -1$, also $f(x)$ is not defined at $x = 1$, thus, $D_f = \mathbb{R} \sim \{1, -1\}$

(ii) $y = \dfrac{1}{x^2 - 1} - \dfrac{1}{x^2 - 3x + 2}$ takes $\infty - \infty$ form at $x = 1$, and hence, domain would not contain $x = 1$. Domain for this function is given by $D_f = \mathbb{R} \sim \{1, -1, 2\}$

(iii) $y = \sec x - \tan x$ takes $\infty - \infty$ form at $x = (2n+1)\dfrac{\pi}{2}$, $n \in \mathbb{Z}$, and hence, domain would not contain odd integral multiples of $\dfrac{\pi}{2}$. Here domain of function is given by, $D_f = \mathbb{R} \sim \{(2n + 1)\dfrac{\pi}{2}; n \in \mathbb{Z}\}$

(iv) $y = (\sin x)^{\tan x}$ takes 1^{∞} form at $x = (2n+1)\dfrac{\pi}{2}$, $n \in \mathbb{Z}$, and hence, domain would not contain odd integral multiples of $\frac{\pi}{2}$; $D_f = \mathbb{R} \sim \left[\left(\bigcup_{n\in\mathbb{Z}}[(2n+1)\pi,(2n+2)\pi]\right)\cup\left\{(2n+1)\frac{\pi}{2}; n\in\mathbb{Z}\right\}\right]$

(v) $y = (\sin x)^{\tan x}$ takes 0^0 form at $x = n\pi$; D_f would not contain $\{n\pi;\ n \in \mathbb{Z}\}$, i.e., domain would not contain the real numbers $n\pi$, $n \in \mathbb{Z}$. Domain of the function is given in above remark.

(vi) $y = (\tan x)^{\cot x}$ takes ∞^0 form at $x = (2n+1)\frac{\pi}{2}$, and hence, domain would not contain odd integral multiples of $\frac{\pi}{2}$. Here domain of function is given by

$$D_f = \bigcup_{n\in\mathbb{Z}}\left[\left(2n\pi,(2n+1)\frac{\pi}{2}\right)\cup\left((2n+1)\pi,(4n+3)\frac{\pi}{2}\right)\right]$$

i.e., we excluded those real numbers for which $\tan x$ becomes negative and those for which $\cot x$ becomes infinite. Also at $x = (2n + 1)\pi$; $y = (0) - 1 = 1/0$, not defined. Also we excluded those real numbers for which the function takes indeterminate form ∞^0.

i.e., excluding from the set of real numbers the set $\left[(2n+1)\frac{\pi}{2},(2n+1)\pi\right]\cup\left[(4n+3)\frac{\pi}{2},(2n+2)\pi\right]$ $\forall\ n \in \mathbb{Z}$

(vii) To find range of a function first of all domain D_f is determined and the range consists of images $f(x)$ for each $x \in D_f$.

ILLUSTRATION 6: Let $X = \{1, 2, 3, 4\}$ and $Y = \{2, 3, 6, 8, 10, 12\}$. Define a relation f from X to Y as $f(x) = \frac{6}{(x-1)}$. Find domain, co-domain and range of f so that it becomes a function from its domain set to set Y.

SOLUTION: $f(x) = \frac{6}{(x-1)}$

$\therefore$ $f(1)$ is not defined and $f(2) = 6$; $f(3) = 3$; $f(4) = 2$

$\therefore$ Domain set $= D_f = \{2, 3, 4\}$ and co-domain $= Y = \{2, 3, 6, 8, 10, 12\}$; $R = \{2, 3, 6\}$.

Clearly for each $x \in D_f$; $f(x)$ is real, unique and is in Y, therefore $f(x)$ is a function from D_f to Y.

ILLUSTRATION 7: Let $X = \{1, 2, 3, 4\}$ and $Y = \{0, 1, 2, 3, 4\}$, define a relation f from X to Y as $f(x) = \sqrt{\quad}$; then find domain, co-domain and range of f so that it becomes a function from its domain set to set Y.

SOLUTION: $f(x) = \sqrt{x-3}$

$\therefore$ $f(x)$ is real for $x - 3 \geq 0$ $\Rightarrow x \geq 3$

But $x \in X = \{1, 2, 3, 4\}$ $\therefore$ Domain of $f = D_f = \{3, 4\}$ and co-domain of $f = Y = \{0, 1, 2, 3, 4\}$; $R_f = \{f(3), f(4)\} = \{0, 1\} \subseteq Y$.

Clearly for each $x \in D_f$, $f(x)$ is unique, real and is in Y, therefore $f(x)$ is a function from D_f to Y.

ILLUSTRATION 8: Define a function $g(x) : A \to \mathbb{R}$ defined by $g(x) = \sqrt{8x+3}$; where $A = \{x \in \mathbb{R}: 2f(x) = f(3x); f(x) = x^2 + 3x\}$. Find the domain and range of function $g(x)$.

SOLUTION: $A = \{x \in \mathbb{R}: 2f(x) = f(3x); f(x) = x^2 + 3x\}$

Now $2f(x) = f(3x)$

$\Rightarrow 2(x^2 + 3x) = (3x)^2 + 3(3x)$ $\Rightarrow 2x^2 + 6x = 9x^2 + 9x$

$\Rightarrow 7x^2 + 3x = 0$ $\Rightarrow x(7x + 3) = 0$

$\Rightarrow x = 0, -3/7$ $\therefore A = \{0, -3/7\}$

For domain of $g(x) = \sqrt{8x+3}$, $8x + 3 \geq 0$

$\Rightarrow$ $x \geq -3/8$

$\Rightarrow$ Domain of $g(x) = D_g = \{0\}$ as $-\frac{3}{7} < -\frac{3}{8}$ $\Rightarrow$ Range of $g(x) = R_g = \{f(0)\} = \{\sqrt{3}\}$.

ILLUSTRATION 9: Find the domain and range of $f(x) = 3 - \sqrt{x-4}$

SOLUTION: For $f(x) = 3 - \sqrt{x-4}$ to be meaningful the square root of $x - 4$ must be make sense; that means domain must consists of all real numbers x such that $x - 4 \geq 0$ or, equivalently, $x \geq 4$. That is, the domain is the interval $[4, \infty)$. As x varies from 4 to larger numbers, $f(x)$ decreases from $f(4) = 3 - \sqrt{4-4} = 3$ to arbitrarily smaller values. Thus, the range of f consists of all numbers less than or equal to 3, that is, the interval $(-\infty, 3]$.

ILLUSTRATION 10: Find the domain and range for the following functions:

(i) $f(x) = \sqrt{x-5}$ (ii) $f(x) = \sqrt{5-x}$

(iii) $f(x) = \sqrt{x^2 - 5x + 6}$ (iv) $f(x) = \sqrt{x^2 + x + 1}$

(v) $f(x) = \dfrac{x-1}{x-4}$ (vi) $f(x) = \dfrac{1}{\sqrt{(x-1)(x-2)}}$

(vii) $f(x) = \sqrt{\sin x}$ (viii) $f(x) = \sqrt{\sin(2x+3)}$

SOLUTION: (i) $f(x) = \sqrt{x-5}$; for domain $x - 5 \geq 0$

$\Rightarrow$ $D_f = [5, \infty)$

Let $y = \sqrt{x-5}$

$\Rightarrow$ $y^2 = x - 5$ $\Rightarrow$ $x = y^2 + 5$

But $x \geq 5$ $\Rightarrow$ $y^2 + 5 \geq 5$

$\Rightarrow$ $y^2 \geq 0$ $\Rightarrow$ $y \in \mathbb{R}$

But $f(x) = y = \sqrt{x-5} \geq 0$ (principal square root is always non-negative)

$\Rightarrow$ $y \geq 0$ $\Rightarrow$ $R_f = [0, \infty)$

(ii) $f(x) = \sqrt{5-x}$; for domain $5 - x \geq 0$

$\Rightarrow$ $x \leq 5$

$\Rightarrow$ $D_f = (-\infty, 5]$

Let $y = \sqrt{5-x}$ $\Rightarrow$ $y^2 = 5 - x$

$\Rightarrow$ $x = 5 - y^2$

As $x \leq 5$ $\Rightarrow$ $5 - y^2 \leq 5$

$\Rightarrow$ $y^2 \geq 0$ $\Rightarrow$ $y \in \mathbb{R}$

But $y = \sqrt{5-x} \geq 0$ (principal square root is always non-negative)

$\Rightarrow$ $y \geq 0$ $\Rightarrow$ Range of function $= R_f = [0, \infty)$.

(iii) $f(x) = \sqrt{x^2 - 5x + 6}$; for domain $x^2 - 5x + 6 \geq 0$

$\Rightarrow$ $(x - 2)(x - 3) \geq 0$

$\Rightarrow$ $x \leq 2$ or $x \geq 3$ ($\because$ either both factors are positive or both non-negative)

$\Rightarrow$ $D_f = (-\infty, 2] \cup [3, \infty)$

Also $y = f(x) = \sqrt{x^2 - 5x + 6} = \sqrt{(x-2)(x-3)}$; for $x \geq 3$; $(x - 2)(x - 3) \in [0, \infty)$; also for $x \in (-\infty, 2]$, $(x - 2)(x - 3) \in [0, \infty)$

$\Rightarrow\ y=\sqrt{(x-2)(x-3)}\in[0,\infty)$ $\qquad\Rightarrow R_f = [0, \infty)$

(iv) $f(x) = \sqrt{x^2+x+1} = \sqrt{x^2+x+\frac{1}{4}+\frac{3}{4}} = \sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$;

As $\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2 > 0\ \forall\ x\in\mathbb{R}$

$\Rightarrow$ $f(x)$ is real for all values of x $\qquad\Rightarrow D_f = \mathbb{R} = (-\infty, \infty)$

Also $\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2 \geq \frac{3}{4}\ \forall\ x\in\mathbb{R}$

$\Rightarrow\ y = f(x) \geq \sqrt{\frac{3}{4}}\ \ \forall\ x\in\mathbb{R}$

$\Rightarrow$ Range of function $f(x) = R_f = \left[\frac{\sqrt{3}}{2},\infty\right)$

(v) $f(x) = \frac{x-1}{x-4}$; y is not defined for $x = 4$

$\Rightarrow\ D_f = \mathbb{R} \sim \{4\}$

Now y = ——

$\Rightarrow\ xy - 4y = x - 1$ $\qquad\Rightarrow x(y-1) = 4y - 1$

$\Rightarrow\ x=\frac{4y-1}{y-1}$

$\therefore$ for $x\in\mathbb{R}, y\neq 1$ $\qquad\therefore R_f = \mathbb{R}\sim\{1\}$

(vi) $f(x) = \frac{1}{\sqrt{(x-1)(x-2)}}$; For domain $(x-1)(x-2) > 0$

$\Rightarrow\ x < 1$ or $x > 2$

$\Rightarrow\ D_f = (-\infty, 1)\cup(2,\infty)$

Now for $x > 2$, $(x-1)(x-2)\in(0,\infty)$

$\Rightarrow\ \sqrt{(x-1)(x-2)}\in(0,\infty)$

$\Rightarrow\ \frac{1}{\sqrt{(x-1)(x-2)}}\in(0,\infty)$

$\therefore$ Range of function $R_f = (0, \infty)$

(vii) $f(x) = \sqrt{\sin x}$. For domain of function, $\sin x \geq 0$

$\Rightarrow\ x\in[2n\pi, (2n+1)\pi]$; $n\in\mathbb{Z}$

$\Rightarrow\ D_f = \bigcup\limits_{n\in\mathbb{Z}}[2n\pi,(2n+1)\pi]$

Now for $y = \sqrt{\sin x}$ to be real, $\sin x\in[0, 1]$

$\Rightarrow\ \sqrt{\sin x}\in[0, 1]$

$\therefore$ Range of function $= R_f = [0, 1]$.

(viii) $f(x) = \sqrt{\sin(2x+3)}$. For domain of function $\sin(2x+3)\geq 0$

$\Rightarrow\ (2x+3)\in[2n\pi, (2n+1)\pi]$; $n\in\mathbb{Z}$

$\Rightarrow$ $2x \in [2n\pi - 3, (2n+1)\pi - 3]$; $n \in \mathbb{Z}$

$\Rightarrow$ $x \in \left[n\pi - \frac{3}{2}, n\pi + \left(\frac{\pi}{2} - \frac{3}{2}\right)\right]$; $n \in \mathbb{Z}$

$\therefore$ Domain $= D_f = \bigcup_{n\in\mathbb{Z}} \left[n\pi - \frac{3}{2}, n\pi + \left(\frac{\pi}{2} - \frac{3}{2}\right)\right]$; $n \in \mathbb{Z}$

Clearly for $x \in D_f$, $\sin(2x+3) \in [0, 1]$

$\Rightarrow$ $f(x) = \sqrt{\sin(2x+3)} \in [0, 1]$

$\Rightarrow$ Range of function $= R_f = [0, 1]$.

ILLUSTRATION 11: $f(x) = \log_a x$ is defined only for $x > 0$; $a > 0$ and $a \neq 1$, and it is known that $\log_a x$ increases from $-\infty$ to ∞ as x increases from 0 onwards to ∞ for $a > 1$ and $\log_a x$ decreases from ∞ to $-\infty$ as x increases from 0 onwards to ∞ for $a \in (0, 1)$. On the basis of above information, find the domain and range of following logarithmic functions.

(a) (i) $\log_2(x-1)$ (ii) $\log_3\sqrt{x-4}$

(iii) $\log_{1/3}(x+4)$ (iv) $\log_4 x^2$.

(v) $\log_e|x|$

(b) Find the domain of $f(x) = \log_{5-x}(x-2)$

SOLUTION: (a) (i) Given that $\log_a x$ is defined only for $x > 0$, $a > 0$, $a \neq 1$

$\therefore$ For domain of $f(x) = \log_2(x-1)$; $x - 1 > 0$ $\Rightarrow$ $x > 1$

$\therefore$ $D_f = (1, \infty)$

Since base $a = 2 > 1$, value of $\log_2(x-1)$ increases from $-\infty$ to ∞ as $(x-1)$ increases from 0 to ∞.

As x approaches to 1, $(x-1)$ approaches to 0, and hence, $\log_2(x-1)$ approaches to $-\infty$.

$\therefore$ Range of function $f(x) = R_f = (-\infty, \infty)$

(ii) $\log_3\sqrt{x-4}$; For domain, $x - 4 > 0$ $\Rightarrow$ $x > 4$ $\therefore$ $D_f = (4, \infty)$

Here $a = 3 > 1$ $\therefore$ Range of function $= f(x) = R_f = (-\infty, \infty)$

(iii) $f(x) = \log_{1/3}(x+4)$; For domain, $x + 4 > 0$ $\Rightarrow$ $x > -4$ $\Rightarrow$ $D_f = (-4, \infty)$

Here $a = 1/3 \in (0, 1)$, $f(x)$ decreases from ∞ to $-\infty$ as $x + 4$ increases from 0 to ∞.

$\therefore$ Range of function $= R_f = (-\infty, \infty)$

(iv) $f(x) = \log_4 x^2$

Here $x^2 \geq 0\ \forall\ x \in \mathbb{R}$. For domain $x^2 > 0$ $\Rightarrow$ $D_f = \mathbb{R} \sim \{0\}$

As $a = 4 > 1$, $f(x)$ increases from $-\infty$ to ∞ as x increases from 0 to $\pm\infty$.

$\therefore$ Range of function $= R_f = (-\infty, \infty)$

(v) $f(x) = \log_e|x|$; $|x| > 0\ \forall\ x \in \mathbb{R} \sim \{0\}$

$\therefore$ $D_f = \mathbb{R} \sim \{0\}$

Also range of function $= R_f = (-\infty, \infty)$

(b) $f(x) = \log_{5-x}(x-2)$

For domain $(x-2) > 0$; $5 - x > 0$; $5 - x \neq 1$

$\Rightarrow$ $x \in (2, \infty)$; $x < 5$; $x \neq 4$

$\Rightarrow$ $x \in (2, 5) \sim \{4\}$

$\therefore$ Domain of function $= D_f = (2, 5) \sim \{4\}$.

ILLUSTRATION 12: Express the length b of one-side of a right angled triangle as a function of the length a of the other with a constant hypotenuse $c = 5$. Graph this function and find its domain and range.

SOLUTION: $c = (a^2 + b^2)^{1/2}$ $\quad\Rightarrow\quad b^2 = (c^2 - a^2)$

$\Rightarrow b^2 = 25 - a^2$ $\quad\Rightarrow\quad b = \pm(25 - a^2)^{1/2}$

$\Rightarrow b = (25 - a^2)^{1/2}$ (Since b being side length is non-negative)

Thus, $b = f(a) = \sqrt{25 - a^2}$

For domain $a \in [-5, 5]$; but $0 < a < 5$ (otherwise triangle is impossible)

Also a being the length of side of triangle can never be negative, therefore $a \in (0, 5)$.

Therefore domain of f is $(0, 5)$.

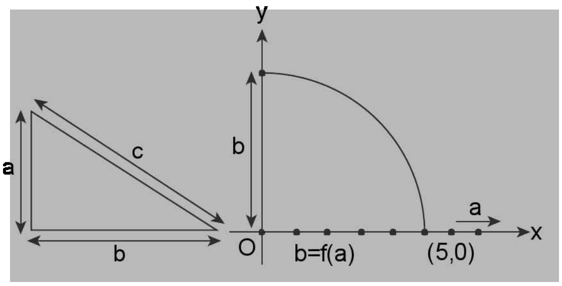


FIGURE 2.7

Also range of function consists of all values attained by b when $a \in (0, 5)$.

$\therefore$ Range of $f = R_f = (0, 5)$. Figure 2.8 represents the function $b = \sqrt{25 - a^2}$

ILLUSTRATION 13: A right circular cone is inscribed in a sphere of radius R. Find the functional relationship between the lateral surface area S of the cone and its generatrix x. Indicate the domain of definition of this function and its range.

SOLUTION: $S = \pi rl = \pi rx$ $\qquad (\because l = x)$

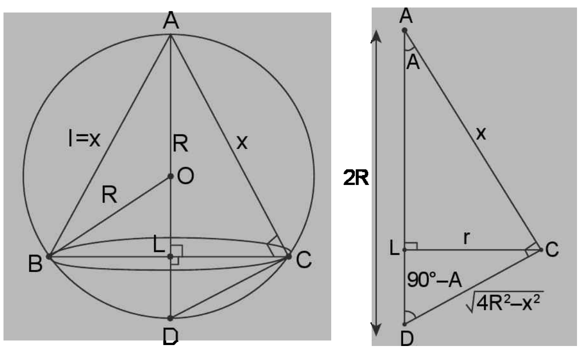


FIGURE 2.8

Now in ΔACL, $\sin A = \dfrac{r}{x}$ .... (i)

And in ΔCDL, $\operatorname{cosec}(90° - A) = \dfrac{\sqrt{4R^2 - x^2}}{r}$

$\Rightarrow \sec A = \dfrac{\sqrt{4R^2 - x^2}}{r}$ .... (ii)

Now, $\cos^2 A = \dfrac{1}{\sec^2 A}$ $\Rightarrow 1 - \sin^2 A = \dfrac{r^2}{4R^2 - x^2}$

$\Rightarrow 1 - \dfrac{r^2}{x^2} = \dfrac{r^2}{4R^2 - x^2}$ $\Rightarrow r = \dfrac{x\sqrt{4R^2 - x^2}}{2R}$ (Using (i) and (ii))

$\therefore$ $S = \pi r x = \dfrac{\pi x^2}{2R}\sqrt{4R^2 - x^2}$; $0 < x < 2R \Rightarrow S = f(x) = \dfrac{\pi x^2\sqrt{4R^2 - x^2}}{2R}$

$\therefore$ Domain of function $S = f(x) = (0, 2R)$.

($\because$ $4R^2 - x^2 \geq 0 \Rightarrow x \in [0, 2R]$ but for $x = 0, 2R$, cone cannot be constructed)

$\therefore$ **Range of function:** As $f'(x) = \dfrac{\pi}{2R}x\dfrac{\left[8R^2 - 3x^2\right]}{\sqrt{4R^2 - x^2}} \Rightarrow f'(x) = 0$ for $x = \dfrac{2\sqrt{2}R}{\sqrt{3}}$ and $x = 0$

At $x = 0$, $f(x)$ attains minimum value 0, where as at $x = \dfrac{2\sqrt{2}R}{\sqrt{3}}$, $f(x)$ attains its

maximum value $= \dfrac{\pi}{2R}\left(\dfrac{8}{3}R^2\right)\sqrt{4R^2 - \dfrac{8}{3}R^2} = \dfrac{8\pi R^2}{3\sqrt{3}}$. Thus, range $= \left(0, \dfrac{8\pi R^2}{3\sqrt{3}}\right]$.

ILLUSTRATION 14: In the Figure 2.9 express y analytically as a continuous function of x with natural domain. Also find the range of function.

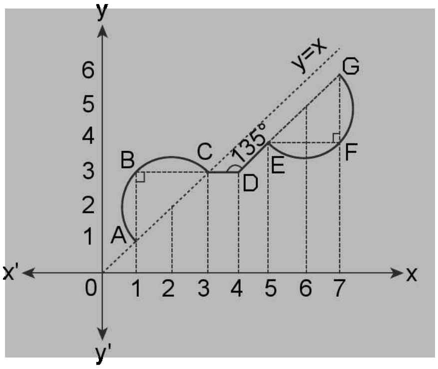


FIGURE 2.9

SOLUTION: $\overset{\frown}{ABC}$ is a semi circle with ends of diameter as (1,1) and (3,3).

$\therefore$ Equation of Arc $\overset{\frown}{ABC}$ is $(x-2)^2 + (y-2)^2 = 2$ or $y - 2 = \pm\sqrt{4x - x^2 - 2}$ ...(1)

Clearly for $x \in \left[2 - \sqrt{2}, 1\right]$ $f(x)$ is not a function as it has two output for each input given by above analytical relation in equation (1). Therefore to convert it to function, isolating

the two branches, we get $y = \begin{cases} 2 + \sqrt{4x - x^2 - 2} & ; y \geq 2 \\ 2 - \sqrt{4x - x^2 - 2} & ; y \leq 2 \end{cases}$ ...(2)

Now f to be continuous function rejecting the negative branch, i.e., for $y \leq 2$, we get

$$f(x) = \begin{cases} 2 + \sqrt{4x - x^2 - 2}; 2 - \sqrt{2} \leq x \leq 3 \\ 3; 3 < x \leq 4 \end{cases}$$

For $x \in [4,5]$ f is a linear polynomial given by a straight line through (4, 3) of slope 1, i.e., $f(x) = x - 1$

$$\therefore \quad f(x) = \begin{cases} 2+\sqrt{4x-x^2-2}; & 2-\sqrt{2} \le x \le 3 \\ 3 & ;\ 3 < x \le 4 \\ x-1 & ;4 < x \le 5 \end{cases}$$

Similarly for the interval $x \in [5, 6+\sqrt{2}]$ $f(x)$ is given by an arc of the lower semicircle (i.e., $y \le 5$ with centre (6, 5) and radius $\sqrt{2}$).

Now equation of Arc $\widehat{EFG}$: $(x - 6)^2 + (y - 5)^2 = 2$

$$\Rightarrow \quad y = \begin{cases} 5-\sqrt{12x-x^2-34}; & y \le 5 \\ 5+\sqrt{12x-x^2-34}; & y \ge 5 \end{cases}$$

Thus, the complete analytical formula for $f(x)$ in its natural domain is given by

$$f(x) = \begin{cases} 2+\sqrt{4x-x^2-2} & ;\ 2-\sqrt{2} \le x \le 3 \\ 3 & ;\ 3 < x \le 4 \\ x-1 & ;\ 4 < x \le 5 \\ 5-\sqrt{12x-x^2-34} & ;\ 5 < x \le 6+\sqrt{2} \end{cases}$$

Domain of $f(x)$ is $[2-\sqrt{2}, 6+\sqrt{2}]$ and range of $f(x)$ is $[f(2-\sqrt{2}), f(6+\sqrt{2})]$, i.e., [2, 5].

TEXTUAL EXERCISE–1: (SUBJECTIVE)

1. If $f(x) = \sqrt{x^2-4}$, find the values of $f(1), f(2), f(-3)$, if they exist (i.e., real).

2. If $f(x) = \dfrac{(x-2)}{(x-4)(x-5)}$, find the real numbers for which $f(x)$ is not defined (not real and finite).

3. If $f(x) = \dfrac{1}{(x-1)}$; then find the real numbers for which $f(x)$ is not defined. Also find the real number(s) which the function $f(x)$ cannot attain.

4. If $f(x) = \sqrt{(x-1)(x-4)}$, then find the interval

(i) For which $f(x)$ is not defined

(ii) For which $\dfrac{1}{f(x)}$ is not defined

(iii) Containing all values of $f(x)$

(iv) Containing all values of $\dfrac{1}{f(x)}$

5. Find the set of all values of x, for which the following functions (expressions) are defined:

(a) $\dfrac{1}{x(x-1)(x-2)}$ (b) $\dfrac{x^2-x}{x^2+x}$

(c) $\dfrac{x+1}{x^2+x+1}$ (d) $\dfrac{1}{\sqrt{x-3}}$

(e) $\sqrt{x-4}$ (f) $\sqrt{x-2}+\sqrt{x+2}$

(g) $\sqrt{x-4}+\sqrt{7-x}$ (h) $\dfrac{1}{\sqrt{x-4}-\sqrt{6-x}}$

(i) $\sqrt{x-7}-\sqrt{3-x}$

6. Find the domain of definition of the following functions:

(a) $f(x) = \dfrac{\sqrt{x-1}}{\sqrt{6-x}}$ (b) $f(x) = \sqrt{\dfrac{x-1}{6-x}-1}$

(c) $f(x) = \dfrac{\sqrt{x-1}}{6-x}$ (d) $\left(\sqrt{x}-1\right)\left(6-\sqrt{x}\right)$

(e) $\left(\sqrt{x}-1\right)\sqrt{(6-x)}$

7. Find the domain of definition the following functions:

(a) $f(x) = \sqrt{x-1}\sqrt{6-x}$ (b) $f(x) = \sqrt{(x-1)(6-x)}$

(c) $f(x) = \sqrt{x-x^2}$

8. Find the domain of the following functions:

(i) $\sqrt{5-x}+\sqrt{x-2}$

(ii) $\sqrt{3-x}+\sqrt{x-5}$

9. Find the domain and range of the function $f(x)$ defined by $f(x) = \sqrt{x^2+x+1}$ from a subset of set $A = \{0, 1, 2, 3\}$ to B = set of irrational numbers.

10. Find pairs of functions from the following function(*s*) whose domain and range are same?

(i) $f(x)=\sqrt{1-x^2}$ (ii) $g(x) = \dfrac{1}{x}$

(iii) $h(x) = \dfrac{1}{\sqrt[3]{x}}$ (iv) $\phi(x) = \sqrt[4]{1-x^2}$

11. Find the domain and range of the function $f(x)$ from a subset of set $A = \{x : x$ is a +ve integer divisor of $36\}$ to set of real numbers ($\mathbb{R}$) defined by

$$f(x) = \frac{x}{(x-1)(x-2)(x-3)....(x-10)}$$

12. Let $A = \{9, 10, 11, 12, 13\}$ and let $f : A \to \mathbb{N}$ be defined by $f(n)$ = the highest prime factor of n. Find the range of f.

13. A cylinder is inscribed in a sphere of radius R. Express the volume V of the cylinder as a function of its height x. Indicate the domain of definition of this function.

14. O is a fixed point on a circle of unit radius whose centre is P. A chord perpendicular to OP at a distance x from O cuts the circle at A and B. Express the area of the segment AOB as a function of x.

15. In the Figure 2.10, $BC = b$ and $AH = h$; where $AH \perp BC$. If $EF = x$, where $EF \perp BC$, then express the area and perimeter of the rectangle $DEFG$ as functions of x.

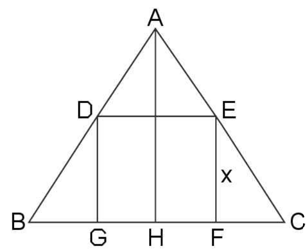


FIGURE 2.10

Answer Keys

1. $f(1)$ does not exist; $f(2) = 0$, $f(-3) = \sqrt{5}$

2. 4, 5

3. $f(x)$ is not defined for $x = 1$; $f(x)$ cannot attain 0 value.

4. (i) $(1, 4)$ (ii) $[1, 4]$ (iii) $[0, \infty)$ (iv) $(0, \infty)$

5. (a) $\mathbb{R} \sim \{0, 1, 2\}$ (b) $\mathbb{R} \sim \{0, -1\}$ (c) $\mathbb{R}$ (d) $(3, \infty)$
(e) $[4, \infty)$ (f) $[2, \infty)$ (g) $[4, 7]$ (h) $[4, 6] \sim \{5\}$ (i) ϕ

6. (a) $[1, 6)$ (b) $[7/2, 6)$ (c) $[1, \infty) \sim \{6\}$ (d) $[0, \infty)$ (e) $[0, 6]$

7. (a) $[1, 6]$ (b) $[1, 6]$ (c) $[0, 1]$

8. (i) $[2, 5]$ (ii) ϕ

9. Domain = $\{1, 2, 3\}$ and Range = $\{\sqrt{3}, \sqrt{7}, \sqrt{13}\}$

10. $(g(x), h(x))$ and $(f(x), \phi(x))$

11. Domain = $\{12, 18, 36\}$; Range = $\left\{\dfrac{12}{11!}, \dfrac{18\times 7!}{17!}, \dfrac{36\times 25!}{35!}\right\}$

12. Range = $\{3, 5, 11, 13\}$

13. $0 < x < 2R$; $\pi x\left(R^2-\dfrac{x^2}{4}\right)$

14. $\cos^{-1}(1-x) - (1-x)\sqrt{2x-x^2}$, $0 \le x \le 1$ and $\pi - \cos^{-1}(x-1) + (x-1)\sqrt{2x-x^2}$, $1 < x \le 2$.

15. $A(x) = \dfrac{bx(h-x)}{h}, p(x) = 2\left\{x+\dfrac{b(h-x)}{h}\right\}$

TEXTUAL EXERCISE–1: (OBJECTIVE)

1. If $f(x)=\frac{1}{1-x}$, then $f\left(\frac{1}{x}\right)$ is

(a) $\frac{x}{x-1}$ (b) $\frac{1}{x-1}$
(c) $\frac{1}{x+1}$ (d) None of these

2. The domain of function $f(x)=\log(1-x)+\sqrt{x^2-1}$ is

(a) $[-1, 1]$ (b) $(1, \infty)$
(c) $(-\infty, -1]$ (d) None of these

3. If $f(x) = ax^2 + bx + c$, then the values of a and b for which the identity $f(x + 1) - f(x) = 8x + 3$ is satisfied are

(a) $a = 4$ and $b = -1$ (b) $a = 1$ and $b = 4$
(c) $a = -1$ and $b = 4$ (d) None of these

4. Let $p(x) = a^2 + bx$, $q(x) = lx^2 + mx + n$. If $p(1) - q(1) = 0$, $p(2) - q(2) = 1$ and $p(3) - q(3) = 4$, then $p(4) - q(4)$ equals

(a) 2 (b) 5
(c) 9 (d) None of these

5. The domain of the function $f(x)=\sqrt{\log(\log x)-\log(4-\log x)-\log 3}$; (base of log is 10)

(a) $[10^3, 10^4)$ (b) $(10^3, 10^4)$
(c) $[10^3, \infty)$ (d) None of these

6. The domain of the function $y = f(x)$ given by $3^y+2^{x^4}=2^{4x^2-3}$ is

(a) $(-\sqrt{3}, \sqrt{3})$ (b) $(-\sqrt{3}, -1) \cup (1, \sqrt{3})$
(c) $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$ (d) None of these

7. The domain of definition of the function $\sqrt[3]{\frac{2x+1}{x^2-10x-11}}$ is given by

(a) $\mathbb{R} \sim \{-1, 11\}$ (b) $\mathbb{R}$
(c) $(0, \infty)$ (d) None of these

8. The domain of the function $f(x)=\sqrt{\log_5(\cos(\sin x))}$ is

(a) $\mathbb{R}$ (b) $(0, \infty)$
(c) $\{n\pi, n \in \mathbb{Z}\}$ (d) None of these

9. The domain of the function $f(x)=\frac{1}{\sqrt{x^6-13x^4+36x^2}}$ is

(a) $(0, 2) \cup (3, \infty)$
(b) $(-\infty, -3) \cup (-2, 0) \cup (0, 2) \cup (3, \infty)$
(c) $\mathbb{R} \sim \{-3, -2, 0, 2, 3\}$
(d) None of these

10. The range of the function $f(x)=x^2+\frac{1}{x^2+1}$ is

(a) $[1, \infty)$ (b) $(0, \infty)$
(c) $[2, \infty)$ (d) None of these

11. The domain of the function $f(x) = \log_4(\log_5(\log_3(18x - x^2 - 77)))$ is

(a) $(8, 10)$ (b) $(-8, 10)$
(c) $(-\infty, 8) \cup (10, \infty)$ (d) None of these

12. Let the function $f : D \to \mathbb{R}$, $f(x) = \log_5(\log_{1/3} \log_8(2x + 1))$; where D is the natural domain of f. If S represents the sum of the absolute values of all integers from D, then the value of S is

(a) 4 (b) 5
(c) 6 (d) None of these

Answer Keys

1 (a)	**2.** (c)	**3.** (a)	**4.** (c)	**5.** (a)	**6.** (b)	**7.** (a)	**8.** (c)	**9.** (b)	**10.** (a)
11. (a)	**12.** (c)								

■ REPRESENTATION OF FUNCTIONS

Analytical Representation: When a function is denoted as $y = f(x)$ or $f(x, y) = 0$, then it is called analytical representation. e.g., $y = \sqrt{x^2+1}$, $f(x) = \frac{\ln x+e^x}{\sin x}$ or $f(x) = \frac{ax^2+bx+c}{e^{2x}\sin^{-1}x}$.

The advantage of analytical representation is the compactness and the possibility of evaluating $f(x)$ for any value of x in the domain. This is the most effective way of representation of a function which allows application of techniques of calculus.

Most of the functions encountered at basic level are defined by means of a single equation.

For example, $f(x) = x^2 - 2$; $f(x) = \sqrt{x+1}$. However, it is not true that a function in analytical representation always use a single equation. We may require more than one equation to represent a function analytically, depending upon its domain.

Example:

(i) $f(x) = \begin{cases} \sqrt{x-1} & \text{for } x \geq 1 \\ 2x-2 & \text{for } x < 1 \end{cases}$

(ii) $g(x) = \begin{cases} \sin x & \text{for } x < -\pi \\ x+\pi & \text{for } -\pi \leq x \leq \pi \\ \tan x & \text{for } x > \pi \end{cases}$

(iii) $h(x) = \begin{cases} x\sin\frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$ etc

The function defined above using more than one equations are called piece-wise defined function.

A piece-wise defined function has different analytical expressions (formulae) on different parts of its domain as illustrated in above examples. In a piece-wise defined function, the domain is the union of the sub-intervals. To evaluate the function at a particular value of x, we choose the appropriate formula.

For instance, for the function $f(x)$ illustrated above $f(-3) = 2(-3) - 2 = -8$ and $f(4) = \sqrt{4-1} = \sqrt{3}$.

The independent variable used in an analytical representation may be continuous or discrete. In the examples given above $f(x)$, $g(x)$ and $h(x)$ have continuous independent variable.

Now consider the function $f(x) = {}^nC_x + {}^nP_x$; where C and P are combination and permutation has discrete independent variable x and having domain $\{0, 1, 2, 3, \ldots, n\}$ consisting of first $(n + 1)$ whole numbers.

Analytical representation may use either explicit form of representation or implicit form of representation. i.e., in the form $y = f(x)$ or $f(x, y) = 0$ e.g., $y = x^2 - 1$ or $y - x^2 + 1 = 0$

Generally we deal with functions in which the dependent variable y is explicitly related to independent variable x. Such functions are called explicit function.

A function is an implicit function, if it is defined by an equation not solved for the dependent variable y. For example $y^3 - x^2 = 5x$; $2xy - 5\tan^2 y = 0$ etc.

Here, y is not directly expressed in terms of x. However, in some cases, an implicit function can be converted into an explicit function.

For example, $y^3 - x^2 = 5x$; can be expressed as $y = (x^2 + 5x)^{1/3}$. Sometimes, the conversion can change the domain. But we must consider the domain of the original expression.

e.g., $\log(x^3 - 1) + \log y - 5 = 0$

$\Rightarrow \log(x^3 - 1)y = 5$

$\Rightarrow (x^3 - 1)y = e^5$

$$\Rightarrow y = \frac{e^5}{(x^3-1)} \qquad \ldots (1)$$

having its domain $\mathbb{R} \sim \{1\}$, but if we observe the original equation, for $\log y$ and $\log(x^3 - 1)$ to be defined $x^3 - 1 > 0$, $y > 0$

$\Rightarrow (x - 1)(x^2 + x + 1) > 0$

$\Rightarrow x > 1$, i.e., the domain of original function expressed in implicit form is $(1, \infty)$.

Thus, the domain is changed, so care must be taken while converting the function given in explicit form to function in implicit form and defining the domain of converted function.

REMARK

Some of the implicit equations do not represent functions. For example, the equation $9x^2 + 4y^2 = 36$ *represents an ellipse and on solving it for y, we get* $y^2 = \dfrac{36-9x^2}{4} \Rightarrow y = \pm\dfrac{\sqrt{36-9x^2}}{2}$

Thus, corresponding to each input x, we get two outputs $\dfrac{\sqrt{36-9x^2}}{2}$ *and* $-\dfrac{\sqrt{36-9x^2}}{2}$.

Thus, the given equation does not represent a function. In fact it represents a many–many relation.

However, we can define two functions here:

$\phi = \{(x, y) : y = \dfrac{\sqrt{36-9x^2}}{2}, x \in [-2, 2]\}$ *and* $y = \{(x, y) : y = -\dfrac{\sqrt{36-9x^2}}{2}, x \in [-2, 2]\}$

Such a relation is sometimes called a multi-valued function.

ILLUSTRATION 15: Consider a circle of radius *r*. Let $f(x)$ be the length of a chord AB of circle at a distance x from the centre of the circle. Express the function $f(x)$ analytically and find its domain of definition.

SOLUTION: Let M be the midpoint of the chord AB and let C be the centre of the circle. Note that $CM = x$ and $CB = a$. By Pythagoras theorem, from right angled ΔBCM

we have, $BM = \sqrt{a^2 - x^2}$. Hence, $AB = 2\sqrt{a^2 - x^2}$

Thus, $f(x) = 2\sqrt{a^2 - x^2}$ is the required function in algebraic form

Now $f(x)$ produces real values for $a^2 - x^2 \geq 0 \Rightarrow x \in [-a, a]$

But x being distance cannot be negative. $\Rightarrow x \in [0, a]$. For $x = a$, length of chord = 0, i.e., no chord. Hence, the domain of definition of the function f is $[0, a]$.

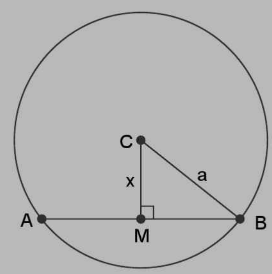


FIGURE 2.11

■ FUNCTION AS A SET OF ORDERED PAIRS

Let A and B be two non-empty sets. A relation f from A to B, i.e., a subset of

$A \times B$ is called a function (or a mapping or a map) from A to B iff

(a) For each $a \in A$ there exists $b \in B$ such that ordered pair $(a, b) \in f$,

(b) Also if $(a, b) \in f$ and $(a, c) \in f \quad \Rightarrow \quad b = c$

Thus, a non-empty subset f of $A \times B$ is a function from A to B if each element of A appear in some ordered pair in f and no two ordered pairs in f have the same first element. If $(a, b) \in f$, then b is called the image of a under f and a is called pre-image of b under f.

Therefore, a function $y = f(x)$ can be expressed as a set of ordered pairs.

That is, if $f : A \to B$ is a function, then $f = \{(a, f(a)) : a \in A$ and $f(a) \in B\}$ is representation of function as a set of ordered pairs.

If $f : X \to Y$ is a function defined as $f(x) = 2x^2 + 1$ such that $X = \{0, 1, 2, 3\}$ and $Y = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$ can be expressed as a set of ordered pairs given by $f = \{(0, 1), (1, 3), (2, 9), (3, 19)\}$.

Graphical Representation: Being a set of ordered pairs, a function can be represented graphically on (x, y) plane, where each point (x, y) on the graph represents an input x and the corresponding output y of the function. The use of graph gives a visual representation and shows the various characteristics of graph viz. concavity, convexity, monotonicity and the domain of definition of the function.

For example:

1. The graph of $f(x) = \begin{cases} \sqrt{x-1}; & \text{if } x \geq 1 \\ 2x-2; & \text{if } x < 1 \end{cases}$ is shown in Figure 2.12.

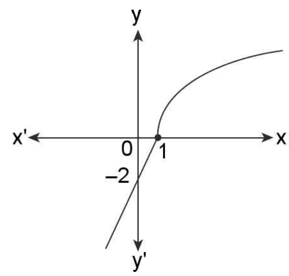


FIGURE 2.12

Corresponding to real numbers less than one the graph of function is a straight line $y = 2x - 2$ and for real numbers greater than or equal to 1, the graph of function is a portion of parabola $y^2 = x - 1$. Thus, the given function is a combination of a straight line and a parabola. Clearly the function is defined that is function takes real and finite values $\forall\, x \in \mathbb{R}$. Thus, the domain of function is $\mathbb{R}$. Also the function takes all real values from $-\infty$ to ∞, i.e., the range of function is $\mathbb{R}$.

From the graph of given function we see that the function is monotonically increasing on $\mathbb{R}$. Also the function is concave downwards on $(1, \infty)$. The function is neither concave upwards nor concave downwards, i.e., linear for $x \in (-\infty , 1]$.

2. $f(x) = 2x^2 + 1$ can be represented graphically as shown in Figure 2.13.

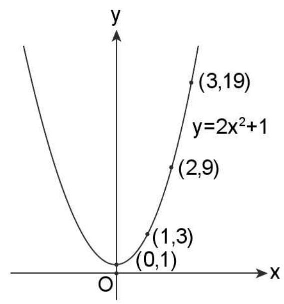


FIGURE 2.13

Observing the graph of given function we find that the domain of function is $\mathbb{R}$ and the range is $[1, \infty)$. Clearly the graph is concave upwards for all $x \in \mathbb{R}$. The graph of function is monotonically decreasing for $x \in (-\infty, 0)$ and monotonically increasing for $x \in (0, \infty)$. Also $x = 0$ is a stationary point, i.e., the point in the domain of function corresponding to which the point on the function has horizontal tangent.

REMARK

Most of the functions can be represented graphically but there are some functions which cannot be represented by a graph. For example:

(i) The Drichlet-Function which is defined as $f(x) = \begin{cases} 0, & \text{when } x \text{ is rational} \\ 1, & \text{when } x \text{ is irrational} \end{cases}$ *cannot be graphed since there exist infinite number of rationals as well as irrationals between any two real numbers.*

(ii) Consider the Euler's totient or Euler's phi function $\phi(n)$ = Number of positive integers less than or equal to n and co-prime to n; where n is a natural number.

The domain of ϕ is the set of positive integers. Its range is the set of positive integers {1, 2, 3, ...}.

We cannot represent this function analytically. A portion of the graph of $\phi(n)$ is shown below for the understanding of the function.

Since g.c.d. (1, 1) = 1 $\Rightarrow$ $\phi(1) = 1$.

Now g.c.d. (1, 2) = 1 $\Rightarrow$ $\phi(2) = 1$

Further g.c.d. (1, 3) = g.c.d. (2, 3) = 1 and 1, 2 < 3 $\Rightarrow$ $\phi(3) = 2$

Similarly, g.c.d. (4, 1) = g.c.d. (4, 3) = 1 $\Rightarrow$ $\phi(4) = 2$,

and g.c.d. = (1, 5) = g.c.d. (2, 5) = g.c.d. (3, 5) = g.c.d. (4, 5) = 1 $\Rightarrow$ $\phi(5) = 4$

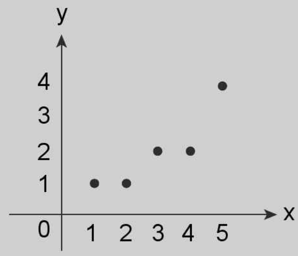


FIGURE 2.14

(iii) Consider another function called prime number function defined by f(x) = number of prime numbers less than or equal to x; where x is positive real number.

Then domain of f(x) is $(0, \infty)$ and range is the set of non-negative integers, i.e., {0, 1, 2, 3, ...,}.

The graph of function is shown below

As x increases, the function f(x) remains constant until x reaches a prime, at which the graph of function jumps by 1. Therefore, the graph of f consists of horizontal line segments. This is an example of a class of function called step functions.

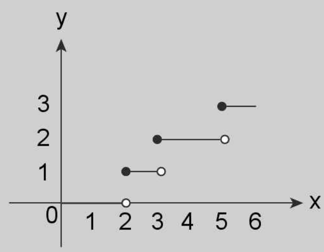


FIGURE 2.15

(iv) Another function, which is so complicated that it is impossible to draw its graph,

$$h(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ -\dfrac{x}{2}, & \text{if } x \text{ is irrational.} \end{cases}$$

As we know that between any two real numbers, there lie infinitely many rationals and irrational numbers, so it is impossible to draw its graph.

■ PARAMETRIC REPRESENTATION

If each of the dependent and independent variables of a function can be represented as a separate function of a third variable called parameter, then the function represented by dependent and independent variables as functions of parameter is called in parametric representation.

i.e., if $y = f(x)$ is a function such that

$$\left.\begin{aligned} x &= \phi(t) \\ y &= \psi(t) \end{aligned}\right\} \qquad \ldots(1)$$

where t assumes values that lies in the interval $[a, b]$. To each value of t there corresponds values of x and y (the function ϕ and ψ are assumed to be single valued). If the ordered pair (x, y) be regarded as the coordinates of a point in a coordinate xy-plane, then to each value of t there correspond a definite point in the plane. And when t varies from a to b, $t$he point will describe a certain curve giving us the graph of function $y = f(x)$.

Equations (1) are called parametric equations of this curve, t is the parameter, and the curve is represented parametrically by equations (1).

The direct relationship between the variables x and y (not involving the parameter t) from equation (1), can be obtained by eliminating t from equations (1).

i.e., the function $x = \phi(t)$ has the inverse, $t = \phi^{-1}(x)$, then $y = \psi(t)$

$$\Rightarrow \quad y = \psi[\phi^{-1}(x)] \qquad \ldots(2)$$

Thus, equation (1) defines y as a function of x, and we say that the function y of x is represented parametrically.

For example, consider the function defined parametrically by $x = t + 2$; $y = t^2 - 1$; $t \in \mathbb{R}$

We have $t = x - 2$ and $y = t^2 - 1$

$$\Rightarrow \quad y = (x - 2)^2 - 1$$

$$\Rightarrow \quad y = x^2 - 4x + 3$$

which represents an upward parabola with its vertex at (2, –1) as shown in Figure 2.16..

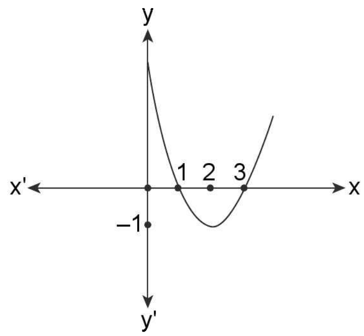


FIGURE 2.16

A parametric representation of a function (or of a curve) is sometimes more convenient than other ways of representing the function as the parametric equation may express a complicated relationship between x and y in a simpler manner.

REMARKS

(*i*) *If at least one of the functions of parametric equation is constant, then it is impossible to eliminate the parameter t to obtain a functional relationship y = f(x). For instance, the equations x = 2, y = cost would not provide us a functional relationship between x and y. Here they would contain the ordered pairs (2, cost); t ∈ ℝ.*

i.e., the graph is a straight line segment joining the points (2, –1) and (2, 1).

Let us consider the function represented parametrically as $x = at^2$, $y = 2at$; $t \in \mathbb{R}$

Analytically the above relation represents a right handed parabola $y^2 = 4ax$ *as shown in Figure 2.17.*

The geometric significance of the parameter t in the above equations is as follows:

When the parameter t ranges from 0 to ∞ the variable point traverses the upper half portion of parabola from the point (0, 0) to the point (∞,∞) and if t varies from 0 to –∞ it describes the lower half portion of parabola from (0, 0) to (∞, –∞). Hence, if t continuously varies from –∞ to ∞, the moving point traverse the parabola from (∞, –∞) to (∞, ∞) continuously in clockwise directions as represented by arrows on the graph of function (parabola).

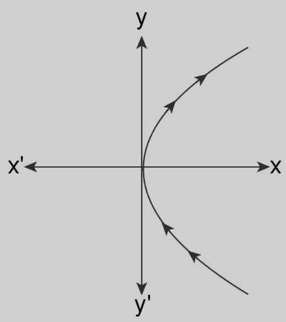


FIGURE 2.17

(*ii*) *Parametric representation of curves is widely used in mechanics. If in the xy-plane there is a certain object in motion and if we know its laws of motion horizontally and vertically,* i.e., *the relationship between distance traveled horizontally and vertically with time.*

$$\text{i.e.,} \quad \left.\begin{aligned} x &= \phi(t) \\ y &= \psi(t) \end{aligned}\right\} \qquad \ldots(3)$$

where the parameter 't' is the time. Then equation (3) *are parametric equations of the trajectory of the moving object. Eliminating from these equations the parameter t, we get the equation of the trajectory in the form y = f(x) or f(x, y) = 0.*

Consider some examples of curves in parametric form.

1. *For a circle with centre at the origin and radius r. Let t denotes the positive angle between x-axis and the radius vector at some point P(x, y) of the circle as shown in Figure 2.18.*

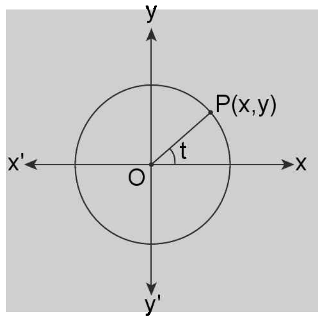


FIGURE 2.18

Then the coordinates of point P on the circle is expressed in terms of the parameter t as follows. $\left.\begin{array}{l} x = r\cos t \\ y = r\sin t \end{array}\right\}, 0 \le t \le 2\pi$

These are the parametric equations of the circle. If we eliminate the parameter t from these equations, we will have an equation of the circle containing only x and y. Squaring the parametric equations and adding, we get $x^2 + y^2 = r^2(\cos^2 t + \sin^2 t)$ *or* $x^2 + y^2 = r^2$.

2. *Given the equation of the ellipse in standard form* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ .... (1)

Let $x = a\cos t$

Putting this value of x in equation (1), *we get* $y = b\sin t$

The equations $\left.\begin{array}{l} x = a\cos t \\ y = b\sin t \end{array}\right\}; 0 \le t \le 2\pi$ *are the parametric equations of the ellipse* (1).

3. *The Asteroid is a curve which is a many–many relation and not a function as shown in Figure 2.19.*

The parametric equations of Asteroid are $\left.\begin{array}{l} x = a\cos^3 t \\ y = a\sin^3 t \end{array}\right\}; 0 \le t \le 2\pi$ ... (1)

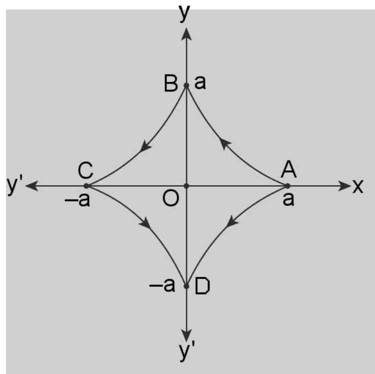


FIGURE 2.19

when t varies from 0 to $\dfrac{\pi}{2}$, *point on curve traces the portion AB, from A to B,* i.e., *x decreases from a to 0 where as y increases from 0 to a as is clear from the parametric equation* (1)

When t varies from $\dfrac{\pi}{2}$ *to* π, *point on curve traces the portion BC.* i.e., *x decreases from 0 to –a and y also decreases from a to 0.*

Similarly when t varies from π *to* $\dfrac{3\pi}{2}$ *and* $\dfrac{3\pi}{2}$ *to* 2π *the point on curve traces the portions CD and DA respectively. During the movement of point along CD, x increases from –a to 0 and y decreases from 0 to –a and during the movement of point along DA, x increases from 0 to a and y increases from –a to 0.*

Raising the terms of both equations (1) *to the power 2/3 and adding, we get the following relationship between x and y:* $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}(\cos^2 t + \sin^2 t)$ *or* $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

$$\Rightarrow y^{\frac{2}{3}} = a^{\frac{2}{3}} - x^{\frac{2}{3}} \qquad \Rightarrow y^2 = \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^3 \qquad \Rightarrow y = \pm\sqrt{\left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^3}$$

Thus, an asteroid is a combination of two functions $\phi = \left\{(x, y) : y = \sqrt{\left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^3}\,;\, x \in [-a, a]\right\}$ *and*

$$\psi = \left\{(x, y) : y = -\sqrt{\left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^3}\,;\, x \in [-a, a]\right\}$$

Here ϕ *represents upper half portion of curve, where as* ψ *represents the lower half portion of curve and* $\psi = -\phi$.

■ DIAGRAMMATIC REPRESENTATION

(By using arrow diagram): If $f: X \to Y$ is a function, then we represent X and Y by two circles or ellipses containing independent and dependent variables respectively and join the inputs with their corresponding outputs (images) by using arrows. The function $f(x) = 2x^2 + 1$ from set $X = \{0, 1, 2, 3\}$ to set $Y = \{1, 3, 5, 7, 9, 11, 13, 19\}$ is represented diagrammatically as shown below.

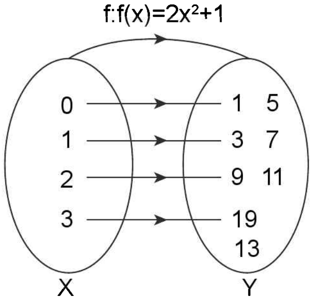


FIGURE 2.20

Methods of Testing a Relation to be a Function

Method 1: *When the relation to be tested is represented analytically*: A relation $f: X \to Y$ defined as $y = f(x)$ will be function iff $x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$ since otherwise an element of X would have two different images.

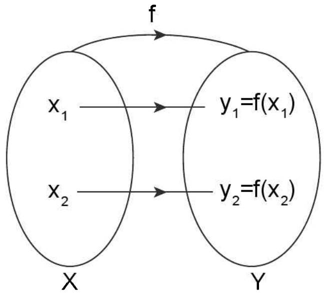


FIGURE 2.21

ILLUSTRATION 16: Test whether following relations $f: \mathbb{R} \to \mathbb{R}$ are function or not

(a) $y = x^3 + 8$ (b) $y^2 = 4x$.

SOLUTION: (a) Let $x_1 = x_2$

$\Rightarrow x_1^3 = x_2^3$ $\Rightarrow x_1^3 + 8 = x_2^3 + 8$ $\Rightarrow f(x_1) = f(x_2)$

So, $f(x) = x^3 + 8$ is a function.

(b) Let $x_1 = x_2$ $\Rightarrow 4x_1 = 4x_2$ $\Rightarrow y_1^2 = y_2^2$

$$\Rightarrow \begin{cases} y_1 = y_2 \\ \text{or} \\ y_1 = -y_2 \end{cases}$$

So, $y^2 = 4x$ is not a function but one–many relation. If the function had been $y = \sqrt{4x}$; $x \geq 0$, then it would be a function.

ILLUSTRATION 17: Which of the following correspondences can be called a function?

(i) $f_1 : \{-1, 0, 1\} \to \{0, 1, 2, 3\}; f_1(x) = x^5$.

(ii) $f_2 : \{0, 4, 9\} \to \{-3, -2, 0, 2, 3\}; f_2(x) = \pm\sqrt{x}$.

(iii) $f_3 : \{0, 1, 9\} \to \{-3, -1, 0, 1, 3\}; f_3(x) = \sqrt{x}$.

(iv) $f_4 : \{0, 4, 9\} \to \{-3, -2, 0, 2, 3\}; f_4(x) = -\sqrt{x}$.

SOLUTION: $f_3(x)$ and $f_4(x)$ are functions as definition of function is satisfied by $f_3(x)$ and $f_4(x)$.

The given correspondence $f_1(x)$ is not a function, as $f_1(-1) = -1 \notin$ co-domain.

Hence, definition of function is not satisfied.

The given relation $f_2(x)$ is not a function, as $f_2(1) = \pm 1$ and $f_2(4) = \pm 2$, $f_2(9) = \pm 3$

i.e., every element of domain is related with two elements of co-domain.

Hence, the definition of function is not satisfied.

Method 2: *When the relation to be tested is represented as a set of ordered pairs:* A relation $f : X \to Y$ represented as a set of ordered pairs will be function from X to Y iff

- Set of abscissa of all ordered pairs is equal to X
- No two ordered pairs should have same abscissa.

REMARK

Because f is a relation from X → Y, therefore abscissa of ordered pairs must belong to X where as ordinates of ordered pairs must belong to Y.

ILLUSTRATION 18: Test whether relation f_i : $\{1,2,3,4\} \to \{2,3,4,5,6\}$ as defined below is a function or not?

(a) $f_1 = \{(1, 2), (2, 4), (4, 3), (3, 0)\}$ (b) $f_2 = \{(1, 3), (2, 4), (3, 5)\}$

(c) $f_3 = \{(1, 3), (2, 5), (3, 4), (1, 6)\}$ (d) $f_4 = \{(1, 6), (2, 5), (3, 4), (4, 2)\}$

SOLUTION: (a) f_1 is not a relation from first set to second set because 0 is not in co-domain, and hence, not a function.

(b) f_2 is not a function from first set to second set as 4 has no image under f_2, however, it is a function from set $\{1, 2, 3\}$ to set $\{2, 3, 4, 5, 6\}$.

(c) f_3 is a not a function from first set to second set as 1 has two images 3 and 6.

(d) f_4 is a function from first set to second set as each element of first set has a unique image in second set.

ILLUSTRATION 19: If $A = \{a, b, c, d, e\}$ and $B = \{p, q, r, s, t\}$, then which of the following subset(s) of $A \times B$ is/are function(s) from A to B.

(i) $f_1 = \{(a, r), (b, r), (b, s), (d, t), (e, q), (c, q)\}$

(ii) $f_2 = \{(a, r), (b, p), (c, t), (d, q)\}$

(iii) $f_3 = \{(a, p), (b, t), (c, r), (d, s), (e, q)\}$

(iv) $f_4 = \{(a, r), (b, r), (c, r), (d, r), (e, r)\}$

SOLUTION: We check for the two conditions of the functions

(i) Since b has two outputs (images) namely r and s, f_1 is not a function

(ii) Since e, an element of A, does not have any image, it is also not a function

(iii) Since every element of A has only one output, it is a function.

(iv) Even if every element's output is r, it is a function, as every element of A is participating and has a unique image r.

ILLUSTRATION 20: Decide whether or not the following are functions from A to B; where $A = \{1, 2, 3, 4, 5\}$ and $B = \{a, b, c, d, e\}$. If they are functions give the range of each. If they are not, explain why?

(i) $f_1 = \{(1, a), (2, b), (3, b), (5, e)\}$

(ii) $f_2 = \{(1, e), (5, d), (3, a), (2, b), (1, d), (4, a)\}$

(iii) $f_3 = \{(5, a), (1, e), (4, b), (3, c), (2, d)\}$

SOLUTION: (i) Since the element $4 \in A$ is not associated to any element of B, therefore, f_1 is not a function from A to B.

(ii) The element $1 \in A$ is associated to two different elements e and d of B. Therefore, f_2 is not a function from A to B.

(iii) Each element of A is associated to a unique element of B. Therefore f_3 is a function from A to B.

The range of f_3 is the set of f_3 images of all elements of A. So range of f_3 is $f_3(A) = \{a, b, c, d, e\} = B$.

ILLUSTRATION 21: Let $A = \{1, 2, 4\}$ and $B = \{1, 3\}$. Let $R = \{(x, y) : x + y$ is an even number, $x \in A, y \in B\}$. Does R represent a function?

SOLUTION: We find all possible ordered pairs such that the sum of abscissa and ordinate is even $R = \{(1, 1), (1, 3)\}$

From the mapping it is clear that R is a relation but not a function.

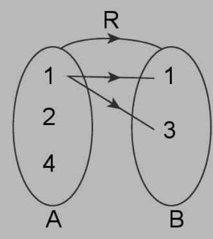


FIGURE 2.22

ILLUSTRATION 22: If $A = \{2, 3, 4, 5\}$, then which of the following relations is a function from A to itself?

(a) $f_1 = \{(x, y) : y = x + 1\}$ (b $f_2 = \{(x, y) : x + y > 6\}$

(c) $f_3 = \{(x, y) : y < x\}$ (d) $f_4 = \{(x, y) : x + y = 7\}$

SOLUTION: Here, $f_1 = \{(2, 3), (3, 4), (4, 5)\}$

$f_2 = \{(2, 5), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 2), (5, 3), (5, 4), (5, 5)\}$

$f_3 = \{(3, 2), (4, 2), (4, 3), (5, 2), (5, 3), (5, 4)\}$

and $f_4 = \{(2, 5), (3, 4), (4, 3), (5, 2)\}$

Here, only f_4 is a function from A to itself.

Note that f_4 is a bijection.

$\left(\because D_{f_1} \neq A; f_2(3), f_2(4), f_2(5) \text{ are unique}; D_{f_3} \neq A\right)$.

Method 3: *When the relation to be tested is represented graphically*: relation $f : X \to Y$; $y = f(x)$ is function iff all the straight line $x = \alpha$; $\forall\ \alpha \in X$ intersect the graph of function exactly once as shown below.

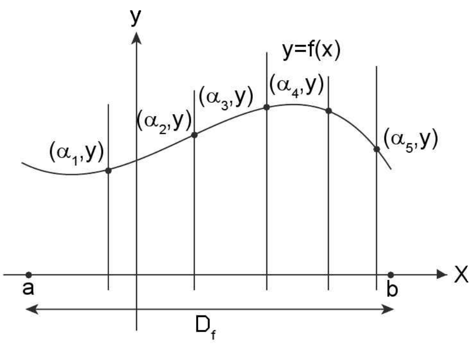


FIGURE 2.23

A relation $f : X \to Y$ will not be a function in the following two conditions.

1. If for some $\alpha \in X$, line $x = \alpha$ does not cut the curve $y = f(x)$. e.g., in the graph of function shown below the line $x = \alpha$ does not cut the graph of function and $\alpha \in X$ $(D_f) = [a, b]$, i.e., no output for input $x = \alpha$.

$\Rightarrow$ $f(x)$ is not a function from X to Y.

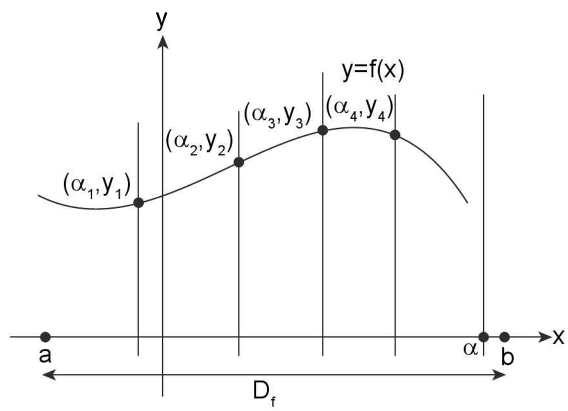


FIGURE 2.24

2. If for atleast one $\alpha \in X$, *l*ine $x = \alpha$ intersects $y = f(x)$ more than once, i.e., there exists an input having more than one output say at (α, y_1) , (α, y_2) and (α, y_3).

$\Rightarrow$ For input $x = \alpha$, $f(x)$ has three outputs y_1 , y_2 as well as y_3 as shown below

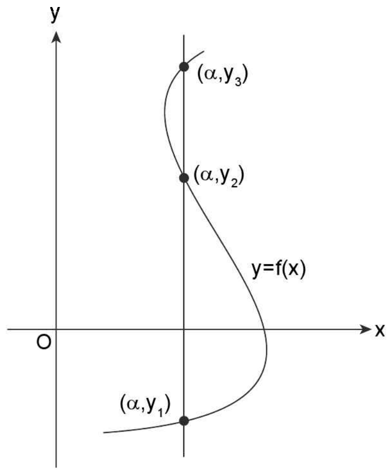


FIGURE 2.25

Hence, $f(x)$ is not function.

It is one–many relation and one–many relations are never functions.

Method 4: When the relation to be tested is represented diagrammatically: A relation $f : X \to Y$ is a function if no input has two or more outputs in Y and no $x \in X$ is un-related. e.g., define a relation from set X having fathers x_1, x_2 and x_3 to set Y having sons y_1 and y_3 and a daughter y_2. Let it be represented diagrammatically as shown.

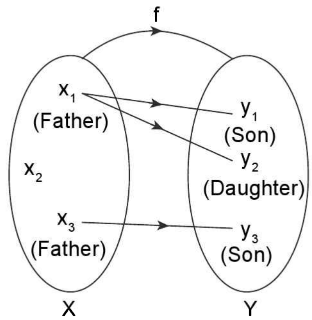


FIGURE 2.26

Because father x_1 has son y_1 and daughter y_2 i.e., input x_1 has two images i.e., f is one–many relation, and hence, not a function from set X to set Y. Moreover, x_2 has no child i.e., x_2 has no image under f, and hence, also from this point of view, f is not a function.

ILLUSTRATION 23: Check whether following sets of ordered pairs are functions or not?

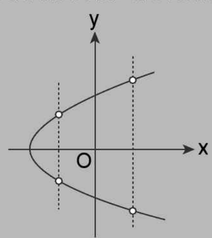
(a) $\{(x, y)$: x is a person and y is an ancestor of $x\}$

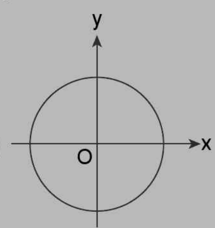
(b) $\{(x, y)$: x is a person and y is the father of $x\}$

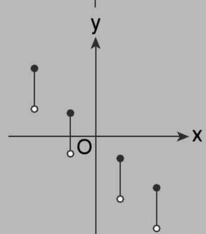
SOLUTION: (a) Because each person on this earth has more than one ancestor, therefore the given relation is one–many and hence, is not a function.

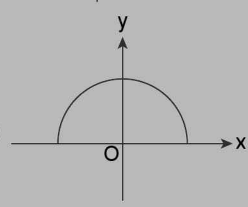
(b) The set of ordered pairs is $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots\}$. Because each person has only one father, therefore the relation would not contain any two ordered pairs having same first element, therefore the given relation is not one–many but it can be many–one as two different persons may have same father and hence, the given relation is a function.

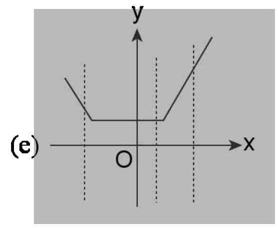
ILLUSTRATION 24: Check whether the following relations whose graphs are shown below are functions or not?

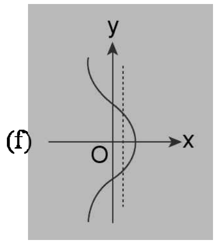
(a)


(b)


(c)


(d)






SOLUTION: For (d) and (e)

In the given domain each line drawn parallel to y-axis intersects the graph only at a single point and hence, the corresponding relations are functions in their domains.

For (a), (b), (c) and (f)

There exist lines parallel to y-axis intersecting the corresponding graphs in more than one point, and hence, the corresponding relations are not functions.

ILLUSTRATION 25: Let $A = \{x : 1 \leq x \leq 5\}$ and $B = \{y : 1 \leq y \leq 4\}$. Define $\phi = \{(x, y) : y \geq x + 1, x \in A, y \in B\}$. Plot ϕ on the $x - y$ plane and determine whether it represent a function?

SOLUTION: Here $A = \{x : 1 \leq x \leq 5\}$ and $B = \{y : 1 \leq y \leq 4\}$; $\phi = \{(x, y) : y \geq x + 1, x \in A$ and $y \in B\}$

Clearly $\phi \subseteq A \times B = [1, 5] \times [1, 4]$, i.e., the graph is entirely contained in the rectangular region R bounded by straight lines $x = 1$, $x = 5$ and $y = 1$, $y = 4$

To obtain ϕ, first of all draw the straight line $y = x + 1$ with in the above defined bounded region R shown below

Now the region above and on the line segment EF of straight line $y = x + 1$ and within the rectangle represents the region ϕ defined in the given problem. This region is shown in above diagram by shaded portion.

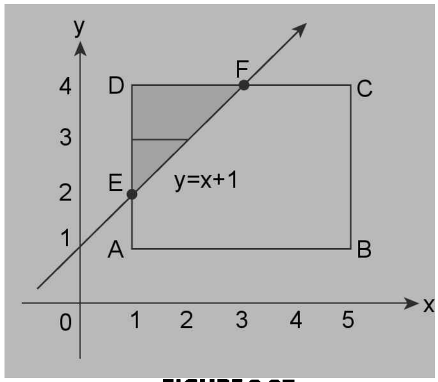
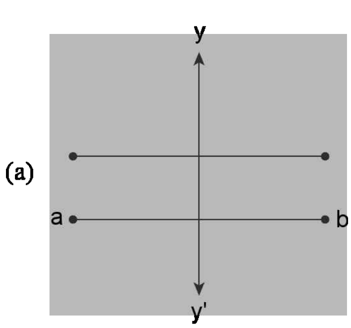
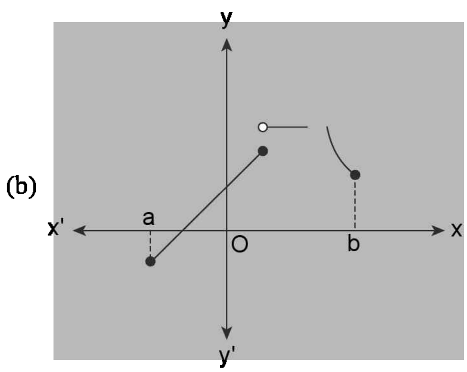


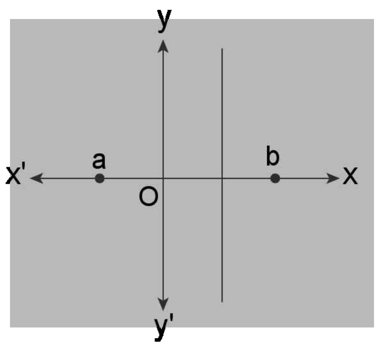
FIGURE 2.27

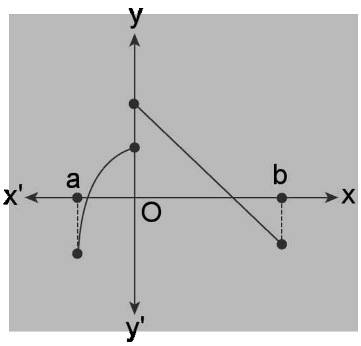
Clearly the above region is one–many relation.

ILLUSTRATION 26: Which of the following is not the graph of a function on set $A = [a, b]$?





(c) 

(d) 

SOLUTION: No vertical line in the domain can meet the graph of a function in more than one point. In (c), the curve shown is a vertical line and in (d), y-axis is meeting the graph in two distinct points.

In graph (b) there are some points in $[a, b]$ having no image under f, however except for those points graph represents the function on the remaining interval of $[a, b]$. In graph (a) each element of domain $[a, b]$ has a unique image. Thus, the relation shown in Figure (a) is a function.

ILLUSTRATION 27: Check whether the following functions expressed in analytical form are functions or not?

(i) $f_1 : [2, 7] \to \mathbb{R}$ defined by $f_1(x) = \dfrac{(x-1)}{(x-3)}$

(ii) $f_2 : [-5, 5] \to \mathbb{R}$ defined by $f_2(x) = \sqrt{x-2}$

(iii) $f_3 : [0, \infty) \to \mathbb{R}$ defined by $[f_3(x)]^2 = x$

(iv) $f_4 : [0, \infty) \to \mathbb{R}$ defined by $f_4(x) = \sqrt{x}$.

(v) $f_5 : \mathbb{R} \to \mathbb{R}$ defined by $f_5(x) = |x - 4|$.

(vi) $f_6 : \mathbb{R} \to \mathbb{R}$ defined by $f_6(x) = \sin^2 x$.

SOLUTION: (i) $\because$ $f_1(3)$ is not defined

$\therefore$ f_1 is not a function from $[2, 7]$ to $\mathbb{R}$; however, f_1 is a function from $[2, 7] \sim \{3\}$ to $\mathbb{R}$

(ii) $f_2(x)$ is real for $x - 2 \geq 0$

$\Rightarrow$ $x \geq 2$

$\therefore$ $f_2(x)$ is not defined from $[-5, 5]$ to $\mathbb{R}$; however, $f_2(x)$ is a function from $[2, \infty)$ to $\mathbb{R}$

(iii) $[f_3(x)]^2 = x$

$\Rightarrow$ $f_3(x) = \pm\sqrt{x}$

$\therefore$ for each $x \in [0, \infty)$, $f_3(x)$ is real but for each x, $f_3(x)$ has two images, therefore $f_3(x)$ is not a function.

(iv) $f_4(x) = \sqrt{x} \in \mathbb{R}\ \ \forall x \in [0,\infty)$ and $f_4(x)$ is unique for each $x \in [0, \infty)$

$\therefore$ $f_4(x)$ is a function from $[0, \infty) \to \mathbb{R}$.

(v) $f_5 : \mathbb{R} \to \mathbb{R}$ defined by $f_5(x) = |x - 4|$ is a function as $f_5(x)$ is real and unique for each $x \in \mathbb{R}$

(vi) $f_6(x) = \sin^2 x \in \mathbb{R}$ and is unique for each $x \in \mathbb{R}$

$\Rightarrow$ $f_6(x)$ is a function from $\mathbb{R}$ to $\mathbb{R}$

TEXTUAL EXERCISE–2: (SUBJECTIVE)

1. Represent all possible functions defined from $\{\alpha, \beta\}$ to $\{1, 2\}$ in the form of ordered pairs.

2. Find the explicit form of the following functions (or relations):

(a) $\log_5 x + \log_5 (y + 2) = 8$

(b) $3^{x+y}(x^2 - 3) = x^3 - 27$

(c) $y^2 = 8x - 4y$. Also find their domain.

3. Consider the functions defined by the following equations. In each case obtain y as an explicit function of x, and state for what values of x it is defined.

(a) $y^2 - 2y - x^2 = 0, y \geq 1$

(b) $y^2 - 2y + x^2 = 0, y \leq 1$

4. Give three examples of algebraic functions which cannot be expressed in explicit form.

5. Test which of the following relations are functions (and also identify whether they are injective or not.) on their respective domain?

(a) $y^{2/3} = 3x + 4$ (b) $y = 2x^2 + 1$

(c) $y^2 = 2\tan x + 5$ (d) $y = \sqrt{3x+2}$

(e) $y^3 = 2x^2 - 1$ (f) $y = \dfrac{x-1}{x+1}$

(g) $y^2 = 4x - 3$ (h) $y = ax^2 + bx + c$

(i) $y^3 = 3x + 1$

6. Which of the following represent functions?

(i) $y = \sqrt{x}$

(ii) $x^2 + 2y^2 = 8$

(iii) $x^2 + y^2 = 4, y \geq 0$

(iv) $x^2 + y^2 = 4, x \geq 0$

(v) $y = \begin{cases} 2x+5; & x \geq 0 \\ \dfrac{-x^2}{4}; & x < 0 \end{cases}$

(vi) $y^3 = x$

(vii) $y^4 = 2x$

7. (a) The relation $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 2x+1\ ; & x \leq 4 \\ x+4\ ; & x > 4 \end{cases}$

(b) The relation h is defined by $h(x) = \begin{cases} x^2; & 0 \leq x \leq 3 \\ 3x; & 3 \leq x \leq 10 \end{cases}$

(c) The relation g is defined by $g(x) = \begin{cases} x^2; & 0 \leq x \leq 2 \\ 3x; & 2 \leq x \leq 10 \end{cases}$

Show that f and h are functions and g is not a function.

8. Give an analytical formula for the function $f(x)$ equal to length of the path from A to B and from B to C for $0 \leq x \leq 4$ as represented by the following Figure 2.28.

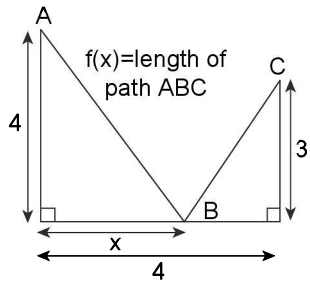


FIGURE 2.28

9. Let $A = \{x : 0 \leq x \leq 4\}$ and $B = \{y : 1 \leq y \leq 3\}$. Define $\phi = \{(x, y) : y \geq x, x \in A, y \in B\}$.
Plot ϕ on the $x - y$ plane and determine whether it represents a function?

10. Draw the graph of the following functions:

(i) $f(x) = \begin{cases} x^2+4 & \text{if } |x| > 0 \\ 3 & \text{if } x = 0 \end{cases}$

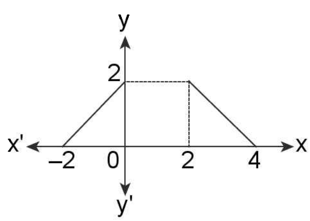
(ii) $g(x) = \begin{cases} x^2+2 & \text{if } x \geq 0 \\ -x-2 & \text{if } x < 0 \end{cases}$

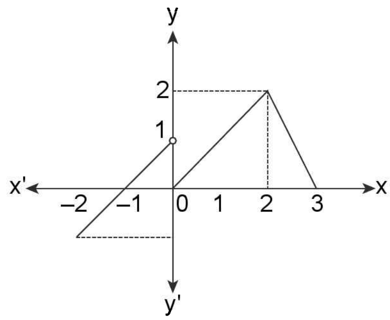
11. Draw the graph of the following functions:

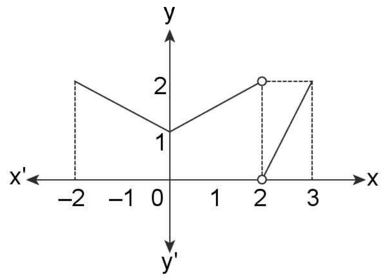
(i) $f(x) = \begin{cases} 2x+2 & \text{if } x < -1 \\ x^2-1 & \text{if } -1 \leq x \leq 1 \\ 2x-3 & \text{if } x > 1 \end{cases}$

(ii) $f(x) = \begin{cases} x^2-1 & \text{if } x \leq -1 \\ 4x^3 & \text{if } -1 \leq x \leq 1 \\ 5-x^2 & \text{if } 1 \leq x \leq 2 \end{cases}$

12. Define the following functions analytically:

(a) 

(b) 

(c) 

13. For which of the following functions is $f(a + b)$ equal to $f(a) + f(b)$ for all positive numbers a and b?

(a) $f(x) = x^2$ (b) $f(x) = 5x$
(c) $f(x) = -6x$ (d) $f(x) = \sqrt{x}$
(e) $f(x) = 3x + 1$ (f) $f(x) = \log x$

14. For which of the following functions is $f(ab)$ equal to $f(a)\,f(b)$ for all positive numbers a and b?

(a) $f(x) = 8x$ (b) $f(x) = x^5$
(c) $f(x) = 1/x^3$ (d) $f(x) = \sqrt{x}$
(e) $f(x) = x + 2$

15. Let f have as its domain the set of all integers. Assume that $f(x + y) = f(x) + f(y)$ for all integers x and y, and $f(1)= 3$.

(a) Show that $f(2) = 6$
(b) Show that $f(0) = 0$
(c) What can you say about $f(-1)$?
(d) Find a possible formula for f

16. Let f be the subset of $\mathbb{Z} \times \mathbb{Z}$ defined by $f = \{(ab, a + b) : a, b \in \mathbb{Z}]$. Is f a function from $\mathbb{Z}$ to $\mathbb{Z}$? Justify your answer.

Answer Keys

1. $f_1 = \{(\alpha, 1), (\beta, 1)\}$; $f_2 = \{(\alpha, 2), (\beta, 2)\}$; $f_3 = \{(\alpha, 1), (\beta, 2)\}$; $f_4 = \{(\alpha, 2), (\beta, 1)\}$

2. (a) $y = \dfrac{(5)^8}{x} - 2$; Domain = $(0, \infty)$ (b) $y = \log_3\left(\dfrac{x^3 - 27}{x^2 - 3}\right) - x$; Domain = $(-\sqrt{3}, \sqrt{3}) \cup (3, \infty)$

(c) $y = -2 \pm \sqrt{8x + 4}$; Domain = $\left[-\dfrac{1}{2}, \infty\right)$. It represents a relation that is a combination of two functions.

3. (a) $y = 1 + \sqrt{x^2 + 1}$; $x \in \mathbb{R}$ (b) $y = 1 - \sqrt{1 - x^2}$; $x \in [-1, 1]$

4. $y^5 - y - x = 0$; $y^3 + y + 2x = 0$; $y^2 + y \sin x + 2x = 0$.

5. b, *d, e, f, h, i* (functions) ; *d* and *f, i* are injective.

6. (i), (iii), (v), (vi) **8.** $f(x) = \sqrt{x^2 - 8x + 25} + \sqrt{x^2 + 16}$

9. The shaded region represents ϕ. Clearly ϕ is not a function, in fact ϕ is a many–many relation.

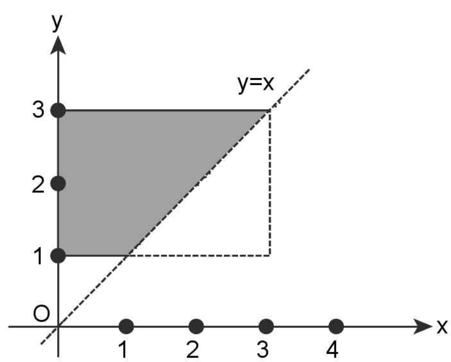
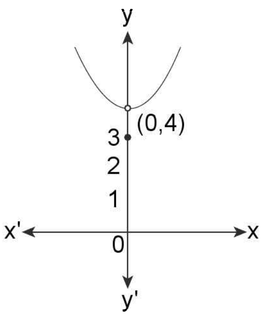
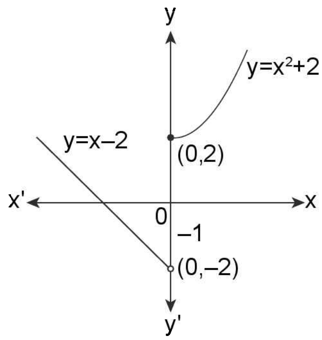
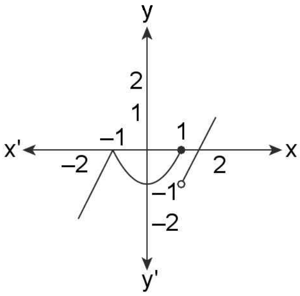
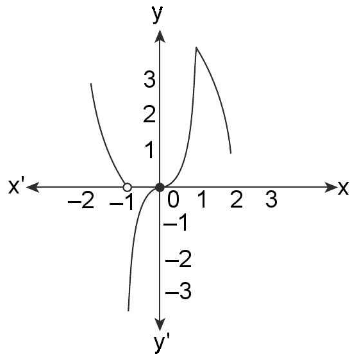


FIGURE 2.29

10. (i) 

(ii) 

11. (i) 

(ii) 

12. (a) $\begin{cases} x+2; & -2 \le x < 0 \\ 2; & 0 \le x \le 2 \\ 4-x; & 2 < x \le 4 \end{cases}$ (b) $\begin{cases} x+1; & -2 \le x < 0 \\ x; & 0 \le x \le 2 \\ 6-2x; & 2 < x \le 3 \end{cases}$ (c) $\begin{cases} 1-\dfrac{x}{2}; & -2 \le x < 0 \\ 1+\dfrac{x}{2}; & 0 \le x < 2 \\ 2x-4; & 2 < x \le 3 \end{cases}$

13. (b) and (c) **14.** (b), (c) and (d) **16.** No

TEXTUAL EXERCISE–2: (OBJECTIVE)

1. Which of the following relations are function from set X to Y; where $X = \{1, 3, 5, 7\}$ and set $Y = \{2, 4, 6, 8\}$?

(a) $\{(3, 2), (3, 4), (5, 4), (7, 4), (1, 8)\}$
(b) $\{(1, 2), (5, 8), (3, 6)\}$
(c) $\{(1,4), (3, 8), (5, 2), (7, 6)\}$
(d) $\{(1,4), (3,4), (5, 8), (7, 8)\}$

2. Which of the following is/are function(s) from set $\mathbb{N}$ to set W (set of whole numbers)?

(a) $f = \{(x, y) : y = x + 2$
(b) $f = \{(x, y) : y = x - 4\}$
(c) $f = \{(x, y) : y^2 = x + 2; x, y \in \mathbb{N}\}$
(d) $f = \{(x, y) : |y| = x - 1; y \in \mathbb{R}$

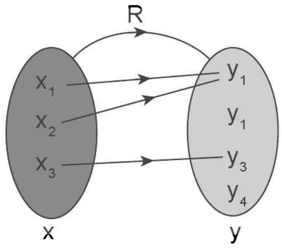
3. Which of the following correspondences can be called a function?

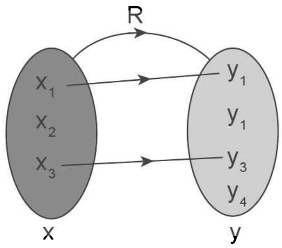
(a) $f : \{-2, 0, 2\} \to \{0, 1, 8, 3\}; f(x) = x^3$
(b) $f : \{0, 1, 4\} \to \{-2, -1, 0, 1, 2\}; f(x) = \pm\sqrt{x}$
(c) $f : \{0, 1, 9\} \to \{-3, -1, 0, 1, 3\}; f(x) = \sqrt{x}$
(d) $f : \{0, 1, 9\} \to \{-3, -1, 0, 1, 3\}; f(x) = -\sqrt{x}$

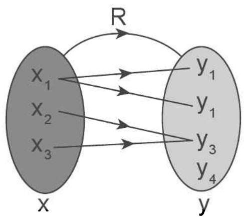
4. Which of the following are functions?

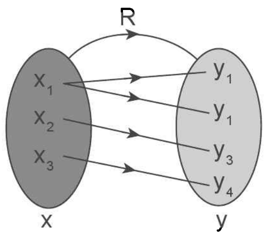
(a) $f = \{(x, y) : y = x + 3, x \in \mathbb{N}, y \in W\}$
(b) $f = \{(x, y) : y = x - 1, x \in \mathbb{N}, y \in W\}$
(c) $f = \{(x, y) : y^2 = x + 1, x \in \mathbb{N}, y \in W\}$
(d) None of these

5. Identify which of the following relations is/are function(s) from $X \to Y$?

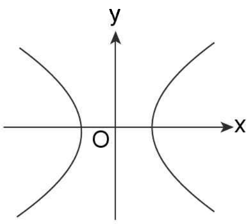
(a) 

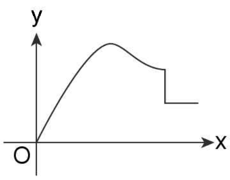
(b) 

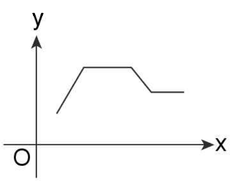
(c) 

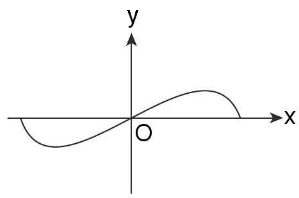
(d) 

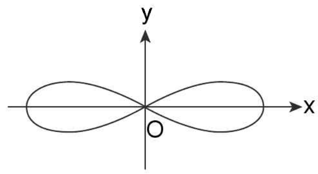
6. Which of the following graphs is/are function?

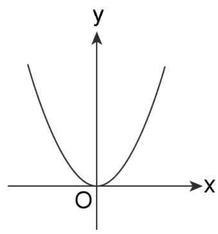
(a) 

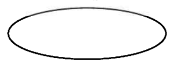
(b) 

(c) 

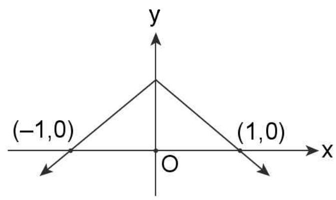
(d) 

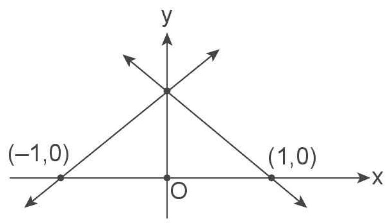
(e) 

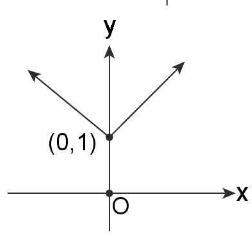
(f) 

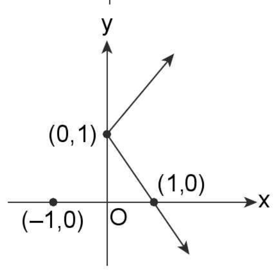
(g) 

7. Which one of the following graphs represents the function $y = 1 + |x|$ for all $x \in \mathbb{R}$.

(a) 

(b) 

(c) 

(d) 

8. The graph of a relation $f(x)$ is as shown below

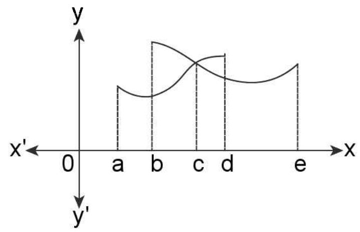


FIGURE 2.30

The domain in which the given relation represents a function is

(a) $[a, c] \cup [d, e]$ (b) $[a, b] \cup [d, e]$
(c) $[a, b) \cup (d, e]$ (d) $(a, b) \cup (d, e)$

9. The graph of a relation $f(x)$ is as shown below

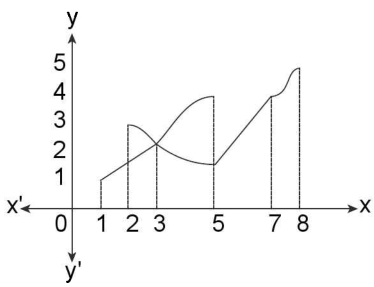


FIGURE 2.31

The domain corresponding to which the given relation represents a function with co-domain $[0.5, 3] \cup [4, 6]$ is

(a) $[1, 2) \cup (5, 8]$ (b) $[1, 2) \cup [7, 8]$
(c) $[1, 2) \cup [5, 8]$ (d) None of these

10. Which of the following equations represent a function?

(a) $|x| + |y| = 2$ (b) $|x + y| = 4$
(c) $|y| = x^2 + \sin x$ (d) $y = |x|^2 - x$

11. A is a point on the circumference of a circle. AB and AC divide the area of the circle into three equal parts. Let $\angle BAC = \alpha$, then which is/are correct?

(a) α satisfies the function $f(x) = c.\ (x + \sin x - \pi/3) = 0$

(b) α satisfies the function $f(x) = \left(x + \sin x - \dfrac{\pi}{3}\right)^n$ $n \in \mathbb{N}$

(c) $\alpha + \sin\alpha = \dfrac{\pi}{3}$

(d) None of these

12. Consider the function $f : \mathbb{R} - \{0\} \to \mathbb{R}$, defined by $f(x) = \dfrac{1}{x}$. Which of the following statements are always true?

(a) $f\left(\dfrac{a^n}{b^n}\right) = \dfrac{f(a^n)}{f(b^n)}$; $n \in \mathbb{N}$

(b) $f(a^n + b^n) = f(a^n) + f(b^n)$; $n \in \mathbb{N}$

(c) $f(a^n) = (f(a))^n$; $n \in \mathbb{N}$

(d) $f(a^n).f\left(\dfrac{1}{a^n}\right) = 1$; $n \in \mathbb{N}$

Answer Keys

1. (c,d) **2.** (a) **3.** (c, d) **4.** (a, b) **5.** (a) **6.** (c, d, f) **7.** (c) **8.** (c,d) **9.** (b)
10. (d) **11.** (a,b,c)
12. (i) True (ii) False (iii) True (d) True

■ MATHEMATICAL TOOLS USED TO FIND THE DOMAIN AND RANGE OF FUNCTIONS

Laws of Inequality

R(1): Adding same real number to both sides does not alter the nature of inequality.
i.e., $a > b \Rightarrow a + x > b + x\ \forall\ x \in \mathbb{R}$ and $a < b$
$\Rightarrow \quad a + x < b + x\ \forall\ x \in \mathbb{R}$

R(2): Transition property: $a > b$ and $b > c$ $\Rightarrow a > c$.

R(3): Multiplying by a real number k to both sides of inequality has the following effects on the nature of inequality $a > b \Rightarrow \begin{cases} ak > bk & \text{if } k > 0 \\ ak = bk & \text{if } k = 0, \\ ak < bk & \text{if } k < 0 \end{cases}$

ILLUSTRATION 28: If x is a variable described as below, then find the range of variation of $(2x + 3)$.

(i) $x \in [2, 6]$ (ii) $x \in (-3, -1)$.

SOLUTION: (i) $x \in [2, 6]$

$\Rightarrow 2 \le x \le 6$ $\Rightarrow 4 \le 2x \le 12$

$\Rightarrow 4 + 3 \le 2x + 3 \le 15$ $\Rightarrow$ Range of $2x + 3 = [7, 15]$.

(ii) $x \in (-3, -1)$ $\Rightarrow -3 < x < -1$

$\Rightarrow -6 < 2x < -2$ $\Rightarrow -6 + 3 < 2x + 3 < -2 + 3$

$\Rightarrow$ Range of $2x + 3 = (-3, 1)$.

ILLUSTRATION 29: If x is a variable described as below, then find the range of variation of $(4 - 3x)$.

(i) $x \in [3, 5]$ (ii) $x \in (-2, 3)$.

SOLUTION: (i) $x \in [3, 5]$

$\Rightarrow 3 \le x \le 5$ $\Rightarrow -9 \ge -3x \ge -15$

$\Rightarrow -9 + 4 \ge 4 - 3x \ge 4 - 15$ $\Rightarrow$ Range of $4 - 3x = [-11, -5]$.

(ii) $x \in (-2, 3)$ $\Rightarrow -2 < x < 3$

$\Rightarrow 6 > -3x > -9$ $\Rightarrow 6 + 4 > 4 - 3x > 4 - 9$

$\therefore$ Range of $(4 - 3x) = (-5, 10)$.

R(4): Law of addition: If $a_1 > b_1$; $a_2 > b_2$; … ; $a_n > b_n$; $n \in \mathbb{N}$

$\Rightarrow (a_1 + a_2 + a_3 + \ldots + a_n) > (b_1 + b_2 + b_3 + \ldots + b_n)$

R(5): Law of multiplication: If $a_1 > b_1 > 0$ ….; $a_2 > b_2 > 0$ …… ; $a_n > b_n > 0$; $n \in \mathbb{N}$

$\Rightarrow (a_1 . a_2 . a_3 . \ldots . a_n) > (b_1 . b_2 . b_3 \ldots b_n)$

ILLUSTRATION 30: If $x - 3 > 2 + 6x$; $y + 6 > 5 - 12x$. Then prove that $2x + y > 9$.

SOLUTION: If $x - 3 > 2 + 6x$; ... (i)

$y + 6 > 5 - 12x$ ... (ii)

from (i) $2x - 6 > 4 + 12x$ .... (iii)

(ii) + (iii) gives $2x + y > 9$, hence, proved.

ILLUSTRATION 31: If given that $x_k \ge k \ \forall \ k \in \mathbb{N}$; then show that

(i) $x_1 + x_2 + x_3 + \ldots + x_n \ge \dfrac{n(n+1)}{2}$

(ii) $x_1 . x_2 . x_3 . \ldots x_n \ge n!$

(iii) $x_1 + 2x_2 + 3x_3 + \ldots + nx_n \ge \dfrac{n(n+1)(2n+1)}{6}$

SOLUTION: (i) $x_1 \ge 1$; $x_2 \ge 2$; $x_3 \ge 3$….. ; $x_n \ge n$, adding above inequalities, we get

$$x_1 + x_2 + x_3 + \ldots + x_n \ge 1 + 2 + 3 + \ldots + n = n\left(\frac{n+1}{2}\right)$$

$$\therefore \quad x_1 + x_2 + x_3 + \ldots + x_n \ge \frac{n(n+1)}{2}$$

(ii) $x_1 \ge 1$; $x_2 \ge 2$; $x_3 \ge 3$ ..... $x_n \ge n$; multiplying the above inequalities, we get

$$x_1 . x_2 . x_3 . \ldots . x_n \ge 1.2.3 \ldots n = n! \qquad \Rightarrow x_1 . x_2 . x_3 \ldots . x_n \ge n!$$

(iii) $x_1 \ge 1$; $x_2 \ge 2$; $x_3 \ge 3$ ..... $x_n \ge n$;

$\Rightarrow x_1 \ge 1$; $2x_2 \ge 2^2$; $3x_3 \ge 3^2$; ..... $nx_n \ge n^2$. Adding above inequalities, we get

$$\Rightarrow x_1 + 2x_2 + 3x_3 + \ldots + nx_n \geq 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\Rightarrow x_1 + 2x_2 + 3x_3 + \ldots + nx_n \geq \frac{n(n+1)(2n+1)}{6}$$

ILLUSTRATION 32: Given two inequalities $x + 1 > 3 > 5 - 2x$ and $x + 1 > 5 - 2x$. Solve them separately for x and what conclusion you can draw regarding transition property of inequality?

SOLUTION: If $x + 1 > 3$ and $3 > 5 - 2x$; $x + 1 > 5 - 2x$ $\Rightarrow$ $x > 2$ and $2x > 2$; $3x > 4$
$\Rightarrow$ $x > 2$ and $x > 1$; $x > 4/3$ $\Rightarrow$ $x \in (2, \infty)$; $x \in (4/3, \infty)$

Conclusion: if $a > b$ and $b > c$ $\Rightarrow$ $a > c$, but the converse is not true

That is, for any three real numbers a, b, c if $a > c$, then it is not necessary that $c < b < a$.

R(6): Law of square/square root:

(a) $a > b$

$$\Rightarrow \begin{cases} a^2 > b^2 \ ; & \text{if both } a \ \& \ b \text{ are non-negative} \\ a^2 = b^2 \ ; & \text{if } |a| = |b| \\ a^2 < b^2 \ ; & \text{if } a \ \& \ b \text{ are both negative} \end{cases}$$

(b) If a and b have opposite signs and $a > b$, i.e., $a > 0$ and $b < 0$, then $\begin{cases} a^2 > b^2 \text{ iff } |a| > |b| \\ a^2 = b^2 \text{ iff } |a| = |b| \\ a^2 < b^2 \text{ iff } |a| < |b| \end{cases}$

This law can be extended for any even natural power ($2n$).

(c) If a and b both are non-negative and

$$a > b \Rightarrow \begin{cases} \sqrt{a} > \sqrt{b} \\ \sqrt[2n]{a} > \sqrt[2n]{b} \end{cases}$$

ILLUSTRATION 33: If x is a variable described as below, then find the range of variation of x^2.
(i) $x \in [2, 6]$ (ii) $x \in (-3, -1)$.

SOLUTION: (i) $x^2 \in [4, 36]$ (ii) $x^2 \in (1, 9)$

Conclusion

If x is a variable such that $a \leq x \leq b$, then to find range of variation of x^2:

Case I: If a and b are non-negative

$\Rightarrow a^2 \leq x^2 \leq b^2$ $\Rightarrow x^2 \in [a^2, b^2]$

Case II: If a and b are non-positive

$\Rightarrow a \leq x \leq b$

$\Rightarrow b^2 \leq x^2 \leq a^2$ $\Rightarrow x^2 \in [b^2, a^2]$

Case III: If $a < 0$ and $b > 0$

$\because a \leq x \leq b$

$\Rightarrow a \leq x \leq 0$ or $0 \leq x \leq b$

$\Rightarrow a^2 \geq x^2 \geq 0$ or $0 \leq x^2 \leq b^2$

$\Rightarrow x^2 \in [0, a^2] \cup [0, b^2]$

$$\Rightarrow \begin{cases} x^2 \in [0, a^2] & \text{if } |a| > |b| \\ x^2 \in [0, b^2] & \text{if } |b| > |a| \end{cases}$$

$\Rightarrow x^2 \in [0, \max\{a^2, b^2\}]$

ILLUSTRATION 34: If x is a variable described as below, then find the range of variation of x^2.
(i) $x \in [2, 6]$ (ii) $x \in (-3, -1)$
(iii) $x \in [-2, 6]$ (iv) $x \in (-4, 2)$

SOLTUION: (i) $x^2 \in [4, 36]$ (ii) $x^2 \in (1, 9)$
(iii) $x^2 \in [0, 36]$ (iv) $x^2 \in [0, 16)$

R(7): Law of cube/cube root: $a > b \Rightarrow a^3 > b^3$ and $a^{1/3} > b^{1/3}\ \forall\ a, b \in \mathbb{R}$

$a < b \Rightarrow a^3 < b^3$ and $a^{1/3} < b^{1/3}\ \forall\ a, b \in \mathbb{R}$

This law can be extended for any odd natural power $(2n + 1)$ and odd root.

That is, $a > b \Leftrightarrow a^{2n+1} > b^{2n+1}$ and $a^{1/2n+1} > b^{1/2n+1}\ \forall\ a, b \in \mathbb{R}$

ILLUSTRATION 35: If x is a variable described as below, then find the range of variation of x^3.

(i) $x \in [1, 3]$ (ii) $x \in (-3, 2)$

SOLUTION: (i) $x^3 \in [1, 27]$

(ii) $x^3 \in (-27, 8)$

R(8): Law of reciprocal: Given two real numbers a and b such that $a > b$

$$\Rightarrow \begin{cases} \dfrac{1}{a} < \dfrac{1}{b} \text{ iff } a \text{ and } b \text{ have same sign} \\ \dfrac{1}{a} > \dfrac{1}{b} \text{ iff } a \text{ and } b \text{ are of opposite signs, i.e., } a > 0, b < 0 \end{cases}$$

Conclusion

Case I: Given $x \in [a, b]$: a and b are of same sign

$$\Rightarrow \frac{1}{x} \in \left[\frac{1}{b}, \frac{1}{a}\right]$$

Proof: $\because\ a \le x \le b$

$$\Rightarrow \frac{1}{a} \ge \frac{1}{x} \ge \frac{1}{b} \quad \Rightarrow \quad \frac{1}{b} \le \frac{1}{x} \le \frac{1}{a}$$

ILLUSTRATION 36: If x is a variable described as below, then find the range of variation of $1/x$.

(i) $x \in [3, 10]$ (ii) $x \in (-1, -1/3)$.

SOLUTION: (i) $1/x \in [1/10, 1/3]$

(ii) $1/x \in (-3, -1)$

Case II: If $x \in [a, b]$; where $a < 0$ and $b > 0$ then, $\dfrac{1}{x} \in \left(-\infty, \dfrac{1}{a}\right] \cup \left[\dfrac{1}{b}, \infty\right)$

Proof: Breaking the interval at $x = 0$, we get $x \in [a, 0) \cup [0, b]$; $x \neq 0$

(otherwise $1/x$ is not defined)

$$\Rightarrow a \le x < 0 \text{ or } 0 < x \le b$$

$$\Rightarrow \frac{1}{a} \ge \frac{1}{x} > -\infty \text{ or } \infty > \frac{1}{x} \ge \frac{1}{b}$$

$$\Rightarrow -\infty < \frac{1}{x} \le \frac{1}{a} \text{ or } \frac{1}{b} \le \frac{1}{x} < \infty$$

$$\Rightarrow \frac{1}{x} \in \left(-\infty, \frac{1}{a}\right] \cup \left[\frac{1}{b}, \infty\right)$$

ILLUSTRATION 37: If x is a variable described as below, then find the range of variation of $1/x$.

(i) $x \in (-1/3, 2)$

(ii) $x \in [-2, 2]$.

SOLUTION: (i) $1/x \in (-\infty, -3) \cup (1/2, \infty)$

(ii) $\dfrac{1}{x} \in \left(-\infty, -\dfrac{1}{2}\right] \cup \left[\dfrac{1}{2}, \infty\right)$

POLYNOMIAL EXPRESSION

An algebraic expression of type $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n$ is called polynomial in variable x, provided powers of x are whole numbers.

Terms: Quantities separated by positive or negative signs.

Coefficient: $a_0, a_1, a_2, a_3, \ldots. a_n$, are called coefficients of polynomial.

Leading terms/leading Coefficient: The coefficient of highest power of variable x is called leading coefficient (because it governs the sign of $f(x)$ when $x \to \infty$). For example, a_nx^n is leading term and a_n is the leading coefficient.

If $f(x) = a_0 + a_1x + a_2x^n + a_3x^n + \ldots + a_nx^n$, then

$$f(x) = x^n\left(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \frac{a_2}{x^{n-2}} + \ldots.. + \frac{a_{n-1}}{x} + a_n\right)$$

$$\text{as } x \to \infty \approx \begin{cases} \to \infty\,; & a_n > 0 \\ \to -\infty; & a_n < 0 \end{cases}$$

∴ Sign of polynomial expression at $x \to \infty$ is same as that of leading coefficient.

SOLVING RATIONAL INEQUALITIES

While solving rational inequalities the following facts must always be kept in mind:

(i) $\underbrace{\dfrac{f(x)}{g(x)} > 0}_{f(x)\text{ and } g(x) \text{ have same signs}}$ $\Rightarrow f(x)\,.\,g(x) > 0;$

e.g., $\dfrac{2}{x-1} > 0 \Rightarrow 2(x-1) > 0 \Rightarrow x > 1$

(ii) $\underbrace{\dfrac{f(x)}{g(x)} < 0}_{f(x)\text{ and } g(x) \text{ have opposite signs}}$ $\Rightarrow f(x)\,.\,g(x) < 0;$

e.g., $\dfrac{x}{x-1} - 1 < 0 \Rightarrow \dfrac{x-(x-1)}{x-1} < 0$

$\Rightarrow \dfrac{1}{x-1} < 0 \Rightarrow x - 1 < 0 \Rightarrow x < 1$

(iii) $\underbrace{\dfrac{f(x)}{g(x)} \geq 0}_{f(x)\text{ and } g(x) \text{ are of same signs or } f(x)=0}$ $\Rightarrow \begin{cases} f(x).g(x) > 0 \\ \text{or} \\ f(x) = 0 \text{ and } g(x) \neq 0 \end{cases}$

e.g., $\dfrac{x-1}{x-2} \geq 0 \Rightarrow \dfrac{x-1}{x-2} > 0$ or $x - 1 = 0$

$\Rightarrow (x-1)(x-2) > 0$ or $x = 1$

$\Rightarrow x \in (-\infty, 1) \cup (2, \infty)$ or $x = 1$

$\Rightarrow x \in (-\infty, 1] \cup (2, \infty)$

(iv) $\underbrace{\dfrac{f(x)}{g(x)} \leq 0}_{f(x)\text{ and } g(x) \text{ are of opposite signs or } f(x)=0}$

$\Rightarrow \begin{cases} f(x).g(x) < 0 \\ or \\ f(x) = 0 \text{ and } g(x) \neq 0 \end{cases}$ e.g., $\dfrac{x-1}{x-2} \leq 0$

$\Rightarrow (x-1)(x-2) \leq 0; x \neq 2$ $\Rightarrow x \in [1, 2)$

WAVY CURVE METHOD

To find the set of solution for inequality $f(x) > 0$, $f(x) \geq 0$, $f(x) < 0$ or $f(x) \leq 0$; where $f(x)$ is a polynomial (if the given inequality is rational inequality, then convert it into polynomial inequality as discussed in previous article).

Step 1: Factorize the polynomial and find all the roots and arrange the factors in the increasing order of roots.

e.g., $f(x) = (x-\alpha)^3(x-\beta)^2(x-\gamma)(x-\delta)^5$; $\alpha < \beta < \gamma < \delta$.

Step 2: Locate the roots (with their multiplicity) on real number line.

Step 3: Keep the sign of expression in the right-most interval (same as that of leading coefficient).

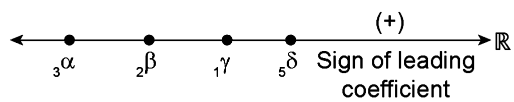


FIGURE 2.32

Step 4: Moving towards left and on changing the sign of expression across the root with multiplicity odd, and retain the same sign across the root with multiplicity even.

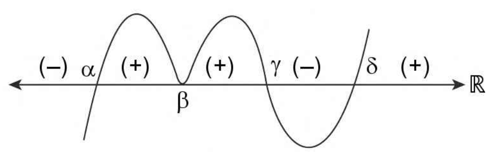


FIGURE 2.33

∵ $f(x) > 0$

$\Rightarrow x \in (\alpha, \beta) \cup (\beta, \gamma) \cup (\delta, \infty)$

Also for $f(x) \geq 0$

$\Rightarrow x \in [\alpha, \beta] \cup [\beta, \gamma] \cup [\delta, \infty)$

$\Rightarrow x \in [\alpha, \gamma] \cup [\delta, \infty)$

Similarly for $f(x) < 0$

$\Rightarrow x \in (-\infty, \alpha) \cup (\gamma, \delta)$ and $f(x) \leq 0$

$\Rightarrow x \in (-\infty, \alpha) \cup (\gamma, \delta) \cup (\alpha, \beta, \gamma, \delta)$

$\Rightarrow x \in (-\infty, \alpha] \cup [\gamma, \delta] \cup \{\beta\}$

ILLUSTRATION 38: Solve the following inequalities by using wavy curve method:

(i) $(x - 1)^2 (x + 2)^4 (x - 3)^3 (x + 5) > 0$

(ii) $(x - 1)^3 (x + 3)^4 (2x - 5)^3 (5 - x)^3 \geq 0$

(iii) $(2x - 9)^2 (x - 2)^2 (3 - x) \leq 0$

(iv) $(x - 5) (x - 1)^2 (2x - 5)^3 < 0$

SOLUTION: (i) Given inequality is $(x - 1)^2(x + 2)^4 (x - 3)^3 (x + 5) > 0$ ... (i)

Equating each factor to zero, we get $x = 1, - 2, 3, -5$. Arranging the above roots along real line, as shown in Figure 2.34.

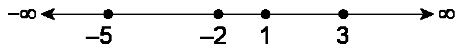


FIGURE 2.34

Sign of leading coefficient is positive and arranging the factors according to increasing order of their corresponding roots, the given inequality becomes

$(x + 5) (x + 2)^4 (x - 1)^2 (x - 3)^3 > 0$ ... (ii)

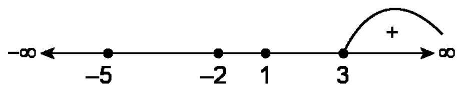


FIGURE 2.35

Now in the first interval (from right end) take the sign of leading term of inequality positive, that is, take the curve above the line and from now onwards go on changing the sign of wave for each odd powered factor of inequality observing from right of inequality (ii) and keeping same as that of previous for each even powered factor, we get the wave pattern as given in Figure 2.36.

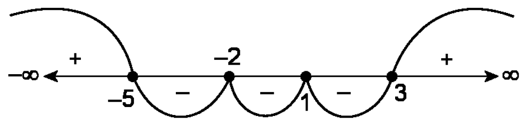


FIGURE 2.36

$\therefore$ $x \in (-\infty, -5) \cup (3, \infty)$.

Note: $-5, -2, 1$ and 3 are excluded, as at these points inequality vanishes.

(ii) On same steps as for part (i) the wave pattern for the inequality $(x - 1)^3 (x + 3)^4 (2x - 5)^3 (5 - x)^3 \geq 0$ will be as shown in Figure 2.37.

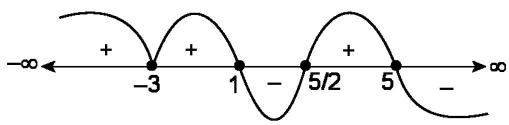


FIGURE 2.37

Ascending arrangement of factors of inequality is $(x + 3)^4 (x - 1)^3 (2x - 5)^3 (5 - x)^3 \geq 0$

$\therefore$ $x \in (-\infty, -3] \cup [-3, 1] \cup \left[\frac{5}{2}, 5\right]$ or $x \in (-\infty, 1] \cup \left[\frac{5}{2}, 5\right]$

(iii) Ascending arrangement of factors of given inequality is $(x-2)^2(3-x)(2x-9)^2 \le 0$ and the wave pattern is as shown in Figure 2.38.

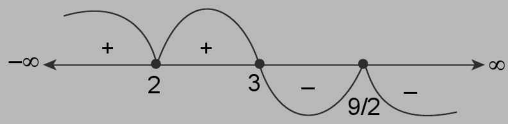


FIGURE 2.38

$$\therefore \quad x \in \left[3, \frac{9}{2}\right] \cup \left[\frac{9}{2}, \infty\right) \equiv [3, \infty)$$

(iv) Ascending arrangement of factors of given inequality is $(x-1)^2(2x-5)^3(x-5) < 0$ and the corresponding wave pattern is as shown in Figure 2.39.

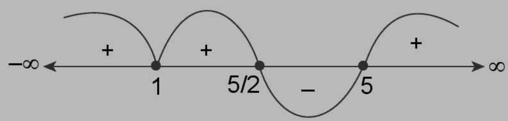


FIGURE 2.39

$$\therefore \quad x \in \left(\frac{5}{2}, 5\right)$$

TEXTUAL EXERCISE–3: (SUBJECTIVE)

1. Solve the following linear inequations and express the result in the form of intervals:

(a) $x^3 - 6x \ge 0$ (b) $x^6 - 9x^3 + 8 > 0$

(c) $x^4 - 2x^2 - 8 \ge 0$

2. Solve the following linear inequations and express the result in the form of intervals:

(a) $(x^2-9)^2(x+1)(x^2-2x-3)(x-1) \ge 0$

(b) $(x^2+3x+1)(x^2+3x-3) \ge 5$

(c) $(x-2)^2(x-3)(x-4)^4(x-6) \ge 0$

3. Find the range of variation of y for the following functions:

(a) $y = \dfrac{1}{3x-2}$ if $-2 < x < 4$

(b) $y = \dfrac{x-1}{x+2}$ if $-1 < x < 3$

(c) $y = \dfrac{1}{x^2+4}$ if $2 \le x \le 3$

(d) $y = \dfrac{x^2-1}{x^2+1}$ if $-1 < x < 3$

4. Solve the following inequalities for x:

(a) $\dfrac{(x+1)^5(x-1)(x+4)^2(x-3)^4}{(x+8)(2x-3)(x-6)^2} < 0$

(b) $\dfrac{(x+1)(x-8)(x-6)^2(x-5)^4}{(x-3)^2(x-1)} < 0$

(c) $(x^2-16)(x^2-5x+4)(x^2-1)(x+3)(x+2) \le 0.$

5. Solve the following inequality for the variable x:

(a) $x^3 - 6x \le 0$

(b) $x^4 - 2x^2 - 8 \ge 0$

(c) $x^6 - 9x^3 + 8 \le 0$

(d) $x^4 + x^3 - x - 1 > 0$

(e) $(x^2-9)^2(x+1)(x^2-2x-3)(x-1) \le 0$

(f) $x^2 - x - 6 \ge 0$ and $x^2 - 4x \le 0$

(g) $4 < x^2 < 9$

(h) $(x+1)(x-3)^2(x-5)(x-4)^2(x-2) < 0$

(i) $(x-3)(x+1)(x+2)(x-9) \ge 0$

(j) $x^3 - 7x^2 + 14x - 8 < 0$

(k) $\dfrac{(x-1)(x-2)(5-x)}{(2x+5)} < 0$

6. Find the largest negative integer and smallest positive integer which satisfy:

(a) $x^2 - 4x - 12 \ge 0$ (b) $3x^2 + 5x - 2 < 0$

(c) $\dfrac{x^2-5x+4}{x^2-1} \le 1$

7. Find the number of positive integers satisfying $\frac{x^2-3x+2}{x-3} \le 0$

8. Find the number of integers satisfying $\frac{x^2(x+2)}{x-1} \le 0$

Answer Keys

1. (a) $(-\sqrt{6},0]\cup[\sqrt{6},\infty)$ (b) $(-\infty,1)\cup(2,\infty)$ (c) $(-\infty,-2]\cup[2,\infty)$
2. (a) $(-\infty,1]\cup[3,\infty)$ (b) $(-\infty,-4]\cup[-2-1]\cup[1,\infty)$ (c) $(-\infty,3]\cup\{4\}\cup[6,\infty)$
3. (a) $\left(-\infty,-\frac{1}{8}\right)\cup\left(\frac{1}{10},\infty\right)$ (b) $\left(-2,\frac{2}{5}\right)$ (c) $\left[\frac{1}{13},\frac{1}{8}\right]$ (d) $\left[-1,\frac{4}{5}\right)$
4. (a) $(-8,-4)\cup(-4,-1)\cup\left(1,\frac{3}{2}\right)$ (b) $(-\infty,-1)\cup(1,3)\cup(3,5)\cup(5,6)\cup(6,8)$ (c) $[-4,-3]\cup[-2,-1]$
5. (a) $(-\infty,-\sqrt{6}]\cup[0,\sqrt{6}]$ (b) $[-2,2]$ (c) $[1,2]$
   (d) $(-\infty,-1)\cup(1,\infty)$ (e) $\{-3,-1\}\cup[1,3]$ (f) $[3,4]$
   (g) $(-3,-2)\cup(2,3)$
   (h) $(-\infty,-1)\cup(2,3)\cup(3,4)\cup(4,5)$ (i) $(-\infty,-2]\cup[-1,3]\cup[9,\infty)$ (j) $x\in(-\infty,1)\cup(2,4)$
   (k) $(-\infty,-5/2)\cup(1,2)\cup(5,\infty)$
6. (a) –2, 6 (b) –1, does not exist (c) does not exist, 2 **7.** 2 **8.** 3

TEXTUAL EXERCISE–3: (OBJECTIVE)

1. Solution of the linear inequations $\frac{x}{5} < \frac{3x-2}{4} - \frac{5x-3}{5}$ expressed in the form of interval is given by
   (a) $(-\infty,2/9)$ (b) $(-\infty,2/3)$
   (c) $(-\infty,1/9)$ (d) None of these

2. Solution of the linear inequations $\frac{2x+3}{5} - 2 \le \frac{3(x-2)}{5}$ expressed in the form of interval is given by
   (a) $[-2,\infty)$ (b) $[2,\infty)$
   (c) $[-1,\infty)$ (d) None of these

3. Solution of the linear inequations $\frac{-x+3}{2} - 2 \le \frac{3(x+2)}{5} \le \frac{x}{6}$ expressed in the form of interval is given by
   (a) $[-2,\infty)$ (b) ϕ
   (c) $(3,\infty)$ (d) None of these

4. Solution of system of inequations $x + 5 > 2(x + 1)$ and $2 - x < 3(x + 2)$ expressed in the form of intervals is given by
   (a) (–1, 3) (b) (–1, 2)
   (c) (–2, 3) (d) None of these

5. Solution of system of inequations $2(x - 6) \le 3x - 7$ and $11 - 2x \le 6 - x$ expressed in the form of intervals is given by
   (a) $(-\infty, 5)$ (b) $[5, \infty)$
   (c) $[-5, \infty)$ (d) None of these

6. The range of y, when $-9 \le x \le 5$, $y = x^3$ is
   (a) [125, 729] (b) (–729, 125)
   (c) [–729, 125] (d) None of these

7. The range of y, when $2 < x < 5$, $y = \frac{1}{x}$ is
   (a) (1/5, 1/2) (b) (–1/2, 1/5)
   (c) $(1/5, \infty)$ (d) None of these

8. The range of y, when $-6 < x < 6$, $y = \frac{1}{x^2}$ is
   (a) (–1/36, 1/36) (b) $(1/36, \infty)$
   (c) (0, 1/36) (d) None of these

9. The range of y, when $9 < x^2 < 25$, $y = x$ is
   (a) $(-5, -3)\cup(3, 5)$ (b) (–5, – 3)
   (c) (9, 25) (d) None of these

10. Solution of the inequation $x^2 - 3x + 2 > 0$ expressed in the form of intervals is given by

(a) $(-\infty, 1) \cup [2, \infty)$ (b) $(-\infty, 1) \cup (2, \infty)$
(c) $(-\infty, -1) \cup (2, \infty)$ (d) None of these

11. Solution of the inequation $x^2 - 3x - 4 \le 0$ expressed in the form of intervals is given by
(a) $[-1, 4]$ (b) $[1, 4]$
(c) $(-1, 4)$ (d) None of these

12. Solution of the inequation $x^2 - 3x + 2 < 0$ expressed in the form of intervals is given by
(a) $[1, 2]$ (b) $(1, 2)$
(c) $(-\infty, 1) \cup (2, \infty)$ (d) None of these

13. Solution of the inequation $\frac{4x+3}{2x-5} \le 6$ expressed in the form of intervals is given by
(a) $\left(-\infty, -\frac{5}{2}\right) \cup \left[\frac{33}{8}, \infty\right)$
(b) $\left(-\infty, \frac{2}{5}\right) \cup \left[\frac{33}{8}, \infty\right)$
(c) $\left(-\infty, \frac{5}{2}\right) \cup \left[\frac{33}{8}, \infty\right)$
(d) None of these

14. Solution of the inequation $\frac{x}{x-5} \ge \frac{1}{2}$ expressed in the form of intervals is given by
(a) $(-\infty, -5] \cup (5, \infty)$ (b) $(-\infty, -5]$
(c) $(-5, 5]$ (d) None of these

15. Solution of the inequality $\frac{(x-1)(x-2)(5-x)}{(2x+5)} < 0$ expressed in the form of intervals is given by
(a) $\left(-\infty, -\frac{5}{2}\right) \cup (1,2) \cup (5,\infty)$
(b) $\left(-\infty, -\frac{5}{2}\right) \cup (5,\infty)$
(c) $\left(-\infty, -\frac{5}{2}\right) \cup (-1,2) \cup [5,\infty)$
(d) None of these

16. Solution of the inequality $\frac{x^2 - 2x + 5}{3x^2 - 2x - 5} > \frac{1}{2}$ expressed in the form of intervals is given by
(a) $[-5,-1] \cup \left(\frac{5}{3}, 3\right)$ (b) $(-5,-1) \cup \left(\frac{5}{3}, 3\right)$
(c) $(-5,1) \cup \left(\frac{5}{3}, 3\right)$ (d) None of these

17. Solution of the inequality $\frac{(x-1)^2(x+1)^3}{x^4(x-2)} \le 0$ expressed in the form of intervals is given by
(a) $[-1, 2)$ (b) $(-\infty, 0) \cup (0, 2)$
(c) $[-1, 0) \cup (0, 2)$ (d) None of these

18. Solution of the inequality $\frac{x(x+1)(x-3)}{(x+4)} > 0$ and $\frac{(5-x)(x+2)}{(x-8)} < 0$ expressed in the form of intervals is given by
(a) $(-1, 0) \cup (3, 5) \cup (8, \infty)$
(b) $(0, 1]$
(c) $(-\infty, 0) \cup [1, \infty)$
(d) None of these

19. The value of x satisfying the inequalities hold $\frac{(2x-1)(x-1)^4(x-2)^4}{(x-2)(x-4)^4} \le 0$
(a) $\left[\frac{1}{2}, 2\right)$ (b) R
(c) ϕ (d) $\left(\frac{1}{2}, 2\right)$

20. The set of all values of x for which $(x-2)^3 (x-3) < 0$ is
(a) $(2, 3)$ (b) $[2, 3)$
(c) $(0, 3)$ (d) $(2, 3]$

21. $(x - 2)^4 (x - 3)^3 (x - 4)^2 (1 - x) \le 0$.
(a) $(1,3)$
(b) $(-\infty, 1) \cup (3, \infty)$
(c) $(-\infty, 1] \cup [3, \infty) \cup \{2\}$
(d) None of these

22. If $c < d$, $x^2 + (c + d)x + cd < 0$
(a) $(-d, -c]$ (b) $(-d, -c)$
(c) $\mathbb{R}$ (d) ϕ

23. If $a, b, c > 0$ and $a(1 - b) > \frac{1}{4}$, $b(1 - c) > \frac{1}{4}$ and $c(1 - a) > \frac{1}{4}$, then
(a) never possible (b) always true
(c) cannot be discussed (d) None of these

Answer Keys

1. (a)	**2.** (c)	**3.** (b)	**4.** (a)	**5.** (b)	**6.** (c)	**7.** (a)	**8.** (b)	**9.** (a)	**10.** (b)
11. (a)	**12.** (b)	**13.** (c)	**14.** (a)	**15.** (a)	**16.** (b)	**17.** (c)	**18.** (a)	**19.** (a)	**20.** **(a)**
21. (c)	**22.** (b)	**23.** (a)							

■ INCREASING AND DECREASING FUNCTION

(i) **Increasing Function:** *A* function $y = f(x)$ is called increasing function if its output $f(x)$. (i.e., y) increases by increasing the input x (it may happen that by increasing the input x, output $f(x)$ remains same for few but not all inputs $x \in D_f$)

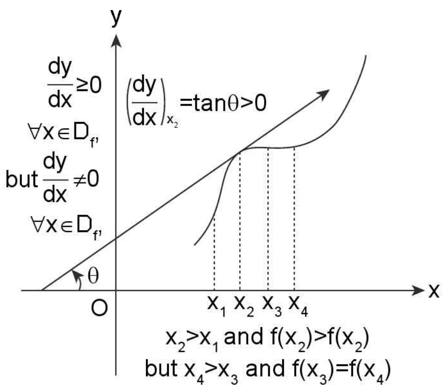


FIGURE 2.40

That is, $x_2 > x_1 \Leftrightarrow f(x_2) \geq f(x_1)$ for distinct values of x_1 and x_2.

$\therefore\ f(x_2) - f(x_1) \geq 0 \Leftrightarrow x_2 - x_1 > 0$

$\Rightarrow\ \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} \geq 0$

$\Rightarrow\ \dfrac{\Delta y}{\Delta x} \geq 0$ ($\because$ when $(x_2 - x_1) = \Delta x \to 0$; $(f(x_2) - f(x_1)) = \Delta y \to 0$ or $\Delta y = 0$)

$\Rightarrow$ That is, instantaneous rate of change of $f(x)$ always remains non-negative.

(ii) **Decreasing Function:** *A* function $y = f(x)$ is called decreasing function if its output $f(x)$, (i.e., y) decreases by increasing the input x. (it may happen that by increasing the input x output $f(x)$ remains same for few but not all inputs $x \in D_f$)

That is, $x_2 > x_1 \Leftrightarrow f(x_2) \leq f(x_1)$ for distinct values of x_1 and x_2.

$\therefore\ f(x_2) - f(x_1) \leq 0 \Leftrightarrow x_2 - x_1 > 0$

$\Rightarrow\ \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} \leq 0$

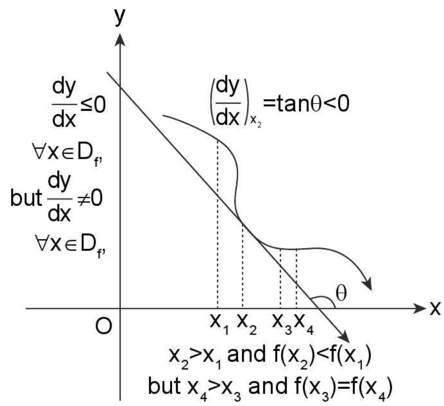


FIGURE 2.41

$\Rightarrow\ \dfrac{\Delta y}{\Delta x} \leq 0$ ($\because$ when $(x_2 - x_1) = \Delta x \to 0$; $(f(x_2) - f(x_1)) = \Delta y \to 0$ or $\Delta y = 0$)

$\Rightarrow\ \dfrac{dy}{dx} = f'(x) \leq 0$. That is, instantaneous rate of change of $f(x)$ always remains non-positive.

REMARKS

f and f^{-1} have same monotonic nature. That is, either both increasing or both decreasing.

By application of increasing function to both sides of inequality the sign of inequality remains unchanged. Same is true for removal of the function (i.e., application of its inverse function)

By application of decreasing function to both sides of inequality the sign of inequality gets reversed. Same is true for removal of the function (i.e., application of its inverse function).

ILLUSTRATION 39: Test whether the following functions are increasing or decreasing.

(i) $y = x^2$ (ii) $y = x^3$

(iii) $y = \sqrt{x}$ (iv) $y = \sqrt[3]{x}$

SOLUTION: (i) $\frac{dy}{dx} = 2x = \begin{cases} +; x > 0 \uparrow \\ -; x < 0 \downarrow \end{cases}$

$\therefore$ $f(x)$ is monotonically increasing for $x \geq 0$ and monotonically decreasing for $x \leq 0$.

(ii) $\frac{dy}{dx} = 3x^2 \geq 0 \;\; \forall \; x \in \mathbb{R}$

$\therefore$ $f(x)$ is monotonically increasing $\forall \; x \in \mathbb{R}$.

(iii) $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} > 0 \;\; \forall \; x > 0$

$\therefore$ $f(x)$ is monotonically increasing $\forall \; x > 0$.

(iv) $\frac{dy}{dx} = \frac{1}{3}(x)^{-\frac{2}{3}} = \frac{1}{3(x^{1/3})^2} > 0; \; \forall \; x \in \mathbb{R} \sim \{0\}$

$\therefore$ $f(x)$ is monotonically increasing $\forall$ x$\in \mathbb{R} \sim \{0\}$.

MONOTONIC FUNCTION

A function $f(x)$ is called Monotonic iff either it increases $\forall \; x \in D_f$ or decreases $\forall \; x \in D_f$ wherever it is defined.

i.e., $f(x) \geq 0 \; \forall \; x \in D_f$ or $f(x) \leq 0 \; \forall \; x \in D_f$

e.g., (i) $y = x$, $y = x^3$ are monotonic functions.

(ii) $y = x^2$ is not a monotonic function as it decreases for $x \leq 0$ and increases for $x \geq 0$.

INTERVAL OF MONOTONICITY

An interval belonging to domain of function in which function $f(x)$ is monotonic, is called interval of monotonicity.

e.g., for $y = x^2$; $f'(x) = 2x = \begin{cases} +ve & \text{for } x > 0 \\ -ve & \text{for } x < 0 \end{cases}$ and $f'(x) = 0$ only at $x = 0$;

$\Rightarrow$ $x \in [0, \infty)$ is the interval of monotonic increase and $x \in (-\infty, 0]$ is the interval of monotonic decrease.

TO FIND INTERVAL OF MONOTONICITY FOR $y = f(x)$

Step 1: Find the derivative of function, i.e., $f'(x)$. Let $f'(x) = (x - \alpha)(x - \beta)(x - \gamma)(x - \delta)(x - k)$.

Step 2: Find the values of $x \in D_f$ (critical points) where $f'(x) = 0$ or it is not defined.

Step 3: Locate these critical points on (number line).

Step 4: Find the sign of $\frac{dy}{dx}$, i.e., $f'(x)$ in these intervals obtained.

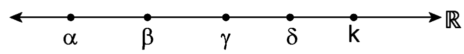


FIGURE 2.42

Suppose $f'(x) \geq 0 \; \forall \; x \in (\alpha, \beta] \cup [\gamma, \delta] \cup [k, \infty)$ and $f'(x) \leq 0 \; \forall \; x \in (-\infty, \alpha] \cup [\beta, \gamma] \cup [\delta, k)$, then $f(x)$ is monotonically increasing for $x \in (\alpha, \beta] \cup [\gamma, \delta] \cup [k, \infty)$ and $f(x)$ is monotonically decreasing for $x \in (-\infty, \alpha] \cup [\beta, \gamma] \cup [\delta, k)$

ILLUSTRATION 40: Find the interval of monotonicity of function $f(x) = x^4 - 2x^2 + 6$

SOLUTION: $f(x) = x^4 - 2x^2 + 6 \quad \Rightarrow \quad f'(x) = 4x^3 - 4x = 0 \quad \Rightarrow \quad x(x - 1)(x + 1) = 0$

$\Rightarrow$ $x = 0, -1, 1$ are the critical points

$\Rightarrow \; f'(x) \text{is} \begin{cases} \geq 0 & \forall \; x \in [-1, 0] \cup [1, \infty) \\ \leq 0 & \forall \; x \in (-\infty, -1] \cup [0, 1] \end{cases}$

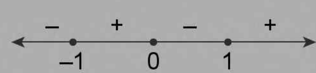


FIGURE 2.43

$\therefore$ $f(x)$ is monotonically increasing for $x \in [-1, 0] \cup [1, \infty)$

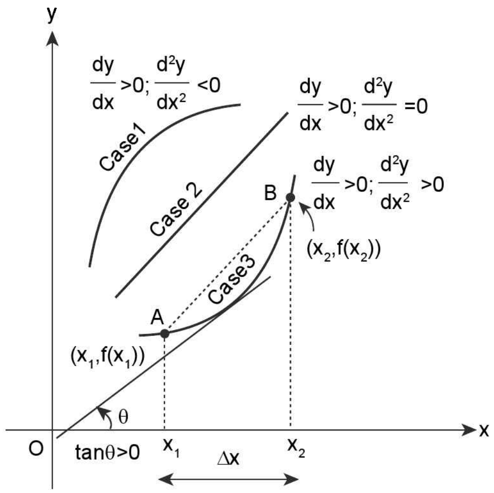
and $f(x)$ is monotonically decreasing for $x \in (-\infty, -1] \cup [0, 1]$

■ STRICTLY MONOTONIC FUNCTIONS

These are of two types:

(i) Strictly increasing function
(ii) Strictly decreasing function

(i) **Strictly Increasing Functions:** A function $y = f(x)$ is said to be strictly increasing on a set S(or D_f) if $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$; $x_1, x_2 \in S$(or D_f)



Case 1: Increases with decreasing rate of increase.
Case 2: Increases with constant rate of increase.
Case 3: Increases with increasing rate of increase.

FIGURE 2.44

i.e.,

$$f'(x) = \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^+} \frac{f(x-h)-f(x)}{-h}.$$

$\because$ $x + h > x \Rightarrow f(x + h) > f(x)$

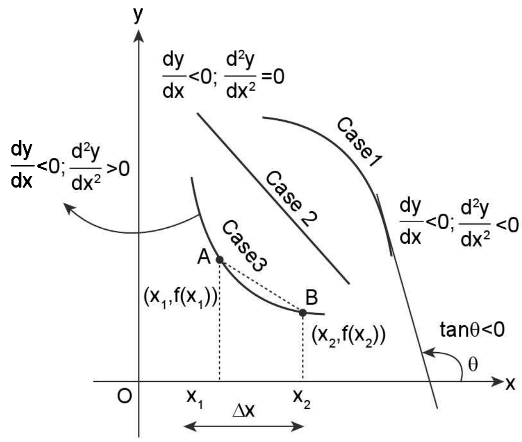
$\therefore$ $\dfrac{f(x+h)-f(x)}{h} > 0$

Also $x - h < x \Rightarrow f(x - h) < f(x)$

$\therefore$ $\dfrac{f(x-h)-f(x)}{-h} > 0 \Rightarrow f'(x) \geq 0$

$\therefore$ *A* function $f(x)$ is said to be strictly increasing if $f'(x) \geq 0$ and $f'(x) = 0$ occurs only at isolated points i.e., $f(x)$ does not has a line segment parallel, to x-axis as a part of its graph.

(ii) Strictly decreasing functions: *A* function $y = f(x)$ is said to be strictly decreasing on a set S(or D_f) if $x_2 > x_1$ $\Rightarrow$ $f(x_2) < f(x_1)$; $x_1, x_2 \in S$(or D_f)



Case 1: decreases with increasing rate of decrease.
Case 2: decreases with constant rate of decrease.
Case 3: decreases with decreasing rate of decrease.

FIGURE 2.45

i.e.,

$$f'(x) = \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^+} \frac{f(x-h)-f(x)}{-h}$$

$\because$ $x + h > x \Rightarrow f(x + h) < f(x)$

$\therefore$ $\dfrac{f(x+h)-f(x)}{h} < 0$

Also $x - h < x \Rightarrow f(x - h) > f(x)$

$\therefore$ $\dfrac{f(x-h)-f(x)}{-h} < 0 \Rightarrow f'(x) < 0$

$\therefore$ *A* function $f(x)$ is said to be strictly decreasing if $f'(x) \leq 0$ and $f'(x) = 0$ occurs only at isolated points, i.e., $f(x)$ does not has a line segment parallel to x-axis as a part of its graph.

ILLUSTRATION 41: Find the intervals in which the function $f(x) = x^3 - 7x^2 + 14x - 8$, is strictly increasing and strictly decreasing.

SOLUTION: Given function is $f(x) = x^3 - 8 - 7x^2 + 14x = (x^3 - 8) - 7x(x - 2)$

$= (x - 2)(x^2 + 4 + 2x) - 7x(x - 2) = (x - 2)[x^2 + 2x + 4 - 7x]$

$= (x - 2)(x^2 - 5x + 4) = (x - 2)(x - 1)(x - 4)$

$f'(x) = 0 \Rightarrow x = 1, 2, 4$; are the critical points

$\therefore$ $f'(x) > 0 \Leftrightarrow x \in (1, 2) \cup (4, \infty)$

Also $f'(x) = 0$ at isolated points $x = 1, 2$ and 4

$\therefore$ $f(x)$ is strictly increasing on $[1, 2] \cup [4, \infty)$ and $f'(x) < 0 \Leftrightarrow x \in (-\infty, 1) \cup (2, 4)$

and $f'(x) = 0$ at isolated points $x = 1, 2, 4$.

$\therefore$ $f(x)$ is strictly decreasing on $(-\infty, 1] \cup [2, 4]$.

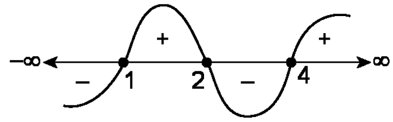


FIGURE 2.46

ILLUSTRATION 42: Find the intervals of monotonicity of the function $f(x) = \dfrac{4x}{4-x^2}$.

SOLUTION: Given function is $f(x) = \dfrac{4x}{4-x^2}$ and $f'(x) = \dfrac{(4-x^2)(4) - 4x(-2x)}{(4-x^2)^2}$

$$= \frac{4[4-x^2+2x^2]}{(4-x^2)^2} = \frac{4(4+x^2)}{(4-x^2)^2} > 0 \ \forall\ x \in \mathbb{R} \sim \{-2, 2\}$$

$\therefore$ $f(x)$ is strictly increasing on its domain $\mathbb{R} \sim \{-2, 2\}$. Graph of $f(x) = \dfrac{4x}{4-x^2}$ is as shown in Figure 2.47.

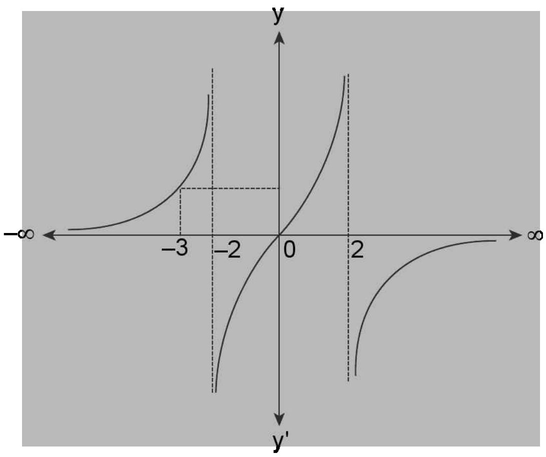


FIGURE 2.47

ILLUSTRATION 43: Find the interval of monotonicities of function $f(x) = \dfrac{x^2}{x^2-1}$

SOLUTION: Given function is $f(x) = \dfrac{x^2}{x^2-1}$

Clearly domain of $f(x) = \mathbb{R} \sim \{\pm 1\}$

Now $f'(x) = \dfrac{(x^2-1)(2x) - x^2(2x)}{(x^2-1)^2}$

$$= \frac{2x[x^2-1-x^2]}{(x^2-1)^2} = \frac{-2x}{(x^2-1)^2}$$

$\therefore$ $f'(x) \geq 0$ for $x \in (-\infty, 0] \sim \{-1\}$ and $f'(x) \leq 0$ for $x \in [0, \infty) \sim \{1\}$

Thus, $f(x)$ is strictly increasing on $(-\infty, 0] \sim \{-1\}$ and is strictly decreasing on $[0, \infty) \sim \{1\}$. Graphically, it is shown in the Figure 2.48.

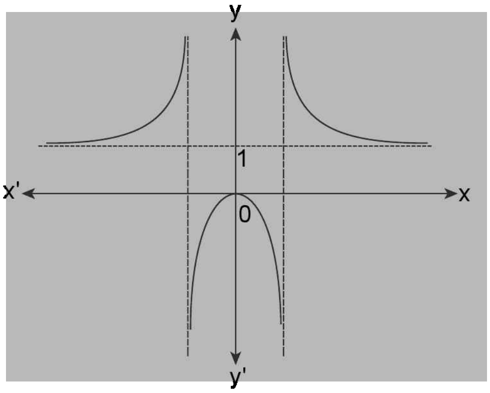


FIGURE 2.48

ILLUSTRATION 44: Find the intervals of monotonoicities of the function $f(x) = \dfrac{(x+2)}{(x-1)(x+1)}$

SOLUTION: Given function is $f(x) = \dfrac{(x+2)}{(x-1)(x+1)}$

Clearly domain of $f(x)$ is $\mathbb{R} \sim \{-1, 1\}$

$$\therefore \quad f'(x) \; \frac{(x-1)(x+1)(1)-(x+2)(2x)}{[(x-1)(x+1)]} = \frac{x^2-1-2x^2-4x}{(x^2-1)^2} = \frac{-x^2-4x-1}{(x^2-1)^2}$$

$\therefore \quad f'(x) \geq 0 \;\Rightarrow\; x^2 + 4x + 1 \leq 0;\ x \neq \pm 1$

$$\Rightarrow \quad x = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$$

$\therefore \quad x^2 + 4x + 1 \leq 0 \;\Rightarrow\; x \in [-2 - \sqrt{3}, -2 + \sqrt{3}] \sim \{-1\}$

$\therefore \quad f(x)$ is strictly increasing on $x \in [-2 - \sqrt{3}, -2 + \sqrt{3}] \sim \{-1\}$

and $f'(x) \leq 0, x \neq \pm 1$ implies $x \in (-\infty, -2 - \sqrt{3}] \cup [-2 + \sqrt{3}, \infty) \sim \{1\}$

That is, $f(x)$ is strictly decreasing on $(-\infty, -2 - \sqrt{3}[\cup [-2 + \sqrt{3}, \infty) \sim \{1\}$

Graph of $f(x)$ is as shown in Figure 2.49.

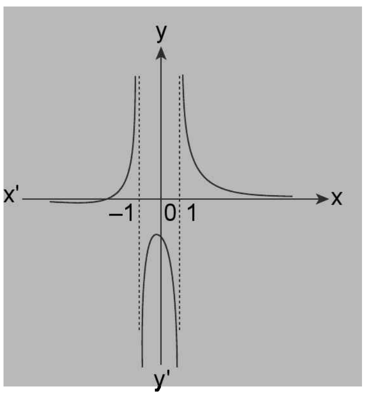


FIGURE 2.49

ILLUSTRATION 45: Find the intervals of monotonicity of the function $f(x)=\dfrac{x^2-5x+6}{x^2-16}$

SOLUTION: Given function is $f(x)=\dfrac{x^2-5x+6}{x^2-16}$; Clearly domain of function is $\mathbb{R} \sim \{4, -4\}$

and $f'(x)=\dfrac{(x^2-16)(2x-5)-(x^2-5x+6)(2x)}{(x^2-16)^2}$

$$=\frac{2x^3-5x^2-32x+80-2x^3+10x^2-12x}{(x^2-16)^2}=\frac{5x^2-44x+80}{(x^2-16)^2}$$

$\therefore$ $f'(x) > 0 \Rightarrow 5x^2 - 44x + 80 \geq 0$

Corresponding quadratic equation is $5x^2 - 44x + 80 = 0$

$$x=\frac{44\pm\sqrt{(-44)^2-(20)(80)}}{2(5)}=\frac{44\pm\sqrt{16\times121-1600}}{10}=\frac{44\pm4\sqrt{21}}{10}$$

$$=\frac{22-2\sqrt{21}}{5},\frac{22+2\sqrt{21}}{5}$$

$\therefore$ $5x^2 - 44x + 80 \geq 0$

$$\Rightarrow\ x\in\left(-\infty,\frac{22-2\sqrt{21}}{5}\right]\cup\left[\frac{22+2\sqrt{21}}{5},\infty\right)\sim\{-4\}$$

That is, $f(x)$ is strictly increasing on $\left(-\infty,\dfrac{22-2\sqrt{21}}{5}\right]\cup\left[\dfrac{22+2\sqrt{21}}{5},\infty\right)\sim\{-4\}$

and $f(x)$ is strictly decreasing on $\left[\dfrac{22-2\sqrt{21}}{5},\dfrac{22+2\sqrt{21}}{5}\right]\sim\{4\}$

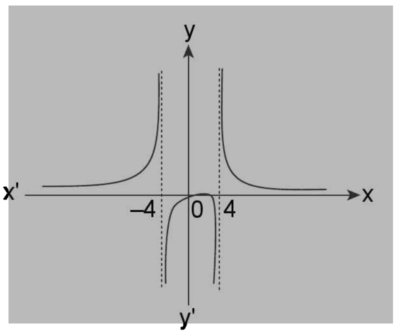


FIGURE 2.50

NOTE

Detail discussion of Derivatives and Monotonicity of Functions will be given in the chapters Methods of Derivatives and Application of Derivatives II of volume VI.

TEXTUAL EXERCISE–4: (SUBJECTIVE)

1. (a) Show that the function $y = x^3 + x$ increases every where.
   (b) Find the intervals of monotonicity of the function $y = x^4 - 2x^2 - 5$.

2. Find out the monotonic nature of the function $y = \sin x + \cos x$ in the interval $(0, \pi)$. Verify whether the inequality $(\sin 23° + \cos 23°) > (\sin 53° + \cos 53°)$ is true?

3. Find out the behaviour of the function $\dfrac{\sin x}{x}$ in the interval $(0, \pi)$. Verify whether $\dfrac{\sin\alpha}{\alpha} > \dfrac{\sin\beta}{\beta}$ $\forall\ 0 < \alpha < \beta < \pi$.

4. Show that the function $y = 2x^3 + 3x^2 - 12x + 1$ decreases in the interval $(-2, 1)$.

5. Show that the function $y = \sqrt{2x - x^2}$;increases in the interval (0, 1) and decreases in the interval (1, 2). Construct the rough sketch of the function.

6. Show that the function $y = \dfrac{x^2 - 1}{x}$ increases in any interval not containing the point $x = 0$.

7. Find the intervals of monotonicity of the given functions.
   (a) $y = (x - 2)^5 (2x + 1)^4$
   (b) $y = x - e^x$
   (c) $y = x^2 e^{-x}$
   (d) $y = \dfrac{x}{\ln x}$
   (e) $y = x - 2 \sin x$; where $0 \le x \le 2\pi$
   (f) $y = x + \cos 2x$; where $0 \le x \le 2\pi$

8. Test whether following are increasing or decreasing.
   (i) $f(x) = 2x + 3$
   (ii) $f(x) = a^x$; $a > 1$
   (iii) $f(x) = x^3$
   (iv) $f(x) = \sqrt{x}$
   (v) $f(x) = \sqrt[3]{x}$
   (vi) $f(x) = \log_a x$; $a > 1$

9. Show that the function $f(x) = x^3 - 3x^2 + 4x$, is strictly increasing function in its domain?

10. Find the interval in which the following functions increase
    (a) $f(x) = \sin 3x : x \in \left[0, -\right]$
    (b) $(x + 1)^3 (x - 3)^3$
    (c) $2x^2 - 3x$

Answer Keys

1. (b) ↑ for $x \in [-1, 0]$ or $[1, \infty)$; ↓ for $x \in (-\infty, -1]$ or $[0, 1]$
2. ↑ for $x \in (0, \pi/4]$; ↓ for $x \in [\pi/4, \pi)$; (false)
3. decreasing
7. (a) ↑ for $x \in (-\infty, -1/2] \cup [11/18, \infty)$ and ↓ for $x \in [-1/2, 11/18]$
   (b) ↑ for $x \in (-\infty, 0]$ and ↓ for $x \in [0, \infty)$
   (c) ↑ for $x \in [0, 2]$ and ↓ for $x \in (-\infty, 0] \cup [2, \infty)$
   (d) ↑ for $x \in [e, \infty)$ and ↓ for $x \in (0, e] - \{1\}$
   (e) ↑ for $x \in \left[\dfrac{\pi}{3}, \dfrac{5\pi}{3}\right]$ and ↓ for $x \in \left[0, \dfrac{\pi}{3}\right] \cup \left[\dfrac{5\pi}{3}, 2\pi\right]$
   (f) ↑ for $x \in \left[0, \dfrac{\pi}{12}\right] \cup \left[\dfrac{5\pi}{12}, \dfrac{13\pi}{12}\right] \cup \left[\dfrac{17\pi}{12}, 2\pi\right]$ and ↓ for $x \in \left[\dfrac{\pi}{12}, \dfrac{5\pi}{12}\right] \cup \left[\dfrac{13\pi}{12}, \dfrac{17\pi}{12}\right]$
8. (i) increasing (ii) increasing (iii) increasing (iv) increasing
   (v) increasing (vi) increasing
10. (a) $[0, \pi/6]$ (b) $x \in [1, \infty)$ (c) $[3/4, \infty)$

TEXTUAL EXERCISE–4: (OBJECTIVE)

1. The monotonic nature of following functions:

Column I

(i) $\ln(x+\sqrt{x^2+1})$

(ii) a^x, $a > 1$

(iii) $x^3 + x$

(iv) $\tan x$

(v) $x^2 - 1$

(vi) $\cot x$

Column II

(a) increasing

(b) decreasing

(c) non-increasing

(d) non-decreasing

(e) non-monotonic

(f) cannot said

Correct match is

	(i)	(ii)	(iii)	(iv)	(v)	(vi)
(a)	a	a	b	b	e	f
(b)	a	a	a	a	e	b
(c)	a	b	c	d	e	f
(d)	f	a	a	a	e	b

2. The function $f(x) = x^2 (x-2)^2$

(a) monotonically increases

(b) decreases $\forall\, x \in (-\infty, 0) \cup (1, 2)$

(c) increasing $\forall\, x \in (0, 1) \cup (2, \infty)$

(d) None of these

3. The function $f(x) = \log(1 + x) - \dfrac{2x}{2+x}$

(a) decreases $\forall\, x \in (-1, \infty)$

(b) increasing $\forall\, x \in (-1, \infty)$

(c) increasing $\forall\, x \in (1, 2)$

(d) None of these

4. $x - \sin x$ is increasing

(a) for all real numbers

(b) for +ve real numbers only

(c) for –ve real numbers only

(d) None of these

5. $\log(\sin x)$ is

(a) decreasing in $(0, \pi/2)$ and $(\pi/2, \pi)$

(b) always increasing

(c) increasing in $(0, \pi/2)$ and decreasing in $(\pi/2, \pi)$

(d) None of these

6. $a^{kx} - a^{-kx}\ \forall\ a > 1$

(a) Increasing for $k > 0$

(b) decreasing for $k < 0$

(c) increasing for $k < 0$

(d) decreasing for $k > 0$

7. $\sin(\cos x)$ defined in $[\pi, 2\pi]$ is

(a) monotonically increasing

(b) monotonically decreasing

(c) non-monotonic

(d) None of these

8. $\cos(\sin x)$ defined for $(0, \pi/2)$ is

(a) monotonically increasing

(b) monotonically decreasing

(c) non-monotonic

(d) None of these

9. State which of the following does not represent the correct order of the functions

(a) $x < x^2 < x^3 < x^4 < \ldots$; where $x \in (1, \infty)$

(b) $x > x^2 > x^3 > x^4 > \ldots$; where $x \in (0, 1)$

(c) $x < x^3 < x^5 < \ldots$ (–ve values); $x^2 > x^4 > x^6 > \ldots$ (+ve values); where $x \in (-1, 0)$

(d) $x > x^3 > x^5 > \ldots$ (–ve values); $x^2 > x^4 > x^6 > \ldots$ (+ve values); where $x \in (-\infty, -1)$

10. State which of the following does not represent the correct order of the functions

(a) $\dfrac{1}{x} > \dfrac{1}{x^2} > \dfrac{1}{x^3} > \dfrac{1}{x^4} > \ldots$; where $x \in (1, \infty)$

(b) $\dfrac{1}{x} < \dfrac{1}{x^2} < \dfrac{1}{x^3} < \dfrac{1}{x^4} < \ldots$;where $x \in (0,1)$

(c) $\dfrac{1}{x^2} < \dfrac{1}{x^4} < \dfrac{1}{x^6} < \ldots$ (+ve values) and $\dfrac{1}{x} < \dfrac{1}{x^3} < \dfrac{1}{x^5} < \ldots$ (–ve values); where $x \in (-1, 0)$

(d) $\dfrac{1}{x^2} > \dfrac{1}{x^4} > \dfrac{1}{x^6} \ldots$ (+ve values) and $\dfrac{1}{x} < \dfrac{1}{x^3} < \dfrac{1}{x^5} \ldots$ (-ve values); where $x \in (-\infty, -1)$

11. State which of the following does not represent the correct order of the functions

(a) $x^{1/2} > x^{1/3} > x^{1/4} > x^{1/5} \ldots$; where $(1, \infty)$

(b) $x^{1/2} < x^{1/3} < x^{1/4} < x^{1/5} \ldots$; where $(0, 1)$

(c) $x^{1/3} > x^{1/5} > x^{1/7} \ldots$; where $x \in (-1, 0)$

(d) $x^{1/3} > x^{1/5} > x^{1/7} \ldots$; where $x \in (-\infty, -1)$

12. The function $\dfrac{e^{2x}-1}{e^{2x}+1}$ is

(a) decreasing

(b) increasing

(c) odd

(d) even

Answer Keys

1. (b) **2.** (b), (c) **3.** (b), (c) **4.** (a) **5.** (c) **6.** (a), (b) **7.** (a) **8.** (b) **9.** (d) **10.** (c)
11. (d) **12.** (b), (c)

SOME STANDARD FUNCTIONS AND THEIR PROPERTIES

Algebraic Function

Functions consisting of finite number of terms involving powers and roots of the independent variable and the four fundamental operations (+, –, ×, ÷) are called algebraic functions.

e.g., $x^{3/2} + 5x$, $\frac{\sqrt{x+1}}{x-1}$, $x + 1$, $3x^4 - 5x + 7$.

ILLUSTRATION 46: Which of the following functions are algebraic and find their corresponding domains.

(i) $f(x)=\frac{x^2+1}{x-1}$ (ii) $f(x)=\sqrt{\frac{x}{x^2-1}}$

(iii) $f(x)=\frac{\sqrt{\sin x}+1}{x^2-1}$ (iv) $f(x)=\frac{x^2-5x+6}{(x-2)}$

(v) $f(x) = x\tan x$ (vi) $f(x) = \sin x + \cos^2 x$

SOLUTION: (i), (ii), (iv) are algebraic function whereas (iii), (v) and (vi) are non-algebraic functions as they do not involve all terms contianing powers of x.

(i) $f(x)=\frac{x^2+1}{x-1}$; $D_f = \mathbb{R} \sim \{1\}$

(ii) $f(x) = \sqrt{\frac{x}{x^2-1}}$; $\frac{x}{x^2-1}\geq 0$; $x^2-1\neq 0$

$\Rightarrow x(x^2 - 1) \geq 0; x \neq \pm 1$ $\Rightarrow (x + 1)(x)(x - 1) \geq 0; x \neq \pm 1$

$\Rightarrow x \in (-1, 0] \cup (1, \infty) = D_f$

(iv) $f(x)=\frac{x^2-5x+6}{(x-2)}=\frac{(x-2)(x-3)}{(x-2)} = x - 3$ for $x \neq 2$; $D_f = \mathbb{R} \sim \{2\}$

(v) $f(x) = x\tan x$ is defined for all $\mathbb{R}$ except for odd integral multiples of $\frac{\pi}{2}$

Thus, domain $= \mathbb{R} \sim \{(2n + 1)\frac{\pi}{2}; n \in \mathbb{Z}\}$.

(vi) $f(x) = \sin x + \cos^2 x$ is defined for all real numbers.

$\therefore$ $D_f = \mathbb{R}$; $f(x)$ being trigonometric function is non-algebraic.

POLYNOMIAL FUNCTION

A map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$; where $x \in \mathbb{R}$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}$ is said to be a polynomial function. If $a_n \neq 0$, then n is called the degree of the polynomial.

e.g., $P(x) = x^7 + 2x^5 + 4$ (a polynomial of degree 7); $Q(x) = x^2 - 3x + 2$ (a polynomial of degree 2); $R(x) = 3 = 3x^0$ (a polynomial of degree 0).

REMARKS

1. *A polynomial of degree one with no constant term is called an odd linear function, i.e., $f(x) = ax, a \neq 0$.*
2. *Every power of x occurring in polynomial function is a whole number. That is, if any power of x is not whole number then the function is not a polynomial function.*

   e.g., $f(x) = x + \frac{1}{x}$; $f(x) = 2x^2 + 5\sqrt{x} + 7$ *etc.*
3. *There are two polynomial functions, satisfying the relation;* $f(x)\, f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$.

   They are (i) $f(x) = x^n + 1$ *and (ii)* $f(x) = 1 - x^n$, *where n is a positive integer. That is,* $f(x) = \pm x^n + 1$.
4. *All polynomial functions are algebraic but converse is not true, i.e., there are some algebraic functions which are not polynomials* e.g., $f(x) = x + \frac{1}{x}$; $f(x) = 2x^2 + 5\sqrt{x} + 7$ *etc.*
5. *The function defined as* $f(x) = 0\ \forall\ x \in \mathbb{R}$ *is called* ***zero function****. Domain of zero function =* $\mathbb{R}$ *and range of zero function =* $\{0\}$.

ILLUSTRATION 47: Which of the following functions are polynomial functions?

(i) $f(x) = x^2 - 3x + 1/5$ (ii) $f(x) = x^2 + 1/x$

(iii) $f(x) = 3x^2 + \sqrt{x} - 5$ (iv) $f(x) = \sqrt[3]{x} + \sqrt{x} + 2$

(v) $f(x) = 5x^4 - 3x^2 + 2x - 7$ (vi) $f(x) = \sqrt{5}\, x^2 + \sqrt{2}\, x + 3$

(vii) $f(x) = \tan^2 x - 5\tan x + 6$

SOLUTION: (i), (v) and (vi) are polynomial functions as they contain whole number powers of x, for (ii) $f(x) = x^2 + x^{-1}$; $-1 \notin W$

(iii) $f(x) = 3x^2 + \sqrt{x} - 5 = 3x^2 + (x)^{1/2} - 5$; $\frac{1}{2} \notin W$

(iv) $f(x) = \sqrt[3]{x} + \sqrt{x} + 2 = (x)^{1/3} + (x)^{1/2} + 2; \frac{1}{3}, \frac{1}{2} \notin W$

(vii) $f(x) = \tan^2 x - 5\tan x + 6$ is not a polynomial in x, but a polynomial in $\tan x$.

That is, substituting $\tan x = t$, we get $f(t) = t^2 - 5t + 6$; which is a polynomial in t.

ILLUSTRATION 48: If $p(x)$ is a cubic polynomial such that $p(2) = 2$, $p(3) = 3$, $p(4) = 4$ and $p(5) = 6$, then evaluate p(10)

SOLUTION: Let $g(x) = p(x) - x$, then $g(2) = p(2) - 2 = 2 - 2 = 0$,

$g(3) = p(3) - 3 = 3 - 3 = 0$ and $g(4) = p(4) - 4 = 4 - 4 = 0$

Thus, $g(x)$ has three roots 2, 3 and 4. Also $p(x)$ is cubic implies $g(x) = p(x) - x$ is also cubic.

Thus, $g(x) = k(x - 2)(x - 3)(x - 4)$ for some non-zero real number k.

Now, $g(5) = p(5) - 5 = 6 - 5 = 1$

$\Rightarrow\ 1 = k(3)(2)(1)$

$\Rightarrow\ k = 1/6$

$\therefore\ g(x) = \frac{1}{6}(x - 2)(x - 3)(x - 4)$

$\Rightarrow\ p(x) = g(x) + x = \dfrac{1}{6}(x-2)(x-3)(x-4) + x$

$\therefore\ p(10) = \dfrac{1}{6}(8)(7)(6) + 10 = 66.$

ILLUSTRATION 49: Evaluate $\sqrt{(50)(51)(53)(54)+(52)^2}$ without using calculator.

SOLUTION: Let $50 = x$

$\Rightarrow\ \sqrt{(50)(51)(53)(54)+(52)^2} = \sqrt{x(x+1)(x+3)(x+4)+(x+2)^2}$

$= \sqrt{x(x+4)(x+1)(x+3)+(x+2)^2} = \sqrt{(x^2+4x)(x^2+4x+3)+(x+2)^2}$

$= \sqrt{(x^2+4x)^2+3(x^2+4x)+(x^2+4x)+4}$

$= \sqrt{y^2+4y+4}$; where $y = (x^2+4x)$

$= \sqrt{(y+2)^2} = |y+2| = y + 2 = x^2 + 4x + 2 = (50)^2 + 4(50) + 2$

$= 2500 + 200 + 2 = 2702$

ILLUSTRATION 50: Find a cubic polynomial $p(x)$ such that $(x-1)^2$ divides $p(x) - 1$ and $(x+1)^2$ divides $p(x) - 5$.

SOLUTION: Let $p(x) = ax^3 + bx^2 + cx + d$

$\therefore\ p(x) - 1 = ax^3 + bx^2 + cx + d - 1$ ... (1)

and $p(x) - 5 = ax^3 + bx^2 + cx + d - 5$ ... (2)

Now it is given that $(x-1)^2$ divides $p(x) - 1$ and $(x+1)^2$ divides $p(x) - 5$

That is, $(x-1)^2$ and $(x+1)^2$ are factors of $p(x) - 1$ and $p(x) - 5$ respectively.

$\Rightarrow\ p(1) - 1 = 0$ and $p(-1) - 5 = 0$ $\quad\Rightarrow\ p(1) = 1$ and $p(-1) = 5$

$\Rightarrow\ a + b + c + d = 1$ .... (3)

$\Rightarrow\ d = 1 - a - b - c$ and $-a + b - c + d = 5$ .... (4)

$\Rightarrow\ d = 5 + a - b + c$

$\therefore$ From (1); $p(x) - 1 = a(x^3 - 1) + b(x^2 - 1) + c(x - 1)$

and $p(x) - 5 = a(x^3 + 1) + b(x^2 - 1) + c(x + 1)$

or $p(x) - 1 = (x - 1)[a(x^2 + x + 1) + b(x + 1) + c]$

and $p(x) - 5 = (x + 1)\ [a(x^2 - x + 1) + b(x - 1) + c]$

Further it is given by $(x-1)^2$ divides $p(x) - 1$ and $(x+1)^2$ divides $p(x) - 5$

Thus, $(x - 1)$ divides $a(x^2 + x + 1) + b(x + 1) + c$

and $(x + 1)$ divides $a(x^2 - x + 1) + b(x - 1) + c$

$\Rightarrow\ 3a + 2b + c = 0$ ... (5)

and $3a - 2b + c = 0$ ... (6)

Solving (5) and (6), we get $b = 0$, $c = -3a$

Also solving (3) and (4), $b + d = 3$

$\Rightarrow\ d = 3$, and $a + c = -2$ $\quad\Rightarrow\ a = 1, b = 0, c = -3, d = 3$

$\therefore\ p(x) = x^3 - 3x + 3$ is the required cubic polynomial.

[redacted]

RATIONAL FUNCTION

The function which can be written as the quotient of two polynomial functions (Denominator $\neq 0$) is said to be a rational function. If $P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$; $Q(x) = b_0 + b_1x + b_2x^2 + \ldots + b_mx^m \neq 0$ be two polynomial functions, then $\dfrac{P(x)}{Q(x)}$ will be a rational function of x.

ILLUSTRATION 52: Which of the following functions are rational functions of x?

(i) $f(x) = \dfrac{x^2 - 3x + 1}{x + 5}$

(ii) $f(x) = \dfrac{x^2 - 2x + 5}{\sqrt{x - 5}}$

(iii) $f(x) = \sqrt[3]{x} - 5x + \dfrac{1}{\sqrt[3]{x}}$

(iv) $f(x) = 2x^2 - \dfrac{1}{x} + 5$

(v) $f(x) = \dfrac{\left(\sqrt{2x-1} + x\right)\left(\sqrt{2x-1} - x\right)}{(x-1)}$

SOLUTION: (i) $f(x) = \dfrac{x^2-3x+1}{x+5}$ is a rational function as both numerator and denominator are polynomial functions.

(ii) $f(x) = \dfrac{x^2-2x+5}{\sqrt{x-5}}$ is not a rational function as denominator is not a polynomial function.

(iii) $f(x)$ is not a rational function as $f(x)$ contains cube root of x.

(iv) $\dfrac{2x^3+5x-1}{x}$ is a rational function.

(v) $f(x) = \dfrac{\left(\sqrt{2x-1}+x\right)\left(\sqrt{2x-1}-x\right)}{(x-1)}$

$= \dfrac{2x-1-x^2}{(x-1)} = -\dfrac{(x^2-2x+1)}{(x-1)} = \dfrac{-(x-1)^2}{(x-1)} = -(x-1)$ for $x \neq 1$.

Therefore $f(x)$ is a rational function but with numerator polynomial on restricted domain, i.e., $\mathbb{R} \sim \{1\}$.

IRRATIONAL FUNCTION

The algebraic functions containing terms having rational powers of x (non-integral) are known as irrational functions.e.g., $y = \sqrt{x}$, $y = \dfrac{\sqrt{x^3+1}-\sqrt{x^{11}}}{\sqrt{x^2+x+1}}$,

$y = \dfrac{x^{17/3}+x^{103/7}-x}{\sqrt[3]{x^{17}+x^{15}-3}}$, $y = \dfrac{\sqrt{\sqrt{x^2+5}+x^{16.5}+x^{1/3}}}{\sqrt{x^2-7}-\sqrt{\sqrt{x^{2/3}-1}}}$

If such function contains $(f(x))^{\frac{2p+1}{2m}}$, p, $m \in \mathbb{Z}$, then it is undefined for $f(x) < 0$.

POWER FUNCTION

The function $f(x) = c.x^n$, where c, n are real constants and x is a real variable is known as power function. Usually we denote it as $f(x) = x^n$ by choosing $c = 1$. Its graph is symmetrical about y-axis if n is a rational number of the form $\dfrac{2p}{2q+1}$, where p and q are integers and symmetric about origin if n is a rational number of the form $\dfrac{2p+1}{2q+1}$, where p and q are integers. The graph of power function lies entirely in first quadrant if n is a rational number of the form $\dfrac{2p+1}{2q}$ or n is an irrational number.

Domain of power function depends on the constant n. The following cases are possible:

Case (i): If $n = \dfrac{2p+1}{2q} > 0$; where $(p, q) \in \mathbb{Z}$; $q \neq 0$, then $f(x) = c.(x)^n = c.(x)^{\frac{2p+1}{2q}} = c.\left((x)^{2p+1}\right)^{\frac{1}{2q}}$ which is defined for non-negative values of $\left(x^{2p+1}\right)$, i.e., non-negative values of x.

$\therefore$ Domain of $f(x)$ will be $[0, \infty)$ in this case.

Case (ii): If $n = \dfrac{2p+1}{2q} < 0$; where $(p, q) \in \mathbb{Z}$; $q \neq 0$, then $f(x) = c.(x)^n = c.(x)^{\frac{2p+1}{2q}} = c.\left((x)^{2p+1}\right)^{\frac{1}{2q}}$ which is defined for positive values of $\left(x^{2p+1}\right)$, i.e., positive values of x

$\therefore$ Domain of $f(x)$ will be $(0, \infty)$ in this case.

Case (iii) If $n = \dfrac{p}{2q+1} \geq 0$; where p, $q \in \mathbb{Z}$ and $2q+1 \neq 0$, then $f(x) = c.(x)^n = c\ (x)^{\frac{p}{2q+1}}$ is defined for all real values of x.

$\Rightarrow$ In this case domain of function will be $\mathbb{R}$.

Case (iv): If $n = \dfrac{p}{2q+1} < 0$, then $f(x) = c(x)^n = c(x)^{\frac{p}{2q+1}}$ is defined for all non-zero real numbers.

$\therefore$ In this case domain of function will be $\mathbb{R} \sim \{0\}$.

Case (v) If n is an irrational number (i.e., power function with irrational exponent). Let us discuss this case by using a particular power function when $n = \sqrt{2}$ and $x = 2$.

$\therefore$ $f(2) = (2)^{\sqrt{2}}$. We can find $(2)^{\sqrt{2}}$ by approximating the irrational exponent by rational exponent that are closer to the irrational one. In our example, we need to find the exponent $(2)^{\sqrt{2}}$. We know that to one significant figure $\sqrt{2} = 1$, so the first approximation to $2^{\sqrt{2}}$ is $2^1 = 2$. To two significant figures, $\sqrt{2} = 1.4 = \frac{14}{10} = \frac{7}{5}$, so the second approximation to $(2)^{\sqrt{2}}$ is $(2)^{(7/5)}$ = the 5^{th} root of 2^7 or the 7^{th} power of the 5^{th} root of 2 which is approximately 2.639. To three significant figures, $\sqrt{2} = 1.41 = \frac{141}{100}$, so we have to take a 100^{th} root of 2 and then raise it to the 141^{st} power, or else take the 100^{th} root of 2^{141}, to get approximately 2.65737. Continuing this way, we get the sequence of approximations as given below 2, 2.639, 2.65737, 2.66475, 2.665119, 2.6651375, 2.66514310,...

The above sequence converges to a limit (this is a calculus idea), because the function 2^x is continuous (another calculus concept). That limit is what we call the value of $2^{\sqrt{2}} \approx 2.665144143$. We cannot write down all the digits in the decimal expansion. This is the way how we can find irrational exponents.

So, to find $(x)^n$; where n is an irrational real number we have to replace n by rational numbers closer and closer to n. But by doing so, we are to encounter with rational numbers of the form $\frac{p}{2q+1}$ as well as of the form $\frac{2p+1}{2q}$.

Thus, for using all these rational numbers x must be non-negative for positive irrational number n. Thus, domain for $f(x)$ will be $[0, \infty)$ in this case. Further x must be positive for negative irrational number n. Thus, domain for $f(x)$ will be $(0, \infty)$ in this case.

Thus, from above discussion we conclude that the domain of power function $f(x) = c.x^n$ is $[0, \infty)$ for positive irrational number n and it is $(0, \infty)$ for negative irrational number n.

Graphs of some power functions are as follows:

(a) When $f(x) = (x)^{2n-1}$; $n \in \mathbb{N}$; i.e., the power is an odd natural number. Domain = $\mathbb{R}$

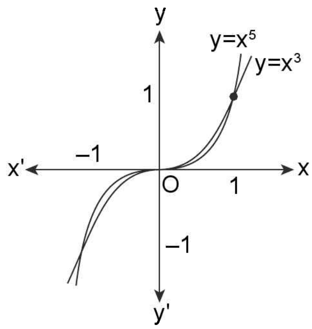


FIGURE 2.51

The graph of function is symmetrical about origin.

For example, $f(x) = x^5$ for $n = 3$.

(b) When $f(x) = x^{-(2n-1)}$; $n \in \mathbb{N}$; i.e., the power is an odd negative integer. Domain = $\mathbb{R} \sim \{0\}$

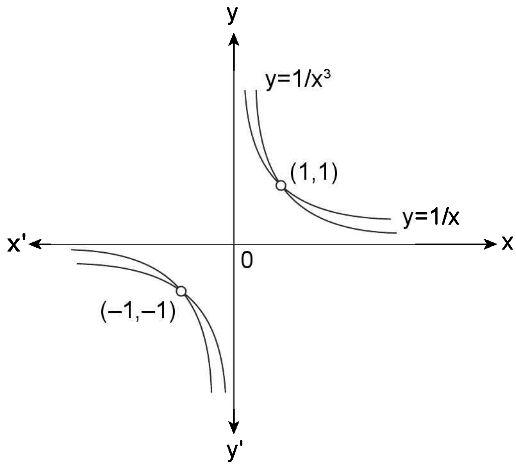


FIGURE 2.52

The graph of function is symmetrical about origin.

For example, $f(x) = x^{-5}$ for $n = 3$.

(c) When $f(x) = (x)^{2n}$; $n \in \mathbb{N}$; i.e., the power is an even natural number. Domain = $\mathbb{R}$

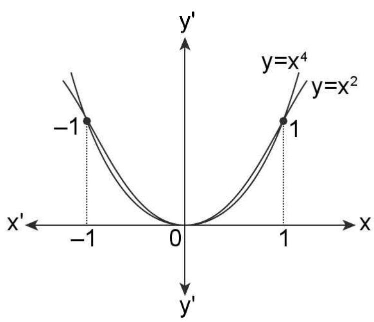


FIGURE 2.53

The graph of function is symmetrical about y-axis

For example $f(x) = x^4$ for $n = 2$.

(d) When $f(x) = (x)^{-2n}$; $n \in \mathbb{N}$; i.e., the power is an even negative integer. Domain = $\mathbb{R} \sim \{0\}$

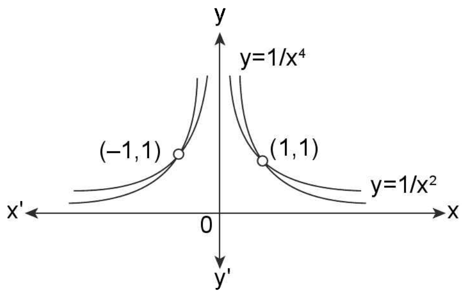


FIGURE 2.54

The graph of function is symmetrical about y-axis

For example $f(x) = x^{-4}$ for $n = 2$.

(e) When $f(x) = (x)^{1/2n-1}$; $n \in \mathbb{N}$; Domain = $\mathbb{R}$

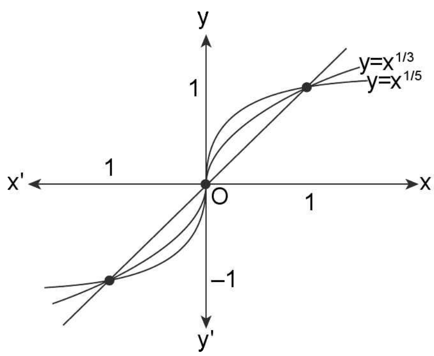


FIGURE 2.55

The graph of function is symmetrical about origin.

For example, $f(x) = x^{1/7}$ for $n = 4$.

(f) When $f(x) = (x)^{-\frac{1}{2n-1}}$; $n \in \mathbb{N}$; Domain = $\mathbb{R} \sim \{0\}$

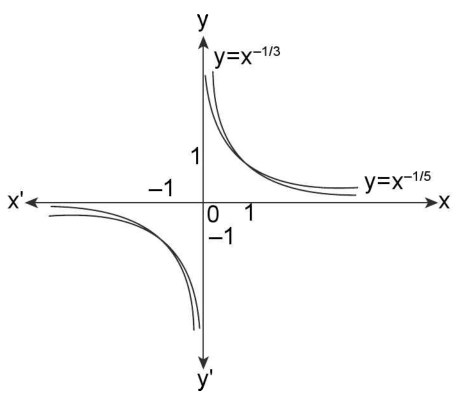


FIGURE 2.56

The graph of function is symmetrical about origin.

For example, $f(x) = x^{-1/7}$ for $n = 4$.

(g) When $f(x) = (x)^{\frac{1}{2n}}$; $n \in \mathbb{N}$; Domain = $[0, \infty)$

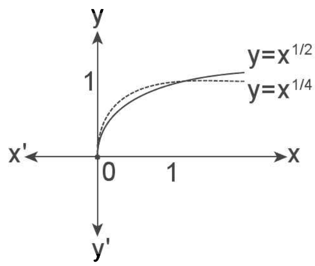


FIGURE 2.57

The graph of function lies only in first quadrant.

For example, $f(x) = x^{1/2} = \sqrt{x}$ for $n = 1$.

(h) When $f(x) = (x)^{-\frac{1}{2n}}$; $n \in \mathbb{N}$; Domain = $(0, \infty)$

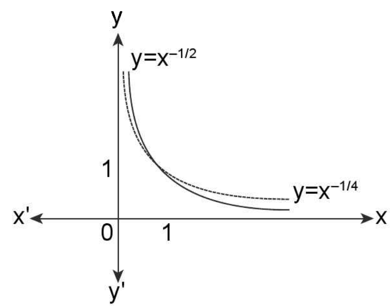


FIGURE 2.58

The graph of function lies only in first quadrant.

For example, $f(x) = x^{-1/4} = \dfrac{1}{\sqrt[4]{x}}$ for $n = 2$.

(i) When $f(x) = (x)^{\frac{2n}{2n+(2m-1)}}$; $n, m \in \mathbb{N}$. Clearly $0 < \dfrac{2n}{2n+(2m-1)} < 1\ \forall\ n, m \in \mathbb{N}$

Domain of function will be $\mathbb{R}$.

Its graph is as shown below as Figure 2.59.

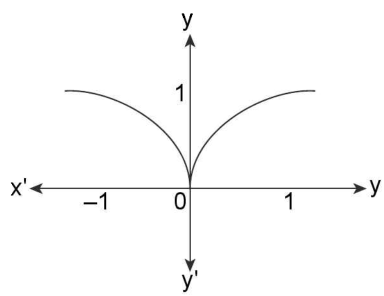


FIGURE 2.59

The graph of function is symmetrical about y-axis

For example, $y = (x)^{2/3}$ for $n = 1, m = 1$

(j) When $f(x) = (x)^{\frac{2n}{2n-(2m-1)}}$; $n, m \in \mathbb{N}$ and $m < \dfrac{2n+1}{2}$.

Clearly $m < \dfrac{2n+1}{2}$

$\Rightarrow 2n - (2m - 1) > 0$ and $2m - 1 \geq 1$

$\Rightarrow 2n > 2n - (2m - 1)$

$\Rightarrow \dfrac{2n}{2n-(2m-1)} > 1$

Domain of function is $\mathbb{R}$.

The graphs of such functions will be as shown in Figure 2.62.

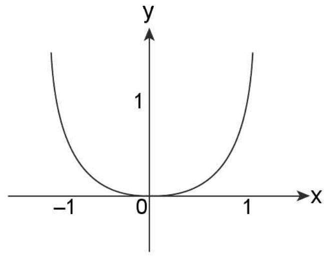


FIGURE 2.60

For example, $y = (x)^{6/5}$ for $n = 3, m = 1$.

(k) When $f(x) = (x)^{\frac{2n-1}{2m-1}}$; $n, m \in \mathbb{N}$; $n < m$; Domain $= \mathbb{R}$.

Clearly $n < m$

$\Rightarrow 2n - 1 < 2m - 1$

$\Rightarrow \dfrac{2n-1}{2m-1} \in (0,1)$.

The graph of function will be as shown in Figure 2.61.

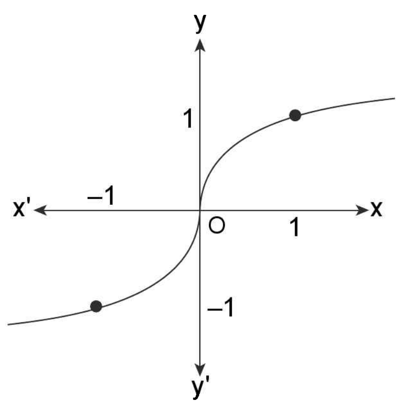


FIGURE 2.61

The graph of function is symmetrical about origin.

For example, $y = (x)^{3/5}$ for $n = 2, m = 3$

(l) When $f(x) = (x)^{\frac{2n-1}{2m-1}}$; $n, m \in \mathbb{N}$; $n > m$; Domain $= \mathbb{R}$.

Clearly $n > m$

$\Rightarrow 2n - 1 > 2m - 1$

$\Rightarrow \dfrac{2n-1}{2m-1} \in (1,\infty)$.

The graph of function will be as shown in Figure 2.62.

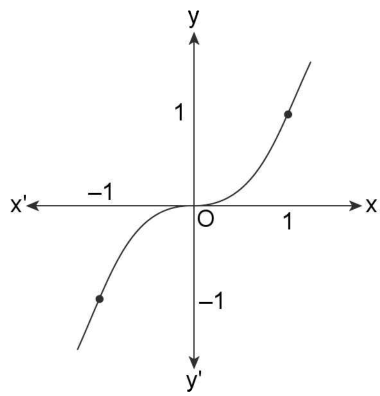


FIGURE 2.62

The graph of function is symmetrical about origin

For example, $y = (x)^{7/3}$ for $n = 4, m = 2$.

(m) When $y = (x)^{-\frac{(2n-1)}{2m}}$; $n, m \in \mathbb{N}$; Domain $= (0, \infty)$

The graph of such functions will be as shown below in Figure 2.63.

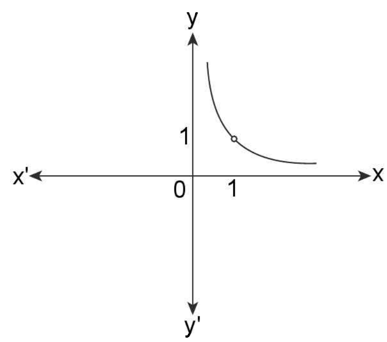


FIGURE 2.63

The graph of function will lie only in first quadrant.

For example, $y = (x)^{-5/6}$ for $n = 3, m = 3$.

(n) When $y = (x)^{-\frac{(2n-1)}{(2m-1)}}$; $n, m \in \mathbb{N}$; Domain $= \mathbb{R} \sim \{0\}$.

The graph of such functions will be as shown below in Figure 2.64.

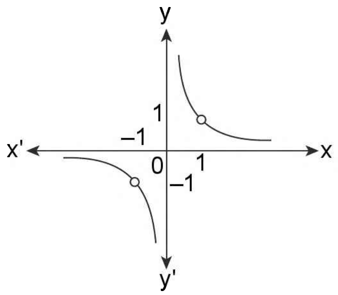


FIGURE 2.64

The graph of function will be symmetrical about origin.

For example, $y = (x)^{-5/7}$ for $n = 3$, $m = 4$.

(o) When $f(x) = (x)^{-\frac{2n}{(2m-1)}}$; $n, m \in \mathbb{N}$; Domain $= \mathbb{R} \sim \{0\}$.

The graph of such functions is shown in Figure 2.65.

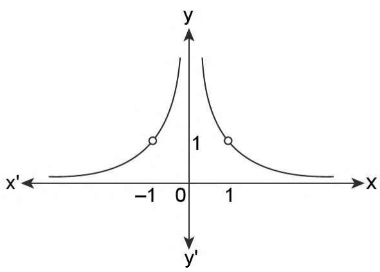


FIGURE 2.65

The graph of function will be symmetrical about y-axis.

For example, $f(x) = (x)^{-4/3}$ for $n = 2$, $m = 2$.

(p) When $f(x) = x^n$; where n is a positive irrational number. Domain $= [0, \infty)$.

The graph of such functions is shown in Figure 2.66.

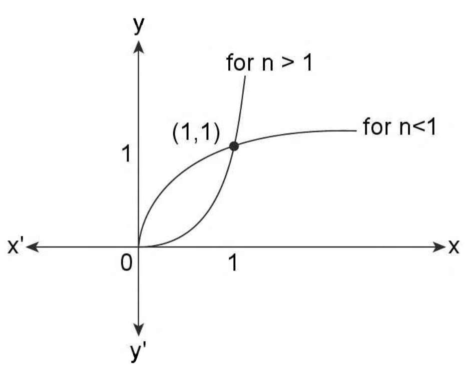


FIGURE 2.66

The graph of function lies in first quadrant.

For example, $f(x) = (x)^{\sqrt{2}}$ for $n = \sqrt{2}$ (> 1) and $f(x) = (x)^{\frac{1}{\sqrt{3}}}$ for $n = \frac{1}{\sqrt{3}} (< 1)$.

(q) When $f(x) = x^n$; where n is a negative irrational number. Domain $= (0, \infty)$.

The graph of such functions wis shown in Figure 2.67.

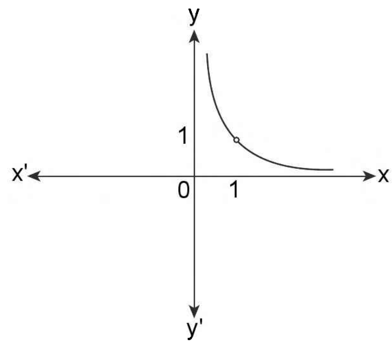


FIGURE 2.67

The graph of function lies in first quadrant.

For example, $f(x) = (x)^{-\sqrt{2}}$ for $n = -\sqrt{2}$

Comparison of the values of $x, x^2, x^3, x^4, \ldots$

For	The relative values are
$x \in (1, \infty)$	$x < x^2 < x^3 < x^4 < \ldots$
$x \in (0, 1)$	$x > x^2 > x^3 > x^4 > \ldots$
$x \in (-1, 0)$	$x < x^3 < x^5 < \ldots$ (–ve values) $x^2 > x^4 > x^6 > \ldots$ (+ve values)
$x \in (-\infty, -1)$	$x > x^3 > x^5 > \ldots$ (–ve values) $x^2 < x^4 < x^6 < \ldots$ (+ve values)

TRANSCENDENTAL FUNCTION

The functions which are not algebraic are called transcendental functions.

e.g., $f(x) = \sin x$, $y = \cos^{-1} x$, $y = \ln x$,

$$y = \sqrt{\ln x - \sin^{-1} x},$$

$$y = \frac{\ln x + \tan x}{\sin^{-1} x + 2^x} \text{ etc.}$$

MODULUS FUNCTION

The modulus function $f(x) = |x|$ is nothing but is magnitude of the real number x and it represents distance of the number

x from zero on the real number line. Since the distance is always non-negative, the modulus of a real number x is always non-negative and is defined

as $|x| = \begin{cases} x & \text{when } x > 0 \\ 0 & \text{when } x = 0 \\ -x & \text{when } x < 0 \end{cases}$

e.g., $|3| = 3$ and $|-6| = -(-6) = 6$.

The graph of modulus function is as shown in Figure 2.68.

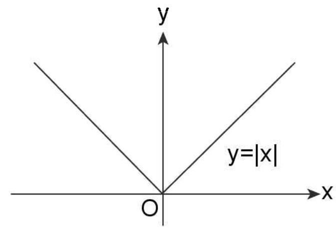


FIGURE 2.68

Clearly, the domain of modulus function is $\mathbb{R}$ and range is the set of non-negative real numbers.

ILLUSTRATION 53: Solve the equation $|x^2 - 4x| + |x - 2| = 6$

SOLUTION: $x^2 - 4x = 0 \quad \Rightarrow x = 0, 4$

and $x - 2 = 0 \quad \Rightarrow x = 2$

Thus, we divide the whole real number line into four parts, i.e., $(-\infty, 0)$; $[0, 2)$, $[2, 4)$, $[4, \infty)$.

For each part let us analyze for the solution.

FIGURE 2.69

Case (i): For $-\infty < x < 0$; we have $x < 0$ and $x < 4$

$\Rightarrow x(x-4) > 0 \quad \Rightarrow x^2 - 4x > 0$

$\Rightarrow |x^2 - 4x| = x^2 - 4x$

Also $x < 0 \quad \Rightarrow x < 2$

$\Rightarrow x - 2 < 0 \quad \Rightarrow |x - 2| = -(x - 2)$

$\therefore$ Given equation become $x^2 - 4x - x + 2 = 6 \quad \Rightarrow x^2 - 5x - 4 = 0$

$\Rightarrow \quad x = \dfrac{5 \pm \sqrt{25+16}}{2} = \dfrac{5 \pm \sqrt{41}}{2}$; but $x < 0 \quad \Rightarrow \quad x = \dfrac{5 - \sqrt{41}}{2}$

Case (ii) For $0 \le x < 2$

we have $x \ge 0$ and $x - 4 < 0$

$\Rightarrow x(x-4) \le 0 \quad \Rightarrow |x^2 - 4x| = -(x^2 - 4x)$

Also $x < 2 \quad \Rightarrow x - 2 < 0 \Rightarrow |x - 2| = -(x - 2)$

$\therefore$ Given equation becomes $-x^2 + 4x - x + 2 = 6$

$\Rightarrow -x^2 + 3x - 4 = 0 \quad \Rightarrow x^2 - 3x + 4 = 0$; which has no real roots.

Case (iii) When $2 \le x < 4$

we have $x > 0$ and $x - 4 < 0$

$\Rightarrow x(x-4) < 0 \quad \Rightarrow (x^2 - 4x) < 0$

$\Rightarrow |x^2 - 4x| = -(x^2 - 4x)$ and $x \ge 2 \quad \Rightarrow x - 2 \ge 0 \quad \Rightarrow |x - 2| = (x - 2)$

$\therefore$ Given equation become $-x^2 + 4x + x - 2 = 6 \Rightarrow -x^2 + 5x - 8 = 0$

$\Rightarrow x^2 - 5x + 8 = 0$ which has no real root.

Case (iv) When $x > 4$

$\Rightarrow x, x - 4 > 0 \quad \Rightarrow x^2 - 4x > 0$

$\Rightarrow |x^2 - 4x| = x^2 - 4x$ and $x > 2 \quad \Rightarrow x - 2 > 0 \quad \Rightarrow |x - 2| = x - 2$

$\therefore$ Given equation become $x^2 - 4x + x - 2 = 6 \quad \Rightarrow x^2 - 3x - 8 = 0$

$$\Rightarrow \quad x = \frac{3 \pm \sqrt{9+32}}{2} \Rightarrow x = \frac{3 \pm \sqrt{41}}{2}\text{; but } x > 4 \quad \Rightarrow x = \frac{3+\sqrt{41}}{2}$$

$\therefore \quad x = \dfrac{5-\sqrt{41}}{2}$ and $x = \dfrac{3+\sqrt{41}}{2}$ are the only solutions of the given equation.

ILLUSTRATION 54: Solve the equation $|x - |2 - x|| - 4x = 8$

SOLUTION: The given equation can be reduced to $|x - (2 - x)| - 4x = 8$ for $x \le 2$ and $|x + (2 - x)| - 4x = 8$ for $x > 2$.

i.e., $|2x - 2| - 4x = 8$ for $x \le 2$ and $2 - 4x = 8$ for $x > 2$ (impossible)

Further, it can be reduced to $2x - 2 - 4x = 8$ for $x \le 2$ and $x \ge 1$

and $-2x + 2 - 4x = 8$ for $x \le 2$ and $x < 1$

i.e., $2x = -10$ for $x \in [1, 2]$ (impossible) and $6x = -6$ for $x \in (-\infty, 1) \Rightarrow x = -1$

Thus, $x = -1$ is the only possible solution of given equation.

ILLUSTRATION 55: Find the values of x satisfying the equation $|||x^2 - 5x + 7| - 1| - 2| = x^2 - 3x - 4$

SOLUTION: $\because$ Disc. of $x^2 - 5x + 7 = 25 - 28 < 0 \quad \Rightarrow \quad x^2 - 5x + 7 > 0\ \forall\ x \in \mathbb{R}$

So, the given equation becomes $||x^2 - 5x + 6| - 2| = x^2 - 3x - 4$

Now $x^2 - 5x + 6 \ge 0$ for $x \in (-\infty, 2] \cup [3, \infty)$ and $x^2 - 5x + 6 < 0$ for $x \in (2, 3)$

$\therefore$ We have $|x^2 - 5x + 4| = x^2 - 3x - 4$ for $x \in (-\infty, 2] \cup [3, \infty)$

and $|x^2 - 5x + 8| = x^2 - 3x - 4$ for $x \in (2, 3)$

But $x^2 - 5x + 4 \ge 0$ for $x \in (-\infty, 1] \cup [4, \infty)$

and $x^2 - 5x + 4 < 0$ for $x \in (1, 4)$ and $x^2 - 5x + 8 > 0\ \forall\ x \in \mathbb{R}$

$\therefore$ The equation reduces to $x^2 - 5x + 4 = x^2 - 3x - 4$ for $x \in (-\infty, 1] \cup [4, \infty)$

and $-(x^2 - 5x + 4) = x^2 - 3x - 4$ for $x \in (1, 2] \cup [3, 4)$

and $x^2 - 5x + 8 = x^2 - 3x - 4$ for $x \in (2, 3)$

$\Rightarrow$ $2x = 8$ for $x \in (-\infty, 1] \cup [4, \infty) \Rightarrow x = 4$ and $2x^2 - 8x = 0$ for $x \in (1, 2] \cup [3, 4)$

That is, no solution and $2x = 12$ for $x \in (2, 3)$, i.e., no solution

$\Rightarrow$ $x = 4$ is the only solution of given equation.

ILLUSTRATION 56: Find the minimum value of the expression $|x - p| + |x - 25| + |x - p - 25|$ for $x \in [p, 25]$ and $p \in [0, 25]$.

SOLUTION: Let $f(x, p) = |x - p| + |x - 25| + |x - p - 25|$ .... (1)

Now we have $0 \le p \le x \le 25$

$\Rightarrow$ $x - p \ge 0$, $x - 25 \le 0$, $x - p - 25 < 0$ as $x - 25 \le 0$ and $p \ge 0$

$\therefore$ $f(x, p) = x - p - x + 25 - (x - p - 25) \quad \therefore f(x, p) = 25 - p - x + p + 25 = 50 - x$

Which has its minimum value 25, when $x = 25$.

ILLUSTRATION 57: Draw the plane region defined by $R = \{(x, y) \in \mathbb{R}^2 : |x + 2| \le 7; |x| \ge 3; |y| \ge 2\}$

SOLUTION: Given $|x + 2| \le 7$

$\Rightarrow$ $-7 \le x + 2 \le 7 \quad \Rightarrow -9 \le x \le 5$ .... (1)

Also $|x| \ge 3 \quad \Rightarrow x \le -3$ or $x \ge 3$ .... (2)

and $|y| \ge 2 \quad \Rightarrow y \le -2$ or $y \ge 2$ .... (3)

$\therefore$ (1), (2); and (3) hold simultaneously on region

$R = \{(x, y) \in \mathbb{R}^2 : x \in [-9, -3] \cup [3, 5]$ and $y \in (-\infty -2] \cup [2, \infty)\}$ as shown in Figure 2.70.

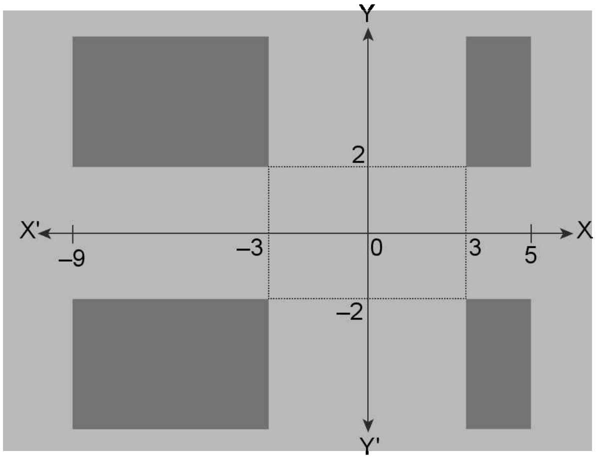


FIGURE 2.70

ILLUSTRATION 58: Solve the following inequalities graphically

(i) $4|x - 1| < (x + 2)$ (ii) $|x^2 - 4x| + |x| \geq 3$

SOLUTION: (i) Let $f(x) = |x - 1|$ and $g(x) = \frac{1}{4}(x+2)$

To solve the inequality $4|x - 1| < (x + 2)$ is equivalent to write $f(x) < g(x)$ i.e., we are to find those values of x for which the graph of $f(x)$ is below the graph of $g(x)$.

This can be done by drawing both the graphs on same $x - y$ plane and finding those points where graph of $f(x)$ is below the graph of $g(x)$ as shown in Figure 2.71.

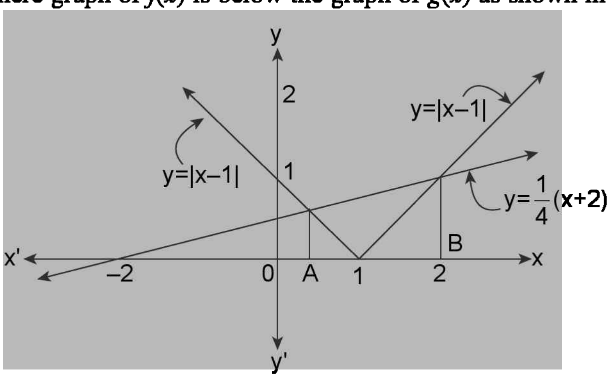


FIGURE 2.71

Clearly graph of $f(x) = |x - 1|$ is below the graph of $g(x) = \frac{1}{4}(x+2)$ for x lying on segment AB of x-axis excluding A and B. And coordinates of A and B are $\left(\frac{2}{5}, 0\right)$ and $(2, 0)$, respectively.

Thus, solution of given inequality is given by $x \in \left(\frac{2}{5}, 2\right)$.

(ii) Given inequality is $|x^2 - 4x| + |x| \geq 3$ or equivalently $|x^2 - 4x| \geq 3 - |x|$

Let $f(x) = |x^2 - 4x|$ and $g(x) = 3 - |x|$. Thus, we are to find those points where $f(x) \geq g(x)$

The graph of $y = f(x)$ and $y = g(x)$ are as shown in Figure 2.72.

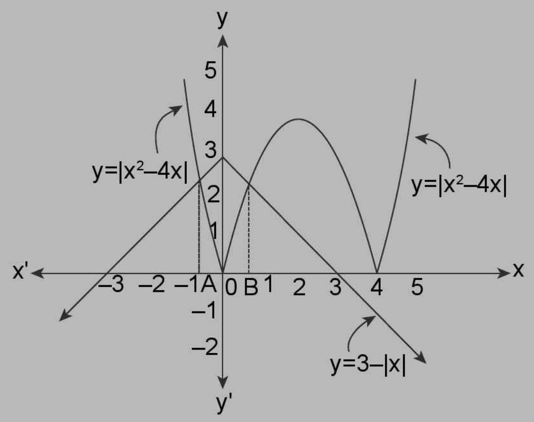


FIGURE 2.72

Clearly for x lying on segment $X'A$ and BX of x-axis, graph of $y = f(x)$ lies on or above the graph of $y = g(x)$. Coordinates of A are given by solving $x^2 - 4x = 3 + x$

$\Rightarrow x^2 - 5x - 3 = 0 \Rightarrow x = \dfrac{5 \pm \sqrt{25+12}}{2} = \dfrac{5 \pm \sqrt{37}}{2}$; But $x < 0$ $\quad \Rightarrow x = \dfrac{5-\sqrt{37}}{2}$

$\therefore$ Coordinates of A are $\left(\dfrac{5-\sqrt{37}}{2}, 0\right)$

Further, coordinates of B are given by solving $-(x^2 - 4x) = 3 - x$

$\Rightarrow x^2 - 5x + 3 = 0 \Rightarrow x = \dfrac{5 \pm \sqrt{25-12}}{2} = \dfrac{5 \pm \sqrt{13}}{2}$, but $x \in (0, 4)$ $\quad \Rightarrow x = \dfrac{5-\sqrt{13}}{2}$

$\therefore$ Coordinates of B are given by $\left(\dfrac{5-\sqrt{13}}{2}, 0\right)$

Thus, the solution of given inequality is given by $\left(-\infty, \dfrac{5-\sqrt{37}}{2}\right) \cup \left(\dfrac{5-\sqrt{13}}{2}, \infty\right)$.

Properties of Modulus of a real number:

1. $|x_1 . x_2 . x_3 . \ldots . x_n| = |x_1| . |x_2| . |x_3| \ldots |x_n| \ \forall \ x_i \in \mathbb{R}$

Proof: Let $x_1, x_2, x_3, \ldots, x_n \in \mathbb{R}$

First of all we shall prove that $|x_1 . x_2| = |x_1| . |x_2|$.... (1)

Let $x_1, x_2 \in \mathbb{R}$, then

$$|x_1.x_2| = \begin{cases} x_1.x_2 & \text{if } x_1.x_2 \geq 0 \\ -(x_1x_2) & \text{if } x_1.x_2 < 0 \end{cases}$$

$$= \begin{cases} (x_1)(x_2) = |x_1||x_2| & \text{if } x_1 \geq 0, x_2 \geq 0 \\ (-x_1)(-x_2) = |x_1||x_2| & \text{if } x_1 < 0, x_2 < 0 \\ (-x_1).(x_2) = |x_1||x_2| & \text{if } x_1 < 0, x_2 \geq 0 \\ (x_1).(-x_2) = |x_1||x_2| & \text{if } x_1 \geq 0, x_2 < 0 \end{cases}$$

Thus, $|x_1 . x_2| = |x_1| . |x_2| \ \forall \ x_1, x_2 \in \mathbb{R}$

Now $|x_1 . x_2 . x_3 \ldots . x_n| = |x_1(x_2 . x_3 \ldots . x_n)|$

$= |x_1| . |x_2 (x_3 \ldots . x_n)|$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$= |x_1| |x_2| |x_3| \ldots . |x_n|$

2. $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|} \ \forall \ x, y \in \mathbb{R}$ and $y \neq 0$.

Proof: $\left|\dfrac{x}{y}\right| = \begin{cases} \dfrac{x}{y} & \text{if } \dfrac{x}{y} \geq 0 \\ -\dfrac{x}{y} & \text{if } \dfrac{x}{y} < 0 \end{cases}$

$$= \begin{cases} \dfrac{(x)}{(y)} & \text{if } x \ge y > 0 \text{ and } \dfrac{-x}{-y} \text{ if } x \le 0, y < 0 \\ \dfrac{(-x)}{(y)} & \text{if } x < 0, y > 0 \text{ and } \dfrac{(x)}{(-y)} \text{ if } x > 0, y < 0 \end{cases}$$

$= \dfrac{|x|}{|y|}$ in each case

Thus, $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$ $\forall\, x, y \in \mathbb{R}$ and $y \ne 0$

3. $|x^n| = |x|^n \ \forall\, n \in \mathbb{Z}$

Proof: Obviously $|x^n| = |x|^n$ holds good for $n = 1$ and $n = 0$ $(x \ne 0)$

Now let $n \in \mathbb{N}$ and $n \ge 2$, then $|x^n| = |\underbrace{x.x.x....x}_{n-\text{times}}|$

$= \underbrace{|x|.|x|.|x|....|x|}_{n-\text{times}} = |x|^n$ [by property (1)]

Next let $n = -m$; $m \in \mathbb{N}$ and $x > 0$, then

$|x^n| = |x^{-m}| = \left|\dfrac{1}{x^m}\right| = \dfrac{|1|}{|x^m|} = \dfrac{1}{|x|^m}$ (by property (2))

$= |x|^{-m} = |x|^n$.

Thus, $|x^n| = |x|^n$ $\forall$ $n \in \mathbb{Z}$ for all real values of x for which LHS and RHS are defined.

REMARK

$|x^t| = |x|^t \ \forall\, t \in \mathbb{Q}$ (set of rational numbers)

$$\left|x^{\frac{p}{q}}\right| = \begin{cases} (x)^{p/q} = |x|^{p/q} & \text{for } x \ge 0; & \dfrac{p}{q} > 0 \\ (x)^{p/q} = |x|^{p/q} & \text{for } x > 0; & \dfrac{p}{q} < 0 \\ (\pm x)^{p/q} = |x|^{p/q} & \text{for } p = \text{even integer}, q \text{ odd integer}, \dfrac{p}{q} > 0; & x \in \mathbb{R} \\ (\pm x)^{p/q} = |x|^{p/q} & \text{for } p = \text{even integer}, q = \text{odd integer}, \dfrac{p}{q} < 0; & x \in \mathbb{R} \sim \{0\} \\ (x)^{p/q} = |x|^{p/q} & \text{for } p = \text{odd integer}, q = \text{even integer}, \dfrac{p}{q} > 0; & x \in [0, \infty) \\ (x)^{p/q} = |x|^{p/q} & \text{for } p = \text{odd integer}, q = \text{even integer}, \dfrac{p}{q} < 0; & x > 0 \\ |x|^{p/q} & \text{for } p, q = \text{odd integer}, \dfrac{p}{q} > 0; & x \in \mathbb{R} \\ (-x)^{p/q} = |x|^{p/q} & \text{for } p, q = \text{odd integer}, \dfrac{p}{q} < 0; & x \in \mathbb{R} \sim \{0\} \end{cases}$$

$= |x|^{p/q}$ *in each case for all those values of x for which* $|x^{p/q}|$ *is defined. Hence, proved.*

4. $|-x| = |x| \ \forall\, x \in \mathbb{R}$

Proof: $|-x| = \begin{cases} -x & \text{iff } -x \ge 0 \\ -(-x) & \text{iff } -x < 0 \end{cases}$

$= \begin{cases} -x & \text{iff } x \le 0 \\ x & \text{iff } x > 0 \end{cases} = |x|$

5. $|x| = \delta \Rightarrow \ x = \delta$ or $x = -\delta$

Proof: Given $|x| = \delta \ge 0$

We know that there exist exactly two real numbers of opposite signs having same positive magnitude.

$\therefore \ |x| = \delta \ge 0 \quad \Rightarrow \quad x = \pm\delta$.

6. $|x| < \delta \ \Rightarrow \ x \in (-\delta, \delta)$ and $|x| > \delta$

$\Rightarrow x \in (-\infty, -\delta) \cup (\delta, \infty)$

Proof: Let $|x| < \delta$

$\Rightarrow$ Absolute value of x is less than δ.

$\Rightarrow$ Distance of x from origin is less than δ.

$\Rightarrow$ x will be at origin or is found either on left side of origin or on right side of origin at a distance less than δ.

$\Rightarrow x > -\delta$ or $x < \delta \Rightarrow x \in (-\delta, \delta)$; Now $|x| > \delta \Rightarrow |x| \nleq \delta$

$\Rightarrow x \notin [-\delta, \delta] \Rightarrow x \in (-\infty, -\delta) \cup (\delta, \infty)$

7. $|x - a| < \delta \Rightarrow x \in (a - \delta, a + \delta)$

Proof: $|x - a| < \delta$

$\therefore$ By above property (5)

$\Rightarrow -\delta < x - a < \delta$

$\Rightarrow a - \delta < x < a + \delta$

$\Rightarrow x \in (a - \delta, a + \delta)$

8. $|x - a| = \delta \quad \Rightarrow \quad x = a + \delta$ or $a - \delta$

Proof: Given $|x - a| = \delta$

By property (4)

$x - a = \pm\delta \quad \Rightarrow \quad x = a \pm \delta$

9. $|x - a| > \delta \quad \Rightarrow \quad x > a + \delta$ or $x < a - \delta$

Proof: Given $|x - a| > \delta$

By property (5)

$x - a < -\delta$ or $x - a > \delta$

$x < a - \delta$ or $x > a + \delta$

10. $\sqrt{x^2} = |x| \ \forall\ x \in \mathbb{R}$

Proof: $\sqrt{x^2}$ = Principal square root of x^2

$= \left|(x^2)^{1/2}\right| = |\pm x| = |x|$

11. $|x| = \max.\{-x, x\} \ \forall\ x \in \mathbb{R}$

Proof: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ .... (i)

when $x \geq 0 \quad \Rightarrow \quad -x \leq 0 \quad \Rightarrow x \geq -x$

$\Rightarrow$ maximum $\{x, -x\} = x$ ... (ii)

and when $x < 0$

$\Rightarrow -x > 0 \quad \Rightarrow -x > x \Rightarrow$ maximum $\{-x, x\} = -x$ (iii)

Thus, from (i), (ii) and (iii), we have $|x|$ = maximum $\{x, -x\}$

12. $|x| = |y| \Leftrightarrow x^2 = y^2$

Proof: Given $|x| = |y|$

$\because \ |x| = |y| \quad \Rightarrow \quad |x|^2 = |y|^2 \quad \Rightarrow \quad x^2 = y^2$

and conversely if $|x|^2 = |y|^2$

$\Rightarrow (|x| - |y|)(|x| + |y|) = 0$

$\Rightarrow |x| = |y|$ or $|x| = -|y|$

(But $|x| = -|y|$ is possible only when $x = y = 0$)

$\Rightarrow |x| = |y| = 0$

Thus, $|x| = |y| \Leftrightarrow |x|^2 = |y|^2$

$\Leftrightarrow x^2 = y^2 \quad (\because |x|^2 = x^2 \ \forall\ x \in \mathbb{R})$

13. $|x + y|$ is not always equal to $|x| + |y|$.

Proof: Given $|x + y|$

$|x + y|$

$= \begin{cases} |x| + |y| & \text{for } x > 0, y > 0 \text{ or } x < 0, y < 0 \\ \big||x| - |y|\big| < |x| + |y| & \text{for } x > 0, y < 0 \text{ or } x < 0, y > 0 \end{cases}$

Thus, $|x + y|$ is not always equal to $|x| + |y|$.

14. (Triangle inequality) $|x + y| \leq |x| + |y|$ for all real x and y, inequality holds if $x.y < 0$. That is, x and y are of opposite signs, equality holds if $x.y \geq 0$. That is, x and y are of same sign or at least one of x and y is zero.

Proof: $x \leq |x|$ and $y \leq |y|$

$\Rightarrow x + y \leq |x| + |y|$ .... (1)

Also $-x \leq |x|$ and $-y \leq |y|$

$\Rightarrow -(x + y) \leq |x| + |y|$

$\Rightarrow (x + y) \geq -(|x| + |y|)$ .... (2)

$\therefore$ from (1) and (2) we have

$-(|x| + |y|) \leq (x + y) \leq (|x| + |y|)$

$\Rightarrow |x + y| \leq |x| + |y| \quad (\because -\delta \leq x \leq \delta \Rightarrow |x| \leq \delta)$

Here $|x + y| = \begin{cases} (x + y) & \text{for } (x + y) \geq 0 \\ -(x + y) & \text{for } (x + y) < 0 \end{cases}$

$= \begin{cases} |x| + |y| & \text{for } x \geq 0, \ y \geq 0 \\ x + y < |x| + |y| & \text{for } x + y > 0 \ \& \ x.y < 0 \\ |x| + |y| & \text{for } x < 0, \ y < 0 \\ -x - y < |x| + |y| & \text{for } x + y < 0 \ \& \ x.y < 0 \end{cases}$

Thus, $|x + y| = |x| + |y|$ for x and y of same sign or at least one of x and y is zero and $|x + y| < |x| + |y|$ for x and y of opposite signs. i.e., $|x + y| = |x| + |y|$ for $x.y \geq 0$ and $|x + y| < |x| + |y|$ for $x.y < 0$

15. $|x - y| \leq |x| + |y|$ for real x and y, inequality holds if $x.y > 0$, i.e., x and y are of same sign, equality holds. If $x.y \leq 0$, That is, x and y are of opposite sign or at least one of x and y is zero.

Proof: By triangle inequality $|x + y| \leq |x| + |y| \ \forall\ x, y \in \mathbb{R}$

$\Rightarrow |x + (-y)| \leq |x| + |-y| \ \forall\ x, -y \in \mathbb{R}$

$\Rightarrow |x - y| \leq |x| + |y| \ \forall\ x, y \in \mathbb{R}$

ILLUSTRATION 59: Solve the following equations/inequations.

(i) $|x^2 - 4x| + |5x - 10| = |x^2 + x - 10|$ (ii) $|x^2 - 3x| + |2x - 6| = |x^2 - 5x + 6|$

(iii) $|x^2 - 4x| + |x + 6| > |x^2 - 5x - 6|$ (iv) $|x^2 - 7x| + |6x - 2| > |x^2 - x - 2|$

SOLUTION: (i) $|x^2 - 4x| + |5x - 10| = |x^2 + x - 10| = |(x^2 - 4x) + (5x - 10)|$

$\because \quad |a| + |b| = |a + b| \Leftrightarrow a.b \geq 0$

$\therefore \quad (x^2 - 4x)(5x - 10) \geq 0$

$\Rightarrow \quad x(x - 4)(x - 2) \geq 0$

$\Rightarrow \quad x \in [0, 2] \cup [4, \infty)$

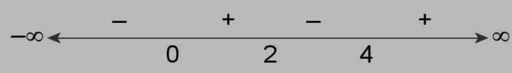


FIGURE 2.73

(ii) $|x^2 - 3x| + |2x - 6| = |x^2 - 5x + 6| = |(x^2 - 3x) - (2x - 6)|$

$\because \quad |a| + |b| = |a - b| \quad \Leftrightarrow a . b \leq 0$

$\therefore \quad (x^2 - 3x)(2x - 6) \leq 0 \quad \Rightarrow x(x - 3)(x - 3) \leq 0 \quad \Rightarrow x(x - 3)^2 \leq 0$

$\Rightarrow \quad x \leq 0 \qquad \therefore \quad x \in (-\infty, 0]$

(iii) $|x^2 - 4x| + |x + 6| > |x^2 - 5x - 6| = |(x^2 - 4x) - (x + 6)|$

$\because \quad |a| + |b| > |a - b| \Leftrightarrow a.b > 0$

$\Rightarrow \quad (x^2 - 4x)(x + 6) > 0$

$\Rightarrow \quad x(x - 4)(x + 6) > 0$

$\Rightarrow \quad x \in (-6, 0) \cup (4, \infty)$

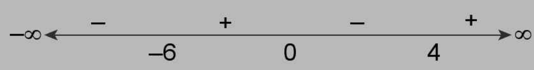


FIGURE 2.74

(iv) $|x^2 - 7x| + |6x - 2| > |x^2 - x - 2| = |(x^2 - 7x) + (6x - 2)|$

$\because \quad |a| + |b| > |a + b|$

$\Leftrightarrow \quad a.b < 0$

$\Rightarrow \quad (x^2 - 7x)(6x - 2) < 0 \quad \Rightarrow \quad x(x - 7)$
$(3x - 1) < 0 \Rightarrow x \in (-\infty, 0) \cup (1/3, 7)$

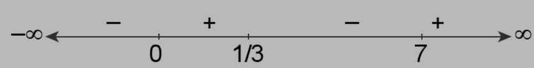


FIGURE 2.75

16. $||x| - |y|| \leq |x + y|$ for real x and y. Equality holds if x and y are of opposite signs and for same sign inequality holds.

Proof: $|x| = |(x + y) + (-y)| \leq |x + y| + |-y|$ (By triangle inequality)

$\Rightarrow \quad |x| \leq |x + y| + |y|$

$\Rightarrow \quad |x| - |y| \leq |x + y| \qquad \ldots (1)$

Also $|y| = |(x + y) + (-x)| \leq |x + y| + |-x|$ (By triangle inequality)

$\Rightarrow \quad |y| \leq |x + y| + |x|$

$\Rightarrow \quad |y| - |x| \leq |x + y| \qquad \Rightarrow \quad -(|x| - |y|) \leq |x + y|$

or $\quad |x| - |y| \geq -|x + y| \qquad \ldots (2)$

$\therefore$ from (1) and (2)

$-|x + y| \leq |x| - |y| \leq |x + y|$

$\Rightarrow \quad ||x| - |y|| \leq |x + y| \qquad (\because -\delta \leq x \leq \delta \Rightarrow |x| \leq \delta)$

$\Rightarrow \quad |x + y| \geq ||x| - |y|| \ \forall\, x, y \in \mathbb{R}$

17. $||x| - |y|| \leq |x - y|$ for real x and y. Equality holds if x and y are of same sign and for opposite signs inequality holds.

Proof: By Property (15), $||x| - |y|| \leq |x + y|$ for real x and y. Replacing y by $-y$, we get $||x| - |-y|| \leq |x - y|$ for real x and $-y$

$\Rightarrow \quad ||x| - |y|| \leq |x - y| \ \forall\, x, y \in \mathbb{R}$

ILLUSTRATION 60: Find the interval in which x can lie in order to satisfy the following inequalities:

(a) $|x - 3| < 5$ (b) $|x + 1/2| \geq \frac{3}{2}$

(c) $|3 - x| \leq \frac{5}{2}$ (d) $|x^2 - 4x + 5| = |x^2 - 3x| + |5 - x|$

SOLUTION: (a) $|x - 3| < 5 \quad \Rightarrow -5 < x - 3 < 5 \quad \Rightarrow -2 < x < 8$

(b) $\left|x + \frac{1}{2}\right| \geq \frac{3}{2} \quad \Rightarrow x + \frac{1}{2} \geq \frac{3}{2}$ or $x + \frac{1}{2} \leq -\frac{3}{2} \quad \Rightarrow x \geq 1$ or $x \leq -2$

(c) $|3-x| \le \frac{5}{2} \Rightarrow -\frac{5}{2} \le 3-x \le \frac{5}{2} \Rightarrow -\frac{5}{2} \le x-3 \le \frac{5}{2}$

$\Rightarrow 3-\frac{5}{2} \le x \le \frac{5}{2}+3 \Rightarrow \frac{1}{2} \le x \le \frac{11}{2}$

(d) $(x^2-3x)(5-x) \ge 0 \Rightarrow x(x-5)(x-3) \le 0 \Rightarrow x \in (-\infty, 0] \cup [3, 5]$

ILLUSTRATION 61: Solve the equation $x^2 - 5|x| + 6 = 0$

SOLUTION: The given equation is equivalent to $\begin{cases} x^2-5x+6=0 & \text{if } x \ge 0 \\ x^2+5x+6=0 & \text{if } x<0 \end{cases} \Rightarrow \begin{cases} (x-2)(x-3)=0 & \text{if } x \ge 0 \\ (x+2)(x+3)=0 & \text{if } x<0 \end{cases}$

Hence, the solutions of the given equation are $x_1 = 2, x_2 = 3, x_3 = -2, x_4 = -3$.

ILLUSTRATION 62: Solve $|x^2 - 4| + |x^2 - 9| = 5$.

SOLUTION: There are three cases:

Case I: When $x^2 < 4$, i.e., $-2 < x < 2$

In this case $|x^2 - 4| = -(x^2 - 4)$ and $|x^2 - 9| = -(x^2 - 9)$

So, the given equation becomes $-(x^2 - 4) - (x^2 - 9) = 5$

$\Rightarrow x^2 = 4 \qquad \Rightarrow x = \pm 2$, no solution

Case II: When $4 \le x^2 < 9$, i.e., $-3 < x \le -2$ or $2 \le x < 3$

In this case $|x^2 - 4| = (x^2 - 4)$ and $|x^2 - 9| = -(x^2 - 9)$

So, the given equation becomes $x^2 - 4 - (x^2 - 9) = 5 \Rightarrow 5 = 5$

Thus, the given equation becomes an identity in this case, i.e., all values of x such that $-3 < x \le -2$ or $2 \le x < 3$ satisfy the given equation.

Case III: When $x^2 \ge 9$, i.e., $x \le -3$ or $x \ge 3$

In this case $|x^2 - 4| = (x^2 - 4)$ and $|x^2 - 9| = (x^2 - 9)$

So, the given equation becomes $(x^2 - 4) + (x^2 - 9) = 5 \Rightarrow x^2 = 9 \qquad \Rightarrow x = \pm 3$

Clearly both of these values satisfy $x \le -3$ or $x \ge 3$

$\therefore$ Solution set $= \{x : -3 \le x \le -2 \text{ or } 2 \le x \le 3\}$

SIGNUM FUNCTION

The function $f(x)$ defined by $f(x) = \begin{cases} \frac{|x|}{x}; & \text{if } x \ne 0 \\ 0; & \text{if } x = 0 \end{cases}$

i.e., $f(x) = \text{sgn}(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x = 0 \\ -1 \text{ if } x < 0 \end{cases}$

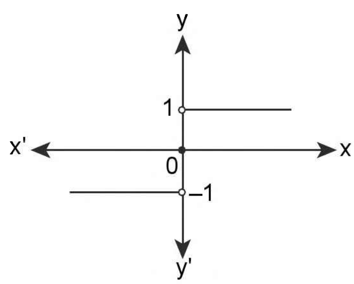


FIGURE 2.76

ILLUSTRATION 63: Solve the following equations:

(i) $(\text{sgn } x)^2 - 1 = 0$

(ii) $(\text{sgn } x) + 1 = 0$

(iii) $\frac{(\text{sgn } x - 1)(\text{sgn } x + 2)}{(\text{sgn } x - 3)} = 0$

(iv) $\text{sgn}(x - 1) = -1$

(v) $\text{sgn}|x^2 - 1| = 0$

SOLUTION : (i) $(\text{sgn } x)^2 - 1 = 0 \Rightarrow \text{sgn}(x) = \pm 1 \Rightarrow x \in \mathbb{R} \sim \{0\}$

(ii) $\text{sgn}x + 1 = 0 \Rightarrow \text{sgn}x = -1 \Rightarrow x < 0 \Rightarrow x \in (-\infty, 0)$

(iii) $\dfrac{(\text{sgn } x-1)(\text{sgn } x+2)}{(\text{sgn } x-3)} = 0 \Rightarrow \text{sgn } x = 1, -2$

$\text{sgn } x = 1, -2$; but $\text{sgn} \in \{-1, 0, 1\}$ $\therefore$ $\text{sgn } x \neq -2$

$\therefore$ $\text{sgn } x = 1 \Rightarrow x > 0 \quad \therefore x \in (0, \infty)$

(iv) $\text{sgn }(x - 1) = -1 \Rightarrow x - 1 < 0 \Rightarrow x < 1$

(v) $\text{sgn } |x^2 - 1| = 0 \Rightarrow |x^2 - 1| = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1.$

CONSTANT FUNCTION

A function $f : A \to B$ is said to be a constant function if every element of A has the same f – image in B. That is, $f : A \to B; f(x) = c, \forall\, x \in A, c \in B$ is a constant function.

A constant function may be one–one (when domain contains only one element), many–one (when domain contains more than one elements), onto (when co-domain or range contains only single element i.e., c), into (when co-domain contains at-least one more element other than c).

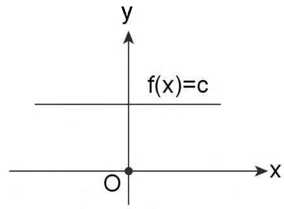


FIGURE 2.77

ILLUSTRATION 64: Which of the following are constant functions on $\mathbb{R}$?. If not on $\mathbb{R}$, then find the domain on which they are constant.

(i) $\text{sgn } x + sgn(-x)$

(ii) $\sin^2 x + \cos^2 x$

(iii) $\text{cosec}^2 x - \cot^2 x$

(iv) $\dfrac{\left(\sqrt{2x-1}+x\right)\left(\sqrt{2x-1}-x\right)}{(x-1)^2}$

(v) $\begin{cases} |x|+x & if\ x<0 \\ |x|-x & if\ x\geq 0 \end{cases}$

(vi) $\ln e^7$

(vii) $e^{ln\,10}$

SOLUTION: (i) $f(x) = sgn\ x + sgn\ (-x) = -1 + 1 = 0$, for $x < 0$ and $f(x) = 0$ for $x = 0$

and $f(x) = 1 + (-1) = 0$ for $x > 0$

$\Rightarrow f(x) = 0\ \forall\, x \in \mathbb{R} \quad \Rightarrow f(x)$ is a constant function on $\mathbb{R}$

(ii) $f(x) = \sin^2 x + \cos^2 x = 1\ \forall\, x \in \mathbb{R}$

$\therefore$ $f(x)$ is a constant function on $\mathbb{R}$.

(iii) $f(x) = \text{cosec}^2 x - \cot^2 x = 1\ \forall\, x \in \mathbb{R} \sim \{n\pi : n \in \mathbb{Z}\}$

$\therefore$ $f(x)$ is not constant function on $\mathbb{R}$; but it is so on $\mathbb{R} \sim \{n\pi\ ;\ n \in \mathbb{Z}\}$

(iv) $f(x) = \dfrac{(\sqrt{2x-1}+x)\,(\sqrt{2x-1}-x)}{(x-1)^2} = \dfrac{(2x-1)-x^2}{(x-1)^2} = -\dfrac{(x-1)^2}{(x-1)^2} = -1$ for $x \in \left[\dfrac{1}{2}, \infty\right) \sim \{1\}$

Therefore $f(x)$ is not a constant function on $\mathbb{R}$, *but* $f(x)$ is constant function on $\left[\dfrac{1}{2}, \infty\right) \sim \{1\}$

(v) $f(x) = \begin{cases} |x|+x & \text{if } x<0 \\ |x|-x & \text{if } x\geq 0 \end{cases} = \begin{cases} -x+x & \text{if } x<0 \\ x-x & \text{if } x\geq 0 \end{cases}$

$\Rightarrow f(x) = 0\ \forall\, x \in \mathbb{R} \quad \Rightarrow f(x)$ is constant function $\forall\, x \in \mathbb{R}$.

IDENTITY FUNCTION

The function $f: A \to B$ defined by $f(x) = x$; $x \in A$; (where A is a subset of B) is called the identity function.

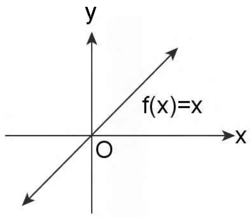


FIGURE 2.78

If $B = A$, then the function is called identity function on set A and is denoted by I_A. Identity function I_A defined on A is a bijection.

ILLUSTRATION 65: Which of the following functions is/are identity function(s) on $\mathbb{R}$?

(i) $f(x) = \dfrac{2}{\pi}(\sin^{-1} x + \cos^{-1} x)x$ (ii) $f(x) = \dfrac{2}{\pi}(\tan^{-1} x + \cot^{-1} x)x$

(iii) $f(x) = e^{\ln x}$ (iv) $f(x) = \ln e^x$

(v) $f(x) = \cos^{-1}(\cos x)$ (vi) $f(x) = \cos(\cos^{-1} x)$

(vii) $f(x) = \tan(\tan^{-1} x)$ (viii) $f(x) = \tan^{-1}(\tan x)$

SOLUTION: (i) $f(x) = \dfrac{2}{\pi}(\sin^{-1} x + \cos^{-1} x)x = \dfrac{2}{\pi}\dfrac{\pi}{2}.x = x \quad \forall\ x \in [-1, 1]$

$\therefore$ $f(x) = x\ \forall\ x \in [-1, 1]$ but not $\forall\ x \in \mathbb{R}$

$\therefore$ $f(x)$ is not identity function on $\mathbb{R}$.

(ii) $f(x) = \dfrac{2}{\pi}(\tan^{-1} x + \cot^{-1} x)x = \dfrac{2}{\pi}\dfrac{\pi}{2}.x = x\ \forall\ x \in \mathbb{R}$

$\therefore$ $f(x) = x\ \forall\ x \in \mathbb{R}$.

$\Rightarrow$ $f(x)$ is an identity function on $\mathbb{R}$.

(iii) $f(x) = e^{\ln x} = x$ for all positive real numbers x

$\therefore$ $f(x) = x$ for all $x > 0$ but not $\forall\ x \in \mathbb{R}$

$\therefore$ $f(x)$ is not an identity function on $\mathbb{R}$.

(iv) $f(x) = \ln e^x = x\ \forall\ x \in \mathbb{R}$

$\therefore$ $f(x) = \ln e^x$ is an identity function on $\mathbb{R}$.

(v) $f(x) = \cos^{-1}(\cos x) = x\ \forall\ x \in [0, \pi]$ but not $\forall\ x \in \mathbb{R}$

$\therefore$ $f(x)$ is not an identity function on $\mathbb{R}$.

(vi) $f(x) = \cos(\cos^{-1} x) = x\ \forall\ x \in [-1, 1]$ but not $\forall\ x \in \mathbb{R}$

$\therefore$ $f(x)$ is not an identity function on $\mathbb{R}$.

(vii) $f(x) = \tan(\tan^{-1} x) = x\ \forall\ x \in \mathbb{R}$

$\therefore$ $f(x)$ is an identity function on $\mathbb{R}$.

(viii) $f(x) = \tan^{-1}(\tan x) = x\ \forall x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ but not $\forall\ x \in \mathbb{R}$

$\therefore$ $f(x)$ is not an identity function on $\mathbb{R}$.

■ EQUAL OR IDENTICAL FUNCTIONS

Two functions f and g are said to be equal if

1. The domain of f = the domain of g.
2. The range of f = the range of g.
3. $f(x) = g(x)$ for every x belonging to their common domain. For example, $f(x) = 1/x$ and $g(x) = x/x^2$ are identical functions.

   $f(x) = \log(x^2)$ and $g(x) = 2\log(x)$ are not-identical functions as domain of $f(x) = (-\infty, \infty) \sim \{0\}$ whereas that of $g(x) = (0, \infty)$.

REMARK

Identical functions have same graph.

ILLUSTRATION 66: State whether the following pair of functions $f(x)$ and $g(x)$ are equal/identical or not. Explain your answer with proper argument.

(a) $f(x)=\dfrac{x^2-4}{x^2+3x+2};\ g(x)=\dfrac{x-2}{x+1}$ (b) $f(x)=\sqrt{x^2-2x+1};\ g(x)=|x-1|$

(c) $f(x)=\sec x.\cot x;\ g(x)=\operatorname{cosec} x$

SOLUTION: (a) Although $f(x)=\dfrac{x^2-4}{x^2+3x+2}$ can be simplified by striking off the common factor $(x + 2)$ from numerator and denominator and one may conclude that $f(x)=\dfrac{(x-2)(x+2)}{(x+2)(x+1)}$ $=\dfrac{x-2}{x+1}=g(x)$, which is indeed not true. Because

$$f(x)=\begin{cases}\dfrac{x-2}{x+1} & \text{if } x\in\mathbb{R}\sim\{-1,-2\}\\ \text{not defined if } x=-1 \text{ or } -2\end{cases},\ \text{whereas}\ g(x)=\begin{cases}\dfrac{x-2}{x+1} & \forall x\in\mathbb{R}\sim\{-1\}\\ \text{not defined for } x=-1\end{cases}$$

Therefore, $f(x)$ and $g(x)$ are not identical functions because of having different natural domains, however, $f(x)$ and $g(x)$ are identical on their common domain, i.e., $\mathbb{R} \sim \{-1, -2\}$

(b) $f(x)=\sqrt{x^2-2x+1}=\sqrt{(x-1)^2}$, which is defined $\forall\ x\in\mathbb{R}$

And further $f(x)=\sqrt{(x-1)^2}=|x-1|\left(\because \sqrt{x^2}=|x|\right)$

Also $g(x) = |x - 1|$ has its domain $\mathbb{R}$

Thus, $f(x)$ and $g(x)$ have same analytical formula and having same natural domain $\mathbb{R}$.

Hence, $f(x)$ and $g(x)$ are identical functions.

(c) $f(x) = \sec x.\ \cot x;\ g(x) = \operatorname{cosec} x$

Hence, $f(x)=\begin{cases}\dfrac{1}{\cos x}.\dfrac{\cos x}{\sin x}; \sin x\neq 0, \cos x\neq 0\\ \text{not defined if either } \sin x=0 \text{ or } \cos x=0\end{cases}$

$$\Rightarrow \begin{cases}\operatorname{cosec} x;\ \sin x\neq 0, \cos x\neq 0\\ \text{not defined; otherwise}\end{cases}$$

$$\Rightarrow \begin{cases}\operatorname{cosec} x; x\in\mathbb{R}\sim\left\{n\pi,(2m+1)\dfrac{\pi}{2}; m,n\in\mathbb{Z}\right\}\\ \text{not defined}; x\in\left\{n\pi,(2m+1)\dfrac{\pi}{2}; m,n\in\mathbb{Z}\right\}\end{cases}$$

Further $g(x) = \text{cosec } x;\ x \in \mathbb{R} \sim \{n\pi;\ n \in \mathbb{Z}\}$

$\therefore$ $f(x)$ and $g(x)$ are not identical due to having different natural domains, however, they are identical on their common domain, i.e., $\mathbb{R} \sim \left\{n\pi, (2m+1)\frac{\pi}{2}; n, m \in \mathbb{Z}\right\}$

TEXTUAL EXERCISE–5: (SUBJECTIVE)

1. For what set of values of x the following equations hold?

(a) $|x| = 5$ (b) $|x| = -2$

(c) $x + |x| = 0$ (d) $x + |x| = 2x$

(e) $\frac{x}{|x|} = -1$ (f) $3x\sqrt{2} = \sqrt{18x^2}$

(g) $x\,|x| - x^2 = 0$ (h) $|x| < -x$

2. Solve the following equations for x:

(a) $|x| + x^2 + 1 = 0$ (b) $|5x^2 - 3| = 2$

(c) $\left|\frac{x+4}{x+2}\right| = 3$ (d) $|x^2 - 4x| = 5$

(e) $|x + 1| + 2 = 2$ (f) $|3x - 4| = 1/2$

(g) $|x + 2| = 2(3 - x)$ (h) $|x| = -3x - 5$

(i) $x^2 + |x - 1| = 1$.

3. Prove that $\sqrt{x^2 - 2x + 1} - \sqrt{x^2 + 2x + 1} = \begin{cases} 2 \text{ if } x \le -1 \\ -2x \text{ if } x \in (-1, 1) \\ -2 \text{ if } x \ge 1 \end{cases}$

4. Solve the following inequalities for x:

(a) $|x| > 2$ (b) $|x - 1| > 3$

(c) $|x - 2| < 1$ (d) $|x + 1| \ge 2$

(e) $|x - 1| < 5$

5. Solve the following equations and inequalities:

(a) $|x - 1| + |x - 3| = 2$

(b) $|x| + |x + 5| = 5$

(c) $|x - 1| + |x - 4| = 2$

(d) $|x^2 - 2x| + |x - 4| = |x^2 - 3x + 4|$

(e) $|x^2 - 2x| + |x - 4| < |x^2 - 3x + 4|$

6. Simplify the expression $\sqrt{9 - 6a + a^2} + \sqrt{9 + 6a + a^2}$ if $a < -3$

7. Which of the following functions are identical functions?

(a) $f(x) = \text{cosec}\, x;\ g(x) = \frac{1}{\sin x}$

(b) $f(x) = \tan x;\ g(x) = \frac{}{\cot}$

(c) $f(x) = \ln e^x;\ g(x) = x$

(d) $f(x) = \sec x;\ g(x) = \frac{1}{\cos x}$

8. Identify whether following pairs of functions are identical or non-identical.

(a) $f(x) = \ln x;\ g(x) = \frac{1}{\log_x e}$

(b) $f(x) = \sqrt{x^2 - 1};\ g(x) = \sqrt{x - 1}\cdot\sqrt{x + 1}$

(c) $f(x) = \sqrt{1 - x^2};\ g(x) = \sqrt{1 - x}\cdot\sqrt{1 + x}$

9. Solve the following equations.

(i) $\text{sgn}\,(x^2 + 2) = 1$

(ii) $\text{sgn}\,(x^2 + 2) = -1$

(iii) $\text{sgn}\,(x^2 - 9) + \text{sgn}\,(x + 3) = 0$

(iv) $|\text{sgn}\,(x - 1)| = 1$.

10. If $f(x)$ is an identity function on $\mathbb{R}$, then solve the following inequations.

(i) $|f(x) - 5| \ge 2$

(ii) $|\text{sgn}\,(f(x) - 5)| > 0$

(iii) $\text{sgn}\,[f(\tan^{-1} x)] = 1$.

Answer Keys

1. (a) $\{-5, 5\}$ (b) $\{\}$ (c) $(-\infty, 0]$ (d) $[0, \infty)$ (e) $(-\infty, 0)$

(f) $[0, \infty)$ (g) $[0, \infty)$ (h) ϕ

2. (a) $\{\}$ (b) $\left\{-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \pm 1\right\}$ (c) $\{-1, -5/2\}$ (d) $\{-1, 5\}$ (e) $x = -1$

(f) $x = 7/6, 3/2$ (g) $x = 4/3$ (h) $x = -5/2$ (i) $x = 0, 1$

4. (a) $(-\infty,-2)\cup(2,\infty)$ (b) $(-\infty,-2)\cup(4,\infty)$ (c) $x\in(1,3)$ (d) $x\in(-\infty,-3]\cup[1,\infty)$
(e) $x\in(-4,6)$

5. (a) [1, 3] (b) [–5, 0] (c) {} (d) $x\in(-\infty,0]\cup[2,4]$ (e) $x\in\{\}$

6. $-2a$

7. (a) Identical (b) Non-identical (c) Identical (d) Identical

8. (a) Non-identical (b) Non-identical (c) Identical

9. (i) $\mathbb{R}$ (ii) ϕ (iii) $(-\infty, 3)$ (iv) $\mathbb{R} - \{1\}$

10. (i) $x \in (-\infty, 3] \cup [7, \infty)$ (ii) $x \in \mathbb{R} \sim \{5\}$ (iii) $x \in (0, \infty)$

TEXTUAL EXERCISE–5: (OBJECTIVE)

1. $\sqrt{2-|x|}+\sqrt{1+|x|}$ is defined for

(a) [–2, 2] (b) (–2, 2)
(c) $(-\infty, -1) \cup (1, 2)$ (d) None of these

2. $f(x)=\sqrt{x^2-|x|-2}$ is defined for

(a) $\mathbb{R} - (-2, 2)$ (b) $\mathbb{R} - [-2\ 2)$
(c) $\mathbb{R} - [-2, 2]$ (d) $\mathbb{R} - (-2, 2]$

3. The range of the function $f(x) = |x - 1| + |x - 2|$, $-1 \le x \le 3$, is

(a) [1, 3] (b) [1, 5]
(c) [3, 5] (d) None of these

4. Which of the following statements is incorrect?

(a) $x \operatorname{sgn} x = |x|$
(b) $|x| \operatorname{sgn} x = x$
(c) $x (\operatorname{sgn} x) (\operatorname{sgn} x) = x$
(d) $|x| (\operatorname{sgn} x)^3 = |x|$

5. If $f(x)=\left\{(\operatorname{sgn} x)^{\operatorname{sgn} x}\right\}^n$; n is an odd integer. Then

(a) $f(x)$ is an odd function
(b) $f(x)$ is an even function
(c) $f(x) = 0$
(d) None of these

6. If $f(x)=\left(\dfrac{x}{1-|x|}\right)^{1/2001}$, then D_f is

(a) $(-\infty, -1) \cup (1, \infty)$ (b) $(-1, 1)$
(c) $\mathbb{R}$ (d) $\mathbb{R} - \{-1, 1\}$

7. If $f(x)=\left(\dfrac{x}{1-|x|}\right)^{1/2002}$ then D_f is

(a) $\mathbb{R} - \{-1, 1\}$ (b) $(-\infty, 1)$
(c) $(-\infty, -1) \cup [0, 1)$ (d) None of these

8. If $f(x)=\dfrac{\sqrt{|\tan x|+\tan x}}{\sqrt{3x}}$ is defined for

(a) $\mathbb{R}$
(b) $\mathbb{R}-\{1/3\}$
(c) $\mathbb{R}^+-\left\{n\pi-\dfrac{\pi}{2}; n\in W\right\}$
(d) None of these

9. In which of the following pairs, the functions are identical?

(a) $f(x)=\sqrt{x^2}$; $g(x)=\left(\sqrt{x}\right)^2$
(b) $f(x)=\dfrac{1}{\sqrt{x^2}}$; $g(x)=\dfrac{x}{x^2}$
(c) $f(x) = \log(x - 1) + \log(x - 2)$; $g(x) = \log(x - 1)(x - 2)$
(d) $f(x) = \sin^2 x + \cos^2 x$; $g(x) = 1$

Answer Keys

1. (a) **2.** (a) **3.** (b) **4.** (d) **5.** (a) **6.** (d) **7.** (c) **8.** (c) **9.** (d)

EXPONENTIAL FUNCTION

If a is positive real number except for 1, then a^x; where $x \in \mathbb{R}$ is always positive and it is called exponential function of x. a is called base and x is called index.

The graph of exponential functions are as shown in Figure 2.80.

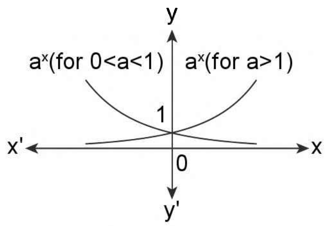


FIGURE 2.79

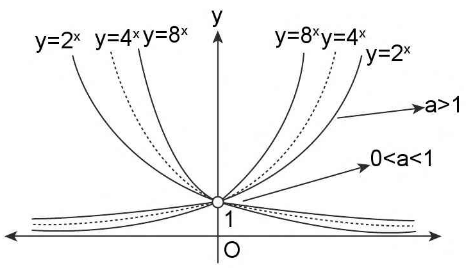


FIGURE 2.80

PROPERTIES OF EXPONENTIAL FUNCTION

(i) Domain of exponential function $y = a^x$; $D_f : (-\infty, \infty)$;
Range of exponential function $= R_f : (0, \infty)$

$\because$ a^x is defined $\forall\ x \in \mathbb{R}$ and $a^x > 0$

(ii) $\left.\begin{array}{l} f(x+y) = f(x)\,.\,f(y) \\ f(x-y) = f(x)\,/\,f(y) \end{array}\right\} \forall\ x, y \in \mathbb{R}$

Proof: $f(x).f(y) = a^x.\, a^y = a^{x+y} = f(x+y)$ and

$f(x)/f(y) = \dfrac{a^x}{a^y} = a^{x-y} = f(x-y)$

(iii) $a^x\,.\,b^x = (ab)^x$ and $\dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x$

(iv) $(a^x)^y = a^{xy} = (a^y)^x$

(v) a^x, where $a > 1$ (say $a = 2$) behaves as an increasing natured function as is clear from the table and graph given below for $y = 2^x$

x	–3	–2	–1	0	1	2	3	4	5
a^x	1/8	1/4	1/2	1	2	4	8	16	32

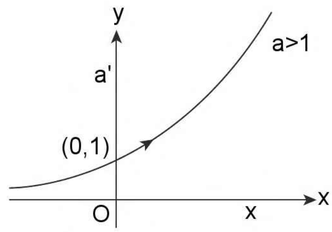


FIGURE 2.81

(vi) If $0 < a < 1$ (say $a = 1/2$) behaves like decreasing natured function as is clear from the table and graph given in Figure 2.82 for $y = (1/2)^x$

x	–5	–4	–3	–2	–1	0	1	2	3
a^x	32	16	8	4	2	1	1/2	1/4	1/8

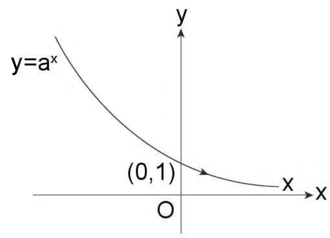


FIGURE 2.82

ILLUSTRATION 67: Find the range of the values of x for which the following functions are defined:

(a) $f(x) = \sqrt{a^x - 1}$; where $a > 1$ and what if $a < 1$

(b) $f(x) = \sqrt[2n]{2 - a^x}$; where $a < 1$

SOLUTION: (a) $f(x)$ to be defined, $a^x - 1 \geq 0$, i.e., $a^x \geq 1$

when $a > 1$: $a^x \geq 1$ $\Rightarrow x \geq 0 \Rightarrow x \in [0, \infty)$

when $a < 1$: $a^x \geq 1$ $\Rightarrow x \leq 0 \Rightarrow x \in (-\infty, 0]$

(b) $f(x)$ to be defined $2 - a^x \geq 0$ $\Rightarrow a^x \leq 2$

when $a < 1$: $a^x \leq 2$

Taking log to the base a both side: $x \geq \log_a 2 \Rightarrow x \in [\log_a 2, \infty)$.

SOLVING EXPONENTIAL EQUATIONS

To solve an exponential equation, we use the following two properties:

(i) $a^x = a^y$

$\Rightarrow$ $x = y$ or $a = 1$

(ii) $a^x = b^x$

$\Rightarrow$ either $x = 0$ or $a = b$

ILLUSTRATION 68: To solve the equation $2^{x^2-1} = 256;\ x \in \mathbb{R}$

SOLUTION: Given $2^{x^2-1} = 256 \quad \Rightarrow 2^{x^2-1} = 2^8 \quad \Rightarrow x^2 - 1 = 8 \quad \Rightarrow x^2 = 9$

$\Rightarrow$ $x = \pm 3$. Therefore set of solution is $\{-3, 3\}$

SOLVING EXPONENTIAL INEQUALITY

To solve an exponential inequality, we use following properties of exponential functions.

1. The value of a^x increases as the value of x increases, when base $a \in (1, \infty)$.
2. The value of a^x decreases as the value of x increases, when base $a \in (0, 1)$

i.e., $a^x \ge a^y \Leftrightarrow \begin{cases} x \ge y & \text{if } a > 1 \\ x \le y & \text{if } a \in (0,1) \end{cases}$

ILLUSTRATION 69: Solve the following inequalities:

(i) $2^{x-4} \ge 8^{3x-8}$ (ii) $\left(\frac{1}{2}\right)^{2x-5} \ge (16)^{-x+5}$

(iii) $(\cos\theta)^{\frac{1}{x-4}} \le (\cos\theta)^{\frac{-x}{x+2}}$ for some fixed $\theta \in (0,\pi/2)$

SOLUTION: $2^{(x-4)} \ge 8^{(3x-8)}$

$\Rightarrow (2)^{x-4} \ge 2^{3(3x-8)} \quad \Rightarrow (2)^{(x-4)} \ge (2)^{9x-24} \quad (\because a = 2 > 1)$

$\Rightarrow (x-4) \ge 9x - 24 \Rightarrow 8x \le 20 \Rightarrow x \le 5/2 \Rightarrow x \in \left(-\infty, \frac{5}{2}\right]$

Thus, $(-\infty, 5/2]$ is the solution set of given inequality

(ii) $\left(\frac{1}{2}\right)^{2x-5} \ge (16)^{-x+5} \Rightarrow \left(\frac{1}{2}\right)^{2x-5} \ge \left(\frac{1}{2}\right)^{4x-20}$

$\Rightarrow (2x-5) \le 4x - 20$ [as base $a = 1/2 \in (0,1)$] $\Rightarrow 2x \ge 15$

$\Rightarrow x \ge 15/2 \Rightarrow x \in \left[\frac{15}{2}, \infty\right)$. Thus, $\left[\frac{15}{2}, \infty\right)$ is the solutions set of given inequality

(iii) $\because$ for $\theta \in (0, \pi/2)$, $\cos\theta \in (0,1)$

$\Rightarrow$ Base of exponential function belongs to $(0,1)$

$\therefore (\cos\theta)^{\frac{1}{x-4}} \le (\cos\theta)^{\frac{-x}{x+2}} \Rightarrow \frac{1}{x-4} \ge \frac{-x}{x+2} \Rightarrow \frac{1}{x-4} + \frac{x}{x+2} \ge 0$

$\Rightarrow \frac{x+2+x^2-4x}{(x-4)(x+2)} \ge 0 \Rightarrow \frac{x^2-3x+2}{(x-4)(x+2)} \ge 0 \Rightarrow \frac{(x-1)(x-2)}{(x-4)(x+2)} \ge 0$

$\Rightarrow (x+2)(x-1)(x-2)(x-4) \ge 0;\ x \ne -2, 4$

$\Rightarrow x \in (-\infty, -2) \cup [1, 2] \cup (4, \infty)$ which is the required solution set of given inequality.

COMPOSITE EXPONENTIAL FUNCTION

A function of the form $y = [f(x)]^{g(x)}$; i.e., a function in which both base and exponent are functions of x

e.g., $y = x^x$, $y = (\sin x)^{\tan x}$, $y = (\ln x)^x$, $y = (x)^{\ln x}$ etc.

Composite exponential function is also known as exponential power function. Composite exponential function. $y = (f(x))^{g(x)}$ is defined for those values of x for which $f(x)$ and $g(x)$ both are defined (i.e., x must be in common domain of $f(x)$ and $g(x)$) and $f(x) > 0$.

Thus, domain of $y = [f(x)]^{g(x)} = \{x \in \mathbb{R} : x \in D_f \cap D_g$ and $f(x) > 0\}$.

For example:

(i) $y = x^{\sin x}$ has its domain $= \{ x \in \mathbb{R} : x > 0\} = (0, \infty)$

(ii) $y = (\sin x)^x$ has its domain $= \{x \in \mathbb{R} : \sin x > 0\}$ $= \{x \in \mathbb{R} : x \in (2n\pi, (2n + 1)\pi);\ n \in \mathbb{Z}\}$ $= \bigcup_{n\in\mathbb{Z}} ((2n\pi),(2n+1)\pi)$

(iii) $y = (\ln x)^x$ has its domain $= \{x \in \mathbb{R} : \ln x > 0\}$ $= \{x \in \mathbb{R} : x > 1\} = (1, \infty)$

(iv) $y = (\sin x)^{\tan x}$ has its domain $= \{x \in \mathbb{R} : \sin x > 0$ and $x \neq \left\{(2n+1)\frac{\pi}{2}; n \in \mathbb{Z}\right\}$

$= \bigcup_{n\in\mathbb{Z}} \left[((2n\pi),(2n+1)\pi) \sim \left\{(2n+1)\frac{\pi}{2}\right\}\right]$

> **NOTE**
>
> $y = \sqrt[x]{f(x)}$ *is defined for* $x \in \{2, 3, 4, \ldots\}$ *and* $f(x) > 0$ *where as* $y = (f(x))^{1/x}$ *is defined for* $x \neq 0$ *and* $f(x) > 0$.
>
> ***Example:***
>
> (i) $y = \sqrt[x]{\sin x}$ *is defined for* $x \in \{2, 3, 4, \ldots.\}$ *and* $\sin x > 0$ $\Rightarrow$ $x \in \{2, 3, 4, \ldots.\}$ *and* $x \in \bigcup_{n\in W}(2n\pi,(2n+1)\pi)$
>
> $\Rightarrow$ $x \in \{2, 3, 4, \ldots\}$ *and* $x \in (0, \pi) \cup (2\pi, 3\pi) \cup (4\pi, 5\pi)\ldots..$ $\Rightarrow$ $x \in \{2, 3, 7, 8, 9, \ldots\}$.
>
> (ii) *Domain of* $y = (\cos\quad)^{\overline{\sin}}$ *is defined for* $\cos x > 0$ *and* $\sin x \neq 0$ $\Rightarrow$ $x \in \bigcup_{n\in\mathbb{Z}}\left[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right] \sim 2n\pi$.

HYPERBOLIC FUNCTIONS

There are six hyperbolic functions as discussed below:

(i) **Hyperbolic Sine:** It is denoted by $\sinh x$ and is defined as

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$= \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right)$$

$$= \frac{1}{2}\left\{2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \ldots.\right\} = \left\{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots.\right\}$$

Thus, $\sinh x = \dfrac{e^x - e^{-x}}{2} = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \ldots..$

Domain of $\sinh x = \mathbb{R}$, set of all real numbers.

Since $\dfrac{d}{dx}(\sinh x) = \left(1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \ldots\right) > 0\ \forall\ x \in \mathbb{R}$

$\Rightarrow$ Graph of $\sinh x$ is strictly increasing $\forall\ x \in \mathbb{R}$

Also $\dfrac{d^2}{dx^2}(\sinh x) = \left(x + \dfrac{x^3}{3!} + \ldots\right)$ which is positive for $x > 0$ and negative for $x < 0$.

Also $\dfrac{d^2}{dx^2}(\sinh x) = 0$ for $x = 0$.

$\Rightarrow$ Graph of $\sinh x$ is concave upwards for $x > 0$ and concave downwards for $x < 0$ and has a point of inflexion at $x = 0$.

Clearly $\sinh x$ being continuous and strictly increasing and $\sinh x \to -\infty$ as $x \to -\infty$ and $\sinh x \to \infty$ as $x \to \infty$, has its range $= \mathbb{R}$.

Thus, the graph of $\sinh x$ would be as shown in Figure 2.83.

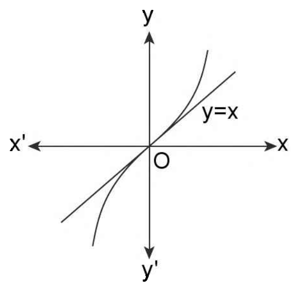


FIGURE 2.83

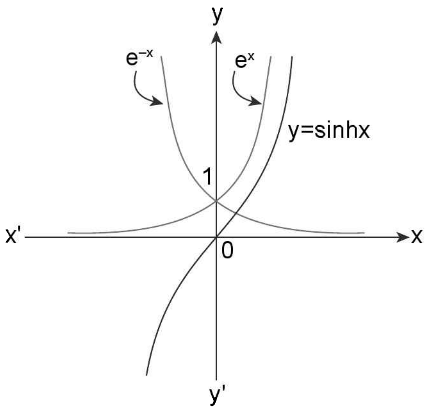


FIGURE 2.84

(ii) **Hyperbolic Cosine:** It is denoted by cosh x and is defined as $\cosh x = \dfrac{e^x + e^{-x}}{2}$

$$= \frac{1}{2}\left\{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right)\right\}$$

$$= \frac{1}{2}\left\{\left(2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \ldots\right)\right\} = \left\{\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right)\right\}$$

$\geq 1 \ \forall\, x \in \mathbb{R}$ $\quad\Rightarrow\quad$ Range of cosh x is $[1, \infty)$

Clearly cosh x is defined $\forall\, x \in \mathbb{R}$

Also $\dfrac{d}{dx}(\cosh x) = \left(x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \ldots.\right) > 0$

for $x > 0$ and < 0 for $x < 0$ and $= 0$ for $x = 0$.

$\Rightarrow$ cosh x is a decreasing function for $x < 0$, increasing function for $x > 0$ and has a stationary point at $x = 0$

Further $\dfrac{d^2}{dx^2}(\cosh x) = \left(1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \ldots.\right) > 0$

$\forall\, x \in \mathbb{R}$.

Graph of cosh x is concave upwards $\forall\, x \in \mathbb{R}$. Collecting all above conclusions, the graph of cosh x would be as shown in Figure 2.85.

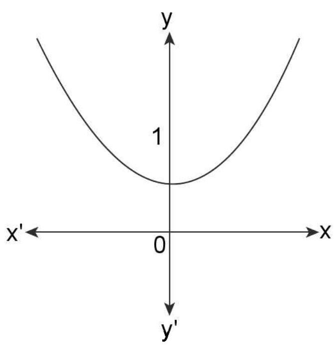


FIGURE 2.85

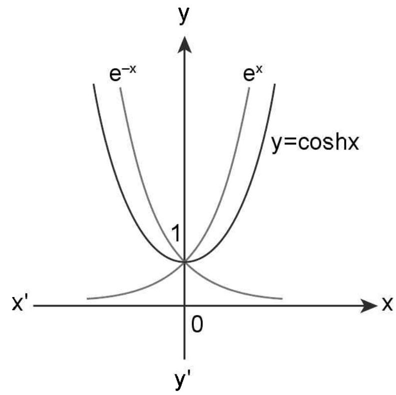


FIGURE 2.86

(iii) **Hyperbolic Tangent:** It is denoted by tanh x and is defined as $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Clearly tanh x is defined $\forall\, x \in \mathbb{R}$.

Also $\tanh x = \dfrac{e^{2x} - 1}{e^{2x} + 1} \to -1$ as $x \to -\infty$ and

$\tanh x = \dfrac{1 - e^{-2x}}{1 + e^{-2x}} \to 1$ as $x \to \infty$

Thus, range of tanh x is $(-1, 1)$.

Next $\tanh x > 0 \Leftrightarrow \dfrac{e^{2x} - 1}{e^{2x} + 1} > 0 \Leftrightarrow e^{2x} > 1$

$\Leftrightarrow$ $2x > \ln 1 \quad \Leftrightarrow \quad x > 0$

Thus, $\tanh x > 0$ for $x \in (0, \infty)$

Similarly $\tanh x < 0$ for $x \in (-\infty, 0)$ and $\tanh x = 0$ for $x = 0$

Now $\dfrac{d}{dx}(\tanh x) = \dfrac{(e^{2x} + 1)(2e^{2x}) - (e^{2x} - 1)(2e^{2x})}{(e^{2x} + 1)^2}$

$$= \frac{4e^{2x}}{(e^{2x} + 1)^2} > 0 \ \forall\, x \in \mathbb{R}$$

$\Rightarrow$ tanh x is strictly increasing $\forall\, x \in \mathbb{R}$ and

$$\frac{d^2}{dx^2}(\tanh x) = \frac{(e^{2x} + 1)^2 . 8e^{2x} - 4e^{2x} . 2(e^{2x} + 1)(2e^{2x})}{(e^{2x} + 1)^4}$$

$$= \frac{8(e^{2x} + 1)e^{2x}(1 - e^{2x})}{(e^{2x} + 1)^4} > 0 \text{ for } x < 0$$

< 0 for $x > 0$ and $= 0$ for $x = 0$

$\Rightarrow$ The graph of tanh x is concave upwards for $x < 0$, concave downwards for $x > 0$ and has a point of inflexion at $x = 0$.

Thus, from above discussion we conclude that the graph of tanh x would be as shown in Figure 2.87.

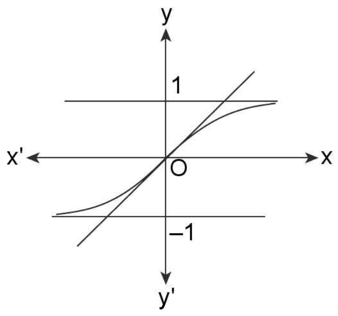


FIGURE 2.87

(iv) **Hyperbolic Cotangent:** It is denoted by $\coth x$ and is defined as $y = \coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}} = \dfrac{e^{2x}+1}{e^{2x}-1}$

$\to -1$ as $x \to -\infty$ and $\to -\infty$ as $x \to 0$ from negative side.

Also $\coth x = \dfrac{1+e^{-2x}}{1-e^{-2x}} \to 1$ as $x \to \infty$ and $\to \infty$ as $x \to 0$ from positive side.

Thus, range of $\coth x$ is $(-\infty, -1) \cup (1, \infty)$.

Clearly domain of $\coth x$ is $\mathbb{R} \sim \{0\}$.

Now $\dfrac{dy}{dx} = \dfrac{(2e^{2x})(-2)}{(e^{2x}-1)^2} < 0 \ \forall\, x \in \mathbb{R} \sim \{0\}$

$\Rightarrow$ $\coth x$ is a strictly decreasing function $\forall\, x \in \mathbb{R} \sim \{0\}$

Next $\dfrac{d^2}{dx^2}(\coth x) = \dfrac{8e^{2x}(1+e^{2x})(e^{2x}-1)}{(e^{2x}-1)^4} > 0$ for $x > 0$

and < 0 for $x < 0$

$\Rightarrow$ The, graph of $\coth x$ is concave upwards for $x > 0$ and concave downwards for $x < 0$

Thus, the graph of $\coth x$ would be as shown below.

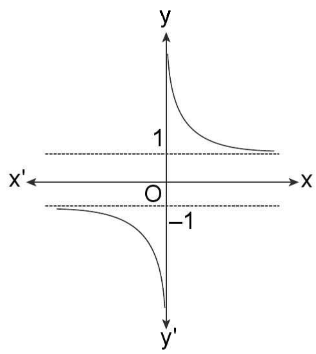


FIGURE 2.88

(v) **Hyperbolic Secant:** It is denoted by sech x and is defined as $\operatorname{sech} x = \dfrac{2}{e^x + e^{-x}} = \dfrac{1}{\cosh x}$

or $\operatorname{sech} x = \dfrac{2e^x}{e^{2x}+1}$ which is defined $\forall\, x \in \mathbb{R}$

$\Rightarrow$ Domain of sech x is $\mathbb{R}$.

Also $\operatorname{sech} x = \dfrac{2y}{y^2+1}$; where $y = e^x$

It can be easily observed that $\dfrac{2y}{y^2+1} \in [-1, 0]$ for $y \in (-\infty, 0]$ and $\dfrac{2y}{y^2+1} \in (0, 1]$ for $y \in (0, \infty)$

$\Rightarrow$ Range of sech x would be $(0, 1]$

Now $\dfrac{d}{dx}(\operatorname{sech} x) = 2e^x \left[\dfrac{1-e^{2x}}{(e^{2x}+1)^2}\right] < 0$ for $x > 0$,

> 0 for $x < 0$ and 0 for $x = 0$.

$\Rightarrow$ sech x is strictly decreasing for $x > 0$, strictly increasing for $x < 0$ and $x = 0$ is a stationary point.

Further $\dfrac{d^2}{dx^2}(\operatorname{sech} x)$

$$= \frac{2e^x(e^x+1)\left[e^{2x} - \left(\dfrac{7-\sqrt{33}}{2}\right)\right]\left[e^{2x} - \left(\dfrac{7+\sqrt{33}}{2}\right)\right]}{(e^{2x}+1)^4}$$

< 0 for $x \in \left(\dfrac{1}{2}\ln\left(\dfrac{7-\sqrt{33}}{2}\right), \dfrac{1}{2}\ln\left(\dfrac{7+\sqrt{33}}{2}\right)\right)$ and > 0

for $x \in \left(-\infty, \dfrac{1}{2}\ln\left(\dfrac{7-\sqrt{33}}{2}\right)\right) \cup \left(\dfrac{1}{2}\ln\left(\dfrac{7+\sqrt{33}}{2}\right), \infty\right)$

$\Rightarrow$ $f(x)$ is concave downwards for $x \in \left(\dfrac{1}{2}\ln\left(\dfrac{7-\sqrt{33}}{2}\right), \dfrac{1}{2}\ln\left(\dfrac{7+\sqrt{33}}{2}\right)\right)$

and concave upwards for $x \in \left(-\infty, \dfrac{1}{2}\ln\left(\dfrac{7-\sqrt{33}}{2}\right)\right) \cup \left(\dfrac{1}{2}\ln\left(\dfrac{7+\sqrt{33}}{2}\right), \infty\right)$

Thus, the graph of hyperbolic secant is as shown in Figure 2.89.

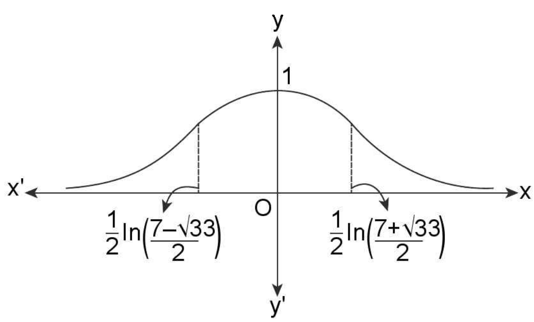


FIGURE 2.89

(vi) **Hyperbolic Cosecant:** It is denoted by cosech x and is defined as $\operatorname{cosec} hx = \dfrac{2}{e^x - e^{-x}} = \dfrac{2e^x}{e^{2x}-1}$

Clearly domain of cosech x is $\mathbb{R} \sim \{0\}$.

It can be easily verified that function $\dfrac{2y}{y^2-1}$ decreases from 0 to $-\infty$ as y increases from 0 to 1 and decreases from ∞ to 0 as y increases from 1 to ∞ with discontinuity at $y = 1$.

Thus, range of function is $(-\infty, \infty) \sim \{0\}$ for $y \in (0, \infty)$.

Similarly cosech x takes its range $(-\infty, \infty) \sim \{0\}$ for $x \in \mathbb{R} \sim \{0\}$

Now $\dfrac{d}{dx}(\operatorname{cosec} hx) = 2\left[\dfrac{-e^x - e^{3x}}{(e^{2x}-1)^2}\right] < 0 \;\forall\, x \in \mathbb{R} \sim \{0\}$

$\Rightarrow$ cosech x is a decreasing function on its domain $\mathbb{R} \sim \{0\}$.

$\Rightarrow$ Further $\dfrac{d^2}{dx^2}(\text{cosech } x) < 0$ for $x \in (-\infty, 0)$ and > 0 for $x \in (0, \infty)$.

Thus, the graph of cosech x is concave downwards for $x \in (-\infty, 0)$ and concave upwards for $x \in (0, \infty)$ as shown below.

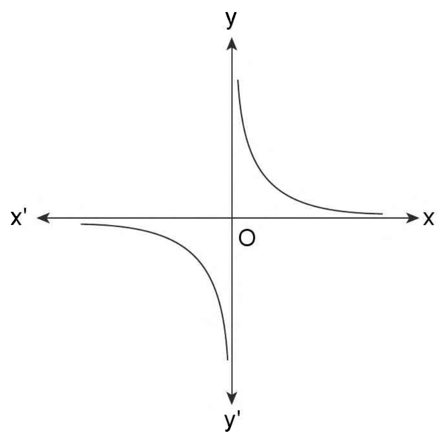


FIGURE 2.90

LOGARITHMIC FUNCTION

Logarithm of a positive number x to the base a is defined as a number y, which when raised to base a the resulting number becomes equal to x.

i.e., $\log_a x = y \Leftrightarrow a^y = x$, when $a > 0$ and $a \neq 1$; $x > 0$ as a^y is positive

e.g., $\log_7 49 = 2$ as $(7)^2 = 49$; $\log_3 81 = 4$ as $(3)^4 = 81$; $\log_{10} 100000 = 5$ as $(10)^5 = 100000$ etc.

Since logarithmic function $y = \log_a x$ is defined only for positive real numbers, and hence, its domain is $(0, \infty)$.

Also logarithmic function is continuous (i.e., its graph never breaks when drawn corresponding to positive real numbers x) and attains each real number, and hence, its range is $\mathbb{R}$, set of real numbers.

In general $y = \log_a x$ is an increasing function for $a > 1$ as has been illustrated below for $y = \log_{10} x$ and $y = \log_2 x$

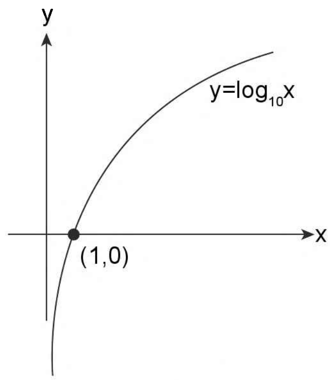


FIGURE 2.91

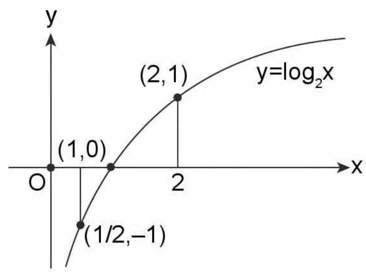


FIGURE 2.92

x	0.0001	0.001	0.01	0.1	1	10	100	1000
$\log_{10} x$	–4	–3	–2	–1	0	1	2	3

And, in general, logarithmic function is a decreasing function for $a \in (0, 1)$ as been illustrated below for $y = \log_{1/2} x$

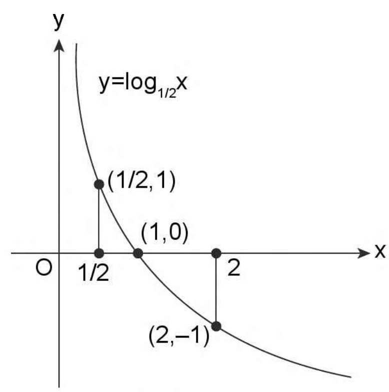


FIGURE 2.93

x	1/16	1/8	1/4	1/2	1	2	4	8
$\log_2 x$	–4	–3	–2	–1	0	1	2	3
$\log_{1/2} x$	4	3	2	1	0	–1	–2	–3

For a given $x \in (0, 1)$ as the base of $\log_a x$ increases the value of logarithm also increases and for given $x \in (1, \infty)$ as the base of $\log_a x$ increases the value of logarithm decreases as has been shown in graph given below.

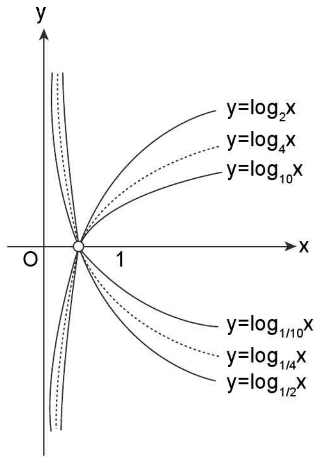


FIGURE 2.94

REMARK

If a = 10, then we write log b rather than $\log_{10} b$. If a = e, we write lnb rather than $\log_e b$.

■ PROPERTIES OF LOGARITHMIC FUNCTIONS

P.1. $\log_a x$ is inverse function of exponential function to base a

$\Rightarrow$ $\log_a x$ and a^x are inverse function, i.e., x when operated by function.

That is, log to the base a and then operated by inverse (exponential function with base a)

i.e., $a^{\log_a x}$ the result is x.

P.2 $\log_a x$ is defined iff $a > 0$, $a \neq 1$ and $x > 0$

$\Rightarrow$ Domain of function $D_f = (0, \infty)$

and Range of function $= R_f = \mathbb{R}$.

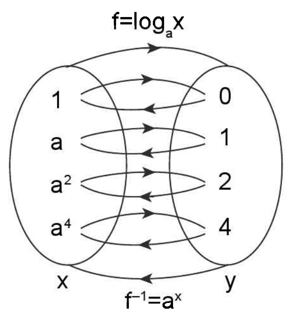


FIGURE 2.95

P.3 $\log_a 1 = 0$ as $a^0 = 1$

P.4 $\log_a a = 1$ as $a^1 = a$

ILLUSTRATION 70: Express log of x to the base 5 in each of the following case

(a) $(25)^{-\frac{1}{3}} = 5x$ (b) $(125)^{-\frac{4}{7}} = 25x$

SOLUTION: (a) $(25)^{-\frac{1}{3}} = 5x \Rightarrow \frac{(25)^{-\frac{1}{3}}}{5} = x \Rightarrow 5^{-\frac{5}{3}} = x \Rightarrow \log_5 x = -\frac{5}{3}$

(b) $(125)^{-\frac{4}{7}} = 25x \Rightarrow \frac{(125)^{-\frac{4}{7}}}{25} = x \Rightarrow 5^{-\frac{26}{7}} = x \Rightarrow \log_5 x = -\frac{26}{7}$

P.5. $\log_a|x| + \log_a|y| = \log_a|x.y|$ and $\log_a xy = \log_a|x| + \log_a|y|$

Proof: LHS $= \log_a|x| + \log_a|y|$

Let $\log_a|x| = m$ and $\log_a|y| = n$

$\Rightarrow a^m = |x|$ and $a^n = |y|$

$\Rightarrow |x.y| = a^{m+n}$

$\Rightarrow \log_a|x| + \log_a|y| = \log_a|xy|$

$[\because a^b = c \Rightarrow \log_a c = b]$ ... (1)

Now $\log_a xy$ to be defined $xy > 0$

$\Rightarrow xy = |xy|$

$\therefore \log_a xy = \log_a|xy| = \log_a|x| + \log_a|y|$ by (1)

P.6. $\log_a|x| - \log_a|y| = \log_a\left|\frac{x}{y}\right|$ and

$$\log_a\frac{x}{y} = \log_a|x| - \log_a|y|$$

Proof: LHS $= \log_a|x| - \log_a|y|$

Let $\log_a|x| = m$ and $\log_a|y| = n$.

$\Rightarrow a^m = |x|$ and $a^n = |y|$

$$\Rightarrow \frac{|x|}{|y|} = \frac{a^m}{a^n}$$

$$\Rightarrow \left|\frac{x}{y}\right| = a^{m-n}$$

$$\Rightarrow \log_a\left|\frac{x}{y}\right| = m - n$$

$$\Rightarrow \log_a|x| - \log_a|y| = \log_a\left|\frac{x}{y}\right| \quad \text{... (1)}$$

Now $\log_a\frac{x}{y}$ to be defined $\frac{x}{y}$ should be positive

$$\Rightarrow \frac{x}{y} = \left|\frac{x}{y}\right|$$

$$\Rightarrow \log_a\frac{x}{y} = \log\left|\frac{x}{y}\right| = \log_a|x| - \log_a|y| \quad \text{(by (1))}$$

Therefore, $\log_a\frac{x}{y} = \log_a|x| - \log_a|y|$

Caution! $\log(x \pm y) \neq \log x \pm \log y$

$\log(x.y) \neq \log x.\log y$

$$\log(x/y) \neq \frac{\log x}{\log y}$$

P.7 $\log_y x = \frac{\log_a x}{\log_a y}$ [Base transformation $a > 0, a \neq 1$]

Proof: It is sufficient to prove that $\log_y x \,.\, \log_a y = \log_a x$

Let $\log_y x = m$

$\Rightarrow x = y^m$ and $\log_a y = n \quad \Rightarrow \quad y = a^n$

$\Rightarrow x = a^{nm} \Rightarrow \quad \log_a x = mn = \log_y x.\, \log_a y.$

Hence, proved.

$$\text{e.g., } \log_{128} 512 = \frac{\log_2 512}{\log_2 128} = \frac{\log_4 512}{\log_4 128} = \ldots$$

Further to solve $\log_{128}512$, we can proceed as follows:

$$\log_{128} 512 = \frac{\log_2 512}{\log_2 128} = \frac{\log_2 2^9}{\log_2 2^7} = \frac{9}{7}$$

P.8. **Chain rule:** $\log_a x = \log_y x.\log_z y.\log_u z.\log_a u$

Proof: RHS $= \log_y x \,.\, \log_z y \,.\, \log_u z \,.\, \log_a u$

$$= \frac{\log_b x}{\log_b y}.\frac{\log_b y}{\log_b z}.\frac{\log_b z}{\log_b u}.\frac{\log_b u}{\log_b a} \quad \text{[By property (7)]}$$

$$= \frac{\log_b x}{\log_b a} = \log_a x = \text{LHS.}$$

P.9. **Reciprocal property:** In a logarithm if the number and base is interchanged, then result is reciprocal of the original. i.e., $\log_y x = \frac{1}{\log_x y}$

Proof: $\log_y x.\log_x y = \frac{\log_t x}{\log_t y}\frac{\log_t y}{\log_t x} = 1$

$$\Rightarrow \log_y x = \frac{1}{\log_x y}$$

P.10. $\log_a(x^\alpha) = \alpha \log_a|x|$

Proof: $\log_a(x^\alpha)$ to be defined $x^\alpha > 0$

Case (i) when $x > 0$, then $\log_a(x^\alpha) = m$ (say)

$\Rightarrow x^\alpha = a^m \Rightarrow x = (a^m)^{1/\alpha} \Rightarrow x = a^{m/\alpha} \quad (\because x > 0)$

$\Rightarrow \log_a x = \frac{m}{\alpha} \Rightarrow m = \alpha\log_a x \Rightarrow m = \alpha \log_a|x|$

$(\because x > 0)$

$\Rightarrow \log_a(x^\alpha) = \alpha \log_a|x|$

Case (ii) When $x < 0$, α = an even integer, then $\log_a(x^\alpha) = m$(say)

$\Rightarrow x^\alpha = a^m \Rightarrow x = -(a^m)^{1/\alpha} \Rightarrow |x| = a^{m/\alpha} \ (\because x < 0)$

$\Rightarrow \log_a|x| = \frac{m}{\alpha} \Rightarrow m = \alpha \log_a|x|$

$\Rightarrow \log_a(x^\alpha) = \alpha \log_a|x|$

P.11. $\log_{a^\beta} x = \frac{1}{\beta}\log_{|a|} x$

Proof: By base condition $a^\beta > 0$ and $a^\beta \neq 1$

There arise two cases:

Case (i) When $a > 0$

Now let $\log_{a^\beta} x = m$

$\Rightarrow$ $x = (a^\beta)^m$

$\Rightarrow$ $x = a^{\beta m}$

$\Rightarrow$ $\log_a x = \beta m$ $\quad (\because a > 0)$

$\Rightarrow$ $\log_a x = \beta \log_{a^\beta} x$

$\Rightarrow$ $\log_{a^\beta} x = \dfrac{1}{\beta}\log_a x = \dfrac{1}{\beta}\log_{|a|} x$ $\quad (\because a > 0)$

Case (ii) When $a < 0$; β is an even integer

Now $\log_{a^\beta} x = m$

$\Rightarrow$ $x = (a^\beta)^m$

$\Rightarrow$ $x = a^{\beta m}$

$\Rightarrow$ $a = -(x)^{1/\beta m}$ $\quad (\because a < 0)$

$\Rightarrow$ $-a = (x)^{1/\beta m}$

$\Rightarrow$ $|a| = (x)^{1/\beta m}$

$\Rightarrow$ $x = |a|^{\beta m}$

$\Rightarrow$ $\log_{|a|} x = \beta m$

$\Rightarrow$ $m = \dfrac{1}{\beta}\log_{|a|} x$

Hence, $\log_{a^\beta} x = \dfrac{1}{\beta}\log_{|a|} x$

P.12 $a^{\log_x b} = b^{\log_x a}$

Proof: Let $a^{\log_x b} = k$; taking $\log_b$ of both sides, we get $\log_b\left(a^{\log_x b}\right) = \log_b k$

$\Rightarrow$ $\log_x b . \log_b a = \log_b k$ $\qquad \Rightarrow$ $\log_x a = \log_b k$

$\Rightarrow$ $k = b^{\log_x a}$ $\qquad \Rightarrow$ $a^{\log_x b} = b^{\log_x a}$

P.13 If a function $f : (0, \infty) \to \mathbb{R}$ satisfies

$$\left.\begin{aligned} f(x.y) &= f(x) + f(y) \\ f\left(\frac{x}{y}\right) &= f(x) - f(y) \end{aligned}\right\} \forall\, x, y \in (0, \infty)$$

$\Rightarrow$ $f(x) = \log_a x$ for $a > 0, \neq 1$

Proof: Given in functional equation topic of differentiability in our volume differential calculus.

P.14 If a is a positive real number, then

(i) For $a > 1$ the logarithmic functions follows the inequality $\log_a x > \log_a y \Leftrightarrow x > y$.
and for $0 < a < 1$ the inequality $\log_a x > \log_a y \Leftrightarrow 0 < x < y$

(ii) If $0 < a < 1$, then $\log_a x > b \Leftrightarrow 0 < x < a^b$

(iii) If $a > 1$, then $\log_a x > b \Leftrightarrow x > a^b$

REMARKS

1. *The base of logarithm can be any positive number other than 1, but usually two bases are mostly used they are 10 and e (≈ 2.7183).*
Logarithms of numbers to the base 10 are named as common logarithms; where as the logarithms of the numbers to the base e are called as natural or Napierian logarithms.

$\therefore$ *we find* $\log_e a = \log_{10} a . \log_e 10$ *or* $\log_{10} a = \dfrac{\log_e a}{\log_e 10} = 0.434 \log_e a$

$\therefore$ *To convert natural logarithm of a number to common logarithm of that number we multiply the natural logarithm by 0.434. i.e.,* $\log_{10} a = 0.434 \log_e a$
Also $\log_e a = \log_{10} a . \log_e 10 = 2.303 \log_{10} a$
Thus, to convert common logarithm of a number to natural logarithm of that number we multiply the common logarithm by 2.303, i.e., $\log_e a = 2.303 \log_{10} a$*.This result is useful in different branches of sciences. For example, chemistry where we need to find natural logarithm of a given positive real number. We find first of all common logarithm of that number with the help of logarithmic table and then multiply the result by 2.303*
e.g., $\log_{10} 2 = 0.3010$

$\Rightarrow$ $\ln 2 = 2.303 \times 0.3010 = 0.693$
$\log_{10} 3 = 0.4771$

$\Rightarrow$ $\ln 3 = 2.303 \times 0.4771 = 1.098$

2. *The integer part of logarithm of a number is called characteristic and the decimal part is called mantissa.*
e.g., $\log_{10} 30 = 1.4771$*, then characteristic is 1 and mantissa is 0.4771.*

3. *Detail discussion of logarithm of a function, logarithmic inequalities and logarithmic equations and their solutions has been given in our book Algebra II Fundamental of Mathematics.*

ILLUSTRATION 71: Find the characteristic and mantissa of $\log_{10}(0.12)^{15}$; given $\log_{10}2 = 0.3010$, $\log_{10}3 = 0.4771$. Hence, find the number of zeros after decimal and before first significant digit in decimal representation of $(0.12)^{15}$.

SOLUTION: $\log_{10}(0.12)^{15} = 15\log_{10}(0.12)$

$= 15\log_{10}\left[\frac{12}{100}\right] = 15\log_{10}\left[\frac{3}{25}\right] = 15[\log_{10}3 - \log_{10}25]$

$= 15\log_{10}3 - 15\log_{10}25 = 15\log_{10}3 - 30\log_{10}5 = 15\log_{10}3 - 30\log_{10}\left(\frac{10}{2}\right)$

$= 15\log_{10}3 - 30[\log_{10}10 - \log_{10}2] = 15\log_{10}3 - 30\log_{10}10 + 30\log_{10}2$

$= 15\log_{10}3 - 30 + 30\log_{10}2 = 15(0.4771) + 30(0.3010) - 30$

$= 16.1865 - 30 = 16 + 0.1865 - 30$

$= -14 + 0.1865 = \overline{14}.1865$

$\therefore$ Characteristic $= -14$, Mantissa $= 0.1865$

The number of zeros after decimal and before first significant digit in decimal representation of a number $x \in (0, 1)$ is given by $|\mathrm{ch}(x) + 1|$.

$\therefore$ Number of such zeros in the decimal representation of $(0.12)^{15}$ will be $= |-14 + 1| = 13$

ILLUSTRATION 72: Find the domain of given logarithmic functions

(i) $f(x) = \log_4(x^2 - 1)$ (ii) $f(x) = \log_{3x}(x^2 - 4)$

(iii) $f(x) = \log_{(5-x)}\left\{\frac{(x^2-9)}{(x+1)(x+6)}\right\}$ (iv) $f(x) = \frac{\log(x-4)(x-5)}{\log(x-3)}$

(v) $f(x) = \log(x - 2) + \log(8 - x)$

SOLUTION: (i) $f(x) = \log_4(x^2 - 1)$.

For $f(x)$ to be defined $(x^2 - 1) > 0$ $\Rightarrow x^2 > 1$

$\Rightarrow x \in (-\infty, -1) \cup (1, \infty)$

$\therefore$ Domain of $f(x) = (-\infty, -1) \cup (1, \infty)$

(ii) $f(x) = \log_{3x}(x^2 - 4)$

By base condition, $3x > 0$; $3x \neq 1$ $\Rightarrow x \in (0, \infty) \sim \left\{\frac{1}{3}\right\}$ .... (1)

Also $x^2 - 4 > 0 \Rightarrow x^2 > 4 \Rightarrow x < -2$ or $x > 2$ ($\because \log f(x)$ is defined for $f(x) > 0$)

$\Rightarrow x \in (-\infty, -2) \cup (2, \infty)$ .... (2)

$\therefore$ x must satisfy (1) as well as (2) $\Rightarrow x \in (2, \infty)$

$\therefore$ Domain of function is $(2, \infty)$.

(iii) $f(x) = \log_{(5-x)}\left\{\frac{(x^2-9)}{(x+1)(x+6)}\right\}$.

For base condition $5 - x > 0; \neq 1 \Rightarrow x < 5, x \neq 4 \Rightarrow x \in (-\infty, 5) \sim \{4\}$ .... (1)

Also $\frac{x^2-9}{(x+1)(x+6)} > 0 \Rightarrow (x^2 - 9)(x + 1)(x + 6) > 0 \Rightarrow (x + 6)(x + 3)(x + 1)(x - 3) > 0$

$\therefore$ By wavy-curve method $x \in (-\infty, -6) \cup (-3, -1) \cup (3, \infty)$ ... (2)

$\therefore$ Domain is the intersection of (1) and (2), i.e., $(-\infty, -6) \cup (-3, -1) \cup (3, 5) \sim \{4\}$

(iv) $f(x)=\dfrac{\log(x-4)(x-5)}{\log(x-3)}$

Here we must have $(x-4)(x-5)>0$ and denominator $\log(x-3)\neq 0$; $(x-3)>0$

$\Rightarrow$ $x\in(-\infty,4)\cup(5,\infty)$ and $x-3\neq 1$; $x-3>0$

$\Rightarrow$ $x\in(-\infty,4)\cup(5,\infty)$ and $x\in(3,\infty)\sim\{4\}$ $\Rightarrow$ $x\in(3,4)\cup(5,\infty)=D_f$

(v) $f(x)=\log(x-2)+\log(8-x)$.

Each linear function on which log is operated, must be positive.

$\Rightarrow$ $(x-2)>0$ and $8-x>0$ $\Rightarrow$ $x>2$ and $x<8$ $\Rightarrow$ $x\in(2,8)=D_f$

ILLUSTRATION 73: Find the number of digits in $(42875)^{12}$, given $\log 2 = 0.3010$, $\log 7 = 0.8451$

SOLUTION: Here $\log(42875)^{12} = 12\log(42875) = 12\log(125\times 343)$

$= 12\log[(5)^3\times(7)^3] = 12[\log(5)^3+\log(7)^3] = 12[3\log 5+3\log 7]$

$= 36\log 5 + 36\log 7 = 36\log\left(\dfrac{10}{2}\right)+36\log(7)$

$= 36 - 36\log 2 + 36\log 7 = 36 - 36(0.3010)+36(0.8451)$ $(\because \log 10 = 1)$

$= 36 - 10.836 + 30.4236 = 55.5876$

$\Rightarrow$ Characteristic = 55 and mantissa = 0.5876.

As the number of digits of a positive integer is one more than the characteristic of logarithm of number $\Rightarrow$ $(42875)^{12}$ has 56 digits.

ILLUSTRATION 74: Find the domain of logarithmic functions involved in the given logarithmic equation, and hence, solve the given equations.

(a) $\log_4(x+5)=3$ (b) $\log_{(x+2)}64=2$

(c) $(0.4)^{\log^2 x+1}=(6.25)^{2-\log x^3}$ (d) $x^{0.5\log_{\sqrt{x}}(x^2-x)}=(4)^{\log_9 3}$

SOLUTION: (a) Given equation is $\log_4(x+5)=3$ .... (1)

For domain of logarithmic function on LHS of (1), $x+5>0$

$\Rightarrow$ $x\in(-5,\infty)$ = Domain of logarithmic function

Now $\log_4(x+5)=3$ $\Rightarrow$ $(x+5)=(4)^3=64$

$\Rightarrow$ $x=59$ which belongs to domain. Thus, $x=59$ is the required solution.

(b) Given equation is $\log_{(x+2)}64=2$

For domain of logarithmic function $(x+2)>0$ and $\neq 1$

$\Rightarrow$ $x>-2$ and $x\neq -1$

$\therefore$ Domain of function $=(-2,\infty)\sim\{-1\}$

Now $64=(x+2)^2$ $\Rightarrow$ $x+2=\pm 8$ $\Rightarrow$ $x=-10$ or 6

but $-10\notin(-2,\infty)\sim\{-1\}$ $\Rightarrow$ $x=6$ is the only solution.

(c) Given equation is $(0.4)^{\log^2 x+1}=(6.25)^{2-\log x^3}$ ... (1)

For domain of logarithmic function on LHS of (1), $x>0$ and for domain of logarithmic function on RHS of (1), $x^3>0$, i.e., $x>0$.

Thus, logarithmic functions involved on both sides of equation (1) are defined for $x>0$, and hence, the solution of equation (1) must lie on this common domain, i.e., $(0,\infty)$

Now, $6.25=\dfrac{25}{4}=\left(\dfrac{5}{2}\right)^2$ and $0.4=\left(\dfrac{2}{5}\right)$

$\therefore$ From equation (1), we get $\left(\dfrac{2}{5}\right)^{\log^2 x+1}=\left[\left(\dfrac{5}{2}\right)^2\right]^{(2-\log x^3)}=\left(\dfrac{2}{5}\right)^{2(\log x^3-2)}$

$\Rightarrow \log^2 x + 1 = 2(3\log x - 2)$ $\quad\Rightarrow (\log x)^2 - 6(\log x) + 5 = 0$

$\Rightarrow \log x = 1$ or $\log x = 5$ $\quad\Rightarrow x = (10)^1$ or $x = (10)^5$.

Both belongs to $(0, \infty)$. Thus, $x = 10$, $x = 10^5$ are the required solutions.

(d) Given equation is $x^{0.5\log_{\sqrt{x}}(x^2-x)} = 4^{\log_9 3}$ .... (1)

For domain of logarithmic function on L.H.S. of above equation (1), $\sqrt{x} > 0; \neq 1$ and $(x^2 - x) > 0$ $\Rightarrow x > 0; \neq 1$ and $x(x-1) > 0 \Rightarrow x > 0; \neq 1$ and $x \in (-\infty, 0) \cup (1, \infty)$.

$\Rightarrow x \in (1, \infty)$ = domain of logarithmic function on LHS of (1).

Now $x^{0.5(2)\log_x(x^2-x)} = (4)^{\log_9 3}$ $\quad\left(\because \log_{a^N}(b) = \dfrac{1}{N}\log_a(b)\right)$

$\Rightarrow x^{\log_x(x^2-x)} = (4)^{\log_9 3}$

$\Rightarrow (x^2 - x) = (3)^{\log_9 4}$ $\quad\left(\because a^{\log_a x} = x \text{ and } a^{\log_N b} = b^{\log_N a}\right)$

$\Rightarrow x^2 - x = (3)^{\log_3 (4)^{1/2}}$ $\quad\Rightarrow x^2 - x = (4)^{1/2}$ $\quad\Rightarrow x^2 - x = 2$

(In fact $(4)^{1/2} = \pm 2$ but $\log (4)^{1/2}$ $\quad\Rightarrow (4)^{1/2} = 2$)

$\Rightarrow x^2 - x - 2 = 0$ $\quad\Rightarrow (x - 2)(x + 1) = 0$

$\Rightarrow x = 2$ or $x = -1$, but $-1 \notin D_f = (1, \infty)$ $\quad\Rightarrow x = 2$ is the only solution.

ILLUSTRATION 75: Solve the logarithmic inequality $\frac{1}{2}\log(2x-1) + \log\sqrt{x-9} > 1$

SOLUTION: Given inequality is $\frac{1}{2}\log(2x-1) + \log\sqrt{x-9} > 1$ .... (1)

For domain of logarithmic functions involved on LHS of inequality (1), $2x - 1 > 0$ and $x - 9 > 0$ $\Rightarrow$ $x > \dfrac{1}{2}$ and $x > 9$.

$\Rightarrow$ Solutions of inequality (1) must belong to $(9, \infty)$

Now from equation (1), we get $\log(2x - 1)^{1/2} + \log\sqrt{x-9} > 1$

$\Rightarrow \log(2x - 1)^{1/2}(x - 9)^{1/2} > 1$ $\quad\Rightarrow (2x - 1)^{1/2}(x - 9)^{1/2} > (10)^1$

$\Rightarrow (2x - 1)(x - 9) > 100$ $\quad\Rightarrow 2x^2 - 19x - 91 > 0$

$\Rightarrow (x - 13)(2x + 7) > 0$ $\quad\Rightarrow x < -\dfrac{7}{2}$ or $x > 13$

Due to domain restriction solution set of logarithmic inequality is $(13, \infty)$.

ILLUSTRATION 76: If x is positive, then show that $\ln(1 + x) < x$ and $\ln(1 + x) > \dfrac{x}{1+x}$. Hence, show that $\ln(1 + n) < 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \ldots + \dfrac{1}{n}$ for $n \in \mathbb{N}$.

SOLUTION: Given $x > 0$

$\ln(1 + x) < x$

$\Leftrightarrow (1 + x) < e^x$

$\Leftrightarrow (1 + x) < 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots..$ which is true as $x > 0$

$\Leftrightarrow \ln(1 + x) > \dfrac{x}{1+x}$ $\Leftrightarrow (1 + x) > e^{\left(\frac{x}{1+x}\right)}$

$$\Leftrightarrow \left(\frac{1}{\left(\frac{1}{1+x}\right)}\right) > e^{\left(\frac{x}{1+x}\right)} \Leftrightarrow \left(\frac{1}{1-\frac{x}{1+x}}\right) > e^{\left(\frac{x}{1+x}\right)}$$

$$\Leftrightarrow \left(1-\frac{x}{1+x}\right)^{-1} > e^{\left(\frac{x}{1+x}\right)} \Leftrightarrow 1+\frac{x}{1+x}+\left(\frac{x}{1+x}\right)^2+.... > 1+\frac{x}{1+x}+\frac{1}{2!}\left(\frac{x}{1+x}\right)^2+...$$

which is true.

Now $\ln(1+x) < x$ for $x > 0$ $\quad\Rightarrow\quad \ln\left(1+\frac{1}{y}\right) < \frac{1}{y}$ for $y > 0$

$\Rightarrow \ln\left(\frac{1+y}{y}\right) < \frac{1}{y}$ $\quad\Rightarrow\quad \log(1+y) - \log y < \frac{1}{y}$ .... (1)

Substituting $y = 1, 2, 3,., n$ in (1), we get $\ln 2 - \log 1 < \frac{1}{1}$

$\ln 3 - \ln 2 < \frac{1}{2}$, $\ln 4 - \ln 3 < \frac{1}{3}$, ..., $\ln(n+1) - \ln n < \frac{1}{n}$

Adding above inequalities, we get $-\log 1 + \log(n+1) < 1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}$

$\Rightarrow \log(n+1) < 1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}$

ILLUSTRATION 77: If $x \in (0, 1)$, prove that $(1+x)^{1+x}(1-x)^{1-x} > 1$. Hence, show $a^a b^b > \left(\frac{a+b}{2}\right)^{a+b}$ for $a > b > 0$.

SOLUTION: Let $y = (1+x)^{(1+x)}(1-x)^{1-x}$ .... (1)

Taking ln on both sides of (1), we get $\ln y = (1+x)\ln(1+x) + (1-x)\ln(1-x)$

$\Rightarrow \ln y = \ln(1+x) + \ln(1-x) + x[\ln(1+x) - \ln(1-x)]$ ... (2)

Now $\ln(1+x) + \ln(1-x)$

$$= \left[x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+...\right]+\left[-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-...\right] = -2\left[\frac{x^2}{2}+\frac{x^4}{4}+...\right] \quad .... (3)$$

and $\ln(1+x) - \ln(1-x)$

$$= \left[x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+...\right]-\left[-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-...\right] = 2\left[x+\frac{x^3}{3}+\frac{x^5}{5}+...\right] \quad ... (4)$$

Using (3) and (4) in (2), we get

$$\ln y = -2\left(\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^6}{6}+...\right)+2x\left[x+\frac{x^3}{3}+\frac{x^5}{5}-...\right] = \left(x^2+\frac{1}{6}x^4+\frac{1}{15}x^6+...\right) > 0$$

Thus, $\ln y > 0 \Rightarrow y > 1$ $\quad\Rightarrow (1+x)^{(1+x)}(1-x)^{(1-x)} > 1$

Thus, for $x \in (0, 1)$, $(1+x)^{(1+x)}(1-x)^{(1-x)} > 1$. ... (5)

Next, we are to show that $a^a b^b > \left(\frac{a+b}{2}\right)^{a+b}$ for $0 < b < a$ and it would hold iff $\dfrac{a^a b^b}{\left(\frac{a+b}{2}\right)^{a+b}} > 1$

$$\Leftrightarrow \left[\frac{a}{\left(\frac{a+b}{2}\right)}\right]^a \cdot \left[\frac{b}{\left(\frac{a+b}{2}\right)}\right]^b > 1 \Leftrightarrow \left(\frac{2a}{a+b}\right)^a \cdot \left(\frac{2b}{a+b}\right)^b > 1$$

$$\Leftrightarrow \left(\frac{a+b+a-b}{a+b}\right)^a \cdot \left(\frac{(a+b)-(a-b)}{a+b}\right)^b > 1 \quad \Leftrightarrow \left(1+\frac{a-b}{a+b}\right)^a \cdot \left(1-\frac{(a-b)}{a+b}\right)^b > 1$$

$$\Leftrightarrow (1+x)^a . (1-x)^b > 1 \text{ for } x = \frac{a-b}{a+b}$$

$$\Leftrightarrow (1+x)^{\frac{2a}{a+b}} . (1-x)^{\frac{2b}{a+b}} > (1)^{\frac{2}{a+b}} \text{ (Raising to power } \frac{2}{a+b} \text{ both sides)}$$

$$\Rightarrow (1+x)^{(1+x)} . (1-x)^{(1-x)} > 1 \text{ for } x = \frac{a-b}{a+b}.$$

which is true by (5) as $x = \frac{a-b}{a+b} = \frac{a+b-2b}{a+b} = 1 - \frac{2b}{a+b} \in (0,1)$ as $a > b > 0$ (given)

ILLUSTRATION 78: Solve the equation $\log_{\cos x} 3 . \log_{\cos^2 x} a + 1 = 0$.

SOLUTION: $\frac{1}{\log_3 \cos x} . \frac{\log_3 a}{\log_3 \cos^2 x} + 1 = 0$; $\cos x > 0$ and $\neq 1$, $a > 0$

$\Rightarrow \frac{\log_3 a}{2(\log_3 \cos x)^2} = -1$ $\quad\Rightarrow 2(\log_3 \cos x)^2 = -\log_3 a$ .... (1)

Clearly L.H.S. > 0 as $\cos x \neq 1$,

$\Rightarrow \log_3 a < 0$ $\quad\Rightarrow 0 < a < 1$

$\therefore$ From equation (1), $(\log_3 \cos x)^2 = -\frac{\log_3 a}{2}$

$\Rightarrow \log_3 \cos x = \pm\sqrt{\frac{-\log_3 a}{2}}$

But $\cos x \in (0, 1)$ $\quad\Rightarrow$ L.H.S. < 0

$\Rightarrow \log_3 \cos x = -\sqrt{\frac{-\log_3 a}{2}}$ $\quad\Rightarrow \cos x = (3)^{-\sqrt{-\log_3 \sqrt{a}}}$

$\Rightarrow x = 2n\pi \pm (3)^{-\sqrt{-\log_3 \sqrt{a}}}$; $n \in \mathbb{Z}$

TEXTUAL EXERCISE–6: (SUBJECTIVE)

1. Simplify and compute the following expressions:

(a) $\frac{(15)^3.(21)^3}{(35)^2.(3)^4}$ (b) $(-1.4)^3.\left(3\frac{4}{7}\right)^3$

(c) $\frac{5.2^{k-2}+10.2^{k-1}}{10^{k+2}}$ (d) $\left(\frac{\sqrt[4]{ab}-\sqrt{b}}{\sqrt{a}-\sqrt[4]{ab}}\right)^{-4}$

2. Which of the given pair of numbers is greater?

(a) 2^{300} or 3^{200} (b) 54^4 or 21^{12}

(c) $(0.4)^4$ or $(0.8)^3$ (d) 10^{20} and 40^{10}

3. Perform the indicated operations and simplify the given expression:

(a) $\left(\left(x^{\frac{2}{3}}y^{-1}\right)^2 \left(x^2 y^{-1}\right)^{\frac{1}{2}} \left(y^{\frac{2}{3}}\right)^{\frac{3}{2}}\right)^6$

(b) $\sqrt{\frac{a}{b}\sqrt{\frac{a}{b}.\sqrt[3]{\frac{a}{b}}}}.a^{-\frac{1}{3}}.b^{\frac{1}{3}}$

4. Solve the following exponential equation for x;

(a) $4^{2x^2-1} = 2$ (b) $5^{x-1} = 10^x.2^{-x}.5^{x+1}$

(c) $5^{2x-1} + 5^{x+1} = 250$ (d) $9^x + 6^x = 2.4^x$

(e) $2^{x+1}\,5^x = 200$ (f) $6^x + 6^{x+1} = 2^x + 2^{x+1} + 2^{x+2}$

5. Solve the following exponential inequalities for x;

(a) $2^{3-8x} > 1$ (b) $16^x > 0.125$

(c) $(0.3)^{2x^2-3x+6} < 0.00243$ (d) $\left(\frac{1}{3}\right)^{\sqrt{x+2}} > 3^{-x}$

(e) $\sqrt[x+1]{3} > 9$ (f) $8^{\sqrt{8^x}} > 4096$

(g) $\left(\frac{2}{5}\right)^{\frac{6-5x}{2+5x}} < \frac{25}{4}$ (h) $2^x + 2^{-x+1} - 3 < 0$

6. Express log of x to the base 2 in each of the following cases:

(a) $32^{\frac{1}{3}} = 4x$ (b) $64^{\frac{1}{4}} = 8x$

7. Prove that $\frac{\log_a N}{\log_{ab} N} = 1 + \log_a b$ and indicate the permissible values of the letters.

8. Find the domain of following logarithmic functions:
(a) $f(x) = \log_x 3$
(b) $f(x) = \log_2(x - 1)$
(c) $f(x) = \log_{2x-3}(x - 1)(x - 4)$
(d) $f(x) = \log_2\left[\dfrac{(x-4)(x-5)}{(x-6)}\right]$

9. Find the smallest possible value of the expression $\ln(x^3 - 5x^2 - 6x + 54) - \ln(x + 3)$

10. Solve the logarithmic equations
(i) $\log_{(x+3)}(x + 4) = 2$ (ii) $2^{\log_4(x+5)} = (x+5)$

11. Find the integer ordered pair solutions of equation $\log_{(x+4)}16 = (y + 2)$

12. Solve the equation $\sqrt[4]{|x-3|^{x+1}} = \sqrt[3]{|x-3|^{x-2}}$.

13. Find the solutions of the equation $\log_{\sin x} 5 . \log_{\sin^2 x} a + 1 = 0$.

14. Solve the inequality $\log_{x-2} 8 + \log_{x+2} 8 > \log_{x-2} 8 . \log_{x+2} 8$.

15. If $x > 0$, $y > 0$ and $z > 0$, prove that $x^{(\ln y - \ln z)} + y^{(\ln z - \ln x)} + z^{(\ln x - \ln y)} \geq 3$.

16. If $x = \dfrac{1}{2}$ satisfies the inequality $\log_a(x^2 - x + 2) - \log_a(-x^2 + 2x + 3) > 0$, then find all real solutions of the inequality.

Answer Keys

1. (a) 315 (b) –125 (c) $\dfrac{5^{-k}}{16}$ (d) $\dfrac{a}{b}$

2. (a) 3^{200} (b) 21^{12} (c) $(0.8)^3$ (d) 10^{20}

3. (a) $x^{14}.y^{-9}$ (b) $\left(\dfrac{a}{b}\right)^{\frac{1}{2}}$

4. (a) $\pm\dfrac{\sqrt{3}}{2}$ (b) $\{-2\}$ (c) 2 (d) 0 (e) $\{2\}$ (f) 0

5. (a) $(-\infty, 3/8)$ (b) $(-3/4, +\infty)$ (c) $(-\infty, 1/2) \cup (1, 8)$ (d) $(2, +\infty)$
(e) $x \in \phi, x \in \mathbb{N}$ (f) $(4/3, +\infty)$ (g) $(-\infty, -2) \cup (-2/5 + \infty)$ (h) $(0, 1)$

6. (a) –1/3 (b) –3/2

7. $a \neq 1, a > 0; b > 0; ab \neq 1$ $ab > 0; N > 0, N \neq 1$

8. (a) $(0, \infty) \sim \{1\}$ (b) $(1, \infty)$ (c) $(4, \infty)$ (d) (e) $(4, 5) \cup (6, \infty)$

9. ln2 **10.** (i) $\left(\dfrac{-5+\sqrt{5}}{2}\right)$ (ii) –4

11. (0, 0); (–2, 2); (12, –1) **12.** 2, 3, 4, 11

13. $x = n\pi + (-1)^n \sin^{-1} a^{\sqrt{-\log_a \sqrt{5}}}$; $n \in \mathbb{Z}$ for $a \in (0, 1)$ and $x = n\pi + (-1)^n . \sin^{-1}\left(a^{-\sqrt{-\log_a \sqrt{5}}}\right); a > 1$

14. $x \in (2, \sqrt{5}) \cup (3, \infty)$ **16.** $\left(\dfrac{3-\sqrt{17}}{4}, \dfrac{3+\sqrt{17}}{4}\right)$

TEXTUAL EXERCISE–6: (OBJECTIVE)

1. The value of $4^{2\log_9 3}$ is
(a) 4
(b) 2
(c) 3
(d) 1

2. If $\log_7 2 = n$, then $\log_{49} 28$ is equal to
(a) $\dfrac{1+2\eta}{4}$ (b) $\dfrac{1+2\eta}{2}$
(c) $\dfrac{1+2\eta}{3}$ (d) None of these

3. If $a^2 + 4b^2 = 12ab$, then log (a + 2b) is

(a) $\frac{1}{2}(\log a + \log b - \log 2)$

(b) $\log\frac{a}{2} + \log\frac{b}{2} + \log 2$

(c) $-(\log a + \log b + 4\log 2)$

(d) $\frac{1}{2}(\log a - \log b - 4\log 2)$

4. If $\log_a ab = x$, then $\log_b ab$ is equal to

(a) $\frac{1}{x}$ (b) $\frac{x}{1+x}$

(c) $\frac{x}{x-1}$ (d) $\frac{x}{1-x}$

5. If $\frac{\log_e a}{y-z} = \frac{\log_e b}{z-x} = \frac{\log_e c}{x-y}$ then $a^{y^2+yz+z^2} b^{z^2+zx+x^2} c^{x^2+xy+y^2}$ is equal to

(a) 2 (b) 0

(c) 1 (d) None of these

6. The value of $81^{\frac{1}{\log_5 3}} + 27^{\log_9 36} + 3^{\frac{4}{\log_7 9}}$ is equal to

(a) 49 (b) 625

(c) 216 (d) 890

7. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$, then

(a) $f(x_1).f(x_2) = f(x_1 + x_2)$

(b) $f(x + 2) - 2f(x + 1) + f(x) = 0$

(c) $f(x) = f(x + 1) = f(x^2 + x)$

(d) $f(x_1) + f(x_2) = f\left(\frac{x_1 + x_2}{1 + x_1 x_2}\right)$

8. If $y = 2^{\frac{1}{\log_x 4}}$, then x is equal to

(a) $\sqrt{y}$ (b) y

(c) y^2 (d) y^4

9. If $\frac{1}{\log_a x} + \frac{1}{\log_c x} = \frac{2}{\log_b x}$, then a, b, c are in

(a) A.P. (b) G.P.

(c) A.P. and G.P. both (d) None of these

10. The solution of equation $\log_{10}(x^2 - x - 6) - x = \log_{10}(x+2) - 4$ is

(a) $x = 5$ (b) $x = 3$

(c) $x = 4$ (d) None of these

11. The number of real values of the parameter k for which $(\log_{16} x)^2 - (\log_{16} x) + (\log_{16} k) = 0$ with real coefficients will have exactly one solution is

(a) 2 (b) 1

(c) 4 (d) None of these

12. If $x + \log_{10}(1 + 2^x) = x\log_{10} 5 + \log_{10} 6$, then x is equal to

(a) 4 (b) 0

(c) 1 (d) None of these

13. If $\log_{1/\sqrt{2}} \sin x > 0$, $x \in [0, 4\pi)$ then the number of values of x which are integral multiples of $\pi/4$ is

(a) 6 (b) 12

(c) 3 (d) None of these

14. If $\log_{1/2}(x+1) \le \log_2(2-x)$, then x belongs to the interval

(a) $\left(0, \frac{1-\sqrt{5}}{2}\right]$ (b) $\left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right]$

(c) $\left[\frac{1+\sqrt{5}}{2}, \infty\right)$ (d) None of these

15. If $\log_{0.3}(x - 1) < \log_{0.09}(x - 1)$, then x lies in the interval

(a) $(2, \infty)$ (b) $(-2, -1)$

(c) $(1, 2)$ (d) None of these

16. The number of values of $x \in [0, n\pi]$, $n \in \mathbb{Z}$ that satisfy $\log_{|\sin x|}(1 + \cos x) = 2$ is

(a) 0 (b) n

(c) $2n$ (d) None of these

17. If $\log_{\cos x} \sin x \ge 0$ and $0 \le x \le 3\pi$, then $\sin x$ is in the interval

(a) $\left[\frac{\sqrt{5}-1}{2}, 1\right]$ (b) $\left(0, \frac{\sqrt{5}-1}{2}\right]$

(c) $\left[0, \frac{1}{2}\right]$ (d) None of these

18. If $\log_{\sqrt{3}}(\sin x + 2\sqrt{2}\cos x) \ge 2$, $-2\pi < x \le 2\pi$, then the number of solutions of x is

(a) 0 (b) infinite

(c) 3 (d) None of these

Answer Keys

1. (a)	**2.** (b)	**3.** (c)	**4.** (c)	**5.** (c)	**6.** (d)	**7.** (d)	**8.** (c)	**9.** (b)	**10.** (c)
11. (b)	**12.** (c)	**13.** (a)	**14.** (b)	**15.** (a)	**16.** (a)	**17.** (d)	**18.** (d)		

GREATEST INTEGER FUNCTION (BRACKET FUNCTION)

Greatest integer function of x is the largest integer which does not exceed x, it is denoted as $[x]$. And also called as integral part of x

Origin: If a and b are positive integers, such that $a = qb + r,\ 0 \le r < b$

Then $\dfrac{a}{b} = q + \dfrac{r}{b}$, where $0 \le \dfrac{r}{b} < 1$

That is, $\left[\dfrac{a}{b}\right]$ is the quotient in the division of a by b.

$[x]$ = greatest integer $\le x$ = the nearest integer to left to

$$x = \begin{cases} x; & x \in \mathbb{Z} \\ x; & x \notin \mathbb{Z} \end{cases}$$

$\therefore\quad [x] = n,\ n \le x < n + 1$

$$\Rightarrow\quad [x] = \begin{cases} -2; & \text{if } -2 \le x < -1 \\ -1; & \text{if } -1 \le x < 0 \\ 0; & \text{if } 0 \le x < 1 \\ 1; & \text{if } 1 \le x < 2 \end{cases}\text{; Graph of gint } x$$

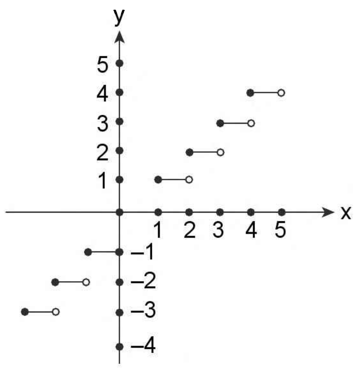


FIGURE 2.96

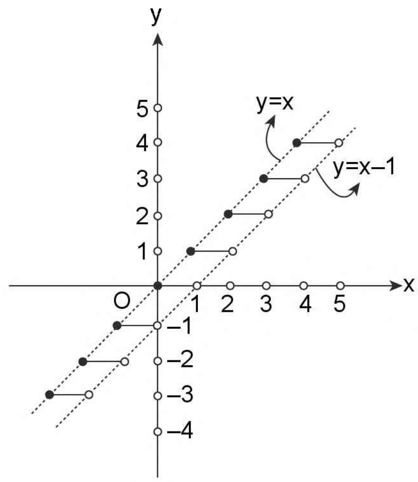


FIGURE 2.97

Graph of gint x is represented by thick horizontal unit length line segments at unit distances on, above, and below x-axis bounded between the lines $y = x$ and $y = x - 1$ as shown in Figure 2.97. The left end point of line segments is included whereas right end point is excluded in the graph.

PROPERTIES OF GREATEST INTEGER FUNCTION (BRACKET FUNCTION)

(i) Domain of $[x]$: $\mathbb{R}$; Range of $[x]$: $\mathbb{Z}$

(ii) $[[x]] = [x]$

Proof: We know that $[x] = n\ \forall\ n \in \mathbb{Z}$

Let $[x] = m$

$\Rightarrow\quad [[x]] = [m] = m = [x]$

Thus, $[[x]] = [x]\ \forall\ x \in \mathbb{R}$

ILLUSTRATION 79: Find the set of all values of $[x]$ for which the function $f(x) = \dfrac{1}{\sqrt{6[[x]] - 5[x] - 4}}$ is defined, and hence, solve the equation $f(x) = \dfrac{1}{4}$ for $[x]$ showing that this equation has infinitely many solutions x.

SOLUTION: For $f(x)$ to be defined $6[[x]] - 5[x] - 4 > 0 \quad \Rightarrow 6[x] - 5[x] - 4 > 0 \quad \Rightarrow [x] > 4$

$\Rightarrow\quad [x] \ge 5 \qquad \Rightarrow\quad [x] \in \{5, 6, 7, 8, \ldots\} \qquad (\because [[x]] = [x])$

Thus, $\{5, 6, 7, 8, \ldots\}$ is the set of all values of $[x]$ for which $f(x)$ is defined

Now $f(x) = \frac{1}{4} \Rightarrow \frac{1}{\sqrt{6[[x]]-5[x]-4}} = \frac{1}{4} \Rightarrow \sqrt{6[x]-5[x]-4} = 4$

$\Rightarrow \sqrt{[x]-4} = 4 \Rightarrow [x] - 4 = 16 \Rightarrow [x] = 20$

Thus, $f(x) = \frac{1}{4}$ holds for $[x] = 20$.

Now equation $f(x) = \frac{1}{4}$ holds only when $[x] = 20$. But $[x] = 20$ holds for all real numbers which are greater than or equal to 20 but less than 21.

That is, $x \in [20, 21)$ which contains infinitely many solutions x.

(iii) $[x + m] = [x] + m$ provided $m \in \mathbb{Z}$

Proof: Let $x = n + f$, where $0 \le f < 1$ and $n \in \mathbb{Z}$

$\therefore [x + m] = [(n + m) + f]$

$= n + m = [x] + m$ $\quad (\because x = n + f \Rightarrow [x] = n)$

Hence, proved.

ILLUSTRATION 80: Solve the following equations:

(a) $[x + 4] + 2[x] = 10$ (b) $2[x - 4 + [x]] = 6 - 3[x]$

SOLUTION: (a) $[x + 4] + 2[x] = 10$

$\Rightarrow [x] + 4 + 2[x] = 10$ $\quad (\because [x + m] = [x] + m$ for $x \in \mathbb{R}, m \in \mathbb{Z})$

$\Rightarrow 3[x] = 6 \Rightarrow [x] = 2 \Rightarrow x \in [2, 3)$ $\quad (\because [x] = n \Rightarrow x \in [n, n + 1))$

Thus, [2, 3) is the required solutions set.

(b) $2[x - 4 + [x]] = 6 - 3[x]$

$\Rightarrow 2([x] - 4 + [x]) = 6 - 3[x] \Rightarrow 4[x] - 8 = 6 - 3[x]$

$\Rightarrow 7[x] = 14 \Rightarrow [x] = 2 \Rightarrow x \in [2, 3)$

Thus, [2, 3) is the required solution set.

ILLUSTRATION 81: Let $S_k = \sum_{k=1}^{n} k!$, then evaluate $\sum_{n=6}^{10}\left(S_n - 6\left[\frac{S_n}{6}\right]\right)$; where [.] is gint function.

SOLUTION: Given $S_k = \sum_{k=1}^{n} k! = 1! + 2! + 3! + \ldots + n!$

For $n = 5$,

$S_5 = 1! + 2! + 3! + 4! + 5! = 1 + 2 + 6 + 24 + 120 = 153$

For $n \ge 6$,

$S_n = 1! + 2! + 3! + 4! + 5! + 6! + 7! + \ldots. + n! = 153 + 6m; m \in \mathbb{Z} = 6m + 150 + 3$

$= 6(m + 25) + 3 = 6r + 3; r \in \mathbb{Z}$

$\therefore$ for $n \ge 6$, $S_n - 6\left[\frac{S_n}{6}\right] = (6r+3) - 6\left[\frac{6r+3}{6}\right]$

$= (6r+3) - 6\left[r + \frac{1}{2}\right] = (6r+3) - 6\left(r + \left[\frac{1}{2}\right]\right) = 6r + 3 - 6r = 3$

$\therefore \sum_{n=6}^{10}\left(S_n - 6\left[\frac{S_n}{6}\right]\right) = 5(3) = 15$

(iv) $[x + [y + [z]]] = [x] + [y] + [z]$

Proof: LHS $= [x + [y + [z]]] = [y + [z]] + [x]$

($\because$ By property $[x + m] = m + [x]; x \in \mathbb{Z}$)

$= [z] + [y] + [x]$

(By same property)

Thus, $[x + [y + [z]]] = [x] + [y] + [z] \ \forall\ x, y, z \in \mathbb{R}$

ILLUSTRATION 82: Solve the following equations

(a) $[[x + [x + [x]]]] = 12$ (b) $\dfrac{4[[x+2]]}{[[x+[x+[x]]]]-12} = \dfrac{1}{3}$

SOLUTION: (a) $[[x + [x + [x]]]] = 12$

$\Rightarrow [[x] + [x] + [x]] = 12$ ($\because [x + [y + [z]]] = [x] + [y] + [z]$)

$\Rightarrow 3[x] = 12 \Rightarrow [x] = 4 \Rightarrow x \in [4, 5)$ ($\because 3[x] \in \mathbb{Z}$ and $[m] = m\ \forall\ m \in \mathbb{Z}$)

(b) $\dfrac{4[[x+2]]}{[[x+[x+[x]]]]-12} = \dfrac{1}{3} \Rightarrow \dfrac{4[[x]+2]}{3[x]-12} = \dfrac{1}{3}$

($\because [x + m] = [x] + m\ \forall\ n \in \mathbb{R}, m \in \mathbb{Z}$ and by part (a))

$\Rightarrow \dfrac{4([x]+2)}{3[x]-12} = \dfrac{1}{3}$ ($\because [x] + 2 \in \mathbb{Z}$ and $[m] = m\ \forall\ m \in \mathbb{Z}$)

$\Rightarrow 12[x] + 24 = 3[x] - 12 \Rightarrow 9[x] = -36 \Rightarrow [x] = -4 \Rightarrow x \in [-4, -3)$

(v) $[x] > n; n \in \mathbb{Z}$

$\Rightarrow [x] \in \{n + 1, n + 2, n + 3, \dots\}$

$\Rightarrow x \in [n + 1, \infty)$

Proof: Given $[x] > n$

$\Rightarrow$ $[x]$ is an integer greater than or equal to $(n + 1)$.

i.e., $[x] \geq (n + 1)$ .... (1)

but $[x] \leq x\ \forall\ x \in \mathbb{R}$ .... (2)

Thus, $(n + 1) \leq [x] \leq x$ [from (1) and (2)]

$\Rightarrow x \geq (n + 1)$

$\Rightarrow x \in [(n + 1), \infty)$.

Hence, proved

ILLUSTRATION 83: Find the domain of the functions

(a) $f(x) = \dfrac{1}{\sqrt{[x]-\sin\theta}}$; $\theta \in \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$ (b) $f(x) = \dfrac{\log([x]-1)}{\sqrt[3]{[x]-2}}$

SOLUTION: (a) $f(x) = \dfrac{1}{\sqrt{[x]-\sin\theta}}$; $\theta \in \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$

for $f(x)$ to be defined $[x] > \sin\theta$

$\Rightarrow [x] > 1 \Rightarrow [x] \in \{2, 3, 4,..\}$ $\left(\because \theta \in \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right] \Rightarrow \sin\theta \in [-1, 1]\right)$

$\Rightarrow x \in [2, \infty)$ ($\because [x] > n \Rightarrow x \in [n + 1, \infty)$)

$\therefore$ Domain of given function $= D_f = [2, \infty)$

(b) $f(x) = \dfrac{\log([x]-1)}{\sqrt[3]{[x]-2}}$

For $f(x)$ to be defined, $[x] - 1 > 0$ and $[x] \neq 2$

$\Rightarrow [x] > 1$ and $[x] \neq 2 \Rightarrow [x] \in \{3, 4, \dots\} \Rightarrow x \in [3, \infty) \Rightarrow D_f = [3, \infty)$

(vi) $[x] \geq n \Rightarrow x \in [n, \infty)$

Proof: Given $[x] \geq n$ .... (1)

Also $[x] \leq x \ \forall \ x \in \mathbb{R}$ ..... (2)

$\therefore$ From (1) and (2)

$x \geq [x] \geq n \Rightarrow x \geq n \Rightarrow x \in [n, \infty)$

ILLUSTRATION 84: Find the domain of $f(x)=\dfrac{1}{\sqrt{[x]-2}}+\sqrt{[x]-4}$

SOLUTION: For $f(x)$ to be defined, $[x] > 2$ and $[x] \geq 4$

$\Rightarrow [x] \geq 4 \qquad \Rightarrow [x] \in [4, \infty)$

$\therefore$ Domain of function $f(x) = [4, \infty)$

ILLUSTRATION 85: Find the domain of definition of the function $f(x)=\dfrac{\left(\sec^{-1} x\right)}{\sqrt{x-[x]}}$

SOLUTION: Given function is $f(x)=\dfrac{\left(\sec^{-1} x\right)}{\sqrt{x-[x]}}$;

which is defined for $x \in (-\infty, -1] \cup [1, \infty)$ for which $x - [x] \neq 0$, i.e., $\{x\} \neq 0$

$\Rightarrow$ Domain of function is $\mathbb{R} \sim \{(-1, 1) \cup \mathbb{Z}\}$.

(vii) $[x] < n \Rightarrow x \in (-\infty, n)$

Proof: Let $x = [x] + f$, where $0 \leq f < 1$.

Given $[x] < n$

$\Rightarrow [x] \leq (n-1)$

$\Rightarrow x - f \leq (n-1) \qquad (\because [x] = x - f)$

$\Rightarrow x \leq n - 1 + f$

$\Rightarrow x \leq n - (1-f) < n$ as $(1-f) > 0$

$\Rightarrow x < n$

$\Rightarrow x \in (-\infty, n)$

ILLUSTRATION 86: Find the domain of following functions

(a) $f(x)=\sqrt{[x]-4}+\dfrac{1}{\sqrt{6-[x]}}$ (b) $f(x) = \sin^{-1}[x+1]$

SOLUTION: (a) $f(x)=\sqrt{[x]-4}+\dfrac{1}{\sqrt{6-[x]}}$

For $f(x)$ to be defined $[x] - 4 \geq 0$; $6 - [x] > 0 \Rightarrow x \in [4, 6)$

$\Rightarrow [x] \geq 4$ and $[x] < 6$

$\Rightarrow x \in [4, \infty)$ and $x \in (-\infty, 6) \Rightarrow x \in (-\infty, 6) \cap [4, \infty)$

(b) $f(x) = \sin^{-1}[x+1]$

For $f(x)$ to be defined $[x+1] \in [-1, 1] \Rightarrow [x+1] = -1, 0, 1$

$\Rightarrow x + 1 \in [-1, 2) \qquad \Rightarrow x \in [-2, 1)$

ILLUSTRATION 87: Find the domain of the function $f(x)=\dfrac{1}{[|x-1|]+[|x-6|]-5}$; where [] represents greatest integer function.

SOLUTION: Given $f(x) = \dfrac{1}{[|x-1|]+[|x-6|]-5}$ is defined when $[|x-1|] + [|x-6|] - 5 \neq 0$ .... (1)

Here we have three cases:

Case 1: When $x \leq 1$

$[1 - x] + [6 - x] - 5 \neq 0$,

$\Rightarrow$ $1 + [-x] + 6 + [-x] - 5 \neq 0$,

$\Rightarrow$ $2[-x] + 2 \neq 0$ $\Rightarrow$ $[-x] \neq -1$ $\Rightarrow$ $-x \notin [-1, 0)$ $\Rightarrow$ $x \notin (0, 1]$ $\Rightarrow x \in (-\infty, 0]$ .... (2)

Case II: When $1 < x \leq 6$

$\Rightarrow$ $[x - 1] + [6 - x] - 5 \neq 0$

$\Rightarrow$ $[x] - 1 + 6 + [-x] - 5 \neq 0$ $\Rightarrow$ $[x] + [-x] \neq 0$ $\Rightarrow$ $x \notin \mathbb{Z}$

$\Rightarrow$ $x \notin \{2, 3, 4, 5, 6\}$ .... (3)

Case III: When $x > 6$

$\Rightarrow$ $[x - 1] + [x - 6] - 5 \neq 0$

$\Rightarrow$ $[x] - 1 + [x] - 6 - 5 \neq 0$ $\Rightarrow$ $2[x] \neq 12$

$\Rightarrow$ $[x] \neq 6$ $\Rightarrow$ $x \notin [6, 7)$ $\Rightarrow$ $x \in [7, \infty)$

Thus, for $f(x)$ to be defined $x \in (-\infty, 0] \cup (1, 6) \cup [7, \infty) \sim \{2, 3, 4, 5\}$

or $x \in \mathbb{R} \sim \{(0, 1] \cup [6, 7) \cup \{2, 3, 4, 5\}\}$.

(viii) $[x] \leq n \Rightarrow n \in (-\infty, n + 1)$

Proof: Let $x = [x] + f$, where $0 \leq f < 1$

Given $[x] \leq n$

$\Rightarrow$ $x - f \leq n$ $\Rightarrow$ $x \leq n + f < n + 1$ as $0 \leq f < 1$

$\Rightarrow$ $x < n + 1$ $\Rightarrow$ $x \in (-\infty, n + 1)$

ILLUSTRATION 88: Find the domain of function $f(x) = \sqrt{([x]-2)}.\sqrt{6-[x]}$

SOLUTION: For $f(x)$ to be defined $([x] - 2), (6 - [x]) \geq 0$

$\Rightarrow$ $[x] \geq 2$ and $6 \geq [x]$ $\Rightarrow$ $[x] \geq 2$ and $[x] \leq 6$

$\Rightarrow$ $x \in [2, \infty)$ and $x \in (-\infty, 7)$ $\Rightarrow$ $x \in (-\infty, 7) \cap [2, \infty) = [2, 7)$

$\therefore$ Domain of function $f(x) = [2, 7)$

(ix) $[-x] = \begin{cases} -[x] = -x & \text{if } x \in \mathbb{Z} \\ -1-[x] & \text{if } x \notin \mathbb{Z} \end{cases}$

Proof: We shall take two cases as given below

Case (i): If $x \in \mathbb{Z}$, then $-x \in \mathbb{Z}$

Thus, $[-x] = -x$ and $[x] = x$

$\Rightarrow$ $[-x] = -[x] = -x$

Case (ii) If $x \notin \mathbb{Z}$

Let $x = [x] + f$, where $0 < f < 1$

$\because$ $-x = -[x] - f = -[x] - 1 + 1 - f$

$= -[x] - 1 + f'$; where $1 - f = f' \in (0, 1)$

$\Rightarrow$ $[-x] = [(-[x] - 1) + f'] = -[x] - 1$ as $f' \in (0\ 1)$.

Hence, proved

ILLUSTRATION 89: Find the domain and range of the given the functions

(a) $f(x)=\dfrac{1}{\sqrt[3]{[x]+[-x]+1}}$ (b) $f(x)=\dfrac{\log([x]+[-x]+1)}{\sqrt{(x-2)(6-x)}}$

SOLUTION: (a) $f(x)=\dfrac{1}{\sqrt[3]{[x]+[-x]+1}}$

For $f(x)$ to be defined $[x] + [-x] + 1 \neq 0$

$\Rightarrow$ $[x] + [-x] \neq -1$

But we know that $[x]+[-x]=\begin{cases} 0 & \text{for } x\in\mathbb{Z} \\ -1 & \text{for } x\notin\mathbb{Z}\end{cases}$

$\therefore$ For $f(x)$ to be defined, $x \in \mathbb{Z}$

$\therefore$ $D_f = \mathbb{Z}$ = set of all integers. Further when $x \in D_f = \mathbb{Z}$.

$[x] + [-x] = 0 \Rightarrow f(x)=\dfrac{1}{\sqrt[3]{1}}=1$ $\therefore$ Range of $f(x) = \{1\}$

(b) $f(x)=\dfrac{\log([x]+[-x]+1)}{\sqrt{(x-2)(6-x)}}$

For $f(x)$ to be defined $[x] + [-x] + 1 > 0$; $(x - 2)(6 - x) > 0$

$[x] + [-x] = -1$, i.e., $[x] + [-x] + 1 = 0$ for $x \notin \mathbb{Z}$

and $[x] + [-x] = 0$, i.e., $[x] + [-x] + 1 = 1 > 0$ for $x \in \mathbb{Z}$

Thus, $[x] + [-x] + 1 > 0$ for $x \in \mathbb{Z}$

Also $(x - 2)(6 - x) > 0$ $\Rightarrow x \in (2, 6)$

$\therefore$ x must be an integer in $(2, 6)$ $\Rightarrow x \in \{3, 4, 5\}$

Domain of function $f(x) = D_f = \{3, 4, 5\}$ and

Range $(R_f) = \{f(3), f(4), f(5)\} = \{0\}$ $(\because f(3) = f(4) = f(5) = 0)$

(x) $x - 1 < [x] \leq x$; equality holds iff $x \in \mathbb{Z}$

Proof: To prove $x - 1 < [x] \leq x$

Consider two cases as given below

Case (i) If $x = n$; $n \in \mathbb{Z}$

$\therefore$ $[x] = n$

$\therefore$ $x - 1 = n - 1$; $[x] = n$ and $x = n$

$\therefore$ $(n - 1) < n = x$

$\Rightarrow$ $(x - 1) < [x] = x$

i.e., equality holds for $x \in \mathbb{Z}$.

Case (ii) If $x = n + f$, $0 < f < 1$

$\therefore$ $x - 1 = (n - 1) + f$, $[x] = n$

$\therefore$ $(n - 1) + f = n - (1 - f) < n < n + f$ as f, $(1 - f) \in (0, 1)$

Thus, $(x - 1) < [x] < x$

e.g., for $x = 2.5$

$x - 1 = 1.5$, $[x] = 2$

$\therefore$ $(x - 1) < [x] < x$

$\therefore$ inequality holds for $x \notin \mathbb{Z}$ and for $x = 5$

$(x - 1) = 4$; $[x] = 5$

$\Rightarrow$ $(x - 1) < [x] = x$

$\therefore$ equality holds for $x \in \mathbb{Z}$.

(xi) $[x] \leq x < [x] + 1$

Proof: To prove $[x] \leq x < [x] + 1$

From above property $x - 1 < [x] \leq x$

$\Rightarrow$ $[x] \leq x$ and $x - 1 < [x]$ ..... (1)

$\Rightarrow$ $x < [x] + 1$ ..... (2)

Combining (1) and (2), we have $[x] \leq x < [x] + 1$

Hence, proved

e.g., for $x = 2.5$

$[x] = 2$, $[x] + 1 = 3$

$\therefore$ $[x] < x < [x] + 1$

$\therefore$ inequality holds for $x \notin \mathbb{Z}$

and for $x = 5$; $[x] = 5$, $[x] + 1 = 6$

$\Rightarrow$ $[x] = x < [x] + 1$

$\therefore$ equality holds for $x \in \mathbb{Z}$

ILLUSTRATION 90: If $\left[\sqrt{n^2+1}\right]=\left[\sqrt{n^2+\lambda}\right]$; where $n, \lambda \in \mathbb{N}$ and [.] is gint function, then show that $2n$ values of λ are possible, and hence, solve $\left[\sqrt{26}\right]=\left[\sqrt{25+\lambda}\right]$ for $\lambda \in \mathbb{N}$.

SOLUTION: Given $\left[\sqrt{n^2+1}\right]=\left[\sqrt{n^2+\lambda}\right]$; $n, \lambda \in \mathbb{N}$ ... (1)

Now $n^2 < n^2 + 1 < n^2 + 2n + 1 \ \forall\ n \in \mathbb{N}$

$\Rightarrow n^2 < n^2 + 1 < (n + 1)^2 \ \forall\ n \in \mathbb{N} \Rightarrow n<\sqrt{n^2+1}<n+1 \ \forall\ n \in \mathbb{N} \Rightarrow \left[\sqrt{n^2+1}\right]=n$.(2)

$\therefore$ From (1) and (2), we have $n=\left[\sqrt{n^2+\lambda}\right]$

$\Rightarrow n\le\sqrt{n^2+\lambda}<(n+1)$, but $\lambda \in \mathbb{N}$ $\Rightarrow \sqrt{n^2+\lambda}\ne n$

$\Rightarrow n<\sqrt{n^2+\lambda}<(n+1) \Rightarrow n^2 < n^2 + \lambda < n^2 + 2n + 1 \Rightarrow 0 < \lambda < (2n + 1)$

$\Rightarrow \lambda \in \{1, 2, 3, \ldots, 2n\}$ i.e., λ can take $2n$ values

Now $\left[\sqrt{26}\right]=\left[\sqrt{25+\lambda}\right]$; $\lambda \in \mathbb{N}$ $\Rightarrow \left[\sqrt{25+1}\right]=\left[\sqrt{25+\lambda}\right]$

$\Rightarrow \left[\sqrt{(5)^2+1}\right]=\left[\sqrt{(5)^2+\lambda}\right] \Rightarrow n = 5 \Rightarrow \lambda \in \{1, 2, 3, \ldots, 10\}$.

(xii) $\left[\frac{[x]}{c}\right]=\left[\frac{x}{c}\right]$ for $c \in \mathbb{N}$ and $x \in \mathbb{R}$

Proof: Let $[x] = n$

$\Rightarrow x = n + r,\ 0 \le r < 1$ .... (i)

Also let LHS $=\left[\frac{[x]}{c}\right]=\left[\frac{n}{c}\right]=m$ ... (ii)

RHS $=\left[\frac{x}{c}\right]=\left[\frac{n+r}{c}\right]$

$=\left[\frac{mc+cs+r}{c}\right]=\left[m+\frac{cs+r}{c}\right]$;where $0\le s<1$. (iii)

(Putting the value of n from (ii))

Now, $0 \le cs \le (c - 1)$ and $0 \le r < 1 \Rightarrow 0 \le cs + r < c - 1 + 1$

$\therefore\ 0\le\frac{cs+r}{c}<1$

$\therefore$ From (iii) RHS $=\left[m+\frac{cs+r}{c}\right]=m$

$\therefore$ LHS = RHS

ILLUSTRATION 91: Solve the following equations

(a) $\left[\frac{[x]}{4}\right]=16$ (b) $\sin\left(\frac{\pi}{\left[\frac{[x]}{4}\right]}\right)=\frac{1}{2}$

SOLUTION: (a) Given $\left[\frac{[x]}{4}\right]=16 \Rightarrow \left[\frac{x}{4}\right]=16$ $\left(\because \left[\frac{[x]}{c}\right]=\left[\frac{x}{c}\right] \text{ for } x\in\mathbb{R} \text{ and } c\in\mathbb{N}\right)$

$\Rightarrow \frac{x}{4}\in[16,17)$ $\Rightarrow x\in[64,68)$

(b) Given $\sin\left(\frac{\pi}{\left[\frac{[x]}{4}\right]}\right)=\frac{1}{2}$ .... (1)

Now $\left[\frac{[\ ]}{\ }\right] = \ \in \mathbb{Z}$, $n \notin 0$ and we know that $\sin\frac{\pi}{6}=\frac{1}{2}$.

$\Rightarrow \left[\frac{[x]}{4}\right] = 6$ and no other integer value of $\left[\frac{[x]}{4}\right]$ satisfies equation (1) as negative (–ve) integer values make LHS –ve whereas positive (+ve) integer values make LHS function injective.

$\therefore \left[\frac{[x]}{4}\right]=6 \Rightarrow \left[\frac{x}{4}\right]=6 \Rightarrow \frac{x}{4}\in[6,7) \Rightarrow x \in [24, 28)$

(xiii) $[x] + [y] \le [x + y] \le [x] + [y] + 1$

Proof: To prove $[x] + [y] \le [x + y] \le [x] + [y] + 1$

Let us discuss three possible cases.

Case (i) When $x = n \in \mathbb{Z}$ and $y = m \in \mathbb{Z}$

$\Rightarrow [x] + [y] = n + m$; $[x + y] = [n + m] = n + m$

and $[x] + [y] = [x + y] < [x] + [y] + 1 = n + m + 1$

Thus, $[x] + [y] = [x + y] < [x] + [y] + 1$

Case (ii): When $x = n \in \mathbb{Z}$ and $y = m + f$, $0 < f < 1$

(Without loss of generality we can assume that one of x and y is integer and other non-integer)

$\therefore [x] + [y] = n + m$; $[x + y] = [n + m + f] = n + m$

and $[x] + [y] + 1 = n + m + 1$

Clearly $[x] + [y] = [x + y] < [x] + [y] + 1$

Case (iii): When $x = n + f$ and $y = m + f'$, where $n, m \in \mathbb{Z}$ and $f, f' \in (0, 1)$, then

$[x] + [y] = n + m$; $[x + y] = [(n + m) + (f + f')]$

$$= \begin{cases} n+m & \text{if } f+f'\in(0,1) \\ n+m+1 & \text{if } f+f'\in[1,2) \end{cases}$$

and $[x] + [y] + 1 = n + m + 1$

Clearly, $[x] + [y] \le [x + y] \le [x] + [y] + 1$.

(xiv) $[x] = \left[\frac{x}{2}\right]+\left[\frac{x+1}{2}\right] \ \forall x \in \mathbb{R}$

Proof: ***Case I:*** Let $x = 2m + y$, where m is an integer and $0 \le y < 1$

$$\Rightarrow [x]=2m,\ \left[\frac{x}{2}\right]=m$$

$$\Rightarrow \left[\frac{x+1}{2}\right]=\left[\frac{2m+1+y}{2}\right]=m$$

$$\Rightarrow \left[\frac{x}{2}\right]+\left[\frac{x+1}{2}\right]=2m=[x] \quad \left[\because \frac{1}{2}\le\frac{1+y}{2}<1\right]$$

Case II: Let $x = (2m + 1) + y$, where m is an integer and $0 \le y < 1$

$\Rightarrow [x] = 2m + 1$

Then, $\left[\frac{x}{2}\right]=m,\ \left[\frac{x+1}{2}\right]=\left[\frac{(2m+2)+y}{2}\right]=m+1$

$$\therefore \left[\frac{x}{2}\right]+\left[\frac{x+1}{2}\right]=2m+1=[x]$$

ILLUSTRATION 92: Solve the equation $\left[\frac{x+8}{2}\right]+\left[\frac{x+7}{2}\right]+\left[\frac{x+6}{2}\right]+\ldots\ldots+\left[\frac{x+1}{2}\right]+[x]=21$

SOLUTION: We know that $\left[\frac{x+1}{2}\right]+\left[\frac{x}{2}\right]=[x]$

$$\therefore \text{LHS} = \left(\left[\frac{x+8}{2}\right]+\left[\frac{x+7}{2}\right]\right)+\left(\left[\frac{x+6}{2}\right]+\left[\frac{x+5}{2}\right]\right)+\left(\left[\frac{x+4}{2}\right]+\left[\frac{x+3}{2}\right]\right)+\ldots + \left(\left[\frac{x+2}{2}\right]+\left[\frac{x+1}{2}\right]\right)+[x]$$

$= [x + 7] + [x + 5] + [x + 3] + [x + 1] + [x] = 21$ (By given equation)

$$= ([x] + 7) + ([x] + 5) + ([x] + 3) + ([x] + 1) + [x] = 21$$

$$(\because [x + m] = [x] + m \text{ for } x \in \mathbb{R},\ m \in \mathbb{Z})$$

$$\Rightarrow\ 5[x] + 16 = 21 \ \Rightarrow\ 5[x] = 5 \ \Rightarrow\ [x] = 1 \ \Rightarrow\ x \in [1, 2)$$

ILLUSTRATION 93: If n be any positive integer, then show that $\left[\frac{n+1}{2}\right]+\left[\frac{n+2}{4}\right]+\left[\frac{n+4}{8}\right]+\left[\frac{n+8}{16}\right]+\ldots=n$

SOLUTION: Applying the formula $[x]=\left[\frac{x}{2}\right]+\left[\frac{x+1}{2}\right]$ to $x = n,\ \frac{n}{2},\frac{n}{4},\frac{n}{8},\frac{n}{16},\ldots$

$$\Rightarrow\ [n]=\left[\frac{n}{2}\right]+\left[\frac{n+1}{2}\right];\ \left[\frac{n}{2}\right]=\left[\frac{n}{4}\right]+\left[\frac{(n/2)+1}{2}\right];\ \left[\frac{n}{4}\right]=\left[\frac{n}{8}\right]+\left[\frac{(n/4)+1}{2}\right] \text{ and}$$

$$\left[\frac{n}{8}\right]=\left[\frac{n}{16}\right]+\left[\frac{(n/8)+1}{2}\right],\ldots.$$

Adding corresponding sides and canceling out $\left[\frac{n}{2}\right],\left[\frac{n}{4}\right],\left[\frac{n}{8}\right],\ldots.$ from both sides,we get

$$[n]=\left[\frac{n+1}{2}\right]+\left[\frac{n+2}{4}\right]+\left[\frac{n+4}{8}\right]+\ldots. \text{ i.e., } n=\left[\frac{n+1}{2}\right]+\left[\frac{n+2}{4}\right]+\left[\frac{n+4}{8}\right]+\ldots.$$

(xv) The number of positive integers less than or equal to n and divisible by m is given by $\left[\,-\,\right]$; $m, n \in \mathbb{N}$.

Proof: Let $n, m \in \mathbb{N}$

$\therefore$ By division algorithm, there exists non-negative integers q and r such that $n = m(q) + r$, where $0 \le r < m$

$$\Rightarrow\ \frac{n}{m}=q+\frac{r}{m}$$

$$\Rightarrow\ \left[\frac{n}{m}\right]=\left[q+\frac{r}{m}\right]=q \qquad \left(\because 0\le r<m \Rightarrow 0\le\frac{r}{m}<1\right)$$

and we know that quotient q defines the number of possible integer multiples of m which are less than or equal to n, i.e., $1.m, 2.m, 3.m, \ldots, q.m$

Thus, there will be $\left[\frac{n}{m}\right]$ number of positive integer multiples of m less than or equal to n.

i.e., $\left[\frac{n}{m}\right]$ defines the number of positive integers less than or equal to n and divisible by m.

e.g., let $n = 29$, $m = 3$, then $\left[\frac{n}{m}\right]=\left[\frac{29}{3}\right]=[9.\bar{6}]=9$, and hence, there are exactly 9 positive integers less than or equal to 29, divisible by 3 which are 3, 6, 9, 12, 15, 18, 21, 24, 27.

ILLUSTRATION 94: If n and k are positive integers and k is greater than 1, then $\left[\frac{n}{k}\right]+\left[\frac{n+1}{k}\right]\le\left[\frac{2n}{k}\right]$.

SOLUTION: Let $n = qk + r$, q and r are integers and $0 \le r \le k - 1$ or $0 \le r \le k$.

Then $\frac{n}{k}=q+\frac{r}{k},\ \frac{n+1}{k}=q+\frac{r+1}{k},\ \frac{2n}{k}=2q+\frac{2r}{k}$.

Case I: If $r < k - 1$, then $\left[\frac{2n}{k}\right]=2q+\left[\frac{2(k-1)}{k}\right]=2q+1$

$$\therefore\ \left[\frac{n}{k}\right]=q,\ \left[\frac{n+1}{k}\right]=q,\ \left[\frac{2n}{k}\right]=2q+1 \Rightarrow \left[\frac{n}{k}\right]+\left[\frac{n+1}{k}\right]<\left[\frac{2n}{k}\right].$$

Case II: If $r = k - 1$, then $\left[\frac{n}{k}\right] = q, \left[\frac{n+1}{k}\right] = q+1, \left[\frac{2n}{k}\right] = \left[2q + \frac{2(k-1)}{k}\right]$

$= \left[2q + \left(2 - \frac{2}{k}\right)\right] = 2q+1 \qquad \Rightarrow \left[\frac{n}{k}\right] + \left[\frac{n+1}{k}\right] = \left[\frac{2n}{k}\right]$

Combining, the above cases, we get the desired result.

(xvi) If p is a prime number and e is the largest exponent of p such that, p^e divides $n!$, then $e = \sum_{k=1}^{\infty}\left[\frac{n}{p^k}\right]$

Proof: We know that $\left[\frac{n}{p}\right] = q$ (say) denotes the number of positive integers less than or equal to n divisible by p which are $p, 2p, 3p, \ldots, qp$ and $(q+1)p > n$

we are to find the highest exponent of p (say e) in $n!$ i.e., p^e divides $n!$ and p^{e+1} does not divide $n!$.

Out of first n consecutive integers the positive integers less than or equal to n divisible by p are p, $2p$, $3p$, ..., $\left[\frac{n}{p}\right]p$. Their product equals $\left[\frac{n}{p}\right]!.p^{\left[\frac{n}{p}\right]}$.

Thus, the above product would contribute $\left[\frac{n}{p}\right]$ in e out of these numbers some may be again divisible by p.

Such number will be $p^2, 2p^2, 3p^2, \ldots, \left[\frac{n}{p^2}\right]p^2$.

These numbers are clearly included in $\left[\frac{n}{p}\right]$ numbers divisible by p, so they would contribute $\left[\frac{n}{p^2}\right]$ in e. Further there would be numbers among them which are divisible by p^3 and they would contribute $\left[\frac{n}{p^3}\right]$ in e.

Continuing in this way, we have

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots = \sum_{i=1}^{\infty}\left[\frac{n}{p_i}\right]$$

Alternatively $E_p(n!) = E_p(1.2.3\ldots.n)$

$$= E_p\left(1.2.3\ldots p.(p+1)(p+2)\ldots 2p.(2p+1)\ldots\left[\frac{n}{p}\right]p\ldots.n\right)$$

$= E_p\left(p.2p\ldots.\left[\frac{n}{p}\right]p\right)$ ($\because$ other numbers are not divisible by p)

$$= E_p\left(\left[\frac{n}{p}\right]!(p)^{\left[\frac{n}{p}\right]}\right) = = E_p\left(\left[\frac{n}{p}\right]!\right) + E_p\left((p)^{\left[\frac{n}{p}\right]}\right)$$

$$= E_p\left(\left[\frac{n}{p}\right]!\right) + \left[\frac{n}{p}\right] \qquad \ldots. (1)$$

Let $\left[\frac{n}{p}\right] = k$, then

$$E_p(n!) = \left[\frac{n}{p}\right] + E_p(k!) = \left[\frac{n}{p}\right] + \left[\frac{k}{p}\right] + E_p\left(\left[\frac{k}{p}\right]!\right)$$

(using result (1) again)

$$= \left[\frac{n}{p}\right] + \left[\frac{[n/p]}{p}\right] + E_p\left(\left[\frac{k}{p}\right]!\right)$$

$$= \left[\frac{n}{p}\right] + \left[\frac{n/p}{p}\right] + E_p\left(\left[\frac{k}{p}\right]!\right)$$

$\left(\because \left[\frac{[x]}{c}\right] = \left[\frac{x}{c}\right] \text{ for } x \in \mathbb{R} \text{ and } c \in \mathbb{N}\right)$

$$= \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + E_p\left(\left[\frac{k}{p}\right]!\right)$$

Continuing in this way, we have

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots.. = \sum_{k=1}^{\infty}\left[\frac{n}{p^k}\right]$$

ILLUSTRATION 95: Find

(a) Exponent of 3 in 50! i.e., $E_3(50!)$

(b) Exponent of 5 in (1500)! i.e., $E_5(1500!)$

SOLUTION: (a) Exponent of 3 in 50! $= E_3(50!) = \sum_{k=1}^{\infty}\left[\frac{50}{(3)^k}\right] = \left[\frac{50}{3}\right] + \left[\frac{50}{3^2}\right] + \left[\frac{50}{3^3}\right] + \left[\frac{50}{3^4}\right] + \ldots..$

$= 16 + 5 + 1 + 0 + 0 + \ldots\ldots = 22$

$\therefore$ $E_3(50!) = 22$, i.e., $3^{22} | 50!$ but $3^{23} \nmid 50!$

(b) Exponent of 5 in (1500)! $= E_5(1500!) = \sum_{k=1}^{\infty}\left[\frac{1500}{(5)^k}\right]$

$= \left[\frac{1500}{5}\right]+\left[\frac{1500}{5^2}\right]+\left[\frac{1500}{5^3}\right]+\left[\frac{1500}{5^4}\right]+\left[\frac{1500}{5^5}\right]+\ldots..$

$= 300 + 60 + 12 + 2 + 0 + 0 + \ldots\ldots = 374$

$\therefore$ $(5)^{374}$ divides (1500)!, but $(5)^{375}$ does not divide (1500)!.

ILLUSTRATION 96: If $f(x) = x - [x]$; $x \in \mathbb{R} \sim \{0\}$, where is [.] gint function, then show that the equation $f(x)+f\left(\frac{1}{x}\right)=1$; has infinitely many solutions.

SOLUTION: Given $f(x)+f\left(\frac{1}{x}\right)=1 \Rightarrow x-[x]+\frac{1}{x}-\left[\frac{1}{x}\right]=1 \Rightarrow x+\frac{1}{x} = [x]+\left[\frac{1}{x}\right]+1$ ... (1)

Now RHS of (1) is an integer for each $x \in \mathbb{R} \sim \{0\}$,

$\Rightarrow$ LHS. of (1) must also be an integer.

So, let $x+\frac{1}{x} = [x]+\left[\frac{1}{x}\right]+1 = k$ (an integer)

$\Rightarrow$ $x+\frac{1}{x}=k$ $\Rightarrow$ $x^2 - kx + 1 = 0$

$\Rightarrow$ $x=\frac{k\pm\sqrt{k^2-4}}{2}$ $\Rightarrow$ For real x, $k^2 - 4 \geq 0$, i.e., $k \in (-\infty,-2] \cup [2, \infty)$

For $k = 2$; $x+\frac{1}{x}=2$ $\Rightarrow$ $x = 1$

$\therefore$ From (1); $2 = 1 + 1 + 1$, which is false and for $k = -2$, $\Rightarrow x+\frac{1}{x}=-2$ $\Rightarrow x = -1$

$\therefore$ From (1), $-2 = -1 - 1 + 1$, which is false.

$\therefore$ Solution does not exit for $k = \pm 2$

$\therefore$ Whenever $[x]+\left[\frac{1}{x}\right]+1>2$ or < -2, there exists a real x for which

$x+\frac{1}{x} = [x]+\left[\frac{1}{x}\right]+1$ and $[x]+\left[\frac{1}{x}\right]+1>2$ or <-2 can hold for infinitely many x.

For example, if $x \in [4, 5)$, then $[x] = 4$ and $\frac{1}{x}\in\left(\frac{1}{5},\frac{1}{4}\right]$

$\Rightarrow$ $[x]+\left[\frac{1}{x}\right]+1=5$ $\Rightarrow$ $x=\frac{5\pm\sqrt{25-4}}{2}=\frac{5\pm\sqrt{21}}{2}$ $\Rightarrow$ $x=\frac{5-\sqrt{21}}{2}$ as $x\in[4,5)$

Similarly if $x\in\left(\frac{1}{4},\frac{1}{3}\right)$ $\Rightarrow$ $[x] = 0$ and $\frac{1}{x}\in(3,4)$ $\Rightarrow$ $\left[\frac{1}{x}\right]=3$

$\therefore$ $[x]+\left[\frac{1}{x}\right]=3$ $\Rightarrow$ $[x]+\left[\frac{1}{x}\right]+1=4$ $\Rightarrow$ $x=\frac{4\pm\sqrt{16-4}}{2}=2\pm\sqrt{3}$

$\Rightarrow$ $x=2+\sqrt{3}$ as $x\in(3,4)$

Thus, the given equation has infinitely many solutions.

LEAST INTEGER FUNCTION

It is also called the ceiling of x and it is represented by $\lceil x \rceil$. It is the least integer greater than or equal to the number x. Therefore $y = \lceil x \rceil = I + 1$ if $I < x \le I + 1$

e.g., $\lceil 1.5 \rceil = 2, \lceil 2.9 \rceil = 3, \lceil -2.3 \rceil = -2, \lceil -0.6 \rceil = 0, \lceil 0.25 \rceil = 1$

Thus, $[x]$ converts $x = (I + f)$ into I while $\lceil x \rceil$ converts it into $I + 1$, but when x is an integer $[x] = x = \lceil x \rceil$

Hence, $\lceil x \rceil = -2$ for $-3 < x \le -2$

$\lceil x \rceil = -1$ for $-2 < x \le -1$

$\lceil x \rceil = 0$ for $-1 < x \le 0$

$\lceil x \rceil = 1$ for $0 < x \le 1$ and so on

It can be expressed graphically as shown in Figure 2.98.

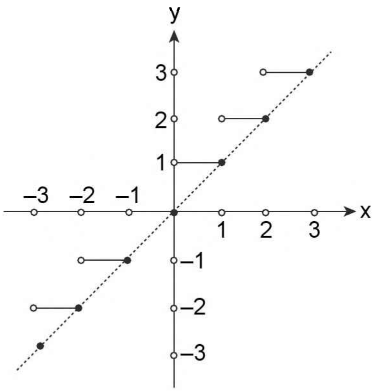


FIGURE 2.98

Properties of Least Integer Function

1. The domain of the function is: $(-\infty, +\infty)$
2. The range is the set of all integers.
3. $[x]$ converts $x = (I + f)$ into I while $\lceil x \rceil$ converts it into $I + 1$.

   e.g., If $x = 2.4$, then $2 < x < 3$

   $\Rightarrow \lceil x \rceil = 3 = I + 1$
4. When x is an integer $[x] = x = \lceil x \rceil$
5. $\lceil x + n \rceil = \lceil x \rceil + n$, where n is an integer.

Proof: Let $x \notin \mathbb{Z}$ and $x = m + f$, $m \in \mathbb{Z}$ and $f \in (0, 1)$; then $m < x < m + 1 \quad \Rightarrow \quad \lceil x \rceil = m + 1$ .... (1)

Now $m < x < m + 1 \quad \Rightarrow \quad m + n < x + n < m + n + 1$

$\Rightarrow \lceil x + n \rceil = m + n + 1 = n + \lceil x \rceil$ (from (1))

Thus, $\lceil x + n \rceil = n + \lceil x \rceil$, where $x \notin \mathbb{Z}$

If $x \in \mathbb{Z}$, then $\lceil x + n \rceil = x + n = \lceil x \rceil + n$

$\therefore \lceil x + n \rceil = \lceil x \rceil + n \ \forall\, x \in \mathbb{R}, n \in \mathbb{Z}$.

FRACTIONAL PART FUNCTION

Since every real number x can be written as the sum of its integer part and fractional i.e., $x = \underbrace{I}_{\text{integer part of } x} + \underbrace{f}_{\text{fractional part}}$;

where $f \in [0, 1)$, we denote $f = \{x\}$ and is called fractional part of x.

$\Rightarrow$ Fractional part function is denoted as $\{x\}$ and defined as $\{x\} = x - [x]$

$$\Rightarrow \{x\} = \begin{cases} x+2; & -2 \le x \le -1 \\ x+1; & -1 \le x < 0 \\ x; & 0 \le x < 1 \\ x-1; & 1 \le x < 2 \end{cases} \quad \text{and so on.}$$

Graph of fractional part function $\{x\}$ is as shown below:

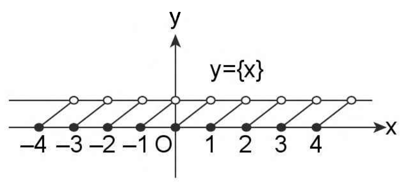


FIGURE 2.99

Properties of Fractional Part Function

(i) Domain of fractional part function $= D_f = \mathbb{R}$; Range of fractional part function $= R_f = [0, 1)$

ILLUSTRATION 97: Find the domain and range of function $f(x) = \dfrac{x - [x]}{1 - x + [x]}$

SOLUTION: $f(x) = \dfrac{x - [x]}{1 - x + [x]} = \dfrac{\{x\}}{1 - (x - [x])} = \dfrac{\{x\}}{1 - \{x\}}$

For domain of function $1 - \{x\} \neq 0 \Rightarrow \{x\} \neq 1$. Which always hold $\forall x \in \mathbb{R} \Rightarrow D_f = \mathbb{R}$

Now $f(x) = \dfrac{\{x\}}{1-\{x\}} = -\left(\dfrac{1-\{x\}-1}{1-\{x\}}\right) = -1 + \dfrac{1}{1-\{x\}}$

Now $-1 < -\{x\} \leq 0 \Rightarrow 0 < 1 - \{x\} \leq 1 \Rightarrow 1 \leq \dfrac{1}{1-\{x\}} < \infty$

$\Rightarrow 0 \leq -1 + \dfrac{1}{1-\{x\}} < \infty \Rightarrow 0 \leq f(x) < \infty \Rightarrow$ Range of function $f(x) = R_f = [0, \infty)$

ILLUSTRATION 98: Solve the inequality $x[x] - x^2 - 5[x] + 5x < 0$

SOLUTION: Given inequality is $x[x] - x^2 - 5[x] + 5x < 0$

$\Rightarrow x(x - \{x\}) - x^2 - 5(x - \{x\}) + 5x < 0$

$\Rightarrow x^2 - x\{x\} - x^2 - 5x + 5\{x\} + 5x < 0$

$\Rightarrow -x\{x\} + 5\{x\} < 0 \qquad \Rightarrow (\{x\})(5 - x) < 0$ ...(1)

Now for $x \in \mathbb{Z}$, $\{x\} = 0$, therefore (1) would not hold, so $x \notin \mathbb{Z}$

$\Rightarrow \{x\} \in (0, 1)$, thus, from (1), we have $(5 - x) < 0 \qquad \Rightarrow x > 5$

Thus, solution of given inequality is every non-integer real number greater than 5.

ILLUSTRATION 99: Prove that $[x] + \left[x + \dfrac{1}{2}\right] = [2x]$

SOLUTION: Let $x = [x] + \{x\}$

$\therefore$ LHS. of given equation becomes, $[x] + \left[[x] + \{x\} + \dfrac{1}{2}\right] = [x] + [x] + \left[\{x\} + \dfrac{1}{2}\right]$.

Let us deal with the following two cases:

Case (i): If $0 \leq \{x\} < \dfrac{1}{2} \Rightarrow \dfrac{1}{2} \leq \{x\} + \dfrac{1}{2} < 1 \Rightarrow \left[\{x\} + \dfrac{1}{2}\right] = 0$

$\therefore$ LHS $= [x] + [x] + 0 = 2[x]$ and RHS $= [2x] = [2[x] + 2\{x\}] = 2[x] + [2\{x\}] = 2[x]$

Case (ii): If $\dfrac{1}{2} \leq \{x\} < 1$, i.e., $1 \leq 2\{x\} < 2$ and $\{x\} + \dfrac{1}{2} \in \left[1, \dfrac{3}{2}\right)$

$\therefore$ LHS $= [x] + [x] + 1 = 2[x] + 1$ and RHS $= 2[x] + [2\{x\}] = 2[x] + 1$.

(ii) $\{x\}$ is periodic function with period 1.

Proof: *A* function $f(x)$ is periodic with period T if $f(x + T) = f(x)\ \forall x \in D_f$

Let x be any real number such that $x = n + f'$; where $n \in \mathbb{Z}$ and $0 \leq f' < 1$

Now $f(x) = \{x\} = \{n + f'\} = f'$, then $f(x + 1) = \{x + 1\}$
$= \{(n + 1) + f'\} \qquad (\because\ x = n + f')$
$= f' = \{x\} = f(x)$

Thus, $f(x + 1) = f(x)\ \forall x \in \mathbb{R}$ (domain of $\{x\}$)

Hence, fractional part function $\{x\}$ is periodic with period 1.

ILLUSTRATION 100: Find the domain and range of functions

(a) $f(x) = [\cos \{x\}]$ (b) $f(x) = [\tan \{x\}]$

(c) $f(x) = \{\tan x\}$

SOLUTION: (a) $\cos \theta$ is defined $\forall \theta \in \mathbb{R}$ and $\{x\} \in [0, 1)\ \forall x \in \mathbb{R}$

Thus, x can take any real value

$\Rightarrow$ Domain of function $= \mathbb{R}$

Now $0 \le \{x\} < 1$

$\Rightarrow$ $\cos 0 \ge \cos\{x\} > \cos 1$ $\quad$ ($\because$ for $\theta \in [0, 1]$ $\cos\theta$ decreases)

$\Rightarrow$ $1 \ge \cos\{x\} > \cos 1$

$\Rightarrow$ $[1] \ge [\cos\{x\}] \ge [\cos 1]$ $\quad$ ($\because$ [ ] is an increasing function)

$\Rightarrow$ $1 \ge [\cos\{x\}] \ge 0$ $\quad$ $\left(1 > \cos 1 > \cos\frac{\pi}{2} = 0\right)$

$\Rightarrow$ $f(x) \in [0, 1]$; but $f(x)$ can take only integer values

$\Rightarrow$ $R_f = \{0, 1\}$

(b) $\tan\theta$ is defined for $\theta \in \mathbb{R} \sim \left\{(2n+1)\frac{\pi}{2}; n \in \mathbb{Z}\right\}$

$\Rightarrow$ Range of $\{x\} = [0, 1)$ is contained in domain of $\tan\theta$

$\Rightarrow$ $\tan\{x\}$ is defined $\forall\, x \in \mathbb{R}$

$\Rightarrow$ Domain of $f(x) = [\tan\{x\}]$ is $\mathbb{R}$.

$\Rightarrow$ But $0 \le \{x\} < 1 \;\Rightarrow\; \tan 0 \le \tan\{x\} < \tan 1$ $\quad$ ($\because$ $\tan\theta$ is increasing for $\theta \in [0, 1)$)

$\Rightarrow$ $0 \le \tan\{x\} < \tan 1 \;\Rightarrow\; [0] \le [\tan\{x\}] \le [\tan 1] \;\Rightarrow\; 0 \le f(x) \le 1$

$$\left\{\because \frac{\pi}{4} < 1 < \frac{\pi}{3} \Rightarrow \tan\frac{\pi}{4} < \tan 1 < \tan\frac{\pi}{3} \Rightarrow 1 < \tan 1 < \sqrt{3} \;\Rightarrow [\tan 1] = 1\right\}$$

But $f(x)$ can take only integer values $\quad\Rightarrow$ Range of $f(x) = \{0, 1\}$.

(c) $f(x) = \{\tan x\}$, $\tan x$ is defined for $x \in \mathbb{R} \sim \left\{(2n+1)\frac{\pi}{2}; n \in \mathbb{Z}\right\}$

$\therefore$ Domain of $f(x) = \mathbb{R} \sim \left\{(2n+1)\frac{\pi}{2}; n \in \mathbb{Z}\right\}$

But $\tan x \in (-\infty, \infty)$ and $\{.\}$ is periodic with period 1 and has range $= [0, 1)$

$\Rightarrow$ $\{\tan x\} \in [0, 1) =$ Range of $f(x) = R_f$

ILLUSTRATION 101: Solve the following equations

(i) $\{\sin x\} + \{\cos x\} = 2$

(ii) $\{\sin x\} + \{\cos x\} = 0$

(iii) $\{x\} + \{\sin x\} + \{\cos x\} = 0$, where { } is fractional part function.

SOLUTION: (i) Since $\{\sin x\}, \{\cos x\} \in [0, 1)$

$\Rightarrow$ $\{\sin x\} + \{\cos x\} \in [0, 2)$ $\quad\Rightarrow$ $\{\sin x\} + \{\cos x\} \ne 2$

$\Rightarrow$ There exists no solution for the given equation.

(ii) Again since $\{\sin x\} + \{\cos x\} \in [0, 2)$

$\therefore$ $\{\sin x\} + \{\cos x\} = 0 \Leftrightarrow$ both $\sin x$ and $\cos x$ are integers

$\Rightarrow$ $\sin x = \pm 1, \cos x = 0$ $\quad$ or $\quad$ $\sin x = 0, \cos x = \pm 1$

$\Rightarrow$ $x \in \left\{n\pi, (2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\right\}$

(iii) $\{x\} + \{\sin x\} + \{\cos x\} = 0$. Each of $\{x\}, \{\sin x\}, \{\cos x\}$ is non-negative

$\Rightarrow$ $\{x\} = \{\sin x\} = \{\cos x\} = 0$ $\quad\Rightarrow$ x, $\sin x$ as well as $\cos x$ all must be integers.

But $\sin x$ and $\cos x$ take integers values only at $x = n\pi$ or $(2n+1)\frac{\pi}{2}$ which are irrationals for $n \ne 0$, and hence, can never be integers for $n \ne 0$. So, there is only one solution for the given equation, i.e., $x = 0$.

(iii) $[\{x\}] = 0$

Proof: Let $x = n + f$, where $n \in \mathbb{Z}$ and $f \in [0, 1)$, then $\{x\} = \{n + f\} = f$

$\Rightarrow$ $[\{x\}] = [f] = 0$ as $0 \le f < 1$.

Thus, $[\{x\}] = 0 \ \forall \ x \in \mathbb{R}$

ILLUSTRATION 102: If $f(x) = [x]$ and $g(x) = x - [x]$, then which of the following is/are the zero function(x)?

(a) $(f + g)(x)$ (b) $(f \,.\, g)(x)$

(c) $(f - g)(x)$ (d) $(fog)\,(x)$

SOLUTION: (a) $(f + g)(x) = f(x) + g(x) = [x] + x - [x] = x \ \forall \ x \in \mathbb{R}$.

$\Rightarrow$ $(f + g)\,(x)$ is an identity function

(b) $(f.g)\,(x) = f(x)\,.\,g(x) = [x]\,(x - [x]) \neq 0 \ \forall \ x \in \mathbb{R}$.

e.g., if $x = 2.4$; then $(f.g)\,(2.4) = [2.4]\,(2.4 - [2.4]) = (2)\,(2.4 - 2) = 2(.4) = 0.8 \neq 0$

(c) $(f - g)\,(x) = f(x) - g(x) = [x] - x + [x] = 2[x] - x \neq 0 \ \forall \ x \in \mathbb{R}$.

e.g., $(f - g)\,(2.4) = 2(2) - 2.4 = 1.6 \neq 0$

(d) $(fog)\,(x) = f(g(x)) = f(x - [x]) = [x - [x]] = [\{x\}] = 0$ as $\{x\} \in [0, 1) \ \forall \ x \in \mathbb{R}$.

Thus, only $(fog)\,(x)$ is a zero function.

(iv) $\{[x]\} = 0$

Proof: Let $x = n + f$, where $n \in \mathbb{Z}$ and $f \in [0, 1)$, then $[x] = n$

$\Rightarrow$ $\{[x]\} = \{n\} = 0$ as every integer has no fractional part.

Thus, $\{[x]\} = 0$

(v) $\{\{x\}\} = \{x\}$; this result is also true when fractional part function is applied on x on left hand side more than twice.

Proof: Let $x = n + f$, where $n \in \mathbb{Z}$ and $f \in [0, 1)$, then $\{x\} = \{n + f\} = f$

$\Rightarrow$ $\{\{x\}\} = \{f\} = f$ as $0 \le f < 1$

Thus, $\{\{x\}\} = \{x\}$

The above result can be generalized to any finite number of fractional part functions applied on x.

i.e., $\{\{\{\ldots\{x\}\ldots\}\}\} = \{x\} \ \forall \ x \in \mathbb{R}$.

(vi) $\{-x\} = \begin{cases} 0; & x \in \mathbb{Z} \\ 1 - \{x\}; & x \notin \mathbb{Z} \end{cases}$

Proof: $\because$ $-x = [-x] + \{-x\}$

$$\Rightarrow \{-x\} = -x - [-x] = \begin{cases} -x + x = 0; & \text{if } x \in \mathbb{Z} \\ -x - (-1 - [x]); & \text{if } x \notin \mathbb{Z} \end{cases}$$

$$\Rightarrow \{-x\} = \begin{cases} 0; & \text{if } x \in \mathbb{Z} \\ 1 - \{x\}; & \text{if } x \notin \mathbb{Z} \end{cases}$$

ILLUSTRATION 103: Find the domain and range of function $f(x) = \dfrac{1}{\sqrt[5]{\{x\} + \{-x\} - 1}}$

SOLUTION: For $f(x)$ to be defined $\{x\} + \{-x\} - 1 \neq 0$

$\Rightarrow$ $\{x\} + \{-x\} \neq 1$

$\Rightarrow$ $\{x\} + \{-x\} = 0$ as $\{x\} + \{-x\} = \begin{cases} 0 & \text{for } x \in \mathbb{Z} \\ 1 & \text{for } x \notin \mathbb{Z} \end{cases}$

$\therefore$ $f(x) = \dfrac{1}{\sqrt[5]{-1}}$ for $x \in \mathbb{Z}$ $\Rightarrow$ $f(x) = -1$ for $x \in \mathbb{Z}$

$\therefore$ Domain $= D_f = \mathbb{Z}$ and range $= \{-1\}$

(vii) $[x+y]=\begin{cases}[x]+[y]; & 0\le\{x\}+\{y\}<1\\ [x]+[y]+1; & 1\le\{x\}+\{y\}<2\end{cases}$

Proof: $\because$ $[x + y] = [[x] + [y] + \{x\} + \{y\}] = [x] + [y] + [\{x\} + \{y\}]$ ($\because$ $[x + m] = m + [x]\ \forall\ x \in \mathbb{R}$)

Now $0 \le \{x\} < 1$ and $0 \le \{y\} < 1$

$\Rightarrow$ $0 \le \{x\} + \{y\} < 2$

$\Rightarrow$ $[x+y]=\begin{cases}[x]+[y]; & 0\le\{x\}+\{y\}<1\\ [x]+[y]+1; & 1\le\{x\}+\{y\}<2\end{cases}$

ILLUSTRATION 104: Find all possible solutions of equation $[x + y] = 3$

SOLUTION: We know that $[x + y] = \begin{cases}[x]+[y]; & \text{for } \{x\}+\{y\}\in[0,1)\\ [x]+[y]+1; & \text{for } \{x\}+\{y\}\in[1,2)\end{cases}$

$\therefore$ $[x + y] = 3$

$\Rightarrow$ $[x] + [y] = 3$ for $\{x\} + \{y\} \in [0, 1)$ and $[x] + [y] + 1 = 3$ for $\{x\} + \{y\} \in [1, 2)$

$\Rightarrow$ $[x] + [y] = 3$ for $\{x\} + \{y\} \in [0, 1)$ and $[x] + [y] = 2$ for $\{x\} + \{y\} \in [1, 2)$

Case 1: $\{x\} + \{y\} \in [0, 1)$

Let $[x] = m$

$\Rightarrow$ $[y] = 3 - m$ and $\{x\} + \{y\} \in [0, 1)$; $m \in \mathbb{Z}$

$\Rightarrow$ $x = m + f$, $y = (3 - m) + f'$; $f + f' \in [0, 1)$; $m \in \mathbb{Z}$

$\Rightarrow$ $x = m + f$, $y = (3 - m) + f'$; $f' \in [0, 1 - f)$; $f \in [0, 1)$

Case 1: $\{x\} + \{y\} = [1, 2)$, then $[x] = n$

$\Rightarrow$ $[y] = 2 - n$ and $\{x\} + \{y\} \in [1, 2)$

$\Rightarrow$ $x = n + f$; $y = (2 - n) + f'$; $1 \le f + f' < 2$

$\Rightarrow$ $x = n + f$; $y = (2 - n) + f'$; $n \in \mathbb{Z}$; $f \in [0, 1)$; $f' \in [1 - f, 2 - f)$

Thus, all possible solutions are

$x = m + f$; $y = (3 - m) + f'$; $m \in \mathbb{Z}$; $f \in [0, 1)$; $f' \in [0, 1 - f)$

and $x = n + f$; $y = (2 - n) + f'$; $n \in \mathbb{Z}$; $f \in [0, 1)$; $f' \in [1 - f, 2 - f)$

ILLUSTRATION 105: Solve the equation $x^2 - 6x + [x] + 7 = 0$

SOLUTION: Given equation is $x^2 - 6x + [x] + 7 = 0$ ...(1)

$\Rightarrow$ $x^2 - 6x + x - \{x\} + 7 = 0$

$\Rightarrow$ $x^2 - 5x + 7 = \{x\}$

Now RHS belongs to $[0, 1)$

$\Rightarrow$ LHS must also belong to $[0, 1)$

$\Rightarrow$ $0 \le x^2 - 5x + 7 < 1$

$\Rightarrow$ $x^2 - 5x + 7 \ge 0$ and $x^2 - 5x + 6 < 0$

First inequation always hold good as its discriminant < 0 and $x^2 - 5x + 6 < 0$

$\Rightarrow$ $(x - 2)(x - 3) < 0$

$\Rightarrow$ $x \in (2, 3)$

$\Rightarrow$ $[x] = 2$

$\therefore$ From equation (1), we have $x^2 - 6x + 9 = 0$

$\Rightarrow (x-3)^2 = 0$

$\Rightarrow x = 3$

But it does not satisfy the inequality $2 < x < 3$.

Hence, the given equation has no solution.

ILLUSTRATION 106: Solve for x, the system of simultaneous inequations $(2\{x\}-1)(3\{x\}-2) \le 0$ and $(3[x]-4)(2[x]-8) \le 0$; where [.] gint is function and {} is fractional part of function.

SOLUTION: Given in inequations $(2\{x\}-1)(3\{x\}-2) \le 0$ ...(1)

And $(3[x]-4)(2[x]-8) \le 0$ ...(2)

From (1), $\frac{1}{2} \le \{x\} \le \frac{2}{3}$ .... (3)

From (2), $\frac{4}{3} \le [x] \le 4 \Rightarrow [x] \in \{2, 3, 4\}$

$\Rightarrow x \in [2, 5)$ ... (4)

Combining (3) and (4), we note that x is a real number greater than or equal to 2 less than 5, and having its fractional part in the interval $\left[\frac{1}{2}, \frac{2}{3}\right]$.

Thus, the solution set of given simultaneous equations (1) and (2) is

$$\left[2+\frac{1}{2}, 2+\frac{2}{3}\right] \cup \left[3+\frac{1}{2}, 3+\frac{2}{3}\right] \cup \left[4+\frac{1}{2}, 4+\frac{2}{3}\right] \text{ i.e., } x \in \left[\frac{5}{2}, \frac{8}{3}\right] \cup \left[\frac{7}{2}, \frac{11}{3}\right] \cup \left[\frac{9}{2}, \frac{14}{3}\right]$$

■ NEAREST INTEGER FUNCTION

(x) denotes the nearest integer to x. If x is the mean of two consecutive integers, then (x) will be the greater among the two consecutive integers, i.e., $\left(\frac{n+n+1}{2}\right) = \left(\frac{2n+1}{2}\right) = n+1$

Since $(x) \ge [x] \Rightarrow (x)$ is either equal to $[x]$ or $[x] + 1$

Example:

(i) $x = 1.3829 \Rightarrow (x) = (1.3829) = 1$

(ii) $x = 5.43 \Rightarrow (x) = (5.43) = 5$

(iii) $x = 3 \Rightarrow (x) = (3) = 3$

(iv) $x = 7.82 \Rightarrow (x) = (7.82) = 8$

Thus, from definition, we have

$(x) = -2$ for $-\frac{5}{2} \le x < -\frac{3}{2}$

$(x) = -1$ for $-\frac{3}{2} \le x < -\frac{1}{2}$

$(x) = 0$ for $-\frac{1}{2} \le x < \frac{1}{2}$

$(x) = 1$ for $\frac{1}{2} \le x < \frac{3}{2}$

$(x) = 2$ for $\frac{3}{2} \le x < \frac{5}{2}$

Graph o*f*(x) can be expressed graphically as shown in the Figure 2.100.

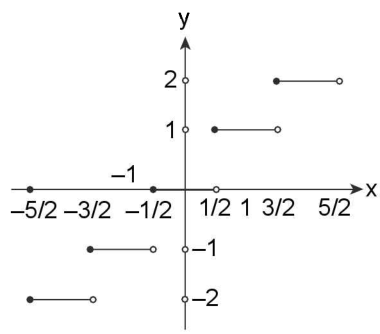


FIGURE 2.100

Properties of Nearest Integer Function

(i) $(x) = \begin{cases} [x] & \text{if } 0 \le \{x\} < \frac{1}{2} \text{ i.e.,} [x] \le x < [x] + \frac{1}{2} \\ [x]+1 & \text{if } \frac{1}{2} \le \{x\} < 1 \text{ i.e.,} [x] + \frac{1}{2} \le x < [x]+1 \end{cases}$

Proof: Let $x = [x] + \{x\}$

Case (i): When $0 \le \{x\} < \frac{1}{2}$

$\Rightarrow$ $[x] \le [x] + \{x\} < [x] + \frac{1}{2}$

$\Rightarrow$ $[x] \le x < [x] + \frac{1}{2} < [x] + 1$

$\Rightarrow$ $[x]$ is the integer nearest to x. Thus, $(x) = [x]$

Case (ii): When $\frac{1}{2} \le \{x\} < 1$

$\Rightarrow$ $[x] + \frac{1}{2} \le [x] + \{x\} < [x] + 1$

$\Rightarrow$ $[x] + \frac{1}{2} \le x < [x] + 1 \Rightarrow [x] < [x] + \frac{1}{2} \le x < [x] + 1$

$\Rightarrow$ $[x] + 1$ is nearest integer to x $\Rightarrow$ $(x) = [x] + 1$.

(ii) $(x + n) = (x) + n$ if $n \in \mathbb{Z}$

Proof: Let $\{x\} \in \left[0, \frac{1}{2}\right)$ $\Rightarrow$ $(x) = [x]$ .... (1)

$\because$ $\{x\}$ is periodic function with period 1

$\therefore$ for any positive integer n, $\{n + x\} = \{x\} \in \left[0, \frac{1}{2}\right)$

$\therefore$ By property (1), $(n + x) = [n + x] = n + [x]$ $= n + (x)$ (By (1)) .... (2)

Again $\{x\} \in \left[\frac{1}{2}, 1\right)$ $\Rightarrow$ $(x) = [x] + 1$ .... (3)

By periodicity of $\{x\}$, $\{n + x\} = \{x\} \in \left[\frac{1}{2}, 1\right)$

$\Rightarrow$ $(n + x) = [n + x] + 1 = n + [x] + 1 = n + (x)$ (By (3))

Thus, from (3) and (4), we conclude that

$(x + n) = n + (x)\ \forall\ x \in \mathbb{R}, n \in \mathbb{Z}$

(iii) $$(-x) = \begin{cases} -(x); & \forall x \in \mathbb{R} \sim \left\{x = \left(\frac{2n+1}{2}\right); n \in \mathbb{Z}\right\} \\ -(x) + 1; & for\ x = \left(\frac{2n+1}{2}\right); n \in \mathbb{Z} \end{cases}$$

Proof: Let $x \in \mathbb{R} \sim \left\{x = \left(\frac{2n+1}{2}\right); n \in \mathbb{Z}\right\}$

Case (i) If $x = n \in \mathbb{Z}$, then $(x) = (n) = n$ and $-x = -n \in \mathbb{Z}$

$\Rightarrow$ $(-x) = (-n) = -n = -(x)$. Thus, $(-x) = -(x)$

Case (ii) If $x \notin \mathbb{Z}$ and $x = n + f; f \in (0, 1) \sim \left\{\frac{1}{2}\right\}$

$\Rightarrow$ $-x = -n - f$ $\Rightarrow -x = -n - 1 + 1 - f$

$\Rightarrow$ $-x = (-n - 1) + (1 - f)$

$\therefore$ If $f \in \left(0, \frac{1}{2}\right)$, then $(1 - f) \in \left(\frac{1}{2}, 1\right)$

$\Rightarrow$ $(x) = n$ and $(-x) = -n = -(x)$ $\therefore$ $(-x) = -(x)$

Now if $f \in \left(\frac{1}{2}, 1\right)$, then $(1 - f) \in \left(0, \frac{1}{2}\right)$

$\Rightarrow$ $(x) = n + 1$ and $(-x) = (-n - 1 + 1 - f) = -n - 1 = -(x)$

Thus, $(-x) = -(x)\ \forall\ x \in \mathbb{R} \sim \left\{x = \left(\frac{2n+1}{2}\right); n \in \mathbb{Z}\right\}$

Case (iii) Let $x = \left(\frac{2n+1}{2}\right)$; $n\ x \in \mathbb{Z}$. Now, $x = n + \frac{1}{2}$

$\Rightarrow$ $(x) = (n + 1)$ and $(-x) = \left(-n - \frac{1}{2}\right) = \left(-n - 1 + \frac{1}{2}\right)$

$= (-n - 1) + 1 = n = -(n + 1) + 1 = -(x) + 1$

METHOD OF SOLVING INEQUALITY INVOLVING X, {X} AND [X]

e.g., $2x + [x] = 3\{x\}$ (say) .... (1)

Step (1): Replace x by $[x] + \{x\}$ in equation (1) and express $[x]$ as function of $\{x\}$.

Given, $2x + [x] = 3\{x\}$ $\Rightarrow$ $3[x] + 2\{x\} = 3\{x\}$

$\Rightarrow$ $3[x] = \{x\}$

Step (2): Apply range of $\{x\}$ i.e., $0 \le \frac{\{x\}}{3} < \frac{1}{3}$

$\Rightarrow$ $0 \le [x] < \frac{1}{3}$ ...(2)

Step (3): Find all possible values of $[x]$ using (2) and corresponding values of $\{x\}$.

Using equation (2), we get $[x] = 0$ and $\{x\} = 0$

$\Rightarrow$ $x = 0$ is the only solution.

ILLUSTRATION 107: Solve the equation $5x + 3\{x\} = 4[2x + 2] - 5$; where [.] is gint function and {} is fractional part function.

SOLUTION: Given equation is $5x + 3\{x\} = 4[2x + 2] - 5$

$\Rightarrow$ $5x + 3\{x\} = 4[2x] + 8 - 5$ ($\because$ $[x + m] = [x] + m$ for $m \in \mathbb{Z}$)

$\Rightarrow$ $5x + 3\{x\} = 4[2x] + 3$ ...(1)

Now $2x = 2([x] + \{x\})$

$\Rightarrow$ $2x = 2[x] + 2\{x\}$

$\Rightarrow$ $[2x] = 2[x] + [2\{x\}]$ ...(2)

Using (2) in (1) we get, $5x + 3\{x\} = 4(2[x] + [2\{x\}]) + 3$

$\Rightarrow$ $5x + 3\{x\} = 8[x] + 4[2\{x\}] + 3$ ...(3)

Let us deal with two cases:-

Case (i): When $0 \le \{x\} < \frac{1}{2}$

$\Rightarrow$ $0 \le 2\{x\} < 1$ $\Rightarrow$ $[2\{x\}] = 0$

$\therefore$ From (3), $5x + 3\{x\} = 8[x] + 3$

$\Rightarrow$ $5[x] + 5\{x\} + 3\{x\} = 8[x] + 3$ (using $x = [x] = \{x\}$)

$\Rightarrow$ $8\{x\} = 3[x] + 3$ $\Rightarrow$ $\{x\} = \frac{3[x]+3}{8}$ ...(4)

Now $\{x\} \in [0, 1/2)$ $\Rightarrow$ $\frac{3[x]+3}{8} \in \left[0, \frac{1}{2}\right)$

$\Rightarrow$ $0 \le 3[x] + 3 < 4$ $\Rightarrow$ $-3 \le 3[x] < 1$ $\Rightarrow$ $-1 \le [x] < \frac{1}{3}$

$\Rightarrow$ $[x] = -1, 0$

Using these values of $[x]$ in (4) we have,

For $[x] = -1$; $\{x\} = 0$ $\Rightarrow$ $x = -1$. For $[x] = 0$; $\{x\} = \frac{3}{8}$ $\Rightarrow$ $x = \frac{3}{8}$

$\therefore$ Solution set in this case is $\left\{-1, \frac{3}{8}\right\}$

Case (ii): When $\frac{1}{2} \le \{x\} < 1$

$\Rightarrow$ $1 \le 2\{x\} < 2$ $\Rightarrow$ $[2\{x\}] = 1$

$\therefore$ From (3), $5x + 3\{x\} = 8[x] + 7$ $\Rightarrow$ $5[x] + 5\{x\} + 3\{x\} = 8[x] + 7$

$\Rightarrow$ $8\{x\} = 3[x] + 7$ $\Rightarrow$ $\{x\} = \frac{3[x]+7}{8}$ ...(5)

Now $\{x\} \in [1/2, 1)$ $\Rightarrow$ $\frac{3[x]+7}{8} \in \left[\frac{1}{2}, 1\right)$

$\Rightarrow$ $4 \le 3[x] + 7 < 8$ $\Rightarrow$ $-3 \le 3[x] < 1$ $\Rightarrow$ $-1 \le [x] < \frac{1}{3}$ $\Rightarrow$ $[x] = -1, 0$

Using these values of $[x]$ in (5) we have,

For $[x] = -1$; $\{x\} = \frac{1}{2}$ $\Rightarrow$ $x = -\frac{1}{2}$

For $[x] = 0$; $\{x\} = \frac{7}{8}$ $\Rightarrow$ $x = \frac{7}{8}$

$\therefore$ Solution set in this case is $\left\{-\frac{1}{2}, \frac{7}{8}\right\}$

Thus, the complete solution set is $\left\{-1, -\frac{1}{2}, \frac{3}{8}, \frac{7}{8}\right\}$

ILLUSTRATION 108: Solve the equation $4x + 5\{x\} = 3[x] - 1$ for x; where [.] is gint function and {} is fractional part function.

SOLUTION: Given equation is $4x + 5\{x\} = 3[x] - 1$

$\Rightarrow\ 4([x] + \{x\}) + 5\{x\} = 3[x] - 1$

$\Rightarrow\ 4[x] + 9\{x\} = 3[x] - 1 \ \Rightarrow\ 9\{x\} = -[x] - 1 \ \Rightarrow\ \{x\} = \dfrac{-[x]-1}{9}$ ... (1)

But $\{x\} \in [0, 1)$ $\Rightarrow\ 0 \le -\dfrac{-[x]-1}{9} < 1$

$\Rightarrow\ 0 \le -[x] - 1 < 9$ $\Rightarrow\ 0 \ge [x] + 1 > -9$

$\Rightarrow\ -1 \ge [x] > -10$ $\Rightarrow\ -1 \ge [x] \ge -9$ ... (2)

Using values of $[x]$ from (2) in (1), we have

For $[x] = -9$; $\Rightarrow\ \{x\} = \dfrac{8}{9}$ $\Rightarrow\ x = -\dfrac{73}{9}$

For $[x] = -8$; $\Rightarrow\ \{x\} = \dfrac{7}{9}$ $\Rightarrow\ x = -\dfrac{65}{9}$

For $[x] = -7$; $\Rightarrow\ \{x\} = \dfrac{2}{3}$ $\Rightarrow\ x = -\dfrac{19}{3}$

For $[x] = -6$; $\Rightarrow\ \{x\} = \dfrac{5}{9}$ $\Rightarrow\ x = -\dfrac{49}{9}$

For $[x] = -5$; $\Rightarrow\ \{x\} = \dfrac{4}{9}$ $\Rightarrow\ x = -\dfrac{41}{9}$

For $[x] = -4$; $\Rightarrow\ \{x\} = \dfrac{1}{3}$ $\Rightarrow\ x = -\dfrac{11}{3}$

For $[x] = -3$; $\Rightarrow\ \{x\} = \dfrac{2}{9}$ $\Rightarrow\ x = -\dfrac{25}{9}$

For $[x] = -2$; $\Rightarrow\ \{x\} = \dfrac{1}{9}$ $\Rightarrow\ x = -\dfrac{17}{9}$

For $[x] = -1$; $\Rightarrow\ \{x\} = 0$ $\Rightarrow\ x = -1$

$\therefore$ The set of solutions is $\left\{-\dfrac{73}{9}, -\dfrac{65}{9}, -\dfrac{19}{3}, -\dfrac{49}{9}, -\dfrac{41}{9}, -\dfrac{11}{3}, -\dfrac{25}{9}, -\dfrac{17}{9}, -1\right\}$

ILLUSTRATION 109: Prove that $[x] + \left[x + \dfrac{1}{n}\right] + \left[x + \dfrac{2}{n}\right] + \ldots + \left[x + \dfrac{n-1}{n}\right] = [nx] \ \ \forall x \in \mathbb{R}$.

SOLUTION: Let $x = [x] + y$, where $0 \le y < 1$. We know that for a given real number, we can always find two consecutive integers, between which the number lies, so let there exists $p \in \mathbb{Z}$ such that $p - 1 \le ny < p$.

Now, $x + \dfrac{q}{n} = [x] + y + \dfrac{q}{n}$ and $p - 1 \le ny < p$

$\Rightarrow\ \dfrac{p-1}{n} \le y < \dfrac{p}{n}$

$\Rightarrow\ \dfrac{p-1}{n} + \dfrac{q}{n} \le y + \dfrac{q}{n} < \dfrac{p}{n} + \dfrac{q}{n}$

$\Rightarrow\ \dfrac{p-1+q}{n} \le y + \dfrac{q}{n} < \dfrac{p+q}{n}$

Now $y + \dfrac{q}{n}$ will be less than 1 for $\dfrac{p+q}{n} \le 1$, i.e., $q \le n - p$,

Consequently $\left[x + \dfrac{q}{n}\right] = \left[[x] + y + \dfrac{q}{n}\right] = [x] + \left[y + \dfrac{q}{n}\right] = [x] + 0 = [x]$

i.e., $\left[x + \dfrac{q}{n}\right] = [x]$ for $q = 0, 1, 2, 3, \ldots, (n - p)$

But $\left[x+\frac{q}{n}\right]=[x]+1$ for $q = n-(p-1), n-(p-2), n-(p-3), \ldots, (n-1)$

$$\therefore \quad [x]+\left[x+\frac{1}{n}\right]+\ldots+\left[x+\frac{n-1}{n}\right] = \underbrace{[x]+[x]+\ldots+[x]}_{(n-p+1)\text{ times}} + \underbrace{([x]+1)+([x]+1)+\ldots+([x]+1)}_{(p-1)\text{ times}}$$

$= (n-p+1)[x] + (p-1)([x]+1) = n[x] + (p-1)$ ..... (i)

Also $[nx] = [n([x]+y)] = n[x] + [ny] = n[x] + p - 1$ .... (ii)

$(\because p-1 \le ny < p)$

$\therefore$ From (i) and (ii), LHS = RHS.

TEXTUAL EXERCISE–7: (SUBJECTIVE)

1. Prove that $[\sin x + [\sin x + [\sin x]]] = 3[\sin x]$.

2. Find the value of $[\sin x + [\tan x + [\cos x + [\sin x]]]]$, when $x \in (0, \pi/4)$.

3. Find the number of solutions of equation $y = \frac{1}{3}[\sin x + [\sin x + [\sin x]]]$ and $[y + [y]] = 2\cos x$; where [ ] denotes greatest integer less than or equal to x.

4. Find the solution set of the following equations (where $[x]$, $\{x\}$ is the greatest integer $\le x$, *f*ractional part of x respectively).

(a) $4\{x\} = x + [x]$ (b) $|[x] - 2x| = 4$

5. Let $[x]$ denotes the greatest integer not larger than x (the 'integer part' of x). For every positive integer n evaluate the sum $\left[\frac{n+1}{2}\right]+\left[\frac{n+2}{2^2}\right]+\ldots+\left[\frac{n+2^k}{2^{k+1}}\right]$.

Answer Keys

2. 0 **3.** 0 **4.** (a) $(0, 5/3)$ (b) $\left\{\pm 4, \frac{7}{2}, \frac{-9}{2}\right\}$

TEXTUAL EXERCISE–7: (OBJECTIVE)

1. The domain of definition of function $\log_{[x]} x^2$

(a) $(2, \infty)$ (b) $[2, \infty)$
(c) $(1, \infty)$ (d) None of these

2. The domain of definition of function $[x]\sin\left(\frac{\pi}{[x+1]}\right)$

(a) $\mathbb{R}$ (b) $\mathbb{R} \sim [-1, 0)$
(c) $(0, \infty)$ (d) None of these

3. The domain of definition of function $\sqrt{\frac{2-[x]}{[x]-3}}$

(a) $[2, 3)$ (b) $(2, 3)$
(c) $(2, \infty)$ (d) None of these

4. The domain of definition of function $\sin^{-1}[2-3x^2]$

(a) $[-1, 1]$ (b) $[0, 1]$
(c) $[-1, 0) \cup (0, 1]$ (d) None of these

5. The domain of definition of function $f(x) = \log\{1/([\cos x] - [\sin x])\}$

(a) $[2n\pi, (2n+1)\pi]$ (b) $\left[(2n+1)\pi+\frac{\pi}{2}, (2n+2)\pi\right]$
(c) $[2n\pi, (2n+2)\pi]$ (d) None of these

6. The domain of definition of function $\sqrt{([x]-1)}+\sqrt{(4-[x])}$ is

(a) (1, 5) (b) (0, 5)
(c) [1, 5) (d) None of these

7. The domain of definition of function $e^x + \sin^{-1}[(x/2)-1] + \log\sqrt{x-[x]}$

(a) (0, 6) – {1,2,3,4,5} (b) (0, 5)
(c) (0, 6) (d) None of these

8. $f : (2, 3) \to (1, 3)$ be defined by $f(x) = x - \{-x\}$, then $f^{-1}(x)$. (where $\{y\}$ denotes fractional part of y) is

(a) $y = 3 + x$ (b) $y = \dfrac{3+x}{2}$
(c) $y = x - 3$ (d) None of these

9. Let $f(x) = \begin{cases} x-[x]; & \text{when } I \le x < I+0.5 \\ [x] & \text{when } I+0.5 \le x < I+1 \end{cases}$, $g(x) = \sin^4 x + \cos^4 x$, then $f\{g(x)\}$ is

(a) 1 (b) –1
(c) 0 (d) None of these

10. The range of the function ([.] denotes the greatest integer function.) $f(x) = \dfrac{x-[x]}{1+x-[x]}$

(a) $\left[0, \dfrac{1}{2}\right)$ (b) [0,1]
(c) $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$ (d) None of these

11. The range of the function $f(x) = \dfrac{e^x}{1+[x]}$, $x \ge 0$ ([.] denotes the greatest integer function.)

(a) $(1, \infty)$ (b) $(-\infty, \infty)$
(c) $[1, \infty)$ as $x \ge 0$ (d) $(-\infty,-1]\ [1,\infty)$

12. Solution of the equation $x^3 - [x] = 3$ (where $[x]$, is the greatest integer $\le x$) is

(a) $(3)^{1/4}$ (b) $(4)^{1/3}$
(c) $\sqrt{2}$ (d) None of these

13. Solution of the equation $(x)^2 + (x + 1)^2 = 25$ (where (x), is the least integer function) is

(a) (–5, 2) (b) (2, 3)
(c) $(-5, -4] \cup (2, 3]$ (d) None of these

14. Solution of the inequation $[x]^2 + (x)^2 \ge 25$ (where $[x]$, and (x), is the greatest integer $\le x$ and least integer $\ge x$, respectively) is

(a) $(-\infty, -4] \cup [4, \infty)$ (b) $(-\infty, -4]$
(c) $[4, \infty)$ (d) None of these

Answer Keys

1. (b)	**2.** (b)	**3.** (a)	**4.** (c)	**5.** (b)	**6.** (c)	**7.** (a)	**8.** (b)	**9.** (c)	**10.** (a)
11. (c)	**12.** (b)	**13.** (c)	**14.** (a)						

BINARY OPERATION

From the very beginning of school days we always come across the four fundamental operations namely addition, subtraction, multiplication, and division. By using these fundamental operations we had been associating two numbers a and b with the unique real numbers $a + b$, $a - b$, $a.b$ and a/b respectively.

If we had to add, subtract, multiply, or divide more than two elements, (say) $a_1, a_2, a_3, \ldots\ a_n$, first of all we operate the desired operation on two elements a_1 and a_2 and the resultant is operated by same operation with the third element a_3 and the result thus obtained is operated with the fourth element a_4 and so on till all the elements are utilised. Thus, these four fundamental operations are called binary operations as they are operated on two elements at a time, as the meaning of binary is two. To cover across all binary operations we need to give a general definition of binary operations and a set X whose elements can be operated by using these binary operations as follows.

Definition of Binary Operations

A binary operation * on a set A is a function from set $A \times A$ to A itself. Thus, * associates each pair $(a_1, a_2) \in A \times A$ to a unique element $(a_1 * a_2)$ of A. Thus, domain of a binary operation defined on set A is $A \times A$ and co-domain is A. Range is subset of A.

For example, let $A = \{-1, 0, 1\}$ and * is a function defined as $*(a_1, a_2) = a_1 . a_2;\ a_1, a_2 \in A$

Now we observe,

$*(-1, -1) = (-1) . (-1) = 1 \in A$;
$*(-1, 0) = (-1) . (0) = 0 \in A$;
$*(-1, 1) = (-1) . (1) = -1 \in A$;
$*(1, 1) = (1) . (1) = 1 \in A$;
$*(1, 0) = (1) . (0) = 0 \in A$;
$*(0, 0) = (0) . (0) = 0 \in A$;

Thus, * operated to every pair $(a_1, a_2) \in A \times A$ gives us a unique element of A.

Hence, the function * defined in the above example is a binary operation on set A.

ILLUSTRATION 110: Prove the following

(i) '+' addition is a binary operation on set of integers $\mathbb{Z}$, set of natural numbers $\mathbb{N}$, set of real numbers $\mathbb{R}$, set of rational numbers $\mathbb{Q}$, *b*ut not on set of irrational numbers.

(ii) '–' subtraction is a binary operation on set of integers $\mathbb{Z}$ but not on set of natural numbers.

(iii) '×' multiplication is a binary operation on set of integers $\mathbb{Z}$, *s*et of natural numbers $\mathbb{N}$, set of rational numbers $\mathbb{Q}$ but not on set of irrational numbers.

(iv) '÷' division is a binary operation on set of non-zero rational numbers but not a binary operation on set of integers.

SOLUTION: (i) For $a, b \in \mathbb{Z}$ $\Rightarrow a + b \in \mathbb{Z}$

For $a, b \in \mathbb{N}$ $\Rightarrow a + b \in \mathbb{N}$

For $a, b \in \mathbb{Q}$ $\Rightarrow a + b \in \mathbb{Q}$

For $a, b \in \mathbb{R}$ $\Rightarrow a + b \in \mathbb{R}$

Thus, '+' is a binary operation on the set of integers, set of natural numbers, set of rational numbers and set of real numbers.

However, '+' is not a binary operation on the set of irrational numbers, e.g., $\sqrt{2}, -\sqrt{2} \in \bar{Q}$ (set of irrational numbers) but $(\sqrt{2}) + \left(-\sqrt{2}\right) = 0 \notin \bar{Q}$.

(ii) As difference of two integers is always an integer, it follows that subtraction '–' is a binary operation on the set of integers.

However, subtraction is not a binary operation on set of natural numbers as is clear from the following counter example: Let $a = 2$, $b = 5$, then $a - b = -3 \notin \mathbb{N}$

(iii) As the product of two integers is an integer, product of two natural number is a natural number and that of two rational numbers is a rational number, multiplication '×' is a binary operation on set of integers $\mathbb{Z}$, set of natural numbers $\mathbb{N}$ and on set of rational numbers. However, the product of two irrational numbers need not be an irrational number, e.g., if $a = \sqrt{8}$ and $b = \sqrt{2}$, then $a \times b = 4$ which is a rational number. Thus, product '×' is not a binary operation on set of irrational numbers.

(iv) Division '÷' is a binary operation on set of non-zero rational numbers as if we divide a rational number by a non-zero rational number, we always obtain a rational number. However, division is not a binary operation on set of integers $\mathbb{Z}$, as if we divide an integer by another non-zero integer, we need not obtain an integer. For example, if $a = 4$, $b = 2$, then $a/b = 2$, however, if $a = 2$, $b = 4$, then $\frac{a}{b} = \frac{1}{2} \notin \mathbb{Z}$.

ILLUSTRATION 111: Prove that the multiplication is a binary operation on set $\{-1, 1, 0, i, -i\}$; where $i = \sqrt{-1}$.

SOLUTION: For proving that 'x' is a binary operation on the given set, we take the help of composition table. It is a rectangular table containing elements of given set in first row and first column keeping the position corresponding to first row and first column for indicating binary operation '*'.

Now if $a * b = c$, then we put 'c' in the position corresponding to row containing 'a' and column containing 'b'. For * to be a binary on given set, 'c' must belong to given set for each ordered pair (a, b).

The composition table for above given set and binary operation * (x) is as given below:

*(×)	–1	1	0	i	$-i$
–1	1	–1	0	$-i$	i
1	–1	1	0	i	$-i$
0	0	0	0	0	0
i	$-i$	i	0	–1	1
$-i$	i	$-i$	0	1	–1

Clearly each entry in the above composition table belongs to given set. Thus, '×' is a binary operation on given set.

ILLUSTRATION 112: Prove that the 'addition modulo 7' i.e., $a * b$ (mod 7) = least non-negative integer obtained when $a + b$ is divided by 7, is a binary operation on set $A = \{0, 1, 2, 3, 4, 5, 6\}$.

SOLUTION: We prove that * (addition modulo 7) is a binary operation on set A by using the composition table as given below:

* ($\oplus_7$)	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

We obtain the above composition table as described below, suppose $a = 1$, $b = 3$, then $a + b = 4$.

Now $4 = 7 \times 0 + 4$, thus, 4 is the least non-negative integer remainder obtained when $a + b(= 4)$ is divided by 7.

Thus, corresponding to row containing 1 and column containing 3, we put 4. Similarly if $a = 4$ and $b = 5$, then $a + b = 9$ and $9 = 1 \times 7 + 2$.

Thus, $a * b = 2$, the least non-negative remainder obtained when 9 is divided by 7. Thus, corresponding to row containing 4 and column-containing 5 we put 2.

From the above composition table it is clear that addition modulo 7 is a binary operation on set {0, 1, 2, 3, 4, 5, 6}. Generally addition modulo 'm' is a binary operation on set $\{0, 1, 2, 3, 4, \ldots, m-2, m-1\}$.

ILLUSTRATION 113: Prove that 'multiplication modulo, 11' is a binary operation on the set $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ where multiplication modulo m is defined as $a * b = a_m b$ = least non-negative integer obtained when $a.\ b$ is divided by m.

SOLUTION: The composition table for the given operation is as given below.

* ($\otimes_{11}$)	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10
2	0	2	4	6	8	10	1	3	5	7	9
3	0	3	6	9	1	4	7	10	2	5	8
4	0	4	8	1	5	9	2	6	10	3	7
5	0	5	10	4	9	3	8	2	7	1	6
6	0	6	1	7	2	8	3	9	4	10	5

7	0	7	3	10	6	2	9	5	1	8	4
8	0	8	5	2	10	7	4	1	9	6	3
9	0	9	7	5	3	1	10	8	6	4	2
10	0	10	9	8	7	6	5	4	3	2	1

Clearly each entry in above composition table belongs to given set A. Thus, multiplication modulo 11 is a binary operation on set A.

ILLUSTRATION 114: Show that the following operations are binary operations on the corresponding sets.

(i) $a*b=\begin{cases}\dfrac{a+b}{2}; & \text{if } a \text{ and } b \text{ are both odd or both even}\\ \dfrac{a+b+1}{2}; & \text{if exactly one of } a \text{ and } b \text{ is odd}\end{cases}$ on $\mathbb{N}$ (set of natural numbers)

(ii) $a*b=\begin{cases}\dfrac{a.b}{2}; & \text{if at least one of } a \text{ and } b \text{ is even}\\ \dfrac{(a-1).b}{2}; & \text{otherwise}\end{cases}$ on $\mathbb{Z}$ (set of integers)

(iii) $a * b = \dfrac{a^2+a+2b}{2}$ on $\mathbb{N}$ (set of natural numbers)

(iv) $a * b = a \,.\, b$ on $A = \{1, \omega, \omega^2\}$; where '$\omega$' is cube root of unity.

(v) $*$ on set $A = \{\pm i, \pm j, \pm k, \pm 1\}$; where $i * i = j * j = k * k = -1$, $i * j = k = -(j * i)$, $(j * k) = i = -(k * j)$, $(k * i) = j = -(i * k)$ and $1 * a = a$, $-1 * a = -a$. $\forall\, a \in A$.

SOLUTION: (i) $a*b=\begin{cases}\dfrac{a+b}{2}; & \text{if } a \text{ and } b \text{ are both odd or both even}\\ \dfrac{a+b+1}{2}; & \text{if one of } a \text{ and } b \text{ is odd and other is even}\end{cases}$

When both a and b are odd or both even, then $(a + b)$ is even natural number ≥ 2.

$\Rightarrow$ $\dfrac{a+b}{2}$ is a natural number ≥ 1.

Again if one of a and b is odd and other is even, their sum $(a + b)$ will be odd natural number ≥ 3.

$\Rightarrow$ $(a + b + 1)$ will be an even natural number ≥ 4

$\Rightarrow$ $\dfrac{a+b+1}{2}$ will be a natural number ≥ 2. Thus, $a * b \in \mathbb{N}\ \forall\ a, b \in \mathbb{N}$

$\Rightarrow$ $*$ is a binary operation on $\mathbb{N}$.

(ii) $a * b = \begin{cases}\dfrac{a.b}{2}; & \text{if at least one of } a \text{ and } b \text{ is even}\\ \dfrac{(a-1).b}{2}; & \text{if none of } a \text{ and } b \text{ is even}\end{cases}$

$\because$ Atleast one of a and b is even $\Rightarrow$ $\dfrac{a.b}{2} \in \mathbb{Z}$

Also if both a and b are odd, $(a - 1)$ is even $\Rightarrow$ $\dfrac{(a-1).b}{2} \in \mathbb{Z}$.

Thus, $(a * b) \in \mathbb{Z};\ \forall\ a, b \in \mathbb{Z}$ $\Rightarrow$ $*$ is a binary operation on $\mathbb{Z}$.

(iii) $a * b = \dfrac{a^2+a+2b}{2} = \dfrac{a(a+1)+2b}{2} = \dfrac{a(a+1)}{2} + b$

Since, a, $a + 1 \in \mathbb{N}$ and we know that the product of two consecutive integers is an even integer.

$\Rightarrow \dfrac{a(a+1)}{2} \in \mathbb{N}$; also $b \in \mathbb{N}$. $\Rightarrow \left[\dfrac{a(a+1)}{2} + b\right] \in \mathbb{N}$ $\Rightarrow a * b \in \mathbb{N}$.

Thus, * is a binary operation on $\mathbb{N}$.

(iv) $a * b = a \,.\, b$ on $A = \{1, \omega, \omega^2\}$; ω being cube root of unity.

Clearly $1 = 1.\,\omega,\ \omega.\,\omega^2 = \omega^2.\omega = 1,\ 1.\,\omega^2 = \omega^2.1 = \omega^2$.

$\Rightarrow$ * is a binary operation on A.

(v) $A = \{\pm i, \pm j, \pm k, \pm 1\}$; $i * i = j * j = k * k = -1$; $i * j = k = -(j * i)$; $(j * k) = i = -(k * j)$; $k * i = j = -(i * k)$;

Based on above given properties of elements of A, the following composition table for * implies that * is a binary operation on set A.

*	$-i$	i	j	$-j$	k	$-k$	1	–1
$-i$	–1	1	$-k$	k	j	$-j$	$-i$	i
i	1	–1	k	$-k$	$-j$	j	i	$-i$
j	k	$-k$	–1	1	i	$-i$	j	$-j$
$-j$	$-k$	k	1	–1	$-i$	i	$-j$	j
k	$-j$	j	$-i$	i	–1	1	k	$-k$
$-k$	j	$-j$	i	$-i$	1	–1	$-k$	k
1	$-i$	i	j	$-j$	k	$-k$	1	–1
–1	i	$-i$	$-j$	j	$-k$	k	–1	1

PROPERTIES OF BINARY OPERATION * ON A SET A

(i) **Closure Property:** Since binary operation * on a set A is a function from $A \times A$ to A, it obeys closure law, i.e., $a * b \in A\ \forall\ a, b \in A$. Also we say that A is closed with respect to binary operation *.

(ii) **Associativity:** Binary operation * on a set A is said to be associative, if $a * (b * c) = (a * b) * c\ \forall\ a, b, c \in A$.

(iii) **Commutativity:** Binary operation * on a set A is said to be commutative if $a * b = b * a\ \forall\ a, b \in A$.

(iv) **Existence of Identity:** An element $e \in A$ is said to be an identity element of set A with respect to binary operation * if $a * e = e * a = a\ \forall\ e \in A$. For example, '+' is a binary operation on set of integer $\mathbb{Z}$. Also $0 \in \mathbb{Z}$ and $x + 0 = 0 + x = x\ \forall\ x \in \mathbb{Z} \Rightarrow$ 0 is an identity element of set of integers $\mathbb{Z}$ with respect to binary operation '+' (addition). Also 0 is called additive identity of set of integers.

Similarly 1 is multiplicative identity of every subset of complex number $\mathbb{C}$.

(v) **Existence of Inverse:** An element $b \in A$ is said to be inverse of element $b \in A$ with respect to binary operation * if $a * b = e = b * a$; where e is the identity element of A with respect to binary operation *. And we denote $b = a^{-1}$.

REMARKS

1. *If a binary operation * on set A is associative and identity element exists in A and every element of A is invertible, then A is said to be a Group with respect to binary operation *.*
2. *In addition to properties given in remark (1) if * is commutative, then set A is said to be an Abelian Group with respect to binary operation *.*
3. *If* $b = a^{-1}$, *then* $a = b^{-1}$.

Proof: *Since* $b = a^{-1}$ $\Rightarrow a * b = e = b * a$

$\Rightarrow b * a = e = a * b$ $\Rightarrow b^{-1} = a$

4. *Identity element if exists is unique.*

Proof: *Let e and e' be two identity elements in set A with respect to binary operation **

$\Rightarrow$ $e, e' \in A$

$\Rightarrow$ $e.e' = e' = e'.e$ (e is identity in A and $e' \in A$)

and $e'.e = e = e.e'$ (e' is identity in A and $e \in A$)

$\Rightarrow$ $e.e' = e' = e$ $\Rightarrow$ $e = e'$

5. *Inverse of an element if exists is unique provided * is associative.*

Proof: *Let if possible* $a^{-1} = b$ and $a^{-1} = c$

$\Rightarrow$ $a * b = e = b * a$ ... (1)

and $a * c = e = c * a$ ... (2)

$\therefore$ from (1) and (2), we get $a * b = a * c = e$ $\Rightarrow$ $b * (a * b) = b * (a * c)$ $\Rightarrow$ $(b * a) * b = (b * a) * c$ ($\because$ * is associative)

$\Rightarrow$ $e * b = e * c$ (By (1)) $\Rightarrow$ $b = c$ ($\because$ e is identity element)

Thus, a^{-1} if exists is unique.

6. *Number of binary operations that can be defined on a set A containing n number of elements is* $(n)^{2n}$.

Proof: Given $n(A) = n$

As we know that binary operation is a function from set $A \times A$ to set A itself.

$\therefore$ Domain of binary operation * is $A \times A$ having n^2 elements.

Also co-domain A consists of n elements. Now each of the n elements of co-domain A has n^2 number of choices for its pre-image in domain $A \times A$.

Thus, by fundamental principle of multiplication of permutations number of possible binary operations that can be defined on set $A = (n^2)^n = (n)^{2n}$.

ILLUSTRATION 115: For each operation * defined below, determine whether * is binary operation, commutative, associative, identity element that exist. Find inverse of element a if the identity exist with respect to given binary operation.

(i) on $\mathbb{Z}$, $a * b = a - b$

(ii) on $\mathbb{Q}$, $a * b = ab + 1$

(iii) on $\mathbb{Q}$, $a * b = \dfrac{ab}{2}$

(iv) on $\mathbb{Z}^+$, $a * b = 2^{ab}$

(v) on $\mathbb{Z}^+$, $a * b = (a)^b$

(vi) on $\mathbb{R} \sim \{-1\}$; $a * b = \dfrac{a}{b+1}$

SOLUTION: (i) $a * b = a - b \in \mathbb{Z}\ \forall\ a, b \in \mathbb{Z}$, as the difference of any two integers is also an integer. Thus, * is a binary operation on set $\mathbb{Z}$. Further $a * b = a - b \neq b - a = b * a$

$\Rightarrow$ * is not commutative in $\mathbb{Z}$.

Also $a * (b * c) = a - (b - c) = (a - b) + c \neq (a - b) - c = (a * b) * c$ unless $c = 0$.

Thus, for non-zero integers associativity does not hold in $\mathbb{Z}$, e.g., $a = 2, b = 4, c = 5$

$\Rightarrow$ $a * (b * c) = 2 - (4 - 5) = 3$, whereas $(a * b) * c = (a - b) - c = (2 - 4) - 5 = -7$.

$\Rightarrow$ * is not associative in $\mathbb{Z}$.

(ii) $a * b = ab + 1 \in \mathbb{Q}\ \forall\ a, b \in \mathbb{Q}$ as the product and sum of two rational numbers is again a rational number. Thus, * is a binary operation on set $\mathbb{Q}$.

Further $a * b = ab + 1 = ba + 1 = b * a\ \forall\ a, b \in \mathbb{Q}$.

$\Rightarrow$ * is commutative on set $\mathbb{Q}$.

Now if $a, b, c \in \mathbb{Q}$, then $(a * b) * c = (ab + 1) * c$

$= (ab + 1)c + 1 = abc + c + 1,$

where as $a * (b * c) = a * (bc + 1) = a(bc + 1) + 1 = abc + a + 1$

$\therefore$ $a * (b * c) \neq (a * b) * c \neq c$ unless $a = c$

Thus, for different rational numbers a, b, c associativity does not hold.

$\Rightarrow$ * is not associative in $\mathbb{Q}$.

(iii) $a * b = \dfrac{ab}{2} \in \mathbb{Q}$ $\forall$ $a, b \in \mathbb{Q}$, *as* the product and division of two rational number ($\neq 0$) is also a rational number.

$\Rightarrow$ * is a binary operation on $\mathbb{Q}$.

Further $a * b = \dfrac{ab}{2} = \dfrac{ba}{2} = b * a$ $\forall$ $a, b \in \mathbb{Q}$.

$\Rightarrow$ * is commutative.

Now for $a, b, c \in \mathbb{Q}$; $a * (b * c) = a * \left(\dfrac{bc}{2}\right) = \dfrac{a\left(\dfrac{bc}{2}\right)}{2} = \dfrac{abc}{4}$

and $(a * b) * c = \left(\dfrac{ab}{2}\right) * c = \dfrac{\left(\dfrac{ab}{2}\right)(c)}{2} = \dfrac{abc}{4}$

$\Rightarrow$ * is associative on $\mathbb{Q}$.

(iv) $a * b = (2)^{ab} \in \mathbb{Z}^+$ as $a, b \in \mathbb{Z}^+$ $\qquad \Rightarrow ab \in \mathbb{Z}^+$

$\Rightarrow$ $(2)^{ab} \in \mathbb{Z}^+$ *as* positive integer raised to the positive integer power is always a positive integer $\quad \Rightarrow$ * is a binary operation on the set of positive integers.

Further $a * b = (2)^{ab} = (2)^{ba} = b * a$ $\qquad \Rightarrow$ * is commutative on $\mathbb{Z}^+$.

Now for $a, b, c \in \mathbb{Z}^+$; $a * (b * c) = a * \left((2)^{bc}\right) = (2)^{a.(2)^{bc}}$ and $(a * b) * c = (2^{ab}) * c = (2)^{(2)^{ab}.c}$

Clearly in general $(2)^{a.(2)^{bc}} \neq (2)^{c.(2)^{ab}}$

For example, if we take $a = 1, b = 2, c = 3$, then $a * (b * c) = (2)^{a.(2)^{bc}} = (2)^{64}$

and $(a * b) * c = (2)^{c.(2)^{ab}} = (2)^{12}$.

(v) $a * b = (a)^b \in \mathbb{Z}^+$ $\forall$ $a, b \in \mathbb{Z}^+$ *as* the product of positive integers is always a positive integer

$\Rightarrow$ * is a binary operation on $\mathbb{Z}^+$, set of positive integers.

Further $a * b = (a)^b$, whereas $b * a = (b)^a$, but $(a)^b \neq (b)^a$ in general.

For example, if $a = 2, b = 3$, then $(a)^b = 8$ and $(b)^a = 9$ $\Rightarrow$ * is not commutative.

Now for $a, b, c \in \mathbb{Z}^+$, $a * (b * c) = a * (b)^c = (a)^{b^c}$ and $(a * b) * c = (a^b) * c = (a^b)^c = (a)^{bc}$

But in general $(a)^{b^c} \neq (a)^{bc}$ for $a, b, c \in \mathbb{Z}^+$, unless $b^c = bc$

So, * is not associative on $\mathbb{Z}^+$.

(vi) $a * b = \dfrac{a}{b+1}$ need not belong to set $\mathbb{R} \sim \{-1\}$ $\forall$ $a, b \in \mathbb{R} \sim \{-1\}$

For example, if we take $a = -2$ and $b = 1$, then $a * b = \dfrac{a}{b+1} = \dfrac{-2}{1+1} = -1 \notin \mathbb{R} \sim \{-1\}$

$\therefore$ * is not a binary operation on $\mathbb{R} \sim \{-1\}$.Further $a * b = \dfrac{a}{b+1}$ and $b * a = \dfrac{b}{a+1}$

$\Rightarrow$ $a * b \neq b * a$ in general $\Rightarrow$ operation * is not commutative.

For $a, b, c \in \mathbb{R} \sim \{-1\}$; $a*(b*c) = a*\left(\dfrac{b}{c+1}\right) = \dfrac{a}{\left(\dfrac{b}{c+1}+1\right)} = \dfrac{a(c+1)}{(b+c+1)}$

And $(a*b)*c = \left(\dfrac{a}{b+1}\right)*c = \dfrac{\left(\dfrac{a}{b+1}\right)}{(c+1)} = \dfrac{a}{(b+1)(c+1)}$.

Clearly $a * (b * c) \neq (a * b) * c$ in general. Thus, operation * is not associative.

ILLUSTRATION 116: Given a non-empty set W, let $*: P(W) \times P(W) \to P(W)$ be defined as $A * B = (A - B) \cup (B - A)$ $\forall A, B \in P(W)$. Show that the empty set ϕ is the identity for the operation * and all the elements A of $P(W)$ are invertible with $A^{-1} = A$. Here $P(W)$ is power set of set W.

SOLUTION: $A * B = (A - B) \cup (B - A)$ $\forall A, B \in P(W)$

Let $A \in P(W)$.

Also $\phi \in P(W)$

Now $A * \phi = (A - \phi) \cup (\phi - A) = A \cup \phi = A$

$\therefore$ $A * \phi = A$ ... (1)

Further $\phi * A = (\phi - A) \cup (A - \phi) = \phi \cup A = A$ ... (2)

Thus, from (1) and (2), $A * \phi = \phi * A = A$

$\Rightarrow$ ϕ is an identity element of $P(W)$ with respect to binary operation *.

Now, let $A \in P(W)$, then $A * A = (A - A) \cup (A - A) = \phi \cup \phi = \phi$

$\Rightarrow$ A is the inverse of itself $\Rightarrow$ $A^{-1} = A$.

ILLUSTRATION 117: Define a binary operation * on the set {0, 1, 2, 3, 4, 5} as $a*b = \begin{cases} a+b, & \text{if } a+b<6 \\ a+b-6, & \text{if } a+b\geq 6 \end{cases}$

Show that zero is the identity for this operation and each element $a \neq 0$ of the set is invertible with $6 - a$ being the inverse of a and inverse of 0 is 0.

SOLUTION: Given set is $A = \{0, 1, 2, 3, 4, 5\}$ and binary operation * is defined by

$$a*b = \begin{cases} a+b, & \text{if } a+b<6 \\ a+b-6, & \text{if } a+b\geq 6 \end{cases}$$

Consider the composition table as given below:

*	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Clearly $a * 0 = a = 0 * a \ \forall \ a \in A$

(See row 1, row 2 and column 1 and column 2).

Further let $a \in A$, $a \neq 0$ and let the inverse of a be $b \in A$.

$\Rightarrow \ a * b = 0 = b * a$. Now there arise two cases.

Case (i): if $a + b < 6$, then $a * b = a + b = b * a$.

$\Rightarrow \ a + b = 0$ $\qquad \Rightarrow \ b = -a$

$\Rightarrow \ b = a = 0$ is the only possibility as a, $b \in A$ and A contains non-negative integers.

So, b cannot be the inverse of a when $a + b < 0$ and $a \neq 0$

Case (ii): let $a + b \geq 6$ $\qquad \Rightarrow \ a * b = 0 = b * a$

$\Rightarrow \ a + b - 6 = 0$ $\qquad \Rightarrow \ b = 6 - a$

Here $A = \{0, 1, 2, 3, 4, 5\}$

$\therefore$ for $a = 0$, $a^{-1} = 0$

for $a = 1$, $a^{-1} = 6 - 1 = 5$

for $a = 2$, $a^{-1} = 6 - 2 = 4$

for $a = 3$, $a^{-1} = 6 - 3 = 3$

for $a = 4$, $a^{-1} = 6 - 4 = 2$

for $a = 5$, $a^{-1} = 6 - 5 = 1$

Thus, inverse of 0 is 0 and inverse of non-zero element $a \in A$ is given by $6 - a$.

ILLUSTRATION 118: Show that the number of binary operations on $\{1, 3\}$ having 1 as identity and having 3 as the inverse of 3 is exactly one.

SOLUTION: A binary operation * on $\{1, 3\}$ is a function from $\{1, 3\} \times \{1, 3\}$ to $\{1, 3\}$,

i.e., a function from $\{(1, 1), (1, 3), (3, 1), (3, 3)\} \rightarrow \{1, 3\}$

Since 1 is the identity for the given binary operation *, $*(1, 1) = 1$, $*(1, 3) = 3$, $*(3, 1) = 3$ and the only choice left is for the pair (3, 3). Since 3 is the inverse of 3, i.e., $*(3, 3)$ must be equal to 1.

Thus, the number of desired binary operation is exactly one.

TEXTUAL EXERCISE–8: (SUBJECTIVE)

1. Determine whether or not each of the definition of * given below gives a binary operation. In the event that * is not a binary operation, give justification for this.

(i) On $\mathbb{N}$, define * by $a * b$ = H.C.F. (a, b)

(ii) On $\mathbb{N}$, define * by $a * b$ = LCM (a, b)

(iii) On $\mathbb{Q}$, define * by $a * b = a + b + ab$

(iv) On $\mathbb{R}$, define * by $a * b = \sqrt{ab}$

(v) On $\mathbb{R}$, define * by $a * b = \sqrt{a+b}$

(vi) On $\mathbb{R}$, define * by $a * b = \sqrt{a^2 - 3ab + 4b^2}$

(vii) On $\mathbb{Z}^+$, define * by $a * b = a - b$

(viii) On $\mathbb{Z}^+$, define * by $a * b = |a - b|$

(ix) On $\mathbb{Z}^+$, define * by $a * b = a.b$

(x) On $\mathbb{R}$, define * by $a * b = ab^2$

(xi) On $\mathbb{Z}^+$, define * by $a * b = a$

2. Let * be a binary operation on the set $\mathbb{Q}$ of rational numbers as follows:

(i) $a * b = a - b$ $\qquad$ (ii) $a * b = a^3 + b^3$

(iii) $a * b = a + 2ab$ $\qquad$ (iv) $a * b = (a - b)^4$

(v) $a * b = \dfrac{ab}{8}$ $\qquad$ (vi) $a * b = 3ab^2$

Find which of the binary operations are commutative and which are associative.

3. Is * defined on the set $A = \{1, 3, 5, 7, 9\}$ by $a * b$ = LCM of a and b a binary operation? Justify your answer.

4. Let * be the binary operation on $\mathbb{N}$ defined by $a * b$ = H.C.F. of a and b. Is * commutative? Is * associative? Does there exist identity for this binary operation on $\mathbb{N}$?

5. Let * be binary operation on $\mathbb{N}$ given by $a * b$ = LCM of a and b. Find
(i) $6 * 7$, $15 * 24$.
(ii) Is * associative?
(iii) Is * commutative?
(iv) Find the identity element w.r.t. *
(v) Find the invertible elements of $\mathbb{N}$ w.r.t. *?

6. Let $A = N \times N$ and * be the binary operation of A defined by $(a, b) * (c, d) = (a + c, b + d)$. Show that * is commutative as well as associative. Find the identity element if exists.

7. Consider a binary operation * on the set {1, 2, 3, 4, 5} given by the following composition table;
(i) Find $(3 * 4) * 5$ and $2 * (3 * 5)$
(ii) Is * commutative?
(iii) Find $(3 * 4) * (4 * 5)$.

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

8. Consider the binary operation Δ on the set {1, 2, 3, 4, 5} defined by $a \Delta b = \min\{a, b\}$. Write the operation table of the operation Δ.

9. (a) Show that * on $\mathbb{R}$ given by $a * b = a + 2b$ is not associative.
(b) Show that * on $\mathbb{R}$ given by $a * b = a + 2b$ is not commutative.

10. Show that $-a$ is not the inverse of a for the addition operation + on $\mathbb{N}$ and $\dfrac{1}{a}$ for multiplication operation × on $\mathbb{N}$, for $a \neq 1$.

11. (i) Show that $* : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $(a, b) \to a + 4b^2$ is a binary operation.
(ii) Show that the $\Delta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $\Delta(a, b) = \min\{a, b\}$ and the $\Delta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $\Delta(a, b) = max\{a, b\}$ are binary operations.
(iii) Let P be the set of all subsets of a given set X. Show that $\cup : P \times P \to P$ given by $\cup(A, B) = A \cup B$ and $\cap : P \times P \to P$ given by $\cap(A, B) = A \cap B$ are binary operations on the set P.

12. Determine which of the following binary operations on the set $\mathbb{R}$ are associative and which are commutative.
(a) $a * b = 4 \; \forall a, b \in \mathbb{R}$
(b) $a * b = \dfrac{(a+b)}{4} \; \forall a, b \in \mathbb{R}$.

13. Given a non-empty set W, consider the binary operation $* : P(W) \times P(W) \to P(W)$ given by $A * B = A \cap B \; \forall A, B$ in $P(W)$ is the power set of W. Show that W is the identity element for this operation and W is the only invertible element in $P(W)$ with respect to the operation *.

14. Consider the binary operations * and o on the set of real numbers $\mathbb{R}$ defined as $a * b = |a - b|$ and $a \, o \, b = a$, $\forall a, b \in \mathbb{R}$. Show that * is commutative but not associative, o is associative but not commutative. Further show that $\forall a, b, c \in \mathbb{R}$ $a * (b \, o \, c) = (a * b) \, o \, (a * b)$. [If it is so, we say that the operation * distributes over the operation o]. Does o distribute over *? Justify your answer.

Answer Keys

1. (i) yes (ii) yes (iii) yes (iv) no (v) no (vi) yes (vii) yes (viii) no (ix) yes (x) yes (xi) yes

2. (ii), (iv), (v) are commutative; (v) is associative

3. no, $3 * 5 = 15 \notin A$

4. * is commutative and associative, identity element does not exist for *.

5. (i) 42, 120 (ii) yes (iii) yes (iv) 1 (v) 1

6. Identity element does not exist

7. (i) 1 and 1 (ii) yes (iii) 1

8.

Δ	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

12. (a) associative and commutative (b) commutative but not associative **14.** no

TEXTUAL EXERCISE–8: (OBJECTIVE)

1. Which of the following operations on set of rational numbers $\mathbb{Q}$ is/are associative?

(a) $a * b = a - b$ (b) $a * b = \dfrac{ab}{4}$

(c) $a * b = \sqrt{a^2b^2}$ (d) $a * b = a^2 - b^2$

2. Which of the following operation on set of integers is/are binary operation(s)?

(a) $a * b = \sqrt{a^2b^2 + b^2 + 2ab^2}$

(b) $a * b = g.c.d\,(a, b)$

(c) $a * b = \dfrac{a-b}{2}$

(d) $a * b = \dfrac{a^2-b^2}{4}$

3. Which of the following operations on set of integers is/are commutative?

(a) $a * b = \sqrt{a^2 + b^2 - 2ab}$

(b) $a * b = a - b$

(c) $a * b = \begin{cases} (a)^b & \text{if } b \ge 0 \\ ab & \text{if } b < 0 \end{cases}$

(d) $a * b = a^2b$

4. If $A = \{1, \omega, \omega^2\}$; where ω is cube root of unity. Define a binary operation * on A by $a * b = ab$, then

(a) ω is identity in A

(b) ω^2 is identity in A

(c) 1 is identity element in A

(d) Identity element in A does not exist

5. If $A = \{1, \omega, \omega^2\}$; ω being cube root of unity. Define a binary operation * on A by $a * b = ab$, then

(a) Inverse of ω is 1 (b) Inverse of ω^2 is 1

(c) Inverse of ω is ω^2 (d) Inverse of ω^2 is ω

6. Number of binary operations defined on set $A = \{a, b, c\}$ will be

(a) 9 (b) 81

(c) $(3)^9$ (d) None of these

7. If * is a binary operation on set $A = \{1, 3, 5\}$ defined by $a * b$ = least non-negative integer obtained when $(a.b)$ is divided by 6, then

(a) Identity in A is 1

(b) $3 * a = 3\ \forall\ a \in A$

(c) Inverse element of 3 does not exist

(d) Inverse element of 5 is 5

8. A binary operation * is defined on set $A = \{2, 4, 6, 8, 10\}$ by the given composition table:

*	2	4	6	8	10
2	10	6	2	4	8
4	6	8	4	10	2
6	2	4	6	8	10
8	4	10	8	2	6
10	8	2	10	6	4

(a) Identity element in A is 6

(b) * is commutative in A

(c) Inverse of 8 is 10

(d) $2 * (4 * 8) \ne (2 * 4) * 8$

Answer Keys

1. (b, c) **2.** (a, b) **3.** (a) **4.** (c) **5.** (c, d) **6.** (c) **7.** (a, b, c, d) **8.** (a, b, c)

DOMAIN OF FUNCTIONS

Domain of Function is defined as: D_f : {x : $f(x)$ is defined, i.e., $f(x) \in$ co-domain Y and is real and finite}.

Principal Domain: Subset of domain in which $f(x)$ takes all its values exactly once, i.e., $f(x)$ is one–one and on-to (Bijective) is called Principal domain. Conventionally it considered nearer to zero and preferably positive.

e.g., if we consider the function $f(x) = \dfrac{x^2}{4a}$, $a > 0$

Clearly domain of function $f(x)$ is $\mathbb{R} = (-\infty, \infty)$ and range $= [0, \infty)$

The graph of $f(x)$ is as given in Figure 2.101.

Clearly the function is many–one in its domain $\mathbb{R}$ and function $f(x)$ takes all values of its range $[0, \infty)$ exactly once in $(-\infty, 0]$ as well as in $[0, \infty)$, and hence, the function $f(x)$ is injective and onto in $(-\infty, 0]$ as well as in $[0, \infty)$.

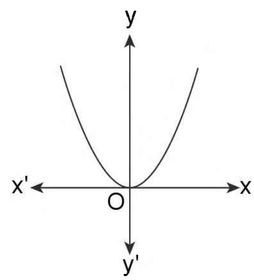


FIGURE 2.101

Among these two domains the domain containing non-negative inputs, i.e., $[0, \infty)$ is taken as its principal domain.

REMARKS

1. *Every function is bijective from its principal domain to its range. Thus, every function is invertible from its principal domain to range set.*
2. *Domain of the function $f^{-1}(x)$ is same as range of the function f(x).*
3. *Range of $f^{-1}(x)$ is same as principal domain of the function f(x).*

PROPERTIES OF DOMAIN

If $f(x)$ and $g(x)$ be two functions such that $f(x)$ has domain D_f and $g(x)$ has domain D_g, then following results always hold good.

Rule 1: Dom $(k \,.\, f(x)) = D_f$ for all $k \in$ set of non-zero real numbers.

Rule 2: $\text{Dom}\left(\dfrac{1}{f(x)}\right) = D_f \sim \{x : f(x) = 0\}$

ILLUSTRATION 119: Find the domain of given functions

(a) $f(x) = \dfrac{3x+4}{x^2-1}$ (b) $f(x) = \dfrac{x^2-9}{x(x-1)(x-2)}$

SOLUTION: (a) $D_f = \mathbb{R} = \sim \{1, -1\}$ (b) $D_f = \mathbb{R} \sim \{0, 1, 2\}$

Rule 3: Dom $(f(x) \pm g(x)) = D_f \cap D_g$.

ILLUSTRATION 120: Find the domain of $f(x) = \sqrt{x-1} + \log(5-x)$

SOLUTION: $x - 1 \geq 0$ and $5 - x > 0$ $\Rightarrow$ $x \geq 1$ and $x < 5$ $\Rightarrow$ $D_f = [1, 5)$

Rule 4: Dom $(f(x) \,.\, g(x)) = D_f \cap D_g$

ILLUSTRATION 121: Find the domain of following functions

(a) $f(x) = \sqrt{x-2}\sqrt{4-x}$ (b) $f(x) = \sqrt{(x-2)(x-4)}$

SOLUTION: (a) $x - 2 \geq 0$ and $x - 4 \geq 0$ $\Rightarrow$ $x \geq 2$ and $x \geq 4$ $\Rightarrow$ $D_f = [4, \infty]$

(b) $(x - 2)(x - 4) \geq 0$ $\Rightarrow$ $x \in (-\infty, 2] \cup [4, \infty)$ $\Rightarrow$ $D_f = (-\infty, 2] \cup [4, \infty)$

Rule 5: Dom $f(g(x)) = \{x : x \in D_g \text{ and } g(x) \in D_f\} = D_g \sim \{x \mid g(x) \notin D_f\}$.

ILLUSTRATION 122: Find the domain of $f(x) = \sin^{-1}(\log_2 x)$.

SOLUTION: $x > 0$ and $\log_2 x \in [-1, 1] \Rightarrow x > 0$ and $x \in [1/2, 2] \Rightarrow x \in [1/2, 2] = D_f$

Rule 6: Domain of even root of $f(x) = \sqrt[2m]{f(x)} = D_f \sim \{x : f(x) < 0\}$

ILLUSTRATION 123: Find the domain of the following functions:

(a) $f(x) = \sqrt{x^2 - 4}$ (b) $f(x) = \sqrt[4]{\sin x}$

SOLUTION: (a) $(x^2 - 4) \geq 0 \Rightarrow x^2 \geq 4 \Rightarrow x \leq -2$ or $x \geq 2 \Rightarrow x \in (-\infty, -2] \cup [2, \infty) = D_f$

(b) $\sin x \geq 0 \Rightarrow x \in \bigcup_{n \in \mathbb{Z}} [2n\pi, (2n + 1)\pi]$

Rule 7: Domain $\sqrt[2m+1]{f(x)} = D_f$

ILLUSTRATION 124: Find the domain of the following functions:

(a) $f(x) = \sqrt[3]{x + 4}$ (b) $f(x) = \sqrt[3]{\sin x}$

SOLUTION: (a) $(x + 4) \in \mathbb{R} \Rightarrow x \in \mathbb{R} = D_f$ (b) D_f = Domain of $\sin x = \mathbb{R}$

Rule 8: Dom $(\log f(x)) = D_f \sim \{x : f(x) \leq 0\}$

ILLUSTRATION 125: Find the domain of the following functions:

(a) $\log\left(\dfrac{x^2 - 1}{27 - x^3}\right)$ (b) $\log\left(\dfrac{4 - x^2}{\sqrt{\sin x}}\right)$ (c) $\log_2(x - 1)$

SOLUTION: (a) $\left(\dfrac{x^2 - 1}{27 - x^3}\right) > 0 \Rightarrow (x^2 - 1)(27 - x^3) > 0 \Rightarrow (x^2 - 1)(x^3 - 27) < 0$

$\Rightarrow (x - 3)(x + 1)(x - 1)(x^2 + 9 + 3x) < 0 \Rightarrow (x + 1)(x - 1)(x - 3) < 0$ ($\because x^2 + 9 + 3x > 0$ as its discriminant is negative) $\Rightarrow x \in (-\infty, -1) \cup (1, 3)$

(b) $\dfrac{4 - x^2}{\sqrt{\sin x}} > 0$ and $\sin x > 0 \Rightarrow 4 - x^2 > 0$ and $x \in (2n\pi, (2n + 1)\pi)$; $n \in \mathbb{Z}$

$\Rightarrow x \in (-2, 2)$ and $x \in (2n\pi, (2n + 1)\pi)$; $n \in \mathbb{Z} \Rightarrow x \in (0, 2)$

(c) $(x - 1) > 0 \Rightarrow x \in (1, \infty)$

ILLUSTRATION 126: Find the domain of $f(x) = \log_{10} \sin x$.

SOLUTION: For existence of $f(x)$, $\sin x > 0$ and from the graph of $y = \sin x$, it is clear that $\sin x$ is positive when $2n\pi < x < (2n + 1)\pi$; $n \in \mathbb{Z}$

Hence, Domain of $f(x) = \bigcup_{n \in \mathbb{Z}} (2n\pi, (2n+1)\pi)$

ILLUSTRATION 127: Find the domain of definition of function $f(x) = \sin^{-1}(\log_2(3^{\cos x}))$

SOLUTION: This is the example for finding domain of composition function of the form, i.e., $(ho\phi o\psi og)$ and the composition to be defined $3^{\cos x} > 0$ (always hold)

and $-1 \leq \log_2(3^{\cos x}) \leq 1 \Rightarrow 1/2 \leq 3^{\cos x} \leq 2$ ...(i)

$\Rightarrow \log_3 \dfrac{1}{2} \le \log_3 3^{\cos x} \le \log_3 2$ ($\because \log_a x$ is increasing for $a > 1$)

$\Rightarrow \log_3 \dfrac{1}{2} \le \cos x \le \log_3 2$ $\Rightarrow -\log_3 2 \le \cos x \le \log_3 2$

$\Rightarrow x \in \bigcup\limits_{n \in \mathbb{Z}} \{[(2n-1)\pi + \theta, (2n\pi - \theta)] \cup [(2n\pi + \theta), (2n+1)\pi - \theta]\}$

Where $\cos\theta = \log_3 2$ and $\theta \in (0, \pi/2)$

ILLUSTRATION 128: Find the domain of the function $y = \sqrt{\dfrac{2x-1}{2x^3+3x^2+x}}$

SOLUTION: (a) For y to be defined:

$\Rightarrow \dfrac{2x-1}{2x^3+3x^2+x} \ge 0 \Rightarrow \dfrac{2x-1}{x(2x^2+3x+1)} \ge 0 \Rightarrow \dfrac{2x-1}{x(2x+1)(x+1)} \ge 0$

By sign scheme, we have:

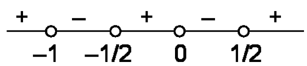


FIGURE 2.102

$\Rightarrow (2x-1)x(2x+1)(x+1) \ge 0; x \ne 0, -1/2, -1$

$\Rightarrow x \in (-\infty, -1) \cup (-1/2, 0) \cup [1/2, \infty)$.

ILLUSTRATION 129: Find the domain of the function $10^x + 10^y = 10$

SOLUTION: Given $10^x + 10^y = 10$ $\Rightarrow 10^y = 10 - 10^x$

$\Rightarrow \log_{10} 10^y = \log_{10}(10 - 10^x)$ $\Rightarrow y = \log_{10}(10 - 10^x)$

For y to be defined $10 - 10^x > 0$ $\Rightarrow 10^x < 10$ $\Rightarrow x < 1$

Hence, domain of y is $(-\infty, 1)$

ILLUSTRATION 130: Find the domain of definition of $f(x) = \sin^{-1}(|x-1| - 2)$

SOLUTION: $f(x) = \sin^{-1}(|x-1| - 2)$. For domain $-1 \le |x-1| - 2 \le 1$

$\Rightarrow 1 \le |x-1| \le 3 \Rightarrow x - 1 \in [-3, -1] \cup [1, 3] \Rightarrow x \in [-2, 0] \cup [2, 4]$

ILLUSTRATION 131: Find the domain of the function $f(x) = \sqrt[6]{4^x + 8^{(2/3)(x-2)} - 52 - 2^{2(x-1)}}$

SOLUTION: For f to be defined, we must have

$4^x + 8^{(2/3)(x-2)} - 52 - 2^{2(x-1)} \ge 0 \Leftrightarrow 2^{2x} + 2^{2(x-2)} - 2^{2(x-1)} \ge 52 \Leftrightarrow 2^{2x}\ [1 + 2^{-4} - 2^{-2}] \ge 52$

$\Leftrightarrow 2^{2x}\left(\dfrac{13}{16}\right) \ge 52 \Leftrightarrow 2^{2x} \ge 64 = 2^6$. Hence, we must have $x \ge 3$. So, the domain of f is $[3, \infty)$

ILLUSTRATION 132: Discuss the domain of definition of the function, $f(x) = \log(ax^2 + bx + c)$ for values of $a, b, c \in \mathbb{R}$.

SOLUTION: ***Case I:*** If $a = 0$, then $f(x) = \log(bx + c)$, then $f(x)$ will be defined for $bx + c > 0$

$\Rightarrow bx > -c$. if $b > 0$, then $x > -\dfrac{c}{b}$ and if $b < 0$, then $x < -\dfrac{c}{b}$

Case II: If $a = b = 0$, then $f(x) = \log(c)$ = constant function and is defined for $\forall\ x \in \mathbb{R}$ for which $c > 0$ and not defined for any real x if $c \le 0$.

Case III: If $a \ne 0$, then $f(x) = \log(ax^2 + bx + c)$.

Sub-case (a) If $D = b^2 - 4ac < 0$, $a > 0$, then $ax^2 + bx + c > 0\ \forall\ x \in \mathbb{R}$

$\Rightarrow$ Domain of $f(x) = \mathbb{R}$

Sub-case (b) If $D = b^2 - 4ac < 0$, $a < 0$, then $ax^2 + bx + c < 0\ \forall\ x \in \mathbb{R}$

$\Rightarrow$ Domain of $f(x) = \{\ \} = \phi$

Sub-case (c) If $D = b^2 - 4ac > 0$, $a > 0$, then $ax^2 + bx + c > 0\ \forall\ x \in (-\infty, \alpha) \cup (\beta, \infty)$;
where α, β are roots of $ax^2 + bx + c = 0$ $\Rightarrow$ Domain $= (-\infty, \alpha) \cup (\beta, \infty)$.

Sub-case (d) If $D = b^2 - 4ac > 0$, $a < 0$, then $ax^2 + bx + c > 0\ \forall\ x \in (\alpha, \beta)$; where α, β are roots of $ax^2 + bx + c = 0$ $\Rightarrow$ Domain $= (\alpha, \beta)$.

Sub-case (e) If $D = b^2 - 4ac = 0$, $a > 0$, then $ax^2 + bx + c = 0$ at $x = \frac{-b}{2a}$ and $ax^2 + bx + c > 0$
$\forall x \in \mathbb{R} \sim \left\{\frac{-b}{2a}\right\}$, which is the domain in this case.

Sub-case (f) If $D = b^2 - 4ac = 0$, $a < 0$, then $ax^2 + bx + c \le 0\ \forall\ x \in \mathbb{R}$

$\Rightarrow$ $f(x)$ is not defined at any real x. $\Rightarrow$ Domain $= \{\ \} = \phi$.

Domains of Some Standard Functions:

f(x)	**Domain : D_f**	*f(x)*	**Domain : D_f**
Polynomial $P(x)$	$\mathbb{R}$	$\cot^{-1} x$	$\mathbb{R}$
Constant function	$\mathbb{R}$	$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$
$\dfrac{P(x)}{Q(x)}$	$\mathbb{R} - \{x : Q(x) = 0\}$	$\text{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$
a^x	$\mathbb{R}$	$\text{sgn}\, x$	$\mathbb{R}$
$\log x$	$(0, \infty)$	$\sin(\sin^{-1}x)$	$[-1, 1]$
$\lvert x\rvert$	$\mathbb{R}$	$\cos(\cos^{-1}x)$	$[-1, 1]$
$\{x\}$	$\mathbb{R}$	$\tan(\tan^{-1}x)$	$\mathbb{R}$
$\sin x$	$\mathbb{R}$	$\cot(\cot^{-1}x)$	$\mathbb{R}$
$\cos x$	$\mathbb{R}$	$\sec(\sec^{-1} x)$	$(-\infty, -1] \cup [1, \infty)$
$\tan x$	$\mathbb{R} - \{(2n+1)\frac{\pi}{2}; n \in \mathbb{Z}\}$	$\text{cosec}(\text{cosec}^{-1}x)$	$(-\infty, -1] \cup [1, \infty)$
$\cot x$	$\mathbb{R} - \{n\pi; n \in \mathbb{Z}\}$	$\sin^{-1}(\sin x)$	$\mathbb{R}$
$\sec x$	$\mathbb{R} - \left\{(2n+1)\frac{\pi}{2}; n \in \mathbb{Z}\right\}$	$\cos^{-1}(\cos x)$	$\mathbb{R}$
$\text{cosec}\, x$	$\mathbb{R} - \{n\pi; n\in\mathbb{Z}\}$	$\tan^{-1}(\tan x)$	$\mathbb{R} - \left\{(2n+1)\frac{\pi}{2}; n \in \mathbb{Z}\right\}$
$\sin^{-1} x$	$[-1, 1]$	$\cot^{-1}(\cot x)$	$\mathbb{R} - \{n\pi; n\in\mathbb{Z}\}$
$\cos^{-1} x$	$[-1, 1]$	$\sec^{-1}(\sec x)$	$\mathbb{R} - \left\{(2n+1)—; n \in \mathbb{Z}\right\}$
$\tan^{-1} x$	$\mathbb{R}$	$\text{cosec}^{-1}(\text{cosec}\, x)$	$\mathbb{R} - \{n\pi; n \in \mathbb{Z}\}$

TEXTUAL EXERCISE–9: (SUBJECTIVE)

1 Find the domain of definition of the following functions:

(a) $f(x)=\dfrac{\sqrt{x-1}+\sqrt{6-x}}{\sqrt{1-x}+\sqrt{x-6}}$

(b) $f(x)=\dfrac{x^2+1}{x-\sqrt{(x+2)}}$

(c) $f(x)=\sqrt{\dfrac{x-2}{x+2}}+\sqrt{\dfrac{1-x}{1+x}}$

2. Find the domain of definition of the following functions:

(a) $f(x)=\sqrt{\dfrac{1-5^x}{7^{-x}-7}}$

(b) $f(x)=\dfrac{\sqrt{5-x}+\sqrt{x-2}}{\sqrt{3-x}-\sqrt{x-1}}$

3. Find the domain of definition of the following functions:

(a) $f(x)=\log_{2\{x\}-3}(x^2-5x+13)$

(b) $f(x)=\log_{10}\left(\dfrac{9-x^2}{x^2-4}\right)$

(c) $f(x)=\sqrt{\log_2\left(\dfrac{5x-x^2}{4}\right)}$

(d) $f(x)=\sqrt{\log_{\frac{1}{2}}\dfrac{5x-x^2}{4}}$

4 Find the domain of definition of the following functions:

(a) $f(x)=\dfrac{1}{\sqrt{x-|x|}}$

(b) $f(x)=\dfrac{1}{\sqrt{|x|-x}}$

(c) $f(x)=\sin^{-1}\left(\dfrac{3-|x|}{2}\right)$

(d) $f(x)=\sqrt{x^2-3|x|+2}$

5. Find the domain of definition of the following functions:

(a) $f(x)=\sqrt{\sin x}+\sqrt{16-x^2}$

(b) $\sqrt{\cos(\sin x)}+\sqrt{\sin(\cos x)}$

6. Find the domain of definition and range of the following functions:

(a) $f(x) = \cos^{-1}[x]$

(b) $f(x)=\sin^{-1}\left(x^2+1\right)$

7. Find the domain of definition of the following functions:

(a) $f(x)=\cos^{-1}\left(\dfrac{2x}{1+x^2}\right)$

(b) $f(x)=\cos^{-1}\left(\dfrac{1+x^2}{2x}\right)$

(c) $f(x)=\tan^{-1}\left(1+x^2\right)$

(d) $f(x)=\sin^{-1}\left(\dfrac{x^2-1}{2(x^2+1)}\right)$

8. Find the domain of definition of the following functions:

(a) $f(x) = \log_2 (3 + 5 \sin x)$

(b) $f(x) = \log_2(3 + 2 \sin x)$

9. Find the domain of definition of the following functions:

(a) $f(x)=\sqrt{\log_x(\cos 2\pi x)}$

(b) $f(x)=\sqrt{\left(x^2-3x-10\right)\ln^2(x-3)}$

(c) $f(x)=\cos^{-1}\left(\dfrac{2-|x|}{4}\right)+(\ln(3-x))^{-1}$

10. Find the domain of definition of the following functions:

(a) $f(x)=\dfrac{\sqrt{\cos x-\frac{1}{2}}}{\sqrt{6+35x-6x^2}}$

(b) $f(x)=\left(\log_{\frac{x-2}{x+3}}2\right)+\sqrt{9-x^2}$

11. Find the domain of definition of the following functions:

(a) $f(x)=\dfrac{1}{[x]}+\log_{1-\{x\}}(x^2-3x+10)+\dfrac{1}{\sqrt{2-|x|}}+\dfrac{1}{\sqrt{\sec(\sin x)}}$

(b) $f(x)=\dfrac{1}{\sqrt{\sin(\cos x)}}+\sin^{-1}\left(\dfrac{2x}{\pi}\right)+\dfrac{1}{\{-x\}}+\dfrac{1}{\ln\left(1-\left[\tan\frac{x}{2}\right]\right)-\left[-\tan\frac{x}{2}\right]}$

Answer Keys

1. (a) {} (b) $(-2, -1) \cup (-1, 2) \cup (2, \infty)$ (c) ϕ
2. (a) $(-\infty, -1) \cup [0, \infty)$ (b) $(2, 3]$
3. (a) ϕ (b) $(-3, -2) \cup (2, 3)$ (c) $[1, 4]$ (d) $(0, 1] \cup [4, 5)$
4. (a) ϕ (b) $(-\infty, 0)$ (c) $x \in [-5, -1] \cup [1, 5]$ (d) $(-\infty, -2] \cup [-1, 1] \cup [2, \infty)$
5. (a) $[-4, -\pi] \cup [0, \pi]$ (b) $\bigcup_{n\in\mathbb{Z}}\left[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right]$
6. (a) $-1 \le x < 2, \left\{\pi, \frac{\pi}{2}, 0\right\}$ (b) $D_f = \{0\}, R_f = \left\{\frac{\pi}{2}\right\}$
7. (a) $\mathbb{R}$ (b) $\{-1, 1\}$ (c) $\mathbb{R}$ (d) $\mathbb{R}$
8. (a) $\left(2n\pi - \sin^{-1}\left(\frac{3}{5}\right), (2n+1)\pi + \sin^{-1}\left(\frac{3}{5}\right)\right)$; $n \in \mathbb{Z}$ (b) $\mathbb{R}$
9. (a) $\left(0, \frac{1}{4}\right) \cup \left(\frac{3}{4}, 1\right) \cup \{2, 3, 4, \ldots\ldots \infty)]$ (b) $\{4\} \cup [5, \infty)$ (c) $[-6, 3) - \{2\}$
10. (a) $\left(-\frac{1}{6}, \frac{\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 6\right)$ (b) $(2, 3]$
11. (a) $(-2, -1) \cup (-1, 0) \cup (1, 2)$ (b) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \{-1, 0, 1\}$

TEXTUAL EXERCISE–9: (OBJECTIVE)

1. Domain of the definition of the function $\sqrt{x(x-1)}$ is
   (a) $[0, 1]$ (b) $[-1, 0]$
   (c) $(-\infty, 0] \cup [1, \infty)$ (d) None of these

2. Domain of the definition of the function $(x)^{\frac{1}{\ln x}}$ is
   (a) $(0, \infty) - \{1\}$ (b) $(0, 1)$
   (c) $[0, \infty) \sim \{1\}$ (d) None of these

3. Domain of the definition of the function $\frac{1}{\sqrt{x-|x|}}$ is
   (a) { } (b) $\mathbb{R}$
   (c) $(0, \infty)$ (d) None of these

4. Domain of the definition of the function $\frac{1}{\sqrt{|x|-x}}$ is
   (a) { } (b) $(-\infty, 0)$
   (c) $(0, \infty)$ (d) None of these

5. Domain of the definition of the function $\frac{1}{2-\cos 3x}$ is
   (a) $\mathbb{R} \sim \cos^{-1}\frac{2}{3}$ (b) $\mathbb{R}$
   (c) $\mathbb{R} \sim \cos^{-1}\frac{3}{2}$ (d) None of these

6. Domain of the definition of the function $\log_{10}(1 + x^3)$ is
   (a) $(-1, \infty)$ (b) $(0, \infty)$
   (c) $\mathbb{R}$ (d) None of these

7. The domain of function $\frac{1}{\sqrt{x(x-2)(x-3)}}$
   (a) $(0, 2)$ (b) $\mathbb{R} \sim \{0, 2, 3\}$
   (c) $(0, 2) \cup (3, \infty)$ (d) None of these

8 The domain of function $\frac{1}{\sqrt[3]{(x-1)(x-2)(x-4)}}$
   (a) $(1, 2) \cup (4, \infty)$ (b) $R - \{1, 2, 4\}$
   (c) $\mathbb{R}$ (d) None of these

9. The domain of definition of the function $y(x)$ given by the equation $2^x + 2^y = 2$ is
   (a) $0 < x \le 1$ (b) $0 \le x \le 1$
   (c) $-\infty < x \le 0$ (d) $-\infty < x < 1$

10. The domain of $f(x) = \sqrt{3-x} + \cos^{-1}\left(\frac{3-2x}{5}\right)$ is equal to
   (a) $[-1, 3]$ (b) $(-1, 3]$
   (c) $[-1, 3)$ (d) None of these

11. The domain of the function $f(x) = \sin^{-1}\left(\frac{x-3}{2}\right) - \log_{10}(4-x)$ is

(a) (1, 4) (b) [1, 4]
(c) [1, 4) (d) (1, 4]

12. The domain of definition of the function $y = \frac{1}{\log_{10}(3-x)} + \sqrt{x+2}$ is

(a) [–2, 3) (b) (–2, 3)
(c) [–2, 0) ∪ (0, 3] (d) None of these

13. The domain of the function $f(x) = \sqrt{\frac{(x+1)(x-3)}{(x-2)}}$ is

(a) [–1, 2) ∪ [3, ∞)
(b) (–1, 2) ∪ [3, ∞)
(c) [–1, 2] ∪ [3, ∞)
(d) None of these

14. The domain of definition of $f(x) = \sqrt{\frac{1-|x|}{2-|x|}}$ is

(a) $\frac{(-\infty, \infty)}{[-1, 1]}$
(b) $\frac{(-\infty, \infty)}{[-2, 2]}$
(c) [–1, 1] ∪ (–∞, –2) ∪ (2, ∞)
(d) None of these

15. The domain of the function $f(x) = \cos^{-1}\left(\frac{3}{4+2\sin x}\right)$ is equal to

(a) $\left[-\frac{\pi}{6}+2n\pi, \frac{\pi}{6}+2n\pi\right]$
(b) $\left(-\frac{\pi}{6}+2n\pi, \frac{\pi}{6}+2n\pi\right)$
(c) $\left(-\frac{\pi}{6}+2n\pi, \frac{\pi}{6}+2n\pi\right]$
(d) None of these

16. If $f(x) = \frac{1}{\sqrt{\sin x}} + \sqrt[3]{\sin x}$, then D_f is

(a) $\bigcup_{k\in\mathbb{Z}} (2k\pi, (2k+1)\pi)$
(b) (0, π)
(c) (π, 2π)
(d) ((2k + 1)π, (2k + 2)π)

17. The domain of $f(x) = \frac{1}{\sqrt{|\cos x| + \cos x}}$ is

(a) $\bigcup_{n\in\mathbb{Z}} [-2n\pi, 2n\pi]$
(b) $\bigcup_{n\in\mathbb{Z}} (2n\pi, (2n+1)\pi]$
(c) $\bigcup_{n\in\mathbb{Z}} \left(\frac{(4n+1)\pi}{2}, \frac{(4n+3)\pi}{2}\right)$
(d) $\bigcup_{n\in\mathbb{Z}} \left(\frac{(4n-1)\pi}{2}, \frac{(4n+1)\pi}{2}\right)$

Answer Keys

1. (c)	**2.** (a)	**3.** (a)	**4.** (b)	**5.** (b)	**6.** (a)	**7.** (c)	**8.** (b)	**9.** (d)	**10.** (a)
11. (c)	**12.** (d)	**13.** (a)	**14.** (c)	**15.** (d)	**16.** (a)	**17.** (d)			

RANGE OF FUNCTIONS

Bounded and Un-bounded Functions

A function *f*(*x*) is called bounded iff there exists two real and finite numbers *m*, *M* such that $m \le f(x) \le M \ \forall \ x \in D_f$; *m* is called lower bound and *M* is upper bound.

Existence of single upper/lower bound means existence of infinitely many upper/lower bounds .... $m_3 < m_2 < m_1 < m < f(x) < M < M_1 < M_2 \ldots$, i.e., every real number less than a lower bound is also a lower bound and every real number greater than an upper bound is also an upper bound.

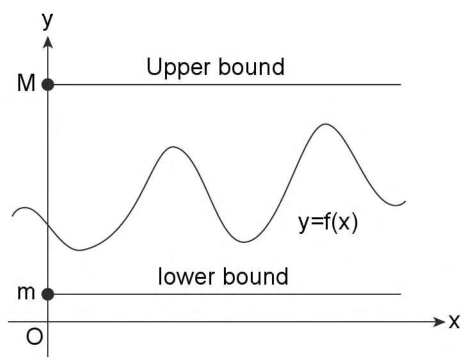


FIGURE 2.103

Greatest Lower Bound

The greatest of all lower bounds is called greatest lower bound denoted by $g.l.b.$ (m_0) and if $f(x_0) = m_0$ for some $x_0 \in D_f$, then $f(x_0) = m_0$ is called minimum value of $f(x)$. However, if $f(x) > m_0$ $\forall\ x \in D_f$ but $f(x) \neq m_0$ for any $x \in D_f$, then m_0 will be the $g.l.b.$ of $f(x)$ but m_0 will not be the minimum value of $f(x)$. For example, $f(x) = 1/x^2$; then $f(x) > 0\ \forall\ x \in \mathbb{R} \sim \{0\}$, therefore $g.l.b.$ of $f(x)$ is zero but it is not attained at any real number x, and hence, $f(x)$ has no minimum value.

That is, minimum value of $f(x) = 1/x^2$ does not exist but it approaches to 0.

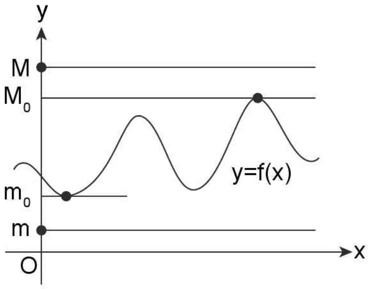


FIGURE 2.104

Conclusion

1. Greatest lower bound may or may not be the minimum value of function.
2. Minimum value of function if exists is the greatest lower bound of function
3. Every real number less than the greatest lower bound will be a lower bound of $f(x)$.
4. Every bounded function has its greatest lower bound but function may or may not attain it at any real number. If it attains its greatest lower bound, then the greatest lower bound will be the minimum value of function.
5. If the range of a function is (a, b) or $[a, b]$, then the greatest lower bound of $f(x)$ will be a. Minimum value of $f(x)$ does not exist in former case whereas minimum value of $f(x) = a$ in later case.

Least Upper Bound

The smallest of all upper bounds is called least upper bound denoted by $l.u.b.$ (M_0) and if $f(x_0) = M_0$ for some x_0, then $f(x_0)$ is called maximum value of $f(x)$. However, if $f(x) < M_0$ $\forall\ x \in D_f$ but $f(x) \neq M_0$ for any $x \in D_f$, then M_0 will be the $l.u.$b. of $f(x)$ but M_0 will not be the maximum value of $f(x)$. e.g., $f(x) = 1 - \dfrac{1}{x^2}$; then $f(x) < 1\ \forall\ x \in \mathbb{R} \sim \{0\}$, therefore, $l.u.$b. of $f(x)$ is 1, but it is not attained at any real number x, and hence, $f(x)$ has no maximum value.

That is, maximum value of $f(x) = 1 - \dfrac{1}{x^2}$ does not exist but it approaches to 1.

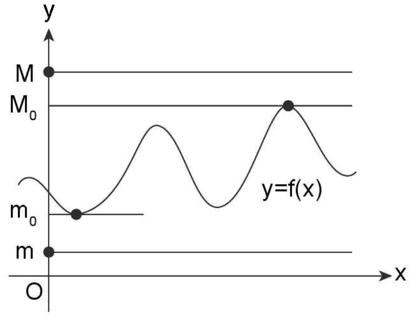


FIGURE 2.105

Conclusion

1. Least upper bound may or may not be the maximum value of function.
2. Maximum value of function if exists is the least upper bound of function.
3. Every real number greater than the least upper bound will be an upper bound of $f(x)$.
4. Every bounded function has its least upper bound but function may or may not attain it at any real number. If it attains its least upper bound, then the least upper bound will be the maximum value of function.
5. If the range of a function is (a, b) or $[a, b]$, then the least upper bound of $f(x)$ will be b. Maximum value of $f(x)$ does not exist in former case whereas maximum value of $f(x) = b$ in later case.

Intermediate Value Theorem

Every continuous and bounded function attains every real number between its bounds. That is, if a function is continuous in its domain and $g.l.b. = a$ and $l.u.b. = b$, then $f(x)$ must take up all real values between a and b. That is, there exists atleast one $\alpha \in D_f$ where $f(\alpha) = k\ \forall\ k \in (a, b)$.

$\Rightarrow$ Range of

$$f(x) = \begin{cases} [a,b]; \text{ if function also attains values } a \text{ and } b. \\ (a,b); \text{ if function does not attain values} \\ a \text{ and } b. \\ [a,b); \text{ if function attains value } a \text{ but does} \\ \text{not attain } b. \\ (a,b]; \text{ if function attains value } b \text{ but does} \\ \text{not attain } a. \end{cases}$$

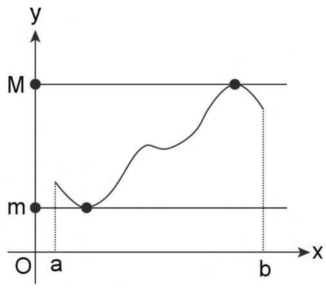


FIGURE 2.106

If $f(x)$ is monotonic and continuous with domain $[a, b]$, then it takes up all values from $f(a)$ and $f(b)$ exactly once.

That is, for each $k \in [a, b]$ there exists exactly one $\alpha \in D_f$ where $f(\alpha) = k$.

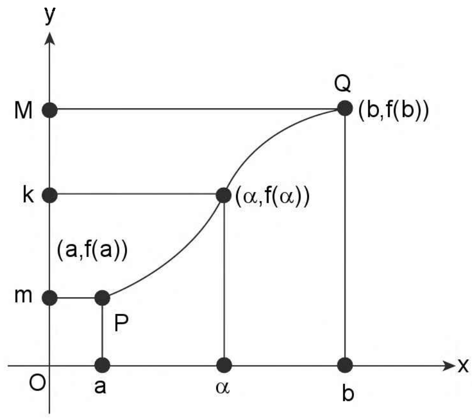


FIGURE 2.107

ILLUSTRATION 133: State whether the following statements are true or false.

(a) A function is called unbounded if at least one of its *g.l.b.* and *l.u.b.* does not exist.

(b) A function is called bounded when it is bounded from both above and below.

(c) A function may be bounded even if its *l.u.b.* and *g.l.b.* both do not exist.

(d) A function may not have its minimum value equal to its *g.l.b.*

SOLUTION: (a) True (b) True

(c) False (d) True

DEFINITION OF RANGE

Let f be a function from X to Y, then the range of function $= R_f = \{f(x) : x \in X\}$

Range of function $y = f(x)$ is defined as set of values of y for which x is real and finite $\forall\ x \in D_f$. It is denoted as R_f

Range is also called domain of variation and it is always a subset of co-domain (Y).

REMARK

So far, we have already discussed tools used to find the range of function, i.e., laws of inequalities, wavy curve method, monotonicity of functions, bounded ness of functions, intermediate value theorem. Now we shall use all these tools to find the range of functions as discussed further.

Methods to Find Range of Functions

Given a function $f : X \to Y$, where $y = f(x)$

Method I:

Step 1: Find domain of $f(x)$ (say) $\alpha \le x \le \beta$

Step 2: Express x in terms of y using equation of function. i.e., $x = f^{-1}(y)$

Step 3: Apply the domain restriction, i.e., $\alpha \le x \le \beta \Rightarrow \alpha \le f^{-1}(y) \le \beta$

Step 4: Find the set of all possible y satisfying the above inequality.

ILLUSTRATION 134: Find the range of function $y = \cos^{-1}(x^2+1)$.

SOLUTION: The domain of function $f(x) = \cos^{-1}(x^2+1)$ *is* $\{0\}$
Therefore range of function is $\{f(0)\}$ i.e., $\{0\}$

Method II:

For composition of continuous functions.

Step 1: Identify the function as composite function of constituent functions f, g and h (say) $\phi(x) = h(f(g(x)))$.

Step 2: Test the monotonicity of f, g and h. Let $g(\uparrow$(increasing)), $f(\downarrow$ (decreasing)), $h(\downarrow$ (decreasing))

Step 3: Find domain of $h(f(g(x)))$, (say) $\alpha \le x \le \beta$.

Step 4: $\because\ \alpha \le x \le \beta$

$\Rightarrow\ g(\alpha) \le g(x) \le g(\beta)$

$\Rightarrow\ f(g(\alpha)) \ge f(g(x)) \ge f(g(\beta))$

$\Rightarrow\ h(f(g(\alpha))) \le h(f(g(x))) \le h(f(g(\beta)))$ .... (1)

Because composition of continuous functions is also continuous, therefore, by (1) and intermediate value theorem, range of function ϕ is given by $R_\phi = [h(f(g(\alpha))), h(f(g(\beta)))]$

ILLUSTRATION 135: Find the range of function $f(x) = \dfrac{x^2-4}{x^2+4}$.

SOLUTION: Given function is $f(x) = \dfrac{x^2-4}{x^2+4} = \dfrac{x^2+4-8}{x^2+4} = 1 - \dfrac{8}{x^2+4}$

Clearly domain of function $= D_f = \mathbb{R}$

$\Rightarrow\ -\infty < x < \infty\ \Rightarrow\ 0 \le x^2 < \infty\ \Rightarrow\ 4 \le x^2+4 < \infty$

$\Rightarrow\ 0 < \dfrac{1}{x^2+4} \le \dfrac{1}{4}\ \Rightarrow\ 0 < \dfrac{8}{x^2+4} \le 2\ \Rightarrow\ -2 \le \dfrac{-8}{x^2+4} < 0\ \Rightarrow\ -1 \le 1 - \dfrac{8}{x^2+4} < 1$

$\Rightarrow\ f(x) \in [-1, 1)\ \Rightarrow$ Range of function is $R_f = [-1, 1)$

ILLUSTRATION 136: Find the range of the following functions:

(i) $f(x) = \sqrt{\sin x}$ (ii) $f(x) = 3^{\sin^{-1}x}$

SOLUTION: (i) $f(x) = \sqrt{\sin x}$

For domain of function $\sin x \ge 0$, but $-1 \le \sin x \le 1$, therefore we must have $0 \le \sin x \le 1$

$\Rightarrow\ 0 \le \sqrt{\sin x} \le 1\ \Rightarrow\ f(x) \in [0, 1]$ $\therefore$ Range of $f(x) = [0, 1]$

(ii) $f(x) = 3^{\sin^{-1}x}$

For domain of function $x \in [-1, 1]$ $\Rightarrow\ -\dfrac{\pi}{2} \le \sin^{-1}x \le \dfrac{\pi}{2}$,

$\Rightarrow\ 3^{-\pi/2} \le 3^{\sin^{-1}x} \le 3^{\pi/2}$; $(\because a \le x \le b \Rightarrow 3^a \le 3^x \le 3^b)$

$\Rightarrow\ 3^{-\pi/2} \le f(x) \le 3^{\pi/2}$ $\therefore$ Range of $f(x) = [3^{-\pi/2}, 3^{\pi/2}]$

Some Remarks and Tips to find the Range of Functions:

1. If domain is a set having only finite number of points, then the range will be the set of corresponding values of $f(x)$.
2. If domain of $y = f(x)$ is $\mathbb{R}$ or $\mathbb{R}$ −{some finite points} or an infinite interval, then with the help of given relation, express x in terms of y and from there find the values of y for which x is defined and belongs to the domain of the function $f(x)$. The set of corresponding values of y constitute the range of function.

ILLUSTRATION 137: To find the range of $y = \dfrac{x^2 - 9}{x^2 + 9}$

SOLUTION: $\because$ Domain is $\mathbb{R}$ consider the equation $y = \dfrac{x^2-9}{x^2+9}$ $\Rightarrow$ $yx^2 + 9y = x^2 - 9$

$\Rightarrow \quad x^2 = \dfrac{9(y+1)}{1-y} \quad \Rightarrow \quad x = \pm 3\sqrt{\dfrac{y+1}{1-y}}$

Thus, for x to be real $\dfrac{y+1}{1-y} \geq 0$ and $y \neq 1 \Rightarrow (y+1)(y-1) \leq 0$ and $y \neq 1$

$\Rightarrow$ $y \in [-1, 1)$. Therefore range of function = $[-1, 1)$

3. If a function is continuous, then find the least (l) and the greatest (g) values of $f(x)$ using monotonicity. Then the range of function will be $[l, g]$.
4. For the quadratic function $f(x) = ax^2 + bx + c$, domain is $\mathbb{R}$ and range is given by $R_f = \begin{cases} \left[-\dfrac{D}{4a}, \infty\right) & \text{for } a > 0 \\ \left(-\infty, \dfrac{-D}{4a}\right] & \text{for } a < 0 \end{cases}$
5. For the quadratic function $f(x) = \sqrt{ax^2 + bx + c}$ domain is given by $D_f = \begin{cases} \mathbb{R} \text{ for } a > 0, D < 0 \\ \phi \text{ for } a < 0, D < 0 \end{cases}$ and range is given by
$$R_f = \begin{cases} [0, \infty) & \text{for } D \geq 0,\ a > 0 \\ \left[\sqrt{\dfrac{-D}{4a}}, \infty\right) & \text{for } D < 0,\ a > 0 \\ \left[0, \sqrt{\dfrac{-D}{4a}}\right] & \text{for } D \geq 0,\ a < 0 \\ \phi & \text{for } D < 0,\ a < 0 \end{cases}$$
6. For odd degree polynomial, domain and range both are $\mathbb{R}$.
7. For even degree polynomial domain is $\mathbb{R}$ and range is given by $[k, \infty)$ if the leading coefficient is positive, where k is the minimum value of polynomial occurring at one of the points of local minima, whereas range is $(-\infty, k]$ if the leading coefficient is negative where k is maximum value of polynomial occurring at one of the points of local maxima.
8. For $\dfrac{\text{Quadratic}}{\text{Quadratic}}$ or $\dfrac{\text{Linear}}{\text{Quadratic}}$ or $\dfrac{\text{Quadratic}}{\text{Linear}}$ expression, put $y = \dfrac{P(x)}{Q(x)}$;. Cross multiply, convert into a quadratic and use the knowledge of quadratic equations.

ILLUSTRATION 138: To find the range of the function : $f(x) = .\dfrac{x^2 - 3x + 4}{x^2 + 3x + 4}$

SOLUTION: $y = \dfrac{x^2 - 3x + 4}{x^2 + 3x + 4}$. Since, denominator could not be zero, therefore domain of above function is a set of real number, hence, by cross multiplying, we get $y(x^2 + 3x + 4) = (x^2 - 3x + 4)$.

$\Rightarrow$ $(y - 1)x^2 + 3(y + 1)x + 4(y - 1) = 0$ (is quadratic with real roots)

$\Rightarrow$ $D \geq 0$ $\Rightarrow$ $9(y + 1)^2 - 16(y - 1)^2 \geq 0$

$\Rightarrow$ $(7y - 1)(7 - y) \geq 0$ $\Rightarrow$ $(7y - 1)(y - 7) \leq 0$ $\Rightarrow$ $y \in [1/7, 7]$

9. For discontinuous functions, only method is to draw the graph and find the range known as graphical method of finding out range.

10. While determining domain and range of the functions involving trigonometric functions we need to keep in mind the following useful results.

(i) General solution of equation $\sin\theta = k \in [-1, 1]$ is given by $\theta = n\pi + (-1)^n\alpha$; $\alpha = \sin^{-1}k$; $n \in \mathbb{Z}$

(ii) General solution of equation $\cos\theta = k \in [-1, 1]$ is given by $\theta = 2n\pi \pm \alpha$; $\alpha = \cos^{-1}k$; $n \in \mathbb{Z}$

(iii) General solution of equation $\tan\theta = k \in \mathbb{R}$ is given by $\theta = n\pi + \alpha$; $\alpha = \tan^{-1}k$; $n \in \mathbb{Z}$

(iv) General solution of equation $\sin\theta = \sin\alpha$ is given by $\theta = n\pi + (-1)^n\alpha$; $n \in \mathbb{Z}$

(v) General solution of equation $\cos\theta = \cos\alpha$ is given by $\theta = 2n\pi \pm \alpha$; $n \in \mathbb{Z}$

(vi) General solution of equation $\tan\theta = \tan\alpha$ is given by $\theta = n\pi + \alpha$; $n \in \mathbb{Z}$

(vii) $\sin\theta$, $\cos\theta$, $\sec\theta$, $\operatorname{cosec}\theta$ are periodic functions with period 2π, whereas $\tan\theta$ and $\cot\theta$ are periodic functions with period π.

(viii) (a) General solution of inequality of the form $\sin\theta \geq k$; $1 \geq k \geq 0$ is given by

$$\theta \in \bigcup_{n\in\mathbb{Z}} [2n\pi + \alpha, (2n+1)\pi - \alpha];\ \alpha = \sin^{-1}k$$

General solution of inequality of the form $\sin\theta \geq k$; $-1 \leq k < 0$ is given by

$$\theta \in \bigcup_{n\in\mathbb{Z}} [(2n\pi - \alpha), (2n+1)\pi + \alpha];\ \alpha = \sin^{-1}|k|$$

General solution of inequality of the form $\sin\theta \leq k$; $1 \geq k \geq 0$ is given by

$$\theta \in \bigcup_{n\in\mathbb{Z}} \{[2n\pi, 2n\pi + \alpha] \cup [(2n+1)\pi - \alpha, (2n+2)\pi]\};$$

$\alpha = \sin^{-1}(k)$

General solution of inequality of the form $\sin\theta \leq k$; $-1 \leq k < 0$ is given by

$$\theta \in \bigcup_{n\in\mathbb{Z}} [(2n+1)\pi + \alpha, (2n+2)\pi - \alpha];\ \alpha = \sin^{-1}|k|$$

(b) General solution of inequality of the form $\cos\theta \geq k$; $1 \geq k \geq 0$ is given by

$$\theta \in \bigcup_{n\in\mathbb{Z}} [2n\pi - \alpha, 2n\pi + \alpha];\ \alpha = \cos^{-1}(k)$$

General solution of inequality of the form $\cos\theta \geq k$; $-1 \leq k < 0$ is given by

$$\theta \in \bigcup_{n\in\mathbb{Z}} \left\{ \begin{array}{l} \left[2n\pi - \dfrac{\pi}{2}, \alpha\right] \cup \\ \left[(2n+2)\pi - \alpha, (2n+2)\pi - \dfrac{\pi}{2}\right] \end{array} \right\};\ \alpha = \cos^{-1}(k)$$

General solution of inequality of the form $\cos\theta \leq k$; $1 \geq k \geq 0$ is given by

$$\theta \in \bigcup_{n\in\mathbb{Z}} \left\{ \begin{array}{l} \left[2n\pi - \dfrac{\pi}{2}, 2n\pi - \alpha\right] \cup \\ \left[2n\pi + \alpha, (2n+2)\pi - \dfrac{\pi}{2}\right] \end{array} \right\};\ \alpha = \cos^{-1}(k)$$

General solution of inequality of the form $\cos\theta \leq k$; $-1 \leq k < 0$ is given by

$$\theta \in \bigcup_{n\in\mathbb{Z}} [2n\pi + \alpha, (2n+2)\pi - \alpha];\ \alpha = \cos^{-1}(k)$$

(c) General solution of inequality of the form $\tan\theta \geq k$; $k \in \mathbb{R}$ is given by

$$\theta \in \bigcup_{n\in\mathbb{Z}} \left[2n\pi + \tan^{-1}(k), 2n\pi + \frac{\pi}{2}\right)$$

General solution of inequality of the form $\tan\theta \leq k$; $k \in \mathbb{R}$ is given by

$$\theta \in \bigcup_{n\in\mathbb{Z}} \left(2n\pi - \frac{\pi}{2}, 2n\pi + \tan^{-1}k\right]$$

(d) General solution of inequality of the form $\cot\theta \geq k$; $k \in \mathbb{R}$ is given by

$$\theta \in \bigcup_{n\in\mathbb{Z}} (2n\pi, 2n\pi + \cot^{-1}(k)]$$

General solution of inequality of the form $\cot\theta \leq k$; $k \in \mathbb{R}$ is given by

$$\theta \in \bigcup_{n\in\mathbb{Z}} [2n\pi + \cot^{-1}k, (2n+1)\pi)$$

11. Range of function $f(x) = a\sin x + b\cos x$ is $\left[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}\right]$

12. Graphs of inverse circular functions and their corresponding domain and range.

(a) Graph of $\sin^{-1}x$; Domain $= [-1, 1]$; Range $= \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

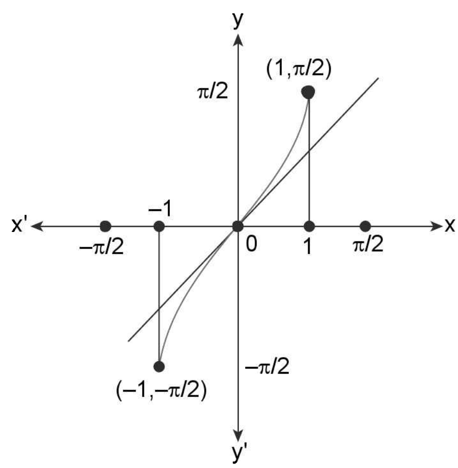


FIGURE 2.108

(b) Graph of $\cos^{-1}x$; Domain = [–1, 1]; Range = [0, π]

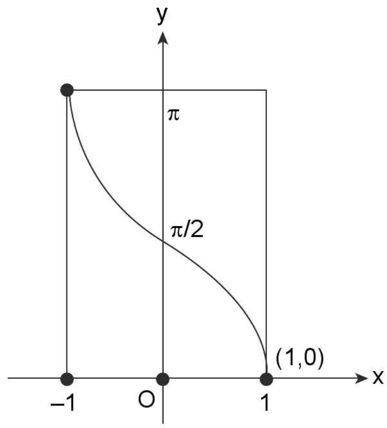


FIGURE 2.109

(c) Graph of $\tan^{-1}x$; Domain = ℝ; Range = $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

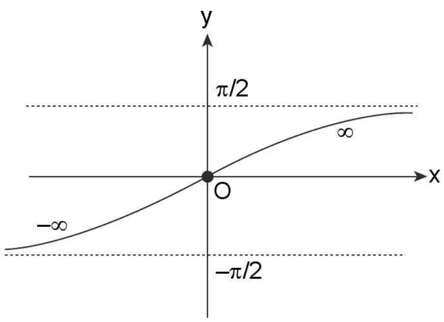


FIGURE 2.110

(d) Graph of $\cot^{-1}x$; Domain = ℝ; Range = (0, π)

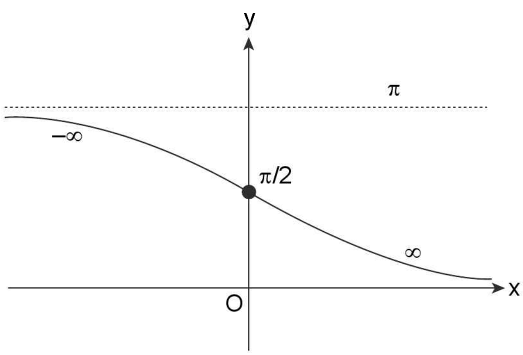


FIGURE 2.111

(e) Graph of $\sec^{-1}x$; Domain = (–∞, –1] ∪ [1, ∞);

Range = $[0, \pi] \sim \left\{\dfrac{\pi}{2}\right\}$

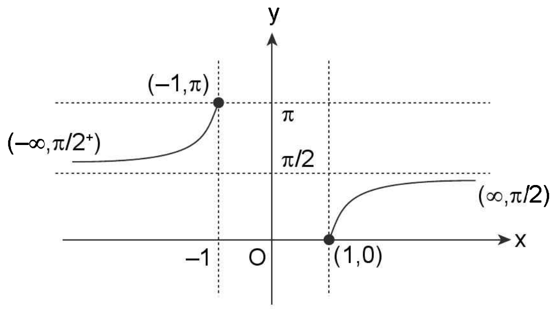


FIGURE 2.112

(f) Graph of $\mathrm{cosec}^{-1}x$; Domain = (–∞, –1] ∪ [1, ∞);

Range = $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] \sim \{0\}$

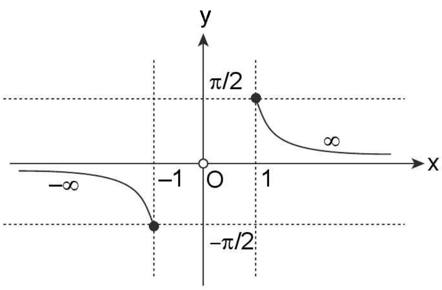


FIGURE 2.113

13. Three important identities of inverse trigonometric functions are given below

(a) $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$ (b) $\tan^{-1}x + \cot^{-1}x = \dfrac{\pi}{2}$

(c) $\sec^{-1}x + \mathrm{cosec}^{-1}x = \dfrac{\pi}{2}$

Range of Some Standard Functions:

f(*x*)	Range : R_f	*f*(*x*)	Range : R_f
Polynomial $P(x)$ of odd degree	ℝ	$\cot x$	ℝ
a^x	(0, ∞)	$\sec x$	(–∞, –1] ∪ [1, ∞)
$\log x$	ℝ	$\mathrm{cosec}\, x$	(–∞, –1] ∪ [1, ∞)
$\lvert x\rvert$	[0, ∞)	$\sin^{-1}x$	[–π/2, π/2]
$[x]$	ℤ	$\cos^{-1}x$	[0, π]
$\{x\}$	[0, 1)	$\tan^{-1}x$	(–π/2, π/2)
$\sin x$	[–1,1]	$\cot^{-1}x$	(0, π)
$\cos x$	[–1,1]	$\sec^{-1}x$	[0, π] – {π/2}
$\tan x$	ℝ	$\mathrm{cosec}^{-1}x$	[–π/2, π/2] – {0}
$\mathrm{sgn}\, x$	{–1, 0, 1}		

ILLUSTRATION 139: Find the range of the following functions:

(a) $y=\left(\dfrac{1}{2-\sin 3x}\right)$ (b) $y = \ln(3x^2 - 4x + 5)$

SOLUTION: (a) Given $\left(\dfrac{\quad}{2\quad \sin 3}\right)$; Domain is $(-\infty, \infty)$, as $2 > \sin 3x\ \forall\ x \in \mathbb{R}$

$\Rightarrow\ — = 2-\sin 3$ $\Rightarrow\ \sin 3x = 2-\dfrac{1}{y}$

Since $-1 \le \sin 3x \le 1$ $\Rightarrow\ -1 \le 2-\dfrac{1}{y} \le 1$

$\Rightarrow\ -3 \le -\dfrac{1}{y} \le -1$ $\Rightarrow\ y \in (-\infty, 0] \cup [1/3, \infty)$ and $y \in (0, 1]$ $\Rightarrow\ y \in \left[\dfrac{1}{3}, 1\right]$

Hence, range is $[1/3, 1]$

(b) Given $y = \ln(3x^2 - 4x + 5)$. For y to be defined $3x^2 - 4x + 5 > 0$

$\Rightarrow\ x^2 - \dfrac{4}{3}x + \dfrac{5}{3} > 0$ $\Rightarrow\ \left(x-\dfrac{2}{3}\right)^2 + \dfrac{11}{9} > 0$; which is true for all x

$\therefore$ Domain is $(-\infty, \infty)$. We have to find out the maximum and minimum value of y.

y is minimum, when $3x^2 - 4x + 5$ is minimum and maximum when it is $3x^2 - 4x + 5$ as $\ln x$ is an increasing function.

Now, $(3x^2 - 4x + 5)$, min $= \dfrac{11}{3}$ (when $x = 2/3$) $\Rightarrow\ y_{min} = \ln \dfrac{11}{3}$

For $y_{max} = (3x^2 - 4x + 5)_{max} = \infty$ $\Rightarrow\ y_{max} = \infty$

Hence, range $= [\ln(11/3), \infty)$.

ILLUSTRATION 140: Find the range of the values of x for which the following functions are defined:

(a) $f(x) = \sqrt{\log_{1/2}\left(\dfrac{5x-x^2}{4}\right)}$ (b) $f(x) = \log_4 \log_2 \log_{1/2}(x)$

SOLUTION: (a) y is defined if $\dfrac{5x-x^2}{4} > 0 \Rightarrow$ $5x - x^2 > 0$ $\Rightarrow$ $0 < x < 5$ ...(i)

$\Rightarrow\ x \in (0, 5)$ and $\log_{1/2}\left(\dfrac{5x-x^2}{4}\right) \ge 0$ $\Rightarrow\ \dfrac{5x-x^2}{4} \le \left(\dfrac{1}{2}\right)^0$ $\Rightarrow\ x^2 - 5x + 4 \ge 0$

$\Rightarrow\ x \le 1$ or $x \ge 4$ ...(ii)

From (i) and (ii), we get $x \in (0, 1] \cup [4, 5)$

(b) y is defined if $\log_2 \log_{1/2}(x) > 0$ $\Rightarrow\ \log_{1/2}(x) > 2^0$

$\Rightarrow\ \log_{1/2}(x) > 1$ $\Rightarrow\ x < 1/2$

$\Rightarrow$ Domain $= (0, 1/2)$.

ILLUSTRATION 141: Find the domain and range of function $f(x) = \sin^{-1} x^2 + \left[\left\{\ell n\sqrt{x-[x]}\right\}\right] + \cot^{-1}\left(\dfrac{1}{1+\sqrt{2x^2}}\right)$.

SOLUTION: Domain of function is $(-1, 1) - \{0\}$, because $x - [x] = 0$ for integral value of x, hence, middle term will not be defined. Also $[\{f\}] = 0$, whenever f is meaningful.

$\therefore$ Value of $f(x) = \sin^{-1} x^2 + \tan^{-1}(1 + \sqrt{2}\, x^2)$ $\left(\cot^{-1} x = \tan^{-1}\dfrac{1}{x} \text{ when } x > 0\right)$

Function is continuous and is even for $x \in (-1, 1) - \{0\}$. Also $f(x)$ is ↓ an $(-1, 0)$ and ↑ on $(0, 1)$

$\therefore$ Least value of the function will occur when $x \to 0$ and is $\dfrac{\pi}{4}$

And maximum value $= \lim\limits_{x\to\pm 1} f(x) = \sin^{-1}1 + \tan^{-1}(1+\sqrt{2}) = \dfrac{\pi}{2} + \dfrac{3\pi}{8} = \dfrac{7\pi}{8}$

$\therefore$ Range of $f(x)$ is $\left(\dfrac{\pi}{4}, \dfrac{7\pi}{8}\right)$; Domain is $(-1, 1) - \{0\}$.

ILLUSTRATION 142: Find the range of $f(x) = \sqrt{\left(\dfrac{x-1}{2-x}\right)}$

SOLUTION: $y = \sqrt{\dfrac{x-1}{2-x}}$ the domain of function is $[1, 2)$

If $y = f(g(x))$ where $f(x) = \sqrt{x}$ (increasing (↑) function $\forall\, x \in [0, \infty)$)

And $g(x) = \dfrac{x-1}{2-x}$ (increasing (↑) function $\forall\, x \in [1, 2)$)

Now $1 \le x < 2$ $\Rightarrow g(1) \le g(x) < g(2)$ as $g(x)$ is increasing (↑)

$\Rightarrow 0 \le g(x) < \infty$ applying f

$\Rightarrow f(0) \le f(g(x)) < f(\infty)$ $\Rightarrow 0 \le \sqrt{\dfrac{x-1}{2-x}} < \infty$ $\Rightarrow y \in [0, \infty)$

Aliter: $y_1 = -\left(\dfrac{1-x}{2-x}\right) = -1 + \dfrac{1}{2-x}$

$\because\ 1 \le x < 2$ $\Rightarrow -2 \le -x \le -1$ $\Rightarrow 0 < 2 - x \le 1$

$\Rightarrow 1 \le \dfrac{1}{2-x} < \infty$ $\Rightarrow 0 \le -1 + \dfrac{1}{2-x} < \infty$ $\Rightarrow y_1 \in [0, \infty)$

$\Rightarrow y \in \sqrt{y_1} \in [0,\infty)$

ILLUSTRATION 143: Find the domain and range of the following functions:

(i) $f(x) = 3 - 4 \sin x$ (ii) $f(x) = \dfrac{1}{2 - 5\sin x}$

(iii) $f(x) = \dfrac{1}{\sqrt{\sqrt{3} - \tan x}}$ (iv) $f(x) = \sqrt{2\sin x - 1}$

SOLUTION: (i) $f(x) = 3 - 4$ sinx is defined for all those real values of x for which $\sin x$ is defined

$\Rightarrow$ Domain of function $= \mathbb{R} = (-\infty, \infty)$

Also $-1 \le \sin x \le 1$

$\Rightarrow -4 \le -4\sin x \le 4$

$\Rightarrow$ Range of $f(x) = [-1, 7]$

$\Rightarrow y\ -1 \le 3 - 4\sin x \le 7$

(ii) $f(x) = \dfrac{1}{2 - 5\sin x}$

For $f(x)$ to be defined $\sin x \ne \dfrac{2}{5}$. From the graph of $\sin x$ given below we conclude that $\sin x \ne \dfrac{2}{5}$

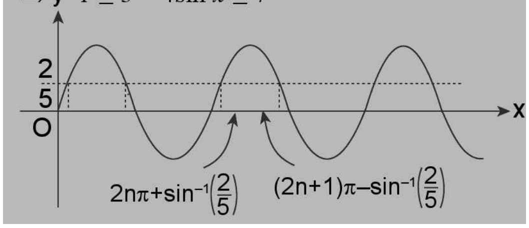


FIGURE 2.114

$\Rightarrow\ x \neq 2n\pi + \sin^{-1}\left(\frac{2}{5}\right),\ (2n+1)\pi - \sin^{-1}\left(\frac{2}{5}\right); n \in \mathbb{Z}$

$\therefore$ Domain $= \mathbb{R} \sim \left\{2n\pi + \sin^{-1}\left(\frac{2}{5}\right), (2n+1)\pi - \sin^{-1}\left(\frac{2}{5}\right); n \in \mathbb{Z}\right\}$

Clearly $-1 \le \sin x \le 1$ $\Rightarrow -5 \le -5\sin x \le 5$

$\Rightarrow -3 \le 2 - 5\sin x \le 7$

But $\sin x \neq \frac{2}{5}$ $\Rightarrow 2 - 5\sin x \in [-3, 7] \sim \{0\}$

$\Rightarrow \frac{1}{2-5\sin x} \in \left(-\infty, -\frac{1}{3}\right] \cup \left[\frac{1}{7}, \infty\right)$ $\therefore$ Range $= \left(-\infty, -\frac{1}{3}\right] \cup \left[\frac{1}{7}, \infty\right)$

(iii) $f(x) = \frac{1}{\sqrt{\sqrt{3} - \tan x}}$

For $f(x)$ to be defined $\sqrt{3} - \tan x > 0$, i.e., $\tan x < \sqrt{3}$

From the graph of tan x given below

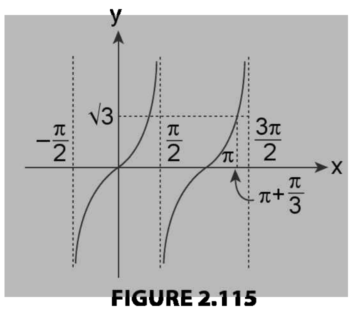


FIGURE 2.115

Since tan x is periodic with period π

Thus, $\tan x < \sqrt{3}$ for $x \in \left(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{3}\right); n \in \mathbb{Z}$

Thus, domain of $f(x) = \bigcup_{n\in\mathbb{Z}} \left(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{3}\right) = D_f$

Also $0 < \sqrt{3} - \tan x < \infty$ for $x \in \left(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{3}\right)$ for each $n \in \mathbb{Z}$ and $\sqrt{3} - \tan x$ is continuous function on $\left(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{3}\right)$ for each $n \in \mathbb{Z}$

$\Rightarrow (\sqrt{3} - \tan x) \in (0, \infty)$ $\Rightarrow \sqrt{\sqrt{3} - \tan x} \in (0, \infty)$

$\Rightarrow \frac{1}{\sqrt{\sqrt{3} - \tan x}} \in (0, \infty)$ $\Rightarrow$ Range of $f(x) = (0, \infty)$

(iv) $f(x) = \sqrt{2\sin x - 1}$

For $f(x)$ to be defined $2\sin x - 1 \ge 0$ $\Rightarrow \sin x \ge \frac{1}{2}$

In $[0, 2\pi]$, $\sin x \ge \frac{1}{2}$, for $x \in \left[\frac{\pi}{6}, \pi - \frac{\pi}{6}\right]$. As sin x is periodic function with period 2π

$\therefore$ Domain of $f(x) = \bigcup_{n\in\mathbb{Z}} \left(2n\pi + \frac{\pi}{6}, (2n+1)\pi - \frac{\pi}{6}\right)$

Also $\frac{1}{2} \le \sin x \le 1$ $\Rightarrow 1 \le 2\sin x \le 2$

$\Rightarrow 0 \le 2\sin x - 1 \le 1$

$\Rightarrow 0 \le \sqrt{2\sin x - 1} \le 1$ $\therefore$ Range of $f(x) = [0, 1]$

ILLUSTRATION 144: Find the domain and range of the following functions

(i) $f(x) = x^2 - 6x + 10$ (ii) $f(x) = x^2 - 8x + 10$

(iii) $f(x) = 3x^7 - 4x^2 + 7$ (iv) $f(x) = 3x^4 + 8x^3 - 6x^2 - 24x + 7$

SOLUTION: (i) $f(x)$ being a polynomial function, has domain = $\mathbb{R}$ = $(-\infty, \infty)$

Now $f(x) = x^2 - 6x + 10 = x^2 - 6x + 9 + 1$

$= (x-3)^2 + 1 \geq 1 \; \forall \; x \in \mathbb{R}$

Also $f(x)$ being a polynomial function is a continuous function and as x tends to infinity, $f(x) = x^2\left(1 - \frac{6}{x} + \frac{10}{x^2}\right)$ tends to $(\infty)^2(1 - 0 + 0) = \infty$, thus, $f(x)$ would attain each and every real number from 1 onwards (intermediate value theorem).

Thus, range of $f(x) = [1, \infty)$

(ii) $f(x) = x^2 - 8x + 10$

Clearly domain = $\mathbb{R}$ and $f(x) = x^2 - 8x + 10 = x^2 - 8x + 16 - 16 + 10 = (x-4)^2 - 6 \geq -6$

$\therefore$ Range of $f(x) = [-6, \infty)$

Alternatively by theory of quadratic equations we know that range of a quadratic polynomial is given by $\left[\frac{-D}{4a}, \infty\right) = \left[\frac{-(64-40)}{4}, \infty\right) = [-6, \infty)$

(iii) $f(x) = 3x^7 - 4x^2 + 7$

Domain = $\mathbb{R}$

Also $f(x)$ being an odd degree polynomial, has its range $\mathbb{R}$, as $f(x)$ is continuous having minimum and maximum values approaching to $-\infty$ and ∞, respectively.

(iv) $f(x) = 3x^4 + 8x^3 - 6x^2 - 24x + 7$

Clearly domain = $\mathbb{R}$ and $x \to \pm\infty, f(x) \to \infty$.

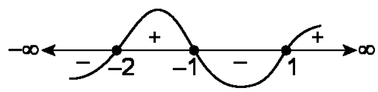


FIGURE 2.116

The minimum value of $f(x)$, would exist at one of the stationary points (where $f'(x) = 0$)

Now, $f'(x) = 12x^3 + 24x^2 - 12x - 24 = 12(x-1)(x+1)(x+2)$

$\therefore \; f'(x) = 0 \qquad \Rightarrow x = -1, 1, -2$

By sign scheme,

$\therefore$ $f(x)$ is decreasing on interval $(-\infty, -2)$, $(-1, 1)$ and increasing on intervals $(-2, -1)$ and $(1, \infty)$

Thus, the graph of $f(x)$ would be of the form as shown below.

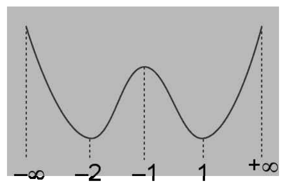


FIGURE 2.117

Thus, $f(x)$ has its minimum value at $x = -2$ or at $x = 1$

Now $f(-2) = 3(-2)^4 + 8(-2)^3 - 6(-2)^2 - 24(-2) + 7 = 15$

And $f(1) = 3(1)^4 + 8(1)^3 - 6(1)^2 - 24(1) + 7 = -12 \Rightarrow$ Range of $f(x)$ will be $= [-12, \infty)$

ILLUSTRATION 145: If $[2 \cos x] + [\sin x] = -3$, then find the range of the function, $f(x) = \sin x + \sqrt{3} \cos x$ in $[0, 2\pi]$. (where [.] denotes greatest integer function)

SOLUTION: $[2 \cos x] + [\sin x] = -3$

Now, $-2 \leq 2 \cos x \leq 2 \Rightarrow [2 \cos x] \in \{-2, -1, 0, 1, 2\}$ and $-1 \leq \sin x \leq 1 \Rightarrow [\sin x] \in \{-1, 0, 1\}$

Now, equation holds true for $[2 \cos x] = -2$ and $[\sin x] = -1$

$\Rightarrow \; -1 \leq \cos x < -\frac{1}{2}$ and $-1 \leq \sin x < 0 \; \Rightarrow \; x \in \left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$ and $x \in (\pi, 2\pi) \Rightarrow \; x \in \left(\pi, \frac{4\pi}{3}\right)$

$\Rightarrow f(x) = \sin x + \sqrt{3}\cos x$

$$\Rightarrow \quad f(x) = 2\sin\left(x+\frac{\pi}{3}\right)$$

Now, $\frac{4\pi}{3} < x + \frac{\pi}{3} < \frac{5\pi}{3}$

$$\Rightarrow \quad -1 \le \sin\left(x+\frac{\pi}{3}\right) < -\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad -2 \le 2\sin\left(x+\frac{\pi}{3}\right) < -\sqrt{3}$$. Hence, range is $\left[-2, -\sqrt{3}\right)$

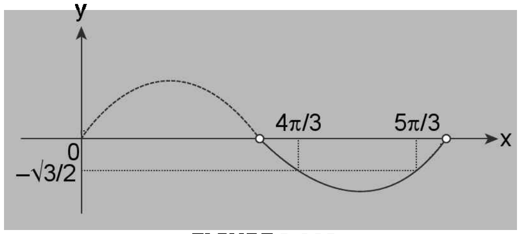


FIGURE 2.118

ILLUSTRATION 146: A function f is defined as $f(x) = x^3 - 5x^2 + 15x + \sin^{-1}\left(\frac{x}{2}\right)$, then $(f_{max.} - f_{min})$ is given by

(a) –40 (b) $\pi - 40$

(c) $76 + \pi$ (d) None of these

SOLUTION: $$f'(x) = \underbrace{3x^2 - 10x + 15}_{\substack{a>0 \\ D<0}} + \frac{1}{2}\frac{1}{\sqrt{1-\frac{x^2}{4}}} > 0$$

$\Rightarrow f(x)$ is monotonically increasing in its domain [–2, 2]

$$\therefore \quad f_{min} = f(-2) = -8 - 5(4) + 15(-2) + \left(-\frac{\pi}{2}\right) = -\left(58 + \frac{\pi}{2}\right) = -\left(\frac{116+\pi}{2}\right)$$

and $f_{max} = f(2) = 8 - 20 + 30 + \frac{\pi}{2} = 18 + \frac{\pi}{2} = \frac{36+\pi}{2} \quad \Rightarrow f_{max} - f_{min} = 76 + \pi$

ILLUSTRATION 147: The range of $f(x) = |\sin x|^2 + 2|\sin x| - |\cos x|$ is given by

(a) [–1, 2] (b) [–1, 1]

(c) [–1, 3] (d) None of these

SOLUTION: $f(x)$ is periodic with period π

$$\Rightarrow \quad f(x) = \begin{cases} \sin^2 x + 2\sin x - \cos x; & \text{for } x \in \left[0, \frac{\pi}{2}\right] \\ \sin^2 x + 2\sin x + \cos x & \text{for } x \in \left[\frac{\pi}{2}, \pi\right] \end{cases}$$

$\Rightarrow f(x)$ is increasing in the interval $[0, \pi/2]$ and decreasing in $\left[\frac{\pi}{2}, \pi\right]$.

Thus, if $0 \le x \le \frac{\pi}{2}$ $\quad \Rightarrow f(0) \le f(x) \le f\left(\frac{\pi}{2}\right) \quad \Rightarrow f(x) \in [-1, 3]$

and if $\frac{\pi}{2} \le x \le \pi$ $\quad \Rightarrow f(\pi) \le f(x) \le f\left(\frac{\pi}{2}\right) \quad \Rightarrow f(x) \in [-1, 3]$

$\Rightarrow$ Range of f is [–1, 3]

ILLUSTRATION 148: Find the range of the expression $\frac{\tan^2\theta - 4\tan\theta - 2}{\tan^2\theta - 2\tan\theta - 5}$, for all permissible values of θ.

SOLUTION: Let $x = \tan\theta$, where $x \in \mathbb{R}$ for all permissible values of θ.

$$\Rightarrow \quad y = \frac{x^2 - 4x - 2}{x^2 - 2x - 5} \qquad \Rightarrow x^2y - 2xy - 5y = x^2 - 4x - 2$$

$\Rightarrow (y-1)x^2 + 2x(2-y) + 2 - 5y = 0$ as $x \in \mathbb{R}$, hence, $D \geq 0$

$\Rightarrow 4(2-y)^2 - 4(y-1)(2-5y) \geq 0$ $\Rightarrow 4(4+y^2-4y) - 4(-5y^2+7y-2) \geq 0$

$\Rightarrow 24y^2 - 44y + 24 \geq 0$ $\Rightarrow 6y^2 - 11y + 6 \geq 0$ .... (1)

Since, coefficient of $y^2 > 0$ and $D = 121 - 144 < 0$

Hence, (1) is true for every real value of y. Therefore range of y is $(-\infty, \infty) = \mathbb{R}$.

ILLUSTRATION 149: Find the range of the function $f(\theta) = \dfrac{(\cot^2\theta+3)(\cot^2\theta+8)}{\cot^2\theta+1}$ for all permissible values of θ.

SOLUTION: Let $x = \cot^2\theta + 1$; where $x \in [1, \infty)$ for all permissible values of θ.

Then $f(\theta) = \dfrac{(x+2)(x+7)}{x} = x + 9 + \dfrac{14}{x}$

$$= \left(\sqrt{x}\right)^2 + \left(\sqrt{\frac{14}{x}}\right)^2 + 9 = \left(\sqrt{x}\right)^2 + \left(\sqrt{\frac{14}{x}}\right)^2 + 2\sqrt{14} + 9 - 2\sqrt{14} = \left(\sqrt{x} - \sqrt{\frac{14}{x}}\right)^2 + 9 + 2\sqrt{14}$$

$\Rightarrow$ Range of $f(\theta) = \left[9 + 2\sqrt{14}, \infty\right)$.

ILLUSTRATION 150: Find the range of the following functions:

(i) $f(x) = \cos^{-1}\left(\dfrac{\sqrt{2x^2+1}}{x^2+1}\right)$ (ii) $g(x) = \log_2(2 - \log_{\sqrt{2}}(4\cos^2 x + 1))$

SOLUTION: (i) Given $f(x) = \cos^{-1}\left(\dfrac{\sqrt{2x^2+1}}{x^2+1}\right)$

$$= \sin^{-1}\left(\sqrt{1 - \frac{(2x^2+1)}{(x^2+1)^2}}\right) = \sin^{-1}\left(\sqrt{\frac{x^4}{(x^2+1)^2}}\right) = \sin^{-1}\left(\frac{x^2}{1+x^2}\right);$$

Now $\dfrac{x^2}{x^2+1} = 1 - \dfrac{1}{x^2+1}$ and $x^2 + 1 \geq 1$ $\Rightarrow \dfrac{1}{x^2+1} \in (0,1]$ $\Rightarrow 1 - \dfrac{1}{x^2+1} \in [0,1)$

$\Rightarrow 0 \leq \sin^{-1}\left(\dfrac{x^2}{x^2+1}\right) < \dfrac{\pi}{2}$ $\Rightarrow y \in \left[0, \dfrac{\pi}{2}\right)$ $\Rightarrow$ The range of $f(x) = \left[0, \dfrac{\pi}{2}\right)$

(ii) Given $g(x) = \log_2(2 - \log_{\sqrt{2}}(4\cos^2 x + 1))$

Now for $g(x)$ to be well defined, $2 - \log_{\sqrt{2}}(4\cos^2 x + 1) > 0$ and $4\cos^2 x + 1 > 0$

The second inequality is true for all $x \in \mathbb{R}$.

$\Rightarrow g(x)$ is well defined for $4\cos^2 x + 1 < 2$, i.e., $1 \leq 4\cos^2 x + 1 < 2$

$\Rightarrow 0 \leq \log_{\sqrt{2}}(4\cos^2 x + 1) < 2$ $\Rightarrow -2 < -\log_{\sqrt{2}}(4\cos^2 x + 1) \leq 0$

$\Rightarrow 0 < 2 - \log_{\sqrt{2}}(4\cos^2 x + 1) \leq 2$ $\Rightarrow -\infty < g(x) \leq \log_2 2$

$\Rightarrow g(x) \in (-\infty, 1]$ $\Rightarrow$ Range of $g(x)$ is $(-\infty, 1]$.

ILLUSTRATION 151: Find the range of the function $f(x) = \sin^{-1} x^4 + \left[\left\{\ln\sqrt{x - [x]}\right\}\right] + \cot^{-1}\left(\dfrac{1}{2+\sqrt{3x^2}}\right)$; where {.} and [.] are fractional part function and greatest integer function, respectively.

SOLUTION: $\sin^{-1}x^4$ is defined for $x^4 \in [-1, 1]$ $\Rightarrow$ $x^4 \in [0, 1]$ $\Rightarrow$ $x \in [-1, 1]$

And $\cot^{-1}\left(\dfrac{1}{2+\sqrt{3x^2}}\right)$ is defined $\forall\, x \in \mathbb{R}$, but $\ln\sqrt{x-[x]} = \ln\sqrt{\{x\}}$ is not defined for $\{x\} = 0$

i.e., $x = -1, 0, 1$ $\therefore$ Domain of given function is $(-1, 1) \sim \{0\}$

Now $\left[\left\{\ln\sqrt{x-[x]}\right\}\right]=0$, since $[\{k\}] = 0$ whenever k is defined

$\therefore\quad f(x) = \sin^{-1}x^4 + \cot^{-1}\left(\dfrac{1}{2+\sqrt{3}x^2}\right)$; $x \in (-1, 1) \sim \{0\}$

or $f(x) = \sin^{-1}x^4 + \tan^{-1}\left(2+\sqrt{3}x^2\right)$ $\qquad$ $\left(\because\ \cot^{-1}\dfrac{1}{x}=\tan^{-1}x \text{ for } x > 0\right)$

Further $\sin^{-1}x^4$ and $\tan^{-1}(2 + \sqrt{3}\,x^2)$ are even continuous and decreases for $x \in (-1, 0)$ and increasing for $x \in (0, 1)$.

Thus, $f(x)_{\min} \to f(0) = \sin^{-1}0 + \tan^{-1}2 = \tan^{-1}2$

and $f(x)_{\max} \to f(\pm 1) = \sin^{-1}1 + \tan^{-1}(2+\sqrt{3}) = \dfrac{\pi}{2}+\dfrac{5\pi}{12}=\dfrac{11\pi}{12}$

$\therefore\quad$ The range of $f(x)$ is $\left(\tan^{-1}2, \dfrac{11\pi}{12}\right)$ or $\left(\tan^{-1}2, \tan^{-1}(2+\sqrt{3})\right)$

ILLUSTRATION 152: Find the range of the function $f(x)=\cos^{-1}\left(\dfrac{\sqrt{1+x^6}}{1+4x^{12}}\right)$.

SOLUTION: Clearly $\dfrac{\sqrt{1+x^6}}{1+4x^{12}}>0\ \forall\ x \in \mathbb{R}$ and as $x \to \pm\infty$, $\dfrac{\sqrt{1+x^6}}{1+4x^{12}} \to 0$ and $\dfrac{\sqrt{1+x^6}}{1+4x^{12}}=1$ for $x = 0$

Also $\dfrac{\sqrt{1+x^6}}{1+4x^{12}}$ is continuous $\forall\ x \in \mathbb{R}$

$\therefore\quad$ By extreme value theorem $\dfrac{\sqrt{1+x^6}}{1+4x^{12}}$ would attain each real number belonging to $(0, 1]$, consequently, $f(x)=\cos^{-1}\left(\dfrac{\sqrt{1+x^6}}{1+4x^{12}}\right)$

$\therefore\quad$ Range of $f(x)=\left[0,\dfrac{\pi}{2}\right)$

Note the function would not be defined for those values of x for which $\dfrac{\sqrt{1+x^6}}{1+4x^{12}}$ exceeds 1.

ILLUSTRATION 153: Find the domain and range of definition of the function $f(x)=\dfrac{1}{x^5-x^3+x^2-1}$

SOLUTION: Given function is $f(x)=\dfrac{1}{x^5-x^3+x^2-1}$

$$=\frac{1}{x^3(x^2-1)+1(x^2-1)}=\frac{1}{(x^2-1)(x^3+1)}=\frac{1}{(x^2-1)(x+1)(x^2-x+1)}$$

Therefore domain of $f(x) = \mathbb{R} \sim \{-1, 1\}$ as $x^2 - x + 1 > 0\ \forall\ x \in \mathbb{R}$

Also range of $x^5 - x^3 + x^2 - 1$ is $(-\infty, \infty)$, therefore range $= (-\infty, \infty) \sim \{0\}$

ILLUSTRATION 154: If $f(x)=\sin\ln\left(\dfrac{\sqrt{4-x^2}}{1-x}\right)$, then find the domain and range of $f(x)$

SOLUTION: $f(x)$ is defined for $\dfrac{\sqrt{4-x^2}}{1-x}>0 \Rightarrow 1 - x > 0;\ 4 - x^2 > 0 \Rightarrow x < 1;\ x^2 < 4 \Rightarrow x \in (-2, 1)$

Now, $f(x)=\sin\ln\left(\dfrac{\sqrt{4-x^2}}{1-x}\right)$

Let $y = \frac{\sqrt{4-x^2}}{1-x}$ (> 0 for $x \in (-2, 1)$); $g'(x) = \frac{4-x}{(1-x)^2} > 0\ \forall\ x \in (-2,1)$

$\Rightarrow$ $g(x)$ is an increasing function on $(-2, 1)$ $\Rightarrow$ $g(x) \in (0, \infty)$, also $\log_e x$ being increasing function, $\ln(g(x)) \in (-\infty, \infty)$

$\Rightarrow$ $\sin\left(\ln\left(\frac{\sqrt{4-x^2}}{1-x}\right)\right)$ $\quad\therefore$ Range of $f(x) = [-1, 1]$

$\therefore$ Domain of $f(x) = (-2, 1)$ and range $= [-1, 1]$

ILLUSTRATION 155: The domain and range of definition of the function $y(x)$ given by the equation $a^x + a^y = a(a > 1)$ is

(a) $0 < x \le 1$ (b) $0 \le x < 1$

(c) $-\infty < x < 1$ (d) $-\infty < x \le 0$

SOLUTION: Given $a^x + a^y = a;\ a > 1$ $\Rightarrow$ $a^y = a - a^x$

$\Rightarrow$ $y = \log_a(a - a^x)$ which is defined for $a - a^x > 0$ $\Rightarrow$ $a^x < a = a^1$

$\because$ $a > 1, x < 1$ $\Rightarrow$ $x \in (-\infty, 1)$

Similarly $y \in (-\infty, 1)$

$\therefore$ Domain as well as range of $y(x)$ is given by $(-\infty, 1)$

TEXTUAL EXERCISE–10: (SUBJECTIVE)

1. Find the domain of the following rational functions and find the maximum/minimum value and range of these functions:
   (a) $\sqrt{x^2+4}$ (b) $\frac{1}{(x-1)(x-5)}$
   (c) $\sqrt[3]{x+8}$ (d) $\frac{1}{x^2+4}$
2. $f(x) = {}^{x+1}C_{2x-8}$ and $g(x) = {}^{2x-8}C_{x+1}$ and $h(x) = f(x) \cdot g(x)$, then find the domain range of $h(x)$.
3. Find the range of definition of the following functions:
   (a) $f(x) = 4\tan x \cos x$ (b) $f(x) = \cos(2\sin x)$
   (c) $f(x) = \sin(\log_2 x)$
4. Find the range of definition of following functions.
   (a) $f(x) = [\{x\}]$ or $\{[x]\}$
   (b) $g(x) = \frac{\tan(\pi[x-\pi])}{x^2-3x+4}$
   (c) $f(x) = \cos 2x - \sin 2x$
   (d) $f(x) = \cot^2\left(x - \frac{\pi}{4}\right)$
5. Find the range of definition of following functions.
   (a) $f(x) = \sin^{-1}\left(\frac{x^2-1}{2(x^2+1)}\right)$
   (b) $f(x) = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$
   (c) $f(x) = \cos^{-1}\left(\frac{1+x^2}{2x}\right)$
   (d) $f(x) = \tan^{-1}(1 + x^2)$
   (e) $f(x) = \log_2(3 + 5\sin x)$
   (f) $f(x) = \tan^{-1}\frac{2x}{1+x^2}$
6. Find the range of the following functions:
   (a) $f(x) = \ln(5x^2 - 8x + 4)$
   (b) $f(x) = \cot^{-1}(\log_{4/5}(5x^2 - 8x + 4))$
   (c) $f(x) = \cot^{-1}\log_{5/4}(5x^2 - 8x + 4)$
7. Find the range of the following functions:
   (a) $f(x) = \frac{x^2-x+1}{x^2+x+1}$ (b) $f(x) = \frac{x}{\ell n x}$
8. Find the domain and range of the function: $f(x) = \sin^{-1}[x^2] + \left[\left\{\ell n\sqrt{x-[x]}\right\}\right] + \cot^{-1}\left(\frac{1}{1+\sqrt{2}x^2}\right)$
9. Find the range of the following functions:
   (i) $\sqrt{x-1}$ (ii) $\sqrt[3]{x-2}$
   (iii) $\sqrt{(x-1)(x-4)}$ (iv) $\frac{}{(x-1)(x-2)(x-3)}$

(v) $\dfrac{x-1}{x^2-4x+3}$ (vi) $\dfrac{x^2-1}{x^2+5x+4}$

(vii) $\dfrac{x^2+2x-3}{x^2-1}$ (viii) $\dfrac{x^2+2x+3}{x}$

(ix) $\dfrac{x+1}{x^2-2x+3}$ (x) $\sqrt{x-1}+\sqrt{5-x}$

(xi) $\sin^2 x - 5\sin x - 6$ (xii) ${}^{7-x}P_{x-3}$

10. Find the range of the following functions:

(i) $\log_e(3x^2 - 4x + 5)$ (ii) $\cot^{-1}(x^2 - 4x + 5)$

(iii) $\cot^{-1}(2x - x^2)$ (iv) $\dfrac{x}{1+|x|}$

(v) $\cos([x]\pi)$ (vi) $f(x) = 4^x - 2^x + 1$

11. Find the domain of definition of the following given function

(i) $\dfrac{1}{(|x|-1)(\cos^{-1}(2x+1))\tan 3x}$

(ii) $\log[1 - \log(x^2 - 4x + 5)]$

(iii) $\log_2\log_{1/3}\log_3(x^2 - 4x + 3)$

(iv) $\log_2\log_3\log_{4/\pi}(\tan^{-1}x)^{-1}$

(v) $\left(\dfrac{x}{1-|x|}\right)^{\frac{1}{2002}}$

12. Find the lower upper bound and greatest lower bound in each case:

(i) $\dfrac{2x}{x^2+1}$

(ii) $\dfrac{x^2-1}{x^2+1}$

(iii) $\dfrac{x^2-3x+4}{x^2+3x+4}$

(iv) $s = \dfrac{1}{2+x^2}$; $-6 \le x \le 4$

(v) $s = \dfrac{3-x}{1-x}$; $x > 0, x \ne 1$

(vi) $s = \{\sqrt{1-x^2} : |x| \le \sqrt{3/2}\}$

(vii) $s = \left\{\dfrac{3x+1}{2x+3} : |x-1| \le 2\right\}$

Answer Keys

1. (a) Domain $\mathbb{R}$, Minimum = 4, Maximum $\to \infty$, Range = $[4, \infty)$

(b) Domain $\mathbb{R} \sim \{1, 5\}$, Minimum = –4, Maximum $\to \infty$, Range = $[-\infty, -1/4) \cup (0, \infty)$

(c) Domain $\mathbb{R}$, Minimum = $\to -\infty$, Maximum $\to \infty$, Range = $\mathbb{R}$

(d) Domain $\mathbb{R}$, Minimum = $\to 0$, Maximum $\to \dfrac{1}{4}$, Range = $\left(0, \dfrac{1}{4}\right]$

2. Domain of $h(x)$= {9} and range of $h(x)$ = {1}

3. (a) $(-4, 4)$ (b) $[\cos 2, 1]$ (c) $[-1, 1]$

4. (a) {0} (b) {0} (c) $[-\sqrt{2}, \sqrt{2}]$ (d) $[0, \infty)$

5. (a) $\left[-\dfrac{\pi}{6}, \dfrac{\pi}{6}\right)$ (b) $[0, \pi]$ (c) $\{0, \pi\}$ (d) $\left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$ (e) $(-\infty, 3]$

(f) $\left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$

6. (a) $[\ln 4, \infty)$ (b) $[\cot^{-1}(\log_{4/5} 4), \pi]$ (c) $\left(0, \cot^{-1}(\log_{4/5} 4)\right]$

7. (a) [1/3, 3]; $f'(x) = \dfrac{2(x^2-1)}{(x^2+x+1)^2} = f(1) = \dfrac{1}{3}; f(-1) = 3;$ (b) Range = $(-\infty, 0) \cup [e, \infty)]$

Domain = $\mathbb{R}$, Range = $\left[\dfrac{1}{3}, 3\right]$

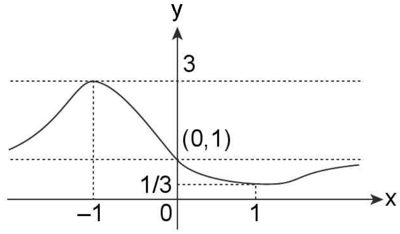


FIGURE 2.119

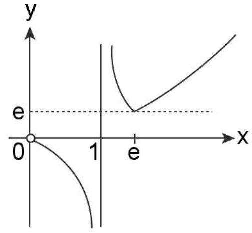


FIGURE 2.120

8. Range: $\left(\frac{\pi}{4}, \frac{7\pi}{8}\right)$; Domain is $(-1, 1) - \{0\}$

9. (i) $[0, \infty)$ (ii) $\mathbb{R}$ (iii) $[0, \infty)$ (iv) $(-\infty$ $) \sim \{0\}$ (v) $\mathbb{R} \sim \left\{0, -\frac{1}{2}\right\}$ (vi) $\mathbb{R} \sim \left\{1, \frac{-2}{3}\right\}$

(vii) $\mathbb{R} - \{1, 2\}$ (viii) $(-\infty, 2 - 2\sqrt{3}] \cup [2\sqrt{3}, \infty)$ (ix) $\left[\frac{1}{2} - \frac{\sqrt{6}}{4}, \frac{1}{2} + \frac{\sqrt{6}}{4}\right]$ (x) $[2, 2\sqrt{2}]$

(xi) $[-10, 0]$ (xii) $\{24, 6, 1\}$.

10. (i) $\left[\log\left(\frac{11}{3}\right), \infty\right), D \in R$ (ii) $\left(0, \frac{\pi}{4}\right]$ (iii) $\left[\frac{\pi}{4}, \pi\right)$ (iv) $(-1, 1)$ (v) $\{-1, 1\}$

(vi) $\left[\frac{3}{4}, \infty\right)$

11. (i) $(-1, 0) - \left\{-\frac{\pi}{6}\right\}$ (ii) $(-1, 5)$ (iii) $(0, 2-\sqrt{2}) \cup (2 \quad \sqrt{2}, 4)$ (iv) $(0, 1)$ (v) $(-\infty, -1) \cup [0, 1)$

12. (i) min: −1 , max: 1 (ii) min: −1, max: d s not exist, *l.u.b.* = 1
(iii) min: 1/7 , max : 7 (iv) *l.u.b.*=1/8, *g.l.b.* 1/38 (v) both *l.u.b.* and *g.l.b.* does not exist
(vi) *g.l.b.* = 0, *l.u.b.* = 1 (vii) *l.u.b.* = 10/9, *g..* = −2

TEXTUAL EXERCISE–10: (OBJECTIVE)

1. Range of the function $y = [x]$ is
(a) $\mathbb{N}$ (b) W
(c) $\mathbb{Z}$ (d) None of these

2. Range of the function $\frac{2x-1}{2x+1}$ is
(a) $\mathbb{R}$ (b) $\mathbb{R} - \{1\}$
(c) $(0, \infty)$ (d) None of these

3. Range of the function $\frac{x^2-1}{x^2+1}$ is
(a) $[-1, 1)$ (b) $(0, 1]$
(c) $[-1, 0]$ (d) None of these

4. Range of the function $\frac{2x}{1+x^2}$ is
(a) $[0, 1]$ (b) $[-1, 1]$
(c) $[-2, 2]$ (d) None of these

5. Range of the function $\frac{x^2-x+1}{x^2+x+1}$ is
(a) $[0, 3]$ (b) $[-3, 3]$
(c) $[1/3, 3]$ (d) None of these

6. The range of the function $\frac{1}{(x-1)(x-2)}$
(a) $(-4, \infty)$ (b) $(-\infty, -4) \cup (0, \infty)$
(c) $(-\infty, -4\,] \cup (0, \infty)$ (d) None of these

7. The range of the function $A \sin x + B \cos x$
(a) $\left[-\sqrt{A^2+B^2}, \sqrt{A^2+B^2}\right]$
(b) $\left[0, \sqrt{A^2+B^2}\right]$
(c) $[A, B]$
(d) None of these

8. The range of the function $y = \frac{x^2}{1+x^2}$ is
(a) $[0, 1)$ (b) $[1, 1]$
(c) $[0, 1]$ (d) None of these

9. The range of the function $y = \frac{1}{2 - \sin 3x}$ is
(a) $(1/3, 1)$ (b) $[1/3, 1)$
(c) $[1/3, 1]$ (d) None of these

10. The minimum and maximum values of the expression $x^2 + x + 1$ are given by
(a) min: 3/4, max : ∞
(b) min: – 3/4, max : ∞
(c) max = 3/4, min $\to \infty$
(d) None of these

11. The minimum and maximum values of the expression $x^2 - 4x + 5$ are given by
(a) min = –1, max $\to \infty$
(b) bounded function
(c) min : 1, max : ∞
(d) None of these

12. The minimum and maximum values of the expression $ax^2 + bx + c$; $a > 0$ are given by

(a) min: $\dfrac{4ac-b^2}{4a}$, max: ∞

(b) min: $\dfrac{3}{4}$, max : ∞

(c) min: 1, max : ∞

(d) None of these

13. The minimum and maximum values of the expression $2 - 3x - x^2$ are given by

(a) min: , max : ∞ (b) min: $-\infty$, max :17/4

(c) min : 1, max : ∞ (d) None of these

14. The minimum and maximum values of the function $\dfrac{x^2-1}{x^2+1}$ are given by

(a) min: –1 , max : 1

(b) min: $-\infty$, max :1

(c) min: –1 , max : tends to 1

(d) None of these

15. The minimum and maximum values of the function $\dfrac{x^2-3x+4}{x^2+3x+4}$ are given by

(a) min: –1, max : 3

(b) min: 1/7, max : 7

(c) min: –7, max : tends to 1

(d) None of these

16. The *l.u.b.* and *g.l.b.* of the function $y = \dfrac{1}{2+x^2}$; $-6 \le x \le 4$ are given by

(a) *l.u.b.* $= \dfrac{1}{38}$, *g.l.b.* $= \dfrac{1}{2}$

(b) *l.u.b.* $= \dfrac{1}{2}$, *g.l.b.* $= \dfrac{1}{38}$

(c) *l.u.b.* $= \dfrac{1}{38}$, *g.l.b.* $= \dfrac{1}{4}$

(d) None of these

17. The *l.u.b.* and *g.l.b* of the function $y = \dfrac{3-x}{1-x}$; $x > 0$, $x \ne 1$ are given by

(a) *l.u.b.* = 3, *g.l.b.* = 1

(b) *l.u.b.* = 3, *g.l.b.* does not exist

(c) both *l.u.b.* and *g.l.b.* does not exist

(d) None of these

18. If $|x - 5| < 1$, then the range of $\dfrac{x}{x+10}$ is

(a) $(2/7\ ,\ 3/8)$ (b) [7/2, 8/3]

(c) [2/7, 3/8] (d) None of these

19. If $e^x = y + \sqrt{1+y^2}$, then y is

(a) $e^x + e^{-x}$ (b) $e^x - e^{-x}$

(c) $\dfrac{e^x - e^{-x}}{2}$ (d) $\dfrac{e^x + e^{-x}}{2}$

20. If $f(x)$ is defined on [0, 1] by the rule $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x, & \text{if } x \text{ is irrational} \end{cases}$, then for all $x \in [0, 1]$, $f(f(x))$ is equal to

(a) Constant (b) $1 + x$

(c) x (d) None of these

21. The range of the function $y = \dfrac{x}{1+x^2}$ is

(a) $\left[0, \dfrac{3}{2}\right]$ (b) $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$

(c) $\left[-\dfrac{3}{2}, 0\right]$ (d) None of these

22. The range of functions $\dfrac{1}{x(x-1)(x-2)}$ is

(a) $\mathbb{R}$ (b) $(0, \infty)$

(c) $\mathbb{R} - \{0\}$ (d) None of these

23. The range of function $\log(e^x - x)$ is

(a) $\mathbb{R}$ (b) $(-\infty, 0)$

(c) $[0, \infty)$ (d) None of these

24. The range of functions $\dfrac{x^2}{1+x^2}$ is

(a) (0, 1) (b) [0, 1)

(c) $(0, \infty)$ (d) None of these

25. Which of the following functions are defined for all x?

(a) $\sin[x] + \cos[x]$ ($[x]$denotes greatest integer $\le x$)

(b) $\sec^{-1}(1 + \sin^2 x)$

(c) $\tan(\log x)$

(d) $\sqrt{\dfrac{9}{8} + \cos x + \cos 2x}$

26. The range of the function $\sin^2 x - 5\sin x - 6$ is

(a) [–10, 0] (b) [–1, –1]

(c) $[0, \pi]$ (d) [–49/4, 0]

Answer Keys

1. (c)	**2.** (b)	**3.** (a)	**4.** (b)	**5.** (c)	**6.** (c)	**7.** (a)	**8.** (a)	**9.** (c)	**10.** (a)
11. (c)	**12.** (a)	**13.** (b)	**14.** (a)	**15.** (b)	**16.** (b)	**17.** (c)	**18.** (a)	**19.** (c)	**20.** (c)
21. (b)	**22.** (c)	**23.** (c)	**24.** (b)	**25.** (a,b,d)	**26.** (a)				

CLASSIFICATION OF FUNCTIONS

Based on their specific properties, functions can be classified in following ways

(i) One–one/many–one functions
(ii) Onto/Into functions
(iii) One–one and onto (Bijective) functions
(iv) Composite functions
(v) Invertible/Non-invertible functions
(v) Even/Odd/Neither even nor odd functions
(v) Periodic/Non-periodic functions

ONE–ONE/MANY–ONE FUNCTIONS

One–one (Injective) Function

$f : X \to Y$ is called injective, when different elements in set X are related with different elements of set Y, i.e., no two elements of domain have same image in co-domain. In other words, we can also say that no element of co-domain is related with two or more elements of domain. Consider the following mapping:

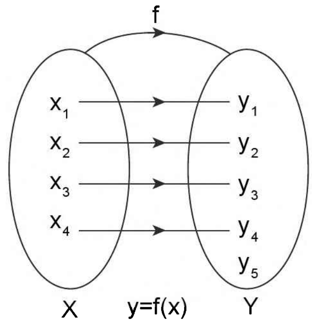


FIGURE 2.121

Clearly it is one–one because each element x_i in X is associated with a distinct element y_i in Y.

e.g., f: $\mathbb{R} \to \mathbb{R}$ such that $f(x) = x^3 + x$, $f(x) = 2^x - 2^{-x}$.

$f(x) = mx + c$, $f(x) = a^x$, $a > 0$ and $a \neq 1$; $f(x) = \log_a x$, $a \neq 1$, $a > 0$ are injective function.

ILLUSTRATION 156: Let $A = \{1, 2, 3, 4\}$; $B = \{1, 2, 3, 4, 5, 6\}$ and f: $A \to B$ be a function defined by $f(x) = x + 2$ for all $x \in A$. Is it one–one?

SOLUTION: Using the analytical formula $f(x) = x + 2$, we get $f(1) = 3$, $f(2) = 4$, $f(3) = 5$, $f(4) = 6$. Therefore the function as a set of ordered pair is obtained as $f = \{(1, 3), (2, 4), (3, 5), (4, 6)\}$. It is clear that different elements in A have different images under function f. So, $f : A \to B$ is an injection.

MANY–ONE FUNCTIONS

$f : X \to Y$ is many–one, when there exist at least two elements in the domain set X which are related with same element of co-domain Y. For example consider a mapping.

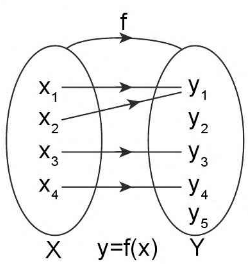


FIGURE 2.122

Clearly it is a many–one mapping as $f(x_1) = f(x_2) = y_1$ whereas x_1 and x_2 are distinct elements.

e.g., f : $\mathbb{R} \to \mathbb{R}$ given by $\sin x$, $\cos x$, $\tan x$, $x^2 + 2x$, $2^x + 2^{-x}$.

METHOD OF TESTING FOR INJECTIVITY

To test a given function for injectivity, we assume two different elements x_1 and x_2 in domain X and their corresponding images $f(x_1)$ and $f(x_2)$ in co-domain Y. Thus, we have to prove that for distinct pre-images x_1 and x_2 their images $f(x_1)$ and $f(x_2)$ remain different. We apply this feature to develop condition for injectivity in different ways.

(a) Analytical Method: A function $f : X \to Y$ is injective (one–one) iff whenever two images are equal then it means that they are outputs of same pre-image. i.e., $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2 \ \forall \ x_1, x_2 \in X$. Or by using contrapositive of the above condition, i.e., $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2) \ \forall \ x_1, x_2 \in X$.

NOTES

1. *If f(x) is not one–one, then it is many–one function .If we go according to definition consider $f(x_1) = f(x_2)$.*

$\Rightarrow$ *x_1 is not necessarily equal to x_2.*

i.e., If two f-images are equal, then their pre-images may or may not be equal.

2. *To test injectivity of f(x), consider $f(x_1) = f(x_2)$ and solve the equation and get x_2 in terms of x_1 .If $x_2 = x_1$ is only solution, then function f is injective, but if other real solutions also exist, then f is many–one function.*

ILLUSTRATION 157: Show that the function $f: \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^2 + x\ \forall\ x \in \mathbb{Z}$, is many–one function.

SOLUTION: Let $x_1, x_2 \in \mathbb{Z}$ such that $f(x_1) = f(x_2)$, then $f(x_1) = f(x_2)$

$\Rightarrow x_1^2 + x_1 = x_2^2 + x_2$ $\Rightarrow (x_1^2 - x_2^2) + (x_1 - x_2) = 0$

$\Rightarrow (x_1 - x_2)(x_1 + x_2 + 1) = 0$ $\Rightarrow x_1 = x_2$ or $x_1 + x_2 + 1 = 0$

Since, $f(x_1) = f(x_2)$ does not provide a unique solution $x_1 = x_2$ but it also provides $x_2 = -x_1 - 1$, this means there may arise a situation when $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.

To verify the second situation, let us analyses the function further.

$f(x) = x^2 + x = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$ $\Rightarrow f(x) = \frac{(2x+1)^2 + 3}{4}$

it is very evident from above that $f(x)$ generates same output for $2x + 1 = \alpha$ or $-\alpha$; where $\alpha \in \mathbb{R}$

i.e., for $x = -\frac{1}{2} - \frac{\alpha}{2}$ or $\frac{-1}{2} + \frac{\alpha}{2}$. For instance when $\alpha = 1$, we get $x = -1$ or 0.

Clearly $f(0) = f(-1) = 1$. Thus, for two distinct inputs $f(x)$ generates same output. Hence, f is many–one function. Also one can make out from the above discussion that infinitely many such pairs of integer inputs are possible by choosing α from set $\{(2n + 1); n \in \mathbb{Z}\}$.

ILLUSTRATION 158: Test whether $f(x) = e^{2x^3+5}$ is an injective function or not.

SOLUTION: Consider $f(x_1) = f(x_2) \Rightarrow e^{2x_1^3+5} = e^{2x_2^3+5} \Rightarrow 2x_1^3 + 5 = 2x_2^3 + 5 \Rightarrow x_1^3 - x_2^3 = 0$

$\Rightarrow (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0$ $(\because x_1^2 + x_1x_2 + x_2^2 \geq 4)$

Consequently the only real solution is $x_1 = x_2$ .Therefore, f is one–one function.

ILLUSTRATION 159: Test the following functions for injectivity:

(a) $f(x) = \frac{x^2 - x - 2}{x^2 - 5x + 6}$ (b) $f(x) = \frac{2x-3}{2x+3}$ (c) $f(x) = \frac{ax^2 - 1}{ax^2 + 1}$; $(a > 0)$

SOLUTION: (a) Given function $f(x) = \frac{x^2 - x - 2}{x^2 - 5x + 6} = \frac{(x+1)(x-2)}{(x-2)(x-3)} = \frac{x+1}{x-3}$, $x \neq 2, 3$

This is clearly a rational linear fractional function and we know that it is an injective function.

(b) Given $f(x) = \frac{2x-3}{2x+3}$

$\therefore f(x_1) = f(x_2) \Rightarrow \left(\frac{2x_1 - 3}{2x_1 + 3}\right) = \left(\frac{2x_2 - 3}{2x_2 + 3}\right)$

Applying componendo and dividendo, we get $\frac{-4x_1}{6} = \frac{-4x_2}{6}$ $\Rightarrow x_1 = x_2$.

$\Rightarrow$ $f(x)$ is one–one function.

(c) Given $f(x)=\dfrac{ax^2-1}{ax^2+1}$; $(a>0)$ $\therefore$ $f(x_1)=f(x_2)$ $\Rightarrow$ $\left(\dfrac{ax_1^2-1}{ax_1^2+1}\right)=\left(\dfrac{ax_2^2-1}{ax_2^2+1}\right)$

Applying componendo and dividendo, we get $-ax_1^2=-ax_2^2$ $\Rightarrow$ $x_1^2=x_2^2$

$\Rightarrow$ $x_1=x_2$ or $x_1=-x_2$ $\therefore$ $f(x)$ is many–one function.

ILLUSTRATION 160: A function $f:\mathbb{R}\to\mathbb{R}$ is defined as $f(x)=\dfrac{2x+1}{x^3+4}$. Find whether f is one–one or many–one function.

SOLUTION: Let $x_1, x_2 \in \mathbb{R}$ such that $f(x_1)=f(x_2)$ $\Rightarrow$ $\dfrac{2x_1+1}{x_1^3+4}=\dfrac{2x_2+1}{x_2^3+4}$

$\Rightarrow$ $(2x_1+1)(x_2^3+4)=(2x_2+1)(x_1^3+4)$ $\Rightarrow$ $2x_1x_2^3+8x_1+x_2^3+4=2x_2x_1^3+8x_2+x_1^3+4$

$\Rightarrow$ $2x_1x_2(x_2^2-x_1^2)+8(x_1-x_2)+(x_2^3-x_1^3)=0$

$\Rightarrow$ $(x_2-x_1)\left[2x_1x_2(x_1+x_2)+x_1^2+x_2^2+x_1x_2-8\right]=0$

$\Rightarrow$ $(x_2-x_1)\left(x_1^2+x_2^2+x_1x_2(2x_1+2x_2+1)-8\right)=0$

$\Rightarrow$ $x_2-x_1=0$ or $x_1^2+x_2^2+x_1x_2(2x_1+2x_2+1)-8=0$ ... (i)

If $x_2=0$, then $x_1^2+x_2^2+x_1x_2(2x_1+2x_2+1)-8=0$ $\Rightarrow$ $x_1^2-8=0$ $\Rightarrow$ $x_1^2=8$ $\Rightarrow$ $x_1=\pm 2\sqrt{2}$

Therefore equation (i), suggests that if $x_2=0$, then x_1 can have two values $2\sqrt{2}$ or – $-2\sqrt{2}$; consequently $x_1 \neq x_2$, but $f(x_1)=f(x_2)$ $\therefore$ f is a many–one function.

(b) Graphical Method:

For one–one: If every line parallel to x-axis, $y=k\in R_f$ cuts the graph of function exactly once, then the function is one–one or injective.

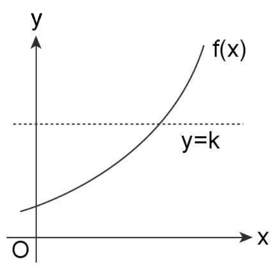


FIGURE 2.123

For Many–one: If there exists a line parallel to x-axis which cuts the graph of function at least twice, then the function is many–one.

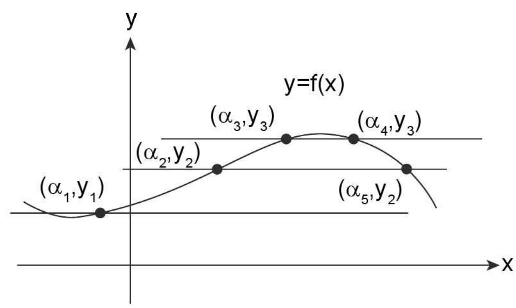


FIGURE 2.124

ILLUSTRATION 161: Function f: $[-3,\infty)\to\mathbb{R}$ is piecewise defined as $f(x)=\begin{cases}\sqrt{9-x^2}; & -3\le x\le 0\\ \dfrac{2}{3^x}+k; & 0<x<\infty\end{cases}$

Determine the range of values of k for which the function f be (a) injective (b) many–one.

SOLUTION: The function f represents quarter circle given by analytical formula $y=\sqrt{9-x^2}$, i.e., $x^2 + y^2 = 9$ for $x \in [-3, 0]$ and $y \in [0, 3]$, as shown in Figure 2.125, where as for the domain $(0,\infty)$ it is a decreasing exponential function such that $f(0) = k + 2$ and $f(\infty) \to k$, to remain injective, no horizontal line should intersect the graph more than once. Therefore

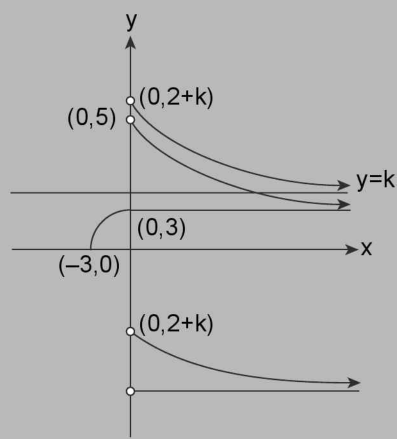


FIGURE 2.125

Case I: Either $f(x)=\dfrac{2}{3^x}+k$ above the quarter circle

$\Rightarrow\ k \geq 3.$

Case II: or it lies below the x-axis

$\Rightarrow\ 2 + k < 0$

$\Rightarrow\ k < -2$

Hence, if $k \in (-\infty,-2) \cup [3, \infty)$, $f(x)$ remains injective function but for $k \in [-2, 3)$, $f(x)$ becomes many–one function.

(c) Method of Monotonicity: The concept of monotonicity can be used to determine the injectivity of function conveniently for continuous functions. For discontinuous functions the study of monotonic behavior of function in the interval concerned helps to decide the injectivity / non-injectivity of function.

For one–one: If a function $f(x)$ is continuous and monotonic on an interval I, then it is always one–one on interval I because any straight line parallel to x-axis $y = k \in I$ intersects the graph of such functions exactly once.

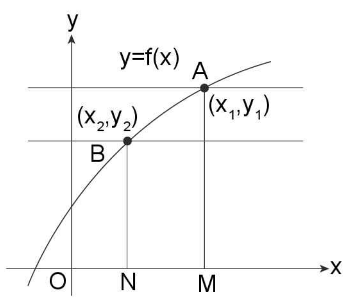


FIGURE 2.126

$f'(x) \geq 0$; $f'(x) = 0$ occurs only at isolated points or

$x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$

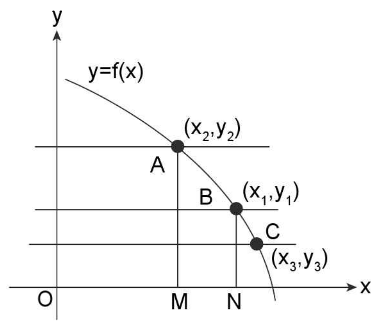


FIGURE 2.127

$f'(x) \leq 0$; $f'(x) = 0$ occurs only at isolated points or

$x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$

For many–one:

(i) If a function is continuous and non-monotonic on interval I, then it must be many–one on interval I.

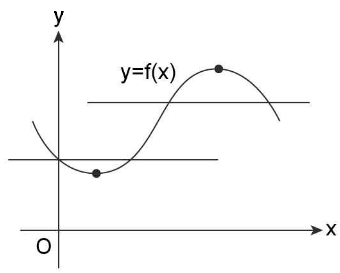


FIGURE 2.128

(ii) If a function is discontinuous and monotonic on interval I, then it can be one–one or many–one on I. Because after getting disconnected if the values of function drops down and again increases as shown in the figure below, then function becomes many–one on I. In case of jump in the value, function may remain injection on I.

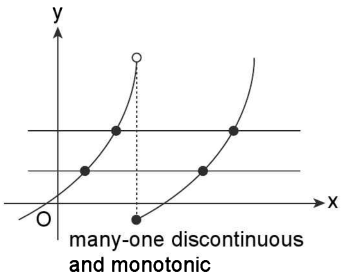


FIGURE 2.129

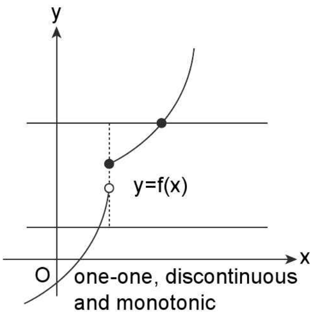


FIGURE 2.130

e.g., $y = \tan x$ is continuous and monotonic on $(-\pi/2, \pi/2)$.

$\therefore$ $y = \tan x$ is injective on $(-\pi/2, \pi/2)$; whereas $y = \tan x$ is discontinuous and monotonic on $(0, \pi)$ and is one–one in $(0, \pi)$ but is many–one in $(0, 2\pi)$.

(iii) Even functions and periodic functions are always many–one in their natural domains. Whereas they are one–one in their principal domain. They can be made one–one by restricting the domain. e.g., $\cos x$ is many–one on $\mathbb{R}$ but is one–one on $[0, \pi]$ or $\left[0, \frac{\pi}{2}\right]$.

Similarly, fractional part function $\{x\}$ is periodic function with period 1. It is many–one on $\mathbb{R}$ but one–one on $[n, n + 1)$ for each integer n.

(iv) If a function is discontinuous and non-monotonic on an interval I, then it can be one–one or many–one on I. It can be understood well by the graph shown below.

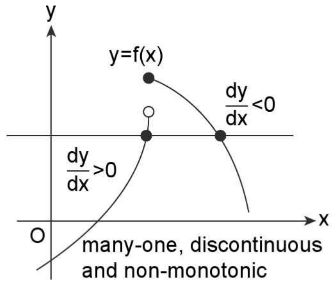


FIGURE 2.131

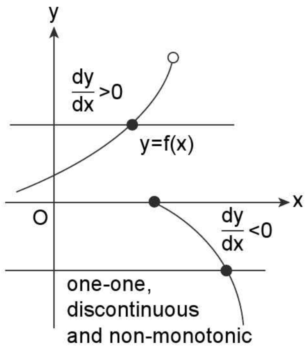


FIGURE 2.132

(v) All polynomials of even degree defined in $\mathbb{R}$ have at least one local maxima or minima and hence, are many–one in the domain $\mathbb{R}$. Polynomials of odd degree can be one–one or many–one in $\mathbb{R}$.
e.g., $f(x) = x^3 + 1$ is one–one in $\mathbb{R}$ as $f'(x) = 3x^2 \geq 0$ and $f'(x) = 0$ only at one isolated point $x = 0$; whereas $f(x) = x^3 + 4x^2 + x - 6$ is many–one in $\mathbb{R}$ as $f'(x) = 3x^2 + 8x + 1$ which changes its sign in $\mathbb{R}$. i.e., $f(x)$ is continuous and non-monotonic in $\mathbb{R}$.

(vi) If f is a rational function, then $f(x_1) = f(x_2)$ will always be satisfied when $x_1 = x_2$ in the domain.
Hence, we can write $f(x_1) - f(x_2) = (x_1 - x_2).g(x_1, x_2)$; where $g(x_1, x_2)$ is some function in x_1 and x_2.
Now if $g(x_1, x_2) = 0$ gives some solution which is different from $x_1 = x_2$ and lies in the domain, then f is many–one else one–one.

ILLUSTRATION 162: If the function f: $\mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, then find whether $f(x)$ is injective or not.

SOLUTION: Given $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ $\Rightarrow$ $f'(x) = \dfrac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} = \dfrac{4}{(e^x + e^{-x})^2} \geq 0 \ \forall \ x \in \mathbb{R}$.

Thus, $f(x)$ is strictly increasing and continuous function, consequently $f(x)$ is an injective function.

ILLUSTRATION 163: Find the set of all real values of parameter t for which the function f: $\mathbb{R} \to \mathbb{R}$ defined as $f(x) = 3\sin x - 4tx + 5$ is an injective function

SOLUTION: A continuous function $f(x)$ to be one–one it should be either monotonically increasing ($f'(x) \geq 0$) or monotonically decreasing ($f'(x) \leq 0$) in its entire domain.

Clearly our function is continuous and $f'(x) = 3\cos x - 4t$, thus, for one–one either $f'(x) \geq 0\ \forall\ x \in \mathbb{R}$ or $f'(x) \leq 0\ \forall\ x \in \mathbb{R}$

Case (i): $f'(x) \geq 0\ \forall\ x \in \mathbb{R}$

$\Rightarrow f'(x) = 3\cos x - 4t \geq 0\ \forall\ x \in \mathbb{R}$

$\Rightarrow\ t \leq \frac{3}{4}\cos x\ \ \forall\ x \in \mathbb{R}\ \ \Rightarrow\ \ t \leq -\frac{3}{4}\ \forall\ x \in \mathbb{R}$ as $-\frac{3}{4} \leq \frac{3}{4}\cos x\ \ \ \Rightarrow\ t \in \left(-\infty, -\frac{3}{4}\right]$

Case (ii): $f'(x) = 3\cos x - 4t \leq 0\ \forall\ x \in \mathbb{R}$

$\Rightarrow\ t \geq \frac{3}{4}\cos x\ \ \Rightarrow\ \ t \geq \frac{3}{4}\ \ \because\ \ \frac{3}{4}\cos x \leq \frac{3}{4}\ \ \Rightarrow\ t \in \left[\frac{3}{4}, \infty\right)$

Concluding the outcome of both the above cases, we conclude that f to be one–one function

$t \in \left(-\infty, \frac{-3}{4}\right] \cup \left[\frac{3}{4}, \infty\right)$

ILLUSTRATION 164: Test whether the following functions are injective or not:

(a) $f(x) = \dfrac{2\sin x - 5}{2\sin x + 5}$ (b) $f(x) = \dfrac{a\log x^2 - 2}{a\log x^2 + 2}$; $(a, x > 0)$

SOLUTION: (a) Given function is $f(x) = \dfrac{2\sin x - 5}{2\sin x + 5}$

$\therefore\ \ f(x_1) = f(x_2)$

$\Rightarrow\ \left(\dfrac{2\sin x_1 - 5}{2\sin x_1 + 5}\right) = \left(\dfrac{2\sin x_2 - 5}{2\sin x_2 + 5}\right)$

Applying componendo and dividendo, we get

$\dfrac{-4\sin x_1}{10} = \dfrac{-4\sin x_2}{10}\ \ \Rightarrow\ \sin x_1 = \sin x_2$

But we know that sin x is periodic function, and hence, is many–one function. Thus, $\sin x_1 = \sin x_2$ would occur at infinitely many pairs (x_1, x_2) such that $x_1 \neq x_2$. Hence, $f(x)$ is many–one function.

(b) Given $f(x) = \dfrac{a\log x^2 - 2}{a\log x^2 + 2}$ $(a, x > 0)$

$\therefore\ \ f(x_1) = f(x_2)\ \ \Rightarrow\ \left(\dfrac{a\log x_1^2 - 2}{a\log x_1^2 + 2}\right) = \left(\dfrac{a\log x_2^2 - 2}{a\log x_2^2 + 2}\right)$

Applying componendo and dividendo, we get $\dfrac{-a\log x_1^2}{4} = \dfrac{-a\log x_2^2}{4}\ \Rightarrow\ \log x_1^2 = \log x_2^2$

But log x is a monotonic and continuous function on $(0, \infty)$, thus, log x is one–one function on $(0, \infty)$.

$\Rightarrow\ x_1^2 = x_2^2\ \Rightarrow\ (x_1 + x_2)(x_1 - x_2) = 0$

$\Rightarrow\ x_1 = x_2$ or $x_1 = -x_2$, but $x_1, x_2 > 0\ \Rightarrow\ \ x_1 \neq -x_2\ \Rightarrow\ x_1 = x_2$

$\Rightarrow$ $f(x)$ is one–one function.

(d) **Hit and Trial Method to Test Many–One Functions:**

Many a times by keen observation it is possible to find an element in the range of function which is f image of two or more than two elements in the domain of function thus, concluding the fact that the function being investigated is many–one function.

For instance $f(x)=\dfrac{x^2-4x+3}{x^2+4x+5}$ is clearly a many–one function since it vanishes at two different values of input x, i.e., $x = 1, 3$ (given by roots of $x^2 - 4x + 3 = 0$). But it is not necessary that to be many–one the numerator must have distinct real roots. Especially for those cases when numerator is non-vanishing we may proceed as illustrated below.

Let us try $f(x) = f(\alpha)$ (where α is some fixed real number such that $f(\alpha)$ is simple fraction, say $\alpha = 0$).

i.e., $f(x) = f(0) \quad \Rightarrow \quad \dfrac{x^2-4x+3}{x^2+4x+5}=\dfrac{3}{5}$

$\Rightarrow\ 5x^2 - 20x + 15 = 3x^2 + 12x + 15$

$\Rightarrow\ 2x^2 - 32x = 0 \quad \Rightarrow \quad 2x(x-16) = 0$

$\Rightarrow\ x = 0, 16 \quad \Rightarrow \quad f(0)=f(16)=\dfrac{3}{5}$

Hence, $f(x)$ is many–one function while carrying out the above process it should be kept in mind that it is hit and trial method and may fail to hit the questions rightly, for instance consider $f(x)=\dfrac{x^2-3x+4}{x^2+3x+4}$

Solving $f(x) = f(0)$, we get $\dfrac{x^2-3x+4}{x^2+3x+4}=1$

$\Rightarrow\ 6x = 0$ i.e., $x = 0$

Thus, we can't arrive at any conclusion from the result, so f may or may not be many–one, to decide either we go by other method or try for some different real output of function say

Let $f(x) = f(1) \quad \Rightarrow \quad \dfrac{x^2-3x+4}{x^2+3x+4}=\dfrac{1}{4}$

$\Rightarrow\ 4x^2 - 12x + 16 = x^2 + 3x + 4$

$\Rightarrow\ 3x^2 - 15x + 12 = 0$ i.e., $x^2 - 5x + 4 = 0$ (which has $D > 0$)

$\Rightarrow\ x = 1,4$

Thus, $f(1) = f(4) = 1/4$, and thus, f is many–one.

ILLUSTRATION 165: Given a function $f:\mathbb{R}\to\mathbb{R}$ defined as $f(x)=\dfrac{x^2-x+12}{x^2+x+8}$. Test whether $f(x)$ is injective function or non-injective.

SOLUTION: ***Method 1:*** **Analytically:**

Let $f(x_1) = f(x_2)$

$$\Rightarrow\ \frac{x_1^2-x_1+12}{x_1^2+x_1+8}=\frac{x_2^2-x_2+12}{x_2^2+x_2+8}$$

Applying componendo and dividendo, we get $\dfrac{4-2x_1}{2x_1^2+20}=\dfrac{4-2x_2}{2x_2^2+20}$

$$\Rightarrow\ (2-x_1)(x_2^2+10)=(2-x_2)(x_1^2+10)$$

$$\Rightarrow\ 2x_2^2+20-x_1x_2^2-10x_1 = 2x_1^2+20-x_2x_1^2-10x_2$$

$$\Rightarrow\ 2(x_2^2-x_1^2)-x_1x_2(x_2-x_1)+10(x_2-x_1)=0 \Rightarrow (x_2-x_1)(2(x_1+x_2)-x_1x_2+10)=0$$

$$\Rightarrow\ x_1 = x_2 \text{ or } 2x_1 + 2x_2 - x_1x_2 + 10 = 0 \quad \Rightarrow\ x_1 = x_2 \text{ or } x_1=\frac{2(x_2+5)}{(x_2-2)}$$

Thus, evidently $f(x_1) = f(x_2)$ does not necessarily implying $x_1 = x_2$. For different inputs x_1 and x_2, $f(x_1)$ and $f(x_2)$ can be equal, consequently f is many–one function.

Method 2: **Monotonically:** $f(x)$ is a continuous function because denominator is non-vanishing polynomial

$$\text{Now, } f'(x)=\frac{(x^2+x+8)(2x-1)-(x^2-x+12)(2x+1)}{(x^2+x+8)^2}=\frac{2x^2-8x-20}{(x^2+x+8)^2}=\frac{2(x^2-4x-5)}{(x^2+x+8)^2}$$

$\because$ Denominator is non-vanishing and perfect square, thus, always positive, whereas determinant of numerator $= D = 36 > 0$, thus, has two real roots.

$$f'(x) = \frac{2(x-5)(x+1)}{(x^2+x+8)^2} = \begin{cases} \text{positive for } x \in (-\infty,-1) \cup (5,\infty) \\ \text{negative for } x \in (-1,5) \end{cases}$$

Due to change in sign of derivative $f(x)$ is clearly continuous but non-monotonic function, so it is non-injective function.

Method 3: **Using hit and trial:**

Let $f(x) = f(0)$

$$\Rightarrow \frac{x^2-x+12}{x^2+x+8} = \frac{3}{2} \Rightarrow 2x^2 - 2x = 3x^2 + 3x \Rightarrow x^2 + 5x = 0 \Rightarrow x = 0 \text{ or } -5$$

So, for distinct inputs 0 and -5; $f(0) = f(-5) = 3/2$. Thus, $f(x)$ is many–one function.

ILLUSTRATION 166: Test the injectivity of the following functions

(a) $f(x) = e^x + e^{-x}$ (b) $f(x) = e^x - e^{-x}$

SOLUTION: (a) **Analytical method:** Consider $f(x_1) = f(x_2)$ $\Rightarrow e^{x_1} + e^{-x_1} = e^{x_2} + e^{-x_2}$

$$\Rightarrow e^{x_1} - e^{x_2} = \frac{1}{e^{x_2}} - \frac{1}{e^{x_1}} \Rightarrow \left(e^{x_1} - e^{x_2}\right) - \frac{e^{x_1} - e^{x_2}}{e^{x_1+x_2}} = 0 \Rightarrow \left(e^{x_1} - e^{x_2}\right)\left(1 - \frac{1}{e^{x_1+x_2}}\right) = 0$$

$$\Rightarrow e^{x_1} = e^{x_2} \text{ or } e^{x_1+x_2} = 1 \qquad \Rightarrow x_1 = x_2 \text{ or } x_1 + x_2 = 0$$

Clearly $x_2 = x_1$ is not the only solution, $x_2 = -x_1$ is also solution. Thus, f is many–one function. The above analysis–also shows that $f(x)$ takes same values for x and $-x$, thus, being an even function, it is many–one function. Consequently, we conclude that if $f(x)$ is defined as $f: [0, \infty) \to \mathbb{R}$; $f(x) = e^x + e^{-x}$, it becomes an injective function.

Method of monotonicity:

$$f(x) = e^x + e^{-x} \Rightarrow f'(x) = e^x - e^{-x} = \frac{e^{2x}-1}{e^x} = \begin{cases} +ve \text{ if } x > 0 \\ -ve \text{ if } x < 0 \end{cases}$$

$\Rightarrow$ $f(x)$ is monotonically decreasing in $(-\infty,0)$ and increasing in $(0,\infty)$, thus, has non-monotonic nature with minima at $x = 0$, also $f(x)$ is continuous function. Consequently $f(x)$ is non-injective function.

Graphically: $f(0) = 2$

Graph of $f(x)$ is symmetric about y-axis and lines $y = k$; $k > 2$, cut the curve twice showing that function is many–one. It is also evident from the graph that restricting the domain of function either to $(-\infty, 0]$ on $[0, \infty)$ converts it to an injective function.

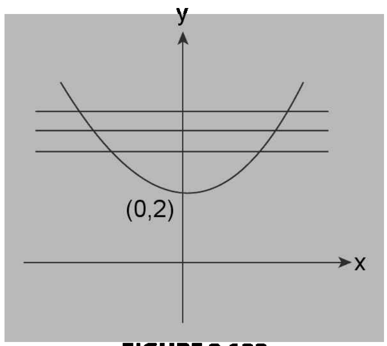


FIGURE 2.133

(b) **Analytical method:** Consider $f(x_1) = f(x_2)$

$$\Rightarrow e^{x_1} - e^{-x_1} = e^{x_2} - e^{-x_2}$$

$$\Rightarrow e^{x_1} - e^{x_2} = e^{-x_1} - e^{-x_2} \Rightarrow e^{x_1} - e^{x_2} = \frac{1}{e^{x_1}} - \frac{1}{e^{x_2}} \Rightarrow \left(e^{x_1} - e^{x_2}\right) + \frac{(e^{x_1} - e^{x_2})}{e^{x_1+x_2}} = 0$$

$$\Rightarrow \left(e^{x_1} - e^{x_2}\right)\left(1 + \frac{1}{e^{x_1+x_2}}\right) = 0 \Rightarrow e^{x_1} = e^{x_2} \text{ or } e^{x_1+x_2} = -1 \text{ (impossible) as } e^x > 0\ \forall\ x \in \mathbb{R}$$

$\Rightarrow$ $x_1 = x_2$. Clearly $x_2 = x_1$ is the only solution. Thus, f is one–one, i.e., injective.

Method of monotonicity: $f(x) = e^x - e^{-x}$

$\Rightarrow f'(x) = e^x + e^{-x} = e^x + \dfrac{1}{e^x} > 0 \ \forall x \in \mathbb{R} \quad \Rightarrow f(x)$ is strictly increasing on $\mathbb{R}$.

Also $f(x)$ being the difference of two exponential functions is continuous on $\mathbb{R}$. Hence, $f(x)$ is injective on $\mathbb{R}$.

Graphically: $f(x) = e^x - e^{-x}$; $f(0) = 0$

$f(x) = e^x - e^{-x} \geq 0 \Leftrightarrow e^{2x} - 1 \geq 0 \Leftrightarrow e^x \leq -1$ or $e^x \geq 1 \Leftrightarrow e^x \in [1, \infty) \Leftrightarrow [0, \infty)$ and $f(x) \leq 0$ for $x \in (-\infty, 0]$. Also $f(-x) = e^{-x} - e^x = -(e^x - e^{-x}) = -f(x)$

$\Rightarrow f(x)$ is symmetric about origin

Moreover, as x increases e^x also increases, where as e^{-x} decreases

$\Rightarrow e^x - e^{-x}$ goes on increasing as x increases.

Thus, $f(x)$ increases with x as shown in Figure 2.134. Clearly $f(x)$ is injective.

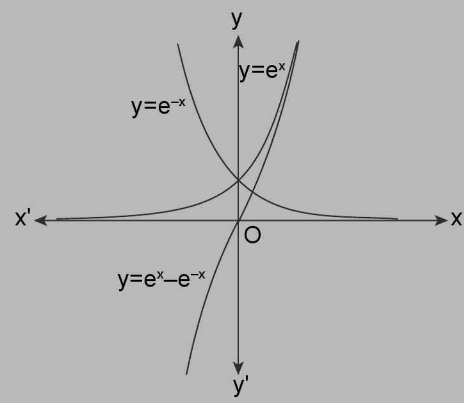


FIGURE 2.134

TEXTUAL EXERCISE–11: (SUBJECTIVE)

1. Find whether the functions defined below are injective or not?
   (i) $f(x) = 5x - 4$
   (ii) $f(x) = x - (1/x)$
   (iii) $f(x) = x^2 - 6x + 15$; $(x < 3)$
   (iv) $f(x) = \dfrac{(x^2+1)}{2x}$; $(x > 0)$
   (v) $f(x) = 2x^5 + 40x^2 + 2$; for $x \in [0,\infty)$
   (vi) $f(x) = \dfrac{x-2}{x+2}$; $x \in \mathbb{R} - \{-2\}$
   (vii) $f(x) = x^4 + x^3 + 1$; domain $\mathbb{R}$
   (viii) $f(x) = x^6 + x^4 + x^2 + 1$; domain $\mathbb{R}$
2. State whether following statements are true/false?
   (a) If f is strictly increasing function, then $[f(x)]$ is always many–one function; where [.] denotes gint function.
   (b) If f is continuous and strictly increasing function, then $[f(x)]$ is non-injective function; where [.] denotes gint function.
   (c) If f is continuous and strictly increasing function, then $\{f(x)\}$ is many–one function; where {} denotes fractional part function.
   (d) If $f(x)$ is continuous and strictly increasing function and range of f is a subset of $[0, 1)$, then $\{f(x)\}$ is injective function.
   (e) If f is continuous and strictly increasing function and range of f is superset of $[0, 1]$, then $\{f(x)\}$ is many–one function.
3. Given a set $A = \{1,2,3,4,5,6\}$. Prove that there is no injective functions $f: A \to A$ possible such that $a + f(a)$ is a perfect square for every $a \in A$. Does the above statement holds good if set A is replaced by B not containing element 1?
4. Let f: $\mathbb{R} \to \mathbb{R}$ where $f(x) = \dfrac{x^2+4x+7}{x^2+x+1}$. Is f one–one? Justify.
5. Find the values of k; if $f(x) = 2\sin x - kx + 3$, is defined from $\mathbb{R} \to \mathbb{R}$ and is one–one.
6. A function $f: D \to \mathbb{R}$ is given by $f(x) = \dfrac{5x - x^2 - 4}{5x - 4x^2 - 1}$; where D is domain of function. Is the function one–one?
7. Let f: $\mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{x^2+3x+c}{x^2+x+1}$. Show that f is a many–one function for all c.

Answer Keys

1. (a) injective (ii) non-injective (iii) injective (iv) non-injective
   (v) non-injective (vi) injective (vii) non-injective (viii) non-injective
2. (a) false (b) true (c) false (d) true (e) true
3. no; only injective function $f: B \to B$ is given as {(2, 2), (3, 6), (4, 5), (5, 4), (6, 3)}
4. non-injective; $f(x)$ has one point of local maxima and one point of local minima
5. $k \in (-\infty, -2] \cup [2, \infty)$
6. yes; (since it is linear rational function)

TEXTUAL EXERCISE–11 (OBJECTIVE)

1. $f(x) = x^3 - 12x + 1$ is
   (a) non-injective on $\mathbb{R}$
   (b) injective on $(-\infty, -2]$ and on $[2, \infty)$
   (c) injective on $[-2, 2]$
   (d) non-injective on $[-2 - k, -2 + k]$ and on $[2 - k, 2 + k]$ for each $k > 0$

2. The function $f(x) = x^4 - 4x^3 + 4x^2 - 1$ is
   (a) injective on $\mathbb{R}$
   (d) non-injective on $(1 - k, 1 + k)$ for each $k > 0$
   (c) non-injective on $[2, \infty)$
   (d) non-injective on $(-\infty, 0]$

3. Let $f: \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) = x^3 + x^2 + 3x + \sin x$. Then
   (a) f is injective
   (b) f is non-injective
   (c) f is increasing
   (d) decreasing

4. The function $y = 5 + e^{x^2 - 2}$ is
   (a) injective on $\mathbb{R}$
   (b) injective $[0, \infty)$
   (c) non-injective on $(-\infty, 0]$
   (d) injective on $[-k, k]$ for each real $k > 0$

5. The function $f(x) = \dfrac{x^2}{x^2 - 1}$ is
   (a) injective in $(-\infty, -1)$
   (b) non-injective in $(-1, 1)$
   (c) non-injective on $\mathbb{R}$
   (d) None of these

6. The function $f(x) = \ln(1 - \ln x)$
   (a) non-injective
   (b) injective
   (c) having range $\mathbb{R}$
   (d) decreasing function in its domain

7. The function $f(x) = \dfrac{x^3 - 2x^2 - 9x + 18}{x^2 - 4}$ is
   (a) injective on its domain
   (b) increasing at every point of its domain
   (c) non-injective on its domain
   (d) decreasing at every point of its domain

8. The function $y = [x] + \sqrt{\{x\}}$; (where [.] and {} are greatest integer function and fractional part function respectively) is
   (a) injective on $\mathbb{R}$ (d) decreasing on $\mathbb{R}$
   (c) non-injective on $\mathbb{R}$ (d) increasing on $\mathbb{R}$

9. If $A = \mathbb{R} - \{3\}$, $B = \mathbb{R} - \{1\}$ and $f: A \to B$ be defined by $f(x) = \left(\dfrac{x-2}{x-3}\right)$. Then
   (a) f is non-injective
   (b) f is one–one
   (c) f is increasing function
   (d) f is decreasing function

Answers

1. (a,b,c,d) 2. (b) 3. (a,c) 4. (b) 5. (a,b,c) 6. (b,c,d) 7. (b,c) 8. (a,d) 9. (b,d)

SURJECTIVE AND NON-SURJECTIVE FUNCTION

As we know that in a function all the elements of domain X must be related to some element in co-domain. When similar restriction is imposed for co-domain as well, then the function obtained is known as onto function.

Onto (Surjective) Function

A function $f : X \rightarrow Y$ is called surjective only when each element in the co-domain is f-image of at least one element in the domain, i.e., $f : X \rightarrow Y$ is onto iff $y \in Y$ there exists $x \in X$ such that $f(x) = y$, i.e., iff R_f = co-domain (Y)

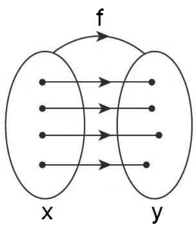


FIGURE 2.135

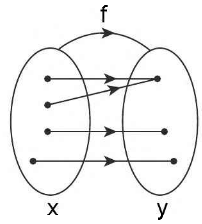


FIGURE 2.136

$\therefore$ Surjective $f : X \rightarrow Y$ reduces the co-domain set to range of function.

Example:

1. If $f(x)$ is any polynomial of odd degree, then it is surjective from $\mathbb{R} \rightarrow \mathbb{R}$
2. If $f(x) = \log_e x$ $(x > 0)$, then it is surjective from $(0, \infty) \rightarrow \mathbb{R}$
3. If $f(x) = \tan x$, then it surjective from $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbb{R}$
4. If $f(x) = \cot x$, then $f(x)$ is surjective from $(0, \pi) \rightarrow \mathbb{R}$.

NOTE

The equality of range of function to co-domain forms the condition to test surjectivity of function. For instance to test surjectivity of f: $[0, \infty) \rightarrow [2, \infty)$ such that $f(x) = x^2 + 2$.

Using the analytic formula we obtain the rule of function for argument of x in terms of y as below:

$\because\ y = x^2 + 2$

$x^2 = y - 2$, i.e., $|x| = \sqrt{y-2}$

$\Rightarrow x = \sqrt{y-2} \qquad \because\ x \geq 0$

Now we check whether the expression of x in forms of y is valid for all elementary co-domain.

If it is so, then f is surjective otherwise it is non-surjective.

Thus, x to be real and positive RHS, i.e., $\sqrt{y-2}$ must be real and positive, thus, $y \in [2, \infty)$. Hence, the given function f is onto.

INTO (NON-SURJECTIVE) FUNCTION

While defining function we have mentioned that there may exist some element in the co-domain which are not related to any element in the co-domain.

$f : X \rightarrow Y$ is into iff there exists at least one $y \in Y$ which is not related with any $x \in X$.

Thus, the range of the into function is proper subset of the co-domain. That is, Range $\subset$ Co-Domain (properly).

Example:

1. If $f(x)$ is any polynomial of even degree, it is into from $\mathbb{R} \rightarrow \mathbb{R}$ as the range of function $= [k, \infty)$ where k is the minimum value of function is proper subset of $\mathbb{R}$.
2. If $f(x) = e^x$, then it is into function from $\mathbb{R} \rightarrow \mathbb{R}$ as the range of function $= (0, \infty)$ is proper subset of $\mathbb{R}$.
3. If $f(x) = \sec x$, then it is an into function from $\mathbb{R} \sim \left\{(2n+1)\frac{\pi}{2}; n \in \mathbb{Z}\right\} \rightarrow \mathbb{R}$ as its range $= (-\infty, -1] \cup [1, \infty)$ is proper subset of $\mathbb{R}$.

4. If $f(x) = \tan^2 x$, then $f(x)$ is into function from $\mathbb{R} \sim \left\{(2n+1)\frac{\pi}{2}; n \in \mathbb{Z}\right\} \to \mathbb{R}$ as its range $= [0, \infty)$ is a proper subset of $\mathbb{R}$.

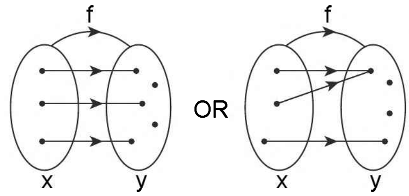


FIGURE 2.137

ILLUSTRATION 167: Find whether the following functions from $\mathbb{R} \to \mathbb{R}$ are onto or into?

(a) $f(x) = \dfrac{2x}{1+x^2}$ (b) $y = x^3$

(c) $y = x\,|x|$ (d) $y = \sin \pi\, x$

SOLUTION: (a) Since range of function is $[-1, 1]$, therefore, range is subset of co-domain (which is set of real number). Hence, $f(x)$ is into function.

(b) $y = x^3$ has range as set of real numbers, so it is onto function

(c) $y = x^2$ when $x \geq 0$ and $y = -x^2$ when $x < 0$

$\Rightarrow$ Range is set of real numbers. So, $f(x)$ is onto function

(d) Range of $\sin \pi x$ is $[-1, 1]$ which is a proper subset of real numbers, so it is an into function

ILLUSTRATION 168: A function $f : X \to Y$ defined as $f(x) = x + \sqrt{3-2x^2}$ is onto. Find the set Y if the set X is its natural domain.

SOLUTION: As $f : X \to Y$ is given as onto, therefore Y must be range of the function. Now to find range

Let $y = x + \sqrt{3-2x^2}$

$\Rightarrow (y-x)^2 = 3 - 2x^2$ $\Rightarrow y^2 - 2xy + x^2 = 3 - 2x^2$

$\Rightarrow 3x^2 - 2xy + y^2 - 3 = 0$. As, $x \in \mathbb{R}$, $D \geq 0 \Rightarrow y^2 - 3(y^2 - 3) \geq 0$

$\Rightarrow 2y^2 - 9 \leq 0$

$\Rightarrow y \in \left[-\dfrac{3}{\sqrt{2}}, \dfrac{3}{\sqrt{2}}\right]$. Hence, the set Y is $\left[-\dfrac{3}{\sqrt{2}}, \dfrac{3}{\sqrt{2}}\right]$.

HIT AND TRIAL METHOD

Sometimes we choose an element of co-domain which may not be an image of any element in domain and we test it for same. If it comes out to be true, then f is into function.

For instance $f : \mathbb{R} \to [0, \infty)$; $f(x) = \dfrac{e^x.(x^2+1)}{3+\sin x}$ is an into function, because $f(x)$ cannot attain zero value for any $x \in \mathbb{R}$, although zero is an element of co-domain.

ILLUSTRATION 169: Given $f : \mathbb{R} \to [-1, 1] : f(x) = \dfrac{3\cos x}{x^2 - 4x + 7}$, test whether $f(x)$ is a surjective function.

SOLUTION: Consider $f(x) = \dfrac{3\cos x}{x^2 - 4x + 7} = \dfrac{3\cos x}{(x-2)^2 + 3}$

Clearly values of $f(x)$ cannot be bygone the set $[-1, 1]$.

Now if we notice the element 1 or -1 in the co-domain

For $f(x) = 1$ $\Rightarrow \dfrac{3\cos x}{(x-2)^2+3} = 1$

$\Rightarrow$ $3\cos x - 3 = (x-2)^2$ $\Rightarrow -6\sin^2\dfrac{x}{2} = (x-2)^2$

LHS. $\in [-6, 0]$, where as RHS $\in [0, \infty)$. Therefore, equality may hold if LHS. = RHS = 0 for same input x, but it is never, so, as RHS vanishes at $x = 2$ but LHS is non-zero at $x = 2$.

$\Rightarrow$ $f(x) \neq 1$, neither $f(x)$ can be -1. Clearly f is an into function.

ILLUSTRATION 170: Show that $f: \mathbb{N} \to \mathbb{N}$ defined as $f(x) = 2x - 1$ is one–one function but not an onto function. Also analyse the nature of above function if it is defined from $\mathbb{R} \to \mathbb{R}$.

SOLUTION: $\because$ $f(x_1) = f(x_2)$ $\Rightarrow$ $2x_1 - 1 = 2x_2 - 1$ $\Rightarrow$ $x_1 = x_2$, thus, f is one–one

But f is not onto function as there does not exist any $x \in \mathbb{N}$ for which $f(x)$ is even.

Hence, no even natural number belonging to co-domain has its pre-image in $\mathbb{N}$.

But if f is a function: $\mathbb{R} \to \mathbb{R}$

Clearly function is one–one and being a polynomial is continuous.

Also $f'(x) = 2 > 0$, i.e., $f(x)$ is monotonic as well.

Also for any real $y \in$ co-domain $\mathbb{R}$ there exists an $x \in \mathbb{R}$ given by $x = \dfrac{y+1}{2}$ such that

$f(x) = f\left(\dfrac{y+1}{2}\right) = y$ consequently f is onto.

Concluding that $f: \mathbb{N} \to \mathbb{N}$ is injective but not surjective, whereas $f: \mathbb{R} \to \mathbb{R}$ is a bijective function.

ILLUSTRATION 171: A function $f: \mathbb{R} \to \mathbb{R}$ is given by: $f(x) = \dfrac{tx^2+3x-4}{t+3x-4x^2}$. Find the interval of values of t for which the function is onto.

SOLUTION: It is given that the co-domain of the function is $\mathbb{R}$. Since, the function is onto, the range of the function is equal to co-domain of the function, i.e., $\mathbb{R}$.

Let $y = \dfrac{tx^2+3x-4}{t+3x-4x^2}$

$\Rightarrow$ $tx^2 + 3x - 4 - ty - 3xy + 4x^2y = 0$

Representing it as a quadratic equation in variable x, we get

$\Rightarrow$ $(t + 4y)x^2 + 3(1 - y)x - (4 + ty) = 0$

Since x is real, $D \geq 0$

$\Rightarrow$ $9(1 - y)^2 + 4(t + 4y)(4 + ty) \geq 0$

$\Rightarrow$ $9(1 + y^2 - 2y) + 4(4t + t^2y + 4ty^2 + 16y) \geq 0$

$\Rightarrow$ $(9 + 16t)y^2 + (4t^2 + 46)y + (9 + 16t) \geq 0$

As the range of the given function has to be $\mathbb{R}$, implying that the above inequality holds for all real values of y

Therefore: Leading coefficient $t > 0$ and $D \leq 0$

$\Rightarrow$ $9 + 16t > 0$ $\Rightarrow$ $t > -9/16$ ..... (i)

$\Rightarrow$ $(4t^2 + 46)^2 - \{2(9 + 16t)\}^2 \leq 0$

$\Rightarrow$ $(4t^2 + 46 + 18 + 32t)(4t^2 + 46 - 18 - 32t) \leq 0$

$\Rightarrow$ $(t^2 + 8t + 16)(t^2 - 8t + 7) \leq 0$

$\Rightarrow\ (t+4)^2 (t-1)(t-7) \le 0$

$\Rightarrow\ (t-1)(t-7) \le 0$ or $t = -4$

$\Rightarrow\ 1 \le t \le 7$ or $t = -4$ ...(ii)

Taking the intersection of the two intervals, we get $t \in [1, 7]$.

ILLUSTRATION 172: Let a function f defined from $\mathbb{R} \to \mathbb{R}$ as $f(x) = \begin{cases} 2x - 3m & \text{for } x \le 2 \\ mx - 11 & \text{for } x > 2 \end{cases}$. If the function is surjective, then find all values of m.

SOLUTION: For the function f to be surjective, its range must be the set of all real numbers, i.e., $\mathbb{R}$

For $x \le 2$, $f'(x) = 2$

$\Rightarrow$ Monotonically increasing for $x \le 2$

Hence, the maximum value of $f(x)$ for $x \le 2$ occurs at $x = 2$.

$\therefore$ The maximum value for $x \le 2$ is $4 - 3m$.

i.e., $f(x) \in (-\infty, 4 - 3m]$ for $x \le 2$

For range to be $\mathbb{R}$, the minimum value of $f(x)$ for $x > 2$, must be less than or equal to $4 - 3m$ and maximum must approach to infinitely.

Now for $x > 2$, $f'(x) = m$. Obviously $m > 0$ .... (1)

($\because$ if $m < 0$, $f'(x) < 0$, i.e., $f(x)$ is decreasing, then the range can't be $\mathbb{R}$ as shown in the given figure)

$\therefore$ for $x > 2$, $2m - 11 \le 4 - 3m$

$\Rightarrow\ 5m \le 15$

$\Rightarrow\ m \le 3$ ... (2)

$\therefore$ From (1) and (2), $m \in (0, 3]$

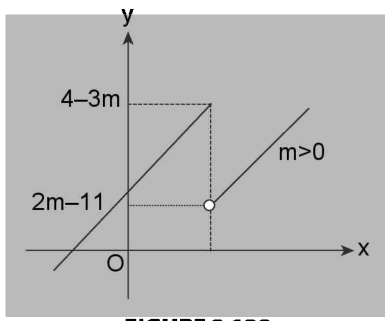


FIGURE 2.138

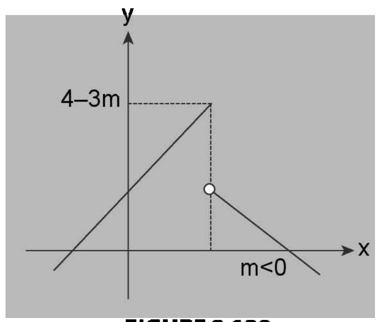


FIGURE 2.139

ILLUSTRATION 173: Prove that $f: \mathbb{R} - \{1\} \to \mathbb{R}$, given by $f(x) = \dfrac{x^{1/3}}{x^{1/3} - 1}$ is injective but not surjective.

SOLUTION: Let $f(a) = f(b)$ $\Rightarrow$ $\dfrac{a^{1/3}}{a^{1/3} - 1} = \dfrac{b^{1/3}}{b^{1/3} - 1}$

$\Rightarrow\ a^{1/3}b^{1/3} - a^{1/3} = a^{1/3}\, b^{1/3} - b^{1/3}$ $\Rightarrow\ -a^{1/3} = -b^{1/3}$ $\Rightarrow\ a = b$

This implies f is injective.

To prove that f is not surjective assume that $f(x) = b$; $b \in \mathbb{R}$.

Then $f(x) = b$

$$\Rightarrow\ \frac{x^{1/3}}{x^{1/3} - 1} = b$$

$$\Rightarrow\ x = \frac{b^3}{(b-1)^3}$$

The expression for x is not a real number when $b = 1$, and therefore, there is not real x such that $f(x) = 1$. Thus, $f(x)$ is not a surjective function.

ONE–ONE ONTO FUNCTION (BIJECTIVE FUNCTION)

If a function is both one–one as well as onto, then $f(x)$ is said to be bijective function or simply bijection. Injection means distinct $x \in X$ are related with distinct $y \in Y$ and surjective mean each $y \in Y$ is related with some $x \in X$.

Thus, taken both conditions together, bijective imposes the most strict condition for a function. It leads to invertibility of the function as all the distinct elements of domain as well as co-domain are related to distinct elements. That is, $\forall\ x, y \in X$ such that $x \neq y$.

$\Rightarrow\ f(x) \neq f(y)$ and $\forall\ y_1, y_2 \in Y$ such that $y_1 \neq y_2$

$\Rightarrow\ x_1 \neq x_2$; where $y_1 = f(x_1)$ and $y_2 = f(x_2)$

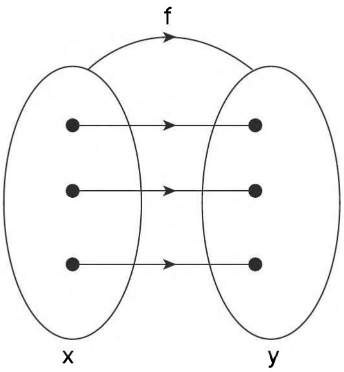


FIGURE 2.140

Thus, a function $f: X \to Y$ can be one of the given four types:

(i) one–one onto function (Injective and Surjective or Bijective):

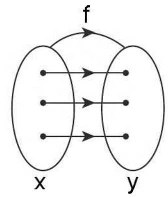


FIGURE 2.141

(ii) one–one into (injective but not surjective)

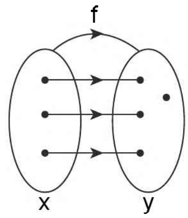


FIGURE 2.142

(iii) many–one onto (surjective but not injective)

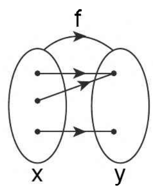


FIGURE 2.143

(iv) many–one into (neither surjective nor injective)

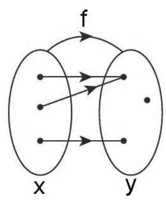


FIGURE 2.144

ILLUSTRATION 174: Show that the function $f(x) = x^2$; $f: A \to B$; where $A = \{-1, 1, 2, -2\}$; $B = \{1, 4\}$ is an onto function.

SOLUTION: $f(A) = \{(-1, 1), (1, 1), (2, 4), (-2, 4)\}$.

Thus, every element of B has a pre-image in A, thus, it is an onto function.

ILLUSTRATION 175: Show that an onto function f: $\{1, 2, 3\} \to \{1, 2, 3\}$ is always one–one.

SOLUTION: Suppose f is not one–one. Then there exists two elements, say 1 and 2 in the domain whose image in the co-domain is same. Also, the image of 3 under f can be only one element.

Therefore, the range set can have at the most two elements of the co-domain $\{1, 2, 3\}$, showing that f is not onto, a contradiction. Hence, f must be one–one.

ILLUSTRATION 176: Show that a one–one function $f: \{1, 2, 3\} \to \{1, 2, 3\}$ must be onto

SOLUTION: Since f is one–one, three elements of $\{1, 2, 3\}$ must be taken to 3 different elements of the co-domain $\{1, 2, 3\}$ under f. Hence, f has to be onto.

ILLUSTRATION 177: Show that f: $\mathbb{Z} \to \mathbb{Z}$, given by $f(x) = \begin{cases} x+5, & \text{if } x \text{ is odd} \\ x-5, & \text{if } x \text{ is even} \end{cases}$ is both one–one and onto.

SOLUTION: Suppose $f(x_1) = f(x_2)$. Note that if x_1 is odd and x_2 is even, then we will have $x_1 + 5 = x_2 - 5$, i.e., $x_2 - x_1 = 10$, which is impossible as the difference of an odd and even integer cannot be an even integer. Similarly, the possibility of x_1 being even and x_2 being odd can also be ruled out, using the similar argument. Therefore, both x_1 and x_2 must be either odd or even. Suppose both x_1 and x_2 are odd, then $f(x_1) = f(x_2)$

$\Rightarrow x_1 + 5 = x_2 + 5$

$\Rightarrow x_1 = x_2$

Similarly, if both x_1 and x_2 are even, then $f(x_1) = f(x_2) \Rightarrow x_1 - 5 = x_2 - 5 \Rightarrow x_1 = x_2$

Thus, f is one–one. Also any odd number $2k + 1$; $(k \in \mathbb{Z})$ in the co-domain $\mathbb{Z}$ is the image of $2k + 6$, in the domain $\mathbb{Z}$ and any even number $2r$ in the co-domain $\mathbb{Z}$ is the image of $2r - 5$ in the domain $\mathbb{Z}$. Thus, f is onto.

ILLUSTRATION 178: Investigate the function $f: (0, \infty) \to \mathbb{R}$ defined as $f(x) = e^{|\ln x|} - x^2$ for bijectivity.

SOLUTION: $f(x) = e^{|\ln x|} - x^2 = \begin{cases} e^{-\ln x} - x^2, & 0 < x < 1 \\ e^{\ln x} - x^2, & x \geq 1 \end{cases}$

$$= \begin{cases} \dfrac{1}{x} - x^2, & 0 < x < 1 \\ x - x^2, & x \geq 1 \end{cases}$$

We obtain the graph of f in (0, 1) by graphical addition as shown below.

From the graph we can see that f is one–one and onto. Hence, f is bijective.

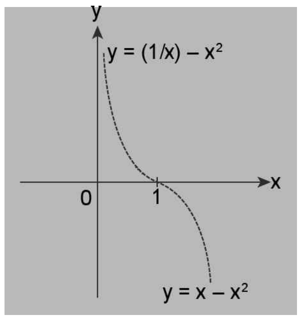


FIGURE 2.145

ILLUSTRATION 179: Let $A = \mathbb{R} - \{3\}$; $B = \mathbb{R} - \{1\}$. Let $f: A \to B$ be defined by $f(x) = \dfrac{x-2}{x-3}$. Is f bijective? Give reasons.

SOLUTION: For one–one (injective): Let $f(x_1) = f(x_2)$

$\Rightarrow \dfrac{x_1 - 2}{x_1 - 3} = \dfrac{x_2 - 2}{x_2 - 3} \Rightarrow x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 2x_1 - 3x_2 + 6$

$\Rightarrow x_1 = x_2$. Hence, it is one–one.

REMARK

In order to convert a function from many–one to an injective function its domain must be transformed to principle domain. In order to convert a function from into to onto, the co-domain of function must be replaced by its range.

ILLUSTRATION 180: A relation $f : \mathbb{R} \to \mathbb{R}$ such that $y = f(x)$ iff $x^2 + y^2 = 1$. Find its domain/range and convert it to

(a) Function
(b) Surjective function but not injective
(c) Injective function but not surjective
(d) Bijective function

SOLUTION: $x^2 + y^2 = 1 \Rightarrow y = \pm\sqrt{1-x^2} \Rightarrow 1 - x^2 \geq 0 \Rightarrow x^2 \leq 1 \Rightarrow x \in [-1, 1]$

$\Rightarrow D_f = [-1, 1]$. Similarly $x = \pm\sqrt{1-y^2} \Rightarrow y \in [-1, 1] \Rightarrow R_f = [-1, 1]$

Because for every value of $x \in [-1, 1]$, there are two values of y.

$\therefore$ $x^2 + y^2 = 1$ is a relation and not a function.

(a) So, to convert it into a function, redefine it as follows:

$f : [-1, 1] \to \mathbb{R}$ defined by $f(x) = \sqrt{1-x^2}$; $D_f = [-1, 1]$; $R_f = [0, 1]$

or $f : [-1, 1] \to \mathbb{R}$ defined by $f(x) = -\sqrt{1-x^2}$; $D_f = [-1, 1]$; $R_f = [-1, 0]$

The above functions represent, respectively, upper half and lower half arc of circle $x^2 + y^2 = 1$ as shown below:

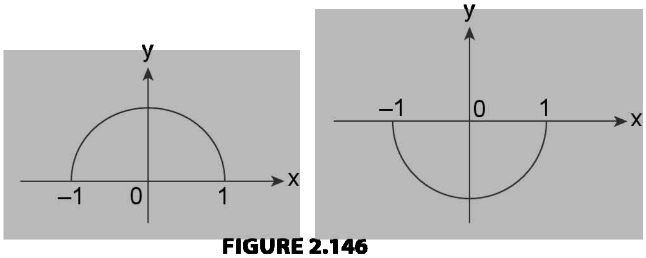


FIGURE 2.146

Clearly these functions are many–one, i.e., non-injective and non-surjective.

(b) However if we redefine the function as given below

$f : [-1, 1] \to [0, 1]$ defined by $f(x) = \sqrt{1-x^2}$; $D_f = [-1, 1]$; $R_f = [0, 1]$

or $f : [-1, 1] \to [0, 1]$ defined by $f(x) = -\sqrt{1-x^2}$; $D_f = [-1, 1]$; $R_f = [-1, 0]$

Clearly these functions represent the upper and lower half of the circle $x^2 + y^2 = 1$ and are surjective but not injective (i.e., many–one), graph of function shown in part (a).

(c) Now let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = \sqrt{1-x^2}$; $D_f = [0, 1]$; $R_f = [0, 1]$ (fig 2.150).

or $f : [-1, 0] \to \mathbb{R}$ be defined by $f(x) = \sqrt{1-x^2}$; $D_f = [-1, 0]$; $R_f = [0, 1]$ (fig 2.151).

or $f : [-1, 0] \to \mathbb{R}$ be defined by $f(x) = -\sqrt{1-x^2}$; $D_f = [-1, 0]$; $R_f = [-1, 0]$ (fig 2.152).

or $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = -\sqrt{1-x^2}$; $D_f = [0, 1]$; $R_f = [-1, 0]$ (fig 2.153).

The above functions represent the arc of circle $x^2 + y^2 = 1$ in first, second, third, and fourth quadrant, respectively, as shown below.

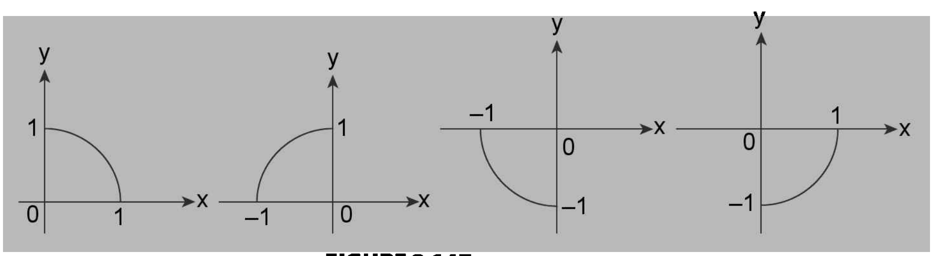


FIGURE 2.147

Clearly this function is injective but not surjective.

(d) Now let $f: [0, 1] \to [0, 1]$ be defined by $f(x) = \sqrt{1-x^2}$; $D_f = [0, 1]$; $R_f = [0, 1]$.
or $f: [-1, 0] \to [0, 1]$ be defined by $f(x) = \sqrt{1-x^2}$; $D_f = [-1, 0]$; $R_f = [0, 1]$.
or $f: [-1, 0] \to [-1, 0]$ be defined by $f(x) = -\sqrt{1-x^2}$; $D_f = [-1, 0]$; $R_f = [-1, 0]$.
or $f: [0, 1] \to [-1, 0]$ be defined by $f(x) = -\sqrt{1-x^2}$; $D_f = [0, 1]$; $R_f = [-1, 0]$.
The above functions represent the arc of circle $x^2 + y^2 = 1$ in first, second, third, and fourth quadrant respectively as shown in part (c).
Clearly these functions are injective as well as surjective. i.e., bijective functions as their Domain = principal domain and range = co-domain

■ NUMBER OF RELATIONS AND FUNCTIONS

Given two finite sets A and B having n and m elements respectively, i.e., $n(A) = n$ and $n(B) = m$.

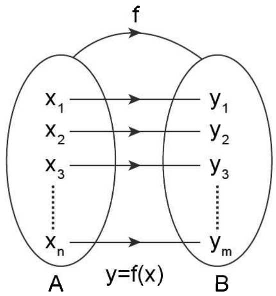


FIGURE 2.148

Number of Relations: No. of relations = Number of subsets of $A \times B = 2^{n(A\times B)} = 2^{nm}$

Number of Functions: Since each element of set A can be mapped in m ways

⇒ Number of ways of mapping all n elements of A

$$= \left(\underbrace{m \times m \times m \times \ldots. \times m}_{n \text{ times}}\right) \text{ ways} = m^n \text{ ways}$$

Conclusion: $2^{nm} \geq m^n \;\forall\; m, n \in \mathbb{N}$

Number of one–one function (injective):

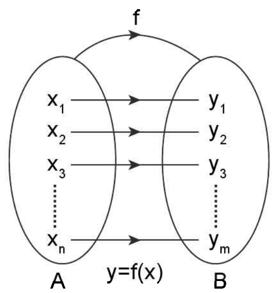


FIGURE 2.149

x_1 can be mapped in m ways
x_2 can be mapped in $(m-1)$ ways
x_3 can be mapped in $(m-2)$ ways
⋮
x_n can be mapped in $(m-(n-1))$ ways

⇒ Number of injective functions

$$= m(m-1)(m-2)\ldots.(m-n+1) = \begin{cases} {}^mP_n & m \geq n \\ 0 & m < n \end{cases}$$

Conclusion: ${}^mP_n \leq m^n$ (total number of functions).

Number of Non-surjective Functions (Into Functions)

Number of into functions (N) = Number of ways of distributing n different objects into m distinct boxes so that at least one box is empty.

⇒ $N = n(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_m)$; where A_i denotes the distributions when i^{th} box is empty.

$= \Sigma n(A_i) - \Sigma n(A_i \cap A_j) + \Sigma n(A_i \cap A_j \cap A_k) - \ldots.$

Now $n(A_i) = (m-1)^n$; $n(A_i \cap A_j) = (m-2)^n$;

$n(A_i \cap A_j \cap A_k) = (m-3)^n$

∴ ${}^mC_1(m-1)^n - {}^mC_2(m-2)^n + {}^mC_3(m-3)^n - \ldots. + (-1)^{m-2}\,.\,{}^mC_{m-1}(1)^n$.

$$\Rightarrow N = \sum_{r=1}^{m} {}^mC_r(-1)^{r-1}(m-r)^n]$$

Number of Surjective Functions

Number of surjective functions = Total number of functions – number of into functions.

$$= m^n - \sum_{r=1}^{m} {}^mC_r(-1)^{r-1}(m-r)^n$$

$$= m^n + \sum_{r=1}^{m} {}^mC_r(-1)^r(m-r)^n = \sum_{r=0}^{m} {}^mC_r(-1)^r(m-r)^n$$

Conclusion: In case when $n(A) = n(B)$, the onto functions will be bijection

Number of onto functions = Number of one–one functions

$$\Rightarrow \sum_{r=0}^{n} {}^nC_r(-1)^r(n-r)^n = n!$$

ILLUSTRATION 181: If a function $f : X \to Y$ be such that $X = \{x, y\}$ and $Y = \{a, b, c\}$, then find :

(a) Number of relations from X to Y. **(b)** Number of functions $f : X \to Y$.
(c) Number of injective functions $f : X \to Y$. **(d)** Number of many–one functions $f : X \to Y$.
(e) Number of into functions $f : X \to Y$.
(f) Number of onto functions (surjective) $f : X \to Y$.

SOLUTION: **(a)** No. of relations from X to Y

$= $ Number of subsets of $X \times Y = (2)^{n(X \times Y)} = 2^{n(X) \times n(Y)} = (2)^{2 \times 3} = (2)^6 = 64$

(b) x can be related to a, b or c, i.e., in three ways, and corresponding to each choice of x, y can be related a, b, c in three ways. Thus, total number of functions $= 3 \times 3 = 9 = (3)^2 = [n(Y)]^{n(X)}$.

(c) Number of injective functions $= {}^{n(Y)}P_{n(X)} = {}^3P_2 = \dfrac{3!}{1!} = 3! = 6$.

Remark: If $n(X) > n(Y)$, then no injective functions can be formed from X to Y as in this case at least one element of Y has to be related to more than one element of X.

(d) Number of many–one functions

$=$ Total number of functions – number of one–one functions $= 9 - 6 = 3$.

(e) Number of into functions (non-surjective)

$\sum_{r=1}^{m} {}^mC_r(-1)^{r-1}(m-r)^n$; $n = n(X) = 2$; $m = n(Y) = 3$

$= {}^3C_1(-1)^0(3-1)^2 + {}^3C_2(-1)^1(3-2)^2 + {}^3C_3(-1)^2(3-3)^2 = 12 - 3 + 0 = 9$

(f) Number of onto functions (surjective)

$= (m)^n -$ Number of into functions $= \sum_{r=0}^{m}(-1)^r\, {}^mC_r(m-r)^n$;

where $n(X) = n$ and $n(Y) = m = (3)^2 - 9 = 0$.

REMARKS

1. *If n(X) < n(Y), then after mapping different elements of X to different elements of Y, we are left with at least one element of Y which is not related with any element of X, and hence, there will be no onto function from X to Y, i.e., all the functions from X to Y will be into.*
2. *If f from X to Y is a bijective function, then n(X) = n(Y)*

ILLUSTRATION 182: Find the number of surjections from X to Y; where $X = \{1, 2, 3, 4\}$, $Y = \{a, b\}$

SOLUTION: Every element of X can be connected to any of the two elements of Y, i.e., 2 choices. So, total number of ways $= 2^4$. But if all the elements of X get connected to a, function will not be onto. Similarly, if all the elements of A get connected to b, function will not be onto. So, number of onto functions is $2^4 - 2 = 14$ or alternatively, $\sum_{r=0}^{2}(-1)^r . {}^2C_r(2-r)^4 = {}^2C_0(2)^4 - {}^2C_1(1)^4 + {}^2C_2(0)^4 = 14$.

ILLUSTRATION 183: Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6, 7\}$. How many functions are there from A to B? How many functions are there from B to A? How many injections are there from A to B? How many surjection are there from B to A.

SOLUTION: There are $(4)^3 = 64$ functions from A to B. There are 4 possibilities for the image of 1, 4 for the image of 2 and 4 for the image of 3. Similarly, there are $(3)^4 = 81$ functions from B to A. Number of surjections from B to A = $\sum_{r=0}^{3}(-1)^r.{}^3C_r(3-r)^4$

$= {}^3C_0(3)^4 - {}^3C_1(2)^4 + {}^3C_2(1)^4 = 81 - 48 + 3 = 36$

By the above result, there are 4.3.2 = 24 injections from A to B

TEXTUAL EXERCISE–12: (SUBJECTIVE)

1. Test the following functions for injectivity and surjectivity.

(a) $y = 5x + 8$ (b) $y = \dfrac{x^2-1}{x^2+1}$

(c) $y = \dfrac{ax+b}{ax-b}$ (d) $y = e^x + e^{-x}$

(e) $y = e^x - e^{-x}$ (f) $y = \log\left(\dfrac{x-1}{x+1}\right)$

(g) $y = \log(x+\sqrt{x^2+1})$ (h) $x + \dfrac{1}{x}$

2. Find number of surjection from A to B where $A = \{1, 2, 3, 4\}$, $B = \{a, b\}$

3. A function $f : X \to Y$ defined as $f(x) = \sqrt{1-2x} + x$ is from $A \to B$ and is onto. If the set A is its natural domain, then find the set B.

4. If the function $f : \mathbb{R} \to A$ given by

(a) $f(x) = \dfrac{e^x - e^{-|x|}}{e^x + e^{-|x|}}$

(b) $f(x) = \dfrac{e^x - e^{-|x|}}{e^x + e^{|x|}}$

(c) $\dfrac{e^x - e^{|x|}}{e^x + e^{-|x|}}$ is surjection, then find A

5. Is the function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = 3x + 5$ surjective.

6. If a function $f : X \to Y$ be such that $X = \{x, y, z\}$ and $Y = \{a, b\}$. Then find :

(a) total number of relations from X to Y.

(b) total number of functions from X to Y.

(c) Number of one–one functions from X to Y.

(d) Number of many–one functions from X to Y.

(e) Number of onto functions from X to Y.

7. Let $f(x) = ax^3 + bx^2 + cx + d \sin x$. Find the condition that $f(x)$ is always one–one function.

8. If $f : \mathbb{R} \to B$, defined by $f(x) = \sin x + \cos x - 4$, is onto, then find set B.

9. There are exactly two distinct linear functions which map $[-1, 1]$ onto $[0, 3]$, then find the point of intersection of the two functions.

10. If $f : \mathbb{R} \to \left[\dfrac{\pi}{6}, \dfrac{\pi}{2}\right)$, $f(x) = \sin^{-1}\left(\dfrac{x^2-a}{x^2+1}\right)$ is an onto function, then find the set of values of 'a'.

11. Let $f : \mathbb{R} \to \left[0, \dfrac{\pi}{2}\right)$ defined by $f(x) = \tan^{-1}(x^2 + x + a)$, then find the set of values of a for which f is onto.

12. For the following given functions $f(x) : D_f \to Y$ (D_f denotes Domain of f): find set Y for which $f(x)$ is surjective (on to) in each case.

(a) $f(x) = \sin^6 x + \cos^6 x$

(b) $f(x) = \log_e(|\sin x|^4 + |\cos x|^4)$

(c) $f(x) = \sqrt{\ln|\operatorname{cosec} x|}$

(d) $f(x) = \log_{\sqrt{2}}\left(\dfrac{\cos x + \sin x + 2\sqrt{2}}{\sqrt{2}}\right)$

13. Let $A = \{x : -k \le x \le k;\ k \in \mathbb{N}\}$ and $B_1 = \left\{\pm\dfrac{m^2}{4}; 0 \le m \le k; m \in \mathbb{Z}\right\}$; $B_2 = A$; $B_3 = \{\pm m^4;$ $0 \le m \le k;\ m \in \mathbb{R}\}$
$B_4 = \{m^2;\ m \in \mathbb{Z};\ 0 \le m \le k\}$; $B_5 = A$. For each of the following functions from A to B_i, find whether it is surjective, injective, or bijective.

(a) $f(x) = \dfrac{x|x|}{4}; f : A \to B_1$

(b) $g(x) = \sqrt{x^2}$; $g : A \to B_2$

(c) $h(x) = x^3|x|$; $h : A \to B_3$

(d) $k(x) = x^2$; $k : A \to B_4$

(e) $\phi(x) = x\cos(2x + 1)\pi x$; $\phi : A \to B_5$

14. Let $f:\mathbb{R} \to \mathbb{R}$ be a function such that $f(x) = x^3 + x^2 + 3x + \sin x$. Then check the injectivity and surjectivity of the function.

15. Which of the following functions from $\mathbb{Z}$ to itself are bijections?
(a) $f(x) = x + 3$ (b) $f(n) = x^5$
(c) $f(x) = 3x + 2$ (d) $f(x) = x^2 + x$

16. If $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} 2x & if\ x \ge 0 \\ 2x-3 & if\ x < 0 \end{cases}$.

Show that f is one–one but not onto.

17. Let $f : \mathbb{R} - \{2\} \to \mathbb{R} - \{1\}$ be defined by $f(x) = \dfrac{x-5}{x-2}$. Is f bijective? Give reasons.

18. Let $A = \mathbb{R} - \{3\}$, $B = \mathbb{R} - \{1\}$ and $f : A \to B$ defined by $f(x) = \dfrac{x-2}{x-3}$. Is f bijective? Give reason.

19. Determine the kind of mapping if $f : [1, 7] \to [2, 27]$, where $f(x) = x^2 - 4x + 6$.

20. For each of the following functions find whether it is one–one or many–one and into or onto.
(a) $f(x) = 2 \cot x; f : (\pi, 2\pi) \to \mathbb{R}$
(b) $f(x) = \dfrac{4}{1+x^2}; f : \mathbb{R} \to \mathbb{R}$
(c) $f(x) = x^4 + \ln x; f : (0, \infty) \to \mathbb{R}$

21. Classify the following functions $f(x)$ defined in $\mathbb{R} \to \mathbb{R}$ as injective, surjective both or none.
(a) $f(x) = \dfrac{x^2+4x+30}{x^2-8x+18}$
(b) $f(x) = x^3 - 6x^2 + 11x - 6$
(c) $f(x) = (x^2 + x + 5)(x^2 + x + 3)$

22. Find the set of values of k for which the function $f(x) = kx + 3\sin x + 5$ is one–one and onto.

23. Let $f(x) = \dfrac{x-1}{m-x^2+1}$, $x \in \mathbb{R}$. If the range of $f(x)$ does not contain the open interval $\left(-1, \dfrac{-1}{3}\right)$, then prove that $m \le \dfrac{-1}{4}$.

24. Verify if $f(x) = \dfrac{x^2-8x+18}{x^2+4x+30}$ is an injective function.

25. Check whether $f(x) = \dfrac{x}{1+|x|}$, $x \in \mathbb{R}$ is injective or surjective or both or none.

26. Prove that $f : (-1, 1) \to \mathbb{R}$ defined by
$f(x) = \begin{cases} \dfrac{x}{1+x}, & -1 < x \le 0 \\ \dfrac{x}{1-x}, & 0 < x < 1 \end{cases}$ is a bijective function.

27. If the function $f : [2, \infty) \to X$ be bijective, where $f(x) = 5 - 4x + x^2$, then find the set X.

28. Let $f : \mathbb{N} \sim \{1\} \to \mathbb{N}$ be defined by $f(n)$ = the highest prime factor of n. Show that f is neither one–one nor onto. Find the range of f.

29. Let $f(x) = 1/(1 - x)$. If $f_2(x)$ denote $f\{f(x)\}$, $f_3(x)$ denote $f\{f\{f(x)\}\}$ and so on, then $f_{3n}(x)$ is ($n \in \mathbb{N}$).

30. Let $f(x) = \dfrac{ax+b}{cx+d}$, $x \ne -\dfrac{d}{c}$. If $d = -a$, show that $f\{f(x)\} = x$, i.e., fof is an identity function.

31. Let $f : R \to [-1, 1]$, $f(x) = \dfrac{\sin x}{x^2+1}$. Find whether f is onto

32. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$ is one–one and onto, where $\mathbb{R}$ is the set of all non-zero real numbers. Is the result true, if the domain $\mathbb{R}$ is replaced by $\mathbb{N}$ with co-domain being same as $\mathbb{R}$?

33. Check the injectivity and surjectivity of the following functions:
(i) $f : \mathbb{N} \to \mathbb{N}$ given by $f(x) = x^2$
(ii) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$
(iii) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$
(iv) $f : \mathbb{N} \to \mathbb{N}$ given by $f(x) = x^3$
(v) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^3$

Answer Keys

1. (a) (a) bijective (b) many–one into (c) one–one into (d) many–one into
(e) one–one onto (f) one–one into (g) one–one onto (h) many–one into

2. 14 **3.**

4. (a) $[0, 1)$ (b) $[0, 1/2)$ (c) $(-\infty, 0)$ **5.**

6. (a) 64 (b) 8 (c) 0 (d) 8 (e) 2

7. $b^2 < 3a(c - d)$; $a > 0$ and $b^2 < 3a(c + d)$; $a < a < 0$ **8.**

9. $y = \dfrac{3x}{2} + \dfrac{3}{2}, \dfrac{-3}{2}x + \dfrac{3}{2}$

10. $x \in \left(-1, -\dfrac{1}{2}\right]$ to be defined and to be onto a = -1/2

11. $x \in \left[\dfrac{1}{4}, \infty\right)$ to be defined and to be onto a = 1/4

12. (a) $[1/4,1]$ (b) $[-\log_e 2, 0]$ (c) $[0, \infty)$ (d) $[0, 2\log_2 3]$

13. (a) bijective (b) non-injective, non-surjective, non-bijective
(c) non-surjective, injective, non-bijective (d) surjective, non-injective, non-bijective
(e) Injective, surjective, bijective

14. f is one–one and onto **15.** $f(x) = x + 3$

17. $f'(x) = \dfrac{3}{(x-2)^2} > 0\ \forall\ x \in \mathbb{R} \sim \{2\}$ $\Rightarrow y$ is injective on $\mathbb{R} \sim \{2\}$ and $y \neq 1$; $y \in \mathbb{R}$ $\Rightarrow y$ is surjective.

18. Yes, $f(x) = f(y)$ $\Rightarrow x = y$ and range = co-domain

19. non-injective, but surjective

20. (a) bijective (b) neither injective nor surjective (c) bijective

21. (a) neither surjective nor injective (b) surjective but not injective
(c) neither injective nor surjective

22. $(-\infty, -3] \cup [3, \infty)$ **24.** no **25.** one–one, not onto **27.** $[1,\infty)$

28. prime numbers **29.** x **31.** into **32.** no

33. (i) Injective but not surjective (ii) Neither injective nor surjective
(iii) Neither injective nor surjective (iv) Injective but not surjective (v) Injective but not surjective

TEXTUAL EXERCISE–12: (OBJECTIVE)

1. Function f: $(-\infty,-1) \to (0, e^5)$ defined by $f(x) = e^{x^3-3x+2}$ is
   (a) many–one and onto (b) many–one and into
   (c) one–one and onto (d) one–one and into

2. Given list of functions in column I and their classification in column II

Column I	Column II
(i) $e^x - e^{-x}$	(a) one–one onto
(ii) $x^3 - 1$	(b) one–one into
(iii) $\dfrac{2x}{1+x^2}$	(c) many–one onto
(iv) a^{x^3}	(d) many–one into
(v) $x^3\|x\|$	(e) not a function
(vi) $\dfrac{x^2-x+1}{x^2+x+1}$	
(vii) $\sin^3 x$	

Which of the following represents the correct match for column I and column II?

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)
(a)	(a)	(c)	(d)	(b)	(a)	(d)	(d)
(b)	(c)	(a)	(d)	(b)	(d)	(a)	(d)
(c)	(a)	(c)	(b)	(d)	(a)	(a)	(d)
(d)	(b)	(a)	(c)	(a)	(d)	(b)	(c)

3. For the following given four functions if the function $f(x)$ $D_f \to Y$ (D_f denotes domain of f):, then $f(x)$ is surjective (onto) for set Y.

   (i) $f(x) = \sqrt{\log_e \dfrac{1}{|\sec x|}}$

   (a) $\mathbb{R}$
   (b) $[0, \infty)$
   (c) $\{0\}$
   (d) $[0,1)$

(ii) $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

(a) $(-1, 1)$ (b) $[-1, 1]$
(c) $\{0\}$ (d) None of these

(iii) $f(x) = \dfrac{x^2}{1+x^4}$

(a) $[0, 1]$ (b) $[1, 2]$
(c) $[3/4, 1]$ (d) None of these

(iv) $f(x) = \sin^4 x + \cos^2 x$

(a) $[3/4, 1]$ (b) $[1/4, 1]$
(c) $[0, 1]$ (d) None of these

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) = x^3 - 6x^2 + 11x - 6$

(a) f is one–one and into
(b) f is many–one and into
(c) f is one–one and onto
(d) f is many–one and onto

5. The range of the function $f(x) = {}^{7-x}P_{x-3}$ is

(a) $[1, 2, 3]$ (b) $\{3, 4, 5, 6\}$
(c) $\{1, 2, 3, 4\}$ (d) $\{1, 2, 3, 4, 5\}$

6. Which of the following functions is not injective ?

(a) $f(x) = |x + 1|$, $x \in [-1, 0)$
(b) $f(x) = x + 1/x$, $x \in [0, \infty)$
(c) $f(x) = x^2 + 4x - 5$
(d) $f(x) = e^{-x}$, $x \in [0, \infty)$

7. A mapping f: $A \to [-1, 1]$ defined by $f(x) = \sin 3x$, is one–one and onto. Which of the following intervals can A be?

(a) $\left[-\dfrac{\pi}{6}, \dfrac{\pi}{6}\right]$ (b) $\left[-\dfrac{\pi}{3}, \dfrac{\pi}{3}\right]$
(c) $\left[\dfrac{\pi}{6}, \dfrac{\pi}{2}\right]$ (d) None of these

8. If $f : \mathbb{N} \to \mathbb{Z}$ is defined by $f(n) = \begin{cases} \dfrac{n-1}{2}, & \text{when } n \text{ is odd} \\ -\dfrac{n}{2}, & \text{when } n \text{ is even} \end{cases}$; then the type of function is

(a) injective
(b) surjective
(c) injective but not surjective
(d) surjective but not injective

9. If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \dfrac{x^2 - 4}{x^2 + 4}$, then the type of function is

(a) injective
(b) surjective
(c) injective but not surjective
(d) surjective but not injective

10. Let $f: (-1, 1) \to Y$ be a function defined by $f(x) = \tan^{-1}\dfrac{2x}{1-x^2}$, then f is bijective when Y is the set

(a) $\left[0, \dfrac{\pi}{2}\right)$ (b) $\left(0, \dfrac{\pi}{2}\right)$
(c) $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ (d) $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

11. Let $g: \mathbb{R} \to \left(0, \dfrac{\pi}{3}\right]$ is defined by $g(x) = \cos^{-1}\left(\dfrac{x^2 - \alpha}{1+x^2}\right)$.

Then the possible values of 'α' for which g is surjective function is/are

(a) $\left(-\dfrac{1}{2}, \dfrac{3}{4}\right]$ (b) $\left(-1, \dfrac{1}{2}\right]$
(c) $\left(-1, \dfrac{3}{4}\right]$ (d) None of these

12. If $f: \mathbb{R} \to \mathbb{R}$, is defined by $f(x) = \begin{cases} x|x| - 6, & x \in Q \\ x|x| - \sqrt{2}, & x \notin Q \end{cases}$, then the type of function is

(a) injective
(b) surjective
(c) injective but not surjective
(d) surjective but not injective

13. If $f(x) = \dfrac{\sin([x]\pi)}{x^2 + x + 1}$; where [.] denotes the greatest integer function, then

(a) f is one–one
(b) f is not one–one and non-constant
(c) f is a constant function
(d) None of these

14. Let $X = \{a_1, a_2, \ldots, a_6\}$ and $Y = \{b_1, b_2, b_3\}$. The number of functions f from X to Y such that it is onto and there are exactly three elements in X such that $f(a_i) = b_1$, is

(a) 75 (b) 90
(c) 100 (d) 120

15. The number of bijective functions $f : Y \to Y$, where $Y = \{4, 5, 6\}$ such that $f(4) \neq 6$, $f(5) \neq 4$, $f(6) \neq 5$, is

(a) 1 (b) 2
(c) 9 (d) None of these

16. Let f: $\mathbb{R} \to \mathbb{R}$ and g: $\mathbb{R} \to \mathbb{R}$ be two one–one and onto functions such that they are the mirror images of each other about the line $y = k$. If $\phi(x) = f(x) + g(x)$, then $\phi(x)$ is

(a) one–one and onto
(b) only one–one and not onto
(c) only onto but not one–one
(d) neither one–one nor onto

17. Let $X = \{a_1, a_3, a_3, a_4, a_5\}$, $Y = \{b_1, b_2, b_3, b_4, b_5\}$. Then the number of one–one mappings from X to Y such that $f(a_i) \neq b_i$, $\forall\ i \in \{1, 2, 3, 4, 5\}$.

(a) 40 (b) 44
(c) 24 (d) 60

Answer Keys

1. (d)	**2.** (a)	**3. (i)** (c) **(ii)** (a)	**(iii)** (c)	**(iv)** (b)	**4.** (c)	**5.** (a)	**6.** (b,c)
7. (a,c)	**8.** (a,b)	**9.** (d)	**10.** (d)	**11.** (b)	**12.** (c)	**13.** (c)	**14.** (d)	**15.** (b)
16. (d)	**17.** (b)							

COMPOSITION OF FUNCTIONS

Composition of function is a process of combining two or more functions by which output of one function is applied to second function whose output in term is applied to third function and so on. For instance, consider two functions f and g such that for some input x the output $f(x)$ lies in the domain of g and therefore the function g assigns a value for the input $f(x)$ called as $g(f(x))$, the above successive operations of functions f over x and then g over $f(x)$ is called as composition and denoted by gof. The composition of function can be easily understood by the following machine model of composite machine $g(f(x))$.

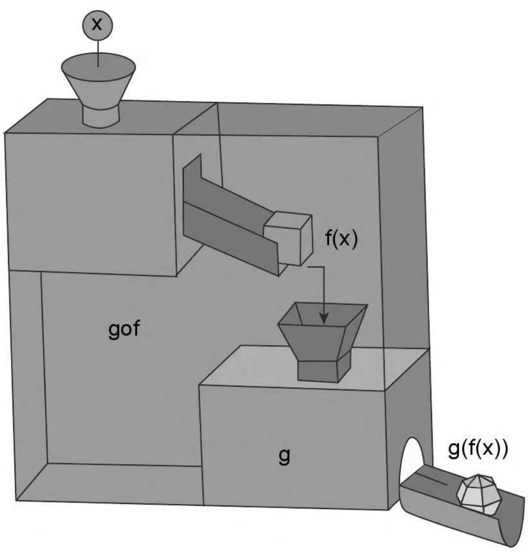


FIGURE 2.150

The output $f(x)$ of f machine is taken as input for g machine and $g(f(x))$ is the final output of composite machine gof.

Let us consider two functions $f : X \to Y$ and $g : Y \to Z$, such that the co-domain of f is domain of g. By composing the function g with f, we mean to define the analytical formula for a new function $h : X \to Z$, by which an element $x \in X$ is associated with an unique element $z \in Z$. The definition can be mathematically described as below.

Composite of Uniformly Defined Functions

Given two functions $f : X \to Y$ and $g : Y \to Z$, then there exists a function $h = gof : X \to Z$ such that $h(x) = (gof)(x) = g(f(x))\ \forall\ x \in X$. It is also called as 'function of function' or 'composite function of g and f' or 'g composed with f' and diagrammatically shown as

$$\xrightarrow[x \in X]{x} \boxed{f} \xrightarrow[y \in Y]{y = f(x)} \boxed{g} \longrightarrow \underbrace{z = g(f(x))}_{z \in Z}$$

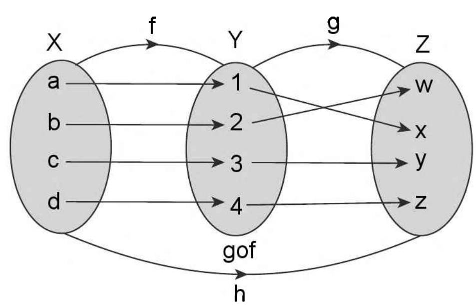


FIGURE 2.151

Thus, the image of every $x \in X$ under the function gof is the g-image of the f-image of x. gof is defined only if $\forall\ x \in X$, $f(x)$ is an element of the domain of g, so that we can take its g-image. To compute $gof(x)$, we first apply f on x, then apply g on the output $f(x)$. Figure 2.152 represents the composition function gof of functions $f: X \to Y$ and $g: Y \to Z$, where $X = \{a, b, c, d\}$; $Y = \{1, 2, 3, 4\}$; $Z = \{w, x, y, z\}$.

ILLUSTRATION 184: Let f: {1, 2, 3, 4} → {2, 3, 5, 7, 11} and g : {2, 3, 5, 7, 11} → {5, 7, 9 } be functions defined as $f(1) = 2, f(2) = 3, f(3) = 5, f(4) = 7$ and $g(2) = g(3) = 5$ and $g(5) = 7, g(7) = 9$. Find gof.

SOLUTION: We have $gof(1) = g(f(1)) = g(2) = 5$, $gof(2) = g(f(2)) = g(3) = 5$, $gof(3) = g(f(3)) = g(5) = 7$ and $gof(4) = g(f(4)) = g(7) = 9$.

Thus, gof; $\{1, 2, 3, 4\} \to \{5, 7, 9\}$ is given by $gof = \{(1, 5), (2, 5), (3, 7), (4, 9)\}$

ILLUSTRATION 185: Given three sets $X = \{0, 1, 2, 3\}$,$Y = \{1, 2, 5, 10, 15\}$ and $Z = \{1, 3, 9, 19, 29, 31, 51, 101, 201\}$. Let $f: X \to Y$ and $g: Y \to Z$ be functions defined as $f(x) = x^2 + 1$ and $g(x) = 2x - 1$. Find $gof: X \to Z$.

SOLUTION: We have $gof(0) = g(f(0)) = g(1) = 1$, $gof(1) = g(f(1)) = g(2) = 3$, $gof(2) = g(f(2)) = g(5) = 9$ and $gof(3) = g(10) = 19$.

Thus, gof; $X \to Z$ is given by $gof = \{(0, 1), (1, 3), (2, 9), (3, 19)\}$

ILLUSTRATION 186: Let two functions be defined as $f = \{(0, 1), (1, 2), (2, 3), (3,4)\}$ and $g = \{(-1, 3), (2, 1), (3, 2), (4, 0)\}$. Define gof.

SOLUTION: Here, Domain of $f = D_1 = \{0, 1, 2, 3\}$; Range of $f = R_1 = \{1, 2, 3, 4\}$

Domain of $g = D_2 = \{-1, 2, 3, 4\}$; Range of $g = R_2 = \{0, 1, 2, 3\}$

Now domain of gof is the set A of those elements of D_1 corresponding to which the output of the function 'f' contains common elements of R_1 and D_2.

Let set $X = R_1 \cap D_2 = \{2, 3, 4\}$. Now $f(1) = 2, f(2) = 3, f(3) = 4$.

Hence, $A = \{1, 2, 3\}$. Clearly A is a subset of D_1. Now, range of gof is the set B of those elements of R_2 which are the outputs of the elements of set X under g. Now $g(2) = 1$, $g(3) = 2$ and $g(4) = 0$. Hence, $B = \{1, 2, 0\}$. Clearly B is a subset of R_2

$\therefore$ gof: $\{1, 2, 3\} \to \{1, 2, 0\}$ and $gof = \{(1, 1), (2, 2), (3, 0)\}$

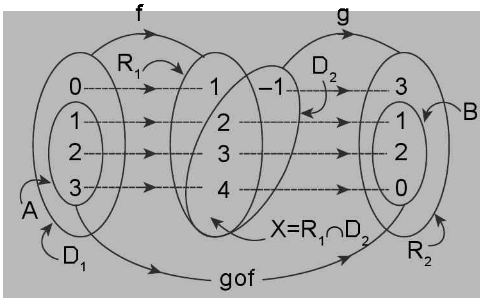


FIGURE 2.152

ILLUSTRATION 187: Show that if $f: \mathbb{R} - \left\{\frac{7}{3}\right\} \to \mathbb{R} - \left\{\frac{5}{3}\right\}$ is defined by $f(x) = \frac{5x+2}{3x-7}$ and $g: \mathbb{R} - \left\{\frac{5}{3}\right\} \to \mathbb{R} - \left\{\frac{7}{3}\right\}$ is defined by $g(x) = \frac{7x+2}{3x-5}$, then $fog = I_A$ and $gof = I_B$; where

$A = \mathbb{R} - \left\{\frac{5}{3}\right\}$, $B = \mathbb{R} - \left\{\frac{7}{3}\right\}$; $I_A(x) = x \in A$, $I_B(x) = x\ \forall\ x \in B$ are called identity functions on set A and B, respectively.

SOLUTION: Here we have $gof(x) = g\left(\frac{5x+2}{3x-7}\right) = \frac{7\left(\frac{5x+2}{3x-7}\right)+2}{3\left(\frac{5x+2}{3x-7}\right)-5} = \frac{35x+14+6x-14}{15x+6-15x+35} = \frac{41x}{41} = x\ \forall\ x \in B$

$\left(\because \quad g\left(\frac{5x+2}{3x-7}\right) \text{is valid for} \frac{5x+2}{3x-7} \neq \frac{5}{3} \text{ i.e., } 6 \neq -35, \text{which is true}\right)$

Similarly, $fog(x) = f\left(\frac{7x+2}{3x-5}\right) = \frac{5\left(\frac{7x+2}{3x-5}\right)+2}{3\left(\frac{7x+2}{3x-5}\right)-7} = \frac{35x+10+6x-10}{21x+6-21x+35} = \frac{41x}{41} = x\ \ \forall\ x \in A$

$\left(\because \quad f\left(\frac{7x+2}{3x-5}\right) \text{is valid for} \frac{7x+2}{3x-5} \neq \frac{7}{3} \text{ i.e., } 6 \neq -35, \text{which is true}\right)$

Thus, $gof(x) = x$, $\forall\ x \in B$ and $fog(x) = x$, $\forall\ x \in A$, which implies that $gof = I_B$ and $fog = I_A$.

ILLUSTRATION 188: Given two linear polynomial functions f and g such that $f(x) = mx + c$ and $g(x) = cx + m$ where m and $c \in \mathbb{N}$. If $f(g(5)) - g(f(5)) = k$, then find all possible ordered pairs (m, c) for

(a) $k = 12$ (b) $k = 35$

SOLUTION: From the definition of f and g; $f(g(x)) = m(g(x)) + c = m(cx + m) + c$

$\Rightarrow f(g(x)) = cmx + m^2 + c$ ..... (1)

Similarly; $g(f(x)) = c(mx + c) + m = cmx + c^2 + m$ .... (2)

Now $f(g(x)) - g(f(x)) = m^2 + c - c^2 - m = (m^2 - c^2) + (c - m) = (m - c)(m + c - 1)$

(a) Now since $f(g(5)) - g(f(5)) = 12$

$\Rightarrow (m - c)(m + c - 1) = 12$ .... (3)

Since $(m - c)$ and $(m + c - 1)$ are of opposite parity and $m - c < m + c - 1$ and $12 = 2^2 \times 3$

$\Rightarrow$ possible factorizations are 1×12 or 2×6 or 3×4

Case (i): $m - c = 1$ and $m + c - 1 = 12$

$\Rightarrow c = 6$ and $m = 7$

Case (ii): $m - c = 2$ and $m + c - 1 = 6$, not possible as $m - c$ and $m + c - 1$ both cannot be even as they are of opposite parity

Case (iii): $m - c = 3$ and $m + c - 1 = 4$. $\Rightarrow c = 1$ and $m = 4$

Thus, the possible ordered pairs (m, c) are (7,6);(4,1)

(b) If $f(g(5)) - g(f(5)) = 35$

$\Rightarrow (m - c)(m + c - 1) = 35$, which is never possible because product of an even and an odd integer can never be odd. Thus, there exists no ordered pair solution (m, c) for $k = 35$.

ILLUSTRATION 189: A function $f: \mathbb{R} \to \mathbb{R}$ is such that $f\left(\frac{4x-8}{x-4}\right) = x$ for all $x \neq 4$; then prove that

(i) $f(f(x)) = x\ \ \forall\ x \in \mathbb{R} - \{4\}$ (ii) $f(f(1/x)) = \frac{1}{x}\ \ \forall\ x \in \mathbb{R} - \left\{0, \frac{1}{4}\right\}$

SOLUTION: (i) If $f\left(\dfrac{4x-8}{x-4}\right)=x$ for all $x\neq 4$ .Let $y=\dfrac{4x-8}{x-4}\Rightarrow x=\dfrac{4y-8}{y-4}\Rightarrow f(y)=\dfrac{4y-8}{y-4}$

$\Rightarrow$ $f(f(x)) = x$ (from given)

(ii) $f(1/x)=\dfrac{4(1/x)-8}{(1/x)-4}=\dfrac{4-8x}{1-4x}$ for $x\neq 0,\dfrac{1}{4}$

$$\therefore\quad f(f(1/x))=f\left(\frac{4-8x}{1-4x}\right)=\frac{4\left(\frac{4-8x}{1-4x}\right)-8}{\left(\frac{4-8x}{1-4x}\right)-4}=\frac{\left(\frac{4-8x}{1-4x}\right)-2}{\left(\frac{1-2x}{1-4x}\right)-1}=\frac{4-8x-2+8x}{1-2x-1+4x}=\frac{2}{2x}=\frac{1}{x}$$

ILLUSTRATION 190: If $f(x) = x+1$; $0\le x\le 2$ and $g(x) = |x|$; $0\le x\le 3$. Calculate $fog(x)$, $fof(x)$.

SOLUTION: $fog\,(x) = f\{\,g(x)\,\} = f(\,|\,x\,|)$; $0\le x\le 3$

$= |\,x\,| + 1;\ 0\le |\,x\,|\le 2$ and $0\le x\le 3$

$= |\,x\,| + 1;\ -2\le x\le 2$ and $0\le x\le 3$

$= |\,x\,| + 1;\ 0\le x\le 2$

$fof(x) = f\{\,f(x)\} = f(x+1);\ 0\le x\le 2$

$= (x+1)+1;\ 0\le (x+1)\le 2$ and $0\le x\le 2$

$= (x+2);\ -1\le x\le 1$ and $0\le x\le 2$

$= (x+2);\ 0\le x\le 1$

ILLUSTRATION 191: Let $f(x) = \sin x$ and $g(x)=\sqrt{|1-x|}$. Find $gof(x)$, $fog(x)$, $gog(x)$ and $fof(x)$.

SOLUTION: $gof(x) = g\{f(x)\,\} = g(\sin x)=\sqrt{|1-\sin x|}$

$fog(x) = f\{\,g(x)\} = f\left\{\sqrt{|1-x|}\right\} = \sin\sqrt{|1-x|}$

$gog(x) = g\,\{\,g(x)\} = g\,\left\{\sqrt{|1-x|}\right\}=\left\{\sqrt{\left|1-\sqrt{|1-x|}\right|}\right\}$

$fof(x) = f(f(x)\} = f(\sin x) = \sin(\sin x)$.

ILLUSTRATION 192: Find fog and gof, if

(i) $f(x) = |x|$; $g(x) = \sin x$ (ii) $f(x) = \sin^{-1} x$; $g(x) = x^2$

(iii) $f(x) = x^2+2$; $g(x) = 1-\dfrac{1}{1-x},x\neq 1$

SOLUTION: (i) $f(x) = |x|$ and $g(x) = \sin x$

$\Rightarrow$ $fog(x) = f(\sin x) = |\sin x|$ and $gof(x) = g(f(x)) = g(|x|) = \sin|x|$

(ii) $f(x) = \sin^{-1}x$ and $g(x) = x^2$

$\Rightarrow$ $fog(x) = f(g(x)) = \sin^{-1}(g(x)) = \sin^{-1}x^2$ and $gof(x) = g(f(x)) = (f(x))^2 = (\sin^{-1}x)^2$

(iii) $f(x) = x^2+2$, $g(x)=\dfrac{x}{x-1}$

$\Rightarrow$ $fog(x) = f(g(x)) = (g(x))^2+2 = \dfrac{x^2}{(x-1)^2}+2=\dfrac{3x^2-4x+2}{(x-1)^2}$

and $gof(x) = g(f(x)) = g(x^2+2) = \dfrac{x^2+2}{(x^2+2)-1}=\dfrac{x^2+2}{x^2+1}$

ILLUSTRATION 193: If $f(x) = \sin^2 x + \sin^2\left(x+\frac{\pi}{3}\right) + \cos x \cos\left(x+\frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right)=1$, then find $(gof)(x)$.

SOLUTION: Given $f(x) = \sin^2 x + \sin^2\left(x+\frac{\pi}{3}\right) + \cos x \cos\left(x+\frac{\pi}{3}\right)$ ...(1)

and $g\left(\frac{5}{4}\right)=1$ ...(2)

We have $f(x) = \sin^2 x + \sin^2\left(x+\frac{\pi}{3}\right)+\cos x\cos\left(x+\frac{\pi}{3}\right)$

$$= \frac{1-\cos 2x}{2}+\frac{1-\cos\left(2x+\frac{2\pi}{3}\right)}{2}+\frac{1}{2}\left[2\cos x\cos\left(x+\frac{\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[1-\cos 2x+1-\cos\left(2x+\frac{2\pi}{3}\right)+\cos\left(2x+\frac{\pi}{3}\right)+\cos\frac{\pi}{3}\right]$$

$$= \frac{1}{2}\left[\frac{5}{2}-\left\{\cos 2x+\cos\left(2x+\frac{2\pi}{3}\right)\right\}+\cos\left(2x+\frac{\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[\frac{5}{2}-2\cos\left(2x+\frac{\pi}{3}\right)\cos\frac{\pi}{3}+\cos\left(2x+\frac{\pi}{3}\right)\right] = \frac{1}{2}\left[\frac{5}{2}-\cos\left(2x+\frac{\pi}{3}\right)+\cos\left(2x+\frac{\pi}{3}\right)\right]$$

or $f(x)=\frac{5}{4}$ for all x. Now, $gof(x) = g\{f(x)\} = g\left(\frac{5}{4}\right)=1$; Hence, $gof(x)=1$, for all x.

ILLUSTRATION 194: **(a)** Find the formula for the function $fogoh$, given $f(x) = \frac{x}{x+1}; g(x)=x^{10}$ and $h(x) = x + 3$.

Find the domain of this composite function. Also compute $(fogoh)(-1)$.

(b) Given $F(x) = \cos^2(x + 9)$. Find the function f, g, h such that $F = fogoh$.

SOLUTION: **(a)** $f(x) = \frac{x}{x+1}$; $g(x) = x^{10}$; $h(x) = x + 3$

$$(fogoh)(x) = f(g(h(x)))=\frac{g(h(x))}{g(h(x))+1}=\frac{(h(x))^{10}}{(h(x))^{10}+1}=\frac{(x+3)^{10}}{(x+3)^{10}+1}$$

$$\Rightarrow (fogoh)(x)=\frac{(x+3)^{10}}{(x+3)^{10}+1} \quad \therefore\ f(g(h(-1)))=\frac{(-1+3)^{10}}{(-1+3)^{10}+1}=\frac{2^{10}}{2^{10}+1}=\frac{1024}{1024+1}=\frac{1024}{1025}$$

(b) $F(x) = \cos^2(x + 9)$

$\therefore$ $F(x) = fogoh(x)$ $\Rightarrow$ $f(x) = x^2$; $g(x) = \cos x$; $h(x) = x + 9$

ILLUSTRATION 195: Let a and b be real numbers and let $f(x) = a \sin x + b\sqrt[3]{x}+4$, $\forall\ x \in \mathbb{R}$. If $f(\log_{10}(\log_3 10)) = 5$, then find the value of $f(\log_{10}(\log_{10} 3))$

SOLUTION: Given $f(x) = a\sin x+b\sqrt[3]{x}+4$ and $f\left(\log_{10}\left(\log_3 10\right)\right)= 5$

Now, let $\log_{10}\log_3 10 = k$; then $\log_{10}\log_{10}3$

$$= \log_{10}\left(\frac{1}{\log_3 10}\right)=(\log_{10}1-\log_{10}\log_3 10)=0-k=-k$$

Now, $\because f(k) = a \sin k + b(k)^{1/3} + 4 = 5 \Rightarrow a \sin k + b(k)^{1/3} = 1$

$\therefore$ $f(-k) = a \sin(-k) + b(-k)^{1/3} + 4 = -(a \sin k + b(k)^{1/3}) + 4 = -1 + 4 = 3.$

PROPERTIES OF COMPOSITION OF FUNCTIONS

(a) Domain: Given two functions $f : X \to Y$ and $g : Y \to Z$. For the composition gof to be defined from set X to set Z, the range of f must be a subset of the domain of g, i.e., $R_f \subseteq Y$.

If range of f is not a subset of domain of g but has some elements common with domain of g, then domain of gof is not the set X but is given by: Domain of $g(f(x)) = \{x : x \in D_f$ and $f(x) \in D_g\}$.

ILLUSTRATION 196: Let $f(x) = \dfrac{2x-3}{x-4}$ and $g(x) = ln(x-1)$, then find the domain of $fog(x)$.

SOLUTION: (i) $fog(x)$

To find the domain of $fog(x)$, we are to find those elements of domain of $g(x)$, whose image under g belongs to the domain of function f.

i.e., domain of $fog(x) = \{x \in D_g : g(x) \in D_f\}$

Now $g(x) = ln(x-1)$; $f(x) = \dfrac{2x-3}{x-4}$ and domain of $g(x)$ is $(1, \infty)$ and domain of $f(x)$ is $\mathbb{R} \sim \{4\}$

$\Rightarrow$ Domain of $fog(x) = \{x \in (1, \infty): ln(x-1) \in \mathbb{R} \sim \{4\}\}$

We know that range of $ln(x-1)$ for $x \in (1, \infty)$ is $(-\infty, \infty)$.

All these values of $\ln(x-1)$ are permissible except for 4.

$\Rightarrow$ $ln(x-1) \neq 4$ $\qquad \Rightarrow$ $(x-1) \neq e^4$

$\therefore$ Domain of $fog(x) = (1, \infty) \sim \{1 + e^4\}$.

Aliter: $fog(x) = f(g(x)) = \dfrac{2g(x)-3}{g(x)-4} = \dfrac{2ln(x-1)-3}{ln(x-1)-4}$

Which is defined for $ln(x-1) - 4 \neq 0$ and $x - 1 > 0$

$\Rightarrow$ $x - 1 \neq e^4$ and $x > 1$ $\qquad \Rightarrow$ $x > 1$ and $x \neq 1 + e^4$

$\therefore$ Domain of $fog(x)$ is given by $(1, \infty) \sim \{1 + e^4\}$.

ILLUSTRATION 197: If $f(x) = \dfrac{1}{1-x}$ Find $f[f\{f(x)\}]$ and draw it's graph and find its domain.

SOLUTION: $f(x) = \dfrac{1}{1-x}$. It is defined when $x \neq 1$. Now, $f\{f(x)\} = f\left(\dfrac{1}{1-x}\right)$

$$\Rightarrow \frac{1}{\left(1-\dfrac{1}{1-x}\right)} = \frac{1}{\left(\dfrac{1-x-1}{1-x}\right)} = \frac{1-x}{-x}\text{; when } x \neq 1\text{; also } x \neq 0.$$

$$= \frac{-1+x}{x}\text{; when } x \neq 0, 1$$

$$\therefore f[f\{f(x)\}] = \frac{1}{1-\left(\dfrac{x-1}{x}\right)} = \frac{x}{x-x+1} = x\text{; when } x \neq 0, 1$$

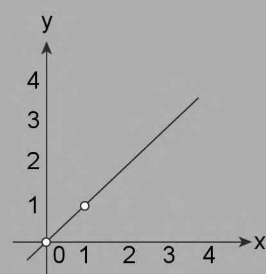


FIGURE 2.153

Therefore, domain of composite function is $\mathbb{R} \sim \{0, 1\}$.

(b) Range: Range of $gof(x) = \{g(b): b \in R_f \cap D_g\}$ and can be obtained as given below:

(i) If $f: A \to B$ and $g: B \to C$; then $gof: A \to C$ has range $= \{gof(a_1), gof(a_2), gof(a_3), \ldots, gof(a_n)\}$; where $A = \{a_1, a_2, a_3, \ldots, a_n\}$, i.e., a finite set.

ILLUSTRATION 198: Let $A = \{2, 3, 4, 5\}$; $B = \{3, 8, 15, 24, 30, 32\}$; $C = \{3, 4/3, 15/13, 12/11, 15/14, 16/15, 17/16\}$

Consider the functions $f: A \to B$ and $g: B \to C$ defined by $f(x) = x^2 - 1$ and $g(x) = \dfrac{x}{x-2}$; find the range of composite function gof.

SOLUTION: $f(A) = \{f(2), f(3), f(4), f(5)\} = \{3, 8, 15, 24\} \subseteq B$ and $g(B) = \{g(3), g(8), g(15), g(24), g(30), g(32)\} = \{3, 4/3, 15/13, 12/11, 15/14, 16/15\} \subseteq C$

Thus, range of f is a subset of domain of g $\Rightarrow$ gof is defined from set A to set C

i.e., A is domain of gof and its range is given by $\{gof(2), gof(3), gof(4), gof(5)\}$

$= \{g(3), g(8), g(15), g(24)\} = \{3, 4/3, 15/13, 12/11\}$

(ii) If $f(x)$ and $g(x)$ are increasing functions in their respective domain, then $gof(x)$ is also an increasing function in its domain. Further, if both $f(x)$ and $g(x)$ are continuous in their respective domain, then gof is also continuous in its domain.

Now if common domain of $f(x)$ and $gof(x)$ is $[\alpha, \beta]$ or (α, β), then range of $f(x)$ is $[f(\alpha), f(\beta)]$, or $(f(\alpha), f(\beta))$ which in turn is domain of $g(x)$. Then range of $fog(x)$ will be $[g(f(\alpha), g(f(\beta))]$ or $(g(f(\alpha), g(f(\beta)))$.

ILLUSTRATION 199: Find the domain and range of $\log(\tan^{-1}x)$.

SOLUTION: Given function is $h(x) = \log(\tan^{-1}x)$.

If we consider $f(x) = \log x$ and $g(x) = \tan^{-1}x$, then $fog(x) = f(g(x)) = \log(g(x)) = \log(\tan^{-1}x)$

Thus, $h(x) = fog(x)$, composition of $f(x)$ and $g(x)$.

Now domain of $f(x)$ is $(0, \infty)$ and $g(x)$ is $\mathbb{R}$.

For $fog(x)$ to be defined, range of $g(x)$ must be in the domain of $f(x)$, i.e., $x \in \mathbb{R}$ such that $g(x) \in (0, \infty)$

$\Rightarrow g(x) > 0$ $\qquad \Rightarrow \tan^{-1}x > 0$

$\Rightarrow \tan^{-1}x \in \left(0, \dfrac{\pi}{2}\right)$ $\qquad \Rightarrow x \in (0, \infty)$.

Thus, domain of $fog(x)$ is $(0, \infty)$

Now $f(x)$ and $g(x)$ both are increasing and continuous functions, hence, $fog(x)$ is also continuous and increasing function on $(0, \infty)$. Further range of $g(x)$ for $x \in (0, \infty)$ is $(g(0), g(\infty))$, i.e., $(\tan^{-1}0, \tan^{-1}\infty)$ i.e., $\left(0, \dfrac{\pi}{2}\right)$. Hence, range of $fog(x)$ will be $\left(f(0), f\left(\dfrac{\pi}{2}\right)\right)$

i.e., $\left(\log 0, \log\left(\dfrac{\pi}{2}\right)\right)$ i.e., $\left(-\infty, \log\left(\dfrac{\pi}{2}\right)\right)$.

(iii) If $f(x)$ and $g(x)$ both are decreasing functions in their respective domain, then gof is also a decreasing function. Further if both $f(x)$ and $g(x)$ are continuous in their respective domain, then gof is continuous and increasing function in its domain. If common domain of $f(x)$ and $gof(x)$ is $[\alpha, \beta]$ or (α, β), then range of $f(x)$ is $[f(\beta), f(\alpha)]$ or $(f(\beta), f(\alpha))$ which in turn in domain of $g(x)$ which is decreasing and continuous function. Thus, range of gof will be $[g(f(\alpha)), g(f(\beta))]$ or $(g(f(\alpha)), g(f(\beta)))$.

ILLUSTRATION 200: Find the domain and range of function $e^{-\cot^{-1}x}$.

SOLUTION: Given function is $e^{-\cot^{-1}x}$

If we consider $g(x) = e^{-x}$ and $f(x) = \cot^{-1}x$, then $gof(x) = g(f(x)) = e^{-f(x)} = e^{-\cot^{-1}x}$

Thus, $h(x) = gof(x)$

Now $f(x)$ and $g(x)$ both are continuous and decreasing functions on their domain $\mathbb{R}$. Now domain of $gof(x)$ would contain those real numbers for which range of $f(x)$ is contained in domain of $g(x)$, i.e., $\mathbb{R}$, which is true for all real values of x. Hence, domain of $gof(x)$ is $\mathbb{R}$.

Further $f(x)$ is continuous and decreasing on $\mathbb{R}$ implies range of $f(x)$ is $(f(\infty), f(-\infty))$, i.e., $(0, \pi)$.

Now in $(0, \pi)$, $g(x) = e^{-x}$ is decreasing implies $gof(x)$ being continuous and decreasing has its range $(g(\pi), g(0))$, i.e., $(e^{-\pi}, e^{-0}) = (e^{-\pi}, 1)$.

Thus, range of gof is $(e^{-\pi}, 1)$.

(iv) If $f(x)$ and $g(x)$ are functions of opposite monotonicity in their respective domain, then gof is a decreasing function on its domain. Further, if $f(x)$ and $g(x)$ are continuous functions, then gof is continuous and decreasing function. If $[\alpha, \beta]$ or (α, β) is a common domain of $gof(x)$ and decreasing function $f(x)$ (say), then range of $f(x)$ is $[f(\beta), f(\alpha)]$ or $(f(\beta), f(\alpha))$ which in turn in domain of $g(x)$. $g(x)$ being continuous and increasing (say), range of $gof(x)$ will be $[g(f(\beta)), g(f(\alpha))]$ or $(g(f(\beta)), g(f(\alpha)))$. Same will be the range of $gof(x)$ if $f(x)$ is increasing and $g(x)$ is decreasing.

ILLUSTRATION 201: Find the domain and range of function $\log(\cot^{-1}x)$

SOLUTION: Given function is $h(x) = \log(\cot^{-1}x)$. Let $g(x) = \log x$ and $f(x) = \cot^{-1}x$, then $g(f(x)) = \log(f(x)) = \log(\cot^{-1}x)$

Domain of $f(x)$ is $(-\infty, \infty)$ and its range is $(0, \pi)$ and domain of $g(x)$ is $(0, \infty)$ and its range is $\mathbb{R}$.

For $gof(x)$ to be defined, range of $f(x)$ must be in domain of $g(x)$.

i.e., $(0, \pi)$ must be in domain of $g(x)$, i.e., $(0, \infty)$, which is true for all real x.

Thus, domain of $gof(x)$ is $\mathbb{R}$. Domain of $f(x)$ is $(-\infty, \infty)$ in which $f(x)$ is continuous and decreasing, and hence, having range $(f(\infty), f(-\infty))$, i.e., $(\cot^{-1}(\infty)), \cot^{-1}(-\infty)$, i.e., $(0, \pi)$.

Now range of $f(x)$ is $(0, \pi)$ in which $g(x)$ is continuous and increasing, and hence, range of $fog(x)$ is $(g(0), g(\pi))$, i.e., $(\log 0, (\log \pi))$ or $(-\infty, \log \pi)$.

(v) If $f(x)$ is an increasing and continuous function in its domain and $g(x)$ is non-monotonic having range $[\alpha, \beta]$ or (α, β), then the range of $fog(x)$ will be $[f(\alpha), f(\beta)]$ or $(f(\alpha), f(\beta))$. Similarly if $f(x)$ is decreasing and continuous function in its domain and $g(x)$ is non-monotonic having range $[\alpha, \beta]$ or (α, β), then the range of $fog(x)$ will be $[f(\beta), f(\alpha)]$ or $(f(\beta), f(\alpha))$.

ILLUSTRATION 202: Define a function $f(x) = 1 + x$ and $g(x) = (\ln x)^2$, then find the domain and range of composite functions $fog(x)$.

SOLUTION: Domain of $f(x) = \mathbb{R}$; range of $f(x) = \mathbb{R}$. Domain of $g(x) = (0, \infty)$; range of $g(x) = [0, \infty)$

Now domain of $fog(x)$ contains those elements of domain of $g(x)$ whose images under g are found in domain of f, i.e., domain of $fog(x) = \{x \in (0, \infty) : g(x) \in \mathbb{R}\} = \{x \in (0, \infty) : (\ln x)^2 \in \mathbb{R}\} = D_g$. ($\because$ $[0, \infty) \subseteq \mathbb{R}$ is true $\forall\ x \in D_g = (0, \infty)$).

Thus, domain of $fog(x)$ is also $D_g = (0, \infty)$ and $fog(x) = f(g(x)) = 1 + (g(x)) = 1 + (\ln x)^2$.

Now $fog(x) : (0, \infty) \to \mathbb{R}$ is the composition of $f(x) = 1 + x$ and $g(x) = (\ln x)^2$

Now on $(0, \infty)$, $f(x)$ is increasing function and range of $g(x)$ is $[0, \infty)$

$\Rightarrow$ range of $fog(x)$ is $[f(0), f(\infty)] = [1, \infty)$

(vi) If $f(x)$ is non-monotonic function and continuous in its domain and $g(x)$ is any function (monotonic or non-monotonic) for which the composition function fog is defined, then range of fog can be obtained by analyzing the behaviour of function $f(x)$ on the range set of function $g(x)$. i.e., by determining the intervals of monotonicity, *l.u.b.*, *g.u.b.* of $f(x)$ in range set of $g(x)$.

ILLUSTRATION 203: Let $f(x) = \sin x$ and $g(x) = \cos^{-1}x$, then find the domain and range of $fog(x)$.

SOLUTION: $fog(x) = f(g(x)) = \sin(g(x)) = \sin(\cos^{-1}x)$

Now $\sin(\cos^{-1}x)$ is defined for every value of $\cos^{-1}x$ which in turn is defined for $x \in [-1, 1]$.

$\therefore$ Domain of $fog(x)$ is $[-1, 1]$

Further range of $g(x)$ is $[0, \pi]$ in which $f(x) = \sin x$ is non-monotonic but continuous.

If we break $[0, \pi]$ into two subintervals $\left[0, \frac{\pi}{2}\right]$ and $\left[\frac{\pi}{2}, \pi\right]$, in interval $\left[0, \frac{\pi}{2}\right]$ $\sin x$ increases and belongs to $[0, 1]$. However in interval $\left[\frac{\pi}{2}, \pi\right]$ $\sin x$ decreases from 1 to 0, and hence, $\sin x$ belongs to $[0, 1]$.

Thus, the range of fog is $[0, 1]$.

ILLUSTRATION 204: Function f and g are defined by $f(x) = \sin x$, $x \in \mathbb{R}$; $g(x) = \tan x$, $x \in \mathbb{R} \sim (2n+1)\frac{\pi}{2}$; $n \in \mathbb{Z}$.

Find the range of the function fog and gof.

SOLUTION: $f(x) = \sin x$; $g(x) = \tan x$.

$\Rightarrow$ $fog(x) = f(g(x)) = \sin(\tan x)$

$\Rightarrow$ Range of $fog(x) = [-1, 1]$ ($\because$ $\tan x \in \mathbb{R}$)

Now $gof(x) = g(f(x)) = \tan(\sin x)$; $-1 \le \sin x \le 1$ and $\tan x$ is continuous and increasing in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Rightarrow$ $\tan(-1) \le \tan(\sin x) \le \tan 1$ $\Rightarrow$ $-\tan 1 \le \tan(\sin x) \le \tan 1$

$\therefore$ Range of $gof(x) = [-\tan 1, \tan 1]$

(vii) If $f(x)$ is monotonic and continuous in its domain and $g(x)$ is non-monotonic for which $fog(x)$ is defined and range of $g(x)$ is $[\alpha, \beta]$ or (α, β), then the range of $fog(x)$ will be $[f(\alpha), f(\beta)]$ or $(f(\alpha), f(\beta))$ if $f(x)$ is increasing and it will be $[f(\beta), f(\alpha)]$ or $(f(\beta), f(\alpha))$ if $f(x)$ is decreasing.

ILLUSTRATION 205: Let $f(x) = \sin x$ and $g(x) = \cos^{-1}x$, then find the domain and range of $gof(x)$.

SOLUTION: $gof(x) = g(f(x)) = \cos^{-1}(f(x)) = \cos^{-1}(\sin x)$

Now $\cos^{-1}(\sin x)$ is defined for $\sin x \in [-1, 1]$ which in turn attains all these values $\forall\ x \in \mathbb{R}$.

So, domain of $\cos^{-1}(\sin x)$ is $\mathbb{R}$.

Range of $\sin x$ is $[-1, 1]$ and $\cos^{-1}(x)$ is decreasing for $x \in [-1, 1]$ and decreases from π to 0.

Thus, range of $gof(x)$ is $[0, \pi]$.

(viii) If $f(x)$ and $g(x)$ both are non-monotonic and continuous, for which $fog(x)$ is defined, then the range of $f(x)$ can be obtained by analyzing the behaviour of $f(x)$ on the range set of $g(x)$ i.e., by determining the intervals of monotonicity, *l.u.b.* and *g.l.b.* of $f(x)$ in the range set of $g(x)$.

ILLUSTRATION 206: If $f(x) = \sin x$ and $g(x) = 2x^2 + 7x + 3$, then determine the domain and range of $fog(x)$ and $gof(x)$.

SOLUTION: Here both $f(x)$ and $g(x)$ are non-monotonic functions. Now $fog(x) = \sin(2x^2 + 7x + 3)$.

Now $fog(x)$ is defined for all real values of $(2x^2 + 7x + 3)$ which in turn is defined for all real values of x.

Thus, domain of $fog(x)$ is $\mathbb{R}$.

Now range of $2x^2 + 7x + 3$ is given by $\left[-\dfrac{D}{4a}, \infty\right) = \left[-\dfrac{(49-24)}{4(2)}, \infty\right)$, i.e., $\left[\dfrac{-25}{8}, \infty\right)$.

Now $\sin x$ being continuous function, when fed to inputs $\left[\dfrac{-25}{8}, \infty\right)$, by intermediate value theorem, $\sin x$ would attain each and every real number from $f\left(\dfrac{-25}{8}\right)$ to $f(\infty)$.

As $x \to \infty$ $\sin x$ attains all real values of interval $[-1, 1]$

Hence, $fog(x) = \sin(2x^2 + 7x + 3)$ would attain each value of interval $[-1, 1]$

Hence, range of fog is $[-1, 1]$

Further $gof(x) = g(f(x)) = 2(f(x))^2 + 7f(x) + 3 = 2(\sin x)^2 + 7\sin x + 3$

Now, $gof(x)$ is defined for all real values of $\sin x$ which in turn is defined for all real values of x. Hence, domain of $gof(x)$ is $\mathbb{R}$

Next, $(gof)(x) = 2\sin^2 x + 7\sin x + 3$

$$= 2\left[\sin^2 x + \frac{7}{2}\sin x\right] + 3 = 2\left[\sin^2 x + \frac{7}{2}\sin x + \frac{49}{16}\right] + 3 - \frac{49}{8} = 2\left[\sin x + \frac{7}{4}\right]^2 - \frac{25}{8}$$

Now $-1 \le \sin x \le 1$

$\Rightarrow \dfrac{3}{4} \le \sin x + \dfrac{7}{4} \le \dfrac{11}{4}$ $\quad\Rightarrow \left(\sin x + \dfrac{7}{4}\right)^2 \in \left[\dfrac{9}{16}, \dfrac{121}{16}\right]$

$\Rightarrow 2\left(\sin x + \dfrac{7}{4}\right)^2 - \dfrac{25}{8} \in [-2, 12]$ $\quad\Rightarrow gof(x)$ has its range $[-2, 12]$

(c) $fog(x)$ is not necessarily equal to $gof(x)$. i.e., generally not commutative. However, equality holds when f and g are functions from set A to set A itself and each is inverse of other or at least one of them is an identity function on set A.

Proof: Let $f : A \to A$ is a bijective function, then $f^{-1} = g : A \to A$ is also a bijective function.

such that $f(x) = y \Leftrightarrow g(y) = x$

Clearly $f^{-1} = g$ and $g^{-1} = (f^{-1})^{-1} = f$, i.e., f and g are inverse of each other.

Now gof and fog both are composite function from set A to set A itself.

Now for any $x \in A$, $gof(x) = g(f(x)) = g(y)$ [where $y = f(x) \Rightarrow g(y) = x$] $= x$ .... (1)

and $fog(x) = f(g(x)) = f(y')$ [where $y' = g(x) \Rightarrow f(y') = x$] $= x$ .... (2)

From (1) and (2), we have $gof(x) = fog(x) = x \;\forall\; x \in A$

$\Rightarrow gof = fog = I_A$, i.e., identity function on set A.

Next let us suppose that $f, g : A \to A$ and $g = I_A$, i.e., an identity function on set A.

$\Rightarrow g(x) = x \;\forall\; x \in A$.

Now fog, i.e., composition of f and g is a function from set A to A itself, and for each $x \in A$, $fog(x) = f(g(x)) = f(x)$ and for each $x \in A$, $gof(x) = g(f(x)) = f(x)$

Thus, $fog(x) = gof(x) = f(x) \;\forall\; x \in A$.

$\Rightarrow fog = gof = f$

Similarly, we can show that $gof = fog = g$ if f is an identity function on set A, i.e., $f = I_A$.

ILLUSTRATION 207: If $f(x) = 1 + x$ and $g(x) = \ln x$, then find the domain and range of composite functions $fog(x)$ and $gof(x)$, and hence, show that $fog \neq gof$.

SOLUTION: $fog(x) = f(g(x)) = 1 + g(x) = 1 + \ln x$, which is defined $\forall\; x > 0$

$\therefore$ Domain of $fog(x)$ is $(0, \infty)$ and $(fog)'(x) = \dfrac{1}{x} > 0 \;\forall\; x \in (0, \infty)$

$\Rightarrow$ $(fog)(x)$ is an increasing function.

Also 1 and $\ln x$ are continuous and $fog(x)$ being sum of two continuous functions is also continuous on $(0, \infty)$.

Thus, $fog(x)$ will have its range $(fog(0), fog(\infty)) = (1+ \ln(0), 1+ \ln(\infty)) = (-\infty,\infty)$

Further, $gof(x) = g(f(x)) = \ln (f(x)) = \ln(1 + x)$ which is defined for $x > -1$

$\therefore$ Domain of gof is $(-1, \infty)$ and $(gof)'(x) = \dfrac{1}{(1+x)} > 0 \;\forall\; x \in (-1, \infty)$

$\Rightarrow$ $gof(x)$ is monotonically increasing on $(-1, \infty)$

$\therefore$ Range of $gof(x)$ will be $(gof(-1), gof(\infty)) = (-\infty, \infty)$

Thus, from above discussion, we observe that range of fog and gof are same, however, they have different domains. Thus, the two functions can't be equal.

Thus, $fog \neq gof$

Aliter: Let us suppose that fog and gof have common domain A.

Now $fog = gof$ iff $fog(x) = gof(x) \;\forall\; x \in A$

Let $x = 1$, clearly fog and gof are both defined at $x = 1 \Rightarrow 1 \in A$, then $fog(1) = 1 + \ln 1 = 1$

and $gof(1) = \ln (1 + 1) = \ln 2$

Now $\ln 2 = 1 \;\Rightarrow\; 2 = e$, which is a contradiction, so we get a counter example showing that $fog(x) \neq gof(x) \;\forall\; x \in A$.

ILLUSTRATION 208: Let $f(x) = x^2 + x + 1$ and $g(x) = \sin x$. Show that $fog \neq gof$

SOLUTION: Given $f(x) = x^2 + x + 1$; $g(x) = \sin x$

$\Rightarrow$ $fog(x) = f(g(x)) = \sin^2 x + \sin x + 1$ and $gof(x) = g(f(x)) = \sin (x^2 + x + 1)$

$\therefore$ $fog(x) \neq gof(x)$

(d) The composition of functions is associative in nature, i.e., if f, g, h are three functions such that $fo(goh)$ and $(fog)oh$ are defined, then $fo(goh) = (fog)\,oh$.

Proof: Let $h : A \to B$; $g : B \to C$ and $f : C \to D$, then

$\Rightarrow$ $goh : A \to C$ and $fo(goh) : A \to D$.

Also $fog : B \to D$ and $(fog)oh : A \to D$

Thus, $fo(goh)$ and $(fog)oh$ has common domain A.

Let $a \in A$

$\Rightarrow$ there exists $b \in B$, $c \in C$ and $d \in D$ such that $h(a) = b$, $g(b) = c$, $f(c) = d$,

Then $[fo(goh)](a) = f[(goh(a)] = f[g(h(a))] = f[g(b)] = f(c) = d$

and $[(fog)\ oh]\ (a) = (fog)\ (h(a)) = (fog)(b) = f[g(b)] = f(c) = d$

Thus, $[fo\ (goh)]\ (a) = [(fog)oh](a)\ \forall\ a \in A$

$\Rightarrow$ $fo(goh) = (fog)oh$

ILLUSTRATION 209: Given $f : \mathbb{N} \to I_0$ (set of non-zero integers); $f(x) = 2x$ and $g : I_0 \to \mathbb{Q}$ (set of rational numbers); $g(x) = \dfrac{1}{x}$ and $h : \mathbb{Q} \to \mathbb{R}$; $h(x) = e^{1/x}$. Show that $[(hog)of](x) = [ho(gof)](x) = e^{2x}\ \forall\ x \in \mathbb{N}$.

SOLUTION: Clearly $[(hog)of](x)$ and $[ho(gof)](x)$ will be the functions from $\mathbb{N} \to \mathbb{R}$

LHS. $= [(hog)of](x) = (hog)(f(x)) = (hog)(2x) = h(g(2x)) = h\left(\dfrac{1}{2x}\right) = e^{2x}$

RHS $= [ho(gof)](x) = ho[g(f(x))] = ho\left[\dfrac{1}{f(x)}\right] = h\left[\dfrac{1}{2x}\right] = e$

Thus, $[(hog)of](x) = [ho(gof)](x) = e^{2x}\ \forall\ x \in \mathbb{N}$.

ILLUSTRATION 210: Let $f(x) = x^2$, $g(x) = \sin x$, $h(x) = \sqrt{x}$, then verify that $[fo(goh)]\ (x)$ and $[(fog)\ oh]\ (x)$ are equal

SOLUTION: Given functions are $f(x) = x^2$; $g(x) = \sin x$; $h(x) = \sqrt{x}$

Now LHS $= [fo(goh)]\ (x) = fo(goh)(x) = f(g(h(x))) = f\left(\sin\sqrt{x}\right) = \left(\sin\sqrt{x}\right)^2 = \sin^2\sqrt{x}$

and RHS $= [(fog)\ oh]\ (x) = (fog)\ oh(x) = fog(\sqrt{x}) = \sin^2\sqrt{x}$

Thus, $[fo(goh)]\ (x) = [(fog)\ oh]\ (x) = \sin^2\sqrt{x}$

(e) The composition of two bijections is a bijection.

Proof: Let $f : A \to B$ and $g : B \to C$ be two bijections such that $gof : A \to C$ exists.

Injectivity: Let $a_1, a_2 \in A$ such that $(gof)(a_1) = (gof)(a_2)$ $\Rightarrow$ $g[f(a_1)] = g\,[f(a_2)]$

$\Rightarrow$ $f(a_1) = f(a_2)$ [$\because$ g is one–one]

$\Rightarrow$ $a_1 = a_2$ [$\because$ f is one–one]

$\therefore$ gof is also one–one function.

Surjectivity: Let $c \in C$,

$\Rightarrow$ There exists $b \in B$ such that $g(b) = c$ and $b \in B$ there exists $a \in A$ such that $f(a) = b$

[$\because$ g and f are onto functions]

Therefore, we can conclude that $\forall\ c \in C$ there exists $a \in A$: $(gof)\ (a) = g(f(a)) = g(b) = c$

$\Rightarrow$ Every element of C is the gof image of some element of A.

$\Rightarrow$ gof is onto function.

$\Rightarrow$ Thus, gof being one–one and onto, is a bijection.

ILLUSTRATION 211: Define a function $f(x) = \dfrac{2x}{x^2+1}$ on $[-1, 1]$ and $g(x) = \cos^{-1}x$, then show that the composite function $gof(x)$ is a bijective function.

SOLUTION: $f(x) = \dfrac{2x}{x^2+1}$; clearly domain of function $f(x) = D_f = \mathbb{R}$

$y = \frac{2x}{x^2+1}$ $\Rightarrow x^2y - 2x + y = 0$

For x to be real, Disc. ≥ 0

$\Rightarrow (-2)^2 - 4y^2 \geq 0 \Rightarrow 1 - y^2 \geq 0 \Rightarrow y^2 \leq 1 \Rightarrow y \in [-1, 1]$

Thus, range of function $f(x)$ is $[-1, 1]$. Further $f(-1) = -1$ and $f(1) = 1$

Since $f(x) = \frac{2x}{x^2+1}$ being a rational function is continuous. Therefore by intermediate value theorem function $f(x)$ will attain each and every real number from $f(-1)$ to $f(1)$, i.e., from -1 to 1.

Now we know that domain of $g(x) = \cos^{-1}x$ is $[-1, 1]$.

Thus, range of $f(x)$ = domain of $g(x)$.

Thus, $gof(x)$ is defined from set $[-1, 1]$ to $[0, \pi]$.

Further, $f'(x) = \frac{2(1-x^2)}{(x^2+1)^2} \geq 0$ for $x \in [-1, 1]$.

$\Rightarrow f(x)$ is strictly increasing on $[-1, 1]$.

$\Rightarrow f(x)$ is a bijective function from set $[-1, 1]$ to $[-1, 1]$.

Also $g'(x) = \frac{-1}{\sqrt{1-x^2}} < 0$ for $x \in [-1, 1]$.

$\Rightarrow g(x)$ is a bijective function from set $[-1, 1]$ to $[0, \pi]$. We shall show that $gof(x)$ is a bijective function from $[-1, 1]$ to $[0, \pi]$.

Let $gof(x_1) = gof(x_2)$ for some $x_1, x_2 \in [-1, 1]$.

$\Rightarrow g(f(x_1)) = g(f(x_2))$

$\Rightarrow \cos^{-1}\left(\frac{2x_1}{x_1^2+1}\right) = \cos^{-1}\left(\frac{2x_2}{x_2^2+1}\right); \frac{2x_1}{x_1^2+1}, \frac{2x_2}{x_2^2+1} \in [-1,1]$.

$\Rightarrow \frac{2x_1}{x_1^2+1} = \frac{2x_2}{x_2^2+1}$ as $\cos^{-1}(x)$ is a bijective function.

$\Rightarrow x_1 = x_2$ as $x_1, x_2 \in [-1, 1]$ and $f(x) = \frac{2x}{x^2+1}$ is a bijective function on $[-1, 1]$.

Thus, $gof(x_1) = gof(x_2)$ $\Rightarrow x_1 = x_2 \ \forall\ x_1, x_2 \in [-1, 1]$.

$\Rightarrow gof(x)$ is a bijective function from set $[-1, 1]$ to $[0, \pi]$.

(f) If gof is one–one, then f is one–one and g need not be one–one.

Proof: Let $gof(x) = gof(y)$, then $x = y$

($\because$ gof is one–one) ...(1)

$\Rightarrow$ Let $f(x) = f(y)$, then $g(f(x)) = g(f(y))$

($\because$ g is a function)

$\Rightarrow gof(x) = gof(y)$

$\Rightarrow x = y$ (by (1))

$\Rightarrow f$ is one–one, however g need not be one–one.

e.g., $y = e^{2x} = (e^x)^2$; let $g(x) = x^2$ and $f(x) = e^x$, then $gof(x) = g(e^x) = (e^x)^2 = e^{2x}$;

Clearly $gof(x)$ is monotonically increasing and hence, is one–one, also $f(x) = e^x$ is one–one but $g(x) = x^2$ is many–one.

(g) If gof is onto, then g is onto but f need not be onto.

Proof: Let $f: X \to Y$ and $g: Y \to Z$, then $gof: X \to Z$

Let $z \in Z$, then $x \in X$ such that $gof(x) = z$

$\Rightarrow g(f(x)) = z$. Let $f(x) = b \in Y$

$\Rightarrow g(b) = z$

$\Rightarrow g$ is onto, however f need not be onto.

e.g., Let $X = \{1, 2, 3, 4, 5\}$; $Y = \{-1, 1, 4, 9, 16, 25\}$ and $Z = \{0, 2\log 2, 2\log 3, 2\log 4, 2\log 5\}$;

Let $f: X \to Y$ and $g: Y \to Z$ defined as $f(x) = x^2$ and $g(x) = \log|x|$, then $gof: X \to Z$ is given by $gof(x) = 2\log|x|$ is onto. Also clearly $g(x)$ is onto but $f(x)$ is not onto as $-1 \in Y$ has no pre-image in X under f.

(h) If $f(x)$ and $g(x)$ are both continuous functions, then $g(f(x))$ is also continuous.

Proof: Let $f : X \to Y$ and $g : Y \to Z$ be two continuous functions.

Then $gof : X \to Z$ is composite function of f and g.

Let $a \in X$; such that $f(a) = b \in Y$, then $f(x)$ is continuous at $x = a$ and $g(x)$ is continuous at $x = b$

$\Rightarrow \lim\limits_{x\to a} f(x) = f(a)$ and $\lim\limits_{x\to b} g(x) = f(b)$ ....(1)

Now $\lim\limits_{x\to a} gof(x) = \lim\limits_{x\to a} g(f(x))$

$= \lim\limits_{y\to f(a)} g(y) = \lim\limits_{y\to b} g(y)$

$= g(b) = g(f(a))$

$= gof(a)$ $\left[\begin{array}{l} \because \lim\limits_{x\to a} f(x) = f(a) \\ \Rightarrow as\ x \to a; f(x) \to f(a) \\ \text{i.e. } y \to f(a) \end{array}\right]$

$\Rightarrow$ $gof(x)$ is continuous at $x = a$

$\therefore$ Composition of two continuous functions is also continuous.

ILLUSTRATION 212: Prove that

(i) $\sin(\tan^{-1}x)$ is continuous function on $\mathbb{R}$

(ii) $\ln\left(t + x^2\right) + e^{\cot^{-1}x}$ is a continuous function on $\mathbb{R}$ for each value of $t > 0$.

SOLUTION: (i) Let $f(x) = \sin x$ and $g(x) = \tan^{-1}x$, then $fog(x) = \sin(\tan^{-1}x)$ which is defined for each $x \in (-\infty, \infty)$

As for $x \in (-\infty, \infty)$, $\tan^{-1}x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ on which $\sin x$ is defined.

Thus, domain of $fog(x) = \sin(\tan^{-1}x)$ is $\mathbb{R}$

Also $\sin x$ and $\tan^{-1}x$ are continuous functions on $\mathbb{R}$.

Thus, their composition function $\sin(\tan^{-1}x)$ is also continuous function on $\mathbb{R}$.

(ii) Let $f(x) = \ln x$, $g(x) = t + x^2$; $h(x) = e^x$; $k(x) = \cot^{-1}x$, then $f(x)$ is continuous $\forall\ x > 0$.

Also $\forall\ x \in \mathbb{R}$ and $t > 0$, $g(x) > 0$

$\Rightarrow$ $f(g(x)) = \ln(t + x^2)$ is continuous on $\mathbb{R}$; Also e^x and $\cot^{-1}x$ are continuous on $\mathbb{R}$.

$\Rightarrow$ Their composition function $e^{\cot^{-1}x}$ is also continuous on $\mathbb{R}$.

Thus, $\ln(1 + x^2) + e^{\cot^{-1}x}$ being the sum of two continuous functions on $\mathbb{R}$ is also a continuous function on $\mathbb{R}$.

(i) Monotonicity of composite function: Composition of two functions having same monotonicity is a monotonically increasing function. i.e., if f is increasing ($\uparrow$) and g is increasing ($\uparrow$), then $f(g(x))$ is increasing ($\uparrow$) and if f is decreasing ($\downarrow$) and g is decreasing ($\downarrow$), then $f(g(x))$ is increasing ($\uparrow$).

Proof:* *Case 1: When f and g both are increasing functions

Let $x, y \in$ domain of fog such that $x > y$

$\Rightarrow$ $x, y \in$ domain of $g(x)$ and $x > y$

$\Rightarrow$ $g(x) > g(y)$ ($\because$ g is increasing)

$\Rightarrow$ $f(g(x)) > f(g(y))$ ($\because$ f is increasing)

$\Rightarrow$ $fog(x) > fog(y)$

$\Rightarrow$ fog is increasing function.

Case 2: When f and g both are decreasing functions

Let $x, y \in$ domain of fog such that $x > y$

$\Rightarrow$ $x, y \in$ domain of $g(x)$ and $x > y$

$\Rightarrow$ $g(x) < g(y)$ ($\because$ g is decreasing)

$\Rightarrow$ $f(g(x)) > f(g(y))$ ($\because$ f is decreasing)

$\Rightarrow$ $fog(x) > fog(y)$

$\Rightarrow$ fog is increasing function.

Aliter: $f(x)$, $g(x)$ increasing shows $f'(x) > 0$ and $g'(x) > 0$ and $\{fog(x)\}' = f'(g(x)).g'(x) > 0$

$\Rightarrow$ fog is also an increasing function

ILLUSTRATION 213: Show that the function $(\tan x)^{1/3}$ and $e^{(\tan x)^{1/3}}$ are monotonically increasing functions in their domains.

SOLUTION: Let $f(x) = (x)^{1/3}$ and $g(x) = \tan x$

$\Rightarrow$ $D_f = \mathbb{R}$ and $D_g = \mathbb{R} \sim \left\{(2n+1)\frac{\pi}{2}; n \in \mathbb{Z}\right\}$. Now, $f'(x) = \frac{1}{3(x)^{2/3}} > 0 \ \forall\, x \in \mathbb{R} \sim \{0\}$

$\Rightarrow$ $f(x)$ is monotonically increasing on $\mathbb{R}$ and $g'(x) = \sec^2 x > 0\ \forall\, x \in \mathbb{R} \sim \left\{(2n+1)\frac{\pi}{2}; n \in \mathbb{Z}\right\}$

$\Rightarrow$ $g(x)$ is monotonically increasing on $\mathbb{R} \sim \left\{(2n+1)\frac{\pi}{2}; n \in \mathbb{Z}\right\}$

Thus, $fog(x) = (\tan x)^{1/3}$ being the composition of two increasing (same monotonicity) functions is also monotonically increasing on its domain $\mathbb{R} \sim \left\{(2n+1)\frac{\pi}{2}; n \in \mathbb{Z}\right\}$.

Next, let $h(x) = e^x$ and $k(x) = (\tan x)^{1/3}$, then $hok(x) = e^{(\tan x)^{1/3}}$ being the composition of two monotonically increasing functions $h(x)$ and $k(x)$ on their respective domains $\mathbb{R}$ and $\mathbb{R} \sim \left\{(2n+1)\frac{\pi}{2}; n \in \mathbb{Z}\right\}$ is also continuous on its domain $\mathbb{R} \sim \left\{(2n+1)\frac{\pi}{2}; n \in \mathbb{Z}\right\}$.

(j) Composition of two functions having opposite monotonicity is a decreasing function.

Proof: Let $f(x)$ be increasing and $g(x)$ be decreasing functions, then $f'(x) > 0$ and $g'(x) < 0$.

Now $\{fog\,(x)\}' = f'(g(x)).g'(x) < 0$

$\Rightarrow$ fog is a decreasing function.

ILLUSTRATION 214: Show that the function $3(\log_{1/2}x)^3 + 4(\log_{1/2}x)^2 + 7(\log_{1/2}x) + 11$ is a monotonically decreasing function on its domain.

SOLUTION: Let $f(x) = 3x^3 + 4x^2 + 7x + 11$ and $g(x) = \log_{1/2}x$

$\Rightarrow$ Domain of $f(x) = \mathbb{R}$; and domain of $g(x) = (0, \infty)$

Now $fog(x) = f(g(x)) = 3(g(x))^2 + 4(g(x))^2 + 7(g(x)) + 11$

$= 3(\log_{1/2}x)^2 + 4(\log_{1/2}x)^2 + 7(\log_{1/2}x) + 11$ and domain of $fog(x)$ is $(0, \infty)$.

Further $f'(x) = 9x^2 + 8x + 7$

Disc. $f'(x) = 64 - 252 = -188 < 0$ $\Rightarrow$ $f'(x) > 0\ \forall\, x \in \mathbb{R}$

$\Rightarrow$ $f(x)$ is monotonically increasing on $\mathbb{R}$.

Also we know that $\log_{1/2}x$ is a decreasing function on $(0, \infty)$, thus, $g(x)$ is monotonically decreasing function on $(0, \infty)$.

Thus, $fog(x)$ being the composition of two monotonic functions of opposite monotonicity is decreasing on its domain $(0, \infty)$.

ILLUSTRATION 215: If $f(x) = \ln(x^2 - x + 2) : \mathbb{R}^+ \to \mathbb{R}$ and $g(x) = \{x\} + 1 : [1, 2] \to [1, 2]$; where $\{x\}$ denotes fractional part of x. Find the domain and range of $f(g(x))$ when defined.

SOLUTION: Given $f(x) = \ln(x^2 - x + 2) : \mathbb{R}^+ \to \mathbb{R}$ and $g(x) = \{x\} + 1 : [1, 2] \to [1, 2]$

$\Rightarrow$ $f(g(x)) = \ln(\{x\}^2 + \{x\} + 2)$

$\Rightarrow$ Domain of $fog(x)$ contains those elements of $[1, 2]$ for which $\{x\}^2 + \{x\} + 2 > 0$ which holds $\forall\, x \in [1, 2]$ as Disc. of quadratic is negative or each term is non-negative.

$\Rightarrow$ Domain of $fog(x) = [1, 2]$, for $x \in (1, 2)$; $\{x\} = x - 1$

$\Rightarrow$ $fog(x) = \ln((x-1)^2 + (x-1) + 2) = \ln[x^2 - x + 2]$

Now since $x^2 - x + 2$ is continuous and increasing on $(1, 2)$

$\Rightarrow$ Range of $fog(x) = (\ln 2, \ln 4)$ for $x \in (1, 2)$ and for $x = 1$ or 2, $\{x\} = 0$

$\Rightarrow$ $fog(x) = \ln 2$ $\Rightarrow$ Range of $fog(x) = [\ln 2, \ln 4)$

COMPOSITION OF NON-UNIFORMLY DEFINED FUNCTIONS

Piecewise or non-uniformly defined functions are those functions whose domain is divided into two or more than two parts so that the function has different analytical formulae in different parts of its domain. We can say that a piecewise defined function is composed of branches of two or more functions.

$$\text{e.g., } f(x) = \begin{cases} 3x+1; & -\infty < x \le -\dfrac{1}{3} \\ \sin 3\pi x; & -\dfrac{1}{3} < x \le 0 \\ \tan x; & 0 < x \le \dfrac{\pi}{4} \\ x; & x > \dfrac{\pi}{4} \end{cases}$$

The domain of function $f(x)$, i.e., $(-\infty, \infty)$ is divided into four parts, i.e., $\left(-\infty, -\dfrac{1}{3}\right]$, $\left(-\dfrac{1}{3}, 0\right]$, $\left(0, \dfrac{\pi}{4}\right]$ and $\left(\dfrac{\pi}{4}, \infty\right)$. Graphically it is shown in Figure 2.154.

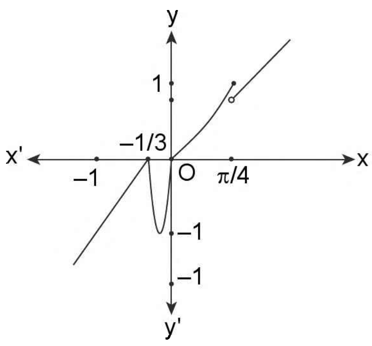


FIGURE 2.154

Now let us discuss the method to find the composition of two non-uniformly defined functions with the help of following example. Consider the functions as defined below

$$f(x) = \begin{cases} 2x-1; & 0 \le x < 2 \\ x^2+1; & 2 \le x \le 4 \end{cases} \;\&\; g(x) = \begin{cases} x+1; & -1 \le x < 1 \\ 2x; & 1 \le x \le 3 \end{cases}$$

Let us find the composite function $fog(x)$. The following steps are involved.

Step 1: Replace $g(x)$ in the place of x in the definition of $f(x)$.

$$\text{i.e., } f(g(x)) = \begin{cases} 2g(x)-1; & 0 \le g(x) < 2 \\ g(x))^2+1; & 2 \le g(x) \le 4 \end{cases}$$

Step 2: Apply the definition of $g(x)$ in the above step (1).

$$\Rightarrow f(g(x)) = \begin{cases} 2(x+1)-1; & -1 \le x < 1 & \& \; 0 \le x+1 < 2 \\ 2(2x)-1; & 1 \le x \le 3 & \& \; 0 \le 2x < 2 \\ (x+1)^2+1; & -1 \le x < 1 & \& \; 2 \le x+1 \le 4 \\ (2x)^2+1; & 1 \le x \le 3 & \& \; 2 \le 2x \le 4 \end{cases}$$

Step 3: Take the intersection of domains and find the final definition.

$$\text{i.e., } f(g(x)) = \begin{cases} 2(x+1)-1; & -1 \le x < 1 \\ 2(2x)-1; & x \in \{\,\} \\ (x+1)^2+1; & x \in \{\,\} \\ (2x)^2+1; & 1 \le x \le 2 \end{cases}$$

$$\Rightarrow \quad fog(x) = \begin{cases} 2x+1; & -1 \le x < 1 \\ 4x^2+1; & 1 \le x \le 2 \end{cases}$$

Thus, the domain of composite function $fog(x)$ is $[-1, 2]$ and graphically as shown in Figure 2.155.

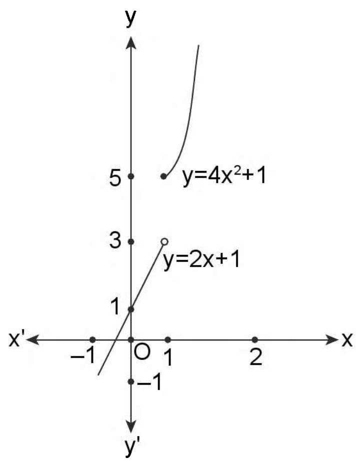


FIGURE 2.155

ILLUSTRATION 216: If $f(x) = -1 + |x - 2|$; $0 \le x \le 4$ and $g(x) = 2 - |x|$; $-1 \le x \le 3$; then find $fog(x)$ and $gof(x)$. Draw rough sketches of the graphs of $fog(x)$ and $gof(x)$.

SOLUTION: Given $f(x) = \begin{cases} -1-(x-2); & 0 \le x < 2 \\ -1+(x-2); & 2 \le x \le 4 \end{cases}$

$$\Rightarrow\ f(x) = \begin{cases} 1-x\,; & 0 \le x < 2 \\ x-3; & 2 \le x \le 4 \end{cases} \text{ and } g(x) = \begin{cases} 2+x\,; & -1 \le x < 0 \\ 2-x; & 0 \le x \le 3 \end{cases}$$

$$fog\,(x) = f(g(x)) = \begin{cases} 1-g(x)\,; & 0 \le g(x) < 2 \\ g(x)-3; & 2 \le g(x) \le 4 \end{cases}$$

$$= \begin{cases} 1-(2+x); & 0 \le 2+x < 2; -1 \le x < 0 \\ 1-(2-x); & 0 \le 2-x < 2;\ 0 \le x \le 3 \\ (2+x)-3; & 2 \le 2+x \le 4; -1 \le x < 0 \\ (2-x)-3; & 2 \le 2-x \le 4;\ 0 \le x \le 3 \end{cases} = \begin{cases} -1-x; & -2 \le x < 0; -1 \le x < 0 \\ x-1; & 0 < x \le 2;\ 0 \le x \le 3 \\ x-1; & 0 \le x \le 2; -1 \le x < 0 \\ -1-x; & -2 \le x \le 0;\ 0 \le x \le 3 \end{cases}$$

$$f(g(x)) = \begin{cases} -1-x; & -1 \le x < 0 \\ x-1; & 0 \le x \le 2 \end{cases}$$; graphically shown in Figure.

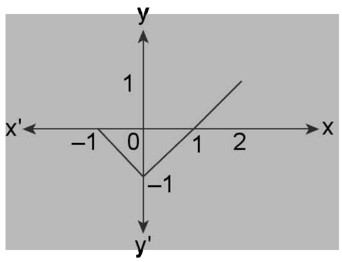


FIGURE 2.156

$$\text{Next } (gof)\,(x) = g(f(x)) = \begin{cases} 2+f(x)\,; & -1 \le f(x) \le 0 \\ 2-f(x); & 0 \le f(x) \le 3 \end{cases}$$

$$= \begin{cases} 2-(x-3); & 0 \le (x-3) \le 3;\ 2 \le x \le 4 \\ 2-(1-x); & 0 \le (1-x) \le 3;\ 0 \le x < 2 \\ 2+(x-3); & -1 \le (x-3) \le 0;\ 2 \le x \le 4 \\ 2+(1-x); & -1 \le (1-x) \le 0;\ 0 \le x < 2 \end{cases}$$

$$= \begin{cases} 5-x; & 3 \le x \le 6;\ 2 \le x < 4 \\ 1+x; & -2 \le x \le 1;\ 0 \le x < 2 \\ x-1; & 2 \le x \le 3;\ 2 \le x \le 4 \\ 3-x; & 1 \le x \le 2;\ 0 \le x < 2 \end{cases}$$

$$(gof)\,(x) = \begin{cases} 1+x; & 0 \le x < 1 \\ 3-x; & 1 \le x < 2 \\ x-1; & 2 \le x < 3 \\ 5-x; & 3 \le x \le 4 \end{cases}$$; graphically shown in Figure.

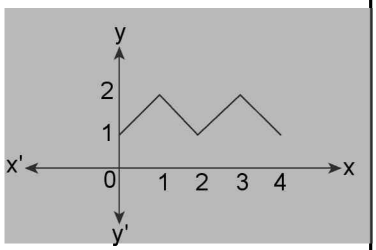


FIGURE 2.157

ILLUSTRATION 217: Composition of piecewise defined functions:

If $f(x) = |\,|x - 3| - 2\,|$; $0 \le x \le 4$

$g(x) = 4 - |2 - x|$; $-1 \le x \le 3$, then find $fog(x)$ and draw rough sketch of $fog(x)$.

SOLUTION: $f(x) = |\,|x - 3| - 2|$; $0 \le x \le 4$

$$\Rightarrow\ f(x) = \begin{cases} |x-1|; & 0 \le x < 3 \\ |x-5|; & 3 \le x \le 4 \end{cases} = \begin{cases} 1-x; & 0 \le x < 1 \\ x-1; & 1 \le x < 3 \\ 5-x; & 3 \le x \le 4 \end{cases}$$

And $g(x) = 4 - |2 - x|$; $-1 \le x \le 3$

$$= \begin{cases} 4-(2-x); & -1 \le x < 2 \\ 4-(x-2); & 2 \le x \le 3 \end{cases} = \begin{cases} 2+x; & -1 \le x < 2 \\ 6-x; & 2 \le x \le 3 \end{cases}$$

$$\therefore \quad fog(x) = \begin{cases} 1-g(x); \ 0 \le g(x) < 1 \\ g(x)-1; \ 1 \le g(x) < 3 \\ 5-g(x); \ 3 \le g(x) \le 4 \end{cases} = \begin{cases} 1-(2+x); & 0 \le 2+x < 1 & \text{and} & -1 \le x < 2 \\ 2+x-1; & 1 \le 2+x < 3 & \text{and} & -1 \le x < 2 \\ 5-(2+x); & 3 \le 2+x \le 4 & \text{and} & -1 \le x < 2 \\ 1-6+x; & 0 \le 6-x < 1 & \text{and} & 2 \le x \le 3 \\ 6-x-1; & 1 \le 6-x < 3 & \text{and} & 2 \le x \le 3 \\ 5-6+x; & 3 \le 6-x \le 4 & \text{and} & 2 \le x \le 3 \end{cases}$$

$$= \begin{cases} -1-x; & -2 \le x < -1 & \text{and} & -1 \le x < 2 \\ 1+x; & -1 \le x < 1 & \text{and} & -1 \le x < 2 \\ 3-x; & 1 \le x \le 2 & \text{and} & -1 \le x < 2 \\ x-5; & -6 \le -x < -5 & \text{and} & 2 \le x \le 3 \\ 5-x; & -5 \le -x < -3 & \text{and} & 2 \le x \le 3 \\ x-1; & -3 \le -x \le -2 & \text{and} & 2 \le x \le 3 \end{cases} = \begin{cases} -1-x; & -2 \le x < -1 & \text{and} & -1 \le x < 2 \\ 1+x; & -1 \le x < 1 & \text{and} & -1 \le x < 2 \\ 3-x; & 1 \le x \le 2 & \text{and} & -1 \le x < 2 \\ x-5; & 5 < x \le 6 & \text{and} & 2 \le x \le 3 \\ 5-x; & 3 < x \le 5 & \text{and} & 2 \le x \le 3 \\ x-1; & 2 \le x \le 3 & \text{and} & 2 \le x \le 3 \end{cases}$$

$$= \begin{cases} 1+x; & -1 \le x < 1 \\ 3-x; & 1 \le x < 2 \\ x-1; & 2 \le x \le 3 \end{cases}$$

Alternate method for finding fog: $g(x) = \begin{cases} 2+x; & -1 \le x < 2 \\ 6-x; & 2 \le x \le 3 \end{cases}$

Graph of $g(x)$ is

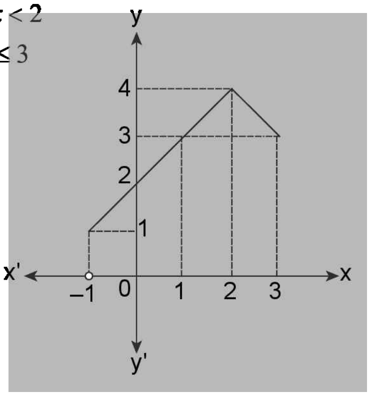


FIGURE 2.158

$$fog(x) = \begin{cases} 1-g(x); \ 0 \le g(x) < 1 \\ g(x)-1; \ 1 \le g(x) < 3 \\ 5-g(x); \ 3 \le g(x) \le 4 \end{cases}$$

From the graph of $g(x)$, we note that no portion of graph of $g(x)$ lies in the interval [0, 1).

Also $g(x) \in [1, 3)$ for $x \in [-1, 1)$ and $g(x) \in [3, 4]$ for $x \in [1, 3]$

$$\therefore \quad fog(x) = \begin{cases} 1-g(x); & \text{for no value} \\ g(x)-1; & -1 \le x < 1 \\ 5-g(x); & 1 \le x \le 3 \end{cases} \quad \ldots. (1)$$

Now for $x \in [-1, 1)$, $g(x) = (2 + x)$; for $x \in [1, 2)$, $g(x) = (2 + x)$ and for $x \in [2, 3]$, $g(x) = (6 - x)$

Therefore from (1), we have

$$fog(x) = \begin{cases} 2+x-1; & -1 \le x < 1 \\ 5-(2+x); & 1 \le x < 2 \\ 5-(6-x); & 2 \le x \le 3 \end{cases}$$

$$= \begin{cases} x+1; & -1 \le x < 1 \\ 3-x; & 1 \le x < 2 \\ x-1; & 2 \le x \le 3 \end{cases}$$; graphically shown in Figure.

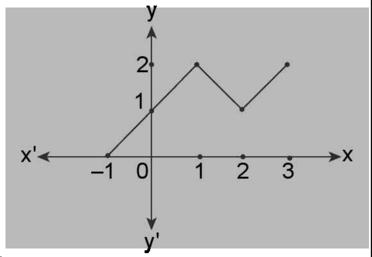


FIGURE 2.159

TEXTUAL EXERCISE–13: (SUBJECTIVE)

1. Let f, g be real valued functions defined as:

$$f(x) = \begin{cases} 7x^2+x-8; & x \le 1 \\ 4x+5; & 1 < x \le 7 \\ 8x+3; & x > 7 \end{cases} \text{ and}$$

$$g(x) = \begin{cases} |x|; & x < -3 \\ 0; & -3 \le x < 2 \\ x^2+4; & x \ge 2 \end{cases}$$

Find the values of composite functions at given points $(fog)(-3)$, $(fog)(7)$, $(fog)(9)$, $(gof)(2)$, $(gof)(0)$, $(gof)(6)$.

2. Let $f(x) = \dfrac{ax}{x+1}$ $x \ne -1$. If $(fof)(x) = x$, find the value of a.

3. The function f is defined on the interval $[0, 1)$, then find the domain of definition of the following functions; where [ ] denotes the greatest integer function.
   (i) $f(|[x-2.5]|)$
   (ii) $f(\cos x)$

4. Define $fog(x)$ and $gof(x)$. Also find their domain and range.
   (i) $f(x) = [x]$, $g(x) = \sin x$
   (ii) $f(x) = \tan x$, $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$; $g(x) = \sqrt{1-x^2}$

5. Find the composite function $fof(x)$ for the following functions
   (a) $f(x) = \begin{cases} 1+x; & 0 \le x \le 2 \\ 3-x; & 2 < x \le 3 \end{cases}$
   (b) $f(x) = -1 + |x-2|$; $0 \le x \le 4$
   (c) $f(x) = 2 - |x|$; $-1 \le x \le 3$
   (d) $f(x) = \begin{cases} x+1; & x \le 1 \\ 5-x^2; & x > 1 \end{cases}$
   (e) $f(x) = \begin{cases} -x; & x < 0 \\ x; & 0 \le x \le 1 \\ 2-x; & x > 1 \end{cases}$.

6. Find composition function $fog(x)$ for the following functions
   (a) $f(x) = \begin{bmatrix} 1+x^2 & x \le 1 \\ x+1 & 1 < x \le 2 \end{bmatrix}$ and $g(x) = 1 - x$; $-2 \le x \le 1$
   (b) $f(x) = \begin{cases} 1+x; & x \le 1 \\ 2x+2; & 1 < x \le 2 \end{cases}$ and
   $g(x) = \begin{cases} x^2; & -1 \le x < 2 \\ x+2; & 2 \le x \le 3 \end{cases}$

7. Find the composite functions $fog(x)$ and $gof(x)$ for the following functions:
   (a) $f(x) = -1 + |x-2|$, $0 \le x \le 4$; $g(x) = 2 - |x|$, $-1 \le x \le 3$
   Draw rough sketch of the graphs of $fog(x)$ and $gof(x)$.
   (b) $f(x) = \begin{cases} x+1; & x \le 1 \\ 2x+1; & 1 < x \le 2 \end{cases}$ and
   $g(x) = \begin{cases} x^2; & -1 < x \le 2 \\ x+2; & 2 < x \le 3 \end{cases}$

8. Give two function $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ such that gof is onto but f is not onto.

9. Show that if for a map $f : X \to Y$, there exists a map $g : Y \to X$ such that $(fog)(y) = y$, for every $y \in Y$, then f is onto.

10. Show that if for $f : X \to Y$, there exists a map $g : Y \to X$ such that $(gof)(x) = x$ for all $x \in X$, then f is one–one.

11. A function is defined as $f : D \to \mathbb{R}$, $f(x) = \cot^{-1}(sgn\ x) + \sin^{-1}(x - \{x\})$ where $\{x\}$ denotes the fractional part function. Find the largest domain and range of the function. State with reasons whether the function is injective or not. Also draw the graph of the function.

12. Given two functions $f(x)$ and $g(x)$ defined as below $f(x) = \begin{cases} x+2; & x \le 1 \\ 2x+1; & 1 < x \le 3 \end{cases}$ and
   $g(x) = \begin{cases} x^2+1; & -1 \le x \le 4 \\ x-1; & 4 < x \le 6 \end{cases}$. Find composite functions $f(g(x))$ and $g(f(x))$ and their domain and range and also find the number of solutions of equations
   (a) $f(g(x)) = 5.01$ (b) $f(g(x)) = 5$
   (c) $f(g(x)) = 1$

13. Let $f(x) = \begin{cases} x+a; & \text{if } x < 0 \\ |x-1|; & \text{if } x \ge 0 \end{cases}$ and
   $g(x) = \begin{cases} x+1; & \text{if } x < 0 \\ (x-1)^2+b; & \text{if } x \ge 0 \end{cases}$; where a and b are

non-negative real numbers. Determine the composite function *gof*. If (*gof*) (x) is continuous for all real x, determine the values of a and b. Further for these values of a and b, is *gof* differentiable at x =0? Justify your answer.

14. Let f be increasing and g be decreasing functions from $[0, \infty)$ to $[0, \infty)$. Let $h(x) = (fog)(x)$ and $h(0) = 0$. Prove that $h(x) = 0 \ \forall \ x \in [0, \infty)$.

15. If $f : [0, 1] \to [1, 2]$, defined by $f(x) = 1 + x$ and $g : [1, 2] \to [0,1]$ defined by $g(x) = 2 - x$, determine *gof*.

16. Find domain and range of the following functions:

(i) $f(x) = \cos^{-1}\sqrt{\log_{[x]}\frac{|x|}{x}}$, where [.] denotes the greatest integer function

(ii) $f(x) = \sqrt{\ell n(\cos(\sin x))}$

Answer Keys

1. $(fog)(-3) = -8$, $(fog)(7) = 427$, $(fog)(9) = 683$, $(gof)(2) = 173$, $(gof)(0) = 8$, $(gof)(6) = 845$.

2. -1 **3.** (i) $\left[\frac{5}{2}, \frac{7}{2}\right)$ (ii) $\bigcup_{n\in\mathbb{Z}}\left[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right] \sim \{2n\pi\}$

4. (i) *gof* = sin [x]; domain : $\mathbb{R}$; range $\{\sin a : a \in \mathrm{I}\}$ *fog* = [sin x]; domain : $\mathbb{R}$; range : $\{-1, 0, 1\}$

(ii) $gof = \sqrt{1-\tan^2 x}$; domain $= \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$; range : [0, 1] $fog = \tan\sqrt{1-x^2}$; domain : $[-1, 1]$; range [0, tan 1]

5. (a) $\begin{cases} x+2 & 0 \le x \le 1 \\ 2-x & 1 < x \le 2 \\ 4-x & 2 < x \le 3 \end{cases}$ (b) $\begin{cases} x & ,0 \le x \le 2 \\ 4-x & ,2 < x \le 2 \end{cases}$

(c) $\begin{cases} -x & ,-1 \le x \le 0 \\ x & ,0 < x \le 2 \\ 4-x & ,2 < x \le 3 \end{cases}$ (d) $\begin{cases} x+2; & x \le 0 \\ 5-(x+1)^2; & 0 < x \le 1 \\ 5-(5-x^2)^2; & 1 < x < 2 \\ (5-x^2)+1; & x \ge 2 \end{cases}$

(e) $\begin{cases} 2+x; & x < -1 \\ -x; & -1 \le x \le 0 \\ x; & 0 \le x \le 1 \\ 2-x; & 1 < x \le 2 \\ -(2-x); & x > 2 \end{cases}$

6. (a) $\begin{cases} 2-2x+x^2; & x \le 1 \\ x+1; & -1 \le x < 0 \end{cases}$ (b) $\begin{cases} 1+x^2; & -1 \le x \le 1 \\ 2x^2+2; & 1 < x \le \sqrt{2} \end{cases}$

7. (a) $fog(x) = \begin{cases} -(1+x) & ,-1 \le x \le 0 \\ x-1 & ,0 < x \le 2 \end{cases}; gof(x) = \begin{cases} x+1 & ,0 \le x < 1 \\ 3-x & ,1 \le x \le 2 \\ x-1 & ,2 < x \le 3 \\ 5-x & ,3 < x \le 4 \end{cases}$

(b) $fog(x) = \begin{cases} x^2+1; & -1 \le x < 1 \\ 2x^2+1; & 1 < x \le \sqrt{2} \end{cases}$; $gof(x) = \{(x+1)^2, -2 \le x \le 1\}$.

8. $f(x) = 2x + 4$; $g(x) = \frac{x-4}{2}$; $f(x) = x^2 + 3$; $g(x) = \sqrt{x-3}$

11. $D = [-1, 2)$; $R = \left\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\right\}$; the function is neither injective nor surjective.

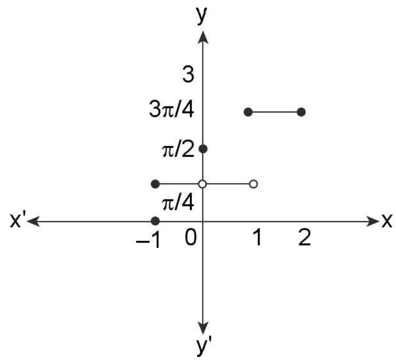


FIGURE 2.160

12. $f(g(x)) = \begin{cases} x^2+3; & x=0 \\ 2x^2+3; & -1 \le x < 0 \text{ or } 0 < x \le \sqrt{2} \end{cases}$; domain of $f(g(x)) = \left[-1, \sqrt{2}\right]$ and range [3, 7].

$gof(x) = \begin{cases} x^2+4x+5; & -3 \le x \le 1 \\ 4x^2+4x+2; & 1 < x \le 3/2 \\ 2x; & 3/2 < x \le 5/2 \end{cases}$; domain of $gof(x) = \left[-3, \frac{5}{2}\right]$ and range [1, 17]

(a) 1 solution (b) 2 solution (c) no solution

13. $gof(x) = \begin{cases} x+a+1; & x<-a \\ (x+a-1)^2+b; & -a \le x < 0 \\ x^2+b; & 0 \le x < 1 \\ (x-2)^2+b; & x \ge 1 \end{cases}$; $a = 1$ and $b = 0$; yes.

15. $(gof)(x) = 1 - x$, $0 \le x \le 1$

16. (i) $D : [2, \infty)$; $R : \{\pi/2\}$ (ii) $D : n\pi$, $n \in I$; $R : \{0\}$

TEXTUAL EXERCISE–13: (OBJECTIVE)

1. If f and g are bijective functions and gof is defined then, gof must be

(a) injective (b) surjective
(c) bijective (d) into only

2. Number of functions which can be defined from $R \to R$ for which the composite function $f(f(x))$ is an identity function for every $x \in \mathbb{R}$, is

(a) zero
(b) exactly one
(c) exactly two
(d) infinite

3. Let $f(x) = \frac{1}{x}$ & $g(x) = \frac{1}{\sqrt{x}}$

(a) $f(g(x))$ and $g(f(x))$ have different domains
(b) $f(g(x))$ and $g(f(x))$ have the same range
(c) $f(g(x))$ is a bijective mapping
(d) $g(f(x))$ is neither odd nor even

4. Let f and g be two functions both being defined from $R \to R$ as follows: $f(x) = \frac{x+|x|}{2}$ and $g(x) = \begin{cases} x \text{ for } x<0 \\ x^2 \text{ for } x \ge 0 \end{cases}$. Then

(a) fog is defined but gof is not
(b) gof is defined but fog is not
(c) both gof and fog are defined but they are unequal
(d) both gof and fog are defined and they are equal.

5. Let $f(x) = \sin x$ and $g(x) = |\ell n\, x|$ if composite functions $fog(x)$ and $gof(x)$ are defined and have ranges R_1 and R_2, respectively, then

(a) $R_1 = \{u : -1 < u < 1\}; R_2 = \{v : 0 < v < \infty\}$
(b) $R_1 = \{u : -\infty < u < 0\}; R_2 = \{v : -1 < v < 1\}$
(c) $R_1 = \{u : 0 < u < \infty\}; R_2 = \{v : -1 < v < 1; v \neq 0\}$
(d) $R_1 = \{u : -1 < u < 1\}; R_2 = \{v : 0 < v < \infty\}$

6. Let $f : [0, 1] \to [1, 2]$ defined as $f(x) = 1 + x$ and $g : [1, 2] \to [0, 1]$ defined as $g(x) = 2 - x$, then the composite function gof is
(a) Injective as well as surjective
(b) Surjective but not injective
(c) Injective but not surjective
(d) Neither injective nor surjective

7. If $f(x) = -1 + |x - 1|$, $-1 \leq x \leq 3$ and $g(x) = 2 - |x + 1|$, $-2 \leq x \leq 2$ then find $(fog)(3/2)$ equals
(a) 0
(b) 1
(c) 1/2
(d) None of these

8. On the interval [0, 1], $f(x)$ is defined as, $f(x) = \begin{bmatrix} x & \text{if } x \in Q \\ 1-x & \text{if } x \notin Q \end{bmatrix}$. Then for all $x \in R$ the composite function $f(f(x))$ is
(a) a constant function
(b) an identity function
(c) an odd linear polynomial
(d) $1 + x$

9. Given the graphs of the two functions, $y = f(x)$ and $y = g(x)$. In the adjacent figure from point A on the graph of the function $y = f(x)$ corresponding to the given value of the independent variable (say x_0), a straight line is drawn parallel to the x-axis to intersect the bisector of the first and the third quadrants at point B. From the point B a straight line parallel to the y-axis is drawn to intersect the graph of the function $y = g(x)$ at C. Again a straight line is drawn from the point C parallel to the x-axis, to intersect the line NN' at D. If the straight line NN' is parallel to y-axis, then the co-ordinates of the point D are

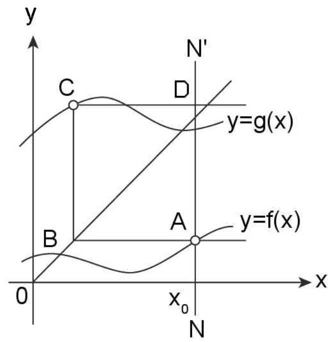


FIGURE 2.161

(a) $f(x_0), g(f(x_0))$
(b) $x_0, g(x_0)$
(c) $x_0, g(f(x_0))$
(d) $f(x_0), f(g(x_0))$

10. If $f(x) = x^2 + 2x + 1$ and $g[f(x)] = |x + 1|$ then the function $g(x)$ is
(a) $g(x) = -x\sqrt{x}$
(b) $g(x) = x\sqrt{x}$
(c) $g(x) = x^{5/2}$
(d) None of these

11. If $f(x) = x^2 + 2x + 1$ and $g[f(x)] = |x + 1|$, then the function $g(x)$ is
(a) $x\sqrt{x}$
(b) $\sqrt{x}$
(c) $\sqrt[3]{x}$
(d) None of these

12. If $f(x) = (ax^3 + b)^5$, such that $f(g(x)) = g(f(x))$, then
(a) $f(g(x)) = x$
(b) $g(x) = \left(\dfrac{x^{1/5} - b}{a}\right)^{1/3}$
(c) $f(g(x) = 2x$
(d) None of these

13. If $f(x) = \sin^2 x$, $g(x) = \sqrt{x}$ x and $h(x) = \cos^{-1} x$, $0 \leq x \leq 1$, then:
(a) $hogof(x) = gofoh(x)$
(b) $gofoh(x) = fohog(x)$
(c) $fohog(x) = hogof(x)$
(d) None of these

Answer Keys

1. (a) **2.** (d) **3.** (b,c,d) **4.** (d) **5.** (d) **6.** (a) **7.** (c) **8.** (b,d) **9.** (c) **10.** (b)
11. (b) **12.** (a,b) **13.** (d)

■ INVERTIBLE/NON-INVERTIBLE FUNCTIONS

While studying function so far we have evolved it as a machine that picks up elements as inputs (pre-images) from domain set D_f and $\forall\ x \in D_f$ generates an output (image) in another set (co–domain) and is denoted as $f(x)$ and f is called connecting rule. A natural question arises, 'whether a machine exists that converts back the output (images) produced by machine f to input (pre-images)?' Of-course yes, such machines are called inverse machines and mathematically denoted as inverse function of $f(x)$, symbolized by f^{-1}: $Y \to X$ .i.e., pre-image(s) = f^{-1} (image)

In some case the invertibility of relation is easily conceivable, for instance the relation 'teacher *of*' is the inverse relation of 'student of'. On the other hand in most of the cases involving functions, the inverse relation is not so explicitly conceivable, for instance the inverse of relation 'a is son of b' is not exact, as b may be father or mother of a. Therefore mathematicians needed to develop a technique (better called algorithm) to obtain inverse function (relation) for a given function.

To obtain inverse function first of all it is necessary to furnish rules by which inversion has to take place, however keeping in mind such rules, f^{1} has to be function from $Y \to X$. This clearly put a restriction that each element y (images under f) of Y must be related with exactly one element x (pre-image under f) of X.

Consequently the inverse function f^{-1} so obtained is always defined from its domain (Y) to its co-domain (X). Thus, we can conclude that domain and co-domain of f is exchanged for f^{-1} and there should be bidirectional restriction on $X \to Y$ that each element of one set is related with exactly one element of other set, i.e., the process of inversion can be carried out for any function $f: X \to Y$ which is bijective. For instance if f is continuous and monotonic on an interval $[a, b] \to [c, d]$, naturally it becomes invertible.

Definition of Inverse of a Function

A function f: $X \to Y$ is said to be invertible iff there exists another function $g : Y \to X$ such that $f(x) = y \Leftrightarrow g(y) = x$, $\forall$ $x \in X$ and $y \in Y$.

Then $g : Y \to X$ is called inverse of $f : X \to Y$ and is denoted by f^{-1}.

$\Rightarrow\quad g = f^{-1} = \{(f(x), x): (x, f(x)) \in f\}$

e.g., let $A = \{2, 3, 5, 7\}$; $B = \{3, 8, 24, 48\}$

Define a function $f : A \to B$ as $f(x) = x^2 - 1$

Then $f = \{(2, 3), (3, 8), (5, 24), (7, 48)\}$. Now if we define another function $g : B \to A$ as $g(x) = \sqrt{x+1}$, then $g = \{(3, 2), (8, 3), (24, 5), (48, 7)\}$. Clearly $(x, y) \in f$ $\Leftrightarrow (y, x) \in g$ for all ordered pairs (x, y) and (y, x)

Thus, $f : A \to B$ is invertible and $f^{-1} = g$ and $g^{-1} = f$ i.e., inverse of function $f(x) = x^2 - 1$ is $g(x) = \sqrt{x+1}$ and inverse of $g(x) = \sqrt{x+1}$ is $f(x) = x^2 - 1$.

ILLUSTRATION 218: Find the inverse of $y = x^2 (x > 0)$ and draw its graph.

SOLUTION: Given function is $f(x) = x^2$, i.e., function converts x (> 0) to x^2.

Thus, $(x, x^2) \in f$ for $x > 0$, which implies $(x^2, x) \in f^{-1}$ for $x > 0$

i.e., $(x, \sqrt{x}) \in f^{-1}$ for $x > 0$, i.e., $y = \sqrt{x}$; $x > 0$ represents the inverse of function $y = x^2 (x > 0)$

The graphs of function $f(x) = x^2$ $(x > 0)$ and $f^{-1}(x) = \sqrt{x}$ $(x > 0)$ are shown below,

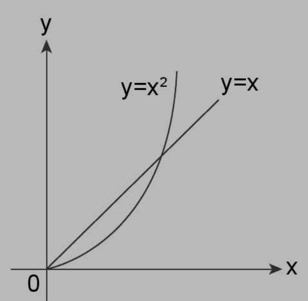


FIGURE 2.162

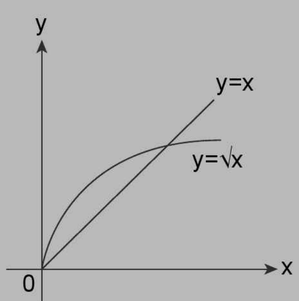


FIGURE 2.163

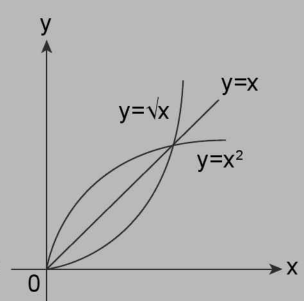


FIGURE 2.164

CONDITIONS OF INVERTIBILITY OF A FUNCTION

For a function to be invertible it must be a bijective function. In case the function loses at least one of the two properties i.e., surjectivity and injectivity, then the function remains no more invertible as is clear from the discussion given below.

Case I: Let f be one–one but not onto.

Consider the function $f : X \to Y$ such that $f = \{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)\}$ (shown in Figure 2.165), where $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5\}$.

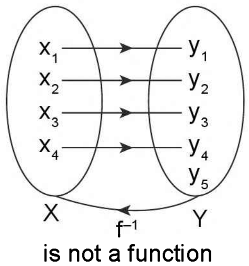


FIGURE 2.165

Clearly the function f is injective as different inputs from X are mapped to different outputs in Y. However, the function is not surjective as $y_5 \in Y$ has no pre image in X under f.

As there is no output for y_5 under the relation g from set Y to X defined by $g(y_i) = x_i \Leftrightarrow f(x_i) = y_i$.

$\Rightarrow$ $g : Y \to X$ is not a function.

$\Rightarrow$ f is not invertible from set X to set Y. Thus, for f to be invertible f should be surjective

Case II: Let f be onto but not one–one.

Consider the function f from set X to set Y such that $f = \{(x_1, y_1), (x_2, y_1), (x_3, y_2), (x_4, y_3)\}$ (shown in Figure 2.166), where $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3\}$.

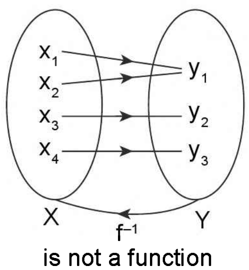


FIGURE 2.166

Clearly the function is surjective as range of function f = co-domain Y. However, the function f is not injective as there is same output y_1 corresponding to different inputs x_1, x_2.

$\Rightarrow$ the relation $g : Y \to X$ defined by $g(y_i) = x_i \Leftrightarrow f(x_i) = y_i$ is not a function h as there are two outputs x_1, x_2 corresponding to input y_1 under the relation g.

$\Rightarrow$ f is not invertible.

$\Rightarrow$ For f to be invertible it must be injective.

CONCLUSION

For a function to be invertible it should be one–one and onto, i.e., bijective function.

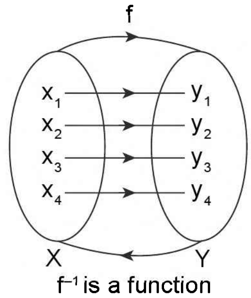


FIGURE 2.167

Method to Find Inverse of a Given Function

Before finding the inverse of a given function $f : X \to Y$ we test the function for bijectivity as if the function is bijective, only then we can find its inverse, otherwise it is not invertible.

We shall demonstrate the procedure to find the inverse of a function with the help of an illustration. Let $f(x) = \dfrac{x-1}{x+2}$ be the given function defined from $\mathbb{R} \sim \{-2\} \to \mathbb{R}$.

Step 1: Check the injectivity (one–one): Take $f(x_1) = f(x_2)$ and show that $x_1 = x_2$ or show that f is continuous and monotonic on its domain.

Here $f(x_1) = f(x_2)$

$$\Rightarrow \quad \frac{x_1-1}{x_1+2} = \frac{x_2-1}{x_2+2}$$

$$\Rightarrow \quad x_1x_2 + 2x_1 - x_2 - 2 = x_1x_2 + 2x_2 - x_1 - 2$$

$$\Rightarrow \quad 3x_1 = 3x_2 \qquad \Rightarrow \qquad x_1 = x_2.$$

Note that if f is non-injective in its domain, then define principal domain (P. D.) of function in which every function is injective, and hence, we can make $f(x)$ injective.

Step 2: Surjectivity (onto):

Find the range of the function (R_f) and compare it with co-domain. If R_f = co-domain, then f is onto.

Here, let $y = \dfrac{x-1}{x+2}$

$\Rightarrow \quad x = \dfrac{1+2y}{1-y}$

$\Rightarrow$ Range of function is $R_f = \mathbb{R} \sim \{1\}$
So, f is an into function

$\Rightarrow f(x) = \dfrac{x-1}{x+2}$ is not invertible from set $\mathbb{R} \sim \{-2\} \to \mathbb{R}$ due to lack of surjectivity.

Note that had the function f been defined as $f: \mathbb{R} \sim \{-2\} \to \mathbb{R} \sim \{1\}$, then the function would have been invertible. If f is not onto, then to make it surjective replace co-domain by Range (R_f).

Step 3: Using equation $y = f(x)$ express x in terms of y.

$$\Rightarrow \quad x = \frac{1+2y}{1-y} \qquad \text{.... (1)}$$

Step 4: Replace x by y and y by x in the obtained relation (1) to get $y = f^{-1}(x)$.

$\Rightarrow \quad y = \dfrac{1+2x}{1-x}$ is required inverse function $f^{-1}(x)$ if $f(x)$ is defined from $\mathbb{R} \sim \{-2\} \to \mathbb{R} \sim \{1\}$ for verification, show that $f(f^{-1}(x)) = x \ \forall x \in Y$, where $f: X \to Y$ is bijective.

$$\text{i.e., } \frac{\dfrac{2x+1}{1-x}-1}{\dfrac{2x+1}{1-x}+2} = \frac{2x+1-1+x}{2x+1+2-2x} = \frac{3x}{3} = x$$

Hence, verified.

REMARK

Since to each $(x, y) \in f$, there exists $(y, x) \in f^{-1}$ and (y, x) and (x, y) are mirror images of each other in the line $y = x$, therefore the graph of $f^{-1}(x)$ is obtained by reflecting the graph of $f(x)$ in the line $y = x$ as shown in Figure 2.168.

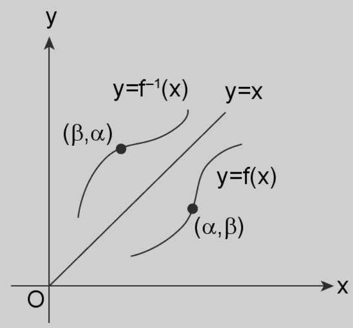


FIGURE 2.168

ILLUSTRATION 219: Find whether the function $f(x) = \dfrac{2x+5}{7}$ from $\mathbb{R} \to \mathbb{R}$ is invertible or not. If invertible, then find its inverse.

SOLUTION: Given function is $f(x) : A \to B$; where $y = \dfrac{2x+5}{7}$; $A = \mathbb{R}$; $B = \mathbb{R}$

The given function $f(x)$ will be invertible iff it is bijective, so we shall verify for the same

Injectivity: Let $x_1, x_2 \in A(=\mathbb{R})$; such that $f(x_1) = f(x_2)$

$$\Rightarrow \frac{2x_1+5}{7} = \frac{2x_2+5}{7} \Rightarrow 2x_1 + 5 = 2x_2 + 5 \Rightarrow x_1 = x_2 \Rightarrow f(x) \text{ is injective.}$$

Surjectivity: Let $y \in B$ $(=\mathbb{R})$; for if possible there exists $x \in A(=\mathbb{R})$ for which $f(x) = y$

$$\Rightarrow \frac{2x+5}{7} = y \Rightarrow x = \frac{7y-5}{2} \in \mathbb{R} \text{ (being defined and finite)} \qquad \text{.... (1)}$$

$$\Rightarrow f\left(\frac{7y-5}{2}\right) = y$$

Thus, every element y of set B has its pre image $\dfrac{7y-5}{2}$ in set A

$\Rightarrow f$ is surjective

$\therefore$ f is bijective, and hence, is invertible from set A to set B.

Now from equation (1), we get $x = \dfrac{7y-5}{2}$ $\Rightarrow f^{-1}(x) = \dfrac{7x-5}{2}$

ILLUSTRATION 220: Find whether the function $f(x) = 2x^2 - 7x + 5$ from $A = \left[\dfrac{7}{4}, \infty\right) \to B = \left[-\dfrac{9}{8}, \infty\right)$ is invertible or not. If invertible, then find its inverse.

SOLUTION: Given function is $f(x) = 2x^2 - 7x + 5$ defined from set A to B; where $A = \left[\dfrac{7}{4}, \infty\right)$; $B = \left[-\dfrac{9}{8}, \infty\right)$

We know by theory of quadratic equations, the quadratic function $f(x) = ax^2 + bx + c$ is one–one in the domain $\left[-\dfrac{b}{2a}, \infty\right)$ i.e., $\left[\dfrac{7}{4}, \infty\right)$ and its range is given by

$\left[-\dfrac{D}{4a}, \infty\right)$ i.e., $\left[-\dfrac{9}{8}, \infty\right) = B$

Thus, $f(x)$ is one–one and onto, i.e., bijective from set A to set B

$\Rightarrow f(x)$ is invertible function from set A to set B

Let $y = 2x^2 - 7x + 5$ $\Rightarrow 2x^2 - 7x + 5 - y = 0$

$$\Rightarrow x = \frac{7 \pm \sqrt{49 - 8(5-y)}}{4} = \frac{7 \pm \sqrt{9+8y}}{4} \Rightarrow x = \frac{7 + \sqrt{9+8y}}{4} \qquad \left(\because x \in \left[\frac{7}{4}, \infty\right)\right)$$

$$\Rightarrow f^{-1}(x) = \frac{7 + \sqrt{9+8x}}{4}$$

ILLUSTRATION 221: Find whether the function $f(x) = |\cos(2x)|$ from $A = \left[0, \dfrac{\pi}{2}\right] \to B = [0, 1]$ is invertible or not. If invertible, then find its inverse.

SOLUTION: Given function is $f(x) = |\cos(2x)|$ from set A to set B, where $A = \left[0, \dfrac{\pi}{2}\right]$ and $B = [0, 1]$

$$\therefore \quad x \in \left[0, \frac{\pi}{2}\right] \Rightarrow \left(\frac{\pi}{2} - x\right) \in \left[0, \frac{\pi}{2}\right]$$

Now $f(x) = |\cos(2x)|$ and $f\left(\dfrac{\pi}{2} - x\right) = |\cos(\pi - 2x)| = |-\cos(2x)| = |\cos(2x)|$

Thus, $f(x) = f\left(\dfrac{\pi}{2} - x\right)$ $\Rightarrow$ Thus, $f(x)$ is not injective

$\Rightarrow f(x)$ is not invertible from set A to set B.

However, range of $|\cos 2x|$ for $2x \in [0, \pi]$ is $[0, 1] = B$ $\Rightarrow f(x)$ is surjective

For making $f(x)$ invertible the domain of function should be selected $\left[0, \dfrac{\pi}{4}\right]$ for which $2x \in \left[0, \dfrac{\pi}{2}\right]$ in which function is bijective.

Now $f(x) = |\cos 2x| = \cos 2x$ $\Rightarrow y = \cos 2x$ $\Rightarrow x = \dfrac{1}{2}\cos^{-1} y$ $\Rightarrow f^{-1}(x) = \dfrac{1}{2}\cos^{-1} x$

ILLUSTRATION 222: Find whether the function $f(x) = |\log x|$ from $A = (0, \infty) \to B = [0, \infty)$ is invertible or not. If invertible, then find its inverse.

SOLUTION: Given function is $f(x) = |\log x|$ from set A to set B, where set $A = (0, \infty)$ and $B = [0, \infty)$

We know that $\log x \in (-\infty, 0]$ for $x \in (0, 1]$ and $\log x \in [0, \infty)$ for $x \in [1, \infty)$

$\Rightarrow$ $|\log x| \in [0, \infty)$ twice, firstly on $(0, 1]$ and secondly on $[1, \infty)$, graphically shown in Figure 2.169.

Thus, $f(x)$ is many–one on $(0, \infty)$ $\Rightarrow$ $f(x)$ is not bijective

$\Rightarrow$ $f(x)$ is not invertible

However, the range of function $f(x) = [0, \infty) = B$ = co-domain

$\Rightarrow$ $f(x)$ is surjective.

Now for making $f(x)$ invertible we shall take its domain as principal domain, i.e., $(0, 1]$ in which $f(x)$ is bijective. In $x \in (0, 1]$, $\log x \le 0 \Rightarrow f(x) = |\log x| = -\log x$

$\Rightarrow$ $y = -\log x$

$\Rightarrow$ $x = 10^{-y}$ $\Rightarrow$ $f^{-1}(x) = 10^{-x}$.

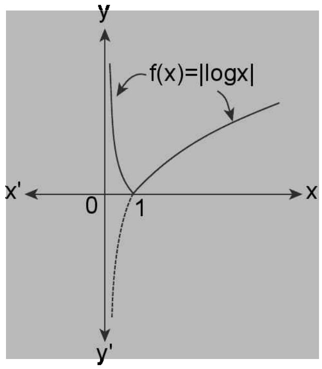


FIGURE 2.169

ILLUSTRATION 223: Test the invertibility, and hence, find the inverse function of the following.

(a) $y = \sin\left(\pi x + \dfrac{\pi}{4}\right)$ (b) $y = x^2 - 4x + 5$

(c) $y = \sqrt{1-x^2}$ (d) $y = x + \dfrac{1}{x}$

SOLUTION: (a) For injectivity $\left(\pi x + \dfrac{\pi}{4}\right)$ must be in principal domain of $\sin x$, i.e., $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

$\Rightarrow$ $-\dfrac{\pi}{2} \le \left(\pi x + \dfrac{\pi}{4}\right) \le \dfrac{\pi}{2}$ $\Rightarrow$ $-\dfrac{3\pi}{4} \le \pi x \le \dfrac{\pi}{4}$

$\Rightarrow$ $x \in [-3/4, 1/4]$ and for surjectivity co-domain equals range of $\sin x$, i.e., $[-1, 1]$

i.e., $f: \left[-\dfrac{3}{4}, \dfrac{1}{4}\right] \to [-1, 1]$. Now $y = \sin\left(\pi x + \dfrac{\pi}{4}\right)$ $\Rightarrow$ $\sin^{-1} y = \pi x + \dfrac{\pi}{4}$

$\Rightarrow$ $x = \dfrac{1}{\pi}\left(\sin^{-1}(y) - \dfrac{\pi}{4}\right)$ $\Rightarrow$ $f^{-1}(x) = y = \dfrac{1}{\pi}\sin^{-1} x - \dfrac{1}{4}$ (interchanging x and y)

(b) $y = x^2 - 4x + 5$

Given function being a quadratic polynomial (even degree polynomial) is many–one on $\mathbb{R}$.

So, for injectivity the domain of function must be the principal domain, i.e., $\left[-\dfrac{b}{2a}, \infty\right)$

i.e., $[2, \infty)$ and for surjectivity co-domain must be equal to range, i.e., $\left[-\dfrac{D}{4a}, \infty\right)$ i.e., $[1, \infty)$;

Now $y = x^2 - 4x + 5$ $\Rightarrow$ $x^2 - 4x + (5 - y) = 0$

$\Rightarrow$ $x = \dfrac{4 \pm \sqrt{4y-4}}{2} = 2 \pm \sqrt{y-1}$ $\Rightarrow$ $x = 2 + \sqrt{y-1}$ $\Rightarrow$ $f^{-1}(x) = 2 + \sqrt{x-1}$ $(\because x \in [2, \infty))$

(c) $y=\sqrt{1-x^2}$; D_f = [–1, 1]

For injectivity domain should be principal domain = [0, 1] and for surjectivity co-domain should be the range of function, i.e., [0, 1]

Now $y=\sqrt{1-x^2}$ $\Rightarrow$ $x^2 = 1 - y^2 \Rightarrow$ $x=\pm\sqrt{1-y^2}$ $\Rightarrow$ $x=\sqrt{1-y^2}$ ($\because x \in [0, 1]$)

$\Rightarrow$ $f^{-1}(x) = \sqrt{1-x^2}$ $\therefore$ $f(x)$ is inverse of itself.

(d) $y=x+\dfrac{1}{x}$; $D_f = \mathbb{R} \sim \{0\}$; Range = $(-\infty, -2] \cup [2, \infty)$

The graph of $y = x + 1/x$ is as shown above.

For injectivity take domain as principal domain = $[-1, 1] \sim \{0\}$ and

For surjectivity take co-domain = Range = $(-\infty, -2] \cup [2, \infty)$

Now, $y=\dfrac{x^2+1}{x}$ $\Rightarrow$ $x^2 - xy + 1 = 0$

$$\Rightarrow \quad x=\frac{y\pm\sqrt{y^2-4}}{2}$$

$$\Rightarrow \quad x=\frac{y+\sqrt{y^2-4}}{2} \text{ for } y \le -2 \text{ and } x=\frac{y-\sqrt{y^2-4}}{2} \text{ for } y \ge 2$$

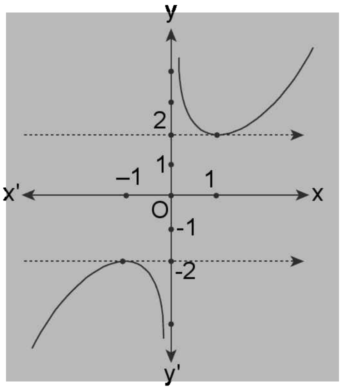


FIGURE 2.170

$$\therefore \quad f^{-1}(x) = \begin{cases} \dfrac{x+\sqrt{x^2-4}}{2}; & \text{for } x \in (-\infty, -2] \\ \dfrac{x-\sqrt{x^2-4}}{2}; & \text{for } x \in [2, \infty) \end{cases}$$

ILLUSTRATION 224: Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \cos(5x + 2)$. Is f invertible? Justify your answer.

SOLUTION: We know that any function $f: X \to Y$ is invertible iff it is bijective. Now $f: \mathbb{R} \to \mathbb{R}$, given by $f(x) = \cos(5x + 2)$ is neither injective nor surjective.

For $-1 \le \cos(5x + 2) \le 1$

$\Rightarrow$ Range of $f \ne \mathbb{R}$ $\Rightarrow$ f is not surjective.

Also $f(x + 2\pi/5) = \cos\{5(x + 2\pi/5) + 2\} = \cos(5x + 2) = f(x)$

So, the function is periodic with period $2\pi / 5$ and all periodic functions, are many–one.

$\therefore$ f is not injective. Thus, f can't be invertible.

ILLUSTRATION 225: If the function $f: [1, \infty) \to [1, \infty)$ is defined by $f(x) = a^{x(x-1)}$, then find $f^{-1}(x)$.

SOLUTION: Injectivity: $f(x) = a^{x^2-x}$ $\Rightarrow$ $f'(x) = a^{x^2-x}.(2x - 1)$

For $f(x)$ to be one–one, it should be strictly increasing or strictly decreasing.

So, $f'(x) > 0$ $\Rightarrow$ $a^{x^2-x}.(2x-1) > 0$, where $a^{x^2-x} > 0$ for all x

$\Rightarrow$ $2x - 1 > 0$ $\Rightarrow$ $x > 1/2$

Thus, for given domain $[1, \infty)$, $f(x)$ is always increasing, hence, one–one.

Surjectivity: As $f(x)$ is strictly increasing and being exponential function is continuous over its given domain.

$\Rightarrow$ Range $= [f(1), f(\infty)) = [1, \infty)$

$\Rightarrow$ Range of $f(x)$ = co-domain of $f(x)$, thus, $f(x)$ is onto.

To find inverse: As $f(x)$ is one–one and onto (bijective), thus, inverse of $f(x)$ can be obtained.

Let $y = f(x)$

$\Rightarrow\ y = a^{x^2-x} \ \Rightarrow\ x^2 - x = \log_a y \ \Rightarrow\ x^2 - x - \log_a y = 0 \ \Rightarrow\ x = \dfrac{1 \pm \sqrt{1+4\log_a y}}{2}$

$\Rightarrow\ x = \dfrac{1+\sqrt{1+4\log_a y}}{2}$ $(\because x \in [1, \infty))$

Hence, $f^{-1}(x) = \dfrac{1+\sqrt{1+4\log_a x}}{2}$ (replacing y by x and x by y).

■ PROPERTIES OF INVERSE OF A FUNCTION

(i) The inverse of a bijection is unique.

Proof: Let $f : A \to B$ be a bijection. If possible let g : $B \to A$ and $h : B \to A$ be two inverse functions of f.

Also let $a_1, a_2 \in A$ and $b \in B$ such that $g(b) = a_1$ and $h(b) = a_2$, then $g(b) = a_1$

$\Rightarrow f(a_1) = b$ and $h(b) = a_2 \quad \Rightarrow \quad f(a_2) = b$.

But f is one–one, therefore $f(a_1) = f(a_2)$

$\Rightarrow a_1 = a_2 \quad \Rightarrow \quad g(b) = h(b), \forall\, b \in B$

$\Rightarrow g(x) = h(x)$, i.e., inverse of f is unique

(ii) The inverse of a bijection is also a bijection.

Proof: Let $f : A \to B$ be a bijection and $g : B \to A$ be its inverse. We have to show that g is one–one and onto.

Injectivity: Let $b_1, b_2 \in B$ such that $g(b_1) = g(b_2)$... (i)

$\because$ $b_1, b_2 \in B$ and g is a function from B to A

$\Rightarrow$ there exists $a_1, a_2 \in A$ such that $g(b_1) = a_1$ and $g(b_2) = a_2$

$\therefore$ from (i) $a_1 = a_2$ and $f : A \to B$ is a function.

$\Rightarrow f(a_1) = f(a_2)$

$\Rightarrow b_1 = b_2$

$\therefore$ g is injective.

$$\left[\begin{array}{l} \because\ g(b_1) = a_1,\ g(b_2) = a_2 \\ \Rightarrow f(a_1) = b_1, f(a_2) = b_2 \\ \text{as } f \ \& \ g \text{ are inverse of each other.} \end{array}\right]$$

Surjectivity: Let $a \in A$

$\Rightarrow$ There exists $b \in B$ such that $f(a) = b$ (by definition of f)

$\Rightarrow a = g(b) \qquad [\because f(a) = b \Rightarrow a = g(b)]$

Which proves that g is onto. Hence, g is also a surjection.

$\therefore$ If $f : A \to B$ is bijective function, then $f^{-1} : B \to A$ is also a bijective function.

(iii) If f and g are two bijections $f : A \to B$, $g : B \to C$, then the inverse of gof exists and $(gof)^{-1} = f^{-1}og^{-1}$.

Proof: Since $f : A \to B$ and $g : B \to C$ are bijections,

$\therefore$ $gof : A \to C$ is also a bijection $\Rightarrow$ $(gof)^{-1} : C \to A$.

Now let $a \in A$, $b \in B$, $c \in C$ such that $f(a) = b$ and $g(b) = c$

$\Rightarrow (gof)(a) = g[f(a)] = g(b) = c$

$\Rightarrow a = (gof)^{-1}(c)$ ...(1)

Now $f(a) = b \Rightarrow \quad a = f^{-1}(b)$ ....(2)

and $g(b) = c \Rightarrow \quad b = g^{-1}(c)$ ....(3)

$\therefore (f^{-1}og^{-1})(c) = f^{-1}(g^{-1}(c)) = f^{-1}(b)$ [by (3)]

$= a$ [by (2)]

$= (gof)^{-1}(c)$ [by (1)]

$\therefore (gof)^{-1} = f^{-1}og^{-1}$

ILLUSTRATION 226: If f: $\mathbb{R} \to \mathbb{R}$; $f(x) = e^x$ and $g : \mathbb{R} \to \mathbb{R}$: $g(x) = 3x - 2$, then show that $(fog)^{-1} = g^{-1}of^{-1}$ and $(gof)^{-1} = f^{-1}og^{-1}$. Also find domain of $(fog)^{-1}$ and $(gof)^{-1}$.

SOLUTION: Given $f(x) = e^x$

$\Rightarrow\ x = \ln f(x) \ \Rightarrow\ f^{-1}(x) = \ln x$ and $g(x) = 3x - 2 \ \Rightarrow\ x = \dfrac{g(x)+2}{3} \ \Rightarrow\ g^{-1}(x) = \dfrac{x+2}{3}$

Now $(fog)(x) = e^{3x-2}$; $(gof)(x) = 3e^x - 2$

$$\Rightarrow (fog)^{-1}(x) = \frac{\ln x + 2}{3} \text{ and } (gof)^{-1} = \ln\left(\frac{x+2}{3}\right)$$

Now $g^{-1}of^{-1}(x) = g^{-1}(f^{-1}(x)) = g^{-1}(\ln x) = \dfrac{\ln x + 2}{3}$ .... (1)

and $f^{-1}og^{-1}(x) = f^{-1}(g^{-1}(x)) = f^{-1}\left(\dfrac{x+2}{3}\right) = \ln\left(\dfrac{x+2}{3}\right)$ .... (2)

Therefore from equations (1) and (2), we get $(fog)^{-1} = g^{-1}of^{-1}$ and $(gof)^{-1} = f^{-1}og^{-1}$.

Domain of $(fog)^{-1} = \mathbb{R}^+$ = range of $fog(x)$ = range of $e^{3x-2} = (0, \infty) = \mathbb{R}^+$;

Domain of $(gof)^{-1}$ = Range of $3e^x - 2 = (-2, \infty)$.

(iv) Inverse of inverse of a given function is the given function itself, i.e., $(f^{-1})^{-1} = f$

Proof: Let $f: A \to B$ is bijective, then $f^{-1}: B \to A$ is also bijective $\quad \therefore (f^{-1})^{-1}: A \to B$ exists.

Let $a \in A \Rightarrow$ there exists $b \in B$ such that $f(a) = b$

$\therefore f^{-1}(b) = a \Rightarrow (f^{-1})^{-1}(a) = b = f(a)$

$\therefore (f^{-1})^{-1}(a) = f(a)\ \forall\ a \in A \quad \Rightarrow \quad (f^{-1})^{-1} = f$.

(v) $f(x)$ and $f^{-1}(x)$ if intersect, then the point of intersection should be on the line $y = x$ or $y = -x + k$ for some real value of k.

Proof: Case (a) If $f(x)$ is a function such that whenever $(\alpha, \beta) \in f$, then $(\beta, \alpha) \notin f$. Now $(\alpha, \beta) \in f$

$\Rightarrow (\beta, \alpha) \in f^{-1}$, but $(\beta, \alpha) \notin f$.

Thus, there is no point common in the graph of function f and its inverse f^{-1}, i.e., function f and its inverse f^{-1} would not intersect each other at any point. i.e., the function would remain strictly on one side of the line $y = x$ as shown in Figure 2.171 and 2.172.

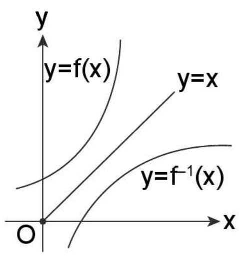


FIGURE 2.171

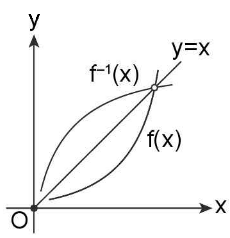


FIGURE 2.172

And if it exists on both sides of the line $y = x$, then there will be a point of discontinuity at the point across which the curve of $f(x)$ transits the line $y = x$.

Case (b) If the function $f(x)$ contains a pair (α, β) such that (β, α) belongs to function $f(x)$, then there exist two sub-cases as given below:

Sub case (i) When $\alpha = \beta$, i.e., if function is anti-symmetric

$\Rightarrow (\alpha, \beta), (\beta, \alpha) \in f \Leftrightarrow \alpha = \beta$

$\Rightarrow (\alpha, \beta) = (\beta, \alpha) = (\alpha, \alpha)$

i.e., function $f(x)$ and its inverse $f^{-1}(x)$ would intersect at the line $y = x$ as shown in Figure 2.173 and 2.174.

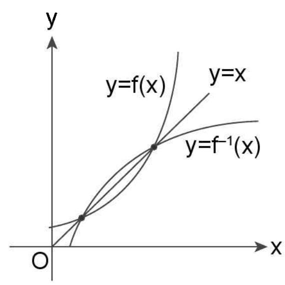


FIGURE 2.173

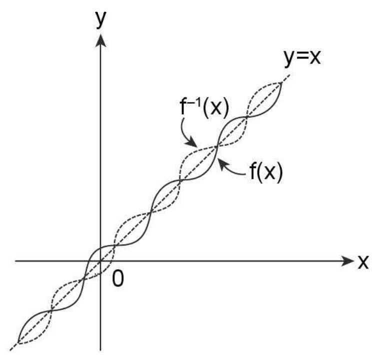


FIGURE 2.174

Sub case (ii) When $\alpha \neq \beta$.

As $f(x)$ contains two different ordered pairs (α, β) as well as (β, α).

It implies inverse function $f^{-1}(x)$ also contains the ordered pairs (β, α) and $(\alpha. \beta)$,

that means the function $f(x)$ and its inverse $f^{-1}(x)$ would intersect each other at two different points (α, β) and (β, α).

Thus, the points of intersection of $f(x)$ and its inverse $f^{-1}(x)$ would lie on the line joining the points (α, β) and (β, α) whose equation is given by

$$y-\beta=\left(\frac{\beta-\alpha}{\alpha-\beta}\right)(x-\alpha)$$

$\Rightarrow$ $y = -(x - \alpha) + \beta \Rightarrow y = -x + k$, where $k = \alpha + \beta$

i.e., $f(x)$ and $f^{-1}(x)$ intersects at line parallel to $y = -x$ as shown in Figure 2.175.

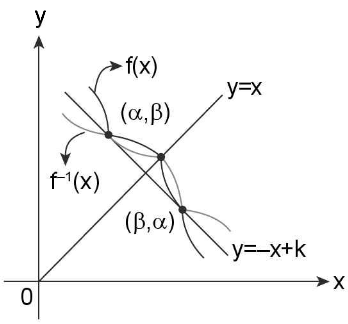


FIGURE 2.175

- If $f(x)$ has more than one points of the form (α, β) for which $(\beta, \alpha) \in f$, then if $\alpha = \beta$, then the point of intersection would lie on the line $y = x$ and if $\alpha \neq \beta$ then indeed the point of intersection would lie on the line $y = -x + k$, where $k = \alpha + \beta$ as shown in Figure 2.176.

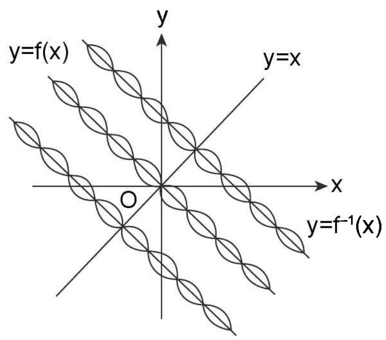


FIGURE 2.176

- If a function is symmetric about the straight line $y = x$ that is $(\alpha, \beta) \in f \Leftrightarrow (\beta, \alpha) \in f\ \forall\ (\alpha, \beta) \in f$, then the function is inverse of itself called as self invertible function graphically shown in Figure 2.177.

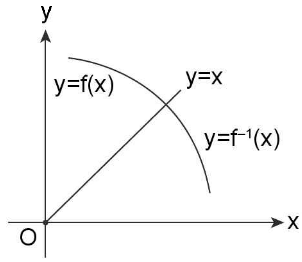


FIGURE 2.177

Further in such cases $f(x) = f^{-1}(x)$ is an identity in x, therefore infinitely many solutions.

- If a function is monotonically increasing, then $f(x)$ and $f^{-1}(x)$ cannot intersect on line $y = -x + k$ unless it is the point of intersection of $y = x$ and $y = -x + k$.

Proof: Let if possible (α, β) and $(\beta, \alpha) \in f(x)$; $\alpha < \beta$ and $f(\alpha) = \beta$ and $f(\beta) = \alpha$ .... (1)

Since $f(x)$ is monotonically increasing, it implies $f(\alpha) \leq f(\beta)$ and $f(\beta) \leq f(\alpha)$

$\Rightarrow$ $\beta \leq \alpha$ and $\alpha \leq \beta$

$\Rightarrow$ $\alpha = \beta$, which is contradiction that $\alpha < \beta$

Thus, whenever (α, β) and $(\beta, \alpha) \in f(x)$, then $\alpha = \beta$

$\Rightarrow$ $f(x)$ can't have two points (α, β) and (β, α) simultaneously for $\alpha \neq \beta$

$\Rightarrow$ $f(x)$ and $f^{-1}(x)$ cannot intersect on any line $y = -x + k$ unless it is the point of intersection of $y = x$ and $y = -x + k$

(vi) $f(x)$ and $f^{-1}(x)$ have same monotonic nature, i.e., either both increasing or both decreasing.

Proof: Let $g(x) = f^{-1}(x)$

$\Rightarrow$ $f(g(x)) = x$ $\Rightarrow$ $f'(g(x)) \,.\, g'(x) = 1$

$$\Rightarrow\ g'(x)=\frac{1}{f'(g(x))}\ \Rightarrow\ g'(x)=\begin{cases}+ve & \text{if } f(x)\uparrow\\ -ve & \text{if } f(x)\downarrow\end{cases}$$

$\therefore$ If $f(x)$ is increasing, then $f^{-1}(x)$ is also increasing and if $f(x)$ is decreasing, then $f^{-1}(x)$ also decreasing

(vii) If $f(x)$ is increasing function, then $f^{-1}(x)$ is also an increasing function but $f(x)$ and $f^{-1}(x)$ have opposite curvatures, i.e., if $f(x)$ is concave upwards, then $f^{-1}(x)$ is concave downwards and if $f(x)$ is concave downwards, then $f^{-1}(x)$ is concave upwards.

Proof: ***Case (a)*** If $f(x)$ is increasing and concave upwards. From the derivative of $g(x) = f^{-1}(x)$, we have

$$g'(x)=\frac{1}{f'(g(x))}$$

$$\Rightarrow\ g''(x)=\underbrace{\frac{-1}{\left(f'(g(x))\right)^2}}_{-ve}\cdot\underbrace{f''(g(x))}_{+ve}\cdot\underbrace{g'(x)}_{+ve} \qquad \text{.... (1)}$$

$\Rightarrow$ $g'(x) < 0 \Rightarrow g(x) = f^{-1}(x)$ is concave downwards graphically shown in Figure 2.178.

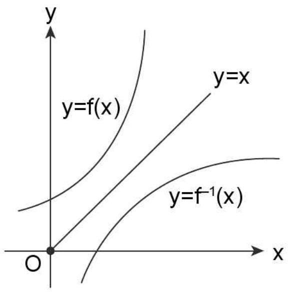


FIGURE 2.178

Case (b) Let $f(x)$ is increasing and concave downwards. From (1), first factor is negative and second factor is negative as $f(x)$ is concave downwards and third term is positive as $f(x)$ and $f^{-1}(x) = g(x)$ both are increasing functions.

$\therefore$ $g'(x) > 0$ i.e., $g(x) = f^{-1}(x)$ is concave upwards graphically shown in Figure 2.179.

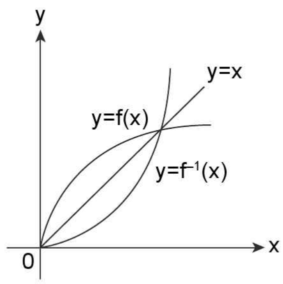


FIGURE 2.179

(viii) If $f(x)$ is a decreasing function, then $f^{-1}(x)$ is also a decreasing function, but $f(x)$ and $f^{-1}(x)$ have same curvatures.

Proof: ***Case (a)*** If $f(x)$ is decreasing and concave upwards.

$$g'(x) = \underbrace{\frac{-1}{(f'(g(x)))^2}}_{-ve} \cdot \underbrace{f'(g(x))}_{+ve} \cdot \underbrace{g'(x)}_{-ve} \qquad \text{.... (i)}$$

$\Rightarrow$ $g'(x) > 0$

$\Rightarrow$ $g(x) = f^{-1}(x)$ is concave upwards graphically shown in Figure 2.180.

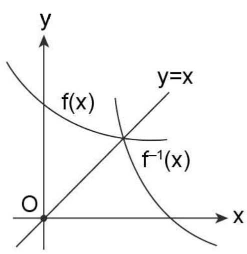


FIGURE 2.180

Case (b) If $f(x)$ is decreasing and concave downwards.

$$g'(x) = \underbrace{\frac{-1}{(f'(g(x)))^2}}_{-ve} \cdot \underbrace{f'(g(x))}_{-ve} \cdot \underbrace{g'(x)}_{-ve}$$

$\Rightarrow$ $g(x) < 0$

$\Rightarrow$ $g(x)$ is concave downwards, graphically shown in Figure 2.181

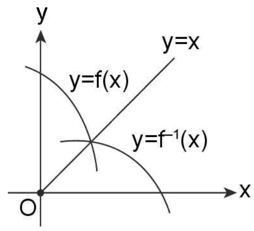


FIGURE 2.181

(ix) If the graph of a function $f(x)$ is symmetric about the line $y = x$, then $f(x)$ and $f^{-1}(\boldsymbol{x})$ are equal functions. i.e., $f(x)$ will be self invertible function or (involution).

$\Rightarrow$ $f(x) = f^{-1}(x)$ is an identity, and therefore has infinitely many solutions.

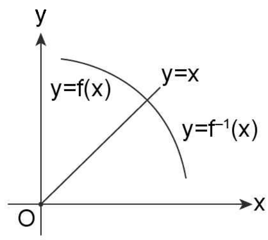


FIGURE 2.182

(x) If $\boldsymbol{f}: A \to B$ is a bijection, then $f^{-1}: B \to A$ is an inverse function of $\boldsymbol{f}$, then $f^{-1}of = I_A$ and $fof^{-1} = I_B$.

Here $\boldsymbol{I_A}$ is an identity function on set $\boldsymbol{A}$, and $\boldsymbol{I_B}$ is an identity function on set $\boldsymbol{B}$.

Proof: Let $f: A \to B$ be a bijection, then $f^{-1}: B \to A$ is also a bijection.

i.e., $f^{-1}of: A \to A$

Let $a \in A$, then there exists $b \in B$ such that $f(a) = b$ and $f^{-1}(b) = a$.

Now $f^{-1}of(a) = f^{-1}(f(a)) = f^{-1}(b) = a$

$\therefore$ $f^{-1}of(a) = a \ \forall \ a \in A$

$\Rightarrow$ $f^{-1}of = I_A$ i.e., identity function on set A

Also $fof^{-1}: B \to B$ is a bijection

Let $b \in B$

$\Rightarrow$ There exists $a \in A$ such that $f^{-1}(b) = a$ and $f(a) = b$

Now $fof^{-1}(b) = f(f^{-1}(\text{b})) = f(a) = b$

$\therefore$ $fof^{-1}(b) = b \ \forall \ b \in B$

$\therefore$ $fof^{-1} = I_B$, i.e., identity function on set B.

ILLUSTRATION 227: If $f(x)$ is an invertible function, and $g(x) = 2f(x) + 5$, then find the value of $g^{-1}(x)$.

SOLUTION: Given equation is $g(x) = 2f(x) + 5$.

Replacing x by $g^{-1}(x)$; we get $g(g^{-1}(x)) = 2f(g^{-1}(x)) + 5$

$\Rightarrow\ x = 2f(g^{-1}(x)) + 5$

$\Rightarrow\ f(g^{-1}(x)) = \dfrac{x-5}{2}$ $\qquad\Rightarrow\ g^{-1}(x) = f^{-1}\left(\dfrac{x-5}{2}\right)$

ILLUSTRATION 228: Compute the inverse of the following functions:

(a) $f(x) = \ell n\left(x+\sqrt{x^2+1}\right)$ (b) $f(x) = 2^{\frac{x}{x-1}}$ (c) $y = \dfrac{10^x - 10^{-x}}{10^x + 10^{-x}}$

SOLUTION: (a) Given $f(x) = \ell n\left(x+\sqrt{x^2+1}\right)$

If $x \ge 0$, then $x + \sqrt{x^2+1} > 0$ and if $x < 0$, then $|x| = \sqrt{x^2} < \sqrt{x^2+1}$

Thus, magnitude of x is less than the magnitude of $\sqrt{x^2+1}$ $\Rightarrow$ $x + \sqrt{x^2+1} > 0$

Thus, for every real values of x, $x + \sqrt{x^2+1} > 0$

$\Rightarrow$ Domain of $f(x)$ is $\mathbb{R}$

Also $x + \sqrt{x^2+1}$ being the sum of two continuous functions is also continuous.

Further, $f'(x) = \dfrac{1}{\sqrt{x^2+1}} > 0\ \forall\ x \in \mathbb{R}$.

$\Rightarrow$ $f(x)$ is strictly increasing function. Thus, $f(x)$ is a bijective function from $\mathbb{R} \to \mathbb{R}$ (range)

Now $y = f(x) = \ln\ (x+\sqrt{x^2+1})$ $\qquad\Rightarrow\ e^y = x+\sqrt{x^2+1}$ ....(1)

Again $y = \ln\ (x+\sqrt{x^2+1})$ $\Rightarrow$ $-y = -\ln\ (x+\sqrt{x^2+1})$ $\Rightarrow$ $-y = \ln\ (x+\sqrt{x^2+1})^{-1}$

$\Rightarrow\ -y = \ln\left(\dfrac{1}{x+\sqrt{x^2+1}}\right)$ $\Rightarrow\ e^{-y} = \dfrac{1}{x+\sqrt{x^2+1}} \times \dfrac{x-\sqrt{x^2+1}}{x-\sqrt{x^2+1}} = -(x-\sqrt{x^2+1})$

$\Rightarrow\ -e^{-y} = x-\sqrt{x^2+1}$ ...(2)

Adding (1) and (2), we get $e^y - e^{-y} = 2x$ or $x = \dfrac{e^y - e^{-y}}{2}$.

Thus, inverse function of $f(x)$ will be $y = \dfrac{e^x - e^{-x}}{2}$

(b) Given $f(x) = 2^{\frac{x}{x-1}}$

Domain of $f(x)$ is $\mathbb{R} - \{1\}$ and range of $\dfrac{x}{x-1}$ is $\mathbb{R} - \{1\}$

Also 2^x is continuous and increasing function, thus, range of $f(x)$ will be $(2^{-\infty}, 2^{\infty}) \sim \{2\}$.

i.e., $(0, \infty) \sim \{2\}$

Thus, $f(x)$ is a bijective function from $\mathbb{R} \sim \{1\}$ to $(0, \infty) \sim \{2\}$, and hence, is invertible when defined from set $\mathbb{R} \sim \{1\}$ to set $(0, \infty) \sim \{2\}$

Now $y = 2^{\frac{x}{x-1}}$ $\Rightarrow\ \log_2 y = \left(\dfrac{x}{x-1}\right)\log_2 2$ $\Rightarrow\ \log_2 y = \dfrac{x}{x-1}$

$\Rightarrow\ x(\log_2 y - 1) - \log_2 y = 0$ $\Rightarrow\ x = \dfrac{\log_2 y}{\log_2 y - 1}$ $\Rightarrow\ f^{-1}(x) = \dfrac{\log_2 x}{\log_2 x - 1}$

(c) Given $f(x) = \dfrac{10^x - 10^{-x}}{10^x + 10^{-x}}$. Clearly $f(x)$ is defined for all real values x, hence, domain of $f(x)$ is $\mathbb{R}$.

Further $f(x) = \dfrac{10^{2x}-1}{10^{2x}+1} = \left(1 - \dfrac{2}{10^{2x}+1}\right)$ which is continuous and having greatest lower bound –1 and least upper bound 1. Thus, range of $f(x)$ is (–1, 1).

Further $f'(x) = \dfrac{4(10)^{2x}\log 10}{(10^{2x}+1)^2} > 0\ \forall x \in \mathbb{R}$ which implies $f(x)$ is monotonically increasing, and hence, is bijective when defined from $\mathbb{R}$ to (–1, 1).

Now $y = \dfrac{10^x - 10^{-x}}{10^x + 10^{-x}} \Rightarrow y = \dfrac{10^{2x}-1}{10^{2x}+1}$

Applying componendo and dividendo, we get $\dfrac{y+1}{y-1} = \dfrac{10^{2x}-1+10^{2x}+1}{10^{2x}-1-10^{2x}-1}$

$$\Rightarrow \frac{y+1}{y-1} = \frac{2.10^{2x}}{-2} \Rightarrow y+1 = (1-y)\,10^{2x} \Rightarrow 10^{2x} = \frac{1+y}{1-y} \Rightarrow 2x = \log_{10}\left(\frac{1+y}{1-y}\right)$$

$$\Rightarrow x = \frac{1}{2}\log_{10}\left(\frac{1+y}{1-y}\right) \Rightarrow f^{-1}(x) = \frac{1}{2}\log_{10}\left(\frac{1+x}{1-x}\right).$$

ILLUSTRATION 229: Find the inverse of $f(x) = 2^{\log_{10} x} + 8$, and hence, solve the equation $f(x) = f^{-1}(x)$.

SOLUTION: Given $y = 2^{\log_{10} x} + 8$

$\log_{10} x$ is defined for $x > 0$, therefore domain of function $f(x)$ is $(0, \infty)$.

Also range of $\log_{10} x$ is $(-\infty, \infty)$ and 2^x being continuous and increasing function, $2^{\log_{10} x}$ has its range $(0, \infty)$.

Thus, range of function $f(x) = 2^{\log_{10} x} + 8$ is $(8, \infty)$. Further $f(x)$ is monotonically increasing (sum of increasing and constant function) when defined from $(0, \infty)$ to set $(8, \infty)$ is bijective, and hence, is invertible.

Now $y = 2^{\log_{10} x} + 8 \Rightarrow \log_{10} x = \log_2(y-8) \Rightarrow x = 10^{\log_2(y-8)}$ $\therefore$ $f^{-1}(x) = 10^{\log_2(x-8)}$

Now, to solve $f(x) = f^{-1}(x)$, we solve $f(x) = x$, and $f(x) = -x + k$.

However, $f(x) = -x + k$ is taken if $f(x)$ has at least two points of symmetric about the line $y = x$, i.e., (α, β), $(\beta, \alpha) \in f$ and $\alpha \neq \beta$.

Here, $f(x) = 2^{\log_{10} x} + 8$ is an increasing function, therefore it would not contain any two different points of symmetry. Thus, $f(x)$ and $f^{-1}(x)$ can intersect only on line $y = x$.

Thus, $f(x) = f^{-1}(x)$ is equivalent to equation $f(x) = x$.

$$\therefore\ 2^{\log_{10} x} + 8 = x \Rightarrow 2^{\log_{10} x} = x - 8 \Rightarrow \log_{10} x = \log_2(x-8) \Rightarrow \log_{x-8} x = \log_2 10$$

$\Rightarrow x = 10$

ILLUSTRATION 230: Let $f: \left[-\sqrt{2}+1, \sqrt{2}+1\right] \to \left[\dfrac{-\sqrt{2}+1}{2}, \dfrac{\sqrt{2}+1}{2}\right]$ be a function defined by $f(x) = \dfrac{1-x}{1+x^2}$.

Is f invertible? If yes, then find its inverse.

SOLUTION: Domain of $f(x) = \left[-\sqrt{2}+1, \sqrt{2}+1\right]$ (given)

Here, $f(x) = \dfrac{1-x}{1+x^2} \Rightarrow f'(x) = \dfrac{x^2-2x-1}{(1+x^2)^2} \le 0$

for $x \in \left[-\sqrt{2}+1, \sqrt{2}+1\right]$

$\Rightarrow$ $f(x)$ is a strictly decreasing function on $\left[-\sqrt{2}+1, \sqrt{2}+1\right]$ graphically as shown in Figure 2.183.

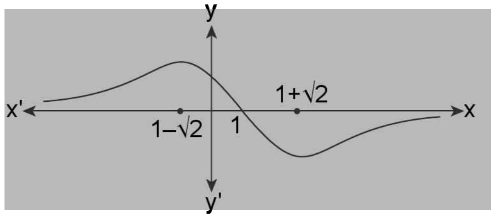


FIGURE 2.183

$\Rightarrow$ $f(x)$ is injective on $[-\sqrt{2}+1, \sqrt{2}+1]$

Also $f(x)$ being a rational function is continuous on $\left[-\sqrt{2}+1, \sqrt{2}+1\right]$ as its denominator is non-vanishing.

Thus, $f(x)$ should have its range $\left[f\left(\sqrt{2}+1\right), f\left(-\sqrt{2}+1\right)\right] = \left[\dfrac{-\sqrt{2}+1}{2}, \dfrac{\sqrt{2}+1}{2}\right]$ = co-domain

$\Rightarrow$ $f(x)$ is surjective. Thus, $f(x)$ is a bijective function and hence, is invertible.

Now $y = \dfrac{1-x}{1+x^2}$ .... (1)

$\Rightarrow$ $x^2y + y = 1 - x$ $\Rightarrow$ $x^2y + x + (y-1) = 0$

$$\Rightarrow \quad x = \frac{-1 \pm \sqrt{1-4y(y-1)}}{2y} = \frac{-1 \pm \sqrt{4y-4y^2+1}}{2y} = \frac{-1 \pm \sqrt{-(4y^2-4y+1)+2}}{2y} \text{ for } y \neq 0$$

Taking negative sign, we have $x = \dfrac{-1-\sqrt{-(4y^2-4y+1)+2}}{2y}$ for $y \neq 0$

Now $\dfrac{1}{2} \in$ co-domain of f (domain of f^{-1}), substituting $y = \dfrac{1}{2}$, we have $x = -1 - \sqrt{2}$

Which does not belong to domain of f (range of f^{-1}),

Hence, rejecting negative sign, we have $x = \dfrac{-1+\sqrt{4y-4y^2+1}}{2y}$ for $y \neq 0$ and for $y = 0$, $x = 1$ [from (1)].

$\therefore$

$$f^{-1}(x) = \begin{cases} \dfrac{-1+\sqrt{4x-4x^2+1}}{2x}; & x \neq 0 \\ 1 & ; x = 0 \end{cases}$$

ILLUSTRATION 231: Let $f: \mathbb{N} \to \mathbb{N}$, where $f(x) = x + (-1)^{x-1}$. Find the inverse of f.

SOLUTION: $y = f(x) = \begin{cases} x+1; & x \text{ an odd natural number} \\ x-1; & x \text{ an even natural number} \end{cases}$ ..... (1)

Injectivity: Let $x_1, x_2 \in \mathbb{N}$ such that $f(x_1) = f(x_2)$

$\Rightarrow$ $x_1 + 1 = x_2 + 1$ if x_1, x_2 are odd natural numbers $\Rightarrow$ $x_1 = x_2$ and

$\Rightarrow$ $x_1 - 1 = x_2 - 1$ if x_1, x_2 are even natural numbers $\Rightarrow$ $x_1 = x_2$

If x_1 and x_2 are opposite in nature (say) x_1 odd and x_2 even, then $f(x_1) = f(x_2)$

$\Rightarrow$ $x_1 + 1 = x_2 - 1$ $\Rightarrow$ $x_2 - x_1 = 2$, which is a contradiction as difference of an odd and even integer can never be an even integer.

Similarly, x_1 even and x_2 odd is impossible. Thus, $f(x)$ is injective function.

Surjectivity: Let n be any natural number.

Now if n is odd say $n = 2m - 1$; $m \in \mathbb{N}$; then $f(2m) = 2m - 1 = n$

and if n is even say $2m$, $m \in \mathbb{N}$; then $f(2m - 1) = 2m = n$.

Thus, f is surjective function. Hence, f is an invertible function.

Now from (1), we have

$$x = \begin{cases} y-1; & \text{if } x \text{ an odd natural number} \\ y+1; & \text{if } x \text{ an even natural number} \end{cases} \text{ or } x = \begin{cases} y-1; & \text{if } y \text{ an even natural number} \\ y+1; & \text{if } y \text{ an odd natural number} \end{cases}$$

$$\Rightarrow f^{-1}(x) = \begin{cases} x-1; & \text{if } x \text{ an even natural number} \\ x+1; & \text{if } x \text{ an odd natural number} \end{cases} \Rightarrow f^{-1}(x) = (x) + (-1)^{x-1} = x + (-1)^{\pm(x\pm1)}.$$

ILLUSTRATION 232: Let $f : D_f \to \mathbb{R}$, where D_f denotes the domain of function f. Find the inverse of f, if it exists, where (i) $f(x) = 1 - 2^{-x}$ (ii) $f(x) = [4 - (x - 7)^3]^{1/5}$

SOLUTION: (i) Given function is $f(x) = 1 - 2^{-x}$ $\Rightarrow$ $f'(x) = 2^{-x}\ln 2 > 0\ \forall\ x \in \mathbb{R}$

$\Rightarrow$ $f(x)$ is strictly increasing and hence, is injective.

Now $2^{-x} \in (0, \infty)$ $\Rightarrow$ $-2^{-x} \in (-\infty, 0)$ $\Rightarrow$ $1 - 2^{-x} \in (-\infty, 1)$

$\therefore$ Range of function is $(-\infty, 1) \neq \mathbb{R}$

$\Rightarrow$ $f(x)$ is not surjective and hence, $f(x)$ is not bijective.

If $\mathbb{R}$ is replaced by range $(-\infty, 1)$, then $f(x)$ is invertible and for its inverse $y = 1 - 2^{-x}$

$$\Rightarrow 2^{-x} = 1 - y \Rightarrow -x = \log_2(1 - y) \Rightarrow x = \log_2\left(\frac{1}{1-y}\right) \Rightarrow f^{-1}(x) = \log_2\left(\frac{1}{1-x}\right)$$

(ii) $f(x) = [4 - (x - 7)^3]^{1/5}$

$$\Rightarrow f'(x) = \frac{1}{5}\left[4-(x-7)^3\right]^{-4/5}.\left[-3(x-7)^2\right] \leq 0$$

$\Rightarrow$ $f(x)$ is a decreasing function $\Rightarrow$ $f(x)$ is an injective function

Also $f(x)$ is continuous function and $\lim\limits_{x\to\infty} f(x) = -\infty$, $\lim\limits_{x\to-\infty} f(x) = \infty$

$\Rightarrow$ Domain of $f(x) = \mathbb{R}$ and Range of $f(x) = \mathbb{R}$ = codomain

$\Rightarrow$ $f(x)$ is a surjective function

$\therefore$ $f(x)$ is bijective function and hence, is invertible.

Now $y = [4 - (x - 7)^3]^{1/5}$ $\Rightarrow$ $x = 7 + (4 - y^5)^{1/3}$ $\Rightarrow$ $f^{-1}(x) = 7 + (4 - x^5)^{1/3}$

ILLUSTRATION 233: Let $f : \left[-\frac{\pi}{3}, \frac{\pi}{6}\right] \to B$ defined by $f(x) = 2\cos^2 x + \sqrt{3}\sin 2x + 1$. Find B such that f^{-1} exists. Also find $f^{-1}(x)$.

SOLUTION: Given $f : \left[-\frac{\pi}{3}, \frac{\pi}{6}\right] \to B$

If f^{-1} exits, then function should be one–one and onto.

Given $f(x) = 2\cos^2 x + \sqrt{3}\sin 2x + 1 = \cos 2x + 1 + \sqrt{3}\sin 2x + 1$.

$$= 2 + \cos 2x + \sqrt{3}\sin 2x = 2 + 2\left[\frac{1}{2}\cos 2x + \frac{\sqrt{3}}{2}\sin 2x\right] = 2 + 2\left(\sin\left(2x + \frac{\pi}{6}\right)\right) \quad \text{..... (1)}$$

Now for $f(x)$ to be invertible, $2x + \frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (principal domain of $\sin x$)

$$\Rightarrow\ x \in \left[-\frac{\pi}{3}, \frac{\pi}{6}\right]; \text{ which is true}$$

Also $f(x) = 2 + 2\sin\left(2x + \frac{\pi}{6}\right) \in [0, 4]$ = Range of $f(x)$ = co-domain (B) $\Rightarrow$ $B = [0, 4]$

Now from (1); $y - 2 = 2\sin\left(2x + \frac{\pi}{6}\right)$ $\Rightarrow$ $\sin^{-1}\left(\frac{y-2}{2}\right) = 2x + \frac{\pi}{6}$

$$\Rightarrow\ x = \frac{1}{2}\left(\sin^{-1}\left(\frac{y-2}{2}\right) - \frac{\pi}{6}\right) \Rightarrow\ f^{-1}(x) = \frac{1}{2}\left[\sin^{-1}\left(\frac{x-2}{2}\right) - \frac{\pi}{6}\right]$$

ILLUSTRATION 234: If $f(x) = \begin{cases} x; & \text{if } x < 1 \\ x^2; & \text{if } 1 \le x \le 4 \\ 8\sqrt{x}; & \text{if } x > 4 \end{cases}$; then find $f^{-1}(x)$, if $f(x)$ is invertible.

SOLUTION: $y = f(x) = \begin{cases} x; & \text{if } x < 1 \\ x^2; & \text{if } 1 \le x \le 4 \\ 8\sqrt{x}; & \text{if } x > 4 \end{cases}$ … (1)

The graph of $f(x)$ is a shown below

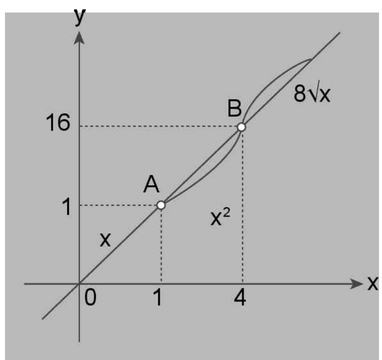


FIGURE 2.184

$$\Rightarrow\ f'(x) = \begin{cases} 1; & \text{if } x < 1 \\ 2x; & \text{if } 1 \le x \le 4 \\ \frac{4}{\sqrt{x}}; & \text{if } x > 4 \end{cases}$$

$\Rightarrow$ $f'(x) > 0\ \forall\ x \in \mathbb{R}$

$\Rightarrow$ $f(x)$ is strictly increasing $\forall\ x \in \mathbb{R}$

$\Rightarrow$ $f(x)$ is injective function $\forall\ x \in \mathbb{R}$

$\Rightarrow$ $f(x)$ is invertible from $\mathbb{R}$ to its range set.

Now from (1); $x = \begin{cases} y; & \text{if } x < 1 \\ \sqrt{y}; & \text{if } 1 \le x \le 4 \\ \frac{y^2}{64}; & \text{if } x > 4 \end{cases}$

$$\Rightarrow\ x = \begin{cases} y; & \text{if } y < 1 \\ \sqrt{y}; & \text{if } 1 \le y \le 16 \\ \frac{y^2}{64}; & \text{if } y > 16 \end{cases}$$

$$\Rightarrow\ f^{-1}(x) = \begin{cases} x & \text{if } x < 1 \\ \sqrt{x} & \text{if } 1 \le x \le 16 \\ \frac{x^2}{64} & \text{if } x > 16 \end{cases}$$

TEXTUAL EXERCISE–14: (SUBJECTIVE)

1. Determine the inverse function and its domain of definition if $f(x)=\begin{cases} x; & -\infty < x < 1 \\ x^2; & 1 \le x \le 4 \\ 2^x; & 4 < x < \infty \end{cases}$

2. Determine f^{-1} (x), if given function is invertible
   (i) f : $(-\infty, -1) \to (-\infty, -2)$ defined $f(x) = -(x+1)^2 - 2$
   (ii) $f:\left[\frac{\pi}{6},\frac{7\pi}{6}\right] \to [-1, 1]$ defined by $f(x) = \sin\left(x+\frac{\pi}{3}\right)$

3. If $f(x)=\begin{cases} x, \text{ if } x \text{ is rational} \\ 0, \text{ if } x \text{ is irrational} \end{cases}$
   $g(x)=\begin{cases} x, \text{ if } x \text{ is rational} \\ 0, \text{ if } x \text{ is irrational} \end{cases}$
   Then prove that the functions $f - g$ and $f + g$ are invertible, whereas $f.g$ is not invertible.

4. Let f: $\mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \dfrac{e^x - e^{-x}}{2}$.
   Is $f(x)$ invertible? If so, find its inverse?

5. Find the inverse of the function (assuming it onto) $y = \log_a\left(x+\sqrt{x^2+1}\right)$; $(a > 1)$

6. If $g(x)$ be the inverse of $f(x)$ and $f'(x) = \dfrac{1}{1+x^3}$, then find $g'(x)$ in term of $g(x)$.

7. If $f(x) = \sin x + \cos x$; $g(x) = x^2 - 4$, then find the domain in which $g(f(x))$ is invertible.

8. If $f(x) = \sin x + \cos x$; $g(x^2) = x^2 + 7$, then find the domain in which $g(f(x))$ is invertible and also find inverse and domain of inverse function.

9. If X and Y are two non-empty sets, where $f: X \to Y$ is a function. Let $C \subseteq X$ and $D \subseteq Y$, we define $f(C) = \{f(x): x \in C\}$. Clearly $f(x) \subseteq Y$ and $f^{-1}(D) = \{x \in X; f(x) \in D\}$. Clearly $f^{-1}(D) \subseteq X$; then show that $f(f^{-1}(B)) = B$ if $B \subseteq f(x)$.

10. Let $g(x)$ be a non-negative function defined on [0, 1]. If the area of the equilateral triangle with two of its vertices at (0,0) and $(x, g(x))$ is $\dfrac{\sqrt{3}}{4}$; then find $g^{-1}(x)$.

11. Let f be an injective function form domain $\{1, i, \omega\}$ to range $\{5, 6, 7\}$. If it is given that exactly one of the following statements is true and other two are false $f(1) = 5$, $f(i) \neq 5$, $f(\omega) \neq 7$, then find $(f^{-1})^{2012}(6)$ $\left(\text{where } i=\sqrt{-1}; \omega = \text{cube root of unity}\right)$

12. Find the solution of equation $2x^2 - 5x - 2 = \dfrac{5-\sqrt{9-8x}}{4}$; where $x < \dfrac{5}{4}$

Answer Keys

1. $f^{-1}(x)=\begin{cases} x; & -\infty < x < 1 \\ \sqrt{x}; & 1 \le x \le 16 \\ \log_2 x; & 16 < x < \infty \end{cases}$

2. (i) $-1-\sqrt{x-2}$ (ii) $\dfrac{2\pi}{3}+\sin^{-1} x$

4. $\log\left(x+\sqrt{x^2+1}\right)$

5. $\dfrac{1}{2}(a^x - a^{-x})$

6. $1 + (g(x))^3$

7. $x \in \left[\dfrac{-\pi}{4},\dfrac{\pi}{4}\right]$

8. $\left[-\dfrac{3\pi}{4},\dfrac{\pi}{4}\right]$, $\sin^{-1}\left(\dfrac{x-7}{\sqrt{2}}\right)-\dfrac{\pi}{4}$, Domain of inverse function is $\left[7-\sqrt{2}, 7+\sqrt{2}\right]$

10. $\sqrt{1-x^2}$; $x \in [0, 1]$

11. ω^2.

12. $x = \dfrac{3-\sqrt{5}}{2}$

TEXTUAL EXERCISE–14: (OBJECITVE)

1. Find $f^{-1}(x)$ if $f : (-\infty, 1] \to (-\infty, 1]$ such that $f(x) = x(2 - x)$, then $f^{-1}(x)$ is given as

(a) $1-\sqrt{1-x}$ (b) $1+\sqrt{1-x}$

(c) $-1-\sqrt{1-x}$ (d) $-1+\sqrt{1-x}$

2. If $f(x) = 3x - 5$ and let $g(x) = f^{-1}(x)$, then $g(4)$ equals

(a) 7

(b) 3

(c) $g(x)$ does not exist as $f(x)$ is not one–one

(d) $g(x)$ does not exist as $f(x)$ is not onto

3. Suppose $f(x) = (x + 2)^2$ for $x \le -2$. If $g(x)$ is the function whose graph is the reflection of the graph of $f(x)$ with respect to the line $y = x$, then $g(x)$ is

(a) $-2-\sqrt{x}; x\ge 0$ (b) $-2+\sqrt{x}; x\ge 0$

(c) $2-\sqrt{x}; x\ge 0$ (d) $2+\sqrt{x}; x\ge 0$

4. If $y = f(x) = \dfrac{x+2}{x+1}$, then

(a) $x = f(y) + 1$

(b) $f^{-1}(4) = 2$

(c) $f^{-1}(0) = 2$

(d) y decreases with x for all $x \ne 1$.

5. If the function $f: \mathbb{R} \to \mathbb{R}$ be such that $f(x) = x = [x]$, where $[y]$ denotes the greatest integer less than or equal to y, then $f^{-1}(x)$ is

(a) $\dfrac{1}{x-[x]}$ (b) $[x] - x$

(c) not defined (d) None of these

6. Let $f : (2, 4) \to (1, 3)$ be a function defined by $f(x) = x-\left[\dfrac{x}{2}\right]$; (where [.] denotes the greatest integer function), then $f^{-1}(x)$ is equal to :

(a) $2x$ (b) $x+\left[\dfrac{x}{2}\right]$

(c) $x + 1$ (d) $x - 1$

7. If $f(x)$: $[0, \infty) \to [0,\infty)$ and $f(x)=\dfrac{x}{1+x}$,then $f(x)$ is

(a) invertible

(b) non-invertible

(c) one–one but not onto

(d) onto but not one–one

8. If f: $[1, \infty) \to [2, \infty)$ is given by $f(x) = x+\dfrac{1}{x}$, then $f^{-1}(x)$ equals

(a) $\dfrac{x+\sqrt{x^2-4}}{2}$ (b) $\dfrac{x}{1+x^2}$

(c) $\dfrac{x-\sqrt{x^2-4}}{2}$ (d) $1+\sqrt{x^2-4}$

9. If $f : [1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, then $f^{-1}(x)$ is given as

(a) $\dfrac{1}{2}\left[1-\sqrt{1+4\log_2 x}\right]$

(b) $\dfrac{1}{2}\left[1+\sqrt{1+4\log_2 x}\right]$

(c) $\left[1-\sqrt{1+4\log_2 x}\right]$

(d) $\left[1-\sqrt{1+4\log_2 x}\right]$

10. Let $f(x) = x^x$, $x \in (0, \infty)$ and let $g(x)$ inverse of $f(x)$, then $g'(x)$ must be

(a) $x(1 + \log x)$ (b) $x(1 + \log f(x))$

(c) $\dfrac{1}{x(1+\log g(x))}$ (d) Does not exist.

11. The inverse function of the function $f(x) = \dfrac{e^x-e^{-x}}{e^x+e^{-x}}$ is

(a) $\dfrac{1}{2}\log\dfrac{1+x}{1-x}$ (b) $\dfrac{1}{2}\log\dfrac{2+x}{2-x}$

(c) $\dfrac{1}{2}\log\dfrac{1-x}{1-x}$ (d) None of these

12. If $A_1 = \{1, 2, 3, 4\}$ and $A_2 = \{5, 6, 7, 8\}$; $A_3 = \{9, 10\}$; $A_4 = \{11, 12, 13, 14\}$. Let n_{ij} = Number of invertible functions from A_i to A_j, then Σn_{ij} equals

(a) 36 (b) 576

(c) 144 (d) None of these

13. Let f be a function defined as $f(x) = \sqrt{3}\sin x - \cos x + 2$. Then $f^{-1}(x)$ is given by

(a) $\sin^{-1}\left(\dfrac{x-2}{2}\right)-\dfrac{\pi}{6}$

(b) $\sin^{-1}\left(\dfrac{x-2}{2}\right)+\dfrac{\pi}{6}$

(c) $\sin^{-1}\dfrac{2\pi}{3}+\cos^{-1}\left(\dfrac{x-2}{2}\right)$

(d) None of these

14. If $f(x) = \sqrt{\sin^{-1}(2x)+\frac{\pi}{6}}$ and $g(x) = f^{-1}(x)$, then domain of $g(x)$ is

(a) $\left[-\sqrt{\frac{2\pi}{3}}, \sqrt{\frac{2\pi}{3}}\right]$ (b) $\left[0, \sqrt{\frac{2\pi}{3}}\right]$

(c) $\left[0, \sqrt{\frac{\pi}{2}}\right]$ (d) None of these

15. If $f(x) = \sqrt{\cos^{-1}(2x)+\frac{\pi}{6}}$ and $g(x) = f^{-1}(x)$, then range of $g(x)$ is

(a) [0,1] (b) $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(c) $\left[\frac{1}{2}, 1\right]$ (d) None of these

16. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and let $m(b)$ be the minimum value of $f(x)$. If b varies $m(b)$ becomes a function of b, then inverse function of $m(b)$ from its principal domain to range is given by $m^{-1}(x)$ equals

(a) $\sqrt{\frac{1-x}{x}}$ (b) $-\sqrt{\frac{1-x}{x}}$

(c) $\sqrt{\frac{x-1}{x}}$ (d) None of these

17. Let $f : (0, \infty) \to (1, \infty)$ be a function such that $F(x) = \int_0^x f(t)\,dt$. If $F(x^2) = x^2(1 + x)$, and $g(x) = f^{-1}(x)$, then g(10) equals

(a) 49 (b) 99

(c) 36 (d) None of these

18. If $f : (0, \infty) \to (1, \infty)$ be a function such that $F(x) = \int_0^x f(t)\,dt$. If $F(x^2) = x^2(1 + x)$ and $g(x) = f^{-1}(x)$, then the point where $g(x)$ and $f(x)$ intersect is

(a) $\left(\frac{1}{4}, \frac{1}{4}\right)$ (b) (4, 2)

(c) (4, 4) (d) Does not exist

Answer Keys

1. (a)	**2.** (b)	**3.** (a)	**4.** (b)	**5.** (c)	**6.** (c)	**7.** (b, c)	**8.** (a)	**9.** (b)	**10.** (c)
11. (a)	**12.** (c)	**13.** (b)	**14.** (b)	**15.** (b)	**16.** (a)	**17.** (c)	**18.** (c)		

■ EVEN AND ODD FUNCTIONS

The knowledge about the symmetry of graph of the function extends great help in study of the behaviour of function and analysis of problems involving these functions. Fundamentally functions possess two kinds of symmetry (i) symmetry about a straight line (say y-axis) (ii) symmetry about a point (say origin).

If a function has its graph symmetric about y-axis, i.e., portion of graph lying to the left of y-axis is mirror image of the portion lying to the right of y-axis, then it is known as an even function. If the graph of the function is symmetric about origin, i.e., it is symmetric in opposite quadrants, then it is known as odd function.

For instance, $f(x) = \cos x$ is symmetric about y-axis and is called an even function and $f(x) = \sin x$ is symmetric about origin and called as odd function graphically shown in the Figures.

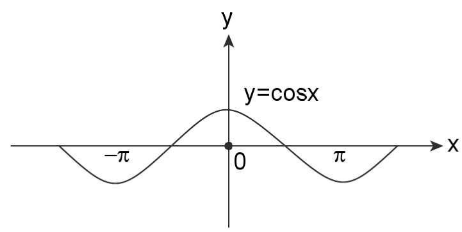


FIGURE 2.185

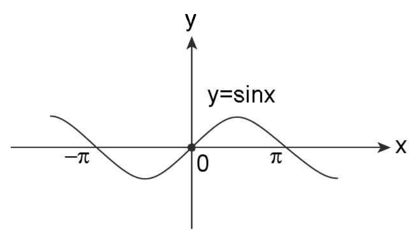


FIGURE 2.186

REMARKS

(i) The functions having no symmetry like odd/even functions are called 'neither even nor odd functions'.

(ii) Before testing the even/odd symmetry of the function it is essential to observe whether the domain of function is symmetric about y-axis, i.e., if the domain is of the type $[-x_0, x_0]$ or $[-x_2, -x_1] \cup [x_1, x_2]$ etc.

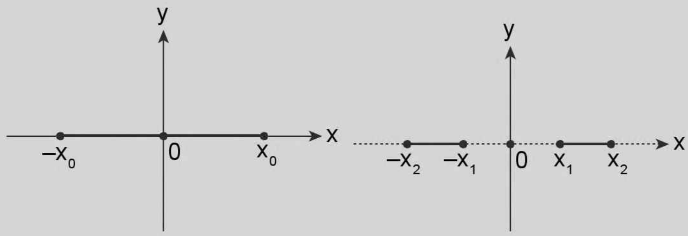


FIGURE 2.187

EVEN FUNCTIONS

A function $f : X \rightarrow Y$ is said to be an even function iff $f(-x) = f(x)\ \forall\ x, -x \in X$ (Domain). i.e., $f(x) - f(-x) = 0$

e.g., x^{2n}, $\sin^2 x$, $\cos x$, $\sec x$, $2^x + 2^{-x}$

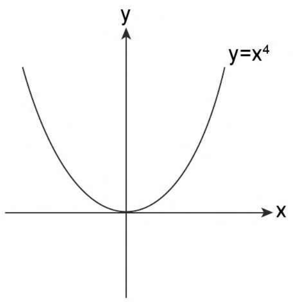


FIGURE 2.188

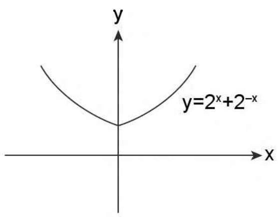


FIGURE 2.189

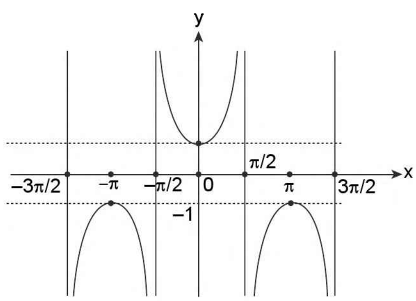


FIGURE 2.190

Properties of Even Functions:

(i) Graph of even function is symmetric about y-axis because if (x, y) lies on the curve, $(-x, y)$ also lies on the curve.

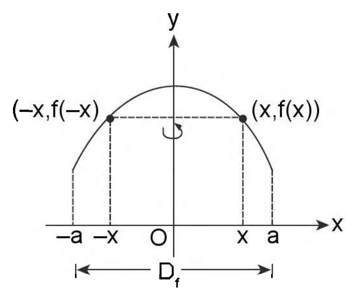


FIGURE 2.191

Proof: Since for even function $f(-x) = f(x)$ $\forall\ x, -x \in X$ (Domain), i.e., $y = f(x) \Rightarrow y = f(-x)$

Consequently $(x, y) \in f \Rightarrow (-x, y) \in f$

It is important to see that if we rotate the curve by 180° about y-axis, then the appearance of the rotated curve is same as the original curve. We can state this alternatively; if we rotate portion of the curve on the left side of y-axis by 180° about y-axis; then we get the portion of graph to the right of y-axis and vice versa.

(ii) For any function $f(x)$, if $g(x) = f(x) + f(-x)$, then $g(x)$ is always an even function.

Proof: $g(-x) = f(-x) + f(x) = g(x)\ \forall\ x, -x \in D_f$

Hence, $g(x)$ is an even function.

(iii) The domain of even function must be symmetric about zero, i.e., D_f: $[-a, a]$ or $(-a, a)$ or $(-a_2, -a_1) \cup (a_1, a_2)$; where a is some real number.

e.g., $f : [-2, 4] \rightarrow \mathbb{R}$ defined as $f(x) = x^2$ is not an even function since domain is not symmetric about zero.

However it will be an even function if we redefine the function $f : [-2, 2] \to \mathbb{R}$ as $f(x) = x^2$ or . $f : [-4, 4] \to \mathbb{R}$ as $f(x) = x^2$

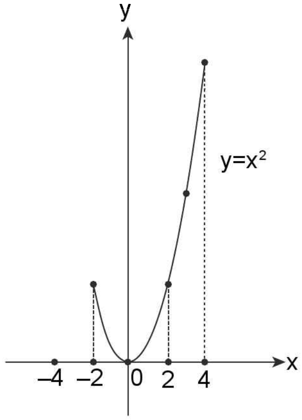


FIGURE 2.192

(iv) Even functions are non-invertible as they can't be strictly monotonic when taken in their natural domain, however, even functions can be made invertible by restricting their domains to either side of y-axis.

Proof: Since $f(-x) = f(x)\ \forall\ x, -x \in D_f$

$\Rightarrow$ $f(x)$ is many–one and hence, not bijective

$\Rightarrow$ $f(x)$ is non-invertible in D_f

(v) If $f(x)$ is even function, then $f'(x)$ is odd function.

Proof: Let $g(x) = f'(x)$, given $f(x)$ an even function. Since $f(-x) = f(x)$; differentiating both sides with respect to x, we have $f'(-x)(-1) = f'(x)$

$\Rightarrow$ $g(-x) = -g(x)$ $\Rightarrow$ $g(x)$ is an odd function.

(vi) $f(x) = c$; where c is a constant defined on symmetrical domain is an even function.

Proof: Since for $x, -x \in D_f$, $f(x) = f(-x) = c$

$\Rightarrow$ $f(x)$ is an even function.

ILLUSTRATION 235: Show that the function $f(x) = x^3 \log\left(\dfrac{1-x}{1+x}\right)$ is an even function.

SOLUTION: To show that $f(x)$ is even we need to prove that $f(-x) = f(x)$.

Since we have domain of $f(x) = (-1, 1)$ which is symmetric about y-axis and $f(x) = x^3 \log\left(\dfrac{1-x}{1+x}\right)$

$$\Rightarrow\ f(-x) = (-x)^3 \log\left(\frac{1-(-x)}{1+(-x)}\right) = -x^3 \log\left(\frac{1+x}{1-x}\right) = -x^3 \log\left(\frac{1-x}{1+x}\right)^{-1} = x^3 \log\left(\frac{1-x}{1+x}\right) = f(x)$$

Thus, $f(x)$ is clearly an even function.

ILLUSTRATION 236: Test whether $f(x) = x(e^x - e^{-x}) + \sqrt{\cot^{-1}(\sin(\sec^{-1} x)}$ is even function or not ?

SOLUTION: To investigate the even nature of function, we need to find the relation between $f(-x)$ and $f(x)$.

Now $f(x) = x(e^x - e^{-x}) + \sqrt{\cot^{-1}(\sin(\sec^{-1} x))}$ $\Rightarrow f(-x) = -x(e^{-x} - e^x) + \sqrt{\cot^{-1}(\sin(\sec^{-1}(-x))}$

$= x(e^x - e^{-x}) + \sqrt{\cot^{-1}(\sin(\sec^{-1}(x)}$ $= f(x)$ $\Rightarrow$ Indeed $f(x)$ is an even function.

ILLUSTRATION 237: Let $f(x) = |x - 4| + |x - 5| + |x - 6|$ and $g(x) = f(x + k)$. Find k so that $g(x)$ is an even function.

SOLUTION: Given $f(x) = |x - 4| + |x - 5| + |x - 6|$ and $g(x) = f(x + k)$

We observe that $x - 4$, $x - 5$ and $x - 6$ are in an $A.P.$

Since the numbers of a group of numbers are equally deviated from the $A.M.$ of the numbers of the group, so, let $y = A.M.$ of $(x - 4, x - 5, x - 6)$ $\Rightarrow$ $y = x - 5$

Replacing x by $y + 5$ in $f(x)$; we get $f(y + 5) = |y + 1| + |y| + |y - 1|$

$\Rightarrow$ $f(-y + 5) = |-y + 1| + |-y| + |-y - 1| = |1 - y| + |y| + |1 + y| = f(y + 5)$

$\therefore$ $g(-y) = g(y)$ for $k = 5$. Hence, value of $k = 5$

ILLUSTRATION 238: If $f(x)$ is a polynomial satisfying $f(x) . f(y) - f(x) - f(y) - f(x.y) + 2 = 0$; $\forall\ x, y \in \mathbb{R}$ and $f(3) = 10$, then show that $f(x)$ is an even function.

SOLUTION: Given $f(x) \,.\, f(y) - f(x) - f(y) - f(xy) + 2 = 0$ ...(1)

Putting $x = 1$ and $y = 1$ in (1), we get $f(1) \,.\, f(1) - f(1) - f(1) - f(1) + 2 = 0$

$\Rightarrow \; (f(1))^2 - 3(f(1)) + 2 = 0 \;\; \Rightarrow \;\; f(1) = 1$ or 2

Case I: Putting $y = 1$ and $f(1) = 1$, we get $f(x)\, f(1) = f(x) + f(1) + f(x) - 2$

$\Rightarrow \; f(x) = 1 \;\forall\, x \in \mathbb{R}$ (rejected, $\because f(3) = 10$)

Case II: Putting $y = 1$ and $f(1) = 2$, we get $f(x).f(1) = f(x) + f(1) + f(x) - 2$

$\Rightarrow \; 2f(x) = 2f(x) + 2 - 2$, which is identically true, hence, $f(1) = 2$

Now; given that $f(x)$ is a polynomial function, therefore putting $y = 1/x$; we get

$\Rightarrow \; f(x)\, f(1/x) = f(x) + f(1/x) + f(1) - 2 \;\; \Rightarrow \;\; f(x).\, f(1/x) = f(x) + f(1/x) \;\; \Rightarrow \;\; f(x) = 1 \pm x^n$

Let $f(x) = 1 + x^n$ and $f(1) = 2; f(3) = 10 \;\; \Rightarrow \;\; n = 2$

$\Rightarrow \; f(x) = 1 - x^n$ is rejected as it does not satisfy $f(1) = 2; f(3) = 10$ for any value of n

$\therefore \; f(x) = 1 + x^2$ and $f(-x) = 1 + (-x)^2 = 1 + x^2 = f(x) \;\; \Rightarrow \;\; f(x)$ is an even function

ILLUSTRATION 239: Prove that in the product $(1 - x + x^2 - x^3 + \ldots - x^{49} + x^{50})\,(1 + x + x^2 + x^3 + \ldots + x^{49} + x^{50})$, after simplification there will be no term containing odd power of x, and hence, show that the above product represents an even function.

SOLUTION: Let $f(x) = (1 - x + x^2 - x^3 + \ldots\ldots - x^{49} + x^{50})\,(1 + x + x^2 + \ldots + x^{50})$

$= [(1 + x^2 + x^4 + \ldots + x^{50}) - (x + x^3 + \ldots + x^{49})]\,.\,[(1 + x^2 + x^4 + \ldots + x^{50}) + (x + x^3 + \ldots + x^{49})]$

$= (1 + x^2 + x^4 + \ldots + x^{50})^2 - (x + x^3 + \ldots + x^{49})^2$

$= (1 + x^2 + x^4 + \ldots + x^{50})^2 - x^2\,(1 + x^2 + \ldots + x^{48})^2$

Now we can observe that the above expression does not contain any term of x having odd degree. Hence, $f(x)$ is an even function.

ODD FUNCTIONS

A function $f : X \to Y$ is said to be an odd function iff $f(-x) = -f(x) \;\forall\, x, -x \in X$

i.e., $f(x) + f(-x) = 0 \;\forall\, x, -x \in X$

e.g., x^3, $\sin x$, $\tan x$, $2^x - 2^{-x}$ are odd functions

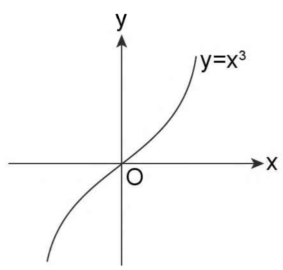


FIGURE 2.193

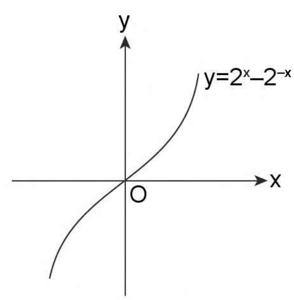


FIGURE 2.194

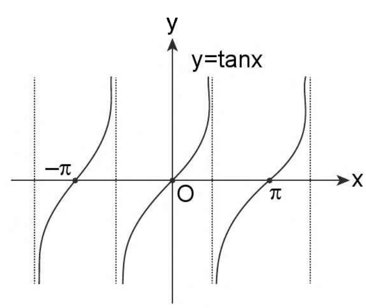


FIGURE 2.195

ILLUSTRATION 240: Prove that $f(x)=4\sqrt{1-x^3}-4\sqrt{1+x^3}$ is an odd function.

SOLUTION: Given $f(x)=4\sqrt{1-x^3}-4\sqrt{1+x^3} \Rightarrow f(-x)=4\sqrt{1-(-x)^3}-4\sqrt{1+(-x)^3}$

$= 4\sqrt{1+x^3}-4\sqrt{1-x^3} = -\left(4\sqrt{1-x^3}-4\sqrt{1+x^3}\right)=-f(x)$

Hence, $f(x)$ is an odd function.

ILLUSTRATION 241: Find whether the following functions are even or odd or none.

(a) $f(x) = \sin x + \tan x$ (b) $f(x) = x \sin^2 x - x^3$ (c) $f(x) = \sin x - \cos x$

SOLUTION: (a) Given $f(x) = \sin x + \tan x \Rightarrow f(-x) = -\sin x - \tan x = -f(x)$

$\therefore f(-x) = -f(x) \Rightarrow f(x)$ is an odd function.

(b) Given $f(x) = x \sin^2 x - x^3 \Rightarrow f(-x) = -x \sin^2 x + x^3 = -f(x)$

$\Rightarrow f(-x) = -f(x) \Rightarrow f(x)$ is an odd function

(c) Given $f(x) = \sin x - \cos x \Rightarrow f(-x) = -\sin x - \cos x \neq f(x), -f(x)$

$\therefore f(x)$ is neither odd nor even function.

ILLUSTRATION 242: Find whether the following functions are even or odd or none.

(a) $f(x)=\dfrac{(x-1)\tan x}{x-1}$ (b) $f(x)=\dfrac{x^5(x^2-1)}{x^2-1}$

SOLUTION: (a) Given function is $f(x)=\dfrac{(x-1)\tan x}{x-1}$

Domain of given function $f(x)$ is $\mathbb{R} \sim \left\{1,(2n+1)\dfrac{\pi}{2};n\in\mathbb{Z}\right\}$.

Here for $x \in D_f$, $f(x) = \tan x$ which seems to be an odd function, but actually it is not so, because the domain of function is not symmetric about the origin as $f(x)$ is defined at $x = -1$, whereas it is not defined at $x = 1$.

(b) Given function is $f(x)=\dfrac{x^5(x^2-1)}{x^2-1}$

Domain of given function $f(x)$ is $\mathbb{R} \sim \{1, -1\}$.

Here for $x \in D_f$, $f(x) = x^5$ which is an odd function since $f(-x) = -f(x)$ and domain of function is symmetric about origin. Therefore, $f(x)$ is an odd function.

■ PROPERTIES OF ODD FUNCTIONS

(i) Graph of odd function is symmetric about origin. Also known as symmetric in opposite quadrants.
i.e., straight lines passing through origin cuts the graph both sides symmetrically.
This is because if (x, y) lies on the curve, $(-x, -y)$ also lies on the curve.

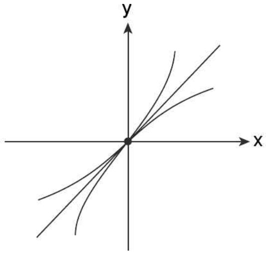


FIGURE 2.196

Proof: Since for an odd function $f(-x) = -f(x)$ $\forall\, x \in X$, i.e., $y = f(x)$

$\Rightarrow -y = f(-x)$ which clearly leads to the fact that $(x, y) \in f$

$\Rightarrow (-x, -y) \in f$

Therefore, it is really interesting to observe that if we rotate the curve by 180° about origin, then the appearance of the rotated curve remains same as the original curve. Further, it is important to notice that the portion of curve lying to the left of y-axis can be obtained by the following two steps:

Step 1: Reflecting (taking mirror image) of the portion of curve lying to the right side of y-axis about y-axis

Step 2: Reflecting (taking mirror image) of 'reflection obtained in step (1)' about x-axis.

(ii) For any function $f(x)$, if $g(x) = f(-x) - f(x)$, then $g(x)$ is always an odd function.

Proof: Since $g(x) = f(-x) - f(x)$

$\Rightarrow g(-x) = f(x) - f(-x) = -g(x)\ \forall\, x, -x \in D_f$

(iii) The domain of odd function must be symmetric about zero. i.e., D_f: $[-a, a]$ or $(-a, a)$ where a is some real number,
e.g., f: $[-\pi, 2\pi] \to \mathbb{R}$ defined as $f(x) = \sin x$ is not an odd function because its domain is not symmetric about zero.

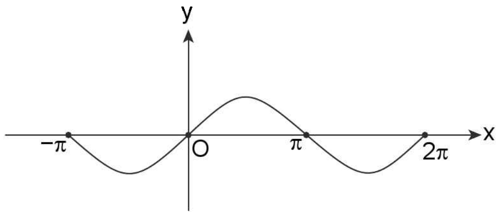


FIGURE 2.197

However $f(x) = \sin x$ is an odd function when defined on set $[-a, a]$ or $(-a, a)$ $\forall\ a \in \mathbb{R}$

(iv) If $f(x)$ is odd, then $f'(x)$ is an even function.

Proof: Let $f'(x) = g(x)$.
Since $f(-x) = -f(x)\ \forall\ x, -x \in D_f$
Differentiating with respect to x, we have $f'(x) = f'(-x)$
$\Rightarrow$ $g(x) = g(-x)\ \ \forall\ x, -x \in D_g$
$\Rightarrow$ $g(x) = f'(x)$ is an even function

(v) If $x = 0$ lies in the domain of an odd function, then $f(0) = 0$, i.e., the graph of an odd function must pass through origin.

Proof: Since $f(x) = -f(-x)\ \forall\ x, -x \in D_f$ and $0 \in D_f$
$\Rightarrow$ $f(0) = -f(0) \Rightarrow\ 2f(0) = 0\ \ \Rightarrow f(0) = 0$
i.e., $f(x)$ passes through (0, 0) (origin)

ILLUSTRATION 243: Prove that $f(x) = \left(\sqrt{1+x} + \sqrt{1-x}\right).\left(\dfrac{a^x - 1}{a^x + 1}\right)$ is an odd function.

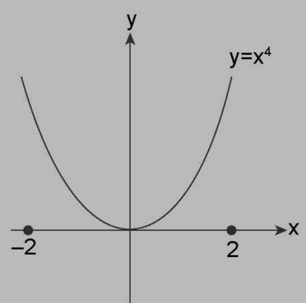
SOLUTION: Clearly the domain of $f(x)$ is $[-1, 1]$ which is symmetric about zero. Therefore, in order to prove that $f(x)$ is odd, we need to only establish that $f(-x) = -f(x)$.

Now $f(-x) = \left(\sqrt{1+(-x)} + \sqrt{1-(-x)}\right).\left(\dfrac{a^{-x} - 1}{a^{-x} + 1}\right) = \left(\sqrt{1-x} + \sqrt{1+x}\right)\left(\dfrac{\frac{1}{a^x} - 1}{\frac{1}{a^x} + 1}\right)$

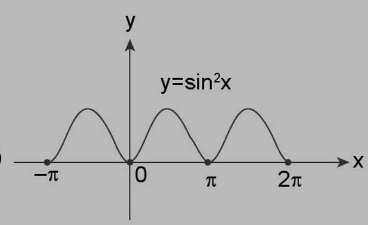
$= -\left(\sqrt{1-x} + \sqrt{1+x}\right)\left(\dfrac{a^x - 1}{a^x + 1}\right) = -f(x)$, consequently $f(x)$ is an odd function

ILLUSTRATION 244: Identify whether the functions graphically shown below are even or odd.

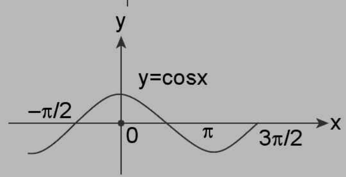
(a)



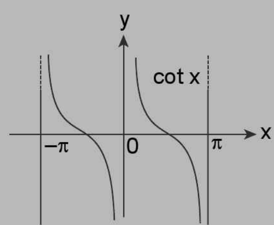
(b)



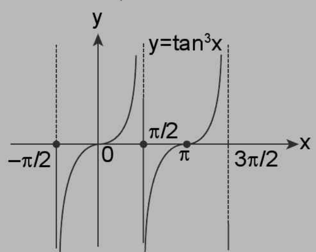
(c)



(d)



(e)



SOLUTION: (a) $f(x)$ is an even function, since the domain is symmetric about origin and the graph is symmetric about y-axis.

(b) $f(x)$ is neither even nor odd, since the domain is not symmetric about origin.

(c) $f(x)$ is neither even nor odd, since the domain is not symmetric about origin.

(d) $f(x)$ is an odd function, since the domain is symmetric about origin and the graph is symmetric in opposite quadrants.

(e) $f(x)$ is neither even nor odd, since the domain is not symmetric about origin.

ILLUSTRATION 245: Find whether the following functions are even or odd or none.

(a) $f(x)=\log(x+\sqrt{1+x^2})$ (b) $f(x)=\dfrac{x(a^x+1)}{a^x-1}$

(c) $f(x)=\dfrac{(1+2^x)^2}{2^x}$ (d) $f(x)=\dfrac{x}{e^x-1}+\dfrac{x}{2}+1$

(e) $f(x) = [(x + 1)^2]^{1/3} + [(x - 1)^2]^{1/3}$

SOLUTION: (a) Given $f(x)=\log(x+\sqrt{1+x^2})$ ...(1)

$\Rightarrow \quad f(-x)=\log(-x+\sqrt{1+x^2})$ ...(2)

Adding (1) and (2), we have $f(-x) + f(x) = \log\ (\sqrt{1+x^2}+x)+\log(\sqrt{1+x^2}-x)$

$= \log [(1 + x^2) - x^2] \ \Rightarrow\ f(-x) + f(x) = 0\ \Rightarrow\ f(-x)=-f(x)\ \Rightarrow\ f(x)$ is an odd function

(b) Given $f(x)=x\dfrac{(a^x+1)}{a^x-1}$

$\Rightarrow \quad f(-x)=-\dfrac{x(a^{-x}+1)}{a^{-x}-1}=-\dfrac{x(1+a^x)}{1-a^x}=\dfrac{x(a^x+1)}{a^x-1}=f(x)$; $f(x)$ is an even function

(c) Given $f(x)=\dfrac{(1+2^x)^2}{2^x}\ \Rightarrow\ f(-x)=\dfrac{(1+2^{-x})^2}{2^{-x}}=\dfrac{\left(1+\dfrac{1}{2^x}\right)^2}{\dfrac{1}{2^x}}=\dfrac{(2^x+1)^2}{2^{2x}.\dfrac{1}{2^x}}=\dfrac{(1+2^x)^2}{2^x}=f(x)$

$\therefore\ f(x)$ is an even function.

(d) Given $f(x)=\dfrac{x}{e^x-1}+\dfrac{x}{2}+1$ .... (1)

$\Rightarrow\ f(-x) = -\dfrac{x}{e^{-x}-1}-\dfrac{x}{2}+1=\dfrac{-x}{\dfrac{1-e^x}{e^x}}-\dfrac{x}{2}+1\ \Rightarrow\ f(-x)=\dfrac{xe^x}{e^x-1}-\dfrac{x}{2}+1$ ... (2)

Subtracting equation (1) from (2), we get $f(-x)-f(x)=\dfrac{x(e^x-1)}{e^x-1}-x=0$

$\Rightarrow\ f(-x) = f(x)\quad \therefore\ f(x)$ is an even function.

(e) Given $f(x) = [(x + 1)^2]^{1/3} + [(x - 1)^2]^{1/3}$

$\Rightarrow\ f(-x) = [(-x + 1)^2]^{1/3} + [(-x - 1)^2]^{1/3} = [(1 - x)^2]^3 + [(1 + x)^2]^{1/3}$

$= [(x - 1)^2]^{1/3} + [(1 + x)^2]^{1/3} = f(x)\qquad \Rightarrow\ f(x)$ is an even function

ILLUSTRATION 246: Let $f(x) = e^{\{e^{|x|}\operatorname{sgn} x\}}$ and $g(x) = e^{[e^{|x|}\operatorname{sgn} x]}$, $x \in \mathbb{R}$; where { } and [ ] denotes the fractional part and integral part functions respectively. Also if $h(x) = \ln\left(f(x)\right) + \ln\left(g(x)\right)$ for all real x, then show that $h(x)$ is an odd function.

SOLUTION: Given $h(x) = \ln\left(f(x)\cdot g(x)\right) = \ln e^{\{y\}+[y]}$; where $y = e^{|x|} \operatorname{sgn} x$

$\Rightarrow\ h(x) = \{y\} + [y] = y = e^{|x|}\operatorname{sgn} x$

$$\therefore\ h(x)=e^{|x|}\operatorname{sgn}x=\begin{cases}e^{x} & \text{if } x>0\\ 0 & \text{if } x=0\\ -e^{-x} & \text{if } x<0\end{cases}\qquad \therefore\ h(-x)=\begin{cases}e^{-x} & \text{if } x<0\\ 0 & \text{if } x=0\\ -e^{x} & \text{if } x>0\end{cases}$$

Clearly $h(x)+h(-x)=0$ for all $x\in\mathbb{R}$ $\Rightarrow$ $h(-x)=-h(x)\ \forall\ x$ $\Rightarrow$ $h(x)$ is an odd function.

ILLUSTRATION 247: Find whether the function $f(x)=\log\left(\dfrac{1+\sin x}{1-\sin x}\right)$ is odd or even or none of these.

SOLUTION: Given function is $f(x)=\log\left(\dfrac{1+\sin x}{1-\sin x}\right)$

$\Rightarrow$ $f(-x)=\log\left(\dfrac{1-\sin x}{1+\sin x}\right)=-\log\left(\dfrac{1+\sin x}{1-\sin x}\right)=-f(x)$ $\Rightarrow$ $f(x)$ is an odd function.

ILLUSTRATION 248: Find whether the function $f(x)=[x]+\dfrac{1}{2}$, $x\notin\mathbb{Z}$ is odd or even or none of these.

SOLUTION: Given function is $f(x)=[x]+\dfrac{1}{2}, x\notin\mathbb{Z}$

$\Rightarrow$ $f(-x)=[-x]+\dfrac{1}{2}=-[x]-1+\dfrac{1}{2}=-\left([x]+\dfrac{1}{2}\right)=-f(x)$ $(\because\ [x]+[-x]=-1$ for $x\notin\mathbb{Z})$

$\therefore$ $f(x)$ is an odd function.

ILLUSTRATION 249: If the graph of the function $f(x)=\dfrac{a^x-1}{x^n(a^x+1)}$; $n\in\mathbb{Z}$ is symmetric about y-axis, then show that n is an odd integer.

SOLUTION: Given function is $f(x)=\dfrac{a^x-1}{x^n(a^x+1)}$; $D_f=\mathbb{R}-\{0\}$

Since graph of function is symmetric about y-axis, it implies that function is an even function

$\Rightarrow$ $f(-x)=f(x)\ \forall\ x\in D_f$ $\Rightarrow$ $\dfrac{1-a^x}{(-x)^n(a^x+1)}=\dfrac{a^x-1}{x^n(a^x+1)}$ $\Rightarrow$ $-(x)^n=(-x)^n=(-1)^n x^n$

$\Rightarrow$ $-1=(-1)^n$ $\Rightarrow$ n must be an odd integer.

■ ALGEBRA OF EVEN–ODD FUNCTIONS

1. $f(x)=0$ (identically zero function) is the only function which is both, an odd and an even function, provided it is defined in a symmetric domain.

Proof: If $f(x)$ is even, then $f(-x)=f(x)$

If $f(x)$ is odd, then $f(-x)=-f(x)$

Comparing $f(-x)$, we get $f(x)=-f(x)$

$\Rightarrow$ $f(x)=0\ \forall\ x\in\mathbb{R}$

ILLUSTRATION 250: Prove that the function defined as $f(x)=\begin{cases}e^{-\sqrt{|\ell n\{x\}|}}-\{x\}^{\sqrt{\frac{1}{|\ell n\{x\}|}}}; & \text{whenever it exists}\\ \{x\}; & \text{otherwise}\end{cases}$; $f(x)$ is odd as well as even; where $\{x\}$ denotes fraction part of x.

SOLUTION: We have to prove that function is odd as well as even. It is possible only when $f(x)=0$.

Consider the following two cases as given below.

Case I: Whenever the expression $e^{-\sqrt{|\ell n\{x\}|}}-\{x\}^{\sqrt{\frac{1}{|\ell n\{x\}|}}}$ exists

Now, $\{x\}\in[0,1)$ and $\ln\{x\}$ is defined for $\{x\}\in(0,1)$. Hence, $\forall\ x\in\mathbb{R}\sim\mathbb{Z}$

Now $f(x) = \ln\{x\} < 0$ for $\{x\} \in (0, 1) \Rightarrow |\ln\{x\}| = -\ln\{x\}$.

$$\therefore \quad e^{-\sqrt{|\ell n\{x\}|}} - \{x\}^{\sqrt{\frac{1}{|\ell n\{x\}|}}} = e^{-(-\ln\{x\})^{1/2}} - e^{\ln\{x\}^{\left(\frac{1}{-\ln\{x\}}\right)^{\frac{1}{2}}}} = e^{-(-\ln\{x\})^{1/2}} - e^{\frac{1}{(-\ln\{x\})^{1/2}}.\ln\{x\}}$$

$$= e^{-(-\ln\{x\})^{1/2}} - e^{-(-\ln\{x\})^{1/2}} = 0$$

Case II: $\{\text{For } x \in \mathbb{Z}\}$

$f(x) = \{x\} = 0$ ($\because$ $\{x\}$ equal to zero for x an integer). Thus, $f(x) = 0 \ \forall \ x \in \mathbb{R}$

$\Rightarrow$ $f(x)$ is even as well as odd function.

ILLUSTRATION 251: It is given that $f(x)$ is an even function and satisfy the relation $f(x) = \dfrac{x f(x^2)}{2 + \tan^2 x . f(x^2)}$, then find the value of $f(10)$.

SOLUTION: Given $f(x) = \dfrac{xf(x^2)}{2 + \tan^2 x \, f(x^2)} \Rightarrow f(-x) = \dfrac{-xf(x^2)}{2 + \tan^2(x) \, f(x^2)}$

$\Rightarrow f(-x) = -f(x) \Rightarrow f(x)$ is an odd function, but $f(x)$ is given to be an even function

$\Rightarrow f(x)$ is odd as well as an even function $\Rightarrow f(x) = 0 \ \forall \ x \in \mathbb{R} \Rightarrow f(10) = 0$

2. A linear combination of two or more even functions is an even function, i.e., in particular for two even functions $f(x)$ and $g(x)$, the function $(\alpha f + \beta g)$ is an even function; where $\alpha, \beta \in \mathbb{R}$.

   Proof: Since $f(x)$ and $g(x)$ are even functions, therefore $f(-x) = f(x)$ and $g(-x) = g(x)$.

   Now let $h = (\alpha f + \beta g)$; where $\alpha, \beta \in \mathbb{R}$

$\Rightarrow h(-x) = (\alpha f + \beta g)(-x) = \alpha f(-x) + \beta g(-x)$

$\Rightarrow h(-x) = \alpha f(x) + \beta g(x) = (\alpha f + \beta g)(x) = h(x)$

$\Rightarrow h(-x) = h(x) \Rightarrow h(x)$ is an even function.

   Applying the above result for more than two functions repeatedly, we can show that linear combination of more than two even functions is also an even function.

3. A linear combination of two or more odd functions is an odd function, i.e., in particular for two odd functions $f(x)$ and $g(x)$, the function $(\alpha f + \beta g)$ is an odd function; where $\alpha, \beta \in \mathbb{R}$.

   Proof: Since $f(x)$ and $g(x)$ are odd functions, therefore $f(-x) = -f(x)$ and $g(-x) = -g(x)$.

   Let $h = \alpha f + \beta g$; where $\alpha, \beta \in \mathbb{R}$

$\Rightarrow h(-x) = (\alpha f + \beta g)(-x) = \alpha f(-x) + \beta g(-x)$

$\Rightarrow h(-x) = -\alpha f(x) - \beta g(x) = -(\alpha f(x) + \beta g(x))$

$= -(\alpha f + \beta g)(x) = -h(x) \Rightarrow h(-x) = -h(x)$

$\Rightarrow h(x)$ is an odd function.

   Applying the above result for more than two functions repeatedly, we can show that linear combination of more than two odd functions is also an odd function.

4. The product of two or more even functions is an even function.

   Proof: Let $f(x)$ and $g(x)$ be two even functions

$\Rightarrow f(-x) = f(x)$ and $g(-x) = g(x)$. Let $h = f.g$

$\Rightarrow h(x) = f(x).g(x)$

$\Rightarrow h(-x) = (f.g)(-x) = f(-x) . g(-x)$

$\Rightarrow h(-x) = f(x) . g(x) = (f.g)(x) = h(x)$

$\Rightarrow h(-x) = h(x) \Rightarrow h = f.g$ is an even function.

   Applying the above result for more than two functions repeatedly, we can show that the product of more than two even function is again an even function.

5. The product of an odd and an even function is an odd function.

   Proof: Let $f(x)$ be an odd function and $g(x)$ be an even function

$\Rightarrow f(-x) = -f(x)$ .... (1)

$g(-x) = g(x)$ .... (2)

Now let $h = f.g \Rightarrow h(x) = (f.g)(x) = f(x) . g(x)$

$\therefore h(-x) = f(-x) . g(-x)$

$= [-f(x)] . [g(x)]$ (by using (1) and (2))

$= -f(x) . g(x) = -(f.g)(x) = -h(x)$

$\therefore h(-x) = -h(x)$

$\Rightarrow h = f.g$, i.e., the product of an odd and an even function is an odd function.

6. The quotient of two even functions (or two odd functions) is an even function.

Proof: Let $f(x)$ and $g(x)$ be two even functions

$\Rightarrow f(-x) = f(x)$ and $g(-x) = g(x)$

Let $h = \dfrac{f}{g}$

$$\Rightarrow h(-x) = \frac{f}{g}(-x) = \frac{f(-x)}{g(-x)} = \frac{f(x)}{g(x)} = \frac{f}{g}(x) = h(x)$$

$\Rightarrow h(-x) = h(x) \Rightarrow h(x)$ is an even function

Similarly if $f(x)$ and $g(x)$ are two odd functions, then

$$h(-x) = \frac{f}{g}(-x) = \frac{f(-x)}{g(-x)} = \frac{-f(x)}{-g(x)} = \frac{f}{g}(x) = h(x)$$

$\Rightarrow h(x)$ is an even function

7. The nature (odd or even) of product of odd functions depends upon the number of functions taken in the product. i.e., product of odd number of odd functions is an odd function and that of even number of odd functions is an even function.

Proof: ***Case I:*** If an even number of odd functions is taken, then their product will be an even function.

Let there be $2n$ number of odd functions. If we make n pairs of all the functions, then the product of functions in each pair is an even function, as the product of two odd functions is an even function.

Thus, our product is reduced to product of n even functions, which is an even function (by property (4)).

Case II: If an odd number of odd functions is taken, then their product will be an odd function.

Let there be $(2n + 1)$ number of odd functions. If we make n pairs of the functions in the product, then we are left with an unpaired odd function. The product of functions in each pair is an even function, as the product of two odd functions is an even function. Thus, our product is reduced to product of n even functions and an odd function which in turn is a product of an even and an odd function, i.e., an odd function. (by property 5).

8. Composition of several functions, $f(g(h...(p(x))...)))$ is odd iff all are odd functions.

Proof: Let $F(x) = f(g(h\ldots p(x))\ldots)))$

$\Rightarrow F(-x) = f(g(h\ldots(p(-x))\ldots))$

$= f(g(h\ldots(-p(x))\ldots.)) = \ldots. = f(g(-(h\ldots(p(x))\ldots)))$

$= f(-g(h\ldots(p(x))\ldots.) = -f(g(h\ldots(p(x))..) = -F(x)$

Thus, $F(-x) = -F(x) \Rightarrow F(x)$ is an odd function.

9. Composition of several functions is even iff atleast one function is even, provided the function composed of either even or odd functions after that even function. i.e., if $F(x) = f(g(h\ldots(p(x))\ldots)))$ and $g(x)$ is an even function, then $h(x), \ldots, p(x)$ all should be either even or odd. For example, $f(x) = e^{\log\tan(\cos\sin(x^3))}$;

then $f(-x) = e^{\log\tan(\cos(\sin(-x)^3))} = e^{\log\tan(\cos(-\sin x^3))}$

$= e^{\log\tan(\cos(\sin x^3))} = f(x)$

10. Any function $f(x)$ can always be written as sum of an even function and an odd function.

i.e., $f(x) = \underbrace{\dfrac{f(x)+f(-x)}{2}}_{even} + \underbrace{\dfrac{f(x)-f(-x)}{2}}_{odd}$;

provided its domain is symmetric about origin.

Proof: Let $g(x) = \dfrac{f(x)+f(-x)}{2}$, then $g(-x) = g(x)$, hence, $g(x)$ is an even function.

and let $h(x) = \dfrac{f(x)-f(-x)}{2}$, then $h(-x) = -h(x)$, hence, $h(x)$ is an odd function.

Thus, $f(x) = g(x) + h(x)$; where $g(x)$ is even and $h(x)$ is an odd function.

NOTE

Consider $f(x) = \dfrac{2}{1-\tan x}$. *Suppose that we express f(x) as* $\left(\dfrac{1}{1-\tan x} + \dfrac{1}{1+\tan x}\right) + \left(\dfrac{1}{1-\tan x} - \dfrac{1}{1+\tan x}\right)$, *then the domain of the new representation is* $\mathbb{R} - \{x: \tan x = \pm 1\}$, *which is different from that of f(x) which is* $\mathbb{R} - \{x: \tan x = 1\}$, *and hence, it is a function different from f(x). This difficulty appears because the domain of f is not symmetric about x = 0. Hence, we cannot express this function as a sum of an even and an odd function*

ILLUSTRATION 252: Express $f(x) = \sin x - \cos x$ as sum of even and odd functions.

SOLUTION: Here domain of f is obviously the set of real numbers, and hence, for each x, $(-x)$ is also in the domain.

Now we can write f as $f(x) = \frac{1}{2}\{f(x) + f(-x)\} + \frac{1}{2}\{f(x) - f(-x)\}$

$= \frac{1}{2}\{(\sin x - \cos x) + (\sin(-x) - \cos(-x))\} + \frac{1}{2}\{(\sin x - \cos x) - (\sin(-x) - \cos(-x))\}$

$= -\cos x + \sin x = h(x) + g(x)$

where $h(x) = -\cos x$ is an even function and $g(x) = \sin x$ is an odd function.

Note that we have followed the standard procedure to express f as a sum of even and odd functions. The same thing could also be done without this if one could sort out all even terms together and odd terms together separately.

ILLUSTRATION 253: Test whether the following functions are even or odd

(a) $f(x) = e^x + e^{-x}$ (b) $(\tan x^5)e^{x^3 \operatorname{sgn} x^7}$

SOLUTION: (a) Given $f(x) = e^x + e^{-x} \Rightarrow f(-x) = e^{-x} + e^{-(-x)} = e^{-x} + e^x \Rightarrow f(-x) = f(x)$;

This clearly shows that the given function is even.

(b) $f(x) = (\tan x^5)e^{x^3 \operatorname{sgn} x^7}$

Since $\tan x$ and x^5 are odd functions, therefore their composition is an odd function.

And x^3, $sgn\ x$, x^7 are all odd functions, therefore sgn x^7 is odd function and $x^3 \times sgn\ x^7$ is an even function. Finally, the given function becomes product of an odd function and even function, so it must be an odd function.

ILLUSTRATION 254: Examine whether the following functions are even, odd or none

(i) $f(x) = \begin{cases} x|x|; & \text{if } x \le -1 \\ [1+x]+[1-x]; & \text{if } -1 < x < 1 \\ -x|x|; & \text{if } x \ge 1 \end{cases}$

(ii) $f(x) = \dfrac{2x(\sin x + \tan x)}{2\left[\dfrac{x+19\pi}{\pi}\right] - 37}$, where [ ] denotes greatest integer function

SOLUTION: (i) Given function is $f(x) = \begin{cases} x|x|; & \text{if } x \le -1 \\ [1+x]+[1-x]; & \text{if } -1 < x < 1 \\ -x|x|; & \text{if } x \ge 1 \end{cases}$, graphically shown below

$\Rightarrow f(-x) = \begin{cases} -(-x)|-x|; & \text{if } x \le -1 \\ [1-x]+[1+x]; & \text{if } -1 < x < 1 \\ (-x)|-x|; & \text{if } x \ge 1 \end{cases}$

$= \begin{cases} x|x|; & \text{if } x \le -1 \\ [1-x]+[1+x]; & \text{if } -1 < x < 1 = f(x) \\ (-x)|x| & \text{if } x \ge 1 \end{cases}$

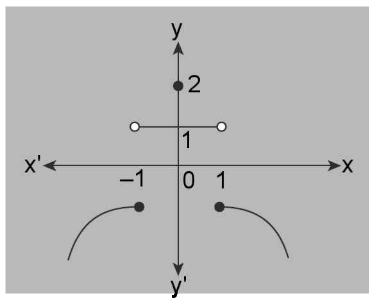


FIGURE 2.198

Hence, $f(x)$ is an even function

(ii) Given $f(x) = \dfrac{2x(\sin x + \tan x)}{2\left[\dfrac{x+19\pi}{\pi}\right] - 37} = \dfrac{2x(\sin x + \tan x)}{2\left[\dfrac{x}{\pi} + 19\right] - 37}$

$$\Rightarrow f(x) = \frac{2x(\sin x + \tan x)}{2\left[\dfrac{x}{\pi}\right] + 1} \quad \ldots (1)$$

$$\Rightarrow\ f(-x)=\frac{2(-x)[\sin(-x)+\tan(-x)]}{2\left[\frac{(-x)}{\pi}\right]+1}=\frac{2(x)[\sin(x)+\tan(x)]}{2\left[\frac{(-x)}{\pi}\right]+1}$$

Now, if $x = n\pi$, $n \in \mathbb{Z}$, then $f(x) = f(-x) = 0 \Rightarrow f(-n\pi) = -f(n\pi) = 0$

So, let $x \neq n\pi$, $n \in \mathbb{Z}$, then $-\frac{x}{\pi} \notin \mathbb{Z} \Rightarrow \left[\frac{-x}{\pi}\right]=-\left(1+\left[\frac{x}{\pi}\right]\right)$ ($\because$ $[x]+[-x] = -1$ for $x \notin \mathbb{Z}$)

$$\Rightarrow\ f(-x)=\frac{2x(\sin x+\tan x)}{2\left\{-1-\left[\frac{x}{\pi}\right]\right\}+1}=\frac{2x(\sin x+\tan x)}{-1-2\left[\frac{x}{\pi}\right]}=-f(x) \Rightarrow f(x) \text{ is an odd function.}$$

ILLUSTRATION 255: If $f(x + y) = f(x) . f(y)$ and $f(0) \neq 0$, then prove that

(i) $g(x)=\dfrac{1-f(x)}{1+f(x)}$ is an odd function

(ii) $h(x)=\dfrac{1-f(x)+f^2(x)}{1+f(x)+f^2(x)}$ is an even function

SOLUTION: (i) Given function is $g(x)=\dfrac{1-f(x)}{1+f(x)}$ .... (1)

And $f(x + y) = f(x) . f(y)$ and $f(0) \neq 0$ .... (2)

Putting $x = y = 0$ in (2), we get $f(0) = (f(0))^2$

$\Rightarrow f(0) = 0$ or $f(0) = 1$

But given that $f(0) \neq 0$.

$\Rightarrow f(0) = 1$ .... (3)

Now putting $y = -x$ in (2); we get $f(x - x) = f(x) . f(-x)$

$\Rightarrow f(0) = f(x) . f(-x) \Rightarrow 1 = f(x) . f(-x) \Rightarrow f(-x)=\dfrac{1}{f(x)}$

Now from (1), we have $g(x)=\dfrac{1-f(x)}{1+f(x)}$

$$\Rightarrow\ g(-x)=\frac{1-f(-x)}{1+f(-x)}=\frac{1-1/f(x)}{1+(1/f(x))}=\frac{f(x)-1}{1+(f(x))}=-g(x) \Rightarrow g(x) \text{ is an odd function}$$

(ii) Given function is $h(x)=\dfrac{1-f(x)+f^2(x)}{1+f(x)+f^2(x)} \Rightarrow h(-x)=\dfrac{1-f(-x)+(f(-x))^2}{1+(f(-x))+(f(-x))^2}$

$$=\frac{1-\frac{1}{f(x)}+\frac{1}{(f(x))^2}}{1+\frac{1}{f(x)}+\frac{1}{(f(x))^2}}=\frac{(f(x))^2-f(x)+1}{(f(x))^2+f(x)+1}=h(x) \Rightarrow h(x) \text{ is an even function.}$$

■ EVEN AND ODD EXTENSION OF FUNCTION

If $f(x)$ is a function $f : [\alpha, \beta] \rightarrow B$ such that α and β are of same sign, then the domain of function can be extended and the function $f(x)$ can be redefined such that it becomes an even or odd function without having any change in the definition of f on $[\alpha, \beta]$.

Even Extension

Extending the domain of function $f(x)$ and redefining such that the function obtained is even.

i.e., $h(x)=\begin{cases} f(x); & \text{if } \alpha \le x \le \beta \\ f(-x); & \text{if } -\beta \le x \le -\alpha \end{cases}$

Clearly for $x \in [\alpha, \beta]$, $h(x) = f(x)$ and $-x \in [-\beta, -\alpha]$

$\Rightarrow \quad h(-x) = f(x).$

Thus, $h(-x) = h(x) = f(x)$ for $x \in [\alpha, \beta]$

Similarly for $x \in [-\beta, -\alpha]$, $h(x) = h(-x) = f(-x)$.

Thus, $h(x)$ defined above is an even function with domain $[-\beta, -\alpha] \cup [\alpha, \beta]$ called even extension of $f(x)$.

e.g., even extension of $f(x) = \begin{cases} \sqrt{x}; & \text{if } 0 \le x \le 1 \\ x; & \text{if } 1 < x \le 2 \end{cases}$ is given by $h(x) = \begin{cases} -x; & -2 \le x < -1 \\ \sqrt{-x}; & -1 \le x \le 0 \\ \sqrt{x}; & 0 \le x \le 1 \\ x; & 1 < x \le 2 \end{cases}$

The graphical interpretation of such extension is that 'the graph of function $f(x)$ is extended by taking the mirror image of $f(x)$ in y-axis, i.e., image about y-axis'.

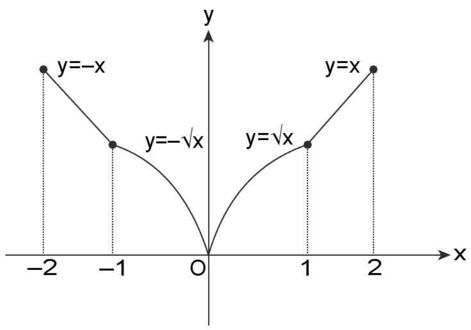


FIGURE 2.199

Odd Extension

Extending the domain of function and redefining it such that the new function obtained becomes odd.

i.e., $h(x) = \begin{cases} f(x); & \text{if } \alpha \le x \le \beta \\ -f(-x); & \text{if } -\beta \le x \le -\alpha \end{cases}$

Clearly for $x \in [\alpha, \beta]$, $h(x) = f(x)$ and $-x \in [-\beta, -\alpha]$

$\Rightarrow \quad h(-x) = -f(x).$

Thus, $h(-x) = -h(x) = -f(x)$ for $x \in [\alpha, \beta]$

Similarly for $x \in [-\beta, -\alpha]$, $h(-x) = -h(x) = -f(-x)$.

Thus, $h(x)$ defined above is an odd function with domain $[-\beta, -\alpha] \cup [\alpha, \beta]$ called odd extension of $f(x)$.

e.g., odd extension of $f(x) = \begin{cases} x^2; & 0 \le x \le 2 \\ 4; & 2 < x \le 4 \end{cases}$ is given by $h(x) = \begin{cases} -4; & -4 \le x < -2 \\ -x^2; & -2 \le x \le 0 \\ x^2; & 0 \le x < 2 \\ 4; & 2 < x \le 4 \end{cases}$

The graphical interpretation of such extension is that 'the graph of function $f(x)$ is extended by taking the mirror image of $f(-x)$ in x-axis'.

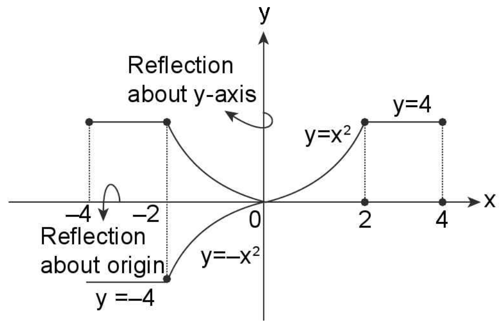


FIGURE 2.200

ILLUSTRATION 256: A function f defined for all real numbers is defined as follows $f(x) = \begin{cases} x; & \text{if } 0 \le x \le 1 \\ 1; & \text{if } x > 1 \end{cases}$

How f should be defined for $x \le 0$ so that

(a) f becomes an even function (b) f becomes an odd function

SOLUTION: (a) For making f an even function we need even extension of $f(x)$ given by $h(x) = \begin{cases} f(x); & \text{if } \alpha \le x \le \beta \\ f(-x); & \text{if } -\beta \le x \le -\alpha \end{cases}$

Thus, $h(x) \begin{cases} x; & \text{if } 0 \le x \le 1 \\ 1; & \text{if } \quad 1 \\ -x; & \text{if } -1 \le x < 0 \\ 1; & \text{if } \quad < -1 \end{cases}$

Thus, $f(x) = 1$ for $x < -1$ and $-x$ for $-1 \le x \le 0$

(b) For making f an odd function we need odd extension of $f(x)$ given by $h(x)=\begin{cases} f(x); & \text{if } \alpha \le x \le \beta \\ -f(-x); & \text{if } -\beta \le x \le -\alpha \end{cases}$

Thus, $h(x)=\begin{cases} x; & \text{if } 0 \le x \le 1 \\ 1; & \text{if } x>1 \\ x; & \text{if } -1 \le x < 0 \\ -1; & \text{if } x < -1 \end{cases}$; Thus, $f(x) = -1$ for $x < -1$ and x for $-1 \le x \le 0$

ILLUSTRATION 257: Given $f(x)=\begin{cases} 1; & x \le -1 \\ -x; & -1 \le x \le 0 \end{cases}$

Find the odd and even extension of the function $f(x)$

SOLUTION: Even extension of $f(x)$ is given by $h(x)=\begin{cases} f(x); & \text{if } \alpha \le x \le \beta \\ f(-x); & \text{if } -\beta \le x \le -\alpha \end{cases}$

$$\therefore \quad h(x)=\begin{cases} 1; & x \le -1 \\ -x; & -1 \le x \le 0 \\ x; & 0 \le x \le 1 \\ 1; & 1 \le x < \infty \end{cases};$$

and odd extension of $f(x)$ is given by $h(x)=\begin{cases} f(x); & \text{if } \alpha \le x \le \beta \\ -f(-x); & \text{if } -\beta \le x \le -\alpha \end{cases}$

$$\therefore \quad h(x)=\begin{cases} 1; & x \le -1 \\ -x; & -1 \le x \le 0 \\ -x; & 0 \le x \le 1 \\ -1; & x \ge 1 \end{cases}$$

ILLUSTRATION 258: Write odd extension of the following functions:

(i) $\sin x + |\sin x|$; for $x \in [0, \infty)$ (ii) $\ln x - \dfrac{(\ln x)^2}{|\ln x|}$

SOLUTION: (i) Given $f(x) = \sin x + |\sin x|$; for $x \in [0, \infty)$

$$\Rightarrow \quad f(x)=\begin{cases} 2\sin x; & \text{if } \sin x \ge 0 \\ 0; & \text{if } \sin x < 0 \end{cases}; \text{ for } x \in [0, \infty)$$

$$\Rightarrow \text{ odd extension of function } f(x) = h(x) = \begin{cases} 2\sin x; & \text{if } \sin x \ge 0 & \text{for } x \ge 0 \\ 0; & \text{if } \sin x < 0 & \text{for } x \ge 0 \\ 2\sin x; & \text{if } \sin x \le 0 & \text{for } x \le 0 \\ 0; & \text{if } \sin x > 0 & \text{for } x \le 0 \end{cases}$$

$$= \begin{cases} 2\sin x; & \text{if } \sin x \ge 0 & \text{for } x \ge 0 \\ 0; & \text{if } \sin x < 0 & \text{for } x \ge 0 \\ \sin x - |\sin x|; & \text{if } \sin x \le 0 & \text{for } x \le 0 \\ \sin x - |\sin x|; & \text{if } \sin x > 0 & \text{for } x \le 0 \end{cases} = \begin{cases} \sin x + |\sin x|; & \text{if } x \ge 0 \\ \sin x - |\sin x|; & \text{if } x \le 0 \end{cases}$$

(ii) Given $y = \ln x - \dfrac{(\ln x)^2}{|\ln x|}$

$$\Rightarrow \quad y=\begin{cases} \ln x - \ln x; & \text{if } \ln x > 0 \\ 2\ln x \quad ; & \text{if } \ln x < 0 \end{cases} \qquad \Rightarrow \quad y=\begin{cases} 0; & \text{if } x > 1 \\ 2\ln x; & \text{if } 0 < x < 1 \end{cases}$$

$$\Rightarrow \text{ odd extension } = h(x) = \begin{cases} 0; & \text{if } x > 1 \\ 2\ln x; & \text{if } 0 < x < 1 \\ -2\ln(-x); & \text{if } -1 < x < 0 \\ 0; & \text{if } x < -1 \end{cases}$$

ILLUSTRATION 259: Let $f(x) = x^2 + 5x - 2$ defined on $A = [0, 2]$. Find even and odd extension of $f(x)$ in $[-2, 2]$.

SOLUTION: Given $f(x) = x^2 + 5x - 2$, $f(-x) = x^2 - 5x - 2$

Let g_e and f_o denote even and odd extension respectively.

$$\therefore \quad g_e(x) = \begin{cases} f(x); & x \in [0,2] \\ f(-x); & x \in [-2,0] \end{cases} \qquad [\because f(-x) = f(x) \text{ for even function } f(x)]$$

$$f_0(x) = \begin{cases} f(x); & x \in [0,2] \\ -f(-x); & x \in [-2,0] \end{cases} \qquad [\because f(-x) = f(x) \text{ for odd function } f(x)]$$

$$\therefore \quad g_e(x) = \begin{cases} x^2 + 5x - 2; & x \in [0,2] \\ x^2 - 5x - 2; & x \in [-2,0] \end{cases} \text{ and } f_0(x) = \begin{cases} x^2 + 5x - 2; & x \in [0,2] \\ -x^2 + 5x + 2; & x \in [-2,0] \end{cases}$$

ILLUSTRATION 260: If $f(x) = \begin{cases} x^3 + x^2; & \text{for } 0 \le x \le 2 \\ x + 2; & \text{for } 2 < x \le 4 \end{cases}$, then find the even and odd extension of the function $f(x)$:

SOLUTION: The even extension of $f(x)$ is given by $h(x) = \begin{cases} f(x); & \text{if } \alpha \le x \le \beta \\ f(-x); & \text{if } -\beta \le x \le -\alpha \end{cases}$

$$\text{as follows } g(x) = \begin{cases} x^3 + x^2; & \text{for } 0 \le x \le 2 \\ x + 2; & \text{for } 2 < x \le 4 \\ -x + 2; & \text{for } -4 \le x \le -2 \\ -x^3 + x^2; & \text{for } -2 \le x \le 0 \end{cases}$$

The odd extension of $f(x)$ is given by $h(x) = \begin{cases} f(x); & \text{if } \alpha \le x \le \beta \\ -f(-x); & \text{if } -\beta \le x \le -\alpha \end{cases}$

$$\text{as follows: } g(x) = \begin{cases} x^3 + x^2; & \text{for } 0 \le x \le 2 \\ x + 2; & \text{for } 2 < x \le 4 \\ x - 2; & \text{for } -4 \le x \le -2 \\ x^3 - x^2; & \text{for } -2 \le x \le 0 \end{cases}$$

TEXTUAL EXERCISE–15: (SUBJECTIVE)

1. Determine whether the following functions are even or odd or neither even nor odd:

(i) $\tan x$

(ii) $\cos x$

(iii) $\sin (x^2 + 1)$

(iv) $x + x^2$

(v) $x - x^3$

(vi) $f(x) = x\left(\dfrac{a^x - 1}{a^x + 1}\right)$

(vii) $f(x) = \log\left(x + \sqrt{x^2 + 1}\right)$

(viii) $f(x) = \sin x + \cos x$

(ix) $f(x) = (x^2 - 1)\,|\,x\,|$

2. Check whether the following functions $f(x)$ are even/odd or neither:

(a) $2^{x^2-x^4}$

(b) $\sin x + \cos x$

(c) $x^2 - |x|$

(d) $\frac{x}{2^x-1}+\frac{x}{2}+1$

(e) $x^{10}.\log\frac{1+\sin x}{1-\sin x}$

(f) $|f(x)| + 1$; $f(x)$ is an odd function

(g) $\frac{(1+2^x)^9}{2^x}$

(h) $\sqrt{1+x+x^2}-\sqrt{1-x+x^2}$

(i) $\phi(x) = [f(x) + f(-x)]\,[g(x) - g(-x)]$

(j) $f(x)=\frac{\sec x + x^2 - 9}{x\sin x}$

3. Determine whether the function $f(x) = (-1)^{[x]}$ is even, odd or neither of the two.

4. Let $\begin{cases} 4; & \text{if } x<-1 \\ -4x; & \text{if } -1\le x\le 0 \end{cases}$. If $f(x)$ is an even function in $\mathbb{R}$, then find the definition of $f(x)$ in $(0, \infty)$.

5. Prove that function $f(x)$ satisfying the property $f(x + y) + f(x - y) = 2f(x).\,f(y)$ is an even function.

6. If $f(x)$ is a real valued function, satisfying $f(x + y) + f(x - y) = 2f(x).\,f(y)\ \forall\ x, y \in \mathbb{R}$. Then prove that the function $f(x)$ is odd if $f(0) = 0$ and even if $f(0) = 1$.

7. Prove that $f(x)=\frac{2x(\sin x+\tan x)}{2\left[\frac{x+21\pi}{\pi}\right]-41}$ is an odd function.

8. Test whether $f(x)$ is even or odd or neither even nor odd, $f(x)=\sqrt{\cos x}$ if $x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and $\sqrt{-\cos x}$ for $x\in\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$.

9. If $f : [-20, 20] \to \mathbb{R}$ defined by $f(x)=\left[\frac{x^2}{a}\right]\sin x+\cos x$, is an even function, then find the set of values of a.
(where [.] denotes greatest integral function)

10. A function defined for all real numbers, is defined for $x > 0$ as follows:
$f(x)=\begin{cases} x|x|, & 0\le x<1 \\ 2x, & x\ge 1 \end{cases}$, how should f be defined for $x < 0$? if
(i) f is an even function
(ii) f is an odd function

11. If $f(x)=\begin{cases} x^2+1; & \text{if } 0<x\le 1 \\ x+1; & \text{if } 1<x\le 2 \end{cases}$. Find even and odd extension of $f(x)$.

12. Extend $f(x) = x^2 + x$ defined in $[0, 3]$ onto the interval $[-3, 3]$ so that the extended function becomes
(i) an even function
(ii) an odd function.

13. If $f(x)=\begin{cases} x^2+\sin x; & \text{if } 0\le x<1 \\ x+e^{-x}; & \text{if } x\ge 1 \end{cases}$, then extend the definition of $f(x)$ for $x \in (-\infty, 0)$ such that $f(x)$ becomes
(i) an even function
(ii) an odd function

Answer Keys

1. (i) odd, (ii) even (iii) even (iv) neither even nor odd
(v) odd (vi) even (vii) odd (viii) neither even nor odd
(ix) even

2. Even: (a), (c), (f), (j); Odd: (e), (h), (i); Neither: (b) , (d), (g)

3. f is an even function when $x \in \mathbb{Z}$ and odd function when $x \notin \mathbb{Z}$

4. $f(x)=\begin{cases} 4x; & \text{if } 0\le x\le 1 \\ 4; & \text{if } x>1 \end{cases}$

8. Neither even nor odd

9. $a\in(400,\infty)$

10. (i) $\begin{cases} x|x|; & \text{if } 0 \le x < 1 \\ 2x; & \text{if } x \ge 1 \\ -2x; & \text{if } x \le -1 \\ -x|x|; & \text{if } -1 < x \le 0 \end{cases}$ (ii) $\begin{cases} x|x|; & \text{if } 0 \le x < 1 \\ 2x; & \text{if } x \ge 1 \\ 2x; & \text{if } x \le -1 \\ x|x|; & \text{if } -1 < x \le 0 \end{cases}$

11. Even extension = $\begin{cases} x^2+1; & \text{if } 0 < x \le 1 \\ x+1; & \text{if } 1 < x \le 2 \\ x^2+1; & \text{if } -1 \le x < 0 \\ -x+1; & \text{if } -2 \le x < -1 \end{cases}$ and odd extension = $\begin{cases} x^2+1; & \text{if } 0 < x \le 1 \\ x+1; & \text{if } 1 < x \le 2 \\ -x^2-1; & \text{if } -1 \le x < 0 \\ x-1; & \text{if } -2 \le x < -1 \end{cases}$

12. (i) $\begin{cases} x^2+x; & \text{if } [0,3] \\ x^2-x; & \text{if } -3 \le x < 0 \end{cases}$ (ii) $\begin{cases} x^2+x; & \text{if } [0,3] \\ -x^2+x; & \text{if } -3 \le x < 0 \end{cases}$

13. (i) $h(x) = \begin{cases} x^2+\sin x; & \text{if } 0 \le x < 1 \\ x+e^{-x}; & \text{if } x \ge 1 \\ x^2-\sin x; & \text{if } -1 < x \le 0 \\ -x+e^{x}; & \text{if } x \le -1 \end{cases}$ (ii) $h(x) = \begin{cases} x^2+\sin x; & \text{if } 0 \le x < 1 \\ x+e^{-x}; & \text{if } x \ge 1 \\ -x^2+\sin x; & \text{if } -1 < x \le 0 \\ x-e^{x}; & \text{if } x \le -1 \end{cases}$

TEXTUAL EXERCISE–15: (OBJECTIVE)

1. Match the functions of column I with column II to correctly define their nature:

Column I

(i) $\dfrac{1-x}{1+x}$

(ii) $\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$

(iii) sin (sin x)

(iv) sin (cos x)

(v) cos (cos x)

(vi) $\dfrac{\sin^4 x+\cos^4 x}{x+x^2\tan x}$

(vii) $x^{17}\left(\dfrac{a^{\sin x}-1}{a^{\sin x}+1}\right)$

Column II

(a) an odd function

(b) an even function

(c) neither odd nor even

(d) cannot be said

Which of the following represents the correct match for column I and column II?

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)
(a)	C	A	A	B	B	A	B
(b)	A	A	C	B	B	A	C
(c)	C	A	A	B	B	B	C
(d)	C	A	B	A	A	B	A

2. Match the functions of column I with column II to correctly define their nature.

Column I

(i) $x^2 \sin\left\{\dfrac{1}{\sin(1/x)}\right\}$

(ii) $\cos x.\log\dfrac{1+\tan x}{1-\tan x}$

(iii) $\left(\dfrac{a^{2x}+1}{a^x}\right)$

(iv) $\dfrac{(1+e^x)^2}{e^x}$

(v) $\sqrt[3]{1+x+x^2}+\sqrt[3]{1-x+x^2}$

(vi) $\sqrt{1+2x+3x^2}-\sqrt{1-2x+3x^2}$

Column II

(a) an odd function
(b) an even function
(c) neither odd nor even
(d) cannot be said

Which of the following represent the correct match for column I and column II?

	(i)	(ii)	(iii)	(iv)	(v)	(vi)
(a)	A	A	B	A	A	B
(b)	A	B	A	B	A	B
(c)	A	A	B	B	B	A
(d)	A	C	D	B	B	A

3. Which of the following functions is/are even?
(a) $f(x) = \ln |x|$
(b) $f(x)=\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$
(c) $f(x)=x\left(\dfrac{a^x-1}{a^x+1}\right)$
(d) $f(x)=\sin\left[\log\left(x+\sqrt{x^2+1}\right)\right]$

4. Which of the following functions is/are odd?
(a) $sgn\, x + x^{2000}$ (b) $|x| - \tan x$
(c) $x^3 \cot x$ (d) $\operatorname{cosec} x^{55}$

5. The function $f(x)=\sec\left[\log\left(x+\sqrt{1+x^2}\right)\right]$ is
(a) odd
(b) even
(c) neither odd nor even
(d) constant

6. The function $f(x)=\sin\left(\tan\left(\log\left(x+\sqrt{x^2+1}\right)\right)\right)$ is
(a) an even function
(b) an odd function
(c) a periodic function
(d) neither an even nor an odd function

7. Let $f(x)=\begin{cases} 0; & \text{for } x=0 \\ x^2\sin\left(\dfrac{\pi}{x}\right); & \text{for } -1<x<1\ (x\neq 0) \\ x|x|; & \text{for } x>1 \text{ or } x<-1 \end{cases}$ then:
(a) $f(x)$ is an odd function
(b) $f(x)$ is an even function
(c) $f(x)$ is neither odd nor even
(d) $f'(x)$ is an even function

8. If $f(x) = 2x^6 + 3x^4 + 4x^2 + 7$, then $f'(x)$ is
(a) even function
(b) an odd function
(c) neither even nor odd
(d) None of these

9. On the interval $[0, 1]$, $f(x)$ is defined as, $f(x)=\begin{cases} x; & \text{if } x\in\mathbb{Q} \\ 1-x; & \text{if } x\notin\mathbb{Q}\end{cases}$. Then for all $x \in \mathbb{R}$ the composite function $f(f(x))$ is:
(a) a constant function
(b) an identity function
(c) an odd linear polynomial
(d) $1 + x$

10. Let $f(x)=\begin{cases} 0; & \text{if } x \text{ is rational} \\ x; & \text{if } x \text{ is irrational}\end{cases}$ and $g(x)=\begin{cases} 0; & \text{if } x \text{ is irrational} \\ x; & \text{if } x \text{ is rational}\end{cases}$; then the function $(f-g)\,x$ is
(a) odd
(b) even
(c) neither odd nor even
(d) odd as well as even

11. If $f(x)$ is an odd function, then:
(a) $\dfrac{f(-x)+f(x)}{2}$ is an even function
(b) $\dfrac{f(x)-f(-x)}{2}$ is neither even nor odd
(c) $[\,|f(x)| + 1]$ is even; where $[x]$ = the greatest integer $\leq x$
(d) None of these

12. If $f(x) = a^{\sin x^3 \operatorname{sgn} x^9}(\tan^3 x)$; $a > 1$ is
(a) an odd function
(b) an even function
(c) neither even nor odd function
(d) None of these

13. If $f(x) = (a - x^n)^{1/n}$; $n \in \mathbb{N}$, then $f(f(x))$ is
(a) an even function
(b) an odd function
(c) neither even nor odd
(d) even as well as odd

14. If $f(x)$ is an even function and $f'(x)$ exists, then $f'(e) + f'(-e)$ is $(e \in \mathbb{R})$
(a) > 0 (b) $= 0$
(c) ≥ 0 (d) < 0

15. A function $f(x)$ not identically zero satisfy the functional equation $f(x + y) = f(x) + f(y)$, then the function $g(x) = a^{f(x)} + a^{-f(x)}$; $a > 1$ is
(a) odd function
(b) even function

(c) neither even nor odd
(d) even as well as odd

16. If $g(f(x)) = |\sin x|$ and $f(g(x)) = (\sin\sqrt{x})^2$; then
(a) $f(x)$ is even and $g(x)$ is odd
(b) $f(x)$ is odd and $g(x)$ is even
(c) $f(x)$ is even and $g(x)$ is neither even nor odd
(d) $f(x)$ is odd and $g(x)$ is neither even nor even.

17. Let $f(x) = x^3 - x^2 + \sin^{-1}\left(\dfrac{2-|x|}{3}\right) + \log(|x| + 1)$, $0 \le x \le 2$, then odd extension of $f(x)$ in $[-2, 2]$ is:
(a) $x^3 - x^2 - \sin^{-1}\left(\dfrac{2-|x|}{3}\right) - \log(|x|+1)$
(b) $x^3 - x^2 + \sin^{-1}\left(\dfrac{2-|x|}{3}\right) + \log(|x|+1)$
(c) $x^3 + x^2 - \sin^{-1}\left(\dfrac{2-|x|}{3}\right) - \log(|x|+1)$
(d) None of these

18. If $f(x) = \sin^2 x + \sin^2\left(x+\dfrac{\pi}{3}\right) + \cos x \,.\, \cos\left(x+\dfrac{\pi}{3}\right)$, then $f(x)$ is
(a) an even function
(d) an odd function
(c) neither even nor odd
(d) even as well as odd

Answer Keys

1. (a)	**2.** (c)	**3.** (a, c)	**4.** (d)	**5.** (b)	**6.** (b)	**7.** (a, d)	**8.** (b)	**9.** (b, c)	**10.** (a)
11. (c)	**12.** (a)	**13.** (b)	**14.** (b)	**15.** (b)	**16.** (c)	**17.** (c)	**18.** (a)		

PERIODIC FUNCTIONS

In our day to day life we usually come across many man made and natural phenomenon-that repeatedly occur after a regular interval of time. For instance, rotation of planets around sun, position of needle of the clock etc. In mathematics the periodicity of functions has same notion as above, except the fact that here it can be observed with respect to any variable. Periodic functions are those who repeat their values after regular interval of values of independent variable in the entire domain. When we observe the graph of periodic functions, indeed we notice that the graph can be constructed by choosing its particular smallest periodic segment and repeating it throughout the domain. The length of such smallest periodic segment (along domain axis) is known as Fundamental Period of the function. e.g., observe the graph of $y = \{x\}$ (fraction part function), the graph can well be constructed by constructing it in an interval of 'one unit length' (say from zero to one) and repeating the same. Thus, making it very clear that we need a smallest segment (along domain axis, i.e., x-axis) of 'one unit length' to construct the entire graph.

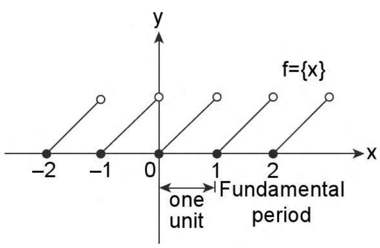


FIGURE 2.201

Observation of periodic symmetry adds further convenience in the analysis of functions due to the fact that it can be analyzed for an interval spread over one period (of length T) anywhere in its domain, as the same analysis prevails in all periods throughout their domain.

Definition of Periodic Function

A function $f(x)$ is said to be a periodic function if there exists a real positive and finite constant T independent of x such that $f(x + T) = f(x)$, $\forall\, x \in D_f$ provided, $(x + T) \in D_f$(domain). Since ... $= f(x-2T) = f(x-T) = f(x) = f(x+T) = f(x + 2T) = \ldots$ Therefore, we conclude $f(x) = f(x + nT)$ $\forall\, n \in \mathbb{Z}$, for which $x + nT \in D_f$. Consequently if a positive constant T is period, then any natural multiple of T within the domain shall also be its period, thus, making it (period) a non-unique quantity.

The least positive value of such T (if exists), is called the *period/principal period or fundamental period* of $f(x)$. e.g., $f(x) = \tan x$, $f(x) = \sin x$ are periodic functions with period π and 2π, respectively.

The functions which are non-periodic are called as non-periodic functions.

Test of Periodicity

In order to test the periodicity of $f(x)$ put $f(x + T) = f(x)$ for all x and find all possible values of T independent of x. If no positive value of T independent of x is possible, then $f(x)$ is said to be aperiodic or non-periodic. If positive value of T independent of x is possible, then $f(x)$ is said to be periodic function and the fundamental period of $f(x)$ will be the least positive value of T.

ILLUSTRATION 261: Test the periodicity of the following functions and if periodic, find their fundamental period.

(a) $f(x) = a\cos(2x + c)$, $a \neq 0$ **(b)** $f(x) = \sin\sqrt{x}$

(c) $f(x) = x - \cos x$ **(d)** $f(x) = \tan\left(\frac{\pi x}{3}\right)$

SOLUTION: As we check periodicity by applying the definition of periodic function, assuming its period to be T and subsequently analysing equation $f(x + T) = f(x)$

(a) Given $f(x) = a\cos(2x + c)$ and $f(x) = f(x + T)$

$\Rightarrow a\cos(2(x + T) + c) = a\cos(2x + c) \Rightarrow \cos(2(x + T) + c) - \cos(2x + c) = 0$

$\Rightarrow 2\sin\left(\frac{2(2x+T)+2c}{2}\right)\sin\left(\frac{2(-T)}{2}\right) = 0 \Rightarrow \sin(T) = 0$ or $\sin\left(2\left(x+\frac{T}{2}\right)+c\right) = 0$

$\Rightarrow T = n\pi$ or $2x + T + c = n\pi;\ n \in \mathbb{Z}$

$\Rightarrow T = n\pi$ or $T = n\pi - 2x - c$ (rejected, as depends on x)

$\therefore T = n\pi;\ n \in \mathbb{Z}$, which is constant ($\because T > 0$) $\Rightarrow T_{min} = \pi$ (fundamental period)

Therefore $f(x) = a\cos(2x + c)$ is periodic function with fundamental period π.

(b) Consider $f(x) = \sin\sqrt{x}$ be periodic with period T.

$\Rightarrow f(x) = f(x + T)$ i.e, $\sin\sqrt{(x+T)} = \sin\sqrt{x} \Rightarrow \sin\sqrt{x+T} - \sin\sqrt{x} = 0$

$$\Rightarrow 2\cos\left(\frac{\sqrt{x+T}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+T}-\sqrt{x}}{2}\right) = 0$$

$$\Rightarrow \left(\frac{\sqrt{x+T}+\sqrt{x}}{2}\right) = \frac{(2n+1)\pi}{2}, \text{ or } \left(\frac{\sqrt{x+T}-\sqrt{x}}{2}\right) = n\pi;\ n \in \mathbb{Z}$$

$$\Rightarrow \sqrt{x+T}+\sqrt{x} = (2n+1)\pi, \text{ or } \sqrt{x+T}-\sqrt{x} = 2n\pi$$

Therefore, T can never be made independent of x, consequently $f(x)$ is a non-periodic function.

Aliter: $\sin\sqrt{(x+T)} = \sin\sqrt{x} \Rightarrow \sqrt{(x+T)} = n\pi + (-1)^n\sqrt{x},\quad n \in \mathbb{Z}$

Squaring both sides and solving for T, we have $(x + T) = (n\pi + (-1)^n\sqrt{x})^2$

$\Rightarrow T = (n\pi + (-1)^n\sqrt{x})^2 - x$

On expanding the right hand side, we observe that the expression of T is not independent of x.

Hence, our assumption is false and therefore the given function is not a periodic function

(c) Consider $f(x) = x - \cos x$ be periodic with period T.

$\Rightarrow f(x) = f(x + T)$, i.e., $(x + T) - \cos(x + T) = x - \cos x$

$$\Rightarrow \cos(x+T) - \cos x = T \Rightarrow -2\sin\left(x+\frac{T}{2}\right)\sin\left(\frac{T}{2}\right) = T$$

Therefore, T can never be made independent of x, consequently $f(x)$ is a non-periodic function.

(d) Let $f(x) = \tan\left(\frac{\pi x}{3}\right)$ be periodic with period T.

$\Rightarrow f(x) = f(x + T)$ $\Rightarrow \tan\left(\frac{\pi}{3}(x+T)\right) = \tan\frac{\pi x}{3}$

$$\Rightarrow \sin\frac{\pi}{3}(x+T)\cos\frac{\pi x}{3} - \cos\left(\frac{\pi}{3}(x+T)\right)\sin\frac{\pi x}{3} = 0$$

$$\Rightarrow \sin\left(\frac{\pi x}{3}+\frac{\pi T}{3}-\frac{\pi x}{3}\right) = 0 \Rightarrow \sin\left(\frac{\pi T}{3}\right) = 0 \Rightarrow \frac{\pi T}{3} = n\pi;\ n \in \mathbb{Z} \Rightarrow T = 3n;\ n \in \mathbb{Z}$$

Hence, $f(x)$ is a periodic function and minimum positive integer value of T, i.e., $T_{min} = 3$

Consequently fundamental period of $\tan\dfrac{\pi x}{3}$ is 3.

ILLUSTRATION 262: Prove that $\cos\left(\dfrac{1}{x}\right)$, $(x \neq 0)$ is a non-periodic function

SOLUTION: Let us assume that the function $f(x) = \cos\left(\dfrac{1}{x}\right)$ be periodic with periodic T, $T > 0$

$\therefore\ f(x + T) = f(x)\ \forall\ x \neq 0$

$$\Rightarrow \cos\left(\frac{1}{x+T}\right) = \cos\left(\frac{1}{x}\right) \Rightarrow \frac{1}{x+T} = 2n\pi \pm \frac{1}{x} \qquad \text{... (1)}$$

Equation (1) is true for every non-zero real number. In particular, it is true for $x = T$ and $x = 2T$.

$\therefore$ Putting $x = T$ and $x = 2T$ in (1), we get $\dfrac{1}{2T} = 2n\pi \pm \dfrac{1}{T}$ ... (2)

And $\left(\dfrac{1}{3T}\right) = 2n\pi \pm \dfrac{1}{2T}$ ... (3)

Subtracting (3) from (2), we get $\dfrac{1}{6T} = \pm\dfrac{1}{2T}$ or $\dfrac{1}{3} = \pm 1$ which is impossible

Hence, $\cos\left(\dfrac{1}{x}\right)$ is a non-periodic function.

ILLUSTRATION 263: Determine whether the function $f(x) = x \sin x$ is periodic?

SOLUTION: To check periodicity, we apply the definition of the periodic functions.

Let us assume that the given function $f(x)$ is periodic with a period T $(T > 0)$, then according to the definition of the periodic functions, we have $f(x + T) = f(x)$.

$\Rightarrow (x + T) \sin (x + T) = x \sin x \Rightarrow T \sin(x + T) = x[\sin x - \sin (x + T)]$

Now, we observe that the left hand side is only a trigonometric function (apart from T, which is a constant), whereas right hand side is a product of an algebraic function and a trigonometric function.

Therefore, there is no algebraic function in x on the left which can cancel out x on the right. Thus, we conclude that T is not independent of x, and hence, our assumption is false and therefore, we conclude that the given function is not periodic.

SOME IMPORTANT FACTS

(a) Trigonometric Functions: The function sin x, cos x, sec x, cosec x are periodic with period 2π, whereas tan x, cot x are periodic functions with period π.

(b) There may be Periodic Functions Which have No Fundamental Period, for example,

(i) Dirichlet Function:

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is rational} \\ 0, & \text{when } x \text{ is irrational} \end{cases}$$

Let $\dfrac{p}{q}$: $p, q \in \mathbb{Z}$; co-prime to each other and $q \neq 0$ be period

$$\Rightarrow f\left(x + \frac{p}{q}\right) = \begin{cases} 1; & x + \dfrac{p}{q} \in \mathbb{Q} \\ 0; & x + \dfrac{p}{q} \notin \mathbb{Q} \end{cases}$$

$$\Rightarrow f\left(x + \frac{p}{q}\right) = \begin{cases} 1; & x \in \mathbb{Q} \\ 0; & x \notin \mathbb{Q} \end{cases} = f(x)$$

$\Rightarrow f(x)$ is periodic with period $\dfrac{p}{q}$ but has no fundamental period, as it is impossible to find a least positive rational number p/q.

(ii) Constant Function: Consider a function $f(x) = c$

Clearly $f(x + T) = f(x) = c$ holds good for all positive values of T. Therefore fulfilling the requirement of periodic function. Although any positive real number may be taken as period of constant function but indeed no such least real number can be predicted. Thus, constant function has no fundamental period.

(c) No Rational Function (except constant function) Can be a Periodic Function.

Let $F(x) = \dfrac{f(x)}{g(x)}$; where $f(x)$ and $g(x)$ are two polynomials and $g(x) \neq 0$ and $F(x)$ be periodic with period T.

$\Rightarrow F(x) = F(x + T)$ ... (1)

$$\Rightarrow \frac{f(x+T)}{g(x+T)} = \frac{f(x)}{g(x)}$$

$\Rightarrow f(x + T).g(x) - g(x + T)f(x) = 0$ is an identity in x therefore substituting $x = 0$, we obtain

$f(T)g(0) - g(T)f(0) = 0$ .... (2)

It is clearly a polynomial in T of degree n (here n is degree of the polynomial having larger degree out of f and g).

Now since equation (2) has degree n but can had $(n + 1)$ roots for instance $0, T, 2T, 3T, \ldots nT$.

Consequently, by fundamental theorem of algebra, $f(x)$ must be identically equal to $0\ \forall\ x \in \mathbb{R}$.

Therefore, $F(x)$ must be indeed a constant function.

(d) Algebraic Function (Except Constant Function) Can Not be a Periodic Function.

(e) A Function Can be Periodic in a Bounded Subset of Its Natural Domain.

We can discuss the periodicity of a function in an interval. For instance $f(x) = \begin{cases} \cos\sqrt{-x}; & x < 0 \\ \cos \pi x; & 0 \le x \le 100 \\ 101 - x; & x > 100 \end{cases}$

Is not a periodic function in its natural domain but it is clearly a periodic function in the interval [0, 98]

Because $f(x + 2) = f(x)$ holds good for all $x \in [0, 98]$.

ILLUSTRATION 264: Draw the graph of $f(x)$ defined as below in the domain [–3, 9]

$$f(x) = \begin{cases} x - 3n; & \text{if } 3n \le x \le 3n+1 \\ 1; & \text{if } 3n+1 \le x \le 3n+2 \\ 3(n+1) - x; & \text{if } 3n+2 \le x \le 3(n+1) \end{cases}$$ and state whether the function $f(x)$ is periodic?

If so, mention its domain of periodicity.

SOLUTION: Consider the graph of function $f(x)$ in the interval [–3, 9]

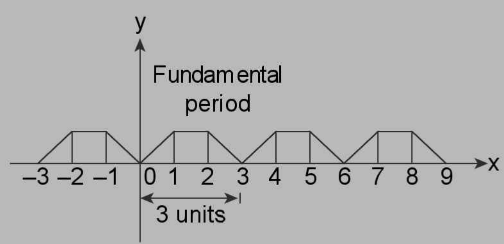


FIGURE 2.202

Clearly $f(x + 3) = f(x)$ holds $\forall\ x \in [-3, 9]$ for which $x + 3 \in [-3, 9]$.

Therefore, $f(x)$ is periodic with fundamental period 3 in the interval [–3, 6].

ILLUSTRATION 265: Let f be an even and periodic function with fundamental period 10 units and the graph of $f(x)$ is shown below for the interval [0, 5]

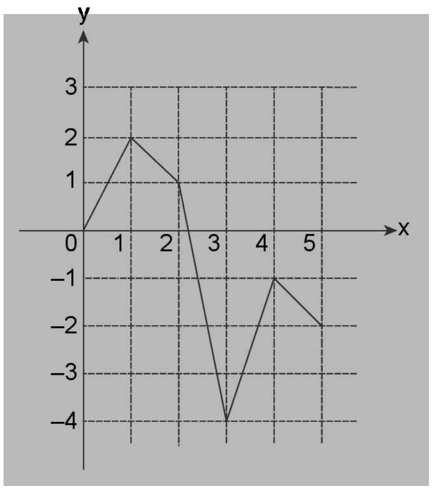


FIGURE 2.203

If [.], ⌈ ⌉ and { } represents the greatest integer/least integer/fractional part function of x; then find the value of $f\left(\frac{3}{4}\right)+3\left[f\left(\frac{-13}{2}\right)\right]+\left\{f\left(\frac{21}{4}\right)\right\}-5\left\lceil f\left(\frac{31}{2}\right)\right\rceil+\cos^{-1}\left(f\left(\frac{-1}{2}\right)\right)+\{f(-273)\}$

SOLUTION: From the graph; we can write the definition of the function for $x \in [0, 5]$

$$f(x)=\begin{cases} 2x; & \forall x\in[0,1] \\ 3-x; & \forall x\in(1,2] \\ 11-5x; & \forall x\in(2,3] \\ 3x-13; & \forall x\in(3,4] \\ 3-x; & \forall x\in(4,5] \end{cases}$$

Now given that the function is even; we can also find the function for $x \in [-5, 0)$

$$f(x)=\begin{cases} -2x; & \forall x\in[-1,0) \\ 3+x; & \forall x\in[-2,-1) \\ 11+5x; & \forall x\in[-3,-2) \\ -3x-13; & \forall x\in[-4,-3) \\ 3+x; & \forall x\in[-5,-4) \end{cases}$$

And given that the function is periodic with period equal to 10; hence, we can find the value of $f(x)\ \forall\ x \in \mathbb{R}$.

Now $f\left(\frac{3}{4}\right)=2\left(\frac{3}{4}\right)=\frac{3}{2}$ and

$$f\left(\frac{-13}{2}\right)=f\left(\frac{-13}{2}+10\right)=f\left(\frac{7}{2}\right)=3\left(\frac{7}{2}\right)-13=\frac{21}{2}-13=\frac{21-26}{2}=-\frac{5}{2}$$

$$\Rightarrow\quad 3\left[f\left(\frac{-13}{2}\right)\right]=3\times(-3)=-9$$

And $f\left(\frac{21}{4}\right)=f\left(\frac{21}{4}-10\right) = f\left(\frac{-19}{4}\right)=f\left(\frac{19}{4}\right)=3-\left(\frac{19}{4}\right)=-\frac{7}{4}$

$\Rightarrow \left\{f\left(\frac{21}{4}\right)\right\}=\left\{\frac{-7}{4}\right\}=\left\{-2+\frac{1}{4}\right\}=\left\{\frac{1}{4}\right\}=\frac{1}{4}$

And $f\left(\frac{31}{2}\right)=f\left(10+\frac{11}{2}\right)=f\left(\frac{11}{2}\right) = f\left(\frac{-9}{2}\right)=f\left(\frac{9}{2}\right)=3-\frac{9}{2}=\frac{-3}{2}$

$\Rightarrow -5\left[f\left(\frac{31}{2}\right)\right]=-5\left[-\frac{3}{2}\right]=-5(-1)=5$ And $\cos^{-1}\left(f\left(\frac{-1}{2}\right)\right)=\cos^{-1}\left(f\left(\frac{1}{2}\right)\right)=\cos^{-1}1=0$

And $f(-273) = f(-270 - 3) = f(-3) = f(3) = -4$

$\Rightarrow \{f(-273)\} = \{-4\} = 0$

Hence, $f\left(\frac{3}{4}\right)+3\left[f\left(\frac{-13}{2}\right)\right]+\left\{f\left(\frac{21}{4}\right)\right\} - 5\left[f\left(\frac{31}{2}\right)\right]+\cos^{-1}\left(f\left(\frac{-1}{2}\right)\right)+\{f(-273)\}$

$=\left(\frac{3}{2}\right)+(-9)+\frac{1}{4}+5+0+0 = -4+\frac{3}{2}+\frac{1}{4}=\frac{-16+6+1}{4}=-\frac{9}{4}$

ILLUSTRATION 266: Using graphs, find whether the following functions are periodic, if so find the period:

(a) $y=\frac{\sin 2x}{|\sin x|}$ (b) $y = \tan x \cot x$

SOLUTION: (a) $y=\frac{\sin 2x}{|\sin x|}=\begin{cases} 2\cos x; & \sin x>0 \\ -2\cos x; & \sin x<0 \end{cases}$

Consider for $x \in (0, 2\pi) \sim \{\pi\}$; we have $y=\begin{cases} 2\cos x; & x\in(0,\pi) \\ -2\cos x; & x\in(\pi,2\pi) \end{cases}$

$\therefore$ The graph of $y=\frac{\sin 2x}{|\sin x|}$ is given by

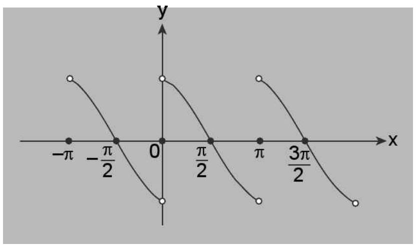


FIGURE 2.204

From the graph, we can see that the function is periodic with period $T = \pi$.

(b) $y = \tan x \,.\, \cot x \Rightarrow y = 1; x \neq \frac{n\pi}{2}$ $\therefore$ The graph of y is given below.

As is evident from the graph the period of $y = \tan x \,.\, \cot x$ is $T = \frac{\pi}{2}$

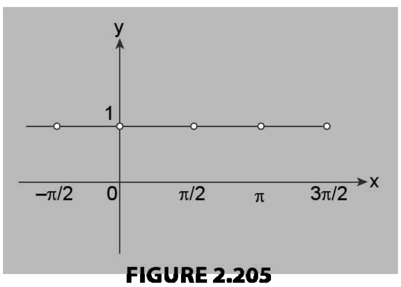


FIGURE 2.205

ILLUSTRATION 267: Check whether the function $f(x)$ defined as $f(x) = \sqrt{x-3k}$; where $3k \le x < 3(k+1)$, $k \in \mathbb{Z}$, is periodic or not.

SOLUTION: Given $f(x) = \sqrt{x-3k}$; where $3k \le x < 3(k+1)$, $k \in \mathbb{Z}$

Putting $k = 0$; we get $f(x) = \sqrt{x}$; $x \in [0, 3)$

Putting $k = 1$; we get $f(x) = \sqrt{x-3}$; $x \in [3, 6)$

Putting $k = 2$; we get $f(x) = \sqrt{x-6}$; $x \in [6, 9)$

Moving on the same pattern; we see that $f(x) = \begin{cases} \sqrt{x+3}; & x \in [-3,0) \\ \sqrt{x}; & x \in [0,3) \\ \sqrt{x-3}; & x \in [3,6) \\ \sqrt{x-6}; & x \in [6,9) \end{cases}$

$\therefore$ The graph of $f(x)$ is as shown below

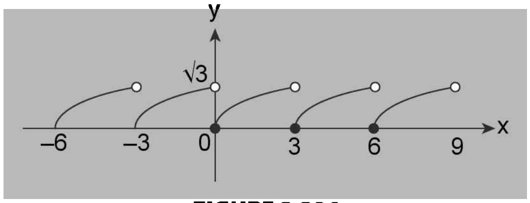


FIGURE 2.206

Now as is evident from the graph; the function is periodic with a period of $T = 3$.

ILLUSTRATION 268: Suppose that f is an even periodic function with period 2, and that $f(x) = -\sin^{-1}\sin\left(\frac{\pi x}{2}\right)$ for all x in the interval $[-1, 0]$. Find the value of $f\left(\frac{11}{3}\right)$.

SOLUTION: Given that $f(x)$ is an even function, therefore $f(x) = \sin^{-1}\sin\left(\frac{\pi x}{2}\right)$; $\forall\ x \in [0, 1]$,

$\therefore$ Graph of $f(x)$ is periodic with period 2 and is shown below

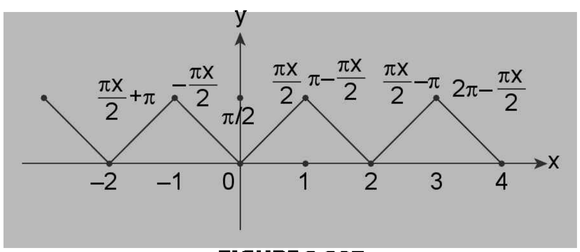


FIGURE 2.207

Now $f\left(\frac{11}{3}\right) = 2\pi - \frac{\pi}{2} \times \left(\frac{11}{3}\right) \Rightarrow f\left(\frac{11}{3}\right) = \frac{\pi}{6}$ $\left[\begin{array}{l} \because 3 < \frac{11}{3} < 4 \text{ and } \forall x \in (3,4) \\ f(x) = 2\pi - \frac{\pi}{2}x \end{array}\right]$

■ PROPERTIES OF PERIODIC FUNCTION

(i) If $f(x)$ is periodic with period T, then $a\,f(x + k) + b$ is also periodic with same period T; where a, b are real constants and $a > 0$.

Proof: Let $g(x) = af(x + k) + b$

$\Rightarrow g(x + T) = af(x + T + k) + \text{b}$ $\qquad (\because f(x + T) = f(x))$

$= af(x + k) + b = g(x)$. Hence, proved.

Alternatively we may put the argument geometrically that by adding a constant (say k) to independent variable (input) the graph of function shifts as a whole horizontally either to the left or right without changing the shape and size of graph, therefore does not affect the periodicity of function.

Similarly, adding a constant to the dependent variable (output) the graph of function shifts as a whole vertically either up or to down without changing the shape and size of graph, thus, not affecting the periodicity of the function. And also it is true that the periodicity of the function is repetition of values of function after a regular interval of inputs x, therefore remains unaffected by transformations in vertical directions along y-axis.

For example, vertical stretching/compression produced in the graph of function due to multiplying $f(x)$ by a constant a also produces no change in periodicity.

(ii) If $f(x)$ is periodic with period T, then $f(kx + b)$ is periodic with period $\frac{T}{|k|}$ provided k is non-zero real number and $b \in \mathbb{R}$

Proof: If $f(x)$ is periodic with period T and let $g(x) = f(kx + b)$

$$\Rightarrow g\left(x+\frac{T}{|k|}\right)=f\left(k\left(x+\frac{T}{|k|}\right)+b\right) = f(kx \pm T + b)$$

$= f(kx + b) = g(x)$

Converse of this property holds good.

Alternatively we may put the argument geometrically as the multiplication or division of independent variable x by a positive constant a, results in change in size with respect to origin in the horizontal direction. The graph shrinks horizontally when independent variable is multiplied by a positive constant greater than 1 by the factor which is equal to the multiplier. This mean periodicity of graph decreases by the same factor, i.e., $|a|$.

The graph stretches horizontally when the independent variable is divided by positive constant greater than 1 by the factor which is equal to the divisor. This means periodicity of graph increases by the same factor, i.e., $|a|$.

We combine these two observations by saying that period of graph decreases by a factor $|a|$.

Note that magnitude of constant a more than 1 represents multiplication and less than 1 represents division.

In the nutshell, if T is the period of $f(x)$, then period of function of the form $y = p.f(ax + b) + q$; $a \neq 0, b, q, p \in \mathbb{R}$. is $\frac{T}{|a|}$

ILLUSTRATION 269: Find the period of function $f(x) = 3 + 2\sin\left(\frac{\pi x+2}{3}\right)$

SOLUTION: Given function is $f(x) = 3 + 2\sin\left(\frac{\pi x+2}{3}\right)$

Rearranging the terms, we have $f(x) = 3 + 2\sin\left(\frac{\pi}{3}x+\frac{2}{3}\right)$ .... (1)

The period of sine function is 2π.

Comparing (1) with the function $pf(ax + b) + q$, magnitude of a i.e., $|a|$ is $\frac{\pi}{3}$. Hence, period of the given function is $T'=\frac{T}{|a|}=\frac{2\pi}{\pi/3}=6$

ILLUSTRATION 270: If $f(x)=\sin\sqrt{[a]}x$; (where [.] denotes the greatest integer function) has π as its fundamental period, then find the set of the values of a.

SOLUTION: Given $f(x)=\sin\left(\sqrt{[a]}\,x\right)$; Period $=\frac{2\pi}{\sqrt{[a]}}=\pi$

$\Rightarrow [a] = 4$ $\qquad \Rightarrow a \in [4, 5)$

■ PERIOD OF COMPOSITE FUNCTIONS

While dealing with periodicity, it is important to understand the changes in the periodicity of a function $f(x)$ that may or may not occur when we compose the function with other functions like modulus or power function, exponential, logarithmic functions. If $f(x)$ is a periodic function with fundamental period T and $g(x)$ is any function for which composite function gof is defined, then gof is also periodic function with one of its period as T (not necessarily fundamental period) .

Theorem: If $f(x)$ is periodic function with fundamental period T and $g(x)$ is monotonic function, over the range of $f(x)$, then $g(f(x))$ is also periodic with fundamental period T.

For example, If $g(x)= e^x$ and $f(x) = \sin x$, then $gof(x) = e^{\sin x}$ is periodic with fundamental period 2π.

Proof: Given the period of $f(x) = T$, then $f(x + T) = f(x)\ \forall\ x \in D_f$

$\Rightarrow$ $g[f(x + T)] = g[f(x)]$

$\Rightarrow$ gof is also periodic with period T ... (1)

Let if possible $k \in \mathbb{R}^+$ and

$k < T$, be the fundamental period of gof

$\Rightarrow$ $gof(k + x) = gof(x)$

$\Rightarrow$ $g(f(k + x)) = g(f(x))$

$\Rightarrow$ $f(k + x) = f(x)$ as $g(x)$ is monotonic, consequently $k < T$ is fundamental period of $f(x)$, which is a contradiction to the fact that $f(x)$ has fundamental period T.

$\therefore$ gof has fundamental period T.

Corollary: If $f(x)$ is periodic with period T, then

(i) $\dfrac{1}{f(x)}$ **is also periodic with same period T.**

Proof: Since $g(x) = \dfrac{1}{x}$ is a decreasing function, therefore $gof(x) = g(f(x)) = \dfrac{1}{f(x)}$ being the composition of a monotonic function over a periodic function with period T is also periodic with same period T.

(ii) $\sqrt{f(x)}$ **is also periodic with same period T.**

Proof: Since $g(x) = \sqrt{x}$ is an increasing function, on its domain $[0, \infty)$, therefore $gof(x) = g(f(x)) = \sqrt{f(x)}$ being the composition of a monotonic function over a periodic function with period T is also periodic with same period T.

NOTE

1. *Composition of a non-monotonic function g(x) over a periodic function f(x) having period T is always a periodic function with period T. (But fundamental period may be less than T)*
   For example, if $g(x) = x^2$ and $f(x)= \cos x$, then $gof(x) = \cos^2 x$ is periodic with period 2π, but its fundamental period is π.
2. *Composition of a periodic function f(x) over a non-periodic function g(x) may be a periodic function. For example, if $g(x) = [x]$ and $f(x) = \cos\pi x$, then $fog(x) = \cos \pi[x]$ is periodic with period 2.*
3. *Composition of two non-periodic functions may be a periodic function*
   For example, Consider $g(x) = 3[x] - 2$ and $f(x) = \begin{cases} \dfrac{x^3-8}{x-2}; & x \notin \mathbb{Z} \\ 3\left(\sin^2 x + \cos^2 x\right); & x \in \mathbb{Z} \end{cases}$ *we have $fog(x) = 3\ \forall\ x \in \mathbb{R}$.*
   which is a periodic function indeed.

ILLUSTRATION 271: Let $f(x) = x - \tan x$ and $g(x) = \cot x$ be two functions, then using these two functions, prove that the composite function of a periodic and a non-periodic function can be a periodic function. Also find the period of that function.

SOLUTION: Let $f(x) = x - \tan x$ and $g(x) = \cot x$

Now $f(x)$ is non-periodic, whereas $g(x)$ is periodic $g(f(x)) = \cot(x - \tan x)$

On replacing x by $x + \pi$ in $g(f(x))$; we get $g(f(x + \pi)) = \cot(x + \pi - \tan(x + \pi)) = \cot(x - \tan x) = g(f(x))$.

Hence, $g(f(x))$ is periodic with period π.

ILLUSTRATION 272: Let $f(x) = x - \cos x$ and $g(x) = \cot x$ be two functions, then using these two functions, prove that the composite function of a periodic and a non-periodic function can be a periodic function. Also find the period of that function.

SOLUTION: Let $f(x) = x - \cos x$ and $g(x) = \cot x$. Now $f(x)$ is non-periodic, whereas $g(x)$ is periodic

$\Rightarrow$ $g(f(x)) = \cot(x - \cos x)$. On replacing x by $x + 2\pi$ in $g(f(x))$; we get

$g(f(x + 2\pi)) = \cot(x + 2\pi - \cos(x + 2\pi)) = \cot(x - \cos x) = g(f(x))$

Hence, $g(f(x))$ is periodic with period 2π. Similarly $\cot(x - \cot x)$ is periodic with period π.

ILLUSTRATION 273: Using the functions $f(x) = \left[\sqrt[3]{x}\right]$; where [ ] represents the greatest integer function and $g(x) = \begin{cases} e^x; & x \notin \mathbb{Z} \\ \sin^2 x + \cos^2 x; & x \in \mathbb{Z} \end{cases}$, show that the composite function of two non-periodic functions can be periodic.

SOLUTION: Now $f(x)$ will always give an integral value. Therefore $g(f(x)) = \sin^2(f(x)) + \cos^2(f(x))$; $\forall f(x) \in \mathbb{Z}$ $\Rightarrow$ $g(f(x)) = 1$ $\forall x \in \mathbb{R}$

Hence, $g(f(x))$ is a constant (periodic with no fundamental period) $\forall x \in \mathbb{R}$

ILLUSTRATION 274: **(a)** If $f(x) = \tan^{-1}x$ and $g(x) = \{x\}$, then show that $fog(x)$ is a periodic function with fundamental period 1.

(b) If $f(x) = \dfrac{2x}{x^2+1}$ and $g(x) = |\sin x|$, then show that $fog(x)$ is a periodic function with period π.

SOLUTION: **(a)** $f(x) = \tan^{-1}x$ $\Rightarrow$ $f'(x) = \dfrac{1}{1+x^2} > 0$ $\forall x \in \mathbb{R}$

$\Rightarrow$ $f(x)$ is monotonically increasing function and $g(x) = \{x\}$.

We know that fundamental period of $\{x\}$ is 1.

Thus, $fog(x)$ represents composition of monotonic function over the periodic function with fundamental period 1

$\therefore$ By above property, $fog(x)$ is periodic function with fundamental period 1.

(b) Now $f(x) = \dfrac{2x}{1+x^2}$

$\Rightarrow$ $f'(x) = \dfrac{2(1-x^2)}{(1+x^2)^2} \geq 0$ for $x \in [-1, 1]$ and < 0 for $x \in (-\infty, -1] \cup [1, \infty)$

$\Rightarrow$ $f(x)$ is a non-monotonic function. Also $g(x) = |\sin x|$ is periodic function with period π.

$\therefore$ $fog(x) = f(g(x)) = \dfrac{2|\sin x|}{1+|\sin x|^2}$

$\Rightarrow$ $fog(x+\pi) = \dfrac{2|\sin(x+\pi)|}{1+|\sin(x+\pi)|^2} = \dfrac{2|\sin x|}{1+|\sin x|^2} = fog(x)$

Thus, $fog(x)$ is a periodic function with period π.

ILLUSTRATION 275: Functions f and g are defined by $f(x) = \sin x$, $x \in \mathbb{R}$; $g(x) = \tan x$, $x \in \mathbb{R} - \left(k+\dfrac{1}{2}\right)\pi$; where $k \in \mathbb{Z}$. Find periods of fog and gof.

SOLUTION: Given $f(x) = \sin x$ and $g(x) = \tan x$. Now, $fog = f(g(x)) = \sin(g(x)) = \sin(\tan x)$

Clearly period of fog is π. Also $gof = g(f(x)) = \tan(f(x)) = \tan(\sin x)$

Period = 2π (period of $\sin x$ is 2π and $\tan x$ is an odd function)

[redacted] PERIODICITY OF MODULUS/POWER OF A FUNCTION

(i) Period if $[f(x)]^{2n+1}$: If the fundamental period of $f(x)$ is T, then the fundamental period of $f(x)^{2n+1}$; $n \in \mathbb{Z}$ will also be T. That is, the fundamental period of function remains same on raising it to an odd integer power.
For example, we know that period of functions $\sin x$, $\cos x$, $\sec x$, $\operatorname{cosec} x$ is 2π, therefore the period of the functions $(\sin x)^{2n+1}$,$(\cos x)^{2n+1}$,$(\sec x)^{2n+1}$,$(\operatorname{cosec} x)^{2n+1}$ is also 2π.

Similarly, the period of $\tan x$, $\cot x$ is π and also the same is the period of $(\tan x)^{2n+1}$ and $(\cot x)^{2n+1}$.

Proof: If T is the period of $f(x)$, then $f(x) = f(x + T)$ ..(1)

On raising both sides of (1) to any odd degree power $(2n+1)$, we get $(f(x))^{2n+1} = (f(x + T))^{2n+1}\ \forall\ n \in \mathbb{Z}$

$\Rightarrow$ T is the period of $(f(x))^{2n+1}\ \forall\ n \in \mathbb{Z}$

Let if possible $k < T$ be the period of $(f(x))^{2n+1}$

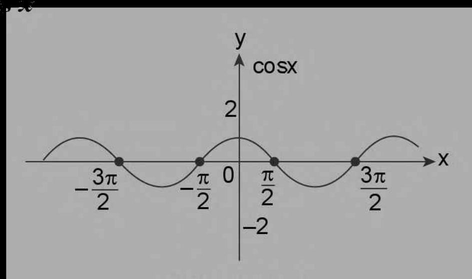
$\Rightarrow$ $(f(x))^{2n+1} = (f(x + k))^{2n+1}\ \forall\ n \in \mathbb{Z}$

$\Rightarrow$ $f(x) = f(x + k)$ $\quad (\because a^{2n+1} = b^{2n+1} \Rightarrow a = b)$

$\Rightarrow$ $k < T$ is period of $f(x)$,which contradicts the fact that T is the fundamental period of $f(x)$.

Thus, T will be the fundamental period of $(f(x))^{2n+1}$.

[redacted]



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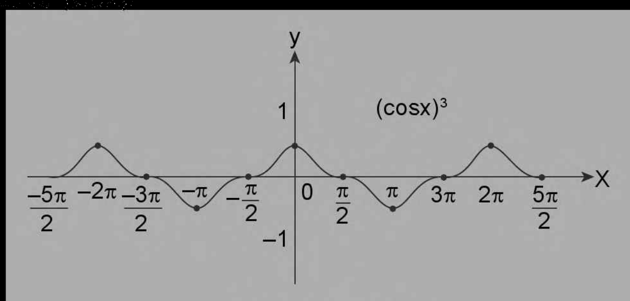

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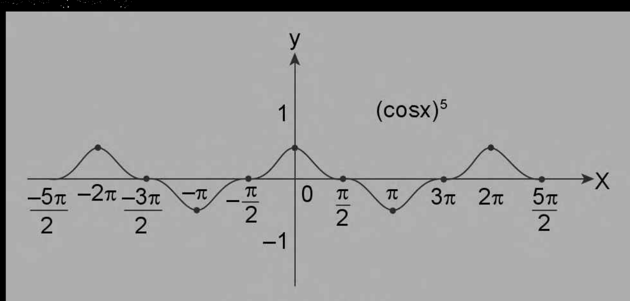


[redacted]

On generalizing, we can say that the periodicity of $\cos x$ does not change on raising its power to an odd degree.

Similarly, with the help of the graphs, we can prove that the above rule holds true for all the functions $f(x)$.

(ii) Period of $[f(x)]^{2n}$: If the fundamental period of $f(x)$ is T, then the fundamental period of $[f(x)]^{2n}$; $n \in \mathbb{Z}$ may not be T.

That is, the fundamental period of function may change on raising it to an even integer power.

For example, we know that the period of the functions $\sin x$, $\cos x$, $\sec x$, $\operatorname{cosec} x$ is 2π and that of $\tan x$, $\cot x$ is π, whereas the period of the functions $(\sin x)^{2n}$, $(\cos x)^{2n}$, $(\sec x)^{2n}$, $(\operatorname{cosec} x)^{2n}$, $(\tan x)^n$, $(\cot x)^n$, $|\sin x|$, $|\cos x|$, $|\tan x|$, $|\cot x|$, $|\sec x|$., $|\operatorname{cosec} x|$ is π.

NOTE

This happens because the modulus operation on a function converts the negative quantity to a positive quantity, while keeping the magnitude same. Raising the power of a function by an even degree is the same as taking the modulus of the function and then raising the power of that function. i.e., $(f(x))^{2n} = |f(x)|^{2n}\ \forall n \in I$

ILLUSTRATION 277: Prove using graphs that the periodicity of $f(x) = \cos x$ gets changed when $\cos x$ is raised to an even power.

SOLUTION: **Graph of $\cos x$**

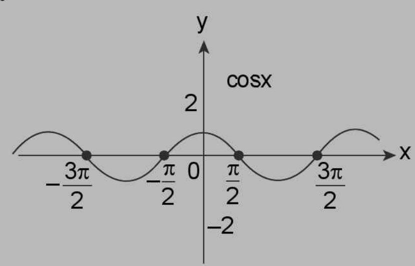


FIGURE 2.211

Graph of $|\cos x|$

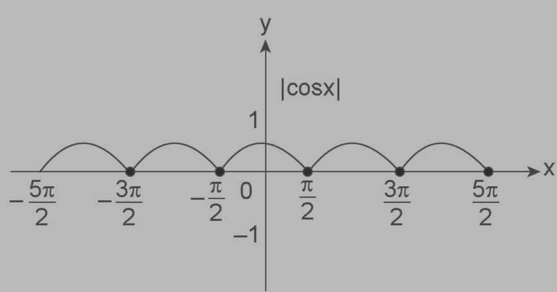


FIGURE 2.212

Graph of (cos x)2

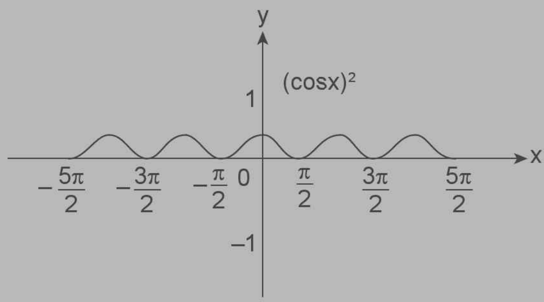


FIGURE 2.213

It is clear from above graphs that the period of cos x is 2π,whereas that of |cos x| and (cos x)2n is π.

On generalizing, we can say that the periodicity of cos x becomes p on raising its power to an even degree.

(iii) Period of [$f(x)$]p/q: where $\frac{p}{q} = \frac{2n+1}{q}$,, where p and q are co-prime, $q \neq 0$ and $n \in Z$.

For example: we know that the period of the functions sin x, cos x, sec x, cosec x is $2p$, therefore the period of the functions (sin x)p/q, (cos x)p/q, (sec x) p/q, (cosec x) p/q is also $2p$.

Similarly, the graph of tan x, cot x is p and also the same is the period of (tan x)p/q and (cot x) p/q .

ILLUSTRATION 278: Using graphs, find the periodicity of $f(x)$ = sin-1sin x, and hence, find the periodicity of (sin-1sin x)3/2 and (sin-1sin x)3/5.

SOLUTION: Graph of sin–1sinx (Periodicity = $2p$)

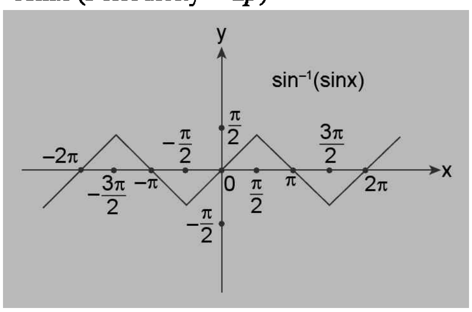


FIGURE 2.214

Graph of (sin–1 sin x)3/2 (Periodicity = $2p$)

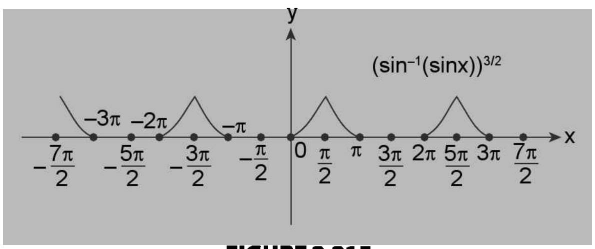


FIGURE 2.215

Graph of (sin–1 sin x)3/5 (Periodicity = $2p$)

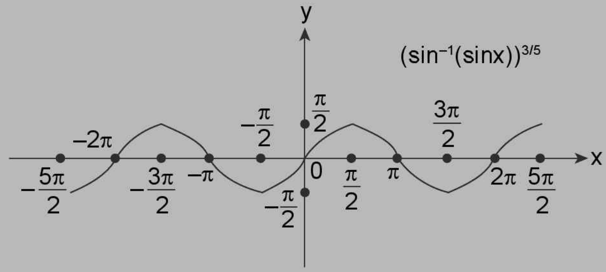


FIGURE 2.216

(iv) Period of $f(x)p/q$: where $\frac{p}{q} = \frac{2n}{q}$, where p and q are co-prime, $q \neq 0$ and $n \in \mathbb{Z}$.

For example, we know that the period of the functions sin x, cos x, sec x, cosec x is 2π and that of tan x, cot x is π, whereas the period of the functions $(\sin x)^{2n}$, $(\cos x)^{2n/p}$, $(\sec x)^{2n/p}$, $(\text{cosec}\, x)^{2n/p}$, $(\tan x)^{2n/p}$, $(\cot x)^{2n/p}$, is π.

ILLUSTRATION 279: Using graphs, find the periodicity of $f(x) = \sin^{-1}\sin x$, and hence, find the periodicity of $(\sin^{-1}\sin x)^{2/3}$ and $(\sin^{-1}\sin x)^{8/5}$.

SOLUTION: **Graph of $\sin^{-1}\sin x$ (Periodicity = 2π)**

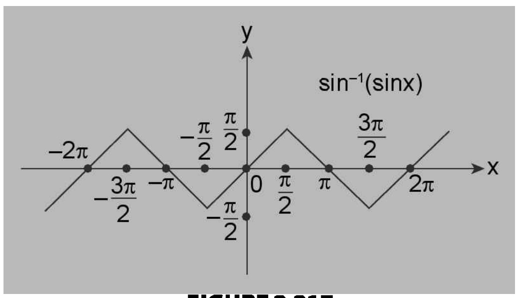


FIGURE 2.217

Graph of $(\sin^{-1}\sin x)^{2/3}$ (Periodicity = π)

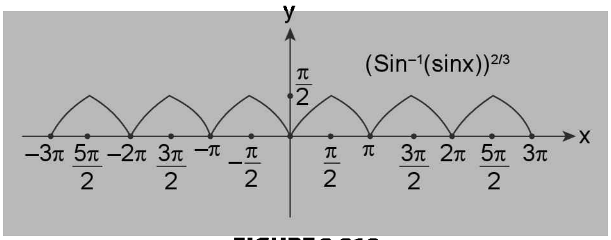


FIGURE 2.218

Graph of $(\sin^{-1}\sin x)^{8/5}$ (Periodicity = π)

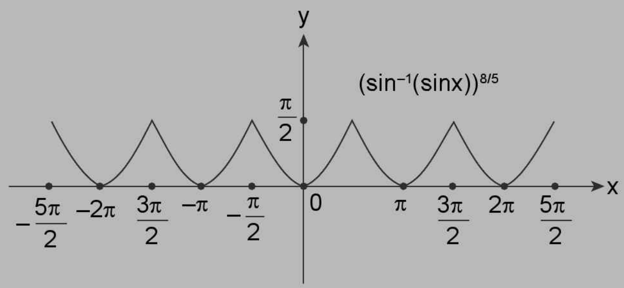


FIGURE 2.219

(v) Period of Linear Combination of Two Functions: If $f(x)$ be periodic with period T_1 and $g(x)$ with period T_2 such that LCM of T_1 and T_2 exist and is equal to T, then $a.f(x) + b.g(x)$ is a periodic function with period T (a and b are non-zeros).

Proof: Let $F(x) = a.f(x) + b.g(x)$ .... (1)

Let T_1 and T_2 be the periods of function $f(x)$ and $g(x)$, respectively.

$\Rightarrow$ $T_1, T_2 > 0$

Let $T = \text{LCM}(T_1, T_2)$

$\Rightarrow$ $T = nT_1$ and $T = mT_2$ for some positive integers n, m.

$\Rightarrow$ $F(x + T) = a.f(x + nT_1) + b.g(x + mT_2) = a.f(x) + b.g(x) = F(x)$

$(\because f(x + nT) = f(x))$

$\Rightarrow$ T is the period of $F(x)$

$\Rightarrow$ If there does not exist $0 < \lambda < T = \text{LCM}(T_1 \text{ and } T_2)$ such that $F(x + \lambda) = F(x)$, then $T = \text{LCM}(T_1 \text{ and } T_2)$ will be the fundamental period of $F(x)$. However if such a λ exists, then T will not be the fundamental period of $F(x)$.

REMARKS

(i) LCM *of two or more fractional numbers* = LCM *of* $\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right) = \frac{L.C.M.\ of\ (a,c,e)}{H.C.F.\ of\ (b,d,f)}$

For example, The LCM *of* $\frac{7}{30}$ and $\frac{3}{20}$ *is* $\frac{L.C.M\ of\ 7\ and\ 3}{H.C.F\ of\ 30\ and\ 20} = \frac{21}{10}$.

(ii) LCM *of rational and irrational number does not exist. For example, the function {x} + cos x is non-periodic, because the period of {x} is 1 and the period of cosx is 2π and the* LCM $(1, 2\pi)$ *does not exist.*

Also the function sin x + tan πx + sin x/3 is not periodic; because LCM *of (2π, 1, 6π) does not exist.*

(iii) *The* LCM *of two irrational quantities may or may not exist. For example, the* LCM *of π and* $\frac{\pi}{2}$ *is π .*

The LCM *of √2 and √3 does not exist.*

Notes: **1.** *The sum / difference of a periodic and a non-periodic function can be periodic.*

For example, consider f(x) = tan x and g(x) = $\begin{cases} 1, & x \in \mathbb{R} - \frac{\pi}{2} \\ 3, & x = \frac{\pi}{2} \end{cases}$ *, then the function f(x) ± g(x) is periodic*

Clearly f(x) is periodic with period π but g(x) is non-periodic function.

The domain of f(x) ± g(x) is $\mathbb{R} \sim \{(2n + 1)\ \pi/2;\ n \in \mathbb{Z}\}$ *hence, f(x) ± g(x) = tan x ± 1 which is periodic function in its natural domain with fundamental period π.*

2. *The sum/difference of two non-periodic functions can be periodic function*

e.g., consider f(x) = x + cos x and g(x) = –x + sin x, then the function f(x) + g(x) = sin x + cos x which is a periodic function with fundamental period of 2π.

ILLUSTRATION 280: Find the fundamental period of $\tan(3\pi x)+\cot\left(\frac{6\pi x}{5}\right)$

SOLUTION: The fundamental period of $\tan(3\pi x)$ is 1/3. And the fundamental period of $\cot\left(\frac{6\pi x}{5}\right)$ is $\frac{5}{6}$.

Therefore, the fundamental period of given function is $L.C.M\left(\frac{1}{3},\frac{5}{6}\right)=\frac{5}{3}$.

ILLUSTRATION 281: Find period of the following functions:

(i) $f(x)=\sin\frac{x}{2}+\cos\frac{x}{3}$ (ii) $f(x) = \{x\} + \sin x$

(iii) $f(x)=\sin\frac{3x}{2}-\cos\frac{x}{3}-\tan\frac{2x}{3}$

SOLUTION: (i) Period of $\sin\frac{x}{2}$ is 4π while period of $\cos\frac{x}{3}$ is 6π. Hence, period of $\sin\frac{x}{2}+\cos\frac{x}{3}$ is 12π. (LCM of 4 and 6 is 12}

(ii) Period of $\{x\} = 1$ and that of $\sin x$ is 2π.

But LCM of 2π and 1 is not possible as their ratio is irrational number

$\therefore$ It is non-periodic

(iii) Period of $f(x)$ is LCM of $\frac{2\pi}{3/2},\frac{2\pi}{1/3},\frac{\pi}{2/3}=$ LCM of $\frac{4\pi}{3},6\pi,\frac{3\pi}{2}=12\pi$

ILLUSTRATION 282: Find period of the function $f(x)=x-[x]+|\sin\pi x|+|\sin 2\pi x|+....+|\sin n\pi x|$

SOLUTION: $x - [x]$ has period 1 and $|\sin x|$ has period π

$\Rightarrow$ $|\sin \pi x|$ has period $\frac{\pi}{\pi}=1$, $|\sin 2\pi x|$ has period $=\frac{\pi}{2\pi}=\frac{1}{2}$

Similarly $|\sin n\pi x|$ has period $\frac{\pi}{n\pi}=\frac{1}{n}$

$\Rightarrow$ period of the function $f(x)=$ LCM $\left\{1,1,\frac{1}{2},\frac{1}{3},...\frac{1}{n}\right\}=1$ $\therefore$ $f(x)$ has period 1.

ILLUSTRATION 283: Find the period of the function $f(x) = 4\sin\{3x\} + 3\cos2\{4x\}$

SOLUTION: Let $f(x) = 4\sin\{3x\} + 3\cos2\{4x\}$

Let $g(x) = 4\sin\{3x\}$ and $h(x) = 3\cos2\{4x\}$

Now period of $g(x)$ is $\frac{1}{3}$ and that of $h(x)$ is $\frac{1}{4}$

Hence, the fundamental period of $f(x)=$ LCM of $\left(\frac{1}{3},\frac{1}{4}\right)=1$

ILLUSTRATION 284: Find the fundamental period of $f(x) = |\sin x| - |\cos x|$.

SOLUTION: Given function $f(x) = |\sin x| - |\cos x|$. The period of both $|\sin x|$ and $|\cos x|$ is π.

So, LCM is π.

ILLUSTRATION 285: Find the period of $\sin\frac{\pi}{4}[x]+\cos\frac{\pi x}{2}+\cos\frac{\pi}{3}[x]$, where $[x]$ denotes the integral part of x.

SOLUTION: Given $\sin\frac{\pi}{4}[x]+\cos\frac{\pi x}{2}+\cos\frac{\pi}{3}[x]$

Let $g(x)=\sin\frac{\pi}{4}[x]$

Since $g(x) = 0, \frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}, 0$ when $x \in [0, 1); [1, 2), [2, 3), [3, 4),$ $[4, 5), [5, 6), [6, 7), [7, 8)$ and $[8, 9)$. $\Rightarrow$ The period of $g(x) = 8$

Similarly the period of function $p(x) = \cos\frac{\pi}{3}[x]$ is 6 and since $h(x) = \cos\frac{\pi x}{2}$, therefore the period of $h(x) = 4$

Now, according to the LCM rule, the period of $f(x)$ = LCM of 8, 4, and 6, i.e., 24.

ILLUSTRATION 286: Find the fundamental period of the function, $f(x) = x + a - [x + b] + \sin \pi x + \cos 2\pi x + \sin 3\pi x + \cos 4\pi x + \ldots\ldots + \sin (2n - 1) \pi x + \cos 2 n\pi x$ for every $a, b \in \mathbb{R}$ is: (where [ ] denotes the greatest integer function)

SOLUTION: $f(x) = x + a - [x + b] + \sin \pi x + \cos 2\pi x + \sin (3\pi x) + \cos (4\pi x) + \ldots + \sin (2n - 1)\pi x + \cos (2n\pi x)$

$\Rightarrow f(x) = \{x + b\} + a - b + \sin (\pi x) + \cos(2\pi x) + \sin(3\pi x) + \cos(4\pi x) + \ldots + \sin(2n - 1) \pi x + \cos(2n\pi x)$

Period of $f(x) = \text{LCM}\left(1, 2, \frac{2}{3}, \frac{2}{4}, \ldots, \frac{2}{2n-1}, \frac{2}{2n}\right) = 2$

$\therefore$ Fundamental period of $f(x) = 2$.

PERIODICITY OF PRODUCT/DIVISION OF FUNCTION

Consider the function $F(x) = \dfrac{\prod_{k=1}^{n} f_k(x)}{\prod_{r=1}^{m} g_r(x)}$; where $f_k(x)$ and $g_r(x)$ are periodic functions with periods T_k and t_r respectively $\forall\, k, r \in \mathbb{N}$ and LCM of T_k and t_r exist and equals to T. Then the function $F(x)$ is periodic with period T.

However, T need not be the fundamental period of $F(x)$. To get the fundamental period, we need to check whether the fundamental period is T or not by applying either,

(a) transformation of $F(x)$ into linear combination of functions, or

(b) analyzing through graph of $F(x)$, or

(c) checking whether T/n is period of $F(x)$ for natural values of n.

NOTE

1. *The product/quotient of a periodic and a non-periodic function can be periodic.*

   For example, *consider f(x) = cot x and* $g(x) = \begin{cases} 1, & x \in \mathbb{R} - \{0\} \\ 3, & x = 0 \end{cases}$*, then the function f(x) . g(x) and* $\frac{f(x)}{g(x)}$ *are periodic*

   Clearly f(x) is periodic with period π but g(x) is non-periodic function.

   The domain of f(x) . g(x) and $\frac{f(x)}{g(x)}$ *is* $\mathbb{R} \sim \{n\pi; n \in \mathbb{Z}\}$*; hence, f(x).* $g(x) = \frac{f(x)}{g(x)}$ *= cot x which is periodic function in its natural domain with fundamental period π.*

2. *The product/quotient of two non-periodic functions can be periodic function*

   For example, *consider* $f(x) = \begin{cases} 1; & x < 0 \\ -1; & x \geq 0 \end{cases}$ *and* $g(x) = \begin{cases} -1; & x < 0 \\ 1; & x \geq 0 \end{cases}$*, then the function f(x).g(x) and* $\frac{f(x)}{g(x)} = -1$ *which being a constant function is a periodic function with no fundamental period.*

ILLUSTRATION 287: Find the fundamental period of the following functions:

(i) $f(x) = \sin x . \tan x$

(ii) $f(x) = \left\{\dfrac{x}{2}\right\}.\cos\left(\dfrac{\pi x}{3}\right)$; ({ } is fractional part function)

SOLUTION: (i) Given $f(x) = \sin x . \tan x$; Period of $f(x)$ = LCM of $(2\pi, \pi) = 2\pi$

(ii) Given $f(x) = \left\{\dfrac{x}{2}\right\}.\cos\left(\dfrac{\pi x}{3}\right)$; Period of $f(x)$ is LCM of $(2, 3) = 6$

EXCEPTION TO LCM RULE

LCM Rule undoubtedly predicts the period of the function but not necessarily the fundamental period. It fails to determine the fundamental period in atleast following two situations:

Case I: If $f(x)$ be periodic with period T_1 and $g(x)$ with period T_2 such that LCM of T_1 and T_2 exists and is equal to T and $f(x)$ and $g(x)$ can be interchanged by adding a least positive constant K $(< T)$.

That is, $f(x + K) = g(x)$ and $g(x + K) = f(x)$, then K is period of $f(x) + g(x)$, otherwise period will be T.

ILLUSTRATION 288: Prove that the function $f(x) = |\sin x| + |\cos x|$ does not follow the LCM rule.

SOLUTION: Let $f(x) = |\sin x| + |\cos x|$

Let $g(x) = |\sin x|$ and $h(x) = |\cos x|$ and Fundamental period of each of $g(x)$ and $h(x)$ is π

$\therefore$ According to the LCM rule; the Fundamental period of $f(x)$ should be π

But if we replace x by $x + \dfrac{\pi}{2}$ in $f(x)$; we get

$$f\left(x+\frac{\pi}{2}\right) = \left|\sin\left(x+\frac{\pi}{2}\right)\right| + \left|\cos\left(x+\frac{\pi}{2}\right)\right| = |\sin x| + |\cos x|; = f(x)$$

[due to the fact that $g(x)$ and $h(x)$ exchanged their values by adding a constant $\dfrac{\pi}{2}$ in x.]

Case II: If $f(x)$ be periodic with period T_1 and $g(x)$ with period T_2 such that LCM of T_1 and T_2 exist and is equal to T, then the period of $F(x) = f(x) \pm g(x)$ or $f(x).g(x)$ or $\dfrac{f(x)}{g(x)}$ is necessarily T but the fundamental period can be given by a positive constant K $(< T)$ if $F(x)$ gets simplified to an equivalent function $F(x + K) = F(x)$.

ILLUSTRATION 289: Find the fundamental period of the following functions:

(i) $f(x) = \sin x . \cos x$

(ii) $f(x) = \cos^2 x \pm \sin^2 x$

(iii) $f(x) = \dfrac{\sin 3x}{\sin 2x.\cos x}$

(iv) $f(x) = 4\cos^3 x - 3\cos x$

(v) $f(x) = \cos x .\cos 3x$

SOLUTION: (i) Given $f(x) = \sin x . \cos x$

Now, according to the LCM rule, period of $f(x)$ = LCM of $(2\pi, 2\pi) = 2\pi$.

But $f(x) = \dfrac{\sin 2x}{2}$, and hence, the fundamental period of $f(x) = \pi$.

(ii) Given $f(x)=\cos^2 x \pm \sin^2 x$

Now, according to the LCM rule, period of $f(x)$ = LCM of $(\pi, \pi) = \pi$.

But for $f(x)=\cos^2 x+\sin^2 x$, the function simplifies to $f(x)=1$ which is a constant function with no fundamental period. And for $f(x)=\cos^2 x-\sin^2 x$, the function simplifies to $f(x) = \cos 2x$, and hence, the fundamental period of $f(x) = \pi$.

(iii) Given $f(x)=\dfrac{\sin 3x}{\sin 2x.\cos x}$

Now, according to the LCM Rule, period of $f(x)$ = LCM of $(2\pi, \pi, 2\pi/3) = 2\pi$.

But $f(x)=\dfrac{3\sin x-4\sin^3 x}{(\cos x)(2\sin x\cos x)}=\dfrac{3-4\sin^2 x}{2\cos^2 x}$, hence, the fundamental period of $f(x) = \pi$.

(iv) Given $f(x)=4\cos^3 x-3\cos x$

Now, according to the LCM Rule, period of $f(x)$ = LCM of $(2\pi, 2\pi) = 2\pi$.

But $f(x) = \cos 3x$, hence, the fundamental period of $f(x) = 2\pi/3$.

(v) Given $f(x) = \cos x .\cos 3x$ $\Rightarrow$ Period of $f(x)$ is LCM of $\left(2\pi, \dfrac{2\pi}{3}\right) = 2\pi$

But 2π may or may not be fundamental period, but fundamental period = $\dfrac{2\pi}{n}$, where $n \in \mathbb{N}$.

Hence, cross-checking for n = 1, 2, 3, … we find π to be fundamental period $f(\pi + x) = (-\cos x)(-\cos 3x) = f(x)$. Hence, fundamental period of $f(x)$ is π.

ILLUSTRATION 290: Find the fundamental period of $f(x)$ defined as follows and explain as to why it does not comply with the LCM rule.

(a) $f(x) = \{x\} + \left\{x+\dfrac{1}{3}\right\}+\left\{x-\dfrac{1}{3}\right\}$

(b) $f(x) = a|\tan mx| + b|\tan nx|$

(c) $f(x) = 16(\sin^6 x + \cos^6 x) - 24(\sin^4 x + \cos^4 x) + 9$

SOLUTION: (a) Let $f(x) = \{x\} + \left\{x+\dfrac{1}{3}\right\}+\left\{x-\dfrac{1}{3}\right\}$

Let $g(x)$, $h(x)$ and $p(x)$ be $\{x\}$, $\left\{x+\dfrac{1}{3}\right\}$ and $\left\{x-\dfrac{1}{3}\right\}$ respectively.

We observe that the fundamental period of each of $g(x)$, $h(x)$ and $p(x)$ is 1

Hence, according to the LCM rule; the fundamental period of $f(x) = g(x) + h(x) + p(x)$ should be 1, but if we replace x by $x + \dfrac{1}{3}$ in $f(x)$,

we get $f\left(x+\dfrac{1}{3}\right) = \left\{x+\dfrac{1}{3}\right\}+\left\{x+\dfrac{1}{3}+\dfrac{1}{3}\right\}+\left\{x+\dfrac{1}{3}-\dfrac{1}{3}\right\}$

$=\left\{x+\dfrac{1}{3}\right\}+\left\{x-\dfrac{1}{3}+1\right\}+\{x\}$

$= \{x\}+\left\{x+\dfrac{1}{3}\right\}+\left\{x-\dfrac{1}{3}\right\} = f(x)$

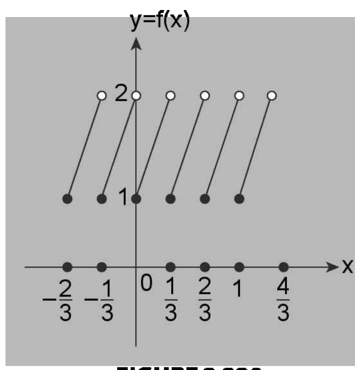


FIGURE 2.220

Hence, fundamental period of $f(x)=\frac{1}{3}$. Same can be concluded from graph.

Here $f(x) = \{x\} + \{x + 1/3\} + \{x - 1/3\}\ \forall\ x \in [0, 1/3) = x + x + 1/3 + x + 2/3 = 3x + 1$

$\therefore$ $f(x)$ is monotonic in $[0,1/3)$ $\Rightarrow$ $f(x)$ cannot have any period smaller than 1/3.

$\therefore$ The graph of $f(x)$ is given by

(b) Given $f(x) = a|\tan mx| + b|\tan nx|$

Let $g(x) = a\ |\tan mx|$ and $h(x) = b\ |\cot nx|$

Now period of $g(x)$ is $\frac{\pi}{m}$ and period of $h(x)$ is $\frac{\pi}{n}$

For $(a \neq b,\ m = n)$; $(a = b,\ m \neq n)$; $(a \neq b,\ m \neq n)$, the fundamental period of $f(x) = \text{LCM of } \left\{\frac{\pi}{m} \text{ and } \frac{\pi}{n}\right\} = \frac{\pi}{H.C.F.(m,n)}$

But when $a = b$ and $m = n$; then $g(x)$ and $h(x)$ are inter convertible, and hence, the fundamental period of $f(x) = \frac{1}{2}\left\{\text{LCM of } \frac{\pi}{m} \text{ and } \frac{\pi}{n}\right\}$ i.e., $\frac{\pi}{2m}$ or $\frac{\pi}{2n}$

(c) Given $f(x) = 16(\sin^6 x + \cos^6 x) - 24(\sin^4 x + \cos^4 x) + 9$

Let $g(x) = 16(\sin^6 x + \cos^6 x) = 16\{(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x \cos^2 x (\sin^2 x + \cos^2 x)\}$

$= 16\left\{1-\frac{3}{4}\sin^2(2x)\right\}$ $\Rightarrow$ Fundamental period of $g(x)$ is $\frac{\pi}{2}$

Let $h(x) = -24(\sin^4 x + \cos^4 x)$

$= -24[(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x] = -24\left(1-\frac{1}{2}\sin^2 2x\right)$

Again Fundamental period of $h(x) = \frac{\pi}{2}$

And $p(x) = 9$, is a constant function (i.e., no fundamental period)

According to the LCM Rule; Fundamental period of $f(x) = \text{LCM of } \left(\frac{\pi}{2},\frac{\pi}{2}\right) = \frac{\pi}{2}$

But $f(x) = 16(\sin^6 x + \cos^6 x) - 24(\sin^4 x + \cos^4 x) + 9(\sin^2 x + \cos^2 x)$

$= (16\sin^6 x - 24\sin^4 x + 9\sin^2 x) + (16\cos^6 x - 24\cos^4 x + 9\cos^2 x)$

$= (4\sin^3 x - 3\sin x)^2 + (4\cos^3 x - 3\cos x)^2 = (\sin 3x)^2 + (-\cos 3x)^2 = 1$

Hence, $f(x)$ is a constant function with no fundamental period.

ILLUSTRATION 291: Find the fundamental period of $f(x) = \sec(\sin x)$.

SOLUTION: Since $\sin x$ is a periodic function with fundamental period 2π, $f(x)$ has a period 2π.

For fundamental period $f(x + \pi) = \sec(\sin(\pi + x)) = \sec(-\sin x) = \sec(\sin x) = f(x)$

$f\left(x+\frac{\pi}{2}\right) \neq f(x)$. Hence, fundamental period is π.

ILLUSTRATION 292: Find the fundamental period of $f(x)$ defined as follows and explain as to why it does not comply with the LCM rule.

(a) $f(x) = \dfrac{|\cos x|+|\sin x|}{\cot x - \tan x}$ (b) $f(x) = \dfrac{\cos x - \cos 3x}{\sin x - \sin 3x}$

(c) $\tan\left(\frac{\pi}{3}-x\right).\tan x.\tan\left(\frac{\pi}{3}+x\right)$ (d) $\dfrac{9\tan\theta - 84\tan^3\theta + 126\tan^5\theta - 36\tan^7\theta + \tan^9\theta}{1-36\tan^2\theta + 126\tan^4\theta - 8\tan^6\theta + 9\tan^8\theta}$

SOLUTION: **(a)** Given $f(x) = \dfrac{|\cos x| + |\sin x|}{\cot x - \tan x}$

Now let $g(x) = |\cos x| + |\sin x|$ and $h(x) = \cot x - \tan x$

And the fundamental period of each of $g(x)$ and $h(x)$ is π

$\therefore$ The fundamental period of $f(x)$ should be π

But if we replace x by $x + \dfrac{\pi}{2}$ in $f(x)$; we get $f\left(x+\dfrac{\pi}{2}\right) = \dfrac{\left|\cos\left(x+\dfrac{\pi}{2}\right)\right| + \left|\sin\left(x+\dfrac{\pi}{2}\right)\right|}{\cot\left(x+\dfrac{\pi}{2}\right) - \tan\left(x+\dfrac{\pi}{2}\right)}$

$= \dfrac{|\cos x| + |\sin x|}{\cot x - \tan x}$. Hence, fundamental period of $f(x)$ is $\dfrac{\pi}{2}$

(b) Given $f(x) = \dfrac{\cos x - \cos 3x}{\sin x - \sin 3x}$

Let $g(x) = \cos x - \cos 3x$ and $h(x) = \sin x - \sin 3x$

Fundamental period of each of $g(x)$ and $h(x)$ is 2π

$\therefore$ According to the LCM rule, the fundamental period of $f(x)$ should be 2π.

But $f(x) = \dfrac{-2\sin 2x \sin(-x)}{-2\sin x \cos 2x} = -\tan 2x$; $\sin x \neq 0$ and the period of $\tan 2x$ is $\dfrac{\pi}{2}$

But as explained earlier, we need to be careful while dealing with domain.

$\sin x \neq 0 \Rightarrow x \neq n\pi \ \forall\ n \in \mathbb{Z}$

Thus, the period of discontinuity of function is π.

$\therefore$ The fundamental period of $f(x)$ will be LCM of π and $\dfrac{\pi}{2}$, i.e., π.

(c) Given $f(x) = \tan\left(\dfrac{\pi}{3} - x\right).\tan x.\tan\left(\dfrac{\pi}{3} + x\right)$

Let $g(x) = \tan\left(\dfrac{\pi}{3} - x\right)$; $h(x) = \tan x$ and $p(x) = \tan\left(\dfrac{\pi}{3} + x\right)$, respectively.

Fundamental period of each of $g(x)$, $h(x)$ and $p(x)$ is π

$\therefore$ According to the LCM rule, the Fundamental period of $f(x)$ should be π.

But $f(x) = \tan\left(\dfrac{\pi}{3} - x\right).\tan x.\tan\left(\dfrac{\pi}{3} + x\right) = \tan 3x$

$\therefore$ Fundamental period of $f(x) = \dfrac{\pi}{3}$.

(d) Given $f(x) = \dfrac{9\tan\theta - 84\tan^3\theta + 126\tan^5\theta - 36\tan^7\theta + \tan^9\theta}{1 - 36\tan^2\theta + 126\tan^4\theta - 84\tan^6\theta + 9\tan^8\theta}$

Let $g(x) = 9\tan\theta - 84\tan^3\theta + 126\tan^5\theta - 36\tan^7\theta + \tan^9\theta$

and $h(x) = 1 - 36\tan^2\theta + 126\tan^4\theta - 84\tan^6\theta + 9\tan^8\theta$

Now Fundamental period of each of $g(x)$ and $h(x) = \pi$

$\therefore$ According to LCM rule; Fundamental period of $f(x) = \pi$

But $f(x) = \dfrac{9\tan\theta - 84\tan^3\theta + 126\tan^5\theta - 36\tan^7\theta + \tan^9\theta}{1 - 36\tan^2\theta + 126\tan^4\theta - 8\tan^6\theta + 9\tan^8\theta} = \tan 9\theta$

Hence, fundamental period of $f(x) = \dfrac{\pi}{9}$

PERIODICITY OF FUNCTIONS EXPRESSED BY FUNCTIONAL EQUATIONS

Many a times we encounter a problem to determine period of a function $f(x)$ which is not defined directly but expressed in the form of equation/equations called as functional equation. In these situations, we often need to manipulate the given relation in such a way that leads to generate an equation of type $f(x + T) = \pm f(x)$ or $f(x + T) = \pm\frac{1}{f(x)}$.

Usually it is achieved by carrying out some suitable substitution in place of independent variable to obtain desired objective.

The process does not involves well defined algorithm (skilled based), thus, can be better understood only by working out on some examples.

(i) If a function $f(x)$ is defined such that $f(x + T) = -f(x)$; where T is a positive constant, then f is periodic with period $2T$. (Converse is not true)

Proof: Given $f(x + T) = -f(x)$

Replacing x by $x + T$, we have $f(x + 2T) = -f(x + T)$

$\Rightarrow$ $f(x + 2T) = -(-f(x))$

$\Rightarrow$ $f(x + 2T) = f(x)\ \forall\ x \in D_f$

e.g., $f(x) = \sin x, \cos x, \sec x, \operatorname{cosec} x$

(ii) If a function $f(x)$ is defined such that $f(x + T) = \frac{1}{f(x)}$ or $f(x + T) = \frac{-1}{f(x)}$; where T is a positive constant, then f is periodic with period $2T$. (Converse is not true)

Proof: Given $f(x + T) = \pm\frac{1}{f(x)}$

Replacing x by $x + T$, we have

$$f(x+2T) = \pm\frac{1}{f(x+T)} = \frac{\pm 1}{\pm(f(x))^{-1}} = f(x)$$

e.g., $f(x) = \tan x, \cot x$

As $f\left(x+\frac{\pi}{2}\right) = \tan\left(\frac{\pi}{2}+x\right) = -\frac{1}{\tan x}$

$\Rightarrow$ Period of $\tan x = 2\left(\frac{\pi}{2}\right) = \pi$.

(iii) If $f(x + \lambda) = g(f(x))$ such that $\underbrace{g\left(g\left(g\left(\ldots\left(g(x)\right)\right)\ldots\right)\right)}_{\text{composed } n \text{ times}} = x$, then prove that $f(x)$ is periodic with period $n\lambda$ (where λ is fixed positive real constant)

Proof: Given $f(x + 2\lambda) = g(f(x + \lambda)) = g(g(f(x))) = g^2(f(x))$

Proceeding similarly $f(x + 3\lambda) = g(f(x + 2\lambda)) = g^2(f(x + \lambda)) = g^2(g(f(x)) = g^3(f(x))$ and $f(x + n\lambda) = g^{n-1}(f(x + \lambda)) = g^{n-1}(g(f(x)) = g^n(f(x)) = f(x)$

$\Rightarrow$ Period of $f(x)$ is $n\lambda$

ILLUSTRATION 293: Let $f: \mathbb{R} \to \mathbb{R} - \{x\}$ be a function such that there exists $T > 0$ for which $f(x + T) = \frac{f(x)-5}{f(x)-3}$ for every $x \in \mathbb{R}$. Prove that $f(x)$ is periodic.

SOLUTION: Given $f(x + T) = \frac{f(x)-5}{f(x)-3}$;

Replacing x by $x + T$, we have $f(x + 2T) = \frac{f(x+T)-5}{f(x+T)-3} = \frac{\frac{f(x)-5}{f(x)-3}-5}{\frac{f(x)-5}{f(x)-3}-3} = \frac{f(x)-5-5f(x)+15}{f(x)-5-3f(x)+9}$

$\Rightarrow$ $f(x + 2T) = \frac{-4f(x)+10}{-2f(x)+4} = \frac{2f(x)-5}{f(x)-2}$;

Replacing x by $x + 2T$, we have $f(x + 4T) = \frac{2f(x+2T)-5}{f(x+2T)-2} = \frac{2\left(\frac{2f(x)-5}{f(x)-2}\right)-5}{\frac{2f(x)-5}{f(x)-2}-2}$

$= \frac{4f(x)-10-5f(x)+10}{2f(x)-5-2f(x)+4} = \frac{-f(x)}{-1} = f(x) \Rightarrow f(x + 4T) = f(x)$

Hence, $f(x)$ is a periodic function with a fundamental period of $4T$.

ILLUSTRATION 294: Find the period of $f(x)$ satisfying the condition:

(i) $f(x+p) = 1 + \{1 - 3f(x) + 3f^2(x) - f^3(x)\}^{1/3}$

(ii) $f(x-1) + f(x+3) = f(x+1) + f(x+5)$

SOLUTION: (i) Given $f(x+p) = 1 + \{1 - 3f(x) + 3f^2(x) - f^3(x)\}^{1/3} = 1 + (1 - f(x))$

$\Rightarrow f(x+p) + f(x) = 2$ ...(1)

$\Rightarrow f(x+2p) + f(x+p) = 2$ ...(2)

From (1) and (2), we get $f(x) = f(x+2p) \Rightarrow$ Period $= 2p$

(ii) Given $f(x-1) + f(x+3) = f(x+1) + f(x+5)$ .....(1)

Replacing x by $x+2$, we have $f(x+1) + f(x+5) = f(x+3) + f(x+7)$ ......(2)

(1) + (2) gives, $f(x-1) = f(x+7)$ $\Rightarrow f(x+8) = f(x)$ $\Rightarrow$ period = 8

ILLUSTRATION 295: Let f be a function that satisfies $f(x+6) + f(x-6) = f(x)$ for all real x. Prove that any such function is periodic with a common positive period p for all of them. Also, find the value of the period p.

SOLUTION: Given $f(x+6) + f(x-6) = f(x)$ .... (1)

Replacing x by $x+6$ in equation (1), we get $f(x+12) + f(x) = f(x+6)$ ... (2)

Substituting the value of $f(x+6)$ from (1) in (2), we get $f(x+12) + f(x) = f(x) - f(x-6)$ ... (3)

Again replacing x by $x+6$ in equation (3), we get $f(x+18) = -f(x)$

Now, replacing x by $x+18$, we get $f(x+36) = -f(x+18)$

$\Rightarrow f(x+36) = f(x) \Rightarrow f(x)$ is periodic with period 36.

ILLUSTRATION 296: Let $y = f(x)$, $x \in \mathbb{R}$ be a function such that the graph of $f(x)$ is symmetrical about the lines $x = \alpha$ and $x = \beta$; where $\alpha < \beta$. Prove that $f(x)$ is periodic, also find the period of $f(x)$.

SOLUTION: We have $f(\alpha - x) = f(\alpha + x)$ ... (1)

and $f(\beta - x) = f(\beta + x)$ ... (2)

[Since the function $f(x)$ is symmetrical about lines $x = \alpha$ and $x = \beta$]

Replacing x by $\alpha - x$ in equation (1), we have $f(x) = f(2\alpha - x)$ ... (3)

Replacing x by $(\beta - x)$ in equation (2), we have $f(x) = f(2\beta - x)$ ... (4)

From equation (3), we have $f(x) = f(2\alpha - x) = f\{2\beta - (2\alpha - x)\} = f\{x + 2(\beta - \alpha)\}$ (from (4))

Hence, $f(x)$ is periodic, having a period $2(\beta - \alpha)$

ILLUSTRATION 297: Let f be real valued function defined for all real numbers x such that for some positive constant a, the equation $f(x+a) = \frac{1}{2} + \sqrt{f(x) - (f(x))^2}$ holds for all x. Prove that the function f is periodic.

SOLUTION: Given $f(x+a) = \frac{1}{2} + \sqrt{f(x) - (f(x))^2}$ or $f^2(x+a) + \frac{1}{4} - f(x+a) = f(x) - f^2(x)$

or $f^2(x) + f^2(x+a) + \frac{1}{4} = f(x) + f(x+a)$ ....(1)

Replacing x by $x+a$, we have $f^2(x+a) + f^2(x+2a) + \frac{1}{4} = f(x+a) + f(x+2a)$ ... (2)

(1) – (2), gives $f^2(x) - f^2(x+2a) = f(x) - f(x+2a)$

$\Rightarrow \{f(x) - f(x+2a)\}\ \{f(x) + f(x+2a)\} = 0$

$\Rightarrow f(x) = f(x+2a)$ or $f(x) = -f(x+2a)$ (rejected, as by given equation, $f(x)$ is positive)

$\Rightarrow$ Period of $f(x) = 2a$

ILLUSTRATION 298: Find the period of the following functions:

(i) $f(x) = sgn\ (e^{-x}) - \sin x + |\sin x| + \min(\sin x, |x|)$

(ii) $\left[x+\dfrac{1}{2}\right]+\left[x-\dfrac{1}{2}\right]+2\ [-x]$, where $[x]$ denotes the greatest integer function

SOLUTION: (i) Given $\operatorname{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$

$\because\ e^{-x} > 0 \Rightarrow sgn\ (e^{-x}) = 1$

$\therefore$ $sgn\ (e^{-x})$ is constant function which is periodic with no fundamental period.

$\therefore\ f(x) = 1 - \sin x + |\sin x| + \sin x$

$\min(\sin x, |x|) = \sin x\ (\because \sin x \le |x|)$

$\therefore\ f(x) = 1 - \sin x + |\sin x| + \sin x$

Hence, $f(x) = 1 + |\sin x|$ which is periodic with period π.

(ii) Given $f(x) = \left[x+\dfrac{1}{2}\right]+\left[x-\dfrac{1}{2}\right]+2[-x]$

Case I: If $x \in \mathbb{Z}$; then $f(x) = [x]+\left[\dfrac{1}{2}\right]+[x]+\left[\dfrac{-1}{2}\right] + 2\ (-[x]) = 0 - 1 = -1$

Case II: If $x \notin \mathbb{Z}$

Sub-case (i): $x = k + \lambda$; where k is an integer and $\lambda \in (0,1/2)$

Hence, $\left[k+\lambda+\dfrac{1}{2}\right] = k\left[\lambda+\dfrac{1}{2}\right]$ and $k < k + \lambda + 1/2 < k + 1 \Rightarrow [x + 1/2] = k$

Also $[x - 1/2] = [k + \lambda - 1/2]$ and $k - 1 < k + \lambda - 1/2 < k$

$\therefore\ [x - 1/2] = k - 1$ and $[-x] = -[x] - 1 = -k - 1$

$\therefore\ \left[x+\dfrac{1}{2}\right]+\left[x-\dfrac{1}{2}\right]+2[-x] = (k) + (k-1) + 2(-k-1) = -3$

Sub-case (ii): If $x = k + \lambda$; where k is an integer and $1/2 \le \lambda < 1$

Hence, $x + 1/2 = k + \lambda + 1/2$ and $k + 1 \le k + \lambda + 1/2 < k + 2$

$\therefore\ [x + 1/2] = k + 1$

Also $(x - 1/2) = k + \lambda - 1/2$ and $k \le k + \lambda - 1/2 < k + 1$

$\therefore\ [x - 1/2] = k$ and $[-x] = -[x] - 1 = -k - 1$

$\therefore\ [x + 1/2] + [x - 1/2] + 2\ [-x] = (k + 1) + k + 2(-k - 1) = -1$

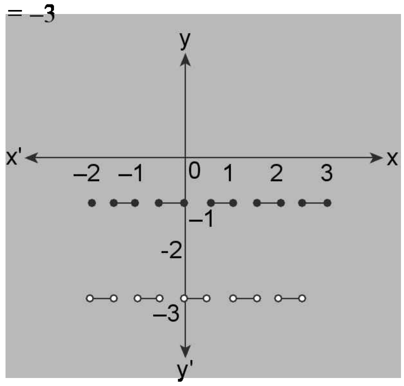


FIGURE 2.221

$\therefore$ The graph of $f(x)$ is as shown in Figure.

Clearly, we can see that the function is periodic with a fundamental period 1.

TEXTUAL EXERCISE–16: (SUBJECTIVE)

1. Find the fundamental period of the following functions:
   (i) $f(x) = 2 + 3\cos(x - 2)$
   (ii) $f(x) = \cos\frac{3}{5}x - \sin\frac{2}{7}x$
   (iii) $f(x) = \sin\frac{\pi x}{4} + \sin\frac{\pi x}{3}$
   (iv) $f(x) = \frac{1}{1-\cos x}$
   (v) $f(x) = \sin(2x + \cos x)$
2. Find the fundamental period of the following functions:
   (i) $f(x) = \sin 3x + \cos 2x + |\tan x|$
   (ii) $f(x) = [\sin 3x] + |\cos 6x|$
   (iii) $f(x) = \frac{\sin 12x}{1+\cos^2 6x}$
   (iv) $f(x) = \sec^3 x + \operatorname{cosec}^3 x$
3. Find the period of the following functions:
   (i) $f(x) = \frac{\sin x + \sin 3x}{\cos x + \cos 3x}$
   (ii) $f(x) = 1 - \frac{\sin^2 x}{1+\cot x} - \frac{\cos^2 x}{1+\tan x}$
   (iii) $f(x) = \tan\frac{\pi}{2}[x]$, where [.] denotes greatest integer function
   (iv) $f(x) = \ln(2 + \cos 3x)$
   (v) $f(x) = e^{\ln \sin x} + \tan^3 x - \operatorname{cosec}(3x - 5)$
   (vi) $f(x) = \sin x + \tan\frac{x}{2} + \sin\frac{x}{2^2} + \tan\frac{x}{2^3} + \ldots + \sin\frac{x}{2^{n-1}} + \tan\frac{x}{2^n}$
4. Find period of the following functions:
   (i) $f(x) = \cos\frac{4x}{5} + \sin\frac{4x}{3} + \tan\frac{8x}{3}$
   (ii) $f(x) = \frac{(1+\sin x)(1+\sec x)}{(1+\cos x)(1+\operatorname{cosec} x)}$
   (iii) $f(x) = \sin\left(2\pi x + \frac{\pi}{3}\right) + 2\sin\left(3\pi x + \frac{\pi}{4}\right) + 3\sin(5\pi x)$
5. Find the period of the following functions:
   (i) $f(x) = \left|\cos^5\left(\frac{2x}{3}\right)\right|$
   (ii) $f(x) = \sin(\pi x) + [x] + 2 - x$
   (iii) $f(x) = x - [x - b]$
6. Prove that the function $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$ is a periodic function with indeterminate period.
7. Find the period of the following functions:
   (i) $f(x) = \frac{1}{2}\left\{\frac{|\sin x|}{\cos x} + \frac{|\cos x|}{\sin x}\right\}$
   (ii) $f(x) = \sin\frac{\pi x}{n!} - \cos\frac{\pi x}{(n+1)!}$
8. Find the period of function $f(x)$ if for a fixed positive real number λ exactly one of the following relations hold.
   (a) $f(x + \lambda) = -f(x)$
   (b) $f(x + \lambda) = \pm\frac{1}{f(x)}$
9. Prove that the function satisfying the following functional equations $f(x+a) = \frac{1+f(x)}{1-f(x)}\ \forall x \in \mathbb{R}$ and a fixed a is periodic function, where a is fixed positive real number.
10. Let $f(x) = \sin x + \cos\left(\sqrt{4-a^2}\right)x$, then find the set of all integer values of a for which $f(x)$ is a periodic function.

Answer Keys

1. (i) 2π	(ii) 70π	(iii) 24	
(iv) 2π	(v) 2π		
2. (i) 2π	(ii) $2\pi/3$	(iii) $\pi/6$	(iv) 2π
3. (i) π	(ii) π	(iii) 2	
(iv) $\frac{2\pi}{3}$	(v) 2π	(vi) $2^n\pi$	
4. (i) $\frac{15\pi}{2}$	(ii) π	(iii) 2	
5. (i) $\frac{3\pi}{2}$	(ii) 2	(iii) 1	

7. (i) 2π (ii) $2((n + 1)!)$
8. (a) 2λ (b) 2λ
9. $4a$
10. $\{-2, 0, 2\}$

TEXTUAL EXERCISE–16: (OBJECTIVE)

1. The period of the function $f(x) = \sin 2\pi x + \sin\left(\frac{\pi x}{3}\right) + \sin\left(\frac{\pi x}{5}\right)$ is
(a) 2 (b) 6
(c) 15 (d) 30

2. The function $f(x) = sgnx \,.\, \sin x$ is
(a) discontinuous no where
(b) an even function
(c) non-periodic
(d) differentiable for all x

3. Which of the following function(s) is/are periodic with period π?
(a) $f(x) = |\sin x|$ (b) $f(x) = [x + \pi]$
(c) $f(x) = \cos(\sin x)$ (d) $f(x) = \cos^2 x$
(where [.] denotes the greatest integer function)

4. Period of $f(x) = nx + n - [nx + n]$; ($n \in \mathbb{N}$; where [ ] denotes the greatest integer function is)
(a) 1 (b) 1/n
(c) n (d) None of these

5. The period of the function $f(x) = \sin(x + 3 - [x + 3])$, where [ ] denotes the greatest integer function is
(a) $2\pi + 3$ (b) 2π
(c) 1 (d) 3

6. Minimum period of the function $f(x) = |\sin^3 2x| + |\cos^3 2x|$ is
(a) π (b) $\frac{\pi}{2}$
(c) $\frac{\pi}{4}$ (d) $\frac{3\pi}{4}$

7. The period of the function $f(x) = \frac{|\sin x| + |\cos x|}{|\sin x - \cos x|}$ is
(a) $\pi/2$ (b) $\pi/4$
(c) π (d) 2π

8. Which one is not periodic?
(a) $|\sin 3x| + \sin^2 x$ (b) $\cos\sqrt{x} + \cos^2 x$
(c) $\cos 4x + \tan^2 x$ (d) $\cos^2 x + \sin x$

9. Which of the following function(s) is/are periodic?
(a) $f(x) = x - [x]$
(b) $g(x) = \sin(1/x)$, $x \neq 0$ and $g(0) = 0$
(c) $h(x) = x \cos x$
(d) $w(x) = \sin^{-1}(\sin x)$

10. If $f(x) = e^{\sin(x - [x]) \cos \pi x}$, then $f(x)$ is ([x] denotes the greatest integer function)
(a) non-periodic
(b) periodic with no fundamental period
(c) periodic with period 2
(d) periodic with period π

11. If $f(x) = 2\tan 3x + 5\sqrt{1 - \cos 6x}$; $g(x)$ is a function having the same time period as that of $f(x)$, then which of the following can be $g(x)$?
(a) $(\sec^2 3x + \operatorname{cosec}^2 3x)\tan^2 3x$
(b) $2 \sin 3x + 3 \cos 3x$
(c) $2\sqrt{1 - \cos^2 3x} + \operatorname{cosec} 3x$
(d) $3 \operatorname{cosec} 3x + 2 \tan 3x$

12. If $n \in \mathbb{N}$, and the period of $\frac{\cos nx}{\sin\left(\frac{x}{n}\right)}$ is 4π, then n is equal to
(a) 4 (b) 3
(c) 2 (d) 1

13. Which one of the following statements is NOT CORRECT?
(a) The derivative of a differentiable periodic function is a periodic function with the same period.
(b) If $f(x)$ and $g(x)$ both are defined on the entire number line and are non-periodic then the function $F(x) = f(x) \,.\, g(x)$ can not be periodic.
(c) Derivative of an even differentiable function is an odd function and derivative of an odd differentiable function is an even function.
(d) Every function $f(x)$ can be represented as the sum of an even and an odd function

14. If $f(x)$ is an odd periodic function with period 2, then $f(4)$ equals
(a) 0
(b) 2
(c) 4
(d) –4

15. The period of the function $f(x) = \dfrac{\sin 8x \cos x - \sin 6x \cos 3x}{\cos 2x \cos x - \sin 3x \sin 4x}$ is
(a) π
(b) 2π
(c) $\dfrac{\pi}{2}$
(d) None of these

16. Suppose that f is a periodic function with period $\dfrac{1}{2}$ and that $f(2) = 5$ and $f\left(\dfrac{9}{4}\right) = 2$ then $f(-3) - f\left(\dfrac{1}{4}\right)$ has the value equal to
(a) 2
(b) 3
(c) 5
(d) 7

Answer Keys

1. (d)	**2.** (a,b,c)	**3.** (a,c,d)	**4.** (b)	**5.** (c)	**6.** (c)	**7.** (c)	**8.** (b)	**9.** (a,d)	**10.** (c)
11. (a)	**12.** (c)	**13.** (b)	**14.** (a)	**15.** (c)	**16.** (b)				

MULTIPLE–CHOICE QUESTIONS

SECTION–I

OBJECTIVE-TYPE SOLVED EXAMPLES

1. Let $f(x)=\dfrac{ax+b}{cx+d}$. Then $fof(x)=x$; provided that

(a) $d=-a$ (b) $d=a$
(c) $a=b=c=d=1$ (d) $a=b=1$

Solution: **(a)** We have $f(x)=\dfrac{ax+b}{cx+d}$.

Therefore, $fof(x)=x \Leftrightarrow f(f(x))=x$

$$\Leftrightarrow f\left(\frac{ax+b}{cx+d}\right)=x \quad \Leftrightarrow \frac{a\left(\frac{ax\quad b}{cx\quad d}\right)\quad b}{c\left(\frac{ax\quad b}{cx\quad d}\right)\quad d}$$

$$\Leftrightarrow \frac{x(a^2+bc)+ab+bd}{x(ac+cd)+bc+d^2}=x\ \forall\ x\in\mathbb{R}$$

Clearly, $d=-a$ satisfies this relation.

2. Let $g(x)=1+x-[x]$ and $f(x)=\begin{cases}-1, & x<0\\ 0, & x=0\\ 1, & x>0\end{cases}$. Then for all x, $f(g(x))$ is equal to; (where [] denotes the greatest integer function)

(a) x (b) 1
(c) $f(x)$ (d) $g(x)$

Solution: (b) $g(x)=1+x-[x]$ is greater than 1, since $x-[x]>0$
$f[g(x)]=1$, since $f(x)=1$ for all $x>0$

3. The set of all values of x for which $|f(x)+g(x)|<|f(x)|+|g(x)|$ is true if $f(x)=x-3$ and $g(x)=4-x$ is given by

(a) $\mathbb{R}$ (b) $\mathbb{R}-]3,4[$ or $\mathbb{R}-(3,4)$
(c) $\mathbb{R}-[3,4]$ (d) None of these

Solution: **(c)** The given inequality is equivalent to $1<|x-3|+|4-x|$
i.e., $1<|x-3|+|x-4|$ ....(1)
For $x<3$, we have $3-x+4-x=7-2x>1$ ....(2)
i.e., $x<3$, which is true in the domain
For $3\le x<4$ (1), gives $x-3+4-x=1>1$,
Which is not true ...(3)
For $4\le x$, we have from (1) $2x-7>1$
$\Rightarrow x>4$, which holds in case domain ...(4)
From (2) and (4), we have the solution of the inequality as $x<3$ or $x>4$
$\therefore$ i.e., $x\in\mathbb{R}-[3,4]$.

4. The domain of $f(x)=\sqrt{\log_{1/4}\left(\dfrac{5x-x^2}{4}\right)}+{}^{10}C_x$ is

(a) $(0,1]\cup[4,5)$ (b) $(0,5)$
(c) $\{1,4\}$ (d) None of these

Solution: (c) Let $f_1=\sqrt{\log_{1/4}\left(\dfrac{5x-x^2}{4}\right)}$ and $f_2={}^{10}C_x$

Clearly f_1 is defined for $\log_{1/4}\left(\dfrac{5x-x^2}{4}\right)\ge 0$ and

$\dfrac{5x-x^2}{4}>0 \quad \Rightarrow 0<\left(\dfrac{5x-x^2}{4}\right)\le 1$

$\Rightarrow \dfrac{5x-x^2}{4}>0$ and $\dfrac{5x-x^2}{4}\le 1$

$\Rightarrow x(x-5)<0$ and $x^2-5x+4\ge 0$
$\Rightarrow x\in(0,5)$ and $x\in(-\infty,1]\cup[4,\infty)$
$\Rightarrow f_1$ is defined for $x\in(0,1]\cup[4,5)=D_{f_1}$ (say)

f_2 is defined for $x\in\{0,1,2,3,4,5,6,7,8,9,10\}$ $=D_{f_2}$ (say)

$f(x)$ is defined for $x\in D_{f_1}\cap D_{f_2}=\{1,4\}$

5. The number of one–one functions $f:\{1,2,3,\ldots,10\}\to\{1,2,3,\ldots,10\}$ such that $f(i)\ne i$ but $f(f(i))=i$ for each $i=1,2,3,\ldots,10$ is

(a) 945 (b) 256
(c) 2560 (d) None of these

Solution **(a)** $f(i)\ne i$ and $f(f(i))=i \Rightarrow f^{-1}=f$
$\Rightarrow$ Functions will be self invertible
$\Rightarrow$ Functions are symmetric, i.e., $(\alpha,\beta)\in f$
$\Rightarrow (\beta,\alpha)\notin f^{-1} \quad \Rightarrow (\beta,\alpha)\in f$ and $\alpha\ne\beta$
Which can be formed by selecting five pairs of 10 in

$\dfrac{{}^{10}C_2\times{}^{8}C_2\times{}^{6}C_2\times{}^{4}C_2\times{}^{2}C_2}{5!}$ ways, i.e., 945 ways.

6. The number of functions f from $\{1, 2, \ldots, 25\}$ onto $\{1, 2, \ldots, 25\}$ such that $f(k)$ is a multiple of 3 whenever k is a multiple of 4 is

(a) $(8)^6 . 19!$ (b) $6!.19!$

(c) $\frac{8!}{2!}\times 19!$ (d) None of these

Solution: (c) $A = \{1, 2, 3, \ldots, 25\}$ contains 6 multiples of 4, i.e., k = 4, 8, 12, 16, 20, 24 and there are 8 multiples of 3

Now out of 8, we can select any 6 in ${}^8C_6$ ways and can be associated with 6 k's in ${}^8C_6 \times 6!$ ways and the remaining 19 elements can associated in 19! ways.

$\therefore$ Total number of required functions

$= {}^8C_6 \times 6! \times 19! = \frac{8!}{2!}\times 19!$

7. The number of bijective functions $f : A \to A$ where $A = \{1, 2, 3, 4)$; such that $f(1) \neq 3, f(2) \neq 1. f(3) \neq 4, f(4) \neq 2$ is

(a) 11 (b) 23

(c) 12 (d) 9

Solution: (d) The number of bijections is equal to number of ways of distributing 4 letters into envelops so that no letter goes to its correct address

$= 4!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right) = 9$.

8. Let f be a function defined on $[-1, 3]$, then the function $g(x) = f(2x^2 - 1)$ has domain

(a) $[0, 2]$ (b) $(-\infty, -2] \cup [2, \infty)$

(c) $[-2, 2]$ (d) None of these

Solution: (c) For the validity of given equation $g(x) = f(2x^2 - 1)$; $-1 \le 2x^2 - 1 \le 3$

$\Rightarrow 0 \le 2x^2 \le 4 \quad \Rightarrow 0 \le x^2 \le 2$

$\Rightarrow x \in [-2, 2]$

9. The largest interval lying in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which the function $f(x) = 3^{-x^2} + \cos^{-1}\left(\frac{1}{2}x-1\right) + \log.\cos x$ is defined, is

(a) $[0, \pi]$ (b) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(c) $[0, \pi/2)$ (d) None of these

Solution: (c) 3^{-x^2} is defined for all $x \in \mathbb{R}$

$\cos^{-1}\left(\frac{1}{2}x-1\right)$ is defined for $-1 \le \left(\frac{x}{2}-1\right) \le 1$

i.e., $0 \le x \le 4$ and log cos x is defined (real) for $\cos x > 0$, i.e., $2n\pi - \frac{\pi}{2} < x < 2n\pi + \frac{\pi}{2}$

Hence, the largest interval lying in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is $\left[0, \frac{\pi}{2}\right)$.

10. Let $f(x) = \sqrt{[\sin 2x]-[\cos 2x]}$ (where [.] denotes the greatest integer function), then range of $f(x)$ will be

(a) $\{0\}$ (b) $\{1\}$

(c) $\{0, 1\}$ (d) $\{0, 1, \sqrt{2}\}$

Solution: (c) We should have $[\sin 2x] \ge [\cos 2x]$

we can have $[\sin 2x] = 1$, $[\cos 2x] = 1, 0, -1$

$[\sin 2x] = 0$, $[\cos 2x] = 0, -1$

$[\sin 2x] = -1$, $[\cos 2x] = -1$ but $[\sin 2x] = 1$,

$[\cos 2x] = 1$ and $[\sin 2x] = 1$, $[\cos 2x] = -1$ are impossible

So, range = $\{0, 1\}$

11. Range of the function $f(x) = \sin^{-1}\left(\left[x^2 - \frac{1}{3}\right]-1\right) + \cos^{-1}\left[x^2 + \frac{2}{3}\right]$ is ([.] denotes the greatest integer function)

(a) $\left[-\frac{\pi}{2}+\frac{3\pi}{2}\right]$ (b) $\left\{-\frac{\pi}{2}\right\}$

(c) $\left\{-\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$ (d) None of these

Solution: (b)

$f(x) = \sin^{-1}\left(\left[x^2 - \frac{1}{3}\right]-1\right) + \cos^{-1}\left[x^2 + \frac{2}{3}\right]$

Since $\cos^{-1}y$ is defined for $-1 \le y \le 1$

So, $\left[x^2 + \frac{2}{3}\right] = -1, 0, 1$ but $\left[x^2 + \frac{2}{3}\right] = -1$ is impossible

$\Rightarrow \left[x^2 + \frac{2}{3}\right] = 0, x^2 + \frac{2}{3} \in \left[\frac{2}{3}, 1\right)$

$\Rightarrow x^2 \in \left[0, \frac{1}{3}\right) \Rightarrow \left[x^2 - \frac{1}{3}\right]$

$\Rightarrow \sin^{-1}\left(\left[x^2 - \frac{1}{3}\right]-1\right)$ is not defined.

$\therefore \left[x^2 + \frac{2}{3}\right] = 1 \Rightarrow \left[x^2 - \frac{1}{3}\right] - 1 = -1$

$\Rightarrow f(x) = \sin^{-1}(-1) + \cos^{-1}(1)$

$\Rightarrow -\frac{\pi}{2} + 0 = -\frac{\pi}{2}$

12. Least value of the expression $\dfrac{1}{2bx-(x^2+b^2+\sin^2 x)}$, $x \in [-1, 0]$, $b \in [2, 3]$ is

(a) $\dfrac{1}{4}$ (b) $-\dfrac{1}{4}$

(c) $\dfrac{-1}{8+\sin^2 1}$ (d) None of these

Solution: (b) Given expression $= \dfrac{1}{-f(x)}$; where

$f(x) = x^2 + b^2 + \sin^2 x - 2bx = (x-b)^2 + \sin^2 x$

Now, $x \in [-1, 0]$ and $b \in [2, 3]$

$\Rightarrow |x-b| \in [2, 4] \Rightarrow (x-b)^2 \in [4, 16]$ and $\sin^2 x \in [0, \sin^2 1]$. As x increases from -1 to 0; $\sin^2 x$ and $(x-b)^2$ both decreases from $\sin^2 1$ to 0 and $(1+b)^2$ to b^2 respectively and $b \in [2, 3]$.

$\Rightarrow f(x) \in [4, 16 + \sin^2 1]$

$$\Rightarrow \frac{1}{f(x)} \in \left[\frac{1}{16+\sin^2 1}, \frac{1}{4}\right]$$

$$\Rightarrow \frac{1}{-f(x)} \in \left[-\frac{1}{4}, -\frac{1}{16+\sin^2 1}\right]$$

So, least value of expression is $\dfrac{-1}{4}$.

13. If $f(x) = |x-1|$, $f: \mathbb{R}^+ \to \mathbb{R}$, $g(x) = e^x$, $g: [-1, \infty) \to \mathbb{R}$. If the function $fog\,(x)$ is defined, then its domain and range respectively are:

(a) $(0, \infty)$ and $[0, \infty)$

(b) $[-1, \infty)$ and $[0, \infty)$

(c) $[-1, \infty)$ and $\left[1-\dfrac{1}{e}, \infty\right)$

(d) $[-1, \infty)$ and $\left[\dfrac{1}{e}-1, \infty\right)$

Solution: (b) Given function is $f(x) = |x-1|$;

$f: \mathbb{R}^+ \to \mathbb{R}$ and $g(x) = e^x$, $g: [-1, \infty) \to \mathbb{R}$

Now $fog(x) = f[g(x)] = |e^x - 1|$

Now $x \in [-1, \infty)$ $\Rightarrow g(x) \in [e^{-1}, \infty)$

$$\Rightarrow (e^x - 1) \in \left[\frac{1}{e}-1, \infty\right) \Rightarrow |e^x - 1| \in [0, \infty)$$

$\therefore$ Range of $fog(x)$ is $[0, \infty)$ and domain $[-1, \infty)$.

14. If $f(x) = \cot^{-1} x : \mathbb{R}^+ \to \left(0, \dfrac{\pi}{2}\right)$ and $g(x) = 2x - x^2$: $\mathbb{R} \to \mathbb{R}$. Then the range of the function $f(g(x))$ wherever defined is

(a) $\left(0, \dfrac{\pi}{2}\right)$ (b) $\left(0, \dfrac{\pi}{4}\right)$

(c) $\left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$ (d) $\left(\dfrac{\pi}{4}\right)$

Solution: (c) Given $f(x) = \cot^{-1} x : \mathbb{R}^+ \to \left(0, \dfrac{\pi}{2}\right)$;

$g(x) = 2x - x^2 : \mathbb{R} \to \mathbb{R} \Rightarrow f(g(x)) = \cot^{-1}(2x - x^2)$

Now $2x - x^2$ is continuous and has its range $(-\infty, 1]$ and $\cot^{-1}(x)$ is defined from $\mathbb{R}^+ \to \left(0, \dfrac{\pi}{2}\right)$ and is continuous and decreasing function.

$\Rightarrow \cot^{-1}(2x - x^2)$ has its range

$$[\cot^{-1}1, \cot^{-1}(-\infty)] = \left[\frac{\pi}{4}, \pi\right) \cap \left(0, \frac{\pi}{2}\right) = \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$$

15. Let $f(x) = 1 + \dfrac{1}{\sqrt[4]{x}}$ and $g(x, y) = \log_x y$, then the domain of $g\left(\dfrac{1}{2}, -g(2, f(x)) - 1\right)$ is

(a) $0 < x < 1$ (b) $0 < x \le 1$

(c) $x \ge 1$ (d) Null set

Solution: (d) $-g(2, f(x)) = -\log_2\left(1 + \dfrac{1}{\sqrt[4]{x}}\right)$

$$\Rightarrow -g(2, f(x)) - 1 = -\log_2\left(1 + \frac{1}{\sqrt[4]{x}}\right) - 1$$

$$\therefore\ g\left(\frac{1}{2}, -g(2, f(x)) - 1\right) = \log_{1/2}\left(-\log_2\left(1 + \frac{1}{\sqrt[4]{x}}\right) - 1\right)$$

$$\Rightarrow \log_2\left(1 + \frac{1}{\sqrt[4]{x}}\right) + 1 < 0 \quad \Rightarrow \quad 0 < 1 + \frac{1}{\sqrt[4]{x}} < 2^{-1}$$

$$\Rightarrow -1 < \frac{1}{\sqrt[4]{x}} < -\frac{1}{2} \quad \Rightarrow x \in \phi \text{ (null set)}$$

16. Let $f(x) = \sin x - \cos x$ and $g(x) = \log_{\sqrt{5}} x$; then the range of $g\left(\sqrt{2} f(x) + 3\right)$ is

(a) $[0, 1]$ (b) $[0, 2]$

(c) $\left[0, \dfrac{3}{2}\right]$ (d) None of these

Solution: (b) $g(\sqrt{2} f(x) + 3)$

$= \log_{\sqrt{5}}\left(\sqrt{2}(\sin x - \cos x) + 3\right)$.

We know that $-\sqrt{2} \le (\sin x - \cos x) \le \sqrt{2}$, $\forall\ x \in \mathbb{R}$

$\left[\because -\sqrt{a^2+b^2} \le a\sin x + b\cos x \le \sqrt{a^2+b^2}\right]$

$\Rightarrow -2 \le \sqrt{2}(\sin x - \cos x) \le 2$

$\Rightarrow 1 \le \sqrt{2}(\sin x - \cos x) + 3 \le 5$

$\Rightarrow 0 \le \log_{\sqrt{}}\left(\sqrt{2}(\sin x - \cos x) + 3\right) \le 2$

($\because \log_a x$ is increasing function for $a > 1$)

Hence, range of $g(\sqrt{2}f(x) + 3)$ is $[0, 2]$.

17. Let $f(x) = 16\sin^2 x + 1$ and $g(x, y) = \log_x y$, then the range of $h(x) = g\left(\sqrt{2}, (2 - g(2, f(x)))\right)$ is

(a) $(-\infty, 1)$ (b) $(-\infty, 2)$

(c) $(-\infty, 1]$ (d) $(-\infty, 2]$

Solution: (d) $h(x) = g\left[\sqrt{2}, (2 - g(2, f(x)))\right]$

$= \log_{\sqrt{}}[2 \quad g(2, f(x))] = \log_{\sqrt{}}[2 \quad \log \quad f(x)]$

$\Rightarrow 0 < 16\sin^2 x + 1 < 4 \quad \Rightarrow 1 < 16\sin^2 x + 1 < 4$

$\Rightarrow 0 \le \sin \quad < \frac{}{16} \quad \Rightarrow 1 \le 16\sin^2 x + 1 < 4$

$\Rightarrow 0 \le \log_2(16\sin^2 x + 1) < 2$

$\Rightarrow 2 \ge 2 - \log \ (16\sin^2 x + 1) > 0$

$\Rightarrow \log_{\sqrt{2}} 2 \ge \log_{\sqrt{2}}\left(2 - \log \ (16\sin \quad + 1)\right) > -\infty$

$\Rightarrow h(x) = (-\infty, 2]$.

18. Domain (D) and Range (R) of $f(x) = \sin^{-1}(\cos^{-1}[\ ])$ is respectively; where [.] greatest integer function.

(a) $D \equiv x \in [1, 2), R \equiv \{0\}$

(b) $D \equiv x \in [0, 1], R \equiv \{-1, 0, 1\}$

(c) $D \equiv x \in [-1, 1], \quad \equiv \left\{0, \sin^{-1}\left(-\right), \sin^{-1}(\ )\right\}$

(d) $D \equiv x \in [-1, 1],\ R \equiv \left\{\frac{\pi}{2}, 0, \frac{\pi}{2}\right\}$

Solution: (a) $f(x) = \sin^{-1}(\cos^{-1}[x])$

$\Rightarrow f(x)$ is defined when $\cos^{-1}[x] \in [-1, 1]$.

But $\cos^{-1}[x] \in [0, \pi] \Rightarrow \cos^{-1}[x] \in [0, 1]$

$\Rightarrow [\cos 1, 1] \qquad \Rightarrow [x] = 1$

$\Rightarrow f(x) = \sin^{-1}(\cos^{-1} 1)$ or $\sin^{-1}0 = 0$

$\Rightarrow$ Range $= \{0\}$ and domain $\{x: [x] = 1\}$

$\Rightarrow x \in [1, 2) \therefore$ Domain $= [1, 2)$ and Range $= \{0\}$

19. The range of the function $f(x) = \sin^{-1}\left[x^2 + \frac{1}{2}\right] + \cos^{-1}\left[x^2 - \frac{1}{2}\right]$, where [.] is the greatest integer function is

(a) $\left\{\frac{\pi}{2}, \pi\right\}$ (b) $\left\{0, \frac{\pi}{2}\right\}$

(c) $\{\pi\}$ (d) $\left(0, \frac{\pi}{2}\right)$

Solution: (c) Given function is $f(x) = \sin^{-1}\left[x^2 + \frac{1}{2}\right] + \cos^{-1}\left[x^2 - \frac{1}{2}\right]$

$= \sin^{-1}\left[x^2 + \frac{1}{2}\right] + \cos^{-1}\left[x^2 + \frac{1}{2} - 1\right]$

$= \sin^{-1}\left[x^2 + \frac{1}{2}\right] + \cos^{-1}\left(\left[x^2 + \frac{1}{2}\right] - 1\right)$

Since $x^2 + \frac{1}{2} \ge \frac{1}{2}$, $\left[x^2 + \frac{1}{2}\right] = 0$ or 1,

$\sin^{-1}\left[x^2 + \frac{1}{2}\right]$ is defined only for these two values.

$\therefore$ When $\left[x^2 + \frac{1}{2}\right] = 0$

$\Rightarrow f(x) = \sin^{-1}0 + \cos^{-1}(-1) = \pi$

and when $\left[x^2 + \frac{1}{2}\right] = 1$

$\Rightarrow f(x) = \sin^{-1}1 + \cos^{-1}0 = \pi$. Therefore range of $f(x) = \{\pi\}$.

20. The range of $f(x) = \cos\left(\sin\left(\ln\left(\frac{x^2 + e}{x^2 + 1}\right)\right)\right) + \sin\left(\cos\left(\ln\left(\frac{x^2 + e}{x^2 + 1}\right)\right)\right)$ is

(a) $[-\sqrt{2}, \sqrt{2}]$

(b) $[-2, 2]$

(c) $[\cos(\sin 1) + \sin\cos 1, 1 + \sin 1]$

(d) $[-1, 1]$

Solution: (c) The function $g(x) = \frac{x^2 + e}{x^2 + 1}$ is continuous with range $(1, e]$.

Since $\ln x$ increases for all $x > 0$.

So, $0 < \ln g(x) \le 1$; again $\sin\theta$ is increasing for $\theta \in \left(0, \frac{\pi}{2}\right]$

So, $\sin(\ln(g(x)))$ increases for $\ln(g(x)) \in (0, 1]$

Thus, $\cos(\sin 1) \le \cos(\sin \ln g(x)) < 1$ ($\cos x$ decreases on $[0, 1]$.

Thus, $\cos\left(\sin\left(\ln\left(\frac{x^2 + e}{x^2 + 1}\right)\right)\right)$ is a decreasing function with domain $\mathbb{R}$.

Similarly, $0 < \ln(g(x)) \le 1$

$\Rightarrow$ $\cos 1 \le \cos(\ln(g(x))) < 1$

$\Rightarrow$ $\sin(\cos 1) \le \sin(\cos(\ln(g(x)))) < \sin 1$.

Now as x increases, $\ln x$ increases

$\Rightarrow$ $\cos(\ln x)$ decreases

$\Rightarrow$ $\sin(\cos(\ln x))$ also decreases

Thus, $\sin\left(\cos\left(\ln\left(\frac{x^2+e}{x^2+1}\right)\right)\right)$ is also a decreasing function on $\mathbb{R}$.

Thus, both constituent functions of $f(x)$ are decreasing.

$\Rightarrow$ range of $f(x)$

$\Rightarrow$ $[\cos(\sin 1) + \sin(\cos 1), 1 + \sin 1]$

21. If $f\left(x+\frac{y}{8}, x-\frac{y}{8}\right) = xy$, then $f(m, n) + f(n, m) = 0$

(a) only when $m = n$ (b) only when $m \ne n$

(c) only when $m = -n$ (d) for all m and n

Solution: (d) Given function is $f\left(x+\frac{y}{8}, x-\frac{y}{8}\right) = xy$,

Putting $\frac{y}{8} = k$; we have $f(x + k, x - k) = 8xk$ … (1)

Putting $x + k = A$ and $x - k = B$

or $x = \frac{A+B}{2}$ and $k = \frac{A-B}{2}$ in (1), we have

$f(A,B) = 8\left(\frac{A+B}{2}\right)\left(\frac{A-B}{2}\right) = 2(A^2 - B^2)$

$\Rightarrow$ $f(m, n) + f(n, m) = 2(m^2 - n^2) + 2(n^2 - m^2) = 0 \ \forall\ m, n$

22. If $f(x + 1) + f(x - 1) = 2f(x)$ and $f(0) = 0$, therefore, $n \in \mathbb{N}$, $f(n)$ equals

(a) $nf(1)$ (b) $(f(1))^n$

(c) 0 (d) None of these

Solution: (a) Putting $x = 1$, $f(2) + f(0) = 2f(1)$

$\Rightarrow$ $f(2) = 2f(1)$ and $f(3) = 3f(1)$

Let us assume $f(n) = nf(1)$; for $n = 1, 2, \ldots, n$.

Now, $f(n + 1) + f(n - 1) = 2f(n)$

$\Rightarrow$ $f(n + 1) + (n - 1)f(1) = 2nf(1)$

$\Rightarrow$ $f(n + 1) = (n + 1)f(1)$

$\therefore$ By principle of mathematical induction, $f(n) = nf(1)$ for all $n \in \mathbb{N}$.

23. Let $f(1) = 1$ and $f(n) = 2.\sum f(r)$. Then $\sum_{n=1}^{m} f(n)$ is equal to

(a) $3^m - 1$ (b) 3^{m-1}

(c) $3^m + 1$ (d) None of these

Solution: (b) Given $f(1) = 1$ and $f(n) = 2.\sum_{r=1}^{n-1} f(r)$

Now $f(n) = 2\{f(1) + f(2) + \ldots + f(n-1)\}$

and $f(n + 1) = 2\{f(1) + f(2) + \ldots + f(n)\}$

$\Rightarrow$ $f(n + 1) = 3f(n)$ for $n \ge 2$

$\Rightarrow$ $f(n+1) - f(n) = 2f(n)$

Also $f(2) = 2f(1) = 2$

$\therefore$ $f(3) = 3f(2) = (3.2)$ and

$f(4) = 3.f(3) = 3.(3.2) = 2.(3)^2$ and so on.

$\therefore$ $\sum_{n=1}^{m} f(n) = f(1) + f(2) + \ldots + f(m)$

$= 1 + 2 + 2.3 + 2.3^2 + \ldots + 2.3^{m-2}$

$= 1 + 2(1 + 3 + 3^2 + \ldots + 3^{m-2}) = 3^{m-1}$

24. If $f(x) = 9.4^x - 24.2^x$ and $g(x) = \cos 2x - 4\sin^2 x - 5\cos^2 x$, then the value of x and y for which $f(x) + 12 = g(y)$ are

(a) $x = 3$, $y = 2n\pi$; $n \in \mathbb{Z}$

(b) $x = \log_2 \frac{4}{3}$; $y = n\pi$; $n \in \mathbb{Z}$

(c) $x = \log_2 \frac{4}{3}$; $y = \frac{2n\pi}{3}$; $n \in \mathbb{Z}$

(d) $x = 2 - \log_4 9$, $y = n\pi$; $n \in \mathbb{Z}$

Solution: (b) $9.4^x - 24.2^x + 12 = \cos 2y - 4\sin^2 y - 5\cos^2 y$. Putting $2^x = t$ we have

$\Rightarrow$ $9t^2 - 24t + 12 = 1 - 2\sin^2 y - 4\sin^2 y - 5\cos^2 y$

Now $9t^2 - 24t + 12$

$= 9\left\{\left(t-\frac{4}{3}\right)^2 - \frac{4}{9}\right\} = 9\left(t-\frac{4}{3}\right)^2 - 4 \ge -4$ and

$1 - 6\sin^2 y - 5\cos^2 y = -4 - \sin^2 y \le -4$.

$\therefore$ Equality holds, i.e., both LHS and RHS are -4 and for this $t - \frac{4}{3} = 0$ and $\sin^2 y = 0$

$\Rightarrow$ $y = n\pi$, $n \in \mathbb{Z}$ and $t = \frac{4}{3}$ $\Rightarrow$ $x = \log_2 \frac{4}{3}$.

25. Let $f(x) = a^x (a > 0)$ be written as $f(x) = g(x) + h(x)$, where $g(x)$ is an even function and $h(x)$ is an odd function. Then the value of $g(x + y) + g(x - y)$ is

(a) $2g(x)\,g(y)$ (b) $2g(x + y)\,g(x - y)$

(c) $2g(x)$ (d) None of these

Solution: (a) Clearly, $g(x) = \frac{1}{2}(a^x + a^{-x})$ and

$h(x) = \frac{1}{2}(a^x - a^{-x})$

Now $g(x+y)+g(x-y)=\frac{1}{2}(a^{x+y} + a^{-(x+y)}) + \frac{1}{2}(a^{x-y} + a^{-x+y})$

$= \frac{1}{2}(a^x a^y + a^x a^{-y} + a^{-x} a^y + a^{-x} a^{-y})$

$= \frac{1}{2}(a^x (a^y + a^{-y}) + a^{-x} (a^y + a^{-y}))$

$= 2\left(\frac{1}{2}\left(a^x+a^{-x}\right)\right)\left(\frac{1}{2}\left(a^y+a^{-y}\right)\right) = 2g(x)\, g(y)$

26. Let $f(x)=([a]^2-5[a]+4)x^3-(6\{a\}^2-5\{a\}+1)x$ – (tan x) sgn x, be an even function for all $x\in\mathbb{R}-\left\{(2n+1)\frac{\pi}{2};n\in\mathbb{Z}\right\}$, then sum of all possible values of a is (where [] and { } denotes greatest integer function and fractional part functions, respectively)

(a) $\frac{17}{6}$ (b) $\frac{53}{6}$

(c) $\frac{35}{3}$ (d) None of these

Solution: (c) $\because f(x)$ is an even function

$\Rightarrow f(x)=f(-x)\ \forall\ x\in\mathbb{R}-\left\{(2n+1)\frac{\pi}{2};n\in\mathbb{Z}\right\}$

$\Rightarrow ([a]^2-5[a]+4)x^3-(6\{a\}^2-5\{a\}+1)x-(\tan x)\, sgn(x)$
$= -([a]^2-5[a]+4)x^3+(6\{a\}^2-5\{a\}+1)x+\tan x\, sgn(-x)$

$\Rightarrow 2([a]^2-5[a]+4)x^3-2(6\{a\}^2-5\{a\}+1)x-\tan x\,(sgn(x)+sgn(-x))=0\ \forall\ x\in\mathbb{R}-\left\{(2n+1)\frac{\pi}{2};n\in\mathbb{Z}\right\}$

But $sgn\, x + sgn\,(-x) = 0\ \forall\ x\in\mathbb{R}$

$\Rightarrow ([a]^2-5[a]+4)x^3-(6\{a\}^2-5\{a\}+1)x=0\ \forall\ x\in\mathbb{R}-\left\{(2n+1)\frac{\pi}{2};n\in\mathbb{Z}\right\}$

$\Rightarrow [a]^2-5[a]+4=0$ and $6\{a\}^2-5\{a\}+1=0$

$\Rightarrow ([a]-1)([a]-4)=0$ and $(3\{a\}-1)(2\{a\}-1)=0$

$\Rightarrow [a]=1$ or $[a]=4$ and $\{a\}=\frac{1}{3}$ or $\{a\}=\frac{1}{2}$

$\therefore\ [a]=1,\{a\}=\frac{1}{3} \Rightarrow a=\frac{4}{3}$

$[a]=1,\{a\}=\frac{1}{2} \Rightarrow a=\frac{3}{2}$

$[a]=4,\{a\}=\frac{1}{3} \Rightarrow a=\frac{13}{3}$

$[a]=4,\{a\}=\frac{1}{2} \Rightarrow a=\frac{9}{2}$

$\therefore$ Sum of all possible values of

$a=\frac{4}{3}+\frac{3}{2}+\frac{13}{3}+\frac{9}{2}=\frac{8+9+26+27}{6}=\frac{35}{3}$

27. If $f(x)$ and $g(x)$ be periodic and non-periodic functions respectively, then $f(g(x))$ is

(a) always periodic
(b) never periodic
(c) periodic, when $g(x)$ is a linear function of x
(d) can't say

Solution: (c) Since we know if $f(x)$ is periodic, then $f(ax+b)$ is also periodic function with period $\frac{T}{|a|}$

Hence, $g(x)$ should be a linear function of x.

28. Consider the following statements:

S_1**:** A function can never be odd and even simultaneously
S_2**:** Domain of the function $f(x)=\log x^2$ is $\mathbb{R}$
S_3**:** Period of $\sin^{99}x$ is 2π
S_4**:** $\sin(e^x)$ is a periodic function.

Which of the following sequences of answers is correct about the above given statements?
(T = true, F = false).

(a) FFTF (b) TFTTT
(c) FTTF (d) TFTF

Solution: (a) S_1 is incorrect as zero function, i.e., $f(x)=0;\ \mathbb{R}\to\mathbb{R}$ is both odd and even simultaneously. S_2 is incorrect as domain of $f(x)$ is $\mathbb{R}\sim\{0\}$; $x=0$ is not in domain. S_3 is correct as $f(x)=\sin^{99}x$

$\Rightarrow f(x+2\pi)=\sin^{99}(x+2\pi)=\sin^{99}x=f(x)$

$\therefore$ Period of $\sin^{99}x$ is 2π.
S_4 is false.
$\because$ Composition of a periodic function, over a non-periodic and non-linear function is non-periodic.

29. Let f be a real valued function such that for any real x, $f(\lambda+x)=f(\lambda-x)$ and $f(2\lambda+x)=-f(2\lambda-x)$ for some $\lambda>0$. Then

(a) f is even and non-periodic
(b) f is odd and periodic
(c) f is odd and non-periodic
(d) f is even and periodic

Solution: (b) Given $f(\lambda+x)=f(\lambda-x)$ ... (1)
$f(2\lambda+x)=-f(2\lambda-x)$ ... (2)

for $\lambda>0$

Replacing x by $\lambda - x$ in $\lambda - x$ in (1), we get

$f(2\lambda - x) = f(x)$ .... (3)

$\therefore$ From (2) and (3), $f(x) = -f(2\lambda + x)$

$\Rightarrow f(x) = -[-f(2\lambda + 2\lambda + x)]$

$\Rightarrow f(x) = f(x + 4\lambda)$ ... (4)

$\Rightarrow f(x)$ is periodic with period 4λ.

Further from (3), replacing x by $-x$, we get

$f(2\lambda + x) = f(-x)$ .... (5)

From (2), (3) and (5), we have

$f(-x) = f(2\lambda + x) = -f(2\lambda - x) = -f(x)$

i.e., $f(-x) = -f(x)$

$\Rightarrow f(x)$ is odd function

Thus, f is odd and periodic function.

30. Consider the following statements:

***S*$_1$:** If $f(x)$ is an even function and $g(x)$ is an odd function, then *fog* and *gof* are even functions

***S*$_2$:** (odd function) × (even function) = odd function

***S*$_3$:** Any function can be expressed as a sum of an even function and an odd function

***S*$_4$:** If f is decreasing odd function, then f^{-1} is also odd and decreasing

Which of the following sequences of answers is correct? (T = true, F = False)

(a) TTFT (b) TTTT

(c) FTTF (d) TTFF

Solution: (a) $\because f(g(-x)) = f(-g(x)) = f(g(x))$ and $g(f(-x)) = g(f(x))$

$\Rightarrow$ *fog* and *gof* both are even functions

$\therefore$ S_1 is true.

S_2 is obviously true since $(fg)(-x) = f(-x)\, g(-x) = -f(x)\, g(x) = -(fg)(x)$

S_3 holds if domain is symmetric about origin, then we can express

$$f(x) = \left(\frac{f(x)+f(-x)}{2}\right) + \left(\frac{f(x)-f(-x)}{2}\right)$$

$$= \underbrace{h(x)}_{\text{even}} + \underbrace{g(x)}_{\text{odd}}$$

Let f be an odd and decreasing function with its domain D_f symmetric about origin, i.e., $x, -x \in D_f$ and $f(-x) = -f(x)$ $\Rightarrow f(x), -f(x) \in R_f$ i.e., range is also symmetric about origin.

Now $f^{-1}(f(x)) = x$ for $x \in$ principal domain of $f(x)$

$\Rightarrow f^{-1}(f(-x)) = -x \in$ Principal domain of $f(x)$

$\Rightarrow f^{-1}(f(-x)) = -f^{-1}(f(x))$, but f is odd

$\Rightarrow f(-x) = -f(x)$ $\quad\therefore f^{-1}(-f(x)) = -f^{-1}(f(x))$

$\Rightarrow f^{-1}(-y) = -f^{-1}(y)$; where $y = f(x), -y = -f(x) \in R_f$

$\Rightarrow f^{-1}$ is an odd function. Also f^{-1} is decreasing.

31. Solution set of $f^{-1}(x) = x$ is a proper subset of the solution set of $f(x) = f^{-1}(x)$; then

(a) Solution set of $f^{-1}(x) = x$, is identical to solution set of $f(x) = x$

(b) $f(x) = f^{-1}(x)$ is identical to solution set of $f(x) = x$

(c) $f(x)$ and $f^{-1}(x)$ would intersect at a point perpendicular to $y = x$, but not lying on $y = x$.

(d) all of the above

Solution: (c) Curves of $y = f(x)$ and $y = f^{-1}(x)$ either intersect on $y = x$ line or intersect on a line perpendicular to $y = x$.

Since solution set of $f^{-1}(x) = x$ is a proper subset of solution set of $f(x) = f^{-1}(x)$, thus, $f(x)$ and $f^{-1}(x)$ must intersect at a line perpendicular to line $y = x$ and point of intersection not lying on line $y = x$.

32. Consider the following statements:

***S*$_1$:** If $f(x)$ is increasing, then $f^{-1}(x)$ is also increasing.

***S*$_2$:** If $f(x)$ is a constant function, then $f^{-1}(x)$ is also a constant function.

***S*$_3$:** If graph of $f(x)$ and $f^{-1}(x)$ are intersecting, then they always intersect on $y = x$ line.

***S*$_4$:** The inverse of $f(x) = \dfrac{x}{1+|x|}$ is $\dfrac{x}{1-|x|}$.

Which of the following sequences of answers is correct about the statements? (T = true, F = false).

(a) TTTF (b) TFFT

(c) FFFT (d) TFTT

Solution: (b) ***S*$_1$:** Obviously true as

$$(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))} > 0 \text{ as } f'(x) > 0 \ \forall\ x \in D_f$$

***S*$_2$:** Since constant function is non-invertible, hence, its inverse does not exist.

$\therefore$ Statement S_2 is invalid, and hence, stands false.

***S*$_3$:** $f(x)$ and $f^{-1}(x)$ can intersect not only on $y = x$ but also they can intersect on a line perpendicular to $y = x$ and not lying on $y = x$.

Hence, the given statement is false.

***S*$_4$:** $$f(x) = \begin{cases} \dfrac{x}{1+x} : & x \ge 0 \\ \dfrac{x}{1-x} : & x < 0 \end{cases}$$

$$\Rightarrow f^{-1}(x) = \begin{cases} \dfrac{x}{1-x} : & x \ge 0 \\ \dfrac{x}{1+x} : & x < 0 \end{cases} = \frac{x}{1-|x|}$$

Hence, S_4 is true.

33. Let $f:\left[\frac{1}{2},1\right]\to[-1,1]$ is given by $f(x) = 4x^3 - 3x$, then $f^{-1}(x)$ is given by

(a) $\cos\left(\frac{1}{3}\cos^{-1}x\right)$ (b) $3\cos(\sin^{-1}x)$

(c) $3\sin^{-1}(\cos x)$ (d) None of these

Solution: (a) $f(x) = 12x^2 - 3 = 3(4x^2 - 1)$
$= 3(2x + 1)(2x - 1) \ge 0$,
for $x\in\left(-\infty,\frac{-1}{2}\right]\cup\left[\frac{1}{2},\infty\right)$

$\Rightarrow$ $f(x)$ is strictly increasing on $\left[\frac{1}{2},1\right]$, i.e., injective

Also range of function on $\left[\frac{1}{2},1\right]$ is $[-1, 1]$ = co-domain.

Thus, $f(x)$ is invertible

Let $g(x)$ be the inverse of f, so $f(g(x)) = x$ $\forall$ $x \in [-1, 1]$

$\Rightarrow$ $4(g(x))^3 - 3g(x) = x$

Let $g(x) = \cos\theta$ and $g(x)\in\left[\frac{1}{2},1\right]\Rightarrow\theta\in\left[0,\frac{\pi}{3}\right]$

$\Rightarrow$ $\cos3\theta = x$ and $3\theta\in[0,\pi]\Rightarrow\cos^{-1}(\cos3\theta)=3\theta$

$\Rightarrow$ $\theta=\frac{1}{3}\cos^{-1}x$, so $g(x)=\cos\left(\frac{1}{3}\cos^{-1}x\right)$

34. If f: $[1, \infty) \to [2, \infty)$ is given by, $f(x)=x+\frac{1}{x}$, then $f^{-1}(x)$ equals

(a) $\frac{x+\sqrt{x^2-4}}{2}$ (b) $\frac{x}{1+x^2}$

(c) $\frac{x-\sqrt{x^2-4}}{2}$ (d) $1-\sqrt{x^2-4}$

Solution: (a) Let $y=x+\frac{1}{x}$ $\Rightarrow$ $y=\frac{x^2+1}{x}$

$\Rightarrow xy = x^2 + 1$ $\Rightarrow$ $x^2 - xy + 1 = 0$

$\Rightarrow$ $x=\frac{y\pm\sqrt{y^2-4}}{2}$ $\Rightarrow$ $f^{-1}(x)=\frac{x\pm\sqrt{x^2-4}}{2}$

Since, the range of the inverse function is $[1, \infty)$, then we take $f^{-1}(x)=\frac{x+\sqrt{x^2-4}}{2}$

$\because$ If we consider $f^{-1}(x)=\frac{x-\sqrt{x^2-4}}{2}$. Then $f^{-1}(x) > 1$

$\Rightarrow$ $\frac{x-\sqrt{x^2-4}}{2}>1$ $\Rightarrow$ $x-2>\sqrt{x^2-4}$ which is possible only if $(x - 2)^2 > x^2 - 4$.

i.e., $x^2 + 4 - 4x > x^2 - 4$.

i.e., if $8 > 4x$ or $x < 2$, whereas in domain of $f^{-1}(x)$; $x \in [2, \infty)$.

Thus, $f^{-1}(x)=\frac{x+\sqrt{x^2-4}}{2}$

35. Suppose $f(x) = (x + 2)^2$ for $x \ge -2$. If $h(x)$ is the function whose graph is the reflection of the graph of $f(x)$ with respect to the line $y = x$ and $g(x)$ is the function whose graph is obtained by shifting 4 units to the left the graph of $h(x)$, then $g(x)$ equals.

(a) $\sqrt{x-4}-2, x\ge4$ (b) $\frac{1}{(x+4)^2}, x\ne-4$

(c) $\sqrt{x-4}, x\ge4$ (d) $\sqrt{x+4}-2, x\ge-4$

Solution: (d) Given $h(x) = f^{-1}(x)$

But $y = f(x) = (x + 2)^2$ $\Rightarrow$ $x + 2 = \pm\sqrt{y}$; $x + 2 \ge 0$

$\Rightarrow$ $x = \sqrt{y} - 2$ $\Rightarrow$ $f^{-1}(x) = h(x) = \sqrt{x} - 2$

Now $g(x) = h(x + 4)$ (given)

$=\sqrt{x+4}-2; x\ge-4$.

36. If $f(x)$ is defined on $[0, 1]$ by $f(x)=\begin{cases} x; & \text{if } x \text{ is rational} \\ 1-x; & \text{if } x \text{ is irrational}\end{cases}$, then for all $x \in [0, 1]$, $f(f(x))$ is

(a) real (b) $1 - x$

(c) x (d) None of these

Solution: (a, c) If x is rational, then $f(x) = x$

$\Rightarrow$ $f(f(x)) = f(x) = x$

If x is irrational, then $f(x) = 1 - x$ = irrational

($\because$ rational-irrational = irrational)

$\Rightarrow$ $f(f(x)) = f(1 - x) = 1 - (1 - x) = x$

Thus, $f(f(x)) = x$ for all x in $[0, 1]$

$\Rightarrow$ $f(f(x)) = x$ (real)

37. The number of functions f from the set $A = \{2, 3, 5\}$ into the set $B = \{1, 3, 5, 7, 9, 11, 13\}$ such that $f(i) \le f(j)$ for $i < j$ and $i, j \in A$.

(a) ${}^9C_3$ (b) ${}^9C_3 + 2({}^9C_2)$

(c) ${}^8C_3 + {}^8C_2$ (d) None of these

Solution: (a,c) A function $f: A \to B$ such that $f(2) \le f(3) \le f(5)$ can be generated in the following four cases.

Case i: $f(2) < f(3) < f(5)$

There are ${}^7C_3$ functions in this case.

Case ii: $f(2) = f(3) < f(5)$

There are $^7C_2$ functions in this case.

Case iii: $f(2) < f(3) = f(5)$

There are $^7C_2$ functions in this case.

Case iv: $f(2) = f(3) = f(5)$

There are $^7C_1$ functions in this case.

Thus, the number of desired functions is $^7C_3 + {}^7C_2 + {}^7C_2 + {}^7C_1 = {}^8C_3 + {}^8C_2 = {}^9C_3$.

Thus, (a) and (c) will be the correct options.

38. Let $f(x) = \cos(\pi/x)$ and $D^+ = \{x : f(x) > 0\}$. Then D^+ contains

(a) $\left(\frac{2}{5}, \frac{2}{3}\right)$ (b) $\left(\frac{2}{9}, \frac{2}{7}\right)$

(c) $\left(\frac{-2}{3}, \frac{-2}{5}\right)$ (d) $\left(\frac{1}{2}, \frac{2}{3}\right)$

Solution: (a, b, c, d)

Given $D^+ = \{x : f(x) > 0\} = \left\{x : \cos\left(\frac{\pi}{x}\right) > 0\right\}$

$= \left\{x : \frac{\pi}{x} \in \mathbb{Z} \bigcup_{n\in \mathbb{Z}} \left(2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right)\right\}$

$= \left\{x : x \in \bigcup_{n\in\mathbb{Z}-\{0\}} \left(\frac{2}{4n+1}, \frac{2}{4n-1}\right)\right\} \cup (-\infty, -2) \cup (2, \infty)$

$\Rightarrow$ D^+ contains $\left(\frac{2}{5}, \frac{2}{3}\right)$ for $n = 1$

D^+ contains $\left(\frac{2}{9}, \frac{2}{7}\right)$ for $n = 2$

D^+ contains $\left(-\frac{2}{3}, -\frac{2}{5}\right)$ for $n = -1$

Also D^+ contains $\left(\frac{2}{5}, \frac{2}{3}\right) \supset \left(\frac{1}{2}, \frac{2}{3}\right)$

D^+ also contains $\left(\frac{1}{2}, \frac{2}{3}\right)$

39. Let $f(x) = 2 + \sqrt{x}$ and $g(x) = \frac{2x}{x^2+1}$, then

(a) Domain $(f + g + 2) = (-1, \infty)$

(b) Domain $(f + g + 2) = [0, \infty)$

(c) Range $f \cap$ range $(g + 2) = [2, 3]$

(d) Range $f \cup$ range $(g + 2) = [1, \infty)$

Solution: (b, c, d) Domain of $f(x) = [0, \infty)$

Domain of $g(x) = (-\infty, \infty)$

$\Rightarrow$ Domain of $(g + 2)(x) = (-\infty, \infty)$

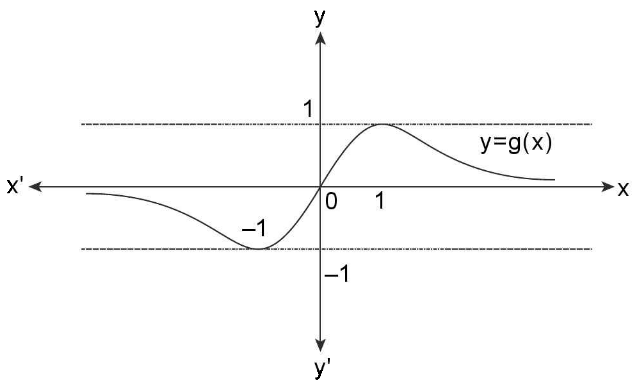
$\therefore$ Domain of $(f + g + 2)(x) = [0, \infty) \cap (-\infty, \infty) = [0, \infty)$

Further range of $f(x) = [2, \infty)$ as $f'(x) = \frac{1}{2\sqrt{x}} > 0 \forall x \in (0, \infty)$.

$\Rightarrow$ $f(x)$ is an increasing and continuous function on $[0, \infty)$.

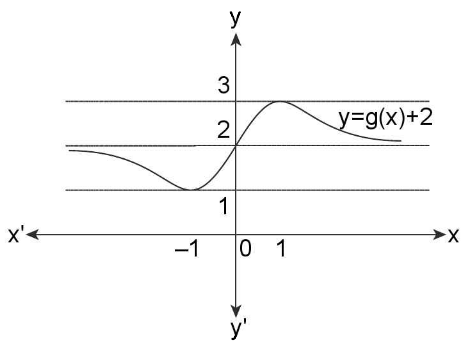
$\therefore$ $f(x)$ has its minimum value $f(0) = 2 + \sqrt{0} = 2$ and maximum tending to infinite.

Now $f(x) = \frac{2x}{1+x^2}$. Graph of $g(x)$ is as shown below



Clearly $g(x)$ has its range $[-1, 1]$

$\Rightarrow$ $(g + 2)(x)$ can be obtained by shifting the graph of $g(x)$ by 2 units above x-axis as shown below



$\therefore$ Range of $(g + 2)(x)$ is $[1, 3]$

$\therefore$ Range of $f(x) \cap$ Range of $(g + 2)(x) = [1, 3] \cap [2, \infty) = [2, 3]$

and range of $f(x) \cup$ range of $(g + 2)(x) = [1, 3] \cup [2, \infty) = [1, \infty)$

40. Which of the following functions have range which is a proper subset of $(-\infty, \infty)$?

(a) $\tan x$ (b) $\cot x$

(c) $\log \sin x$ (d) $\tan x + \cot x$

Solution: (c, d) (a), (b) are false since $\cot x$, $\tan x$ have range as $(-\infty, \infty)$

(c) is true since, $\sin x \in (0, 1]$

$\Rightarrow$ $\log \sin x \in (-\infty, 0]$ which is a proper subset of $(-\infty, \infty)$.

(d) is true since $\tan x + \cot x = \dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}$

$= \dfrac{1}{\sin x \cos x} = 2 \operatorname{cosec} 2x$; $\sin x, \cos x \neq 0$

$\Rightarrow$ Range of function is $(-\infty, -2] \cup [2, \infty)$ which is a proper subset of $(-\infty, \infty)$

41. Let $f(x) = \sin x + \sin(x\sqrt{3})$. Then, which of the following are false?

(a) Maximum value of $f(x)$ cannot be 2

(b) Minimum value of $f(x)$ cannot be -2

(c) $f(x)$ is periodic function with period $2\sqrt{3}\,\pi$

(d) $f(x) > 0\ \forall\ x \in \mathbb{R}$.

Solution: (c,d) If $f(x) = 2$

$\Rightarrow$ $\sin x = 1$, $\sin x\sqrt{3} = 1$

$\Rightarrow$ $x = (4n+1)\dfrac{\pi}{2}$ and $x\sqrt{3} = (4m+1)\dfrac{\pi}{2}$

$\Rightarrow$ $\sqrt{3} = \dfrac{4m+1}{4n+1}$ which is a contradiction as LHS is irrational whereas RHS is rational.

$\therefore$ $f(x)$ cannot attain value 2.

$\therefore$ 2 cannot be the maximum value of $f(x)$.

Similarly $f(x) = -2$.

$\Rightarrow$ $\sin x = \sin x\sqrt{3} = -1$.

Which is again impossible by same reason.

So, (b) is true.

Now period of $f(x) = \text{LCM}\left(2\pi, \dfrac{2\pi}{\sqrt{3}}\right)$ which does not exist as multiples of 2π are $\pm 2\pi, \pm 4\pi, \ldots$ whereas multiples of $\dfrac{2\pi}{\sqrt{3}}$ are $\pm\dfrac{2\pi}{\sqrt{3}}, \pm\dfrac{4\pi}{\sqrt{3}}, \pm\dfrac{6\pi}{\sqrt{3}}$.

Therefore, (c) is false.

Now $f(0) = 0$

$\therefore$ $f(x) > 0\ \forall\ x \in \mathbb{R}$ is false, i.e., (d) is false.

42. If $f(x) = \sin x + \tan x + sgn\,(x^2 - 6x + 10)$, then it is

(a) periodic with period 2π

(b) periodic with period π

(c) non-periodic

(d) periodic with period 4π

Solution: (a, d) Given $f(x) = \sin x + \tan x + sgn\,(x^2 - 6x + 10)$

$= \sin x + \tan x + sgn\,((x-3)^2 + 1)$

$= \sin x + \tan x + 1$

$(\because (x-3)^2 + 1 \geq 1 \Rightarrow sgn\,[(x-3)^2 + 1] = 1)$

$\Rightarrow$ Period of $f(x) = \text{LCM}\,(2\pi, \pi) = 2\pi$, as period of $\sin x$ is 2π and period of $\tan x$ is π, and the period of function is also 4π

43. Let $f(x)$ and $g(x)$ be two real valued functions given by $f(x) = -\ln x$ and $g(x) = e^{-x}$. Let $h(x) = f(x) - x$ and $m(x) = g(x) - x$. Further let the number of solutions of $h(x) = 0$ and $m(x) = 0$ be a and b, then

(a) $f(x)$ and $g(x)$ intersect at $y = x$

(b) $a = b$

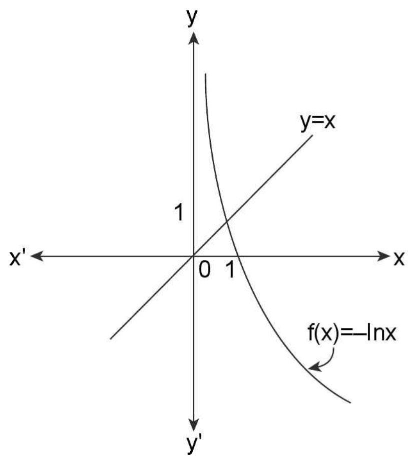
(c) $a = 1$ and $b = 1$

(d) $a \neq b$

Solution: (a, b, c) First we can see that $h(x) = f(x) - x = 0$.

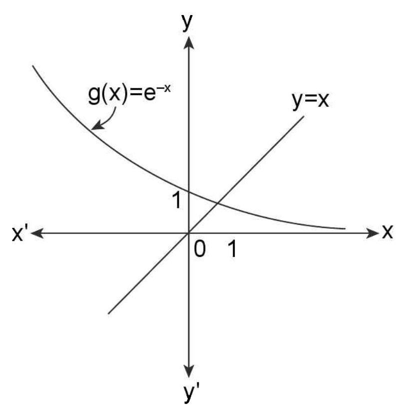
$\Rightarrow$ $f(x) = x$ equation will have only one root

$\because$ We have only one intersection point, and hence, only one solution is possible as is clear from the figure given below.



Similarly for $m(x) = 0$

$\Rightarrow$ $g(x) = x$, we have only one solution as shown in the figure that follows



$\therefore$ $a = 1$ and $b = 1$. Moreover one can see that $f(x)$ and $g(x)$ are inverse of each other and are continuous and cross the line $y = x$

Thus, $y = f(x)$ and $y = g(x) = f^{-1}(x)$ will intersect on line $y = x$.

44. The function $\dfrac{x^2+ax+b}{x^2+cx+d}$ is

(a) defined for all x if $c^2 < 4d$
(b) not invertible for any values of a, b, c, d
(c) invertible if $b = 0, d = 0, a \neq c$
(d) invertible if $a = 0, c = 0, bd \neq c$

Solution: (a, c) (a) is true since if $c^2 < 4d$, then $x^2 + cx + d > 0$ for all x and the function is defined.
(b) is not true since $b = 0, d = 0, a \neq c$

$$\Rightarrow \quad f(x) = \frac{x^2+ax}{x^2+cx} = \frac{x+a}{x+c}$$

which is invertible if defined from $\mathbb{R} - \{0, -c\}$ to $\mathbb{R} - \{1\}$

(c) is true (sec (b))

(d) is not true since $\dfrac{x^2+b}{x^2+d}$ is not one–one.

45. Let $f : \left[-\dfrac{\pi}{3}, \dfrac{2\pi}{3}\right] \to [0, 4]$, be a function defined as $f(x) = \sqrt{3}\sin x - \cos x + 2$. Then $f^{-1}(x)$ is given by

(a) $\sin^{-1}\left(\dfrac{x-2}{2}\right) - \dfrac{\pi}{6}$ (b) $\sin^{-1}\left(\dfrac{x-2}{2}\right) + \dfrac{\pi}{6}$

(c) $\dfrac{2\pi}{3} - \cos^{-1}\left(\dfrac{x-2}{2}\right)$ (d) None of these

Solution: (b, c) Given $f(x) = \sqrt{3}\sin x - \cos x + 2 = 2\sin\left(x - \dfrac{\pi}{6}\right) + 2$ ….. (1)

For $x \in \left[-\dfrac{\pi}{3}, \dfrac{2\pi}{3}\right]$; $\left(x - \dfrac{\pi}{6}\right) \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ and range of $f(x) = [0, 4]$ = co-domain.

$\Rightarrow f(x)$ in one–one and onto, f is invertible

$\Rightarrow f^{-1}$ exists

Now from (1)

$$y = 2\sin\left(x - \frac{\pi}{6}\right) + 2 \Rightarrow \frac{y-2}{2} = \sin\left(x - \frac{\pi}{6}\right)$$

$$\Rightarrow \quad x = \frac{\pi}{6} + \sin^{-1}\left(\frac{y-2}{2}\right)$$

$$\Rightarrow \quad f^{-1}(x) = \sin^{-1}\left(\frac{x-2}{2}\right) + \frac{\pi}{6}$$

$$\text{Also } f^{-1}(x) = \frac{\pi}{2} - \cos^{-1}\left(\frac{x-2}{2}\right) + \frac{\pi}{6} = \frac{2\pi}{3} - \cos^{-1}\left(\frac{x-2}{2}\right)$$

46. Let $f(x) = e^x$; $\mathbb{R}^+ \to \mathbb{R}$ and $g(x) = \sin^{-1} x$; $[-1, 1] \to \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, then

(a) Domain of $fog(x)$ is $[-1, 1]$

(b) Range of $fog(x)$ is $\left(1, e^{\frac{\pi}{2}}\right]$

(c) Domain of $fog(x)$ is $(0, 1]$

(d) Range of $fog(x)$ is $\left[e^{-\frac{\pi}{2}}, e^{\frac{\pi}{2}}\right]$

Solution: (b,c) $fog(x) = e^{\sin^{-1}x}$
Now domain of $fog(x)$ would contain those elements of domain of $\sin^{-1} x$, i.e., $[-1, 1]$
for which $\sin^{-1} x$ belongs to domain of $f(x)$, i.e., $\mathbb{R}^+$
$\Rightarrow$ Domain of $fog(x) = (0, 1]$ and Range of

$$fog(x) = \left(e^{\sin^{-1}0}, e^{\sin^{-1}1}\right] = \left(1, e^{\frac{\pi}{2}}\right]$$

47. If $f(x) = \dfrac{\sin \pi[x]}{\{x\}}$, where $\{x\}$ denotes fractional part function and [.] denotes greatest integer function, then $f(x)$ is:

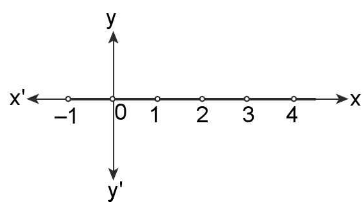
(a) periodic with fundamental period 1
(b) even
(c) range of $f(x)$ is a singleton set
(d) $f(x)$ is identical to $g(x) = \text{sgn}\left(\text{sgn}\dfrac{\{x\}}{\sqrt{\{x\}}}\right) - 1$.

Solution: (a, b, c, d) Given function is $f(x) = \dfrac{\sin \pi[x]}{\{x\}}$, $x \notin \mathbb{Z}$; as otherwise denominator becomes, zero, i.e., $f(x)$ is not defined for $x \in \mathbb{Z}$;

But $\dfrac{\sin \pi[x]}{\{x\}} = 0 \ \forall\, x \notin \mathbb{Z}$;

$$f(x) = \begin{cases} 0; & x \notin \mathbb{Z} \\ \text{not defined}; & x \in \mathbb{Z} \end{cases}$$

$\therefore$ The graph of $f(x)$ is as shown below



(a) It is clear from the above graph that the fundamental period of given function is one

(b) Also $f(-x) = 0 = f(x)\ \forall\ x \notin \mathbb{Z}$

$\Rightarrow f(x)$ is an even function

(c) Range of $f(x) = \{0\}$, which is a singleton set

(d) Now $g(x) = \operatorname{sgn}\left(\operatorname{sgn}\dfrac{\{x\}}{\sqrt{\{x\}}}\right) - 1\,;\ x \notin \mathbb{Z}$

$\because\ \dfrac{\{x\}}{\sqrt{\{x\}}} = \sqrt{\{x\}} > 0$ for $x \notin \mathbb{Z}$;

$\Rightarrow \left(\operatorname{sgn}\dfrac{\{x\}}{\sqrt{\{x\}}}\right) = 1$

$\Rightarrow g(x) = sgn\,(1) - 1 \ \Rightarrow\ g(x) = 1 - 1 = 0$

$\Rightarrow g(x) = 0,\ x \notin \mathbb{Z}$

$\Rightarrow f(x)$ and $g(x)$ have same domain $x \notin \mathbb{Z}$ and $f(x) = g(x) = 0$ for $x \notin \mathbb{Z}$;

$\Rightarrow f(x)$ and $g(x)$ are identical.

48. Let $g(x) = f(x - 1) + f(x + 1)$; where $f(x) = \begin{cases} 1-|x|; & |x| \le 1 \\ 0; & |x| > 1 \end{cases}$. The number of points of intersections with $y = \sin\alpha$; where α is constant

(a) 0; $(\pi < \alpha < 2\pi)$

(b) 4; $(0 < \alpha < \pi)$

(c) infinitely many; $(\alpha = 2\pi)$

(d) None of these

Solution: (a, b, c) $f(x) = \begin{cases} 1-|x|; & |x| \le 1 \\ 0; & |x| > 1 \end{cases}$

$$= \begin{cases} 1-|x|; & -1 \le x \le 1 \\ 0; & x < -1 \text{ or } x > 1 \end{cases} = \begin{cases} 1+x; & -1 \le x < 0 \\ 1-x; & 0 \le x \le 1 \\ 0; & x < -1 \text{ or } x > 1 \end{cases}$$

$$\therefore\ f(x-1) = \begin{cases} 1+(x-1); & -1 \le x-1 < 0 \\ 1-(x-1); & 0 \le x-1 \le 1 \\ 0; & x-1 < -1 \text{ or } x-1 > 1 \end{cases}$$

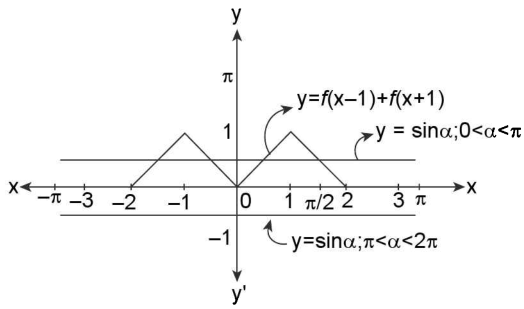
$$\Rightarrow\ f(x-1) = \begin{cases} x; & 0 \le x < 1 \\ 2-x; & 1 \le x \le 2 \\ 0; & x < 0 \text{ or } x > 2 \end{cases} \quad \ldots (1)$$

$$\Rightarrow\ f(x+1) = \begin{cases} 1+(x+1); & -1 \le x+1 < 0 \\ 1-(x+1); & 0 \le x+1 \le 1 \\ 0; & x+1 < -1 \text{ or } x+1 > 1 \end{cases}$$

$$\Rightarrow\ f(x+1) = \begin{cases} 2+x; & -2 \le x < -1 \\ -x; & -1 \le x \le 0 \\ 0; & x < -2 \text{ or } x > 0 \end{cases} \quad \ldots (2)$$

$$\therefore\ f(x-1) + f(x+1) = \begin{cases} 0; & \text{for } x < -2 \text{ or } x > 2 \\ 2+x; & \text{for } -2 \le x < -1 \\ -x; & \text{for } -1 \le x \le 0 \\ x; & \text{for } 0 < x < 1 \\ 2-x; & \text{for } 1 \le x \le 2 \end{cases}$$

The graph of $g(x) = f(x - 1) + f(x + 1)$ is as shown below



Clearly $g(x)$ and $\sin\alpha$ do not intersect for $\pi < \alpha < 2\pi$ as $\sin\alpha < 0$ for $\pi < \alpha < 2\pi$, i.e., $y = \sin\alpha$ lies below x-axis. So, (a) is correct Also $g(x)$ and $\sin x$ intersect at four points for $0 < \alpha < \pi$, as $\sin\alpha \in (0, 1)$, and hence, $y = \sin\alpha$ is a horizontal line between, $y = 0$ and $y = 1$, so (b) is also correct.

Clearly $g(x)$ and $\sin x$ intersect at infinitely many points for $\alpha = 2\pi$ as for $\alpha = 2\pi$, $\sin\alpha = 0$

$\therefore\ y = \sin\alpha$ is equivalent to x-axis which intersect the function $g(x)$ at infinitely many points.

SECTION–II

SUBJECTIVE-TYPE SOLVED EXAMPLES

1. Determine the exactly two distinct linear functions which map $[-1, 1]$ onto $[0, 2]$.

Solution: Let the required function be $f(x) = ax + b$.

If $a > 0$, then $f'(x) = a > 0$

$\Rightarrow f(x)$ is monotonically increasing in $[-1, 1]$

$\Rightarrow f(-1) = 0$ and $f(1) = 2$

That is $-a + b = 0$ and $a + b = 2$ so that $b = 1$ and $a = 1$.

$\Rightarrow f(x) = x + 1$

If $a < 0$, then $f'(x) = a < 0$

$\Rightarrow$ $f(x)$ is monotonically decreasing on $[-1, 1]$

$\Rightarrow$ $f(-1) = 2$ and $f(1) = 0$

That is $-a + b = 2$ and $a + b = 0$ so that $b = 1$ and $a = -1$.

$\Rightarrow$ $f(x) = -x + 1$

Hence, $f(x) = x + 1$ or $f(x) = -x + 1$.

2. Let f: $\{x, y, z\} \to \{a, b, c\}$ be a one–one function. It is known that only one of the following statements is true,

(a) $f(x) \neq b$ (b) $f(y) = b$

(c) $f(z) \neq a$

Determine $f^{-1}(b)$.

Solution: As only one of the given statements is true we have the following possibilities:

	$f(x) \neq b$	$f(y) = b$	$f(z) \neq a$
Case i:	T	F	F
Case ii:	F	T	F
Case iii:	F	F	T

(where T, F stand for True, False respectively)

Case i: In this case we would have $f(x) = a$ (or c); $f(y) = c$ (or a); $f(z) = a$

This possibility is clearly ruled out as f is one–one.

Case ii: In this case we would have $f(x) = b$; $f(y) = b$; $f(z) = a$

Once again this possibility is ruled out as image of x and y are identical.

Case iii: In this case we would have $f(x) = b$; $f(y) = a$ or c; $f(z) = c$ or b

Thus, $f(x) = b$; $f(y) = a$; $f(z) = c$

Hence, $f^{-1}(b) = x$

3. Solve $(x)^2 = [x]^2 + 2x$; where $[x]$ represents greatest integer less than or equal to x and (x) represents integer just greater than or equal to x.

Solution: ***Method 1:***

Case i: Let $x = n \in \mathbb{Z}$

Given equation becomes: $n^2 = n^2 + 2n \Rightarrow n = 0$

Case ii: Let $x \notin \mathbb{Z}$; i.e., $n < x < n + 1$

Given equation becomes: $(n + 1)^2 = n^2 + 2x$

$$\Rightarrow x = n + \frac{1}{2}, n \in \mathbb{Z}$$

$$\therefore x = 0 \text{ or } x = n + \frac{1}{2}; n \in \mathbb{Z}$$

Method 2: ***Case i:*** For $x = n \notin \mathbb{Z}$, the equation holds for $n + \frac{1}{2}$; $x = 0$ as proved in case (I)

So, let $x \notin \mathbb{Z}$

Let $x = [x] + \{x\}$; where $\{x\}$ represents fractional part of x.

But $(x) = [x] + 1 \quad \Rightarrow [x] = (x) - 1$

$\Rightarrow x = (x) - 1 + \{x\} \quad \Rightarrow (x) = x + 1 - \{x\}$

$\therefore$ Using it in given equation we have, $((x + 1) - \{x\})^2 = (x - \{x\})^2 + 2x$ (using the given equation)

$\Rightarrow (x - \{x\})^2 + 1 + 2(x - \{x\}) = (x - \{x\})^2 + 2x$

$$\Rightarrow 1 - 2\{x\} = 0 \quad \Rightarrow \{x\} = \frac{1}{2}$$

$$\Rightarrow x = n + \frac{1}{2}; n \in \mathbb{Z}$$

Thus, $x = 0$, $x = n + \frac{1}{2}$; $n \in \mathbb{Z}$ are required solutions.

4. Let $f : \mathbb{R} \sim \{2\} \to \mathbb{R}$ be a function satisfying the following functional equation:

If $2f(x) + 3f\left(\frac{2x+29}{x-2}\right) = 100x + 80$; $\forall\, x \in \mathbb{R} \sim \{2\}$,

Determine $f(x)$.

Solution: Here, $f(x) = -\frac{3}{2}f\left(\frac{2x+29}{x-2}\right) + 50x + 40$.(1)

Replacing x by $\frac{2x+29}{x-2}$ in the given equation (1), we get $f\left(\frac{2x+29}{x-2}\right)$

$$= -\frac{3}{2}f\left(\frac{2\left(\frac{2x+29}{x-2}\right)+29}{\left(\frac{2x+29}{x-2}\right)-2}\right) + 50\left(\frac{2x+29}{x-2}\right) + 40$$

$$= -\frac{3}{2}f(x) + 50\left(\frac{2x+29}{x-2}\right) + 40 \quad \dots. (2)$$

Substituting the values of $f\left(\frac{2x+29}{x-2}\right)$ from (2) in (1), we get $f(x) = 136 - 40x + \frac{1980}{x-2}$

5. Let $[x]$ denotes greatest integer less than or equal to x. If all the values of x such that the product $\left[x - \frac{1}{3}\right]\left[x + \frac{1}{3}\right]$ is prime, belongs to the set $[\alpha, \beta) \cup [\gamma, \delta)$; (where $\alpha, \beta, \gamma, \delta$ being in lowest form), find the value of (i) $\alpha + \beta + \gamma + \delta$ (ii) $9(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)$

Solution: $\because$ $a \times b$ = a prime number

$$\text{implies}\begin{cases} a = 1, b = \text{prime number} \\ b = 1, a = \text{prime number} \\ a = -1, b = -(\text{prime number}) \\ b = -1, a = -(\text{prime number}) \end{cases}$$

Given $\left[x-\frac{1}{3}\right].\left[x+\frac{1}{3}\right]$ is prime, thus, there arise following four cases.

Case i: $\left[x-\frac{1}{3}\right]=1$; $\left[x+\frac{1}{3}\right]$ = prime number

$\Rightarrow\ x-\frac{1}{3}\in[1,2)$; $x\in\left[\frac{4}{3},\frac{7}{3}\right)$; $x+\frac{1}{3}\in\left[\frac{5}{3},\frac{8}{3}\right)$

Here $\left[x+\frac{1}{3}\right]$ must be a prime number, but only prime number in $\left[\frac{5}{3},\frac{8}{3}\right)$ is 2

$\Rightarrow\ \left[x+\frac{1}{3}\right]=2 \Rightarrow \left(x+\frac{1}{3}\right)\in\left[2,\frac{8}{3}\right) \Rightarrow x\in\left[\frac{5}{3},\frac{7}{3}\right)$

Case ii: $\left[x-\frac{1}{3}\right]$ = prime number and $\left[x+\frac{1}{3}\right]=1$

$\Rightarrow\ x+\frac{1}{3}\in[1,2) \Rightarrow x\in\left[\frac{2}{3},\frac{5}{3}\right) \Rightarrow x-\frac{1}{3}\in\left[\frac{1}{3},\frac{4}{3}\right)$

$\Rightarrow\ \left[x-\frac{1}{3}\right]=0$, 1 which are not a prime number.

So, in this case we have no solution.

Case iii $\left[x-\frac{1}{3}\right]$ = –prime number and $\left[x+\frac{1}{3}\right]=-1$

$\Rightarrow\ x+\frac{1}{3}\in[-1,0) \Rightarrow x\in\left[-\frac{4}{3},-\frac{1}{3}\right)$

$\Rightarrow\ x-\frac{1}{3}\in\left[-\frac{5}{3},-\frac{2}{3}\right) \Rightarrow \left[x-\frac{1}{3}\right]=-2$, which is possible for $\left(x-\frac{1}{3}\right)\in\left[-\frac{5}{3},-1\right)$, i.e, $x\in\left[-\frac{4}{3},-\frac{2}{3}\right)$

Case iv: $\left[x-\frac{1}{3}\right]=-1$ and $\left[x+\frac{1}{3}\right]$ = –prime number

Now $\left[\ --\right]=-$

$\Rightarrow\ x-\frac{1}{3}\in[-1,0) \Rightarrow x\in\left[-\frac{2}{3},\frac{1}{3}\right)$

$\Rightarrow\ x+\frac{1}{3}\in\left[-\frac{1}{3},\frac{2}{3}\right) \Rightarrow \left[x+\frac{1}{3}\right]=-1$ or 0. i.e., we get no –ve prime number.

So, in this case we get no solution.

Thus, from above case analysis we see that the product $\left[x-\frac{1}{3}\right]\left[x+\frac{1}{3}\right]$ is prime number for $x\in\left[-\frac{4}{3},-\frac{2}{3}\right)\cup\left[\frac{5}{3},\frac{7}{3}\right)$

∴ Without loss of generality, we can take

$$\alpha=-\frac{4}{3},\beta=-\frac{2}{3},\gamma=\frac{5}{3} \text{ and } \delta=\frac{7}{3}$$

$\Rightarrow\ \alpha+\beta+\gamma+\delta=-2+4=2$

and $\alpha^2+\beta^2+\gamma^2+\delta^2=\frac{16+4+25+49}{9}=\frac{94}{9}$

$\Rightarrow\ 9(\alpha^2+\beta^2+\gamma^2+\delta^2)=94$

6. Show that there exists no polynomial $f(x)$ with integral coefficients which satisfy $f(a)=b$, $f(b)=c$, $f(c)=a$, where a, b, c are distinct integers.

Solution: Let $f(x)=a_0+a_1x+a_2x^2+\ldots+a_nx^n$; where $a_i\in$ integer ($i=0,1,2,\ldots,n$)

Now $f(a)=a_0+a_1a+a_2a^2+\ldots+a_na^n=b$ ....(1)

$f(b)=a_0+a_1b+a_2b^2+\ldots+a_nb^n=c$ .... (2)

$f(c)=a_0+a_1c+a_2c^2+\ldots+a_nc^n=a$ .... (3)

$\Rightarrow f(a), f(b), f(c)$ are integers

∴ $f(a)-f(b)=(a-b).[a$, function in terms of a and $b]$

$=(a-b).\,f_1(a,b)=b-c$; where $f_1\,(a,b)$ is an integer .... (4)

Similarly $(b-c).f_1(b,c)=c-a$ ....(5)

and $(c-a).f_1(c,a)=a-b$ ....(6)

Multiplying (4), (5) and (6), we get

$f_1(a,b)\,.\,f_1(b,c)\,.\,f_1(c,a)=1$

$\Rightarrow f_1(a,b)=1, f_1(b,c)=1, f_1(c,a)$; [as product of integers is 1, if each is one]

$\Rightarrow a-b=b-c=c-a$

$\Rightarrow a=b=c$ which is not possible [as a, b, c are distinct]

Hence, there exists no polynomial of such type.

7. Determine all values of c so that $f(x)=\frac{x-1}{c-x^2+1}$ does not take any value in the interval $\left[-1,-\frac{1}{3}\right]$

Solution: Given $f(x)=\frac{x-1}{c-x^2+1}$ i.e., $y=\frac{x-1}{c-x^2+1}$

Take $y=-t$, where $t\in\left[1,\frac{1}{3}\right]$ and given function assume the values $(-t)$ at some $x=x_0$, then $f(x_0)=-t$

$\Rightarrow\ -t=\frac{x_0-1}{c-x_0^2+1} \Rightarrow x_0^2-c-1=\frac{x_0-1}{t}$

Now, we have to just make sure that this quadratic in x does not give any real value of x for $t\in\left[1,\frac{1}{3}\right]$

$\Rightarrow$ Discriminant of the quadratic must be negative.

$$\frac{1}{t^2}-4\left(\frac{1}{t}-c-1\right)<0$$

$$\Rightarrow \frac{1}{t^2}-\frac{4}{t}+4<-4c \quad \Rightarrow \quad c+\frac{1}{4}\left(\frac{1}{t}-2\right)^2<0$$

$$\Rightarrow c<-\frac{1}{4}\left(\frac{1}{t}-2\right)^2$$

Now $t\in\left[1,\frac{1}{3}\right];-1\le\frac{1}{t}-2\le 1$

$$\Rightarrow 0\le\frac{1}{4}\left(\frac{1}{t}-2\right)^2\le\frac{1}{4} \quad \Rightarrow \quad -\frac{1}{4}\le-\frac{1}{4}\left(\frac{1}{t}-2\right)^2\le 0$$

Hence, $c\in\left(-\infty,-\frac{1}{4}\right)$

8. Find the domain of the following functions and sketch their corresponding graphs, and hence, find their range

(a) $f(x) = x^4 - 2x^2 + 3$ (b) $f(x)=\dfrac{2x}{1+x^2}$

(c) $f(x) = \sin^2 x - 2\sin x$

Solution: (a) Given $f(x) = x^4 - 2x^2 + 3$

(i) Domain of $f(x)$ is $\mathbb{R}$

(ii) $f(x)$ is even so graph will be symmetrical about y-axis.

(iii) $y = x^4 - 2x^2 + 3 = (x^2 - 1)^2 + 2$

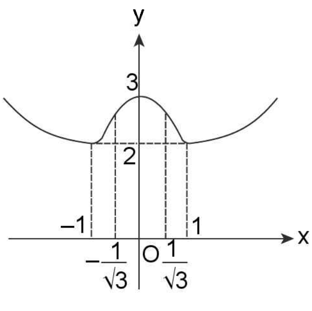
So, minimum value of y is 2 at $x^2 = 1$, Further $f'(x) = 2(x^2 - 1)(2x) \ge 0$ for $x \in [-1, 0] \cup [1, \infty)$

Thus, $f(x)$ is increasing $\forall\, x \in [-1, 0] \cup [1, \infty)$

(iv) Further $f'(x) = 0$ at $x = 0, \pm 1$

(v) When $x = 0$ the value of $y = 3$ and at $x = \pm 1$, $f(x) = 2$

$\therefore$ Graph of $f(x)$ would be as shown below



Clearly the range of function is $[2, \infty)$

(b) $y=f(x)=\dfrac{2x}{1+x^2}$

(i) Domain $= \mathbb{R}$

(ii) $f(x) = -f(x)$, so function is odd the graph is not symmetric about any axis but symmetric about origin.

So, it is sufficient to consider only, $x \ge 0$.

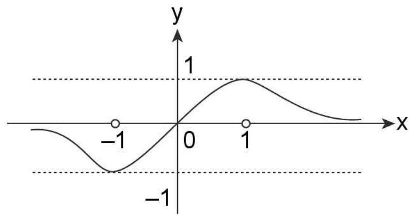
(iii) $y = 0$ when $x = 0$ there is no other point of intersection with coordinate axes

$\Rightarrow$ $x^2 + 1 \ge 2x$

So, $\dfrac{2x}{x^2+1}\le 1$ and equality holds at $x = 1$.

Also from 0 to 1 the function increases and from 1 to ∞ it decreases.

So, graph is as shown in Figure below.



Clearly the range of function is $[-1, 1]$

(c) $y = f(x) = \sin^2 x - 2\sin x$

(i) Domain of y is $\mathbb{R}$

(ii) $0 \le (\sin x - 1)^2 \le 4$

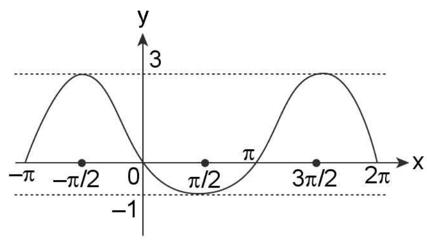
$\Rightarrow$ $0 \le \sin^2 x - 2\sin x + 1 \le 4$

$\Rightarrow$ $-1 \le \sin^2 x - 2\sin x \le 3$

(iii) $f(x)$ has period 2π, so it is sufficient to draw the graph for domain $[0, 2\pi]$

(iv) $y = 0$ for all $x \in n\pi$

Graph is shown in the figure



Clearly the range of function is $[-1, 3]$

9. Find the domain of definition of the function, $f(x)=[2\tan\pi x]^{\left\{\log_{[2\tan\pi x]}\left(\frac{x^2+2x-3}{4x^2-4x-3}\right)\right\}}$; where [.] denotes the greatest integer function.

Solution: $\because\ a^{\log_a b}=b$

$\Rightarrow$ $f(x)=\left(\dfrac{x^2+2x-3}{4x^2-4x-3}\right)$ and $[2\tan \pi x] > 0, \ne 1$

and $\dfrac{x^2+2x-3}{4x^2-4x-3}>0 \quad \Rightarrow \quad [2\tan \pi x] \ge 2$

$\Rightarrow$ $\tan \pi x \ge 1$ i.e., $\dfrac{(x+3)(x-1)}{(2x-3)(2x+1)}>0$

$\Rightarrow$ $x \in (-\infty, -3) \cup (-1/2, 1) \cup (3/2, \infty)$ ...(1)

Also $\tan \pi x \ge 1$

$\Rightarrow \; n\pi + \frac{\pi}{4} \le \pi x < n\pi + \frac{\pi}{2}; \; n \in \mathbb{Z}$

$\Rightarrow \; n + \frac{1}{4} \le x < n + \frac{1}{2};$ .... (2)

$\therefore$ From (1) and (2), we get

$x \in \left[n + \frac{1}{4}, n + \frac{1}{2}\right); n \in \mathbb{Z} - \{-3, -2, -1, 1\}$

Thus, common solutions to (1) and (2) is $\left[n + \frac{1}{4}, n + \frac{1}{2}\right)$; where $n \ge 2$ or $n \le -4$ or $n = 0$.

10. Let $f(x) = x^2 - 1$ and

$$g(x) = \begin{cases} [|f(|x|)|] + 1; & x \in (-1, 0) \cup (0, 1) \\ 1; & \text{otherwise} \end{cases}.$$

Then find the range of $\ln([|g(x)|])$; where [.] denotes the greatest integer function.

Solution: $|f(|x|)| = 1 - x^2; \; x \in (-1, 0) \cup (0, 1)$

$\Rightarrow$ $[|f(|x|)|] = 0; \; x \in (-1, 0) \cup (0, 1)$ and $g(x) = 1$; otherwise

Hence, $g(x) = 1, x \in (-1, 0) \cup (0, 1)$ and 1, otherwise

$\Rightarrow$ $g(x) = 1 \; \forall \; x \in \mathbb{R} \;\; \Rightarrow \;\; [|g(x)|] = 1$

$\Rightarrow$ $\ln[|g(x)|] = 0 \; \forall \; x \in \mathbb{R} \; \Rightarrow$ Range of $\ln[|g(x)|]$ is $\{0\}$.

11. Find the domain and range of $f(x) = \log\left[\cos|x| + \frac{1}{2}\right]$; where [] denotes the greatest integer function.

Solution: We must have $\left[\cos|x| + \frac{1}{2}\right] > 0$; {as $\log x$ is defined for $x > 0$}

$\Rightarrow \; \cos|x| + \frac{1}{2} \ge 1$ ...(1)

But we know; $-1 \le \cos\theta \le 1$; (for $\theta \in \mathbb{R}$}

$\therefore$ $-1 \le \cos|x| \le 1$

or $-1 + \frac{1}{2} \le \cos|x| + \frac{1}{2} \le 1 + \frac{1}{2}$

or $-\frac{1}{2} \le \cos|x| + \frac{1}{2} \le \frac{3}{2}$ ...(2)

$\therefore$ From (1) and (2), we require those values of x which satisfy, $1 \le \cos|x| + \frac{1}{2} \le \frac{3}{2}$

$\Rightarrow \; \left[\cos|x| + \frac{1}{2}\right] = 1$ and $\frac{1}{2} \le \cos|x| \le 1;$

Thus, $f(x)$ is defined for $\frac{1}{2} \le \cos x \le 1$

($\because$ $\cos|x| = \cos x \; \forall \; x \in \mathbb{R}$)

$\therefore$ Domain is $x \in \bigcup_{n=\mathbb{Z}}\left[\left(2n\pi - \frac{\pi}{3}\right), \left(2n\pi + \frac{\pi}{3}\right)\right]$ and range for $f(x)$ is $f(x) = \log(1)$

i.e., $f(x) = 0$ $\qquad\therefore$ Range of $f(x)$ is $\{0\}$

12. Find the range of the following functions:

(i) $f(x) = \ln(\sin x^{\sin x} + 1)$; where $0 < x < \pi/2$

(ii) $f(x) = \ln(2\sin x + \tan x - 3x + 1)$; where $\frac{\pi}{6} \le x \le \frac{\pi}{3}$

Solution: (i) $0 < x < \pi/2$

$\Rightarrow$ $0 < \sin x < 1$

$\therefore$ Range of $\ln(\sin x^{\sin x} + 1)$ for $0 < x < \pi/2$

= Range of $\ln(t^t + 1)$ for $0 < t < 1$

Let $h(t) = t^t + 1 = e^{t\ln t} + 1$

$\therefore$ $h'(t) = e^{t\ln t}(1 + \ln t)$

$\Rightarrow$ $h'(t) > 0$ for $t > 1/e$ and $h'(t) < 0$ for $t < 1/e$

$\therefore$ $h(t)$ has a minima at $t = \frac{1}{e}$ and $h\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{\frac{1}{e}} + 1$

Also $\lim\limits_{t \to 0^+} h(t) = 1 + e^{\lim\limits_{t\to 0}\left(\frac{\ln t}{1/t}\right)} = 1 + e^{\lim\limits_{t\to 0}\left(\frac{1/t}{-1/t^2}\right)}$

$= 1 + e^0 = 2$ and $\lim\limits_{t \to 1^-} h(t) = 2$

$\therefore$ $h(t)$ has its minimum value $= \left(\frac{1}{e}\right)^{\frac{1}{e}} + 1$ and maximum value less than 2

$\therefore$ for $0 < t < 1$

$\Rightarrow \; 1 + \left(\frac{1}{e}\right)^{\frac{1}{e}} \le (t^t + 1) < 2$

$\Rightarrow \; \ln\left(1 + \left(\frac{1}{e}\right)^{1/e}\right) \le \ln(t^t + 1) < \ln 2$

$\Rightarrow \; \ln\left(1 + \left(\frac{1}{e}\right)^{1/e}\right) \le \ln\left((\sin x)^{\sin x} + 1\right) < \ln 2$ for $0 < x < \frac{\pi}{2}$

$\therefore$ Range of $f(x) = \left[\ln\left(1 + e^{\frac{-1}{e}}\right), \ln 2\right)$

(ii) Let $h(x) = (2\sin x + \tan x - 3x + 1)$

$\Rightarrow$ $h'(x) = (2\cos x + \sec^2 x - 3)$

$$= \frac{2\cos^3 x - 3\cos^2 x + 1}{\cos^2 x}$$

$\therefore$ $h'(x) \ge 0$

$\Rightarrow$ $2\cos^3 x - 3\cos^2 x + 1 \ge 0$ and $\cos^2 x \ne 0$

$\Rightarrow \ (\cos x-1)^2(2\cos x+1)\ge 0$

$\Rightarrow \ 2\cos x+1\ge 0; \cos^2 x\ne 0, \text{which holds} \ \forall\, x\in\left[\frac{\pi}{6},\frac{\pi}{3}\right]$

$\Rightarrow$ $h(x)$ is an increasing function of x

$\Rightarrow \ h\left(\frac{\pi}{6}\right)\le h(x)\le h\left(\frac{\pi}{3}\right)$

$\Rightarrow \ \ln h\left(\frac{\pi}{6}\right)\le \ln h(x)\le \ln h\left(\frac{\pi}{3}\right)$

$\Rightarrow \ \ln\left(1+\frac{1}{\sqrt{3}}-\frac{\pi}{2}+1\right)\le f(x)\le \ln\left(\sqrt{3}+\sqrt{3}-\pi+1\right)$

$\Rightarrow$ Range of $f(x)$ is $\left[\ln\left(2+\frac{1}{\sqrt{3}}-\frac{\pi}{2}\right), \ln(1+2\sqrt{3}-\pi)\right]$

13. Functions $f(x)$ and $g(x)$ are defined in $[a, b]$ such that $f(x)$ is continuous and monotonically increasing while $g(x)$ is continuous and monotonically decreasing. It is given that the range of $f(x)$ as well as that of $g(x)$ is a subset of $[a, b]$. Find the range and domain of $h(x) = fog(x) + gof(x)$.

Solution: Domain of the increasing function $f(x)$ is $[a, b]$ $\Rightarrow f'(x) > 0$ in $[a, b]$

Domain of the decreasing function $g(x)$ is $[a, b]$

$\Rightarrow$ $g'(x) < 0$ in $[a, b]$

Now $h'(x) = f(g(x)) + g(f(x))$

$\Rightarrow \ h(x)=\underbrace{g'(x)f'(g(x))}_{-ve}+\underbrace{f'(x)g'(f(x))}_{-ve}<0$

($\because$ $g(x) \in [a, b] \Rightarrow f'(g(x)) > 0$ and $g'(x) < 0$ in $[a, b]$. Similarly $f'(x) > 0$; $g'(f(x)) < 0$)

Hence, $h(x)$ is decreasing function.

Also range of $g(x)$ is subset of $[a, b]$

$\Rightarrow$ Domain of $fog\ (x)$ is $[a, b]$

Similarly, domain of $gof\ (x)$ is $[a, b]$. Hence, the domain of $h(x)$ is $[a, b]$.

Since $h(x)$ is bijective and sum of two continuous functions (composition of continuous functions) is also continuous and monotonically decreasing on domain $[a, b]$.

Therefore, range of $h(x) = [h(b), h(a)]$.

14. Let $f : [0, 1] \to [0, 1]$ defined by $f(x)=\frac{1-x}{1+x}$; $0 \le x \le 1$ and let $g : [0, 1] \to [0, 1]$ defined by $g(x) = 4x\ (1 - x) \le 1$; $0 \le x \le 1$. Determine the composition fog and gof.

Solution: $(fog)\ (x) = f(g(x)) = f[4x(1 - x)]$; $0 \le 4x\ (1 - x) \le 1$; $0 \le x \le 1$

$= \frac{1-4x(1-x)}{1+4x(1-x)}$; $0 \le x \le 1$ and $0 \le 4x - 4x^2 \le 1$

$= \frac{1+4x^2-4x}{1-4x^2+4x}$; $0 \le x \le 1$ and $0 \le (4x - 4x^2) \le 1$

when $0 \le 4x - 4x^2$

$\Rightarrow$ $4x(1 - x) \ge 0$ $\quad\Rightarrow$ $0 \le x \le 1$ and $4x - 4x^2 \le 1$

$\Rightarrow$ $4x^2 - 4x + 1 \ge 0$

$\Rightarrow$ $(2x - 1)^2 \ge 0$; which is true for all x

Hence, $(fog)(x)=\frac{1+4x^2-4x}{1-4x^2+4x}; 0\le x\le 1$

$(gof)(x)=g(f(x))=g\left(\frac{1-x}{1+x}\right); 0\le x\le 1$

$=4\left(\frac{1-x}{1+x}\right)\left(1-\frac{1-x}{1+x}\right); 0\le x\le 1 \ \text{and} \ 0\le\frac{1-x}{1+x}\le 1$

$=\frac{8x(1-x)}{(1+x)^2}; 0\le x\le 1$ and $0 \le 1 - x \le 1 + x$

$=\frac{8x(1-x)}{(1+x)^2}; 0\le x\le 1$ and $0\le x\le 1$

Hence, $(gof)(x)=\frac{8x(1-x)}{(1+x)^2}; 0\le x\le 1$

15. Find the domain and range of $h(x) = g(f(x))$, where $f(x)=\begin{cases}[x], & -2\le x\le -1\\ |x|+1, & -1<x\le 2\end{cases}$ and $g(x)=\begin{cases}[x], & -\pi\le x\le 0\\ \sin x & 0<x\le\pi\end{cases}$; and [.] denotes the greatest integer function.

Solution:

$$h(x)=g(f(x))=\begin{cases}[f(x)]; & - \quad \le f(x)\le 0\\ \sin(f(x)); & 0<f(x)\le\end{cases}$$

$$=\begin{cases}[[x]]; & -\pi\le[x]\le 0; & -2\le x\le -1\\ [|x|+1]; & -\pi\le|x|+1\le 0; & -1<x\le 2\\ \sin[x]; & 0<[x]\le\pi; & -2\le x\le -1\\ \sin(|x|+1); & 0<|x|+1\le\pi; & -1<x\le 2\end{cases}$$

$$=\begin{cases}[x]; & x\in[-3,1)\cap[-2,-1]\\ [|x|+1]; & \text{no value of } x\\ \sin[x]; & x\in[1,4)\cap[-2,-1]\\ \sin(|x|+1); & x\in[-\pi+1,\pi-1]\cap(-1,2]\end{cases}$$

$$=\begin{cases}[x]; & x\in[-2,-1]\\ \sin(|x|+1); & x\in(-1,2]\end{cases}$$

$$=\begin{cases}-2; & x\in[-2,-1)\\ -1; & x=-1\\ [\sin 3,1]; & x\in(-1,2]\end{cases}$$

$\Rightarrow$ Domain of $h(x)$ is $[-2, 2]$ and range of $h(x)$ is $\{-2, -1\} \cup [\sin 3, 1]$

16. If $f(x) = 1 + x$, when $0 \le x \le 2$ and $f(x) = 3 - x$, when $2 < x \le 3$, then determine the following (where [] represents the greatest integer function):

(a) $f(x) = f(f(x))$ (b) $f(f(f(x)))$
(c) $f([x])$ (d) $[f(x)]$

Solution: (a) $f(f(x)) = \begin{cases} 1 + f(x); & 0 \le f(x) \le 2 \\ 3 - f(x); & 2 < f(x) \le 3 \end{cases}$

First consider the case $f(x) = 1 + x$, $0 \le x \le 2$.

$$f(f(x)) = \begin{cases} 1+1+x; & 0 \le 1+x \le 2 \\ 3-1-x; & 2 < 1+x \le 3 \end{cases}$$

$$= \begin{cases} 2+x; & -1 \le x \le 1 \\ 2-x; & 1 < x \le 2 \end{cases}$$

Since our considered domain is $0 \le x \le 2$,

so, $f(f(x)) = \begin{cases} 2+x; & 0 \le x \le 1 \\ 2-x; & 1 < x \le 2 \end{cases}$

Next let us consider the case when $f(x) = 3 - x$, $2 < x \le 3$

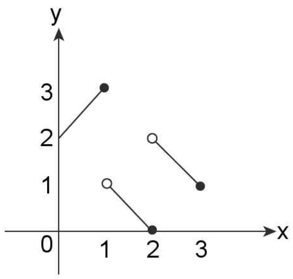
$$f(f(x)) = \begin{cases} 1+3-x; & 0 \le 3-x \le 2 \\ 3-3+x; & 2 < 3-x \le 3 \end{cases}$$

$$= \begin{cases} 4-x; & -3 \le -x \le -1 \\ x; & -1 < -x \le 0 \end{cases} = \begin{cases} 4-x; & 1 \le x \le 3 \\ x; & 0 \le x < 1 \end{cases}$$

Since our considered domain is $2 < x \le 3$, therefore $f(f(x)) = 4 - x$

$$\therefore \quad f(f(x)) = \begin{cases} 2+x; & 0 \le x \le 1 \\ 2-x; & 1 < x \le 2 \\ 4-x; & 2 < x \le 3 \end{cases}$$

Graph of $f(f(x))$ is as shown below



(b) Let $0 \le x \le 1$

$\Rightarrow f(f(f(x))) = f(2 + x)$ where $2 \le 2 + x \le 3$

But we observe that there is no single definition of $f(f(x))$ for this interval.

Therefore, we reduce the interval $0 \le x \le 1$ to $x = 0$ and $0 < x \le 1$ at $x = 0$, $2 + x = 2$

$\therefore f(2 + x) = f(2) = 1 + 2 = 3$

For $0 < x \le 1$; $x + 2 \in (2, 3]$

$\Rightarrow f(x + 2) = 3 - (x + 2) = 1 - x$

Next let $1 < x \le 2$

$\Rightarrow f(f(f(x))) = f(2 - x)$; $0 \le 2 - x < 1$

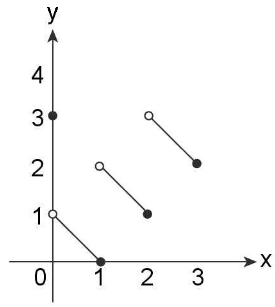
$= 1 + 2 - x = 3 - x$

Now, let $2 < x \le 3 = f(f(f(x))) = f(4 - x)$; $1 \le 4 - x < 2$

$= 1 + (4 - x) = 5 - x$

$$\therefore \quad f(f(f(x))) = \begin{cases} 3; & x = 0 \\ 1-x; & 0 < x \le 1 \\ 3-x; & 1 < x \le 2 \\ 5-x; & 2 < x \le 3 \end{cases}$$

Graph of $f(f(f(x)))$ is as shown below



(c) $f([x])$

Now for $0 \le x < 1$

$\Rightarrow [x] = 0$ $\Rightarrow f[x] = f(0) = 1$;

for $1 \le x < 2$

$\Rightarrow [x] = 1$ $\Rightarrow f[x] = f(1) = 2$;

for $2 \le x < 3$

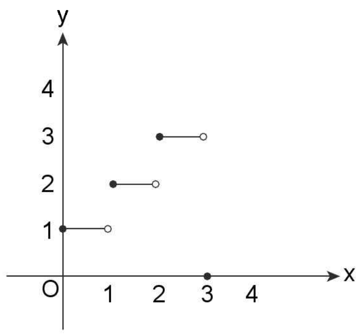
$\Rightarrow [x] = 2$ $\Rightarrow f([x]) = f(2) = 3$;

for $x = 3$

$\Rightarrow [x] = 3$ $\Rightarrow f([x]) = f(3) = 0$

$$\therefore \quad f[x] = \begin{cases} 1; & 0 \le x < 1 \\ 2, & 1 \le x < 2 \\ 3, & 2 \le x < 3 \\ 0, & x = 3 \end{cases}$$

Graph of $f[x]$ is as follows



(d) $y = [f(x)]$, we have for $0 \le x < 1$; $f(x) = x + 1$

$\Rightarrow 1 \le f(x) < 2 \quad \Rightarrow [f(x)] = 1$;

for $1 \le x < 2$; $f(x) = x + 1$

$\Rightarrow 2 \le f(x) < 3 \quad \Rightarrow [f(x)] = 2$;

for $x = 2$

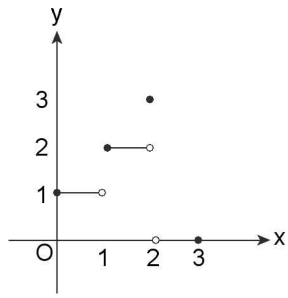
$\Rightarrow f(x) = 3 \quad \Rightarrow [f(x)] = 3$;

For $2 < x \le 3$; $f(x) = 3 - x$

$\Rightarrow 0 \le f(x) < 1 \quad \Rightarrow [f(x)] = 0$

$$\therefore \quad [f(x)] = \begin{cases} 1, & 0 \le x < 1 \\ 2, & 1 \le x < 2 \\ 3, & x = 2 \\ 0, & 2 < x \le 3 \end{cases}$$

Graph of $[f(x)]$ is as shown below



17. Consider $f(x) = (1 - x^n)^{1/n}$ and $g(x) = x^2 + px + q$; $(p, q \in \mathbb{R})$. Let $g(x) = kx$; $(k > 0)$ has imaginary roots, then show that $g(g(g(x))) = k^3 f(f(f(f(x))))$ has no real solution.

Solution: Given $f(x) = (1 - x^n)^{1/n}$

$$\Rightarrow \quad f(f(x)) = \left[1 - \left\{(1 - x^n)^{1/n}\right\}^n\right]^{1/n} = x$$

$\Rightarrow f(f(f(f(x)))) = f(f(x)) = x$

Now as, $g(x) = kx$ has no real roots

It means, $x^2 + px + q - kx > 0$ i.e., $g(x) > k(x)$

So $g(g(x)) > kg(x) > k^2x$

$\Rightarrow g(g(g(x))) > kg\,(g(x)) > k.k^2x$

$\Rightarrow g(g(g(x))) > k^3(x)$

Hence, $g(g(g(x))) = k^3\, f(f(f(f(x))))$ has no real solution.

18. Describe fog and gof wherever is possible for the following functions:

(i) $f(x) = \sqrt{x+3}, g(x) = 1 + x^2$

(ii) $f(x) = \sqrt{x}, g(x) = x^2 - 1$

Solution: (i) Domain of f is $[-3, \infty)$, range of f is $[0, \infty)$.

Domain of g is R, range of g is $[1, \infty)$ for $gof(x)$

Since range of f is a subset of domain of g,

$\therefore$ Domain of gof is $[-3, \infty)$ {equal to the domain of f}

$gof\,(x) = g\{f(x)\} = g(\sqrt{x+3}) = 1 + (x + 3) = x + 4$. Range of gof is $[1, \infty)$

For $fog(x)$

Since range of g is a subset of domain of f,

$\therefore$ Domain of fog is $\mathbb{R}$ {equal to the domain of g}

$\therefore$ $fog\,(x) = f\{g(x)\} = f(1 + x^2) = \sqrt{x^2+4}$; Range of fog is $[2, \infty)$.

(ii) $f(x) = \sqrt{x}$, $g(x) = x^2 - 1$

Domain of f is $[0, \infty)$, range of f is $[0, \infty)$.

Domain of g is $\mathbb{R}$, range of g is $[-1, \infty)$.

For $gof(x)$

Since range of f is a subset of the domain of g.

$\therefore$ domain of gof is $[0, \infty)$ and $g\{f(x)\} = g(\sqrt{x}) = x - 1$.

Range of gof is $[-1, \infty)$.

For $fog(x)$

Since range of g is not a subset of the domain of f.

i.e., $[-1, \infty) \not\subset [0, \infty)$.

$\therefore$ fog is not defined on whole of the domain of g.

Domain of fog is $\{x \in R$, the domain of g: $g(x) \in [0, \infty)$, the range of $f\}$.

Thus, the domain of fog is $D = \{x \in R: 0 \le g(x) < \infty\}$

i.e., $D = \{x \in R: 0 \le x^2 - 1\} = \{x \in R: x \le -1$ or $x \ge 1\} = (-\infty, -1] \cup [1, \infty)$

$fog\,(x) = f\{g(x)\} = f(x^2 - 1) = \sqrt{x^2-1}$.

Its range is $[0, \infty)$.

19. Let $f(x) = x^2 + 3x - 3$; If n points $x_1, x_2, x_3, \ldots, x_n$ are chosen on positive x-axis such that

(a) $\dfrac{1}{n}\sum_{i=1}^{n} f^{-1}(x_i) = f\left(\dfrac{1}{n}\sum_{i=1}^{n}(x_i)\right)$

(b) $\sum_{i=1}^{n} f^{-1}(x_i) = \sum_{i=1}^{n}(x_i)$; where f^{-1} denotes the inverse of f. Find the A.M. of x_i, s

Solution: Given $\dfrac{f^{-1}(x_1) + f^{-1}(x_2) + \ldots\ldots + f^{-1}(x_n)}{n}$

$$= f\left(\frac{x_1 + x_2 + \ldots. + x_n}{n}\right) \quad \ldots.\ (1)$$

and $f^{-1}(x_1) + f^{-1}(x_2) + \ldots. + f^{-1}(x_n) = (x_1 + x_2 + \ldots + x_n)$ ... (2)

Using (2) in (1), we get

$$\frac{x_1 + x_2 + x_3 + \ldots + x_n}{n} = f\left(\frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}\right)$$

$\Rightarrow\ f(\bar{x}) = \bar{x}$; where $\dfrac{x_1 + x_2 + ... + x_n}{n} = \bar{x}$

$\Rightarrow\ \bar{x}^2 + 3\bar{x} - 3 = \bar{x}$ $\quad\Rightarrow\ \bar{x}^2 + 2\bar{x} - 3 = 0$

$\Rightarrow\ \bar{x} = -3, 1$ $\quad\Rightarrow\ \bar{x} = 1$ as $\bar{x} > 0$ ($\because x_i > 0$)

20. Let f be a real valued function satisfying $f(x) + f(x + 4) = f(x + 2) + f(x + 6)$. Prove that $\int_x^{x+8} f(t)\,dt$ is a constant function.

Solution: Given that $f(x) + f(x + 4) = f(x + 2) + f(x + 6)$ ....(1)

Replacing x by $x + 2$, we get

$f(x + 2) + f(x + 6) = f(x + 4) + f(x + 8)$ ....(2)

From (1) and (2), we have $f(x) = f(x + 8)$ .....(3)

Now let $g(x) = \int_x^{x+8} f(t)\,dt$

$\Rightarrow\ g'(x) = f(x + 8) - f(x) = 0$ (from (3))

$\Rightarrow\ g$ is a constant function

21. If $f(r) = 1 + \dfrac{1}{2} + \dfrac{1}{3} + ... + \dfrac{1}{r}$ and $f(0) = 0$, then find $\sum_{r=1}^{n} (2r+1)\, f(r)$

Solution: Since $\sum_{r=1}^{n} (2r+1)\, f(r)$

$= \sum_{r=1}^{n} (r^2 + 2r + 1 - r^2)\, f(r)$

$= \sum_{r=1}^{n} \{(r+1)^2 f(r) - (r+1)^2 f(r+1) + (r+1)^2 f(r+1) - r^2\ f(r)\}$

$= \sum_{r=1}^{n} \{(r+1)^2 \{f(r) - f(r+1)\}\ + \sum_{r=1}^{n} \{(r+1)^2\ f(r+1) - r^2 f(r)\}$

$= -\sum_{r=1}^{n} \dfrac{(r+1)^2}{r+1} + \sum_{r=1}^{n-1} \{(r+1)^2\ f(r+1) + (n+1)^2 f(n+1) - \sum_{r=1}^{n} r^2 f(r)$

$= -\sum_{r=1}^{n} r + 1 + \{2^2 f(2) + 3^2 f(3) + ... n^2 f(n)\} + (n + 1)^2 f(n + 1) - \{1^2 f(1) + 2^2 f(2) + \dots + n^2 f(n)\}$

$= -\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1 + (n+1)^2 f(n+1) - 1^2 f(1)$

$= -\dfrac{n(n+1)}{2} - n + (n+1)^2\ f(n+1) - f(1)$ $\quad(\because f(1) = 1)$

$= (n+1)^2\ f(n+1) - \dfrac{n(n+3n+2)}{2}$.

22. Suppose that function $f: \mathbb{R} \to \mathbb{R}$ satisfies the inequality $\left|\sum_{k=1}^{n} 3^k \{f(x+ky) - f(x-ky)\}\right| \le 1$; for every positive integer n and k, $\forall\ x, y \in \mathbb{R}$. Prove that f is a constant function.

Solution: Given

$$\left|\sum_{k=1}^{n} 3^k \{f(x+ky) - f(x-ky)\}\right| \le 1 \qquad ...(1)$$

Replacing n by $(n - 1)$, we get

$$\left|\sum_{k=1}^{n-1} 3^k \{f(x+ky) - f(x-ky)\}\right| \le 1 \qquad ...(2)$$

Solving inequations (1) and (2), we get

$\left|3^n \{f(x+ny) - f(x-ny)\}\right| \le 2$

$\Rightarrow\ \left|\{f(x+ny) - f(x-ny)\}\right| \le \dfrac{2}{3^n}$

We choose x and y such that $x + ny = u$ and $x - ny = v$; where $u, v \in \mathbb{R}$ and $\left|\{f(u) - f(v)\}\right| \le \dfrac{2}{3^n}$ for arbitrary $n \in \mathbb{N}$

i.e., as $n \to \infty$, $\left|\{f(u) - f(v)\}\right| \le \lim_{n\to\infty} \dfrac{2}{3^n}$

$\Rightarrow\ |f(u) - f(v)| \le 0$ $\quad\Rightarrow\ f(u) = f(v)$

$\Rightarrow\ f$ is a constant function.

23. Show that the inverse of a linear fraction function is always a linear fraction function (except where it is not defined).

Solution: Let, $f(x) = \dfrac{a+bx}{c+dx}$ be the said linear fraction function.

Let at some x it attains value y, so $\dfrac{a+bx}{c+dx} = y$

$\Rightarrow\ a + bx - cy - dxy = 0$

$\Rightarrow\ a - cy + x(b - dy) = 0$

$\Rightarrow\ x = \dfrac{cy-a}{b-dy}$; which is again a linear fraction function defined in R except at $y = \dfrac{b}{d}$ and inverse of the given function is, $y = \dfrac{cx-a}{b-dx}$

24. Find the value of a so that $f(x) = \dfrac{ax+2}{x+4}$ is identical to $f^{-1}(x)$.

Solution: If $f(x) = f^{-1}(x)$, then $fof^{-1}(x) = x$

$\Rightarrow\ fof(x) = x$ $\quad\Rightarrow\ \dfrac{af(x)+2}{f(x)+4} = x$

$$\Rightarrow \frac{a\left(\frac{ax+2}{x+4}\right)+2}{\frac{ax+2}{x+4}+4} = x$$

$\Rightarrow a^2x + 2a + 2x + 8 \equiv x(ax + 2 + 4x + 16)$

$\Rightarrow (a^2 + 2)x + 2a + 8 = (a + 4)x^2 + 18x$

$\Rightarrow a + 4 = 0$ and $a^2 + 2 = 18$ and $2a + 8 = 0$

$\Rightarrow a = -4$ and $a = \pm 4$, $a = -4 \;\Rightarrow\; a = -4$

25. If the function $f(x) = ax + b$ is self invertible, then find the all possible ordered pairs (a, b).

Solution: Let $y = f(x)$

$\Rightarrow x = f^{-1}(y)$ and $y = ax + b$

$\Rightarrow x = \frac{y}{a} - \frac{b}{a} \qquad \Rightarrow f^{-1}(y) = \frac{y}{a} - \frac{b}{a}$

$\Rightarrow f^{-1}(x) = \frac{x}{a} - \frac{b}{a}$ and $f(x) = ax + b$

Now in order that (1) and (2), coincide $a = \frac{1}{a}$ and $b = \frac{-b}{a}$

$\Rightarrow a^2 = 1 \qquad \Rightarrow a = 1$ or -1

if $a = -1$, then $b \in \mathbb{R}$

if $a = 1$, then $2b = 0 \quad \Rightarrow b = 0$

Hence, the ordered pairs (a, b) are $(-1, b)$, $(1, 0)$; where $b \in \mathbb{R}$.

26. Check the invertibility of the function $f(x) = (e^x - e^{-x})$ and then find its inverse.

Solution: We have $f(x) = (e^x - e^{-x})$; $x \in \mathbb{R}$

$\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$

$f'(x) = e^x + e^{-x} > 0;\ \forall\, x \in \mathbb{R}$

$\therefore\ f(x)$ is invertible when defined from $\mathbb{R}$ to $\mathbb{R}$.

Now, $f(x) = y = t - \frac{1}{t}$; [where $t = e^x > 0$]

$\Rightarrow t^2 - 1 = ty \qquad \Rightarrow t^2 - ty - 1 = 0$

$\Rightarrow t = \frac{y + \sqrt{y^2+4}}{2}$ [$\because$ t can't be negative]

Now, $t = e^x$

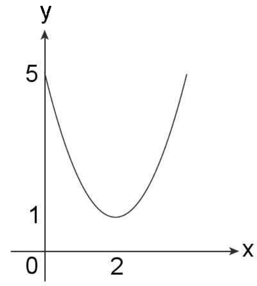
$\Rightarrow e^x = \frac{y+\sqrt{y^2+4}}{2} \qquad \Rightarrow x = \ln\left(\frac{y+\sqrt{y^2+4}}{2}\right)$

Interchanging x and y, we get,

Inverse of $f(x)$ is $f^{-1}(x) = \ln\left(\frac{x+\sqrt{x^2+4}}{2}\right)$

27. Find the set X if the function f: $[2, \infty) \to X$; where $f(x) = 5 - 4x + x^2$ is bijective, and hence, find its inverse.

Solution: $y = x^2 - 4x + 5 = (x - 2)^2 + 1$; when $x = 2$, $y = 1$



As $x \in [2, \infty)$, then $y \in [1, \infty)$.

$\therefore$ i.e., range of function is $[1, \infty)$.

(It is clear from the figure)

$\therefore$ For $f(x)$ to be invertible, range = co-domain

$\Rightarrow$ Set $X \equiv [1, \infty)$

To find the inverse: (using the given rule of function)

$y = 5 - 4x + x^2$

$\Rightarrow x^2 - 4x + 5 - y = 0$

$\Rightarrow x = \frac{4 \pm \sqrt{16 - 4(5-y)}}{2} = 2 \pm \sqrt{y-1}$

$\because$ $x \geq 2$. So, ignoring positive sign, we have $x = 2 + \sqrt{y-1}$

Interchanging x and y, we get $y = 2 + \sqrt{x-1}$

$\Rightarrow f^{-1}(x) = 2 + \sqrt{x-1}$

28. Let $f : \left[\frac{1}{2}, \infty\right) \to \left[\frac{3}{4}, \infty\right)$; where $f(x) = x^2 - x + 1$. Find the inverse of $f(x)$. Hence, or otherwise solve the equation $x^2 - x + 1 = \frac{1}{2} + \sqrt{x - \frac{3}{4}}$.

Solution: $f(x) = x^2 - x + 1 = (x - 1/2)^2 + 3/4$

$\Rightarrow$ Principal domain of $f(x) = \left[\frac{1}{2}, \infty\right)$

and range $= \left[\frac{3}{4}, \infty\right)$

Thus, it is clear that the given function is one–one and onto in the given domain and co-domain.

Thus, its inverse can be obtained.

Clearly $f(f^{-1}(x)) = (f^{-1}(x))^2 - f^{-1}(x) + 1$

$\Rightarrow x = (f^{-1}(x))^2 - (f^{-1}(x)) + 1$

$\Rightarrow (f^{-1}(x))^2 - f^{-1}(x) + (1 - x) = 0$

$\Rightarrow\ f^{-1}(x)=\dfrac{1\pm\sqrt{1-4(1-x)}}{2}=\dfrac{1}{2}\pm\sqrt{x-\dfrac{3}{4}}$

$\Rightarrow\ f^{-1}(x)=\dfrac{1}{2}+\sqrt{x-\dfrac{3}{4}}$

[As range of $f^{-1}(x)$ is $[1/2, \infty)$, we have to consider only the positive sign]

$\Rightarrow\ f^{-1}(x)=\dfrac{1}{2}+\sqrt{x-\dfrac{3}{4}}$

Now the given equation is basically $f(x) = f^{-1}(x)$ and $f(x)$ is increasing on $\left[\dfrac{1}{2},\infty\right)$.

We know that increasing function $f(x)$ and $f^{-1}(x)$ meet on the line $y = x$.
Hence, solution of the given equation will also be solution of $f(x) = x$.

$\Rightarrow\ x^2 - x + 1 = x \qquad \Rightarrow\ x^2 - 2x + 1 = 0$

$\Rightarrow\ (x-1)^2 = 0 \qquad \Rightarrow\ x = 1$

29. Consider a function $f(x)=\dfrac{x+\lambda}{x-1}$, $x \in \mathbb{R} - \{1\}$; where λ is a real constant. If f is not a constant function, then find the following.

(i) f^{-1}, if it exists (ii) the range of f

(iii) $f\left(\dfrac{1}{f(f(x))}\right)-f\left(f\left(f\left(\dfrac{1}{x}\right)\right)\right)$

Solution: (i) Given function is

$y=f(x)=\dfrac{x+\lambda}{x-1},\ (x\neq 1)$

$\Rightarrow\ f'(x)=\dfrac{(x-1)-(x+\lambda)}{(x-1)^2}=\dfrac{-(\lambda+1)}{(x-1)^2}$

Since f is not constant here $\lambda \neq -1$ for $\lambda < -1$, $\lambda + 1 < 0$

$\Rightarrow f'(x) > 0\ \forall\ x \in \mathbb{R} \sim \{1\}$ and for $\lambda > -1$, $\lambda + 1 > 0$

$\Rightarrow f'(x) < 0\ \forall\ x \in \mathbb{R} \sim \{1\}$

$\therefore$ $f(x)$ strictly increasing for $\lambda < -1$ and strictly decreasing for $\lambda > -1$

Hence, $f'(x)$ is either always increasing or decreasing functions.

Therefore f^{-1} is defined $\forall\ \lambda \neq -1$

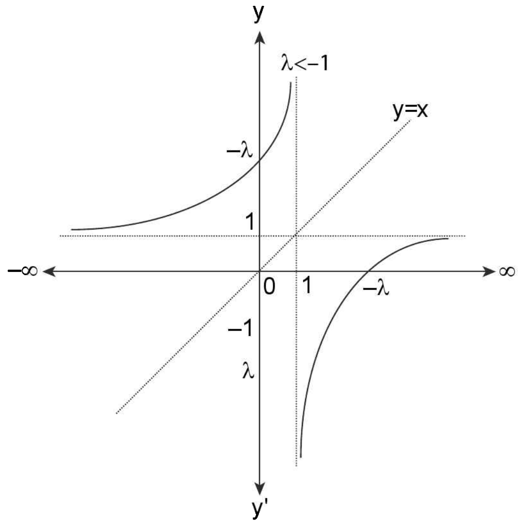
Now, $y=\dfrac{x+\lambda}{x-1}$

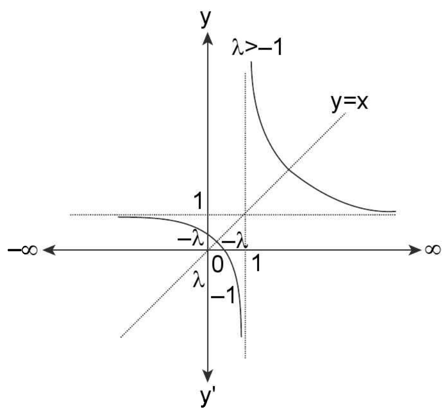
$\Rightarrow yx - y = x + \lambda \qquad \Rightarrow\ x(y-1) = y + \lambda$

$\Rightarrow\ x=f^{-1}(y)=\dfrac{y+\lambda}{y-1} \Rightarrow\ f^{-1}(x)=\dfrac{x+\lambda}{x-1}$

$\Rightarrow$ $f(x)$ is self invertible function for $\lambda \neq -1$

(ii) The graph of f (for $\lambda \neq -1$) is as shown

(a) 

(b) 

Hence, the range of function is $\mathbb{R} - \{1\}$

(iii) Again $f(x)=\dfrac{x+\lambda}{x-1}$

Since $f^{-1}(x) = f(x)$ and $f(f^{-1}(x)) = x$

$\Rightarrow\ f(f(x)) = x \qquad \Rightarrow\ \dfrac{1}{(fof)(x)}=\dfrac{1}{x}$

$\Rightarrow\ f\left(\dfrac{1}{f(f(x))}\right)=f\left(\dfrac{1}{x}\right)$ .... (1)

Similarly $f\left(f\left(\dfrac{1}{x}\right)\right)=\dfrac{1}{x}$

$\Rightarrow\ f\left(f\left(f\left(\dfrac{1}{x}\right)\right)\right)=f\left(\dfrac{1}{x}\right)$ .... (2)

Hence, from (1) and (2),

$f\left(\dfrac{1}{f(f(x))}\right)-f\left(f\left(f\left(\dfrac{1}{x}\right)\right)\right)=0.$

30. If a function $f(x)$ satisfies the equation $2f(x)+xf\left(\frac{1}{x}\right)$ $-\ 2f\left(\left|\sqrt{2}\cos\left(\pi\left(x+\frac{1}{4}\right)\right)\right|\right) = 4\cos^2\frac{\pi x}{2}+x\cos\frac{\pi}{x}$, then show that

(i) $f(2)+f\left(\frac{1}{2}\right)=1$ (ii) $f(1)+f(2)=0$

Solution: Given $2f(x)+xf\left(\frac{1}{x}\right)$ –

$$2f\left(\left|\sqrt{2}\cos\left(\pi\left(x+\frac{1}{4}\right)\right)\right|\right)=4\cos^2\frac{\pi x}{2}+x\cos\frac{\pi}{x} \quad (1)$$

Putting $x = 1$ in (1), we have $3f(1)- 2f(1) = -1$

$\Rightarrow f(1) = -1$

Putting $x = 2$ in (1), we have

$$2f(2)+2f\left(\frac{1}{2}\right)-2f(1)=4$$

$$\Rightarrow f(2)+f\left(\frac{1}{2}\right)=1 \quad (\because f(1)=-1) \quad \ldots (2)$$

Now substituting $x=\frac{1}{2}$ in (1) we get

$$2f\left(\frac{1}{2}\right)+\frac{1}{2}f(2)-2f(1)=2+\frac{1}{2}$$

$$\Rightarrow 2f\left(\frac{1}{2}\right)+\frac{1}{2}f(2)=\frac{1}{2}\Rightarrow 4f\left(\frac{1}{2}\right)+f(2)=1 \quad \ldots(3)$$

Solving (2) and (3), we get

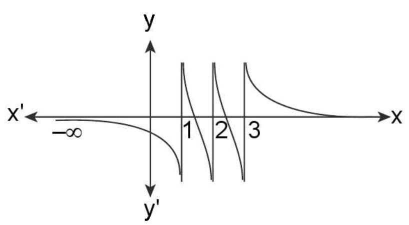
$f\left(\frac{1}{2}\right)=0; f(2)=1$, then $f(2)+f\left(\frac{1}{2}\right)=1$ which proves part (i) and $f(1) + f(2) = 0$, which proves part (ii).

Solved Assertion and Reason

31. A: Let f: $\mathbb{R}$ – {1, 2, 3} → $\mathbb{R}$ be a function defined by $f(x)=\frac{1}{x-1}+\frac{2}{x-2}+\frac{3}{x-3}$. Then f is many–one function.

R: If either $f'(x) > 0\ \forall\ x \in D_f$ or $f'(x) < 0$, $\forall\ x \in D_f$ then $y = f(x)$ is one–one function.

Solution: (c) The graph of given function $f(x)$ is as shown below



From the graph it is clear that $f(x)$ is not one–one. That is, many–one. Therefore, assertion is true.

Also $f'(x) < 0$, $\forall\ x \in D_f$ but the function is not one–one, so the reason is false.

Infact the reason is true if the function $f(x)$ is continuous.

32. A: Domain of definition of function $f(x)=e^x+\cos^{-1}\left(\frac{x}{2}-1\right)$ + $\log\sqrt{x-[x]}$ is (0, 4) – {1, 2, 3}.

R: Domain of $\cos^{-1}\left(\frac{x}{2}-1\right)$ is (0, 4).

Solution: (c) Given function is

$$f(x)=e^x+\cos^{-1}\left(\frac{x}{2}-1\right)+\log\sqrt{x-[x]}$$

Now e^x is defined $\forall\ x \in \mathbb{R}$ ... (1)

$\cos^{-1}\left(\frac{x}{2}-1\right)$ is defined for $-1\le\frac{x}{2}-1\le 1$

i.e., $0\le\frac{x}{2}\le 2 \quad \Rightarrow x \in [0, 4]$ ... (2)

Also $\log\sqrt{x-[x]}=\log\sqrt{\{x\}}$ is defined for $x \notin \mathbb{Z}$ as otherwise $\{x\} = 0$ ... (3)

∴ From (1), (2) and (3) domain must be (0, 4) – {1, 2, 3}

∴ Assertion is true, but reason is false.

∴ Thus, (c) must be the correct option.

33. A: The range of the function $f(x) = \sin^2 x + p \sin x + q$, where $|p| > 2$, will be the set of real numbers between $q-\frac{p^2}{4}$ and $q + p + 1$.

R: The function $g(t) = t^2 + pt + 1$, where $t \in [-1, 1]$ and $|p| > 2$, will attain minimum and maximum value at –1 and 1.

Solution: (d) Given $f(x)=\left(\sin x+\frac{p}{2}\right)^2+q-\frac{p^2}{4}$

If $|p| > 2$, the perfect square term can't vanish, therefore $f(x)$ will be minimum or maximum when $\sin x = 1$ or –1, i.e., $f(x)$ lies between $(-p + q + 1)$ and $(1 - p + q)$.

⇒ Assertion (A) is false (Assertion (A) will be true if $|p| < 2$)

$$g(t)=t^2+pt+\frac{p^2}{4}+1-\frac{p^2}{4} = \left(t+\frac{p}{2}\right)^2+1-\frac{p^2}{4}$$

for $|p| > 2$, $t \in [-1, 1]$

$\Rightarrow |t|\ne\left|\frac{p}{2}\right| \quad \Rightarrow \left(t+\frac{p}{2}\right)^2$ can't vanish

$\Rightarrow$ minimum/maximum value of $g(t)$ occurs at -1 and 1

$\therefore$ Reason is true.

34. A: Let A and B be two sets each with a finite number of elements. Assume that there is an injective mapping from A to B and that there is an injective mapping from B to A. Then there is a bijective mapping A to B.

R: An onto function is not necessarily one–one.

Solution: (b) $\because$ Three is an injective map from A to B

$\Rightarrow$ Number of elements of $A \leq$ Number of elements of B .... (1)

Similarly, there is an injective map from B to A

$\Rightarrow$ Number of elements of $B \leq$ Number of elements of A .... (2)

$\therefore$ From (1) and (2), we have $n(A) = n(B)$

$\Rightarrow$ $f: A \to B$ and $g: B \to A$ are bijective.

Clearly reason is correct but does not support the assertion.

35. A: The function defined by $f(x) = x^3 + ax^2 + bx + c$ is invertible if and only if $a^2 \leq 3b$.

R: A function is invertible if and only if it is bijective

Solution: (a) $f(x)$ will be invertible if $f'(x) \geq 0$ for all x as $f(-\infty) = -\infty$ and $f(\infty) = \infty$

or $3x^2 + 2ax + b \geq 0$; which is true, if $4a^2 - 12b \leq 0$ or $a^2 \leq 3b$ $\Rightarrow$ (a) is true.

36. A: The function $f(x) = \dfrac{2x+a}{bx-2}$ coincides with its inverse if $ab = -4$.

R: $\dfrac{ax+b}{cx+d}$ reduces to a constant if $ad - bc \neq 0$.

Solution: (b) Given $y = \dfrac{2x+a}{bx-2}$

$\Rightarrow$ $bxy - 2y = 2x + a$ $\Rightarrow$ $x(by - 2) = 2y + a$

$\Rightarrow$ $x = \dfrac{2y+a}{by-2}$ $\Rightarrow$ $f^{-1}(x) = \dfrac{2x+a}{bx-2}$

$\Rightarrow$ $f(x)$ is self invertible and $f'(x) = \dfrac{-4-ab}{(bx-2)^2} > 0$

or < 0 for $ab \neq -4$. For $ab = -4$, $f(x)$ is constant

Thus, $f(x)$ is self invertible for $ab \neq -4$.

$\Rightarrow$ Assertion is true and reason is also true, but does not explain assertion completely.

37. A function $y = f(x)$ is represented by equation $y^2 - 2 + \log_3(x - 1) = 0$; where $f(x) \geq 0$ $\forall$ $x \in D_f$

A: The domain of $f(x)$ is $(1, 10]$

R: The domain of inverse of $f(x)$ is $\mathbb{R}$

Solution: (c) The given equation is valid if $x > 1$.

Now $y = \sqrt{2 - \log_3(x-1)}$ is defined if $2 \geq \log_3(x - 1)$

$\Leftrightarrow 9 \geq x - 1$ $\Leftrightarrow$ $x \leq 10$

Hence, the domain of $f(x)$ is $(1, 10]$.

Also $2 - y^2 = \log_3(x - 1)$ $\Rightarrow$ $x = 1 + 3^{2-y^2}$; $y \geq 0$

Hence, the inverse of the function is defined by $f^{-1}(x) = 1 + 3^{2-x^2}$ whose domain is $[0, \infty)$.

38. A: There are exactly two values of n for which the function $f(x) = x^n$ coincides with its inverse.

R: Both the functions x and $1/x$ are inverse of themselves.

Solution: (a) $y = x^n$, $x = y^{1/n}$

$\Rightarrow$ Inverse is $y = x^{1/n}$

At the point of intersection $x^n = x^{1/n}$

$\Rightarrow$ $x^{n-\frac{1}{n}} = 1$ $\Rightarrow$ $n^2 - 1 = 0$ $\Rightarrow$ $n = \pm 1$

$\therefore$ Reason is clearly true and explains the assertion.

Solved Comprehension Passage

A: Consider the functions $f(x) = \begin{cases} 2x + a; & x \geq -1 \\ bx^2 + 3; & x < -1 \end{cases}$

and $g(x) = \begin{cases} x + 4; & 0 \leq x \leq 8 \\ -3x - 2; & -2 \leq x < 0 \end{cases}$; $b > 0$. Based on the given functions answer the following questions:

39. $g(f(x))$ is not defined if

(a) $a \in (10, \infty)$; $b \in (5, \infty)$

(b) $a \in (4, 10)$; $b \in (5, \infty)$

(c) $a\ (\in 10, \infty)$; $b \in (1, 5)$

(d) $a \in (4, 10)$; $b \in (1, 5)$

40. If the natural domain of $g(f(x))$ is $[-1, 4]$, then

(a) $a = 2$; $b > 7$ (b) $a = 0$; $b > 5$

(c) $a = 2$; $b > 10$ (d) $a = 0$; $b \in \mathbb{R}$

41. If $a = 2$ and $b = 3$, then range of $g(f(x))$ is

(a) $(-2, 8]$ (b) $(0, 8]$

(c) $[4, 12]$ (d) None of these

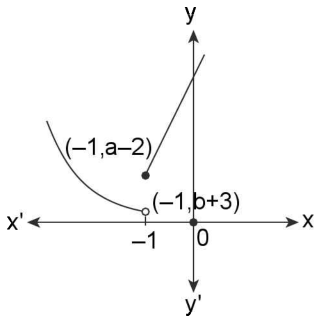
Solution: Given $f(x) = \begin{cases} 2x + a; & x \geq -1 \\ bx^2 + 3; & x < -1 \end{cases}$; $b > 0$

$\Rightarrow$ $f'(x) = \begin{cases} 2; & x > -1 \\ 2bx; & x < -1 \end{cases}$

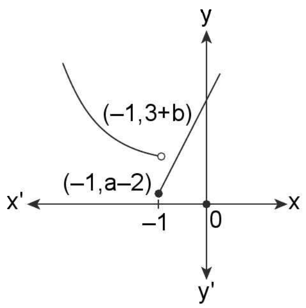
$\Rightarrow$ $f(x)$ is decreasing for $x < -1$ and increasing for $x > -1$

$\Rightarrow$ $f(x)$ has its minimum value $-2 + a$ for $x \geq -1$ and has g.l.b. $(b + 3)$ for $x \leq -1$

$\therefore$ Graph of $f(x)$ would be as shown below



or



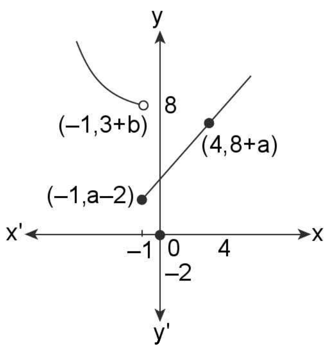
39. (a) For $g(f(x))$ to be undefined range of $f(x)$ should be disjoint from the domain of $g(x)$ which is $(-2, 8]$

$\Rightarrow$ The graph of $f(x)$ must be above the line $y = 8$

$\Rightarrow$ $a - 2 > 8$ and $b + 3 > 8$

$\Rightarrow$ $a > 10$ and $b > 5$ $\Rightarrow$ $a \in (10, \infty)$, $b \in (5, \infty)$

40. (b) Given natural domain of $g(f(x))$ is $[-1, 4]$, i.e., $g(f(x))$ is defined only for $x \in [-1, 4]$ and for $x \in [-1, 4]$; range of $f(x)$ must be entirely contained in the domain of $g(x)$ i.e., situation graphically shown below



$\Rightarrow$ $-2 \le a - 2 \le 8$ and $-2 \le a + 8 \le 8$; $b + 3 > 8$

$\Rightarrow$ $0 \le a \le 10$ and $-10 \le a \le 0$; $b > 5$

$\Rightarrow$ $a = 0$, $b > 5$

41. (c) If $a = 2$, $b = 3$, then $f(x) = \begin{cases} 2x+2; & x \ge -1 \\ 3x^2+3; & x < -1 \end{cases}$ and

$$g(x) = \begin{cases} x+4; & 0 \le x \le 8 \\ -3x-2; & -2 \le x < 0 \end{cases}$$

$$\Rightarrow\ g(f(x)) = \begin{cases} f(x)+4; & 0 \le f(x) \le 8 \\ -3f(x)-2; & -2 \le f(x) < 0 \end{cases}$$

$$= \begin{cases} 2x+6; & 0 \le 2x+2 \le 8; & x \ge -1 \\ 3x^2+7; & 0 \le 3x^2+3 \le 8; & x < -1 \\ -3(2x+2)-2 = -6x-8; & -2 \le 2x+2 < 0; & x \ge -1 \\ -3(3x^2+3)-2 = -9x^2-11; & -2 \le 3x^2+3 < 0; & x < -1 \end{cases}$$

$$= \begin{cases} 2x+6; & -1 \le x \le 3 \\ 3x^2+7; & -\sqrt{\dfrac{5}{3}} \le x < -1 \end{cases}$$

$\Rightarrow$ Range of $g(f(x)) = [4, 12]$

Alternatively: Range of $f(x)$ is $[0, \infty)$ which when fed to function $g(x)$, then range of resulting composition function $gof(x)$ would be $[4, 12]$.

B: If $f(x) = \text{Mid}\{g(x), h(x), p(x)\}$ means the function which will be second in order when values of the three functions at a particular x are arranged in ascending or descending order.

Consider the function:

$$f(x) = \text{Mid}\left\{x-1, (x-3)^2, 3-\frac{(x-3)^2}{2}\right\};\ x \in [1, 4].$$

Now answer the following questions based on above defined function $f(x)$.

42. The value of $fof(2.5))$ is

(a) 2.25 (b) 1.50

(c) 1.875 (d) None of these

43. The greatest value of $f(x)$ in $[1, 4]$ will be

(a) $1 + \sqrt{3}$ (b) $2 + \sqrt{3}$

(c) $3 + \sqrt{3}$ (d) None of these

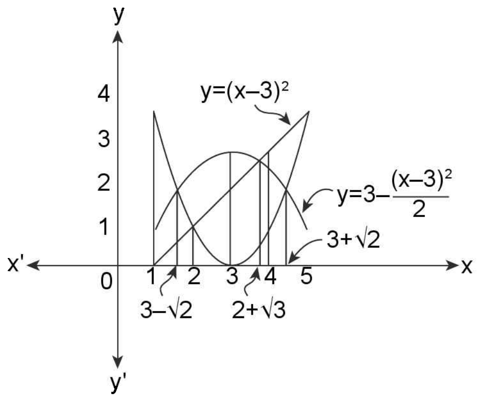
44. Range of $f(x)$ for $x \in [1, 4]$ is

(a) $[\sqrt{3} - 1, 1]$ (b) $[1 + \sqrt{2}, 1 + \sqrt{3}]$

(c) $[1, 1 + \sqrt{3}]$ (d) None of these

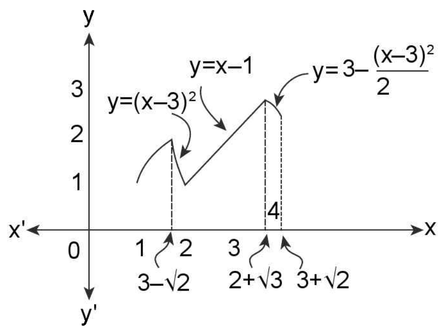
Solution:

The graphs of functions $y = x - 1$, $y = (x - 3)^2$, $y = 3 - \dfrac{(x-3)^2}{2}$ for $x \in [1, 4]$ represented w.r.t. same coordinate axes are shown below.



Now $f(x) = \text{Mid}\left\{x-1,(x-3)^2,3-\frac{(x-3)^2}{2}\right\};\ x\in[1,\ 4]$

can be represented as shown below.



42. (c) At $x = 2.5$, $(x-3)^2 < x-1 < 3-\frac{(x-3)^2}{2}$

$\Rightarrow f(2.5) = 2.5 - 1 = 1.5$

$\therefore\ fof(2.5) = f(f(2.5)) = f(1.5)$;

Further at $x = 1.5$, $x - 1 < 3 - \frac{(x-3)^2}{2} < (x-3)^2$

$\Rightarrow\ f(1.5) = 3-\frac{(1.5-3)^2}{2} = 3-\frac{2.25}{2} = \frac{3.75}{2} = 1.875$

43. (a) Clearly from graph of $f(x)$; $f(x)$ is maximum when

$x-1 = 3-\frac{(x-3)^2}{2} \quad \Rightarrow \quad x+\frac{(x-3)^2}{2} = 4$

$\Rightarrow\ x = \frac{4\pm\sqrt{16-4}}{2} = \frac{4\pm 2\sqrt{3}}{2} = 2\pm\sqrt{3}$

$\Rightarrow$ Clearly $f(x)$ has greatest value at $x > 3$.

$\therefore$ $f(x)$ has greatest value at $x = 2 + \sqrt{3}$ and is $x - 1 = 2 + \sqrt{3} - 1 = 1 + \sqrt{3}$.

44. (c) Maximum of $f(x) = 1 + \sqrt{3}$ and minimum of $f(x) = 1$ for $x \in [1, 4]$

$\therefore$ Range of $f(x)$ is $[1, 1 + \sqrt{3}]$

C: Given a function $f(x) = e^{\tan\left\{\frac{x}{4}\right\}} + \cos\pi\left(\frac{1+2[x]}{2}\right)+\sin\left(\frac{\pi[x]}{2}\right)$ whose fundamental period is p (where {.} and [.] represent fractional part and greatest integral part functions, respectively) and $g(x) = \sqrt{2p+\frac{p}{2}[x]-[x]^2}$ the domain of y is $[q, r)$

One another function $h(x) = \begin{cases} 2+x, x\ge 0 \\ 2-x, x<0 \end{cases}$.

Then on the basis of above information answer the following questions:

45. The period p of $f(x)$ is
(a) an irrational number
(b) rational number
(c) a composite number
(d) 4

46. Value of $r - q - 1$ is equal to
(a) 6 (b) 7
(c) 8 (d) 9

47. Range of $hoh(x)$ in terms of p, q, r is:
(a) $[p, \infty)$ (b) $[q, \infty)$
(c) $[r, \infty)$ (d) $(-\infty, -r) \cup (p, \infty)$

Solution: Period of $e^{\tan\left\{\frac{x}{4}\right\}}$ is 4

$\cos\pi\left(\frac{(1+2[x])}{2}\right) = 0\ \forall\ x\in\mathbb{R}$

Period of $\sin\left(\frac{\pi[x]}{2}\right)$ is 4

45. (c) $\therefore$ Period of $f(x)$ is 4 which is composite and rational.

$\therefore$ (b), (c), (d) are correct options.

46. (a)

$\therefore\ p = 4 \qquad \Rightarrow\ y = \sqrt{8+2[x]-[x]^2}$

$\Rightarrow -[x]^2 + 2[x] + 8 \ge 0$

$\Rightarrow [x]^2 - 2[x] - 8 \le 0$

i.e., $([x] - 4)([x] + 2) \le 0$

$\therefore\ -2 \le [x] \le 4 \qquad \therefore\ -2 \le x < 5$

$\Rightarrow$ Domain of $g(x) = [-2, 5) = [q, r)$

$\Rightarrow q = -2, r = 5$

$\therefore\ r - q - 1 = 5 + 2 - 1 = 6$

$\Rightarrow$ (a) is the correct option

47. (a) $h(x)=\begin{cases}x+2; & x\ge 0\\ 2-x; & x<0\end{cases}$

$\Rightarrow\ hoh(x))=\begin{cases}2+h(x); & h(x)\ge 0\\ 2-h(x); & h(x)<0\end{cases}$

$=\begin{cases}2+x+2; & x+2\ge 0 \text{ and } x\ge 0\\ 2-(x+2); & x+2<0 \text{ and } x\ge 0\\ 2+2-x\,; & 2-x\ge 0 \text{ and } x<0\\ 2-(2-x); & 2-x<0 \text{ and } x<0\end{cases}$

$=\begin{cases}4+x; & x\ge 0\\ 4-x; & x<0\end{cases}$

Range of $hoh(x)$ is $[4, \infty) \cup (4, \infty) = [4, \infty) = [p, \infty)$

$\Rightarrow$ (a) is the correct option

D: Let $y = f(x)$: $A \to B$ is one–one and onto, then $y = f(x)$ is invertible and its inverse $y = f^{-1}(x)$: $B \to A$ exists. If point (α, β) lies on $y = f(x)$, then the point (β, α) lies on $y = f^{-1}(x)$. Let m be the minimum number of points of intersection of $y = f(x)$ and $y = f^{-1}(x)$.

Now answer the following questions:

48. If (2, 3), (3, 2) lie on $y = f^{-1}(x)$ and $y = f(x)$ is continuous, then

(a) $m = 3$ (b) $m = 4$

(c) $m = 6$ (d) $f(x) = f^{-1}(x)\ \forall\ x \in A$

49. If $(\alpha, \beta)\ \forall\ \alpha \in A$ lies on $y = f(x)$ but not on $y = f^{-1}(x)$ (where $\alpha \ne \beta$), then

(a) $y = f(x)$ and $y = f^{-1}(x)$ do not intersect

(b) either $y = f(x)$ and $y = f^{-1}(x)$ do not intersect or $y = f(x)$ and $y = f^{-1}(x)$ intersect on $y = x$.

(c) $y = f(x)$ and $y = f^{-1}(x)$ intersect on $y = x$

(d) $y = f(x)$ and $y = f^{-1}(x)$ can not intersect on $y = x$

50. If $f(x) = -x + \sin x$, then

(a) All point of intersection of $y = f(x)$ and $y = f^{-1}(x)$ lies on $y = x$

(b) All point of intersection of $y = f(x)$ and $y = f^{-1}(x)$ lies on $y = -x$

(c) $y = f(x)$ and $y = f^{-1}(x)$ never intersect except at origin

(d) $f(x) = f^{-1}(x)\ \forall\ x \in \mathbb{R}$

Solutions:

48. (a) If (2, 3), (3, 2) lie on $y = f^{-1}(x)$, then both lie on $y = f(x)$ also.

Further since $y = f(x)$ is continuous

Therefore, the graph must cross the line $y = x$

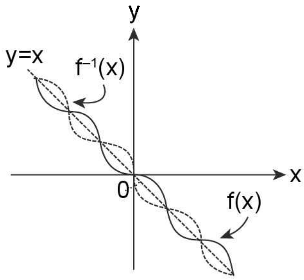
$\therefore$ there are at least 3 points of intersection.

49. (b) Suppose $y = f(x)$ and $y = f^{-1}(x)$ intersect each other at some point (α, β) not lying on $y = x$, then $\alpha \ne \beta$

$\therefore$ (α, β) lies on both $y = f(x)$ and $y = f^{-1}(x)$ which is contrary to the given hypothesis

$\therefore$ the curves either do not intersect or they intersect on $y = x$

50. (b) To solve the equation $f(x) = f^{-1}(x)$, we first draw the graphs of both the functions to observe their points of intersection.



From, the graph, we realize that only the point (0, 0) lies on the line $y = x$. There are other points of intersection $(n\pi, n\pi)$, $n \in \mathbb{Z}$ which lies on the line $y = -x$. Hence, the solution set is $\{n\pi, n \in \mathbb{Z}\}$.

Thus, all the points of intersection lie on $y = -x$.

Solved Column Matching

51. Match the following columns.

Column I

(i) Range of $sgn\ \{x\}$ is (where {.} represents fractional part function)

(ii) Domain of $\sin^{-1} x + \sin^{-1}(1 - x)$ is

(iii) Range of $\sqrt{\dfrac{2\tan^{-1} x}{\pi}}$ is

(iv) Range of $\dfrac{2}{\pi}\ \sin^{-1}[x^2 + x + 1]$ is (where [.] represent greatest integer function)

Column II

(a) {1}

(b) [0, 1)

(c) {0, 1}

(d) [0, 1]

Ans. (i) → (c), (ii) → (d), (iii) → (b), (iv) → (c)

Solution: (i) When $\{x\} = 0$, then $sgn\{x\} = 0$ and when $0 < \{x\} < 1$, then $sgn\ \{x\} = 1$

$\Rightarrow$ Range of $sgn\{x\} = \{0, 1\}$

(ii) For domain $-1 \le x \le 1$ and $-1 \le 1 - x \le 1$

$\Rightarrow$ $-1 \le x \le 1$ and $0 \le x \le 2$ $\Rightarrow$ $0 \le x \le 1$

$\Rightarrow$ Domain of function = [0, 1]

(iii) $-\dfrac{\pi}{2} < \tan^{-1} x < \dfrac{\pi}{2}$ $\forall\, x \in \mathbb{R}$

$\Rightarrow$ $-1 < \dfrac{2}{\pi}\tan^{-1} x < 1$;

for $f(x) = \sqrt{\dfrac{2\tan^{-1} x}{\pi}}$; $\dfrac{2\tan^{-1} x}{\pi} \ge 0$

$\Rightarrow$ $0 \le \dfrac{2\tan^{-1} x}{\pi} < 1$ $\Rightarrow$ $0 \le \sqrt{\dfrac{2}{\pi}\tan^{-1} x} < 1$

$\Rightarrow$ Range of $f(x) = [0, 1)$

(iv) $\dfrac{3}{4} \le x^2 + x + 1 < \infty$, but for $\sin^{-1}[x^2 + x + 1]$;

$-1 \le [x^2 + x + 1] \le 1$ $\Rightarrow$ $\dfrac{3}{4} \le x^2 + x + 1 < 2$

Case I: When $\dfrac{3}{4} \le x^2 + x + 1 < 1$

$\Rightarrow$ $[x^2 + x + 1] = 0$ $\Rightarrow$ $\sin^{-1}[x^2 + x + 1] = 0$

$\Rightarrow$ $f(x) = \dfrac{2}{\pi}\sin^{-1}[x^2 + x + 1] = 0$

Case II: When $1 \le x^2 + x + 1 < 2$,

$\Rightarrow$ $[x^2 + x + 1] = 1$

$\Rightarrow$ $\sin^{-1}\left[x^2 + x + 1\right] = \dfrac{\pi}{2}$ $\Rightarrow$ $f(x) = \dfrac{2}{\pi}\cdot\dfrac{\pi}{2} = 1$

$\therefore$ Range of given function is {0, 1}.

52. Let $f(x) = \ln x$ and $g(x) = x^2 - 1$. Column-I contains composite functions and column-II contains their domain. Match the entries of column-I with their corresponding answer is column-II.

Column-I

(i) *fog*
(ii) *gof*
(iii) *fof*
(iv) *gog*

Column-II

(a) $(1, \infty)$
(b) $(-\infty, \infty)$
(c) $(-\infty, -1) \cup (1, \infty)$
(d) $(0, \infty)$

Ans. (i) → (c) (ii) → (d)
(iii) → (a) (iv) → (b)

Solution: Given $f(x) = \ln x$ and $g(x) = x^2 - 1$

(i) $fog(x) = f(g(x)) = \ln(x^2 - 1)$ which is defined for $x^2 - 1 > 0$

$\Rightarrow$ $(x - 1)(x + 1) > 0$ $\Rightarrow$ $x \in (-\infty, -1) \cup (1, \infty)$

$\Rightarrow$ (i) → (c)

(ii) $gof(x) = g[f(x)] = (\ln x)^2 - 1$ which is defined for $x > 0$

$\Rightarrow$ domain of $gof(x)$ is $(0, \infty)$

$\Rightarrow$ (ii) → (d)

(iii) $fof(x) = f[f(x)] = \ln(\ln x)$ which is defined for $\ln x > 0$ and $x > 0$

$\Rightarrow$ $x > 1$

$\therefore$ Domain of $fof(x)$ is $(1, \infty)$

$\Rightarrow$ (iii) → (a)

(iv) $gog(x) = g[g(x)] = (g(x))^2 - 1 = (x^2 - 1)^2 - 1$, which is defined for all $x \in \mathbb{R}$

$\Rightarrow$ Domain of $gog(x)$ is $\mathbb{R}$ $\Rightarrow$ (iv) → (b)

53. If [.] and {.} represent the greatest integer and fractional part functions respectively, then match the columns given below.

Column I

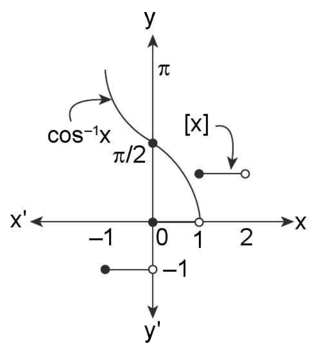
(i) Number of solutions of $[x] = \cos^{-1} x$
(ii) Number of solution of $\sin^{-1}x = sgn(x)$
(iii) Number of solutions of $\{x\} = e^{x^2}$
(iv) Number of solutions of $= \dfrac{\sin^{-1} x + \cos^{-1} x}{2} = \{x\}$

Column II

(a) 3
(b) 2
(c) 1
(d) 0

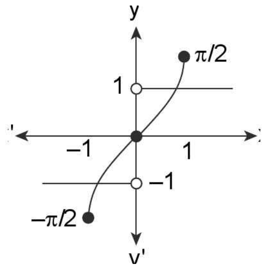
Ans. (i) → (d) (ii) → (a)
(iii) → (d) (iv) → (b)

Solution: (i) The graph of $\cos^{-1}(x)$ and $[x]$ represented with respect to the same coordinate axes as shown in the graph below.



Clearly $[x] = \cos^{-1} x$, has no solution as $f(x) = [x]$ and $g(x) = \cos^{-1} x$ never intersects.

(ii) The graph of $sgn(x)$ and $\sin^{-1}(x)$ represented with respect to the same coordinate axes as shown below.



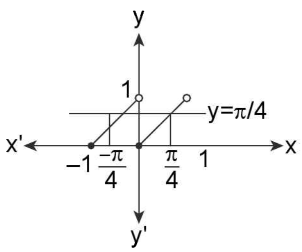
Clearly $sgn(x) = \sin^{-1} x$ has 3 solutions, as $sgn(x)$ and $\sin^{-1} x$ intersect at three points once in $x \in (-1, 0)$. Secondly at $x = 0$ and finally in $x \in (0, 1)$.

(iii) $\{x\} = e^{x^2}$ has no solution.

$\because$ $0 \le \{x\} < 1$ and $e^{x^2} \ge 1$ for all $x \in \mathbb{R}$

$\Rightarrow$ $\{x\}$ and e^{x^2} never intersect.

(iv) The graph of $\dfrac{\sin^{-1} x + \cos^{-1} x}{2} = \dfrac{\pi}{4}$ and $\{x\}$ represented with respect to the same coordinate as shown below



Clearly $\dfrac{\sin^{-1} x + \cos^{-1} x}{2} = \dfrac{\pi}{4}$ and $\{x\}$ intersect at two points, and hence, the equation

$\dfrac{\sin^{-1} x + \cos^{-1} x}{2} = \{x\}$ has exactly two solutions.

Also $\dfrac{\sin^{-1} x + \cos^{-1} x}{2} = \{x\}$

$\Rightarrow$ $\{x\} = \dfrac{\pi}{4}$; where $-1 \le x \le 1$

$\left(\because \sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}\right)$

$\Rightarrow$ $x = -1 + \dfrac{\pi}{4}, \dfrac{\pi}{4}$, i.e., $x = \dfrac{\pi - 4}{4}, \dfrac{\pi}{4}$ are only two solutions.

54. Match the following columns

Column I

(i) Let $X = \{a_1, a_2, ..., a_7\}$ and $Y = \{b_1, b_2, b_3\}$. The number of functions f from X to Y such that it is onto and there are exactly three elements x in X such that $f(x) = b_1$ is

(ii) The number of real solutions for x, y if $y = |\sin x|$ and $y = \sin^{-1}(\sin x)$; where $x \in [-2\pi, 2\pi]$ is

(iii) If a, b and c are distinct positive real numbers such that $a + b + c = 1$, then $\dfrac{(1+a)(1+b)(1+c)}{(1-a)(1-b)(1-c)}$ is greater than

(iv) Period of the function $[8x + 7] + \cos \pi x - 8x$, where [.] denotes the greatest integer function is

Column II

(a) 2

(b) 5

(c) 490

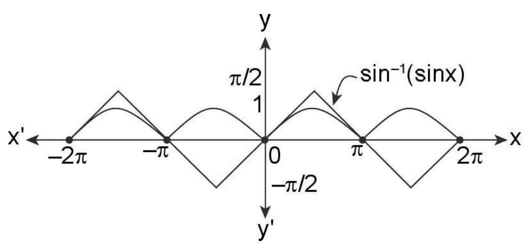
(d) 8

Ans. (i) → (c); (ii) → (b); (iii) → (d) (iv) → (a)

Solution: (i) Since any 3 elements out of 7 can be chosen in ${}^7C_3$ ways which are two be associated with b_1 and each of remaining 4 elements has two choices that is, b_2 and b_3, but there are two functions in which all the four elements will be associated to b_2 or b_3 and other remains unassociated.

So, required number of functions $= {}^7C_3\,(2^4 - 2) = 490$.

(ii) The graph of $|\sin x|$ and $\sin^{-1}(\sin x)$ in $x \in [-2\pi, 2\pi]$ is as that follows:



$\therefore$ From figure it is clear that the two graphs would intersect at five points

$x = -2\pi, -\pi, 0, \pi, 2\pi$, and hence, number of solutions is 5.

(iii) $a = 1 - b - c$

$\Rightarrow$ $1 + a = (1 - b) + (1 - c) > 2\sqrt{(1-b)(1-c)}$ ...(1)

($\because$ $(1 - b) = a + c > 0 \Rightarrow 1 - b > 0$.)

Similarly, $1 - a$, $1 - c > 0$ and $AM > GM$ for positive real numbers) $1 + b > 2\sqrt{(1-c)(1-a)}$ (2)

and $1 + c > 2\sqrt{(1-a)(1-b)}$ .... (3)

$\therefore$ From (1), (2) and (3), we have

$\therefore$ $(1 + a)(1 + b)(1 + c) > 8(1 - a)(1 - b)(1 - c)$

$$\Rightarrow \frac{(1+a)(1+b)(1+c)}{(1-a)(1-b)(1-c)} > 8$$

$\therefore$ Required expression > 8.

(iv) $[8x] + 7 + \cos \pi x - 8x = -\{8x\} + \cos \pi x + 7$

Now fundamental period of $\{8x\}$ is $\frac{1}{8}$ and that of $\cos \pi x$ is 2.

Hence, fundamental period of given function

$= \text{LCM}\left(\frac{1}{8}, 2\right) = 2$.

55. **Column-I**

(i) The period of the function $y = \sin(2\pi t + \pi/3) + 2\sin(3\pi t + \pi/4) + 3\sin 5\pi t$ is

(ii) $y = \{\sin(\pi x)\}$ is a many–one function for $x \in (0, a)$; where a may be

(iii) The fundamental period of the function $y = \frac{1}{2}\left(\frac{|\sin(\pi/4)x|}{\cos(\pi/4)x} + \frac{\sin(\pi/4)x}{|\cos(\pi/4)x|}\right)$ is

(iv) If $f : [0, 2] \to [0, 2]$ is bijective function defined by $f(x) = ax^2 + bx + c$, where a, b, c are non-zero real numbers, then $f(2)$ is equal to

Column II

(a) 1/2

(b) 8

(c) 2

(d) 0

Ans. (i) $\to$ (b, c) (ii) $\to$ (b, c)
(iii) $\to$ (b) (iv) $\to$ (d)

Solution: (i)

$$y = \sin\left(2\pi t + \frac{\pi}{3}\right) + 2\sin\left(3\pi t + \frac{\pi}{4}\right) + 3\sin(5\pi t)$$

Period of $\sin\left(2\pi t + \frac{\pi}{3}\right)$ is $\frac{2\pi}{2\pi} = 1$;

Period of $\sin\left(3\pi t + \frac{\pi}{4}\right)$ is $\frac{2\pi}{3\pi} = \frac{2}{3}$ and period of $\sin(5\pi t)$ is $\frac{2\pi}{5\pi} = \frac{2}{5}$

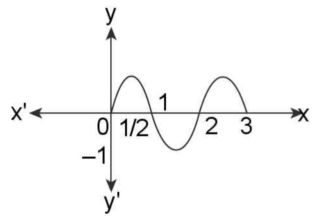
$\therefore$ Period of given function $= \text{LCM}\left(1, \frac{2}{3}, \frac{2}{5}\right)$

$$= \frac{\text{L.C.M.}(1,2,2)}{\text{H.C.F.}(1,3,5)} = \frac{2}{1} = 2$$

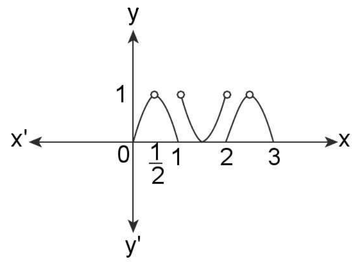
$\Rightarrow$ 8 is also period of given function

$\therefore$ (i) $\to$ (b, c)

(ii) Graph of $\sin \pi x$ is as shown below



Graph of $\{\sin \pi x\}$ is as shown below



Clearly, $\{\sin \pi x\}$ is periodic with period 2 and 8.

$\therefore$ (ii) $\to$ (b, c)

(iii) $$f(x) = \frac{1}{2}\left(\frac{\left|\sin\frac{\pi}{4}x\right|}{\cos\frac{\pi}{4}x} + \frac{\sin\frac{\pi}{4}x}{\left|\cos\frac{\pi}{4}x\right|}\right)$$

$$= \begin{cases} \frac{1}{2}\left(\tan\frac{\pi}{4}x + \tan\frac{\pi}{4}x\right); & \sin\frac{\pi}{4}x, \cos\frac{\pi}{4}x > 0 \\ \frac{1}{2}\left(-\tan\frac{\pi}{4}x - \tan\frac{\pi}{4}x\right); & \sin\frac{\pi}{4}x, \cos\frac{\pi}{4}x < 0 \\ 0; & \sin\frac{\pi}{4}x . \cos\frac{\pi}{4}x \le 0, \text{ and } \cos\frac{\pi}{4}x \ne 0 \\ \text{not defined}; & \cos\frac{\pi}{4}x = 0 \end{cases}$$

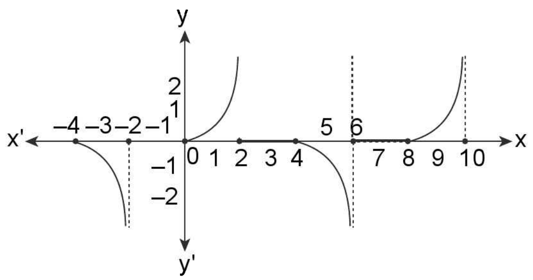
Let us consider $\frac{\pi}{4}x \in [0, 2\pi]$.

$$\Rightarrow f(x) = \begin{cases} \tan\frac{\pi}{4}x; & 0 \le \frac{\pi}{4}x < \frac{\pi}{2} \\ -\tan\frac{\pi}{4}x; & \pi \le \frac{\pi}{4}x < \frac{3\pi}{2} \\ 0; & \frac{\pi}{2} < \frac{\pi}{4}x \le \pi \text{ and } \frac{3\pi}{2} < \frac{\pi}{4}x \le 2\pi \\ \text{not defined}; & \frac{\pi}{4}x = \frac{\pi}{2}, \frac{3\pi}{2} \end{cases}$$

$$\Rightarrow f(x)=\begin{cases}\tan\frac{\pi}{4}x; & 0\le x<2\\ -\tan\frac{\pi}{4}x; & 4\le x<6\\ 0\,; & 2<x\le 4 \text{ and } 6<x\le 8\\ \text{not defined if } x=2,6\end{cases}$$

and so on.

$\therefore$ Graph of $f(x)$ would be as shown below



Clearly, the function is periodic with period 8.

$\therefore$ (iii) $\to$ (b)

(iv) Since $f(x)$ is bijective,

$\therefore$ $f(0) = 0$ or 2 but $f(0) = 0$

$\Rightarrow$ $c = 0$ (which is not true)

$\therefore$ $f(0) = 2$.

We know that a quadratic function is of opposite monotonicity on intervals $\left(-\infty,\frac{-b}{2a}\right]$ and $\left[\frac{-b}{2a},\infty\right)$

$\therefore$ $f(0) = 2$ if $f(x)$ is increasing on [0, 2], then $f(x) > 2$ for $x > 0$.

But range of $f(x) = [0, 2]$

$\Rightarrow$ $f(x)$ must be decreasing on [0, 2]

$\Rightarrow$ $f(2) = 0$

$\therefore$ (iv) $\to$ (d)

Solved Integer-type Questions

56. Let $f(x) = \frac{9^x}{9^x+3}$, then if the value of the sum $f\left(\frac{1}{2006}\right)+f\left(\frac{2}{2006}\right)+f\left(\frac{3}{2006}\right)+\ldots+f\left(\frac{2005}{2006}\right)=S$, then find $2S$.

Solution: Observe the argument of 1st and last term. If argument of 1st term is x, then the argument of the last is $1-x$ and similar relation exists for each pair of terms that are equidistant from the both ends.

Consider, $f(x) + f(1-x)$, i.e., $\frac{9^x}{9^x+3}+\frac{9^{1-x}}{9^{1-x}+3}$

$$=\frac{9^x}{9^x+3}+\frac{9}{9+3\cdot 9^x}=\frac{9^x}{9^x+3}+\frac{3}{3+9^x}=1$$

Now since $f(x) + f(1-x) = 1$

$\therefore$ Sum S becomes $\left[f\left(\frac{1}{2006}\right)+f\left(\frac{2005}{2006}\right)\right]+\left[f\left(\frac{2}{2006}\right)+f\left(\frac{2004}{2006}\right)\right]+\ldots+f\left(\frac{1003}{2006}\right)$

Clearly it contains 2005 terms

$\therefore$ 1002 pairs (each summed up as 1) and an isolated term $f\left(\frac{1003}{2006}\right)$

$$\Rightarrow S = 1002+f\left(\frac{1003}{2006}\right)=1002+f\left(\frac{1}{2}\right)$$

$$\Rightarrow S=1002+\frac{1}{2}=1002+0.5=1002.5.$$

$$\Rightarrow 2S=2\times\frac{2005}{2}=2005$$

57. The set of real values of x satisfying the equality $\left[\frac{3}{x}\right]+\left[\frac{4}{x}\right]=5$; (where [] denotes the greatest integer function) belongs to the interval $\left(a_1,\frac{a_2}{a_3}\right]$; where a_1, a_2, $a_3\in\mathbb{N}$ and $\frac{a_2}{a_3}$ is in its lowest form. Find the value of $\sum a_i+\sum_{i\ne j}a_ia_j+\sum_{i\ne j\ne k}a_ia_ja_k$.

Solution: ***Case i:*** If $x<0$, then $\left[\frac{3}{x}\right]$ and $\left[\frac{4}{x}\right]$ is negative; hence, $\left[\frac{3}{x}\right]+\left[\frac{4}{x}\right]$ can never be equal to 5

Case ii: If $x>0$; we have $\frac{3}{x}<\frac{4}{x}$

$\therefore$ $\left[\frac{3}{x}\right]\le\left[\frac{4}{x}\right]$

Since each of $\left[\frac{3}{x}\right]$ and $\left[\frac{4}{x}\right]$ is a non-negative integer

$\therefore$ 3 possibilities are there (sub-cases)

Sub-case i: $\left[\frac{3}{x}\right]=0$ and $\left[\frac{4}{x}\right]=5$

Sub-case ii: $\left[\frac{3}{x}\right]=1$ and $\left[\frac{4}{x}\right]=4$

Sub-case iii: $\left[\frac{3}{x}\right] = 2$ and $\left[\frac{4}{x}\right] = 3$

As $\left[\frac{3}{x}\right] + \left[\frac{4}{x}\right] = 5$. Now, If $\left[\frac{3}{x}\right] = 0$

$\Rightarrow 0 \le \frac{3}{x} < 1$ $\quad \Rightarrow 0 \le 3 < x\sim$ ...(i)

$\Rightarrow$ Further $\left[\frac{4}{x}\right] = 5$ $\quad \Rightarrow 5 \le \frac{4}{x} < 6$

$\Rightarrow \frac{1}{6} < \frac{x}{4} \le \frac{1}{5}$ $\quad \Rightarrow \frac{2}{3} < x \le \frac{4}{5}$ ...(ii)

These two inequations (i) and (ii) are impossible simultaneously. Hence, no solutions in sub-case (i)

Sub case (ii): Now, $\left[\frac{3}{x}\right] = 1$

$\Rightarrow 1 \le \frac{3}{x} < 2$ $\quad \Rightarrow \frac{1}{2} < \frac{x}{3} \le 1$ $\quad \Rightarrow \frac{3}{2} < x \le 3$ ...(iii)

Further $\left[\frac{4}{x}\right] = 4 \Rightarrow 4 \le \frac{4}{x} < 5$ $\quad \Rightarrow \frac{1}{5} < \frac{x}{4} \le \frac{1}{4}$

$\Rightarrow \frac{4}{5} < x \le 1$ ...(iv)

(iii) and (iv) are impossible, hence, no solution in sub-case (ii)

Sub case (iii): Again if $\left[\frac{3}{x}\right] = 2 \Rightarrow 2 \le \frac{3}{x} < 3$

$\Rightarrow \frac{1}{3} < \frac{x}{3} \le \frac{1}{2}$ $\quad \Rightarrow 1 < x \le \frac{3}{2}$ ...(v)

Further $\left[\frac{4}{x}\right] = 3 \quad \Rightarrow 3 \le \frac{4}{x} < 4$

$\Rightarrow \frac{1}{4} < \frac{x}{4} \le \frac{1}{3}$ $\quad \Rightarrow 1 < x \le \frac{4}{3}$ ...(vi)

Common solution $1 < x \le \frac{4}{3}$

Hence $x \in \left(1, \frac{4}{3}\right]$

$\therefore$ $a_1 = 1, a_2 = 4, a_3 = 3;$

$\therefore$ $\sum a_i + \sum_{i \ne j} a_i a_j + \sum_{i \ne j \ne k} a_i a_j a_k$

$= a_1 + a_2 + a_3 + a_1a_2 + a_2a_3 + a_3a_1 + a_1a_2a_3$

$= 1 + 4 + 3 + 4 + 12 + 3 + 12 = 39$

58. If a, b, are positive real numbers such that $a - b = 4$, then find the smallest value of the constant λ for which $\sqrt{x^2+ax} - \sqrt{x^2+bx} < \lambda$ for all $x > 0$.

Solution: Given $f(x) = \sqrt{x^2+ax} - \sqrt{x^2+bx}$

$\Rightarrow f(x) = \frac{x^2+ax-x^2-bx}{\sqrt{x^2+ax}+\sqrt{x^2+bx}}$

$= \frac{(a-b)x}{\sqrt{x^2+ax}+\sqrt{x^2+bx}} = \frac{a-b}{\sqrt{1+a/x}+\sqrt{1+b/x}}$

$\because$ $x > 0$, as x increases $\frac{1}{x}$ decreases, and denominator of above fraction decreases, and hence, $f(x)$ goes on increasing.

It implies $f(x)$ cannot exceed the value obtained by taking the $\lim_{x\to\infty} f(x) = \frac{a-b}{2}$

$\therefore$ $\lambda = \frac{a-b}{2} = \frac{4}{2} = 2$

59. The range of the function $f(x) = \frac{x+m}{x^2+1}$, $(m \in \mathbb{R})$ contains the interval [0, 1]. If $m \ge \frac{3}{k}$, then find k

Solution: Given function is $y = \frac{x+m}{x^2+1}$

$\Rightarrow yx^2 - x + (y - m) = 0$

we must have $D \ge 0$ for $y \in [0, 1]$

$\Rightarrow 1 - 4y(y - m) \ge 0$ for $y \in [0, 1]$

$\Rightarrow g(y) = 4y^2 - 4my - 1 \le 0$ for $y \in [0, 1]$

$\Rightarrow g(y) \le 0$, for $y \in [0, 1]$ $\Rightarrow g(0) \le 0, g(1) \le 0$

$\Rightarrow -1 \le 0$ and $3 - 4m \le 0$ $\Rightarrow m \ge \frac{3}{4}$ $\Rightarrow k = 4$

60. Let $f(x) = (x + 1)(x + 2)(x + 3)(x + 4) + 5$; where $x \in [-6, 6]$. If the range of the function is $[a, b]$; where $a, b \in \mathbb{N}$, then find the value of $(a + b)$.

Solution: Given $f(x) = (x^2 + 5x + 4)(x^2 + 5x + 6) + 5$

$= [(x^2 + 5x + 5) - 1]\ [(x^2 + 5x + 5) + 1] + 5$

$= (x^2 + 5x + 5)^2 - 1 + 5$

$\Rightarrow f(x) = (x^2 + 5x + 5)^2 + 4$

Hence, $f(x)$ has a minimum value 4 when $x^2 + 5x + 5 = 0$, i.e., $x = \frac{-5\pm\sqrt{5}}{2} \in [-6, 6]$

Also maximum occurs at $x = 6$.

$\Rightarrow f(x)_{\max} = (36 + 30 + 5)^2 + 4 = (71)^2 + 4 = 5041 + 4 = 5045$

Range is $[4, 5045] \equiv [a, b]$ (given)

$\Rightarrow a = 4; b = 5045$ $\quad \Rightarrow a + b = 5049$

61. Suppose $p(x)$ is a polynomial with integer coefficients. The remainder when $p(x)$ is divided by $x - 1$ is 2 and remainder when $p(x)$ is divided by $x - 4$ is 11. If $r(x)$ is the remainder when $p(x)$ is divided by $(x-1)(x-4)$, find the value of $r(2013)$.

Solution: Given $p(x) = (x-1)\,g(x) + 2$

$\Rightarrow$ $p(1) = 2$ and $p(x) = (x-4)\,h(x) + 11$

$\Rightarrow$ $p(4) = 11$

Further let $s(x)$ and $r(x)$ be the quotient and remainder when $p(x)$ is divided by $(x-1)(x-4)$ and let $r(x) = ax + b$

$\Rightarrow$ $p(x) = (x-1)(x-4)\,s(x) + r(x)$

$\Rightarrow$ $p(1) = a + b = 2$ ...(1)

and $p(4) = 4a + b = 11$ ...(2)

Using (1) and (2), we get $a = 3$, $b = -1$

$\therefore$ $r(x) = ax + b$

$\Rightarrow$ $r(x) = 3x - 1$

$\therefore$ $r(2013) = 3(2013) - 1 = 6039 - 1 = 6038$.

62. Find the minimum number of roots of $f(x) = f\left(\dfrac{x+4}{x-2}\right)$.

Solution: Given $f(x) = f\left(\dfrac{x+4}{x-2}\right)$

One of the possibilities is $x = \dfrac{x+4}{x-2}$

$\Rightarrow$ $x^2 - 2x - x - 4 = 0$ $\Rightarrow$ $x^2 - 3x - 4 = 0$

Disc. $= 9 + 16 = 25 > 0$

$\therefore$ Minimum number of roots $= 2$.

63. If $f(2x+1) = 4x^2 + 14x$, then find the sum of the squares of roots of the equation $f(x) = 0$

Solution: Given $f(2x+1) = 4x^2 + 14x$ ...(1)

we are to find the sum of squares of roots of equation $f(x) = 0$. Putting $2x + 1 = y$

$$\Rightarrow x = \frac{y-1}{2} \text{ in (1), we have } f(y) = 4\left(\frac{y-1}{2}\right)^2 + 14\left(\frac{y-1}{2}\right)$$

$\Rightarrow$ $f(y) = y^2 + 1 - 2y + 7y - 7$

$\Rightarrow$ $f(y) = y^2 + 5y - 6$

$\therefore$ $f(x) = 0$

$\Rightarrow$ $x^2 + 5x - 6 = 0$ $\Rightarrow$ $(x+6)(x-1) = 0$

$\Rightarrow$ $x = -6$ or $x = 1$ $\therefore$ $\alpha^2 + \beta^2 = 36 + 1 = 37$

64. If $\alpha = e^{\frac{2\pi i}{13}}$ and $f(x) = 7 + \sum_{k=1}^{50} A_k x^k$, then find the value of $\left(\dfrac{1}{13}\right)\sum_{r=0}^{12} f(\alpha^r x)$.

Solution: Clearly α is a non-real complex root of equation $x^{13} = 1$.

But α is non-real complex, α is a root of $1 + x + x^2 + \ldots + x^{12} = 0$

Now $\sum_{r=0}^{12} f(\alpha^r x) = f(x) + f(\alpha x) + f(\alpha^2 x) + \ldots + f(\alpha^{12} x)$

$$= \left(7 + \sum_{k=1}^{50} A_k x^k\right) + \left(7 + \sum_{k=1}^{50} A_k \alpha^k x^k\right) + \ldots + \left(7 + \sum_{k=1}^{50} A_k \alpha^{12k} x^k\right)$$

$$= 91 + \sum_{k=1}^{50} A\, x^k [1 + \alpha^k + \ldots + \alpha^{12k}]$$

$$= 91 + \sum_{k=1}^{50} A_k x^k \frac{1 - (\alpha^k)^{13}}{1 - \alpha^k} = 91 + \sum_{k=1}^{50} A_k x^k \frac{(1 - \alpha^{13k})}{(1 - \alpha^k)}$$

$= 91 + 0$ ($\because$ α is a root of $x^{13} = 1 \Rightarrow \alpha^{13} = 1$

$\Rightarrow$ $\alpha^{13k} - 1 = 0$)

$\therefore$ $\dfrac{1}{13}\sum_{r=0}^{12} f(\alpha^r x) = 7$

65. If $f(x)$ is a function such that $\dfrac{f(x-1) + f(x+1)}{2f(x)} = \sin 60°$ and $f(5) = 100$, find $\sum_{r=0}^{99} f(5 + 12r)$

Solution: Given $f(x-1) + f(x+1) = 2f(x) \sin 60°$

$\Rightarrow$ $f(x-1) + f(x+1) = \sqrt{3}\, f(x)$ .....(1)

From equation (1)

$f(x+3) + f(x+1) = \sqrt{3}\, f(x+2)$ (replacing x by $x+2$) ...(2)

Adding equations (1) and (2), we get

$f(x-1) + f(x+3) + 2f(x+1) = \sqrt{3}\,\{f(x) + f(x+2)\}$

$= \sqrt{3} \times \sqrt{3}\, f(x+1)$; (from (1) putting $(x+1)$ in place of x)

$= 3\, f(x+1)$

$\therefore$ $f(x-1) + f(x+3) = f(x+1)$ ....(3)

and $f(x+1) + f(x+5) = f(x+3)$ ...(4)

Adding (3) and (4), we get

$f(x-1) + f(x+1) + f(x+3) + f(x+5) = f(x+1) + f(x+3)$

$\Rightarrow$ $f(x-1) + f(x+5) = 0$ $\Rightarrow$ $f(x-1) = -f(x+5)$

$\Rightarrow$ $f(x) = -f(x+6)$ ....(5)

From equation (5) $f(x) = -[-f(x+12)]$

$\Rightarrow$ $f(x+12) = f(x)$

Hence, $f(x)$ is a periodic function with period 12

Now, $\sum_{r=0}^{99} f(5+12r)$ = $f(5)$ + $f(5 + 12)$ + $f(5 + 2.12) + ... + f(5 + 99.12)$
$= f(5) + f(5) + f(5) + ...$ to 100 terms
$= 100f(5) = 100 \times 100 = 10000.$

66. Find the natural number a for which $\sum_{k=1}^{n} f(a+k) = 2n(33+n)$; where the function f satisfies the relation $f(x + y) = f(x) + f(y)$ for all natural numbers x, y and further $f(1) = 4$.

Solution: Since given $f(x + y) = f(x) + f(y)$ for all natural numbers x, y and $f(1) = 4$,
Therefore $f(n) = f(\underbrace{1+...+1}_{n \text{ times}})$

$= \underbrace{f(1)+f(1)+...+f(1)}_{n \text{ times}} = 4n$,for every natural number n.

$\Rightarrow f(n) = 4n$ for $n \in \mathbb{N}$

The given relation can be written as

$\sum_{k=1}^{n} 4(a+k) = 2n(33+n)$

$\Rightarrow 4an + 2n(n + 1) = 2n(33 + n)$
$\Rightarrow 4an + 2n^2 + 2n = 66n + 2n^2 \Rightarrow 4an = 64n$
$\Rightarrow a = 16$

67. Find the natural number a for which $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$; where the function f satisfies the relation $f(x + y) = f(x).f(y)$ for all natural numbers x, y and further $f(1) = 2$

Solution: $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$

$\Rightarrow f(a + 1) + f(a + 2) + ... + f(a + n) = 16\ (2^n - 1).$

Now $f(x + y) = f(x).f(y)$ and $f(1) = 2$ (given)

$\Rightarrow f(2) = f(1 + 1) = f(1).f(1) = (f(1))^2 = (2)^2; f(3) = f(2 + 1) = f(2).f(1) = (2)^2(2) = (2)^3.$

$\Rightarrow f(n) = 2^n\ \forall\ n \in \mathbb{N}.$

Now, $f(a + 1) + f(a + 2) + ... + f(a + n)$
$= 2^a[2 + 4 + ... + 2^n] = 2^a.2(2^n - 1) = 16(2^n - 1)$

$\Rightarrow 16 = 2^{a+1} \qquad \Rightarrow a = 3$

68. $f(x)$ and $g(x)$ are linear functions such that for all x, $f(g(x))$ and $g(f(x))$ are identity functions. If $f(0) = 9$ and $g(5) = 8$, then compute $f(-2012)$.

Solution: Given $f(g(x))$ and $g(f(x))$ are identity functions. Therefore, $f(x)$ and $g(x)$ are inverse to each other.
Let $y = f(x) = ax + b$ be the linear function.

$\Rightarrow y = ax + b \Rightarrow x = \frac{y-b}{a} \Rightarrow f^{-1}(x) = \frac{x-b}{a}$

$\Rightarrow g(x) = \frac{x-b}{a}$

Now $f(0) = 9 \Rightarrow b = 9$ and $g(5) = 8 \Rightarrow a = \frac{-1}{2}$

$\therefore f(x) = -\frac{1}{2}x + 9 \Rightarrow f(-2012) = 606 + 9 = 615.$

69. Let $f(x, y)$ be a periodic function satisfying $f(x, y) = f(2x + 2y, 2y - 2x)$ for all x, y. Define $g(x) = f(2^x, 0)$, then find the period of function g.

Solution: Given $g(x) = f(2^x, 0) = f(2.2^x + 2.0, 2.0 - 2.2^x)$
$= f(2^{x+1}, -2^{x+1})$
$= f(2.2^{x+1} + 2(-2^{x+1}), 2(-2^{x+1}) - 2.2^{x+1})$
$= f(0, -2^{x+3}) = f(-2^{x+4}, -2^{x+4}) = f(-2^{x+6}, 0)$
$= f(-2^{x+7}, 2^{x+7}) = f(0, 2^{x+9}) = f(2^{x+10}, 2^{x+10})$
$= f(2^{x+12}, 0) = g(x + 12) \Rightarrow$ Period of g is 12.

70. Let $g(x) = \frac{e^x - e^{-x}}{2}$ and $g(f(x)) = x$, then evaluate $f\left(\frac{e^{22}-1}{2e^{11}}\right)$.

Solution: Let $y = g(x) = \frac{e^x - e^{-x}}{2}$

$\Rightarrow e^{2x} - 1 = 2ye^x$

Substituting $e^x = t$, we have $t^2 - 2yt - 1 = 0$

$\Rightarrow t = \frac{2y \pm \sqrt{4y^2+4}}{2} = y \pm \sqrt{y^2+1}$

$\Rightarrow e^x = y + \sqrt{y^2+1} \qquad (\because e^x > 0)$

$\Rightarrow x = \ln\left(y + \sqrt{y^2+1}\right)$

$\Rightarrow g^{-1}(y) = \ln\left(y + \sqrt{y^2+1}\right)$

$\therefore g^{-1}(x) = f(x) = \ln\left(x + \sqrt{x^2+1}\right)$

$(\because g(f(x)) = x \Rightarrow g^{-1}(x) = f(x))$

$$\therefore f\left(\frac{e^{22}-1}{2e^{11}}\right) = \ln\left(\frac{e^{22}-1}{2e^{11}} + \sqrt{\left(\frac{e^{22}-1}{2e^{11}}\right)^2 + 1}\right)$$

$$= \ln\left(\frac{e^{22}-1}{2e^{11}} + \sqrt{\left(\frac{e^{22}+1}{2e^{11}}\right)^2}\right) = \ln\,[e^{11}] = 11$$

71. All the values of m for which the function $f(x) = (m + 2)x^3 - 3mx^2 + 9mx - 1$ is invertible lies in the interval $(-\infty, -(\lambda + 1)] \cup [-\lambda + 2, \infty)$, then find the value of λ.

Solution: Given $f(x) = (m + 2)x^3 - 3mx^2 + 9mx - 1$

$\Rightarrow f'(x) = 3(m + 2)x^2 - 6mx + 9m$

$\Rightarrow f(x)$ will be monotonically increasing for all x if $m + 2 > 0$, $36m^2 - 108m\ (m + 2) \le 0$

or $m + 2 > 0$, $m(m + 3) \ge 0$

$\Rightarrow m \in (-2, \infty)$ and $m \in (-\infty, -3] \cup [0, \infty)$

$\Rightarrow m \in [0, \infty)$

Again $f(x)$ will be monotonically decreasing for all x if $m + 2 < 0$ and $36m^2 - 108m\ (m + 2) \le 0$

$\Rightarrow m \in (-\infty, -2)$ and $m \in (-\infty, -3] \cup [0, \infty)$

$\Rightarrow m \in (-\infty, -3]$

$\therefore f(x)$ is monotonically increasing for all x or monotonically decreasing for all x (i.e., invertible) if $m \in (-\infty, -3] \cup [0, \infty)$.

$\Rightarrow \lambda = 2$

72. Let $A = \{1, 2, 3, 4, 5,\ldots,10\}$ and $B = \{1, 2, 3, 4,\ldots, 10\}$, then find the number of functions f from A to B such $f(f(i)) = i\ \forall\ i \in A$

Solution: $f(f(i)) = i\ \forall\ i \in A$

Let $(i, j) \notin f$

$\Rightarrow f(i) = j$

$\Rightarrow f(f(i)) = f(j)$ but $f(f(i)) = i$ (given)

$\Rightarrow f(j) = i \qquad \Rightarrow (i, j)) \in f$

$\Rightarrow f$ is a self invertible function.

Let us deal with two cases as given below.

Case i: When $(i, j) \in f \Leftrightarrow (j, i) \in f$ and $i \ne j$ for every pair $(i, j) \in f$

Case ii: When $(i, j) \in f \Leftrightarrow (j, i) \in f$ and f contains pairs of the form (i, i).

Case i: When $(i, j) \in f \Leftrightarrow (j, i) \in f$ and $i \ne j$ for every pair $(i, j) \in f$

We can generate such functions by making five ordered pairs (i, j) having elements from set A such that $i \ne j$ and if (i, j), (k, l) are any two of the five ordered pairs then $i \ne k$ and $i \ne j$, which is equivalent to distribution of 10 different objects into 5 unnamed groups each group containing 2 objects

$= ({}^{10}C_2 \times {}^8C_2 \times {}^6C_2 \times {}^4C_2 \times {}^2C_2) \div (5!)$

$$= \frac{10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{2\times2\times2\times2\times2\times5!} = 945$$

Now each of the above 945 ways of forming five ordered pairs when taken with the same five pair with their elements interchanged would represent a symmetric function.

A self invertible function not intersecting the line $y = x$. We get 945 functions one of which is $\{(2, 3), (4, 5), (7, 8), (6, 9), (1, 10), (3, 2), (5, 4), (8, 7), (9, 6), (10, 1)\}$

Case ii: When $(i, j) \in f \Leftrightarrow (j, i) \in f$ and f contains pairs of the form (i, i).

If the collection contains only one pair (i, i) and rest of the type (i, j), (j, i); $i \ne j$, then we can't have such function, as leaving 1 element i, we are left with 9 elements (odd) and it is impossible to distribute all of then into graphs each containing two elements similarly we can't have functions having 3 pairs of the form (i, i).

Thus, we can form functions having exactly 2 or 4 or 6 or 8 or 10 pairs of the form (i, i).

Sub-case (a) When functions have exactly two pairs of the form (i, i). Two such pairs can be selected in ${}^{10}C_2$ ways and the remaining 8 elements can be distributed among 4 pairs in $\left\{\frac{1}{4!}\left({}^8C_2 \times {}^6C_2 \times {}^4C_2 \times {}^2C_2\right)\right\}$ ways, so total number of functions in this case would be

$$= \frac{{}^{10}C_2 \times {}^8C_2 \times {}^6C_2 \times {}^4C_2 \times {}^2C_2}{4!}$$

$$= \frac{10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{2\times2\times2\times2\times2\times4!} = 4725.$$

Sub-case (b) When functions contains 4 pairs of the form (i, i). The number of function in this case would be ${}^{10}C_4 \times \frac{{}^6C_2 \times {}^4C_2 \times {}^2C_2}{3!}$

$$= \frac{10\times9\times8\times7}{4!} \times \frac{6\times5\times4\times3\times2\times1}{2\times2\times2\times3!} = 3150$$

Sub-case (c) When functions contain 6 pairs of the form (i, i). The number of function in this case would be

$$= {}^{10}C_6 \times \frac{{}^4C_2 \times {}^2C_2}{2!} = {}^{10}C_4 \times 3$$

$$= \frac{10\times9\times8\times7}{4\times3\times2\times1} \times 3 = 630$$

Sub-case (d) When functions contain 8 pairs of the form (i, i). The number of function in this case would be ${}^{10}C_8 \times {}^2C_2 = {}^{10}C_2 = 45$

Sub-case (e) When functions contains 10 pairs of the form (i, i). There will be only one function, i.e., identity function.

Thus, total number of possible self invertible functions from A to B

$= 945 + 4725 + 3150 + 630 + 45 + 1 = 9496$

TUTORIAL EXERCISE

SECTION–III

(ONY ONE CORRECT ANSWER)

1. If $2f(x^2)+3f\left(\frac{1}{x^2}\right)=x^2-1$ for all $x \in \mathbb{R} \sim \{0\}$. Then $f(x^2)$ is equal to.

(a) $\frac{1-x^4}{5x^2}$ (b) $\frac{(1-x^2)(3+2x^2)}{5x^2}$

(c) $\frac{(1-x^2)(3+2x^2)}{3x^2}$ (d) None of these

2. The domain of definition of the function $f(x)=\sqrt{\frac{2-[x]}{[x]-3}}$ is equal to

(a) [2, 3) (b) (2, 3)

(c) (2, ∞) (d) None of these

3. The domain of definition of the function $f(x) = \sin^{-1}[2-3x^2]$ is equal to

(a) [–1, 1] (b) [0, 1]

(c) [–1, 0) ∪ (0, 1] (d) None of these

4. The real function $f(x) = \cos^{-1}\sqrt{x^2+3x+1}+\cos^{-1}\sqrt{x^2+3x}$ is defined on the set.

(a) {0, 3} (b) interval (0, 3)

(c) {0, –3} (d) interval [–3, 0]

5. Find which of the following statements is/are true $f(x)=\sin^{-1}x+\sqrt{2x-x^2}-\frac{1}{\sqrt{8x-4x^2-3}}$.

(a) Domain is [0, 2] (b) Domain is $\left[-1,\frac{1}{2}\right]$

(c) Domain is $\left(\frac{1}{2},1\right]$ (d) None of these

6. The domain of the function $f(x)=\frac{1}{\sqrt{1-\sqrt{1-\sqrt{1-x^2}}}}$ is

(a) {x: x < 1} (b) (–1, 1)

(c) [–1, 1] (d) None of these

7. The domain of the function $f(x)=\sqrt{3-2^x-2^{1-x}}+\sqrt{\sin^{-1}x}$ is given by.

(a) $\left[0,\frac{1}{2}\right]$ (b) $\left(\frac{1}{2},1\right]$

(c) [0, 1] (d) None of these

8. The domain of definition of $f(x) = \cos^{-1}x + \sqrt{1-log_3(2x^2+6x-5)}$ is equal to

(a) (0, 1] (b) $\left(\frac{1}{2},1\right]$

(c) $\left(\frac{\sqrt{19}-3}{2},1\right]$ (d) None of these

9. The domain of the function $f(x)=\cos^{-1}\left(\frac{3}{4+2\sin x}\right)$ is equal to.

(a) $\bigcup_{n\in\mathbb{Z}}\left((2n+1)\pi+\frac{\pi}{6},(2n+2)\pi-\frac{\pi}{6}\right)$

(b) $\left(-\frac{\pi}{6}+2n\pi,\frac{\pi}{6}+2n\pi\right)$

(c) $\left(-\frac{\pi}{6}+2n\pi,\frac{\pi}{6}+2n\pi\right]$

(d) None of these

10. The domain of definition of function $\frac{1}{\sqrt{x^2-x-2}}+\frac{\sin^{-1}\left(\log_2\left(\frac{x^2}{2}\right)\right)}{\sqrt{5|x|-x^2-6}}$ is

(a) [–2, –1) (b) [–2, 1) ∪ [1, 2]

(c) ϕ (d) None of these

11. The domain of definition of the function $f(x)=e^x+\sin^{-1}\left[\left(\frac{x}{2}\right)-1\right]+\log\sqrt{x-[x]}$ is equal to; [] is gint function in 2nd and 3rd term.

(a) (0, 6) ~ {1, 2, 3, 4, 5}

(b) (0, 5)

(c) (0, 6)

(d) None of these

12. The domain of definition of the function $f(x) = \log\left\{\dfrac{1}{([\cos x]-[\sin x])}\right\}$ is equal to

(a) $\bigcup\limits_{n\in\mathbb{Z}} [2n\pi, (2n+1)\pi]$

(b) $\bigcup\limits_{n\in\mathbb{Z}}\left[(2n+1)\pi+\dfrac{\pi}{2}, (2n+2)\pi\right]$

(c) $\bigcup\limits_{n\in\mathbb{Z}} [(2n+1)\pi, (2n+2)\pi]$

(d) None of these

13. The set of all real numbers x for which $\log_{2004}(\log_4(\log_{2002}(\log_{2001} x)))$ is defined as $\{x | x > c\}$. The value of c is

(a) 0 (b) $(2001)^{2002}$

(c) $(2003)^{2004}$ (d) None of these

14. The domain of the function $f(x) = \sqrt{\sin^{-1}(\log_2 x)} + \sqrt{\cos(\sin x)} + \sin^{-1}\left(\dfrac{1+x^2}{2x}\right)$

(a) $\{x : 1 \le x \le 2\}$

(b) $\{1\}$

(c) not defined for any value of x

(d) $\{-1, 1\}$

15. The domain of the function $f(x) = \sin^{-1}\left(\dfrac{2-|x|}{4}\right) + \cos^{-1}\left(\dfrac{2-|x|}{4}\right) + \tan^{-1}\left(\dfrac{2-|x|}{4}\right)$ is.

(a) [0, 3] (b) [–6, 6]

(c) [–1, 1] (d) [–3, 3]

16. The least value of x for which $f(x) = \dfrac{\sqrt{x^2-4}}{\sin^{-1}(2-x)}$ is defined is

(a) $x = 1$ (b) $x = 2$

(c) $x = 3$ (d) Does not exist

17. If P is a point on a circle of radius a and centre O (origin) the distance of chord AB from P lying along OP is x, then $\angle AOB \le \pi$ written as a function of x has which of the following property?

(a) $2\cos^{-1}\left(\dfrac{1}{a}|x|\right); 0 < x < 2a$

(b) $2\sin^{-1}\left(\dfrac{1}{a}|x-a|\right); 0 < x < 2a$

(c) $2\cos^{-1}\left(\dfrac{1}{a}|x-a|\right); 0 < x < 2a$

(d) None of these

18. Let S be the set of all triangles and $\mathbb{R}^+$ be the set of positive real numbers. Then the function $f: S \to \mathbb{R}^+$, $f(\Delta)$ = area of the Δ, where $\Delta \in S$ is

(a) Injective but not surjective

(b) Surjective but not injective

(c) Injective as well as surjective

(d) Neither injective nor surjective

19. The domain of the real valued function $f(x)$ for which $4^{f(x)} + 4^{1-f(x)} = 4^x$ is

(a) (–1, 1] (b) [1, ∞)

(c) (–∞, 1] (d) (–∞, – 1]

20. Let $f(x) = (x^{12} - x^9 + x^4 - x + 1)^{-1/2}$. The domain of the function is

(a) (–∞, –1) (b) (–1, 1)

(c) (1, ∞) (d) (–∞, ∞)

21. Domain of $\sin^{-1}[\sec x]$ ([.] is greatest integer less than or equal to x) is

(a) $\{(2n + 1)\pi, (2n + 9)\pi\} \cup \{[(2m - 1)\pi, 2m\pi + \pi/3), m \in \mathbb{Z}\}$

(b) $(2n\pi, n \in \mathbb{Z}\} \cup \{[2m\pi, (2m + 1)\pi, m \in \mathbb{Z}\}$

(c) $\{(2n + 1)\pi, n \in \mathbb{Z}\} \cup \{(2m\pi - \pi/3, 2m\pi + \pi/3], m \in \mathbb{Z}\}$

(d) none of these

22. The range of function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x^3 - 50000x^2 + 0.1x + \sin x$ is

(a) $\mathbb{R}$ (b) [–1, 1]

(c) [0, ∞) (d) None of these

23. The range of the function $f(x) = \log_{\sqrt{2}}(2 - \log_2(16\sin^2 x + 1))$ is

(a) (–∞, 1) (b) (–∞, 2)

(c) (–∞, 1] (d) (–∞, 2]

24. Range of $f(x) = \dfrac{1}{\sqrt{\cos(\sin x)}}$ is

(a) $[1, \sqrt{\sec 1}]$ (b) $\left[\sqrt{\cos 1}, \sqrt{\sec 1}\right]$

(c) $\mathbb{R} - \{0\}$ (d) None of these

25. Let $f(x) = [9^x - 3^x + 1]\ \forall\ x \in (-\infty, 1)$, then range of $f(x)$ is ([.] denotes the greatest integer function.)

(a) {0, 1, 2, 3, 4, 5, 6} (b) {0, 1, 2, 3, 4, 5, 6, 7}

(c) {1, 2, 3, 4, 5, 6} (d) {1, 2, 3, 4, 5, 6, 7}

26. The range of the function $f(x) = 6^x + 3^x + 6^{-x} + 3^{-x} + 2$ is

(a) [–2, ∞) (b) (–2, ∞)

(c) (6, ∞) (d) [6, ∞)

27. The image of interval [–1, 2] under the mapping $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x^3 - 6x$ is

(a) [-4, 4) (b) (-4, 4)
(c) [-4, 4] (d) (-4, 4]

28. If $2 < x^2 < 3$, then the number of positive roots of $\{x^2\} = \left\{\frac{1}{x}\right\}$, (where $\{x\}$ denotes the fractional part of x) is

(a) 0 (b) 1
(c) 2 (d) 3

29. The minimum value of the function $f(x) = \frac{\sin x}{\sqrt{1-\cos^2 x}} + \frac{\cos x}{\sqrt{1-\sin^2 x}} + \frac{\tan x}{\sqrt{\sec^2 x - 1}} + \frac{\cot x}{\sqrt{\operatorname{cosec}^2 x - 1}}$ is

(a) 4 (b) –2
(c) 0 (d) 2

30. Solution of the equation $x^3 - [x] = 3$ is (where $[x]$, is the greatest integer $\le x$)

(a) $x = (3)^{1/4}$ (b) $x = (4)^{1/3}$
(c) $x = \sqrt{2}$ (d) None of these

31. Solution set of the equation $(x)^2 + (x + 1)^2 = 25$ is (where (x) represent least integer $\ge x$)

(a) (–5, 2) (b) (2, 3)
(c) (–5, –4] ∪ (2, 3] (d) None of these

32. Solution set of the inequation $[x]^2 + (x)^2 > 25$ is (where $[x]$, (x),} is the greatest integer $\le x$, least integer $\ge x$, respectively)

(a) $(-\infty, -4] \cup [4, \infty)$ (b) $(-\infty, -4]$
(c) $[4, \infty)$ (d) None of these

33. If the function $f(x) = [3.5 + b \sin x]$ (where [.] denotes the greatest integer function) is an even function, then complete set of values of b is:

(a) (– 0.5, 0.5) (b) [–0.5, 0.5]
(c) (0, 1) (d) [–1, 1]

34. The domain of definition of the function $f(x) = \log\sqrt{(10.3^{x-2} - 9^{x-1} - 1)} + \sqrt{\cos^{-1}(2-x)}$ is

(a) [1, 2] (b) [1, 2)
(c) (0, 2) (d) None of these

35. Let

$$f(x) = \sqrt{\sqrt{\cot(5+3x)(\cot(5)+\cot(3x))} - \sqrt{\cot 3x + 1}},$$

then the domain of function $f(x)$ is

(a) $\mathbb{R} - \bigcup_{n\in\mathbb{Z}}\left[\frac{n\pi}{3} + \frac{1}{3}\cot^{-1}\left(\frac{2}{\cot 5 - 1}\right), \frac{1}{3}\left(n + \frac{3}{4}\right)\pi\right]$

(b) $\bigcup_{n\in\mathbb{Z}}\left[\frac{n\pi}{3} + \frac{1}{3}\cot^{-1}\left(\frac{2}{\cot 5 - 1}\right), \frac{1}{3}\left(n + \frac{3}{4}\right)\pi\right)$

(c) $\mathbb{R} - \left\{\frac{n\pi}{3}, \frac{n\pi - 5}{3}\right\}, n \in \mathbb{Z}$

(b) None of these

36. If $f(x) = \log_{100x}\left(\frac{2\log_{10} x + 2}{-x}\right)$; $g(x) = \{x\}$; where $\{x\}$ denotes the fractional part of x. If the function $(fog)(x)$ exists, then the maximum possible range of $g(x)$ is

(a) $(0, 10^{-1})$
(b) $(0, 10^{-2})$
(c) $(0, 10^{-2}) \cup (10^{-2}, 10^{-1})$
(d) None of these

37. If $f(x) \cdot f(y) = f(x) + f(y) + f(xy) - 2\ \forall\ x, y \in \mathbb{R}$ and if $f(x)$ is not a constant function, then the value of $f(1)$ is

(a) 1 (b) 2
(c) 0 (d) –1

38. If a function $f(x)$ is such that $f\left(\frac{2\tan x}{1+\tan^2 x}\right) = \frac{(\cos 2x + 1)(\sec^2 x + 2\tan x)}{2}$, then the value of $f\left(\frac{1}{3}\right)$ equals

(a) $\frac{1}{3}$ (b) $\frac{4}{3}$
(c) $\frac{4}{9}$ (d) None of these

39. Let f and g be two functions defined from $\mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} 0 \text{ if } x \text{ is rational} \\ x \text{ if } x \text{ is irrational} \end{cases} \text{ and}$$

$$g(x) = \begin{cases} 0 \text{ if } x \text{ is irrational} \\ x \text{ if } x \text{ is rational} \end{cases}$$, then the function $f - g$

(a) one–one but not onto
(b) onto–but not one–one
(c) both one–one and onto
(d) neither one–one nor onto

40. Let T be the set of all 2 × 2 matrices with entries from the set of real numbers $\mathbb{R}$. Then the function $f: T \to \mathbb{R}$ defined $f(A) = |A|$, i.e., (det. A) for every $A \in T$, is

(a) one–one and onto
(b) neither one–one nor onto
(c) one–one but not onto
(d) onto but not one–one

41. Let $n(A) = 4$ and $n(B) = 6$. Then the number of one–one functions from $A \to B$, and number of onto functions from $B \to A$ can be given as
(a) 120, 1020 (b) 360, 1560
(c) 24, 120 (d) None of these

42. Let $A = \{1, 2, 3, 4, 5, 6\}$. If f be a bijective function from A to A, then the number of such functions for which $f(\lambda) \neq \lambda, \forall \lambda \in A$ is
(a) 44 (b) 265
(c) 325 (d) 4585

43. Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = (x + 1)^2 - 1, x \geq -1$. The set $S = \{x : f(x) = f^{-1}(x)\}$ is given by
(a) $\{0, 1\}$ (b) $\{0, -1\}$
(c) $\{-1, 1\}$ (d) None of these

44. Let $h(x) = |kx + 5|$ and domain of $f(x)$ and $f(h(x))$ are $[-5, 7]$ and $[-6, 1]$ respectively, then value of k is
(a) 1/3 (b) 4/5
(c) 1 (d) None of these

45. If $f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x . \cos\left(x + \frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right) = 1298$, then 4 $gof(x)$ is equal to
(a) 1298 (b) 2596
(c) 5192 (d) Not defined

46. Let $f(x) = 1 + x^2$ and g be a function such that $f(g(x)) = 1 + x^2 - 2x^3 + x^4$. The value of $g(18)$ is
(a) ±306 (b) 304
(c) –304 (d) None of these

47. The functions $f(x) = \cos^{-1}\sqrt{1 - x^2}$ and $g(x) = -\sin^{-1} x$ are identical for x belonging to
(a) $[-1, 1]$ (b) $[0, 1]$
(c) $[-1, 0]$ (d) None of these

48. Let the function $f(x) = x^2 + x + \sin x - \cos x + \log(1 + |x|)$ be defined on the interval $[0, 1]$. The function $g(x)$ on $[-1, 1]$ satisfying $g(-x) = -f(x)$ is
(a) $x^2 + x + \sin x + \cos x - \log(1 + |x|)$
(b) $-x^2 + x + \sin x + \cos x - \log(1 + |x|)$
(c) $-x^2 + x + \sin x + \cos x + \log(1 + |x|)$
(d) None of these

49. If $f[x]$ and $\{x\}$ represent integral and fractional part of x, then the function defined by $f(x) = [x] + \sum_{r=1}^{1000} \frac{\{x + r\}}{1000}$ equal to.
(a) $2[x] + \{x\}$ (b) $4x$
(c) x (d) $4[x] + 100\{x\}$

50. Let X and Y be two non-empty sets. Let $f: X \to Y$ be a function. For $A \subset X$ and $B \subset Y$, define $f(A) = \{f(x) : x \in A\}$; $f^{-1}(B) = \{x \in X : f(x) \in B\}$, then
(a) $f(f^{-1}(B)) = B$ (b) $f(f^{-1}(B)) \subset B$
(c) $f^{-1}(f(A)) = A$ (d) $f^{-1}(f(A)) \subset A$

51. If $f: \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2 + 1$, then the number of elements in $\{f^{-1}(17)\} \cup \{f^{-1}(3)\}$ is
(a) 2 (b) 4
(c) 6 (d) None of these

52. The set of all integer values of n for which the function $f(x) = \cos nx . \sin\frac{5x}{n}$ is periodic with period 2π is
(a) $\{1, 5, 10\}$ (b) $\{1, 5\}$
(c) $(\pm 1, \pm 5\}$ (d) None of these

53. If $f(x) = 1 - x^3 - x^4 - 2x^5 = g(x) + h(x)$, where g is an even function and h is an odd function. Then $h(-5)$ is equal to
(a) 6375 (b) 3250
(c) 6125 (d) None of these

54. Let $f : \mathbb{R} \to \mathbb{R}$; where $f(x) = \left(\frac{ax + 5}{x^2 + 2}\right)$, then the values of a for which the function is invertible is.
(a) $(0, \infty)$ (b) $(1, \infty)$
(c) $(0, 1)$ (d) None of these

55. The period of the function $f(x) = \frac{1}{2}[\cos(\sin 4x) + \cos(\cos 4x)]$ is
(a) π (b) $\frac{\pi}{2}$
(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{8}$

56. Let $f(x) = \cos\sqrt{k}\, x$; where $k = [m]$ = the greatest integer $\leq m$, if the period of $f(x)$ is π, then
(a) $m \in [4, 5)$
(b) $m = 4, 5$
(c) $m = [4, 5]$
(d) None of these

57. If $f(x)$ and $g(x)$ are periodic functions with periods 7 and 11 respectively, then the period of $F(x) \quad f(x)g\left(-\right) - g(x)f\left(\frac{x}{3}\right)$ is

(a) 177 (b) 222
(c) 433 (d) 1155

58. The period of $\sin(x + 4x + 9x + \ldots + (n + 1)^2x)$ is $\frac{\pi}{7}$, then $n \in \mathbb{N}$ is equal to

(a) 2 (b) 3
(c) 4 (d) None of these

59. The function $f(x) = 2(x - [x]) + \sin 2\pi (x - [x])$ is (where [] denotes greatest integer function)

(a) non-periodic
(b) periodic with period 1
(c) periodic with period 2
(d) None of these

60. If $f(x) = 2\cot 3x + 5\sqrt{1 - \cos 6x}$ and $g(x)$ is a function having the same period as that of $f(x)$, then which of the following can be $g(x)$?

(a) $(\sec^2 3x + \operatorname{cosec}^2 3x)$
(b) $2 \sin 3x + 3 \cos 3x$
(c) $2\sqrt{1 - \cos^2 3x} + \operatorname{cosec} 6x$
(d) None of these

61. Let $f(x) = x(1 - x)$, $0 \le x \le 1$, and $f(x)$ is extended for $x \in \mathbb{R}$ by $f(x + 1) = f(x)$, then f is periodic with period

(a) 1/2 (b) 1
(c) π (d) None of these

62. If $f(x) = \begin{cases} x^2 \sin\left(\frac{\pi x}{2}\right); & \text{when } |x| < 1 \\ x|x|; & \text{when } |x| \ge 1 \end{cases}$, then $f(x)$ is

(a) an even function
(b) an odd function
(c) a periodic function
(d) None of these

63. Let the function $f(x) = x^2 - 1$ and $g(x) = x^3 - 7x^2 + 5x + 6$ be two curves, then number of pairs of points (P, Q) where P lies on $f(x)$ and Q on $g(x)$, such that tangent at P and Q are parallel to each other is equal to

(a) one (b) two
(c) zero (d) infinite

64. Let $f''(x) > 0\ \forall\ x \in \mathbb{R}$ and $g(x) = f(x^2) - f(x^2 + 3)$, then $g(x)$ is

(a) increasing in $(0, \infty)$
(b) decreasing in $(0, \infty)$
(c) increasing in $\mathbb{R}$
(d) decreasing $[-1, 2]$

65. The domain of the function $f(x) = \dfrac{x^3 - 3x^2 + 3x - 7}{\sqrt{(x^{14} - x^{11} + x^{10} - x^7 + x^6 - x^3 + x^2 + 1)}}$ is

(a) $x \in (-\infty, \infty)$
(b) $x \in (-\infty, 0] \cup [1, \infty)$
(c) $(0, \infty)$
(d) None of these

66. The domain of the function $f(x) = \log_{(x^2+x+1)}(3x^2 - 4x + 5)$ is

(a) $\mathbb{R}$
(b) $\mathbb{R} - \{0, -1\}$
(c) [0, 1]
(d) None of these

67. The domain of the function: $f(x) = \sqrt{8 - 3^{x+1} - 3^{1-x}} + \sqrt{\sin^{-1}(2x + 1)}$ is

(a) [0, 1]
(b) (–1/2, 1)
(c) [–1/2, 0]
(d) None of these

68. If the range of the function $f(x) = \dfrac{3\cos^2 x + 3\cos x + 4}{\cos^2 x + \cos x + 1}$ is $[\lambda, \mu]$, then the value of $6\lambda + 9\mu + 2$ is

(a) 23 (b) 20
(c) 61 (d) None of these

69. The quadratic polynomial $f(x) \equiv x^2 - px + q$ has prime zeroes (roots). If $p + q = 11$ and $a = q - p$, then $f(a)$ equals

(a) 3 (b) 2
(c) 7 (d) None of these

70. The maximum vertical distance d between the parabola $y = -x^2 + 5x + 7$ and line $y = 2x + 3$ throughout the region bounded between two curves is

(a) $\frac{5}{2}$ (b) $\frac{25}{9}$
(c) $\frac{25}{4}$ (d) None of these

SECTION–IV

(MORE THAN ONE CORRECT ANSWER)

1. Which of the following functions are periodic?

(a) $sgn(e^{-x})$

(b) $f(x)=\begin{cases} 1 \text{ if } x \text{ is a rational number} \\ 0 \text{ if } x \text{ is an irrational number} \end{cases}$

(c) $f(x)=\sqrt{\dfrac{8}{1+\cos x}+\dfrac{8}{1-\cos x}}$

(d) $\left[x+\frac{1}{2}\right]+\left[x-\frac{1}{2}\right]+2[-x]$; (where [] denotes greatest integer function)

2. Let $f(x) = [x]^2 + [x + 1] - 3$, (where $[x]$ = greatest integer $\le x$), then

(a) $f(x)$ is a many–one and into function

(b) $f(x) = 0$ for infinite number of values of x

(c) $f(x) = 0$ for only two real values of x

(d) None of these

3. Which of the following is true about the function $f(x)=\dfrac{x^2-3x+2}{x^2+x-6}$?

(a) Domain is $\mathbb{R} - \{2, -3\}$

(b) Domain is $\mathbb{R} - \{-3\}$

(c) Range is $\mathbb{R} - \{1\}$

(d) Range is $\mathbb{R}-\left\{\frac{1}{5}, 1\right\}$

4. Which of the statements given below is/are true about $f(x)=\sqrt{2-x}+\sqrt{1+x}$?

(a) Domain is $(-1, 2)$

(b) Range is $[\sqrt{3}, \sqrt{6})$

(c) Domain is $[-1, 2]$

(d) Range is $[\sqrt{3}, \sqrt{6}]$

5. Which of the following is true regarding the relation $\log_x y.\log_{xy} y.\log_{x^2 y} y=\frac{1}{6}$?

(a) It can be written in explicit form as $(x^2 - y^2)(x^2 + y^2 + xy) = 0$

(b) It can be written explicitly as $y = x$

(c) Its domain is $(0, 1) \cup (1, \infty)$

(d) Its domain is $(1, \infty)$

6. If $f(x)=(1+\tan x)\left\{1+\tan\left(\frac{\pi}{4}-x\right)\right\}$ and let $g(x)$ be defined for all real x, then which of the following statements is/are true for $gof(x)$?

(a) domain of $gof(x)$ is $\mathbb{R}$

(b) $gof(x)$ is constant $\forall\ x \in D_f$

(c) $gof(x)$ is monotonically increasing function

(d) $gof(x)$ is non-surjective function

7. The function $f(x)=\sqrt{x}\sqrt{x-1}$ and $g(x)=\sqrt{x(x-1)}$ are

(a) equal function $\forall\ x \in \mathbb{R}$

(b) identical function in their common domain

(c) identical functions for $x \in [1, \infty)$

(d) None of these

8. Let $g(x)$ be a function defined on $[-1, 1]$. If the area of the equilateral triangle with two of its vertices at $(0, 0)$ and $\{x, g(x)\}$ is $\frac{\sqrt{3}}{4}$, then the function $g(x)$ is

(a) $\pm\sqrt{1+x^2}$ (b) $\sqrt{1-x^2}$

(c) $-\sqrt{1-x^2}$ (d) $\sqrt{1+x^2}$

9. Let $f(x) = \sec^{-1}[1 + \cos^2 x]$, where [.] denotes the greatest integer function, then

(a) the domain of f is $\mathbb{R}$

(b) the domain of f is $[1, 2]$

(c) the range of f is $[1, 2]$

(d) the range of f is $\{0, \pi/3\}$.

10. If the function $f(x)=\sin^{-1}\dfrac{2x}{1+x^2}$ and $g(x)=\cos^{-1}\dfrac{1-x^2}{1+x^2}$ are identical functions, then domain D_f and range R_f are given by

(a) $D_f = [0, 1]$ (b) $R_f=\left[0,\frac{\pi}{4}\right]$

(c) $D_f=\left[0,\frac{1}{2}\right]$ (d) $R_f=\left[0,\frac{\pi}{2}\right]$

11. Let X_1 be the set of values of x satisfying the equation $\sin\left(\dfrac{\cos^{-1} x}{y}\right)=1$; for $y \in Y_1 \subseteq \mathbb{Z} - \{0\}$, then

(a) $X_1 = \{-1, 1\}$ (b) $X_1 = \{-1, 0\}$
(c) $Y_1 = \{2, 1\}$ (d) $Y_1 = \{-1, 1\}$

12. Let X_2 be the set of values of x satisfying the equation $\cos\left(\dfrac{\sin^{-1} x}{y}\right) = 0$; for $y \in Y_2 \subseteq \mathbb{Z} - \{0\}$, then

(a) $X_2 = \{-1, 1\}$
(b) $X_2 = \{-1, 0, 1\}$
(c) $Y_2 = \{-1, 1\}$
(d) $Y_2 = \{-1, 1, 0\}$

13. Number of functions from set $\{1, 2, 3,\ldots,n\}$ to set $\{-1, 1\}$ that are one-to-one is
(a) 2 for $n = 1$
(b) 2 for $n = 2$
(c) ${}^nP_2$ for $n \geq 3$
(d) 0 for $n \geq 3$

14. Number of functions from set $\{1, 2, 3,\ldots,n\}$ to set $\{-1, 1\}$ that assign -1 to both 1 and n is
(a) 1 for $n = 1$
(b) $(2)^{n-2}$ for $n \geq 2$
(c) 1 for $n \geq 2$
(d) None of these

15. Number of functions from set $\{1, 2, 3,\ldots,n\}$ to set $\{-1, 1\}$ that assign 1 to exactly one of the positive integer less than n, is
(a) 0 for $n = 1$
(b) $2(n-1)$ for $n \geq 2$
(c) 2 for $n = 2$
(d) $(n-1)$ for $n \geq 1$

16. For the function $f(x) = \sqrt{\log_{1/2}(\log_5([x^2]-3))}$; ([.] denotes greatest integer function)
(a) $[x^2] \in \{5, 6, 7, 8\}$
(b) $x \in (-3, -\sqrt{5}] \cup [\sqrt{5}, 3)$
(c) $[x] \in \{-3, 2\}$
(d) Range of $f(x) = \{\sqrt{\log_{1/2}(\log_5 2)}, \sqrt{\log_{1/2}(\log_5 3)}, \sqrt{\log_{1/2}(\log_5 4)}, 0\}$

17. If $f(x)$ and $g(x)$ are two functions of x such that $f(x) + g(x) = e^x$ and $f(x) - g(x) = e^{-x}$, then
(a) $f(x)$ is an odd function
(b) $g(x)$ is an odd function
(c) $f(x)$ is an even function
(d) $g(x)$ is an even function

18. Let $f(x) = 2x + \cos x$ and $g(x) = \sqrt[3]{x}$, then
(a) range of gof is R
(b) gof is one–one
(c) both f and g are one–one
(d) both f and g are onto

19. Which of the following functions (is) are/are injective map (s)?
(a) $f(x) = |x + 1|, x \in [-1, \infty)$
(b) $g(x) = x + \dfrac{1}{x}; x \in (0, \infty)$
(c) $h(x) = x^2 + 4x - 5, x \in (0, \infty)$
(d) $k(x) = e^{-x}, x \in [0, \infty)$

20. If $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$; where $[x]$ stands for the greatest integer function, then
(a) $f(\pi/2) = -1$
(b) $f(\pi) = 1$
(c) $f(-\pi) = 0$
(d) None of these

SECTION–V

ASSERTION AND REASON-TYPE QUESTIONS

1. A: Exponential functions are defined as $y = k^x$; where $k \in \mathbb{R}^+$ and $k \neq 1$
R: The restriction that the base k for exponential functions be greater than zero is a matter of convention; we could have allowed $k < 0$ without affecting the domain or range of the function.

2. A: The inverse of a strictly increasing exponential function is a logarithmic function that is strictly decreasing
R: ℓnx is inverse of e^x

3. A: Fundamental period of sin x + tan x is 2π

R: If the period of $f(x)$ is T_1 and the period of $g(x)$ is T_2, then the fundamental period of $f(x) + g(x)$ is the LCM of T_1 and T_2.

4. A: $y = -x + \sin x$ and its inverse intersect on the line $y = -x$.

R: Let $f(x)$ be an invertible function and $f^{-1}(x)$ be the inverse of $f(x)$. Then $y = f(x)$ and $y = f^{-1}(x)$ intersect either on $y = x$ or on $y = -x + c$; where $c \in \mathbb{R}$

5. A: $f(x) = \sin x$ is periodic and $g(x) = \cos x$ is also periodic.

R: If the derivative of a function is periodic, then the function will also be periodic.

6. A: function $f(x) = \sin(x + 3\sin x)$ is periodic.

R: $f(g(x))$ is periodic if $g(x)$ is periodic.

7. A: There is no continuous onto function $f(x)$ defined on $[0, 2]$ and whose co-domain is $(1, 3)$.

R: A function defined on a closed interval attains its greatest and least values.

8. A: If $y = f(x)$ is increasing in $[\alpha, \beta]$, then its range is $[f(\alpha), f(\beta)]$.

R: Every increasing function need not to be continuous.

9. A: The greatest value of function $f(x) = \cos(x\, e^{[x]} + 2x^2 - x)$; $x \in (-1, \infty)$ is 1.

R: $\cos\theta$ has its maximum value 1 if $\theta \in (k, \infty)$ for some finite real number k.

10. A: If $f(x) = \begin{cases} \sin x;\ x \neq n\pi;\ n \in \mathbb{Z} \\ 2;\quad \text{otherwise} \end{cases}$ and

$$g(x) = \begin{cases} x^2 + 1; & x \neq 0, 2 \\ 4\,; & x = 0 \\ 5\,; & x = 2 \end{cases}, \text{ then } \lim_{x \to 0} gof(x) \text{ is } 1.$$

R: $g(f(x)) = \begin{cases} 1; & \text{for } x \neq n\pi;\ n \in \mathbb{Z} \\ 5; & \text{for } x = n\pi;\ n \in \mathbb{Z} \end{cases}$.

11. A: If $y + \cos x = \sin x$ has a real solution for given value of y, then range of y is $[-\sqrt{2}, \sqrt{2}]$

R: Range of $f(x) = a\cos x + b\sin x$ is $\left[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}\right]$.

12. A: Range of $4^x + 2^x + 4^{-x} + 2^{-x} + 3$ is $\left[\frac{3}{4}, \infty\right)$.

R: The given equation reduces to $f(z) = z^2 + z + 1$, $z = 2^x + \frac{1}{2^x}$.

13. A: The function $f(x) = x^6 + 4x^4 + 3x + 7$ defined from $\mathbb{R} \to \mathbb{R}$ is non-injective.

R: Every polynomial of even degree is an even function, and hence, is non-injective.

14. A: The function $f(x) = x^4 + 5x^2 + 3x + 1$ defined from $\mathbb{R} \to \mathbb{R}$ is non-invertible.

R: Every polynomial of even degree is many-one, that is, non-injective, and hence, non-invertible.

15. A: If $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)\ \forall\, x \in \mathbb{R}$, then $f(2) - f(1) = -6$

R: $f(0) = -6$ and $f(2) - f(1) = f(0)$.

16. A: If $A = \{1, 2, 3, \ldots, 2n\}$, then the number of self invertible functions f from $A \to A$ such that $f(i) \neq i$ is equal to $({}^{2n}C_2 \times {}^{2n-2}C_2 \times {}^{2n-4}C_2 \times \ldots \times {}^4C_2 \times {}^2C_2)/n!$.

R: Number of ways of distributing $(2n)$ different objects into n unnamed groups each having 2 elements $= \frac{(2n)!}{(2)^n . n!}$.

17. A: If f, g and h are even, odd and odd functions, respectively and each one is a polynomial such that $f(-7) = 9, f(-3) = 0, g(4) = 7, g(-12) = 9, g(2) = 3, h(-2) = -4, h(9) = 12$, then $f(g(h(2))) + g(h(f(7))) + h(f(g(2))) = 0$.

R: For every even continuous function $p(x)$ on $\mathbb{R}$, $p(0) \neq 0$, i.e., there exist even continuous functions $p(x)$ on $\mathbb{R}$ for which $p(0) \neq 0$.

18. A: There are exactly 3 integers in the range of the function $f(x) = \frac{7}{2 + 9a^x}; a > 0 \neq 1$.

R: If the range of a function is (α, β), then the number of integers in the range is $[\beta - \alpha]$; where [.] is the greatest integer function.

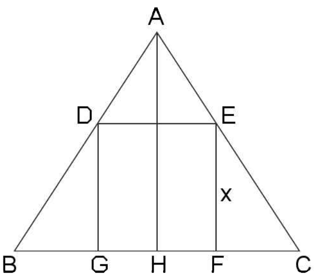
19. A: If a function f satisfies $f(x) + f(5x + y) + 6xy = f(6x - y) + 2x^2 + 1\ \forall\, x, y \in \mathbb{R}$, then $f(5) = -24$.

R: By substituting $y = \frac{x}{2}$, in the given equation, we can get analytical formula for $f(x)$ which gives us the value of $f(5)$.

SECTION–VI

LINKED COMPREHENSION-TYPE QUESTIONS

A: In the given figure, triangle ABC with base $BC = b$ and altitude $AH = h(AH \perp BC)$. If $EF = x$, where $EF \perp BC$, then the area and perimeter of the rectangle $DEFG$ can be expressed as function of x. H is mid point of BC.



1. The perimeter of rectangle $DEFG$ as function of x is given as

(a) $\dfrac{x(h-b)+bh}{x}$ (b) $2\left\{x+\dfrac{b(h-x)}{h}\right\}$

(c) $\dfrac{2b(h-x)}{x}$ (d) None of these

2. Area of rectangle $DEFG$ as function of x is given are

(a) $\dfrac{bx(h+x)}{h}$ (b) $\dfrac{bx^2}{2h}$

(c) $\dfrac{bx(h-x)}{h}$ (d) None of these

3. Values of x for which perimeter and area are numerically equal given that $b = 2$, $h = 4$

(a) $x = 4$ (b) $x = 1, 3$

(c) no real value of x (d) None of these

B: If $f : [0, 2] \to [0, 2]$ is a bijective function defined by $f(x) = ax^2 + bx + c$; where a, b, c are non-zero real numbers, then

4. $f(2)$ is equal to

(a) 2 (b) α where $a \in (0, 2)$

(c) 0 (d) cannot be determined

5. Which of the following is one of the roots of $f(x) = 0$?

(a) $\dfrac{1}{a}$ (b) $\dfrac{1}{b}$

(c) $\dfrac{1}{c}$ (d) $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$

6. Which of the following is not a value of a?

(a) $=--$ (b) $a = \dfrac{1}{2}$

(c) $a = -\dfrac{1}{2}$ (d) $a = 1$

C: Let $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying $f(2 - x) = f(2 + x)$ and $f(20 - x) = f(x)$, $\forall\, x \in \mathbb{R}$. For this function f answer the following questions.

7. If $f(0) = 5$, then minimum possible number of values of x satisfying $f(x) = 5$, for $x \in [0, 170]$, is

(a) 21 (b) 12

(c) 11 (d) 22

8. Graph of $y = f(x)$ is

(a) symmetrical about $x = 18$

(b) symmetrical about $x = 5$

(c) symmetrical about $x = 8$

(d) symmetrical about $x = 20$

9. If $f(2) \neq f(6)$, then

(a) fundamental period of $f(x)$ is 4

(b) fundamental period cannot be 16

(c) period of $f(x)$ cannot be 8

(d) fundamental period of $f(x)$ is 8

D: A tower has the following shape: a truncated right circular cone (one with radii $2R$(the lower base) and R(the upper base), and the height R, bears a right circular cylinder whose radius is R, the height being $2R$. Finally, a hemisphere of radius R is mounted on the cylinder. Suppose that the cross-sectional area S of the tower is given by $f(x)$, where x is the distance of the cross section from the lower base of cone.

10. The domain of the function $f(x)$ is

(a) $(-\infty,\infty)$ (b) $[0, 4R]$

(c) $[0, R]$ (d) $[R, 4R]$

11. For $0 \le x \le$ R, the function $f(x)$ is given by

(a) $\pi\,(2R - x)^2$ (b) $\pi\,(R - x)^2$

(c) πR^2 (d) $4\pi\, x^2$

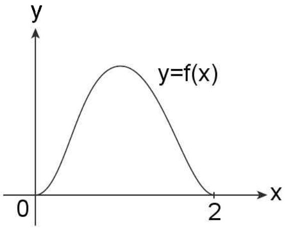
12. The range of $f(x)$ is

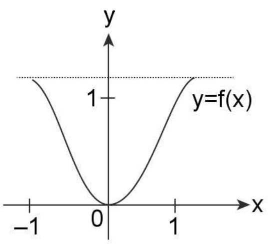
(a) $[0, 4\pi R^2]$ (b) $[\pi R^2, 4\pi R^2]$

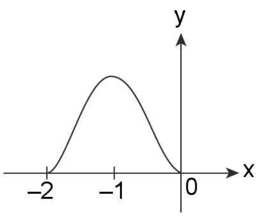
(c) $[0, \pi R^2]$ (d) $\pi R^{2,} 0, 3\pi R^2$

13. The function $f(x)$ is
(a) one–one on $[R, 2R]$
(b) one–one on $[R, 3R]$
(c) one–one on $[0, 4R]$
(d) one–one on $[0, R] \cup [3R, 4R]$

E: The accompanying figure shows the graph of a function $f(x)$ with domain $[0, 2]$ and range $[0, 1]$. Now answer the following questions:

(i)


(ii)


(iii)


14. Figure-(ii) represents the graph of the function
(a) $-f(x)$ (b) $-f(x-1)+1$
(c) $-f(x+1)-1$ (d) $-f(x+1)+1$

15. $[1, 3]$ and $[0,1]$ are the domain and range (respectively) of the function in (figure-i)
(a) $-f(x)$ (b) $f(x-1)$
(c) $-f(x+1)+1$ (d) $-f(x+1)$

16. Figure-(iii) represents the graph of function
(a) $2f(x)$ (b) $f(x-2)$
(c) $f(x+2)$ (d) $f(x-2)+1$

17. The domain and range (respectively) of
(a) $f(-x)$ are $[-2, 0]$ and $[-1, 0]$
(b) $f(x)-1$ are $[0,2]$ and $[0, 1]$
(c) $f(x)+2$ are $[0, 2]$ and $[1, 2]$
(d) $-f(x+1)+1$ are $[-1,1]$ and $[0, 1]$

F: If C_1 is a set of sets and * is an operator which can be operated on each pair of elements of C_1, then we say that.
(i) C_1 is closed w.r.t. (*) if $A * B \in C_1\ \forall A, B \in C_1$
(ii) (*) is associative in C_1 if $(A * B) * C = A * (B * C)\ \forall A, B, C \in C_1$.
(iii) (*) is commutative in C_1 if $(A * B) = (B * A)\ \forall A, B \in C_1$
(iv) C_1 has an identity element, if there exists $I \in C_1$ such that $A * I = A = I * A\ \forall A \in C_1$
(v) * is said to be invertible in C_1, if for each $A \in C_1$, there exists $B \in C_1$ such that $A * B = I = B * A$.
Then, choose the most appropriate alternative in the following

If $C = \{(a, b, c)$: a, b, c are three consecutive even whole number$\}$
Consider the operation * on C defined by $(a, b, c) * (d, e, f) = (a + d, b + e - 2, c + f - 4)$, then answer the following questions:

18. Which of the following statement is correct?
(a) C is closed w.r.t *
(b) C is not closed w.r.t. *
(c) C is closed w.r.t. * but * is not commutative in C
(d) C is closed w.r.t. * as well as * is commutative in C

19. Which of the following statement is correct?
(a) * is associative in C
(b) * is invertible in C
(c) C does not have an identity element
(d) None of these

20. Which of the following statement is correct
(a) C does not have an identity element
(b) C has an identity element
(c) * is not associative in C
(d) All of these

21. Which of the following statement is correct
(a) * is not commutative in C
(b) * is not associative in C
(c) * is not invertible in C
(d) * is not closed in C

G: Let $A = \{2, 4, 6, 8, 10, 12\}$ and $B = \{3, 7, 11\}$ be two sets. Then answer the following questions

22. Number of functions defined from set A to B is
(a) $(6)^3$ (b) $(3)^6$
(c) ${}^6C_3$ (d) ${}^6P_3$

23. Number of functions defined from A to B, such that element 3 in B has exactly 2 pre-images in A is
(a) 80 (b) 160
(c) 240 (d) None of these

24. Number of functions from A to B, such that range does not contain the element 7 is
(a) $(2)^6$ (b) $(6)^2$
(c) ${}^6C_2$ (d) ${}^6P_2$

25. Number of functions such that element 3 of B has two, 7 of B has three and 11 of B has one pre-image in B is
(a) 90 (b) 60
(c) 120 (d) None of these

26. Number of functions such that $f(2) = 7$ is
(a) $(2)^5$ (b) ${}^5C_2$
(c) $(3)^5$ (d) None of these

SECTION–VII

COLUMN MATCHING-TYPE QUESTIONS

1. Match the column

Column I

(i) The number of possible values of k if fundamental period of $\sin^{-1}(\sin kx)$ is $\frac{\pi}{2}$

(ii) Number of points in the domain of $f(x) = \tan^{-1}x + \sin^{-1}x + \sec^{-1}x$

(iii) $f(x) = \sin\left(\frac{\pi x}{2}\right).\operatorname{cosec}\left(\frac{\pi x}{2}\right)$ is periodic with period

(iv) If range of the function $f(x) = \cos^{-1}[5x]$, where [.] denotes greatest integer, is $\{a, b, c\}$, then $a + b + c = k\pi/2$, then k is

Column II

(a) 1
(b) 2
(c) 3
(d) 4

2. Match the column

Column I

(i) Smallest positive integral value of x for which $x^2 - 3x + \sin^{-1}(\sin 2) > 0$ is

(ii) Number of solution of $2[x] = x + 2\{x\}$ is {where $[x]$, $\{x\}$ are greatest integer and fractional part functions respectively}

(iii) If $x^2 + y^2 = 1$, then maximum value of $(x + y)^2$ is

(iv) $f\left(x+\frac{1}{2}\right)+f\left(x-\frac{1}{2}\right)=f(x)$ for all $x \in \mathbb{R}$, then period of $f(x)$ is

Column II

(a) $3\pi/2$
(b) 3
(c) 1
(d) 2

3. **Column I**

(i) If function $f(x)$ is defined in $[-2, 2]$, then domain of $f(|x| + 1)$ is

(ii) Range of the function $f(x)=\dfrac{\sin^{-1}x+\cos^{-1}x+\tan^{-1}x}{\pi}$ is

(iii) Range of the function $f(x) = 3|\sin x| - 4|\cos x|$ is

(iv) Range of $f(x) = (\sin^{-1}x)\sin x$ is

Column II

(a) $\left[\frac{1}{4}, \frac{3}{4}\right]$
(b) $[-1, 1]$
(c) $[-4, 3]$
(d) $\left[0, \frac{\pi}{2}\sin 1\right]$

4. Let $f(x) = \sin^{-1}x$, $g(x) = \cos^{-1}x$ and $h(x) = \tan^{-1}x$. For what interval of variation of x the following are true?

Column I

(i) $f(\sqrt{x}) + g(\sqrt{x}) = \pi/2$

(ii) $f(x)+g\left(\sqrt{1-x^2}\right)=0$

(iii) $g\left(\dfrac{1-x^2}{1+x^2}\right)=2h(x)$

(iv) $h(x)+h(1)=h\left(\dfrac{1+x}{1-x}\right)$

Column II

(a) $[0, \infty)$
(b) $[0, 1]$
(c) $(-\infty, 1)$
(d) $[-1, 0]$

5. Column I

(i) Number of solutions of the equation $sgn(\{x\}) = |1 - x|$ is

(ii) Number of elements in the domain of the function is $f(x) = \sin^{-1}\left(\frac{x^2 - 2x}{3}\right) + \sqrt{[x]+[-x]}$ is

(iii) Let $f(x)$ be a function defined on $\mathbb{R}$ such that $f(x + 2) = f(x - 2)\ \forall\ x \in \mathbb{R}$. If $f(x) = 0$ has exactly three roots in the interval [4, 8] and 8 is one of the three roots, then the number of roots of $f(x) = 0$ in the interval [–8, 12] will be $(k + 7)$, then k equals.

(iv) If $f(x) = \frac{1}{x-1}$, then $f\left(\frac{1}{x-1}\right)$ is not defined at x equal to; {where [.] and{.} represents greatest integer and fractional part functions respectively}.

Column II

(a) 2
(b) 4
(c) 5
(d) 1

6. Column I

(i) $f(x) = \sqrt{-2\sqrt{2}(\cos \pi x) - \sqrt{3} - 1}$ is defined if x lies in

(ii) $\lim\limits_{x \to 1^-} \frac{3}{4}[x] + \frac{7}{8}\{x\} = L_1$ lies in; {where [.] and {.} represent greatest integer and fractional part function respectively}

(iii) If $\sqrt{\log_x (0.75)} < 1$, then x lies in

(iv) If $\frac{|x-1|}{x+2} > 1$, then x lies in

Column II

(a) $(0, 1)$
(b) $(0, 3/4)$
(c) $\left[\frac{11}{12}, \frac{13}{12}\right]$
(d) $\left(-2, -\frac{1}{2}\right)$

SECTION–VIII

INTEGER TYPE QUESTIONS

1. If q is the absolute difference of $\sqrt{x^2 + x + 3}$ $(= y \text{ say})$ and x, for which both x and $\sqrt{x^2 + x + 3}$ are rationals, then x and y are given by $\frac{q^2 - a}{1 - bq}$ and $\frac{q^2 - q + c}{|1 - bq|}$; $\left(q \neq \frac{1}{b}\right)$, then find the number of positive integer divisors of $(a.b.c)$.

2. Find the maximum numerical value of $f(x)$; $f(x) = {}^{15-x}C_{3x-1} + {}^{20-4x}C_{5x-7}$.

3. Let $f(x) = x^{99} + x^{79} - x^{69} + x^9 + 1$. If $f(x)$ is divided by $x^3 - x$, then the remainder is some function of x say $g(x)$. Find the value of $g(20)$.

4. If $f\left(x + \frac{1}{x}\right) = x^3 + x^{-3}$, then find $f(7)$.

5. If $f(x)$ is an even function, then find the number of distinct real numbers x such that $f(x) = f\left(\frac{x+1}{x+2}\right)$.

6. If $f(x + y) = f(x).f(y)\ \forall\ x, y \in \mathbb{N}$ and $f(1) = 2$, then find $\sum_{n=1}^{10} f(n)$.

7. If $g(x) = \left(4\cos^4 x - 2\cos 2x - \frac{1}{2}\cos 4x - x^9\right)^{\frac{1}{9}}$, then find the value of $g(g(2016))$.

8. The fundamental period of the function $\frac{\sin x + \sin 3x + \sin 5x + \sin 7x}{\cos x + \cos 3x + \cos 5x + \cos 7x}$ is $\frac{\pi}{k}$, then find k.

9. If the function $\sin^{-1}(\sin 4\pi x)$ is periodic with period k, then evaluate $4k$.

10. If the period of $f(x)$ satisfying the condition: $f(x + p) = 1 + \{1 - 3f(x) + 3f^2(x) - f^3(x)\}^{1/3}$ is λp, then evaluate λ.

11. Find the period of $f(x)$ satisfying the condition $f(x - 1) + f(x + 3) = f(x + 1) + f(x + 5)$.

12. The function $f(x) = \frac{x^3}{3} + (m-1)x^2 + (m+5)x + 11$ is invertible only for k integer values of m. Then find k.

13. Let $f(x, y)$ be a function satisfying the functional equation: $f(x, y) = f(2x + 2y, 2y - 2x)$ for all real numbers x, y. Define $g(x)$ by $g(x) = f(2^x, 0)$. Also given that $g(x)$ is a periodic function with period k, then find value of k.

14. If the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{ax^2 + 6x - 8}{a + 6x - 8x^2}$ is onto for $a \in [m, n]$, then evaluate $\sqrt{mn - 3}$.

15. Given the function $y = \frac{x-4}{16-\lambda^2}$ and the initial value of the independent variable $x_1 = 4 - \lambda$. At what terminal value x_2 of the independent variable x, the increment Δy equal to $\frac{1}{4-\lambda}$.

16. If $f(x) = \cos^2 x + \cos^2\left(\frac{\pi}{3} + x\right) - \cos x \cos\left(\frac{\pi}{3} + x\right)$ and $g(3/4) = 2$, then find the value of $(gof)(1)$.

17. Find the number of maps f from the set $\{1, 2, 3, 4\}$ into the set $\{1, 2, 3, 4, 5\}$ such that $f(i) \le f(j)$, when ever $i < j$ and $i, j \in \{1, 2, 3\}$.

18. The quadratic polynomial $p(x)$ is such that $p(x) \ge 0$ $\forall\, x \in \mathbb{R}$ and $p(1) = 0$, $p(2) = 2$. Find the value of $p(0) + p(3)$.

19. Let p be the product of non-real roots of the equation $x^4 - 4x^3 + 6x^2 - 4x = 2008$, where [.] denotes the greatest integer function, then find $[p]$.

20. Find the maximum number of points of intersection of the function $f(x) = ax^2 + bx + c$, $a > 0$ and $g(x) = \frac{1}{x^2 - 4}$.

Answer Keys

SECTION–III

1. (b)	**2.** (a)	**3.** (c)	**4.** (c)	**5.** (c)	**6.** (b)	**7.** (c)	**8.** (c)	**9.** (a)	**10.** (c)
11. (a)	**12.** (b)	**13.** (b)	**14.** (b)	**15.** (b)	**16.** (d)	**17.** (c)	**18.** (b)	**19.** (b)	**20.** (d)
21. (c)	**22.** (d)	**23.** (d)	**24.** (a)	**25.** (a)	**26.** (d)	**27.** (c)	**28.** (a)	**29.** (b)	**30.** (b)
31. (c)	**32.** (a)	**33.** (a)	**34.** (b)	**35.** (b)	**36.** (c)	**37.** (b)	**38.** (b)	**39.** (c)	**40.** (d)
41. (b)	**42.** (b)	**43.** (b)	**44.** (d)	**45.** (c)	**46.** (a)	**47.** (c)	**48.** (b)	**49.** (c)	**50.** (b)
51. (b)	**52.** (c)	**53.** (a)	**54.** (d)	**55.** (d)	**56.** (a)	**57.** (d)	**58.** (a)	**59.** (b)	**60.** (c)
61. (b)	**62.** (b)	**63.** (d)	**64.** (b)	**65.** (a)	**66.** (b)	**67.** (c)	**68.** (c)	**69.** (b)	**70.** (c)

SECTION–IV

1. (a,b,c,d)	**2.** (a,b)	**3.** (a,d)	**4.** (c,d)	**5.** (b,c)	**6.** (b,d)	**7.** (b,c)	**8.** (b),(c)
9. (a,d)	**10.** (a,d)	**11.** (b,c)	**12.** (a,c)	**13.** (a,b,d)	**14.** (a,b)	**15.** (a,b,c)	**16.** (a,b,c,d)
17. (b,c)	**18.** (a,b,c,d)		**19.** (a,c,d)	**20.** (a,c)			

SECTION–V

1. (c)	**2.** (d)	**3.** (c)	**4.** (a)	**5.** (c)	**6.** (b)	**7.** (c)	**8.** (d)	**9.** (b)	**10.** (c)
11. (a)	**12.** (d)	**13.** (c)	**14.** (a)	**15.** (c)	**16.** (a)	**17.** (b)	**18.** (c)	**19.** (a)	

SECTION–VI

1. (b)	**2.** (c)	**3.** (c)	**4.** (c)	**5.** (a)	**6.** (c)	**7.** (b)	**8.** (a)	**9.** (c)	**10.** (b)
11. (a)	**12.** (a)	**13.** (b)	**14.** (d)	**15.** (b)	**16.** (c)	**17.** (d)	**18.** (a,d)	**19.** (a)	**20.** (b)
21. (c)	**22.** (b)	**23.** (c)	**24.** (a)	**25.** (b)	**26.** (c)				

SECTION–VII

1. (i) → (b), (ii) → (b), (iii) → (b), (iv) → (c)
2. (i) → (b), (ii) → (b), (iii) → (d), (iv) → (b)
3. (i) → (b), (ii) → (a), (iii) → (c), (iv) → (d)
4. (i) → (b) (ii) → (d) (iii) → (a) (iv) → (c), (d)
5. (i) → (d) (ii) → (c) (iii) → (b) (iv) → (a), (d)
6. (i) → (c) (ii) → (a) (iii) → (a), (b) (iv) → (d)

SECTION–VIII

1. 6	**2.** 1507	**3.** 41	**4.** 322	**5.** 4	**6.** 2046	**7.** 2016	**8.** 4	**9.** 2	**10.** 2
11. 8	**12.** 6	**13.** 12	**14.** 5	**15.** 8	**16.** 2	**17.** 175	**18.** 10	**19.** 45	**20.** 4

HINTS AND SOLUTIONS

TEXTUAL EXERCISE–1: (SUBJECTIVE)

1. $f(x)=\sqrt{x\ -4}$; $f(1)$ doesn't exist

$f(2)=\sqrt{2^2-4}=\sqrt{4-4}=0$

$f(-3)=\sqrt{(-3)^2-4}=\sqrt{9-4}=\sqrt{5}$

2. $f(x)=\dfrac{x-2}{(x-4)(x-5)}$, when $x \to 4, f(x) \to \pm\infty$

And when $x \to 5$ side $f(x) \to \pm\infty$

Therefore, $f(x)$ is not defined at $x = 4$ and $x = 5$.

3. $f(x)=\dfrac{1}{x-1}$; $x = 1$

$\Rightarrow\ f(1)=\dfrac{1}{1-1}=\dfrac{1}{0}$ i.e., not defined, therefore, $f(x)$ is not defined at $x = 1$.

As the numerator is never zero, therefore, $f(x)$ cannot attain the value 0.

4. $f(x)=\sqrt{(x-1)(x-4)}$

(i) For $f(x)$ to be undefined, the quantity inside the square root should be negative, therefore, $(x - 1)(x - 4) < 0$.

$\Rightarrow x \in (1, 4)$

(ii) $\dfrac{1}{f(x)}=\dfrac{1}{\sqrt{(x-1)(x-4)}}$

For $\dfrac{1}{f(x)}$ to be undefined, the quantity inside the square root should be negative or 0, therefore, $(x - 1)(x - 4) \le 0$.

$\Rightarrow x \in [1, 4]$

(iii) $f(x)=\sqrt{(x-1)(x-4)}=\sqrt{x^2-5x+4}=\sqrt{\left(x-\dfrac{5}{2}\right)^2-\dfrac{9}{4}}$

$\Rightarrow\ \left(x-\dfrac{5}{2}\right)^2-\dfrac{9}{4}\ge -\dfrac{9}{4}$, but for $f(x)$ to be defined

$\left(x-\dfrac{5}{2}\right)^2-\dfrac{9}{4}\ge 0$

$\Rightarrow\ \sqrt{\left(x-\dfrac{5}{2}\right)^2-\dfrac{9}{4}}\ge 0$

$\Rightarrow f(x) \ge 0 \qquad \Rightarrow f(x) \in [0, \infty)$

(iv) $\dfrac{1}{f(x)}=\dfrac{1}{\sqrt{(x-1)(x-4)}}$

$\Rightarrow\ (x - 1)(x - 4) = x^2 - 5x + 4 = \left(x-\dfrac{5}{2}\right)^2-\dfrac{9}{4}$

$\Rightarrow\ \left(x-\dfrac{5}{2}\right)^2-\dfrac{9}{4}\ge -\dfrac{9}{4}$

$\Rightarrow\ \left(x-\dfrac{5}{2}\right)^2-\dfrac{9}{4}\in\left[-\dfrac{9}{4},\infty\right)$

For $\dfrac{1}{\sqrt{(x-1)(x-4)}}$ to be defined $\dfrac{1}{\left(x-\dfrac{5}{2}\right)^2-\dfrac{9}{4}}\in(0,\infty)$

$\Rightarrow\ \dfrac{1}{\sqrt{\left(x-\dfrac{5}{2}\right)^2-\dfrac{9}{4}}}\in(0,\infty)$

$\Rightarrow\ \dfrac{1}{f(x)}\in(0,\infty)$

5. (a) $f(x)=\dfrac{1}{x(x-1)(x-2)}$; for $f(x)$ to be defined the denominator should never be zero.

$\Rightarrow x(x - 1)(x - 2) \neq 0 \quad \Rightarrow x \neq 0, 1, 2$

$\Rightarrow x \in \mathbb{R} \sim \{0, 1, 2\}$

(b) $f(x)=\dfrac{x^2-x}{x^2+x}$; for $f(x)$ to be defined the denominator should never be zero.

$\Rightarrow x^2 + x \neq 0 \qquad \Rightarrow x(x + 1) \neq 0$

$\Rightarrow x \neq 0, -1 \qquad \Rightarrow x \in \mathbb{R} \sim \{0, -1\}$

(c) $f(x)=\dfrac{x+1}{x^2+x+1}$; for $f(x)$ to be defined $x^2 + x + 1 \neq 0$

As the discriminate of the quadratic in denominator is $1 - 4 = -3$ is negative, hence, the above quadratic expression has no roots, i.e., $x^2 + x + 1 \neq 0$. Hence, $D_f = \mathbb{R}$.

(d) $f(x)=\dfrac{1}{\sqrt{x-3}}$; for $f(x)$ to be defined $x - 3 > 0$

$\Rightarrow x > 3 \qquad \Rightarrow x \in (3, \infty)$

(e) $f(x)=\sqrt{x-4}$; for $f(x)$ to be defined $x - 4 \ge 0$

$\Rightarrow x \ge 4 \qquad \Rightarrow x \in [4, \infty)$

(f) $f(x)=\sqrt{x-2}+\sqrt{x+2}$; for $f(x)$ to be defined $x - 2 \ge 0$ and $x + 2 \ge 0$

$\Rightarrow x \ge 2$ and $x \ge -2 \qquad \Rightarrow x \in [2, \infty)$ and $x \in [-2, \infty)$

$\Rightarrow x \in [2, \infty)$

(g) $f(x)=\sqrt{x-4}+\sqrt{7-x}$; for $f(x)$ to be defined $x - 4 \ge 0$ and $7 - x \ge 0$

$\Rightarrow x \ge 4$ and $7 \ge x \qquad \Rightarrow x \in [4, \infty)$ and $x \in [-\infty, 7]$

$\Rightarrow x \in [4, 7]$

(h) $f(x)=\dfrac{1}{\sqrt{x-4}-\sqrt{6-x}}$; for $f(x)$ to be defined $x - 4 \ge 0$ and $6 - x \ge 0$ and $\sqrt{x-4}-\sqrt{6-4}\neq 0$

$\Rightarrow x \ge 4$ and $6 \ge x$ and $\sqrt{x-4}\neq\sqrt{6-x}$

$\Rightarrow x \in [4, \infty)$ and $x \in (-\infty, 6]$ and $x - 4 \neq 6 - x$

$\Rightarrow x \in [4, 6]$ and $x \neq 5 \quad \Rightarrow x \in [4, 6] \sim \{5\}$

(i) $f(x)=\sqrt{x-7}-\sqrt{3-x}$; $x - 7 \ge 0$ and $3 - x \ge 0$

$\Rightarrow x \in [7, \infty)$ and $x \in (-\infty, 3]$

$\Rightarrow x \in \phi$

6. (a) $f(x)=\dfrac{\sqrt{x-1}}{\sqrt{6-x}}$; $x - 1 \ge 0$ and $6 - x > 0$

$\Rightarrow x \ge 1$ and $x < 6 \qquad \Rightarrow x \in [1, \infty)$ and $x \in (-\infty, 6)$

$\Rightarrow x \in [1, 6)$

(b) $f(x)=\sqrt{\dfrac{x-1}{6-x}-1}=\sqrt{\dfrac{2x-7}{6-x}}$

$\Rightarrow$ $2x-7=0$ or $(2x-7)(6-x)>0$

$\Rightarrow$ $x=\dfrac{7}{2}$ or $x\in\left(\dfrac{7}{2},6\right)$ $\Rightarrow$ $x\in\left[\dfrac{7}{2},6\right)$

(c) $f(x)=\dfrac{\sqrt{x-1}}{6-x}$; $x-1\geq 0$ and $6-x\neq 0$

$\Rightarrow$ $x\in[1,\infty)$ and $x\neq 6$ $\Rightarrow$ $x\in[1,\infty)\sim\{6\}$

(d) $f(x)=(\sqrt{x}-1)(6-\sqrt{x})$; $x\geq 0$

$\Rightarrow$ $x\in[0,\infty)$

(e) $f(x)=(\sqrt{x}-1)\sqrt{(6-x)}$

$\Rightarrow$ $x\geq 0$ and $6-x\geq 0$ $\Rightarrow$ $x\geq 0$ and $x\leq 6$

$\Rightarrow$ $x\in[0,\infty)$ and $x\in(-\infty,6]$

$\Rightarrow$ $x\in[0,6]$

7. **(a)** $f(x)=\sqrt{x-1}\sqrt{6-x}$

$\Rightarrow$ $x-1\geq 0$ and $6-x\geq 0$

$\Rightarrow$ $x\geq 1$ and $6\geq x$

$\Rightarrow$ $x\in[1,\infty)$ and $x\in(-\infty,6]$

$\Rightarrow$ $x\in[1,6]$

(b) $f(x)=\sqrt{(x-1)(6-x)}$; $(x-1)(6-x)\geq 0$

$\Rightarrow$ $x\in[1,6]$

(c) $f(x)=\sqrt{x-x^2}$ $\Rightarrow$ $x-x^2\geq 0$

$\Rightarrow$ $x(1-x)\geq 0$ $\Rightarrow$ $x\in[0,1]$

8. **(i)** $f(x)=\sqrt{5-x}+\sqrt{x-2}$

$\Rightarrow$ $5-x\geq 0$ and $x-2\geq 0$

$\Rightarrow$ $x\in(-\infty,5]$ and $x\in[2,\infty)$

$\Rightarrow$ $x\in[2,5]$

(ii) $f(x)=\sqrt{3-x}+\sqrt{x-5}$

$\Rightarrow$ $3-x\geq 0$ and $x-5\geq 0$

$\Rightarrow$ $x\leq 3$ and $x\geq 5$ $\Rightarrow$ $x\in(-\infty,3]$ and $x\in[5,\infty)$

$\Rightarrow$ $x\in\phi$

9. $f(x)=\sqrt{x^2+x+1}$; $f: A\to B$; where A = {0, 1, 2, 3} and B = set of irrational numbers

$f(0)=\sqrt{0^2+0+1}=\sqrt{1}=1$; Which is not irrational number, hence, the output doesn't lie in co-domain, therefore, 0 do, not lie in the domain of f.

$f(2)=\sqrt{2^2+2+1}=\sqrt{4+2+1}=\sqrt{7}$; which is irrational and lie in co-domain, therefore, 2 lies in the domain of f.

$f(3)=\sqrt{3^2+3+1}=\sqrt{13}$; which is irrational and lie in co-domain, therefore, 3 lies in the domain of f.

Hence, D_f = {1, 2, 3} and $R_f=\{\sqrt{3},\sqrt{7},\sqrt{13}\}$

10. **(i)** $f(x)=\sqrt{1-x^2}$ $\Rightarrow$ $1-x^2\geq 0$

$\Rightarrow$ $(1-x)(1+x)\geq 0$

$x\in[-1,1]$ $\Rightarrow$ $D_f=[-1,1]$

$\Rightarrow$ $x^2\geq 0$ $\Rightarrow$ $-x^2\leq 0$

$\Rightarrow$ $1-x^2\leq 1$

$\Rightarrow$ $\sqrt{1-x^2}\in[0,1]$ $\Rightarrow$ $R_f=[0,1]$

(ii) $g(x)=\dfrac{1}{x}$ $\Rightarrow$ $D_f=\mathbb{R}-\{0\}$

$\Rightarrow$ $x\in(-\infty,0)\cup(0,\infty)$ $\Rightarrow$ $\dfrac{1}{x}\in(-\infty,0)\cup(0,\infty)$

$\Rightarrow$ $R_f=\mathbb{R}-\{0\}$

(iii) $h(x)$ $\overline{\sqrt{}}$ $\Rightarrow$ $D_f=R-\{0\}$

$\Rightarrow$ $x\in(-\infty,0)\cup(0,\infty)$ $\Rightarrow$ $\sqrt[3]{x}\in(-\infty,0)\cup(0,\infty)$

$\Rightarrow$ $\dfrac{1}{\sqrt[3]{x}}\in(-\infty,0)\cup(0,\infty)$

$\Rightarrow$ $D_f=R-\{0\}$

(iv) $\phi(x)=\sqrt[4]{1-x^2}$ $\Rightarrow$ $1-x^2\geq 0$

$\Rightarrow$ $(1-x)(1+x)\geq 0$ $\Rightarrow$ $D_f=[-1,1]$

$\Rightarrow$ $x\in[-1,1]$

$\Rightarrow$ $x^2\geq 0$ $\Rightarrow$ $-x^2\leq 0$

$\Rightarrow$ $1-x^2\leq 1$

$\Rightarrow$ $\sqrt[4]{1-x^2}\in[0,1]$ $\Rightarrow$ $R_f=[0,1]$

Hence, ($f(x)$, $\phi(x)$) and ($h(x)$, $g(x)$) are the pairs of functions having same range and domain.

11. A = {x: x is a positive integer divisor of 36} = {1, 2, 3, 4, 6, 9, 12, 18, 36}

$$f(x)=\frac{x}{(x-1)(x-2)(x-3)\ldots.(x-10)}$$

$f(x)$ is not defined when $x\in\{1,2,3,4,6,9\}$, but $f(x)$ is defined when $x\in\{12,18,36\}$

Hence, $D_f=\{12,18,36\}$

$$f(12)=\frac{12}{11\times10\times9\times8\times7\times6\times5\times4\times3\times2}=\frac{12}{11!}$$

$$f(18)=\frac{18}{17\times16\times15\times14\times13\times\ldots\times8}=\frac{18\times7!}{17!}$$

$$f(36)=\frac{36}{35\times34\times33\times\ldots.\times26}=\frac{36\times25!}{35!}$$

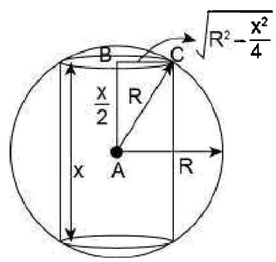
12. $A=\{9,10,11,12,13\}$; $f: A\to\mathbb{N}$

$f(x)$ = The highest prime factor of n

$f(9)=3$; $f(10)=5$; $f(11)=11$; $f(12)=3$; $f(13)=13$

$\therefore$ $R_f=\{3,5,11,13\}$

13. In ΔABC; $AB=\dfrac{x}{2}$ (half of the height of cylinder)



$AC=R$

$BC=\sqrt{R^2-\dfrac{x^2}{4}}$ (Pythagoras theorem)

Volume of cylinder = $\pi \times (BC)^2 \times x = \pi \times \left(R^2 - \frac{x^2}{4}\right) \times x$

$= \pi x \left(R^2 - \frac{x^2}{4}\right)$

As volume of cylinder can never be negative or 0 and height of cylinder can never be negative or 0

$\pi \times x\left(R - \frac{x}{2}\right)\left(R + \frac{x}{2}\right) > 0$ and $x > 0$

$\Rightarrow$ $x\left(R - \frac{x}{2}\right)\left(R + \frac{x}{2}\right) > 0$ and $x > 0$

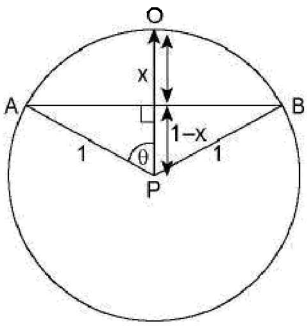
$\Rightarrow$ $x(2R - x)(2R + x) > 0$ and $x > 0$

$\Rightarrow$ $x \in (-\infty, -2R) \cup (0, 2R)$ and $x > 0$

$\Rightarrow$ $x \in (0, 2R)$

$\Rightarrow$ $0 < x < 2R$

14. **Case I:** $x \in (0, 1]$



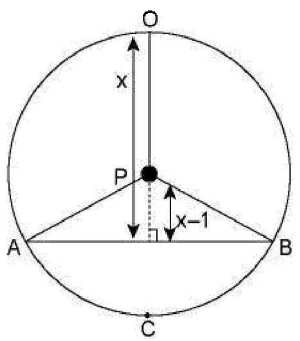
Area of sector $AOB = r^2\theta = 1^2 \times \cos^{-1}(1 - x) = \cos^{-1}(1 - x)$

Area of $\Delta ABP = \frac{1}{2} \times (1 - x) \times 2 \times \sqrt{1 - (1 - x)^2}$

$= (1 - x)\sqrt{1 - 1 - x^2 + 2x} = (1 - x)\sqrt{2x - x^2}$

Area of segment $AOB = \cos^{-1}(1 - x) - (1 - x)\sqrt{2x - x^2}$

Case II: $x \in (1, 2)$

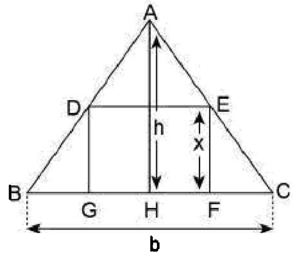


Area of segment $AOB = \pi r^2$ – area of segment ACB

$= \pi(1)^2 - [\cos^{-1}(1 - x) - (1 - x)\sqrt{2x - x^2}]$

$= \pi - \cos^{-1}(1 - x) + (1 - x)\sqrt{2x - x^2}$

15. $HC = \frac{b}{2}$; $\tan C = \frac{AH}{HC} = \frac{EF}{FC}$



$\Rightarrow$ $\frac{h}{\frac{b}{2}} = \frac{x}{FC}$ $\Rightarrow$ $FC = \frac{bx}{2h}$

$\Rightarrow$ $HF = HC - FC = \frac{b}{2} - \frac{b}{2}\frac{x}{h} = \frac{b}{2}\left(1 - \frac{x}{h}\right)$

$\Rightarrow$ $GF = 2HF = 2 \times \frac{b}{2}\left(1 - \frac{x}{h}\right) = b\left(1 - \frac{x}{h}\right)$

Perimeter of rectangle $DEFG = 2\left(x + b\left(1 - \frac{x}{h}\right)\right)$

Area of rectangle $DEFG = x \times b\left(1 - \frac{x}{h}\right) = bx\left(1 - \frac{x}{h}\right)$

TEXTUAL EXERCISE–1: (OBJECTIVE)

1. **(a)** $f(x) = \frac{1}{1 - x}$ $\Rightarrow$ $f\left(\frac{1}{x}\right) = \frac{1}{1 - \frac{1}{x}} = \frac{x}{x - 1}$

2. **(c)** $f(x) = \log(1 - x) + \sqrt{x^2 - 1}$

$1 - x > 0$ and $x^2 - 1 \geq 0$

$x < 1$ and $(x - 1)(x + 1) \geq 0$

$\Rightarrow$ $x \in (-\infty, 1)$ and x $(-\infty, -1] \cup [1, \infty)$

$\Rightarrow$ $x \in (-\infty, -1]$

3. **(a)** $f(x) = ax^2 + bx + c$

$f(x + 1) = a(x^2 + 2x + 1) + bx + b + c = ax^2 + (2a + b)x + a + b + c$

$f(x + 1) - f(x) = 8x + 3$

$ax^2 + (2a + b)x + a + b + c - ax^2 - bx - c = 8x + 3$

$2ax + a + b = 8x + 3$

$2a = 8$ $\Rightarrow$ $a = 4$ and $a + b = 3$

$\Rightarrow$ $b = -1$

4. **(c)** $p(x) = a^2 + bx$

$q(x) = lx^2 + mx + x$

$\therefore$ $p(1) - q(1) = 0$

$\Rightarrow$ $(a^2 + b - l - m - n) = 0$ ......(i)

$\Rightarrow$ $p(2) - q(2) = 1$

$\Rightarrow$ $(a^2 + 2b) - (4l + 2m + n) = 0$ ......(ii)

And $p(3) - q(3) = 4$

$\Rightarrow$ $(a^2 + 3b) - (9l + 3m + n) = 4$ ......(iii)

Solving (i), (ii) and (iii), we get

$l = -1$, $b - m = -2$, $a^2 - n = 1$

$\therefore$ $p(x) - q(x) = (a^2 - n) + (b - m)x - lx^2 = 1 - 2x + x^2 = (x - 1)^2$

$\Rightarrow$ $p(4) - q(4) = (4 - 1)^2 = 9$

5. **(a)** $f(x) = \sqrt{\log(\log x) - \log(4 - \log x) - \log 3}$

$= \sqrt{\log\left(\dfrac{\log x}{3(4 - \log x)}\right)} \Rightarrow \dfrac{\log x}{3(4 \log x)} \geq 1$

$\Rightarrow \dfrac{\log x}{4 - \log x} - 3 \geq 0 \qquad \Rightarrow \dfrac{4\log x - 12}{4 - \log x} \geq 0$

$\Rightarrow \dfrac{\log x - 3}{4 - \log x} \geq 0$

$\Rightarrow$ $\log x = 3$ or $\dfrac{\log x - 3}{4 - \log x} > 0$

$\Rightarrow$ $x = 10^3$ or $\log x \in (3, 4)$

$\Rightarrow$ $x = 10^3$ or $x \in (10^3, 10^4)$

$\Rightarrow$ $x \in [10^3, 10^4)$

6. **(b)** $3^y = 2^{4x^2-3} - 2^{x^4}$; y to be real $3^y > 0$

$\Rightarrow$ $2^{4x^2-3} - 2^{x^4} > 0$

$\Rightarrow$ $4x^2 - 3 > x^4$

$\Rightarrow$ $x^4 - 4x^2 + 3 < 0$

$\Rightarrow$ $(x^2 - 1)(x^2 - 3) < 0$

$\Rightarrow$ $(x - 1)(x + 1)(x - \sqrt{3})(x + \sqrt{3}) < 0$

$\Rightarrow$ $x \in (-\sqrt{3} - 1) \cup (1, \sqrt{3})$

7. **(a)** $f(x) = \sqrt[3]{\dfrac{2x+1}{x^2 - 10x - 11}} = \sqrt[3]{\dfrac{2x+1}{(x-11)(x+1)}}$

$\Rightarrow$ $x \in \mathbb{R} - \{11, -1\}$ $\quad\Rightarrow$ $\mathbb{R} - \{-1, 11\}$

8. $f(x) = \sqrt{\log_5(\cos(\sin x))}$

$\cos(\sin x) \geq 1$ but $\cos(\sin x) \in [-1, 1]$

$\Rightarrow$ $\cos(\sin x) = 1$

$\Rightarrow$ $\sin x = 0, 2\pi, 4\pi$ but $\sin x \in [-1, 1]$

$\Rightarrow$ $\sin x = 0$ $\quad\Rightarrow$ $x = n\pi$, where $n \in \mathbb{Z}$

9. **(b)** $f(x) = \dfrac{1}{\sqrt{x^6 - 13x^4 + 36x^2}} = \dfrac{1}{\sqrt{(x^4 - 13x^2 + 36)x^2}}$

$= \dfrac{1}{\sqrt{(x^2 - 9)(x^2 - 4)x^2}}$

$\Rightarrow$ $x^2(x - 3)(x + 3)(x - 2)(x + 2) > 0$

$\Rightarrow$ $(x - 3)(x + 3)(x - 2)(x + 2) > 0, x \neq 0$

$\Rightarrow$ x $(-\infty, -3) \cup (-2, 0) \cup (0, 2) \cup (3, \infty)$

10. **(a)** $f(x) = x^2 + \dfrac{1}{x^2 + 1} = y; y \geq 0$

$\Rightarrow y = \dfrac{x^4 + x^2 + 1}{x^2 + 1} \qquad \Rightarrow x^4 + (1 - y)x^2 + (1 - y) = 0$

Equation to have real roots, $D \geq 0$

$\Rightarrow$ $(1 - y)^2 - 4(1 - y) \geq 0$

$\Rightarrow$ $(1 - y)(1 - y - 4) \geq 0$

$\Rightarrow$ $(y - 1)(y + 3) \geq 0$ $\quad\Rightarrow$ $y \geq 0$

$\Rightarrow$ $y + 3 \geq 3$ $\quad\Rightarrow$ $y - 1 \geq 0$

$\Rightarrow$ $y \geq 1$ $\quad\Rightarrow$ $y \in [1, \infty)$

11. **(a)** $f(x) = \log_4(\log_5(\log_3(18x - x^2 - 77)))$

$\Rightarrow$ $\log_5(\log_3(18x - x^2 - 77)) > 0$

$\Rightarrow$ $18x - x^2 - 77 > 3$ $\quad\Rightarrow$ $x^2 - 18x + 80 < 0$

$\Rightarrow$ $(x - 10)(x - 8) < 0$ $\quad\Rightarrow$ $x \in (8, 10)$

12. **(c)** $\log_5(\log_{1/3}(\log_8(2x + 1)) = f(x)$

$f(x)$ to be defined

$2x + 1 > 0$ and $\log_{1/3}(\log_8(2x + 1)) > 0$

$\Rightarrow$ $x > -\dfrac{1}{2}$ and $\log_8(2x + 1) < 1$

$\Rightarrow$ $x \in \left(-\dfrac{1}{2}, \infty\right)$ and $2x + 1 < 8$

$\Rightarrow$ $x \in \left(-\dfrac{1}{2}, \infty\right)$ and $x \in \left(-\infty, \dfrac{7}{2}\right)$

$x \in \left(-\dfrac{1}{2}, \dfrac{7}{2}\right)$

$D = \left(-\dfrac{1}{2}, \dfrac{7}{2}\right)$; integers belonging to D are 0, 1, 2, 3

$S = 0 + 1 + 2 + 3 = 6$

TEXTUAL EXERCISE–2: (SUBJECTIVE)

1. $f_1 = \{(\alpha, 1), (\beta, 1)\}$; $f_2 = \{(\alpha, 2), (\beta, 2)\}$; $f_3 = \{(\alpha, 1), (\beta, 2)$; $f_4 = \{(\alpha, 2), (\beta, 1)\}$

2. **(a)** $\log_5 x + \log_5(y + 2) = 8$

$\Rightarrow$ $\log_5 x(y + 2) = 8$ $\quad\Rightarrow$ $x(y + 2) = 5^8$

Domain: for $\log_5 x$ to be defined, $x > 0$ and $y = \dfrac{5^8}{x} - 2$

$\Rightarrow$ $x \in (0, \infty)$ and $y + 2 > 0$

$\Rightarrow \dfrac{5^8}{x} > 0 \qquad \Rightarrow x > 0$

$\Rightarrow$ $x \in (0, \infty)$

(b) $3^{x+y}(x^2 - 3) = x^3 - 27$

$\Rightarrow 3^{x+y} = \dfrac{x^3 - 27}{x^2 - 3} \qquad \Rightarrow x + y = \log_3\left(\dfrac{x^3 - 27}{x^2 - 3}\right)$

$\Rightarrow y = \log_3\left(\dfrac{x^3 - 27}{x^2 - 3}\right) - x$

Domain: $\dfrac{x^3 - 27}{x^2 - 3} > 0$

$\Rightarrow \dfrac{(x - 3)(x^2 + 9 + 3x)}{(x - \sqrt{3})(x + \sqrt{3})} > 0$

$\Rightarrow \dfrac{(x - 3)}{(x - \sqrt{3})(x + \sqrt{3})} > 0$

[$x^2 + 9 + 3x$ is always positive as $D < 0$ and leading coefficient > 0]

$\Rightarrow x \in (-\sqrt{3}, \sqrt{3}) \cup (3, \infty)$

(c) $\Rightarrow$ $y^2 = 8x - 4y$ $\quad\Rightarrow$ $y^2 + 4y + 4 = 8x + 4$

$\Rightarrow (y + 2)^2 = 8\left(x + \dfrac{1}{2}\right)$

$\Rightarrow \quad y = \pm\sqrt{8\left(x+\frac{1}{2}\right)} - 2$

Domain: $\left(x+\frac{1}{2}\right) \geq 0 \quad \Rightarrow \quad x \geq -\frac{1}{2} \Rightarrow x \in \left[-\frac{1}{2}, \infty\right)$

3. **(a)** $y^2 - 2y - x^2 = 0; y \geq 1$

$\Rightarrow y^2 - 2y + 1 = x^2 + 1$

$\Rightarrow (y-1)^2 = x^2 + 1 \quad \Rightarrow \quad y - 1 = \sqrt{x^2+1}$

$\Rightarrow [y - 1 \geq 0 \Rightarrow y - 1 = \sqrt{x^2+1}]$

$\Rightarrow y = 1 + \sqrt{x^2+1} \quad \Rightarrow \quad x^2 + 1 \geq 0$ true for all $x \in \mathbb{R}$

$\Rightarrow D_f = \mathbb{R}$

(b) $y^2 - 2y + x^2 = 0; y \leq 1$

$\Rightarrow (y-1)^2 = 1 - x^2$

$\Rightarrow y - 1 = -\sqrt{1-x^2} \quad \Rightarrow \quad y = 1 - \sqrt{1-x^2}$; $[y - 1 \leq 0]$

$\Rightarrow 1 - x^2 \geq 0$

$\Rightarrow x \in [-1, 1] \quad \Rightarrow \quad D_f = [-1, 1]$

5. **(a)** $y^{2/3} = 3x + 4 \quad \Rightarrow y^2 = (3x+4)^3$

$\Rightarrow y = \pm(3x+4)^{3/2}$

Hence, it is a one–many relation, so, it is not a function.

(b) $y = 2x^2 + 1$

If $x_1 = x_2 \quad \Rightarrow \quad 2x_1^2 = 2x_2^2$

$\Rightarrow 2x_1^2 + 1 = 2x_2^2 + 1 \quad \Rightarrow \quad y_1 = y_2$

Hence, it is a function.

If $y_1 = y_2$

$\Rightarrow 2x_1^2 + 1 = 2x_2^2 + 1 \quad \Rightarrow \quad x_1^2 = x_2^2$

$\Rightarrow x_1 = x_2$ or $x_1 = -x_2$

Since it is a many–one function, it is not injective.

(c) $y^2 = 2\tan x + 5 \quad \Rightarrow \quad y = \pm\sqrt{2\tan x + 5}$

It is a one–many relation.

Hence, it is not a function.

(d) $y = \sqrt{3x+2}$

If $x_1 = x_2 \quad \Rightarrow \quad 3x_1 + 2 = 3x_2 + 2$

$\Rightarrow \sqrt{3x_1+2} = \sqrt{3x_2+2} \quad \Rightarrow \quad y_1 = y_2$

Hence, it is a function

If $y_1 = y_2 \quad \Rightarrow \quad \sqrt{3x_1+5} = \sqrt{3x_2+5}$

$\Rightarrow 3x_1 + 5 = 3x_2 + 5$

$\Rightarrow x_1 = x_2$

Hence, it is a one–one function, i.e., injective.

(e) $y^3 = 2x^2 - 1$

Let $x_1 = x_2 \quad \Rightarrow \quad 2x_1^2 - 1 = 2x_2^2 - 1$

$\Rightarrow y_1^3 = y_2^3 \quad \Rightarrow \quad y_1^3 - y_2^3 = 0$

$\Rightarrow (y_1 - y_2)(y_1^2 + y_2 y_1 + y_2^2)$

$\Rightarrow y_1 = y_2$

Hence, it is a function.

Let $y_1 = y_2$

$\Rightarrow (2x_1^2 - 1)^{\frac{1}{3}} = (2x_2^2 - 1)^{\frac{1}{3}}$

$\Rightarrow 2x_1^2 - 1 = 2x_2^2 - 1$

$\Rightarrow x_1^2 - x_2^2 = 0 \quad \Rightarrow \quad (x_1 - x_2)(x_1 + x_2) = 0$

$\Rightarrow x_1 = x_2$ or $x_1 = -x_2$

Hence, it is a many–one function, so it is not injective.

(f) $y = \frac{x-1}{x+1}$

Let $x_1 = x_2 \quad \Rightarrow \quad x_1 + 1 = x_2 + 1$

$\Rightarrow \frac{1}{x_1+1} = \frac{1}{x_2+1} \quad \Rightarrow \quad \frac{-2}{x_1+1} = \frac{-2}{x_2+1}$

$\Rightarrow 1 - \frac{2}{x_1+1} = 1 - \frac{2}{x_2+1} \quad \Rightarrow \quad \frac{x_1-1}{x_1+1} = \frac{x_2-1}{x_2+1} \Rightarrow y_1 = y_2$

Hence, it is a function.

Let $y_1 = y_2$

$\Rightarrow \frac{x_1-1}{x_1+1} = \frac{x_2-1}{x_2+1} \quad \Rightarrow \quad 1 - \frac{2}{x_1+1} = 1 - \frac{2}{x_2+1}$

$\Rightarrow \frac{-2}{x_1+1} = \frac{-2}{x_2+1} \quad \Rightarrow \quad \frac{1}{x_1+1} = \frac{1}{x_1+1} \Rightarrow x_1 = x_2$

Hence, it is an injective function.

(g) $y^2 = 4x - 3$

Let $x_1 = x_2 \quad \Rightarrow \quad 4x_1 - 3 = 4x_2 - 3$

$\Rightarrow y_1^2 = y_2^2$

$\Rightarrow y_1 = y_2$ or $y_1 = -y_2$

Hence, it is a one–many relation, so, it is not a function.

(h) $y = ax^2 + bx + c$

Let $x_1 = x_2 \quad \Rightarrow \quad x_1 + \frac{b}{2a} = x_2 + \frac{b}{2a}$

$\Rightarrow \left(x_1 + \frac{b}{2a}\right)^2 = \left(x_2 + \frac{b}{2a}\right)^2$

$\Rightarrow a\left(x_1 + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a}\right)$

$= a\left(x_2 + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a}\right)$

$\Rightarrow y_1 = y_2$

Hence, it is a function.

Let $y_1 = y_2$

$\Rightarrow ax_1^2 + bx_1 + c = ax_2^2 + bx_2 + c$

$\Rightarrow a\left(x_1 + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a}\right)$

$= a\left(x_2 + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a}\right)$

$\Rightarrow \left(x_1 + \frac{b}{2a}\right)^2 = \left(x_2 + \frac{b}{2a}\right)^2$

$\Rightarrow x_1 = x_2$ or $x_1 = -x_2 - \frac{b}{a}$

Hence, it is a many–one function, and therefore, it is not injective function.

(i) $y^3 = 3x + 1$

Let $x_1 = x_2 \quad \Rightarrow \quad 3x_1 + 1 = 3x_2 + 1$

$\Rightarrow y_1^3 = y_2^3$

$\Rightarrow (y_1 - y_2)(y_1^2 + y_2^2 + y_1 y_2) = 0$

$\Rightarrow y_1 = y_2$

Hence, it is a function.

Let $y_1 = y_2 \quad \Rightarrow \quad (3x + 1)^{1/3} = (3x_2 + 1)^{1/3}$

$\Rightarrow\ 3x_1 + 1 = 3x_2 + 1$
$\Rightarrow\ x_1 = x_2$
Hence, it is injective.

6. **(i)** $y = \sqrt{x}$
Let $x_1 = x_2 \quad \Rightarrow \quad \sqrt{x_1} = \sqrt{x_2} \Rightarrow y_1 = y_2$
Hence, it is a function.

(ii) $x^2 + 2y^2 = 8 \quad \Rightarrow \quad 2y^2 = 8 - x^2$
$\Rightarrow\ y^2 = \dfrac{8 - x^2}{2}$
Let $x_1 = x_2 \quad \Rightarrow \quad -x_1^2 = -x_2^2$
$\Rightarrow\ \dfrac{8 - x_1^2}{2} = \dfrac{8 - x_2^2}{2}$
$\Rightarrow\ y_1^2 = y_2^2 \quad \Rightarrow \quad y_1 = y_2$ or $y_1 = -y_2$
$\Rightarrow$ Hence, it is not a function.

(iii) $x^2 + y^2 = 4;\ y \geq 0$
$y^2 = 4 - x^2$
Let $x_1 = x_2$
$\Rightarrow\ -x_1^2 = -x_2^2 \quad \Rightarrow \quad 4 - x_1^2 = 4 - x_2^2$
$\Rightarrow\ y_1^2 = y_2^2$
$\Rightarrow\ (y_1 - y_2)(y_1 + y_2) = 0 \quad \Rightarrow \quad y_1 = y_2$ or
$\Rightarrow\ y_1 = y_2 = 0$
Hence, it is a function.

(iv) $x^2 + y^2 = 4;\ x \geq 0 \quad \Rightarrow \quad y^2 = 4 - x^2$
$\Rightarrow$ let $x_1 = x_2 \quad \Rightarrow \quad -x_1^2 = -x_2^2$
$\Rightarrow\ 4 - x_1^2 = 4 - x_2^2$
$\Rightarrow\ y_1^2 = y_2^2 \quad \Rightarrow \quad (y_1 - y_2)(y_1 + y_2) = 0$
$\Rightarrow\ y_1 = y_2$ or $y_1 = -y_2$
Hence, it is not a function.

(v) $y = \begin{cases} 2x + 5 & ;\ x \geq 0 \\ \dfrac{-x^2}{4} & ;\ x < 0 \end{cases}$

For $x \geq 0$; let $x_1 = x_2 \quad \Rightarrow \quad 2x_1 + 5 = 2x_2 + 5$
$\Rightarrow\ y_1 = y_2$
For $x < 0$; let $x_1 = x_2 \quad \Rightarrow \quad -\dfrac{x_1^2}{4} = \dfrac{-x_2^2}{4} \Rightarrow y_1 = y_2$
Hence, it is function.

(vi) $y^3 = x,$
Let $x_1 = x_2 \quad \Rightarrow \quad y_1^3 = y_2^3$
$\Rightarrow\ (y_1 - y_2)(y_1^2 + y_2^2 + y_1y_2) = 0$
$\Rightarrow\ y_1 = y_2$
Hence, it is a function.

(vii) $y^4 = 2x$
let $x_1 = x_2 \quad \Rightarrow \quad 2x_1 = 2x_2$
$\Rightarrow\ y_1^4 = y_2^4$
$\Rightarrow\ (y_1^2 - y_2^2)(y_1^2 + y_2^2) = 0$
$\Rightarrow\ y_1 = y_2$ or $y_1 = -y_2$ or $y_1 = y_2 = 0$
Hence, it is not a function.

7. **(a)** $f: R \rightarrow R;\ f(x) = \begin{cases} 2x + 1; & x \leq 4 \\ x + 4; & x > 4 \end{cases}$
For $x \leq 4$
Let $x_1 = x_2 \quad \Rightarrow \quad 2x_1 + 1 = 2x_2 + 1$
$\Rightarrow\ f(x_1) = f(x_2)$
For $x > 4$
Let $x_1 = x_2 \quad \Rightarrow \quad x_1 + 4 = x_2 + 4$
$\Rightarrow\ f(x_1) = f(x_2)$
$\Rightarrow\ f(x)$ is *a* function.

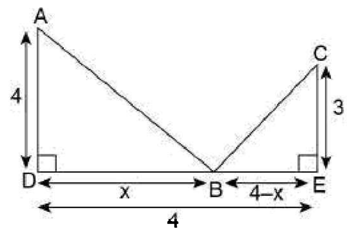
(b) $f(x) = \begin{cases} x^2; & 0 \leq x \leq 3 \\ 3x; & 3 \leq x \leq 10 \end{cases}$

For $0 \leq x \leq 3$
Let $x_1 = x_2 \quad \Rightarrow \quad x_1^2 = x_2^2$
$\Rightarrow\ f(x_1) = f(x_2)$
For $3 \leq x \leq 10$
Let $x_1 = x_2 \quad \Rightarrow \quad 3x_1 = 3x_2$
Hence, $h(x_1) = h(x_2)$
At $x = 3$
$\Rightarrow\ f(x) = x^2 = 3^2 = 9 \quad \Rightarrow \quad f(x) = 3x = 3 \times 3 = 9$
Hence, $h(x)$ is a function.

(c) $g(x) = \begin{cases} x^2 & ;0 \leq x \leq 2 \\ 3x & ;2 \leq x \leq 10 \end{cases}$

For $0 \leq x \leq 2$
Let $x_1 = x_2 \quad \Rightarrow \quad x_1^2 = x_2^2$
$\Rightarrow\ g(x_1) = g(x_2)$
For $2 \leq x \leq 10$
Let $x_1 = x_2 \quad \Rightarrow \quad 3x_1 = 3x_2$
$\Rightarrow\ g(x_1) = g(x_2)$
At $x = 2$
$g(x) = x^2 = 4$ and $g(x) = 3x = 6$
Hence, at $x = 2$ the relation has two values, hence, it is not a function.

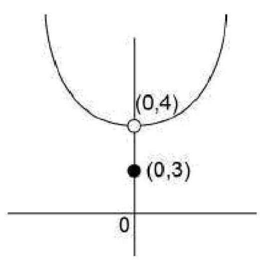
8. Length $AB = \sqrt{4^2 + x^2}$ and length $BC = \sqrt{3^2 + (4 - x)^2}$
From Pythagoras theorem



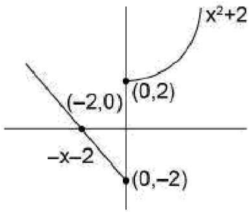
Hence, $f(x) = \sqrt{16 + x^2} + \sqrt{25 + x^2 - 8x}$; for $x \in [0, 4]$

9. See the answer

10. **(i)** $f(x) = \begin{cases} x^2 + 4; & |x| > 0 \\ 3; & x = 0 \end{cases}$



(ii) $g(x)=\begin{cases} x^2+2; & \text{if } x \geq 0 \\ -x-2; & \text{if } x<0 \end{cases}$



11. See the answer

12. See the answer

13. **(a)** $f(x) = x^2$
$\Rightarrow$ $f(a) = a^2; f(b) = b^2$
$\Rightarrow$ $f(a + b) = (a + b)^2 = a^2 + b^2 + 2ab \neq f(a) + f(b)$
(b) $f(x) = 5x$
$\Rightarrow$ $f(a) = 5a;\ f(b) = 5b$
$\Rightarrow$ $f(a) + f(b) = 5a + 5b = 5(a + b) = f(a + b)$
(c) $f(x) = -6x$
$\Rightarrow$ $f(a) = -6a; f(b) = -6b$
$\Rightarrow$ $f(a) + f(b) = -6a - 6b = -6(a + b) = -f(a + b)$
(d) $f(x)=\sqrt{x}$
$\Rightarrow$ $f(a)=\sqrt{a}$; $f(b)=\sqrt{b}$
$\Rightarrow$ $f(a)+f(b)=\sqrt{a}+\sqrt{b}\neq\sqrt{a+b}$
$\Rightarrow$ $f(a) + f(b) \neq f(a + b)$
(e) $f(x) = 3x + 1$
$\Rightarrow$ $f(a) = 3a + 1; f(b) = 3b + 1$
$\Rightarrow$ $f(a) + f(b) = 3a + 1 + 3b + 1 = 3(a + b) + 1 + 1$
$\Rightarrow$ $f(a + b) + 1 \neq f(a + b)$
(f) $f(x) = \log x$
$\Rightarrow$ $f(a) = \log a; f(b) = \log b$
$\Rightarrow$ $f(a) + f(b) = \log a + \log b = \log ab \neq \log (a + b)$
$\Rightarrow$ $f(a) + f(b) \neq f(a + b)$

14. **(a)** $f(x) = 8x$
$\Rightarrow$ $f(a) = 8a; f(b) = 8b$
$\Rightarrow$ $f(a) \,.\, f(b) = 8a \times 8b = 64ab \neq 8ab$
$\Rightarrow$ $f(a) \,.\, f(b) \neq f(ab)$
(b) $f(x) = x^5$
$\Rightarrow$ $f(a) = a^5; f(b) = b^5$
$\Rightarrow$ $f(a) \,.\, f(b) = a^5b^5 = (ab)^5 = f(ab)$
(c) $f(x)=\dfrac{1}{x^3}$
$\Rightarrow$ $f(a)=\dfrac{1}{a^3}$; $f(b)=\dfrac{1}{b^3}$
$\Rightarrow$ $f(a).f(b)=\dfrac{1}{a^3}\times\dfrac{1}{b^3}=\dfrac{1}{(ab)^3}=f(ab)$
(d) $f(x)=\sqrt{x}$
$\Rightarrow$ $f(a)=\sqrt{a}$; $f(b)=\sqrt{x}$
$\Rightarrow$ $f(a).f(b)=\sqrt{a}\sqrt{b}=\sqrt{ab}=f(ab)$
(e) $f(x) = x + 2$
$\Rightarrow$ $f(a) = a + 2; f(b) = b + 2$
$\Rightarrow$ $f(a) \,.\, f(b) = (a + 2)(b + 2) = ab + 4 + 2(a + b) \neq ab + 2$
$\Rightarrow$ $f(a) \,.\, f(b) \neq f(ab)$

15. $f(x + y) = f(x) + f(y); f(1) = 3$ for all integers x and y
(a) $f(x + y) = f(x) + f(y)$
Let $x = y = 1$
$\Rightarrow$ $f(1 + 1) = f(1) + f(1)$
$\Rightarrow$ $f(2) = 3 + 3 = 6$
$\Rightarrow$ $f(2) = 6$
(b) Let $x = 1; y = 0$
$\Rightarrow$ $f(1 + 0) = f(1) + f(0)$ $\Rightarrow$ $f(0) = 0$
(c) Let $x = -1; y = 0$
$\Rightarrow$ $f(-1 + 1) = f(1) + f(-1)$
$\Rightarrow$ $f(0) = f(1) + f(-1)$
$\Rightarrow$ $0 = f(1) + f(-1)$
$\Rightarrow$ $f(-1) = -f(1) = -3$

16. $\because$ $f(6) = f(2 \times 3) = 2 + 3 = 5$; Also $f(6) = f(1 \times 6) = 1 + 6 = 7$
$\Rightarrow$ $f(6)$ does not have a unique image
$\Rightarrow$ $f(x)$ is not a function

TEXTUAL EXERCISE–2: (OBJECTIVE)

1. **(c), (d)**
(a) one–many relation as 3 has two images in Y, i.e., 2 and 4.
(b) Not a function as 7 has no image in Y.
(c) is a function as every element $\in X$ has only one image in Y.
(d) is a function as every element $\in X$ has only one image in Y.

2. **(a)** (a) $\because$ For every $x \in \mathbb{N}, y \geq 3$ and $y \in W$
$\Rightarrow$ f is a function.
(b) $\because$ For $f(2) = -2 \notin W$
$\Rightarrow$ f is not a function.
(c) $\because$ $y = \pm(x + 2)$
$\Rightarrow$ y is one–many, and hence, it is not a function.
(d) $\because$ $y = \pm(x - 1)$
$\Rightarrow$ y is one–many relation (not a function).

3. **(c), (d)**
(a) f: $\{-2, 0, 2\} \to \{0, 1, 8, 3\}; f(x) = x^3$
If $x = -2$, then $f(x) = -8$; which is not present in co-domain, hence, it is not a function.
(b) f: $\{0, 1, 4\} \to \{-2, 1, 0, 1, 2\}$; $f(x)=\pm\sqrt{x}$
If $x = 1$, then $f(x)=\pm\sqrt{x}=\pm 1$
Hence, for one input there are two output, hence, it is not a function.
(c) f: $\{0, 1, 9\} \to \{-3, -1, 0, 1, 3\}$; $f(x)=\sqrt{x}$
If $x = 0$; $f(x)=\sqrt{x}=0$; if $x = 1$, $f(x)=\sqrt{x}=\sqrt{1}=1$
If $x = 9$; $f(x)=\sqrt{x}=\sqrt{9}=3$
Hence, for every input, there is only one output and all of which are present in the co-domain, hence, it is a function.

(**d**) f: $\{0, 1, 9\} \to \{-3, -1, 0, 1, 3\}$; $f(x) = -\sqrt{x}$

If $x = 0$; $f(0) = -\sqrt{0} = 0$, if $x = 1$, $f(1) = -\sqrt{1} = -1$

If $n = 9$; $f(9) = -\sqrt{9} = -3$

Hence, for every input, there is unique output and all of which are present in the co-domain. Hence, it is a function.

4. (**a**), (**b**) (a) For every $x \in \mathbb{N}, y \in W$

$\Rightarrow$ y is a function.

(b) For every $x \in \mathbb{N}, y \geq 0$ and $y \in W$

$\Rightarrow$ y is a function.

(c) $y = \pm\sqrt{x+1}$

$\Rightarrow$ y is a relation not a function.

5. (**a**) (a) Every element of domain has a unique image in co-domain, hence, it is a function.

(b) x_2 do not have an image

(c) x_1 has two images

(d) x_1 has two images

6. (**c**), (**d**), (**f**)

(a) many–many relation

(b) one–many relation

(e) many–many relation

(g) many–many relation

7. (**c**) $\because$ Output is obtained by increasing the magnitude of input by 1 unit.

8. (**c**), (**d**)

(a) From $[b, c]$, the relation is a one–many relation, hence, it not a function.

(b) At $x = b$, the relation has two images.

9. (**b**) $\because$ On $[1, 2) \cup [7, 8]$, $f(x)$ is one–one and $f(x) \in [1, 2) \cup [4, 5]$, i.e., $f(x)$ is in co-domain.

(a) From $(5, 7]$, the function generates values which are not an image in co-domain.

(b) At $x = 5$, the relation has two values, hence, it is not a function.

10. (**d**)

(a) $|y| = 2 - |x|$

$\Rightarrow$ $y = \pm(2 - |x|) = \begin{cases} \pm(2+x) \text{ for } x < 0 \\ \pm(2-x) \text{ for } x \geq 0 \end{cases}$

$\Rightarrow$ Relation represents one–many relation, and hence, not a function.

(b) $x + y = \pm 4$ $\quad\Rightarrow\quad$ $y = -x \pm 4$

$\Rightarrow$ Relation is one–many.

(c) $|y| = x^2 + \sin x$ $\quad\Rightarrow\quad$ $y = \pm(x^2 + \sin x)$

$\Rightarrow$ Relation is one–many.

(d) $|y| = |x|^2 - x = x^2 - x = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4}$

$\because$ $x^1 = x^2$ $\quad\Rightarrow\quad$ $\left(x_1 - \frac{1}{2}\right) = \left(x_2 - \frac{1}{2}\right)$

$\Rightarrow$ $\left(x_1 - \frac{1}{2}\right)^2 = \left(x_2 - \frac{1}{2}\right)^2$

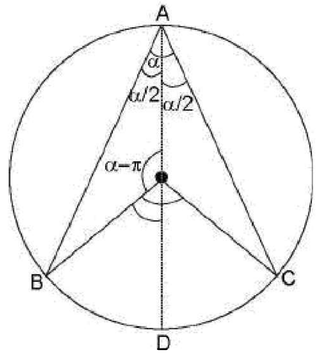
$\Rightarrow$ $\left(x_1 - \frac{1}{2}\right)^2 - \frac{1}{4} = \left(x_2 - \frac{1}{2}\right)^2 - \frac{1}{4}$

$\Rightarrow$ $y_1 = y_2$ $\quad\Rightarrow$ It is a function

11. (**a**), (**b**), (**c**)

AB and AC must be symmetric about AO

Area $(ABDC)$ = 2(area of ΔOAB) + (area of sector $OBDC$)



$$= 2\left[\frac{1}{2}r^2 \sin(\pi - \alpha)\right] + \frac{2\alpha}{2\pi}(\pi r^2) = \frac{\pi r^2}{3}$$

$\Rightarrow$ $\sin\alpha + \alpha = \frac{\pi}{3}$

$\therefore$ α satisfies the function

$\therefore$ $f(x) = c\left(x + \sin x - \frac{\pi}{3}\right)^n \forall c \in \mathbb{R}, n \in \mathbb{R}$

12. f: $\mathbb{R} - \{0\} \to \mathbb{R}$; $f(x) = \frac{1}{x}$

(**a**) $f\left(\frac{a^n}{b^n}\right) = \frac{b^n}{a^n} = \frac{\frac{1}{a^n}}{\frac{1}{b^n}} = \frac{f(a^n)}{f(b^n)}; n \in \mathbb{N}$ (True)

(**b**) $f(a^n + b^n) = \frac{1}{a^n + b^n} \neq \frac{1}{a^n} + \frac{1}{b^n}$

$\Rightarrow$ $f(a^n + b^n) \neq f(a^n) + f(b^n)$ $n \in \mathbb{N}$ (False)

(**c**) $f(a^n) = \frac{1}{a^n} = \left(\frac{1}{a}\right)^n = (f(a))^n; n \in \mathbb{N}$ (True)

(**d**) $f(a^n).f\left(\frac{1}{a^n}\right) = \frac{1}{a^n} \times a^n = 1;$ $n \in \mathbb{N}$ (True)

TEXTUAL EXERCISE–3: (SUBJECTIVE)

1. (**a**) $x^3 - 6x \geq 0$ $x(x - \sqrt{6})(x + \sqrt{6}) \geq 0$

$\Rightarrow$ $x \in [-\sqrt{6}, 0] \cup [\sqrt{6}, \infty)$

(**b**) $x^6 - 9x^3 + 8 > 0$ $\quad\Rightarrow$ $(x^3 - 1)(x^3 - 8) > 0$

$\Rightarrow$ $(x - 1)(x - 2)(x^2 + 1 + x)(x^2 + 4 + 2x) > 0$

$\Rightarrow$ $(x - 1)(x - 2) > 0$

[$\because$ $(x^2 + x + 1)$ and $(x^2 + 4 + 2x)$ are always positive as $D < 0$ and leading co-efficient > 0]

$\Rightarrow$ $x \in (-\infty, 1) \cup (2, \infty)$

(c) $x^4 - 2x^2 - 8 \geq 0 \quad \Rightarrow \quad (x^2 + 2)(x^2 - 4) \geq 0$
$\Rightarrow \quad (x^2 + 2)\,(x - 2)\,(x + 2) \geq 0$
$\Rightarrow \quad x \in (-\infty, -2] \cup [2, \infty)$

2. (a) $(x^2 - 9)^2\,(x + 1)\,(x^2 - 2x - 3)\,(x - 1) \geq 0$
$\Rightarrow \quad (x - 3)^2\,(x + 3)^2\,(x + 1)\,(x - 1)\,(x - 3)\,(x + 1) \geq 0$
$\Rightarrow \quad (x - 3)^3\,(x + 3)^2\,(x + 1)^2\,(x - 1) \geq 0$
$\Rightarrow \quad x \in (-\infty, 1] \cup [3, \infty)$
(b) $(x^2 + 3x + 1)\,(x^2 + 3x - 3) \geq 5$
$\Rightarrow \quad (x^2 + 3x)^2 - 2(x^2 + 3x) - 8 \geq 0$
$\Rightarrow \quad (x^2 + 3x - 4)\,(x^2 + 3x + 2) \geq 0$
$\Rightarrow \quad (x + 4)\,(x - 1)\,(x + 1)\,(x + 2) \geq 0$
$\Rightarrow \quad x \in (-\infty, -4] \cup [-2, -1] \cup [1, \infty)$
(c) $(x - 2)^2\,(x - 3)\,(x - 4)^4\,(x - 6) \geq 0$
$\Rightarrow \quad x \in (-\infty, 3] \cup [6, \infty) \cup \{4\}$

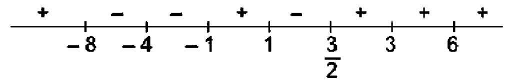
3. (a) $y = \dfrac{1}{3x - 2}; x \in (-2, 4)$
$3x \in (-6, 12)$
$3x - 2 \in (-8, 10)$
$\Rightarrow \quad \dfrac{1}{3x - 2} \in \left(-\infty, -\dfrac{1}{8}\right) \cup \left(\dfrac{1}{10}, \infty\right)$
$y \in \left(-\infty, -\dfrac{1}{8}\right) \cup \left(\dfrac{1}{10}, \infty\right)$

(b) $y = \dfrac{x - 1}{x + 2}; -1 < x < 3$
$y = \dfrac{x + 2 - 3}{x + 2} = 1 - \dfrac{3}{x + 2}$
$\because \quad x \in (-1, 3)$
$\Rightarrow \quad x + 2 \in (1, 5) \quad \Rightarrow \quad \dfrac{1}{x + 2} \in \left(\dfrac{1}{5}, 1\right)$
$\Rightarrow \quad \dfrac{-3}{x + 2} \in \left(-3, -\dfrac{3}{5}\right) \quad \Rightarrow \quad 1 - \dfrac{3}{x + 2} \in \left(-2, \dfrac{2}{5}\right)$
$\Rightarrow \quad \dfrac{x - 1}{x + 2} \in \left(-2, \dfrac{2}{5}\right) \quad \Rightarrow \quad y \in \left(-2, \dfrac{2}{5}\right)$

(c) $y = \dfrac{1}{x^2 + 4}; x \in [2, 3]$
$\Rightarrow \quad x^2 \in [4, 9] \quad \Rightarrow \quad x^2 + 4 \in [8, 13]$
$\Rightarrow \quad \dfrac{1}{x^2 + 4} \in \left[\dfrac{1}{13}, \dfrac{1}{8}\right] \quad \Rightarrow \quad y \in \left[\dfrac{1}{13}, \dfrac{1}{8}\right]$

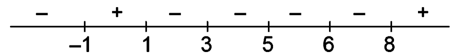
(d) $y = \dfrac{x^2 - 1}{x^2 + 1}; -1 < x < 3$
$\Rightarrow \quad y = \dfrac{x^2 + 1 - 2}{x^2 + 1} = 1 - \dfrac{2}{x^2 + 1}$
$\Rightarrow \quad x = (-1, 3) \quad \Rightarrow \quad x^2 \in [0, 9]$
$\Rightarrow \quad x^2 + 1 \in [1, 10] \quad \Rightarrow \quad \dfrac{1}{x^2 + 1} \in \left(\dfrac{1}{10}, 1\right]$
$\Rightarrow \quad \dfrac{-2}{x^2 + 1} \in \left[-2, -\dfrac{1}{5}\right) \quad \Rightarrow \quad 1 - \dfrac{2}{x^2 + 1} \in \left[-1, \dfrac{4}{5}\right)$
$\Rightarrow \quad y \in \left[-1, \dfrac{4}{5}\right)$

4. (a) $\dfrac{(x + 1)^5 (x - 1)(x + 4)^2 (x - 3)^4}{(x + 8)(2x - 3)(x - 6)^2} < 0$
$\Rightarrow \quad (x + 8)(x + 4)^2(x + 1)^5(x - 1)(2x - 3)(x - 3)^4(x - 6)^2 < 0$

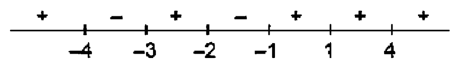


$\Rightarrow \quad x \in (-8, -4) \cup (-4, -1) \cup \left(1, \dfrac{3}{2}\right)$

(b) $\dfrac{(x + 1)(x - 8)(x - 6)^2 (x - 5)^4}{(x - 3)^2 (x - 1)} < 0$
$\Rightarrow \quad (x + 1)(x - 1)(x - 3)^2(x - 5)^4(x - 6)^2(x - 8) < 0$

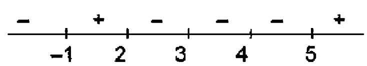


$\Rightarrow \quad x \in (-\infty, -1) \cup (1, 3) \cup (3, 5) \cup (5, 6) \cup (6, 8)$
(c) $(x^2 - 16)\,(x^2 - 5x + 4)\,(x^2 - 1)(x + 3)\,(x + 2) \leq 0$
$\Rightarrow \quad (x - 4)\,(x + 4)\,(x - 1)\,(x - 4)\,(x - 1)\,(x + 1)\,(x + 3)\,(x + 2) \leq 0$
$\Rightarrow \quad (x + 4)(x + 3)\,(x + 2)(x + 1)(x - 1)^2(x - 4)^2 \leq 0$

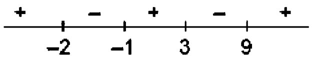


$\Rightarrow \quad x \in [-4, -3] \cup [-2, -1]$

5. (a) $x^3 - 6x \leq 0 \Rightarrow \quad x(x^2 - 6) \leq 0$
$\Rightarrow \quad (x + \sqrt{6})x(x - \sqrt{6}) \leq 0$
$\Rightarrow \quad (-\infty, -\sqrt{6}] \cup [0, \sqrt{6}]$
(b) $x^4 - 2x^2 - 8 \leq 0 \quad \Rightarrow \quad (x^2 - 4)(x^2 + 2) \leq 0$
$\Rightarrow \quad x \in [-2, 2]$
(c) $x^6 - 9x^3 + 8 \leq 0 \quad \Rightarrow \quad (x^3 - 8)(x^3 - 1) \leq 0$
$\Rightarrow \quad x \in [1, 2]$
(d) $x^4 + x^3 - x - 1 > 0 \quad \Rightarrow \quad x^3\,(x + 1) - 1\,(x + 1) > 0$
$\Rightarrow \quad (x - 1)\,(x^2 + x + 1)\,(x + 1) > 0$
$\Rightarrow \quad x \in (-\infty, -1) \cup (1, \infty)$
(e) $(x^2 - 9)^2(x + 1)(x^2 - 2x - 3)(x - 1) \leq 0$
$\Rightarrow \quad x = \pm 3, (x - 1)(x - 3)(x + 1)^2 \leq 0$
$\Rightarrow \quad x = 3, -1, (x - 1)(x - 3) \leq 0$
$\Rightarrow \quad x \in [1, 3] \cup \{-1, -3\}$
(f) $x^2 - x - 6 \geq 0 \quad \Rightarrow \quad (x - 3)(x + 2) \geq 0$
$\Rightarrow \quad x \leq -2$ or $x \geq 3$; $(x^2 - 4x) \leq 0$
$\Rightarrow \quad x \in [0, 4]$
$\Rightarrow \quad x \in [3, 4]$
(g) $4 < x^2 < 9 \quad \Rightarrow \quad x^2 \in (4, 9)$
$\Rightarrow \quad x \in (-3, -2) \cup (2, 3)$
(h) $(x + 1)\,(x - 3)^2\,(x - 5)\,(x - 4)^2\,(x - 2) < 0$
$\Rightarrow \quad (x + 1)(x - 2)\,(x - 3)^2\,(x - 4)^2\,(x - 5) < 0$



$\Rightarrow \quad x \in (-\infty, -1) \cup (2, 3) \cup (3, 4) \cup (4, 5)$
(i) $(x - 3)\,(x + 1)\,(x + 2)\,(x - 9) \geq 0$
$\Rightarrow \quad (x + 2)\,(x + 1)\,(x - 3)\,(x - 9) \geq 0$



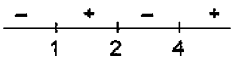
$\Rightarrow\ x \in (-\infty, -2) \cup [-1, 3] \cup (9, \infty)$

(j) $x^3 - 7x^2 + 14x - 8 < 0$

$\Rightarrow\ (x-2)(x^2 + 2x + 4) - 7x(x-2) < 0$

$\Rightarrow\ (x-2)(x^2 - 5x + 4) < 0$

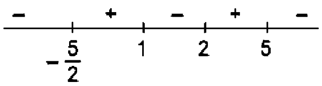
$\Rightarrow\ (x-2)(x-1)(x-4) < 0$



$\Rightarrow\ x \in (-\infty, 1) \cup (2, 4)$

(k) $\frac{(x-1)(x-2)(5-x)}{(2x+5)} < 0$

$\Rightarrow\ -(2x+5)(x-1)(x-2)(x-5) < 0$



$\Rightarrow\ x \in \left(-\infty, -\frac{5}{2}\right) \cup (1,2) \cup (5,\infty)$

6. **(a)** $x^2 - 4x - 12 \ge 0 \quad \Rightarrow \quad (x-6)(x+2) \ge 0$

$\Rightarrow\ x \in (-\infty, -2] \cup [6, \infty)$

Largest negative integer = –2

Smallest positive integer = 6

(b) $3x^2 + 5x - 2 < 0 \quad \Rightarrow\ 3x^2 + 6x - x - 2 < 0$

$\Rightarrow\ (3x-1)(x+2) < 0$

$\Rightarrow\ x \in \left(-2, \frac{1}{3}\right)$

Largest negative integer = –1

Smallest positive integer does not exist.

(c) $\frac{x^2 - 5x + 4}{x^2 - 1} \le 1 \quad \Rightarrow \quad \frac{x^2 - 5x + 4 - x^2 + 1}{x^2 - 1} \le 0$

$\Rightarrow\ \frac{-5(x-1)}{(x^2-1)} \le 0 \quad \Rightarrow \quad \frac{(x-1)}{x^2-1} \ge 0$

$\Rightarrow\ x \in (-1, \infty) - \{1\}$

Largest negative integer does not exist.

Smallest positive integer = 2.

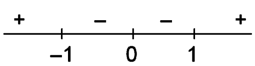
7. $\frac{x\ \ -3x+2}{\ } \le \quad \Rightarrow \quad \frac{(x-1)(x-2)}{x-3} \le 0$

$\Rightarrow\ x \in (-\infty, 1] \cup [2, 3)$

$\Rightarrow$ Number of positive integers satisfying the above condition = 2, i.e., 1 and 2

8. $\frac{x^2(x+2)}{x-1} \le 0 \ \Rightarrow\ x = 0, -2$ or $(x-1)(x+2) < 0$

$\Rightarrow\ x = 0, -2$ or $x \in (-2, 1)$



$\Rightarrow\ x \in [-2, 1)$

$\Rightarrow$ Number of integers satisfying the above inequality $\rightarrow$ 3

TEXTUAL EXERCISE–3: (OBJECTIVE)

1. **(a)** $\frac{x}{5} < \frac{3x-2}{4} - \frac{5x-3}{5}$

$\Rightarrow\ 0 < \frac{3x-2}{4} - \frac{(5x-3+x)}{5}$

$\Rightarrow\ \frac{3x-2}{4} - \frac{(6x-3)}{5} > 0$

$\Rightarrow\ \frac{15x - 10 - 24x + 12}{20} > 0$

$\Rightarrow\ \frac{-9x+2}{20} > 0 \qquad \Rightarrow\ x \in \left(-\infty, \frac{2}{9}\right)$

2. **(c)** $\frac{2x+3}{5} - 2 \le \frac{3x-6}{5}$

$\Rightarrow\ 2x + 3 - 10 \le 3x - 6 \ \Rightarrow\ x + 1 \ge 0$

$\Rightarrow\ x \in [-1, \infty)$

3. **(b)** $\frac{-x+3}{2} - 2 \le \frac{3(x+2)}{5} \le \frac{x}{6}$

$\Rightarrow\ \frac{-x+3}{2} - 2 \le \frac{3x+6}{5}$ and $\frac{3x+6}{5} \le \frac{x}{6}$

$\Rightarrow\ -5x + 15 - 20 \le 6x + 12$ and $18x + 36 \le 5x$

$\Rightarrow\ 11x + 17 \ge 0$ and $13x + 36 \le 0$

$\Rightarrow\ x \in \left[-\frac{17}{11}, \infty\right)$ and $x \in \left(-\infty, -\frac{36}{13}\right]$

$\Rightarrow\ x \in \left[-\frac{17}{11}, \infty\right) \cap \left(-\infty, -\frac{36}{13}\right]$

$\Rightarrow\ x \in \phi$

4. **(a)** $x + 5 > 2x + 2$ and $2 - x < 3x + 6$

$\Rightarrow\ x - 3 < 0$ and $4x + 4 > 0$

$\Rightarrow\ x \in (-\infty, 3)$ and $x \in (-1, \infty)$

$\Rightarrow\ x \in (-\infty, 3) \cap (-1, \infty) \Rightarrow\ x \in (-1, 3)$

5. **(b)** $2x - 12 \le 3x - 7$ and $11 - 2x \le 6 - x$

$\Rightarrow\ x + 5 \ge 0$ and $x - 5 \ge 0$

$\Rightarrow\ x \in [-5, \infty)$ and $x \in [5, \infty)$

$\Rightarrow\ x \in [5, \infty)$

6. **(c)** $-9 \le x \le 5;\ y = x^3 \quad \Rightarrow \quad x^3 \in [-729, 125]$

7. **(a)** $2 < x < 5,\ \ y = \frac{1}{x} \Rightarrow \frac{1}{5} < \frac{1}{x} < \frac{1}{2}$

$\Rightarrow\ y \in \left(\frac{1}{5}, \frac{1}{2}\right)$

8. **(b)** $y = \frac{1}{x^2}; x \in (-6, 6)\ \Rightarrow\ x \in (-6, 6)\ ;\ x^2 \in (0, 36)$

$\Rightarrow\ \frac{1}{x^2} \in \left(\frac{1}{36}, \infty\right) \qquad \Rightarrow\ y \in \left(\frac{1}{36}, \infty\right)$

9. **(a)** $x^2 \in (9, 25) \quad \Rightarrow\ x \in (-5, -3) \cup (3, 5)$

$\Rightarrow\ y \in (-5, -3) \cup (3, 5)$

10. **(b)** $x^2 - 3x + 2 > 0 \ \Rightarrow\ (x-1)(x-2) > 0$

$\Rightarrow\ x \in (-\infty, 1) \cup (2, \infty)$

11. **(a)** $x^2 - 3x - 4 \le 0 \ \Rightarrow\ (x-4)(x+1) \le 0$

$\Rightarrow\ x \in [-1, 4]$

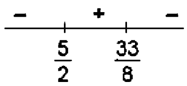
12. **(b)** $x^2 - 3x + 2 < 0 \Rightarrow (x-1)(x-2) < 0$

$\Rightarrow x\ (1, 2)$

13. **(c)** $\dfrac{4x+3}{2x-5} \le 6 \quad \Rightarrow \dfrac{4x+3-12x+30}{2x-5} \le 0$

$\Rightarrow \dfrac{-8x+33}{2x-5} \le 0$

$\Rightarrow -8x + 33 = 0$ or $(-8x+33)(2x-5) < 0$

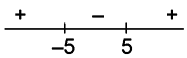


$\Rightarrow x \in \left(-\infty, \dfrac{5}{2}\right) \cup \left(\dfrac{33}{8}, \infty\right) \cup \left\{\dfrac{33}{8}\right\}$

$\Rightarrow x \in \left(-\infty, \dfrac{5}{2}\right] \cup \left[\dfrac{33}{8}, \infty\right)$

14. **(a)** $\dfrac{x}{x-5} \ge \dfrac{1}{2} \quad \Rightarrow \dfrac{2x-x+5}{2(x-5)} \ge 0$

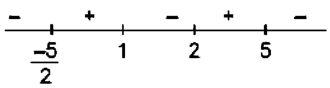
$\Rightarrow \dfrac{x+5}{2(x-5)} > 0$ or $x = -5$



$\Rightarrow x \in (-\infty, -5) \cup (5, \infty) \cup \{-5\}$

$\Rightarrow x \in (-\infty, -5] \cup (5, \infty)$

15. **(a)** $\dfrac{(x-1)(x-2)(5-x)}{(2x+5)} < 0$



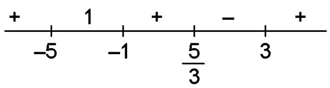
$\Rightarrow x \in \left(-\infty, -\dfrac{5}{2}\right) \cup (1,2) \cup (5, \infty)$

16. **(b)** $\dfrac{x^2-2x+5}{3x^2-2x-5} > \dfrac{1}{2}$

$\Rightarrow \dfrac{2x^2-4x+10-3x^2+2x+5}{3x^2-2x-5} > 0$

$\Rightarrow \dfrac{-x^2-2x+15}{3x^2-5x+3x-5} > 0$

$\Rightarrow \dfrac{(x+5)(x-3)}{(x+1)(3x-5)} < 0$



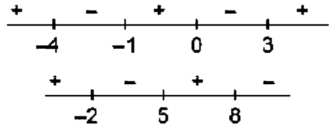
$\Rightarrow x \in (-5, -1) \cup \left(\dfrac{5}{3}, 3\right)$

17. **(c)** $\dfrac{(x-1)^2(x+1)^3}{x^4(x-2)} < 0 \qquad$ or $\dfrac{(x-1)^2(x+1)^3}{x^4(x-2)} = 0$

$\Rightarrow (x+1)^3(x-2) < 0$ or $x = \pm 1;\ x \ne 0, 2$

$\Rightarrow x \in (-1, 2)$ or $x = \pm 1 \Rightarrow x \in [-1, 0) \cup (0, 2)$

18. **(a)** $\dfrac{x(x+1)(x-3)}{(x+4)} > 0$ and $\dfrac{(5-x)(x+2)}{(x-8)} < 0$



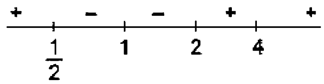
$\Rightarrow x \in (-\infty, -4) \cup (-1, 0) \cup (3, \infty)$ and $x \in (-2, 5) \cup (8, \infty)$

$\Rightarrow x \in (-1, 0) \cup (3, 5) \cup (8, \infty)$

19. **(a)** $\dfrac{(2x-1)(x-1)^4(x-2)^4}{(x-2)(x-4)^4} \le 0$

$\Rightarrow \dfrac{(2x-1)(x-1)^4(x-2)^4}{(x-2)(x-4)^4} < 0$

$\Rightarrow x = \dfrac{1}{2}$ or $x = 1$ and $n \ne 2$ and $x \ne 4$ or $\dfrac{(2x-1)}{(x-2)} < 0$



$\Rightarrow x \in \left(\dfrac{1}{2}, 1\right) \cup (1,2)$ or $x \in \left\{\dfrac{1}{2}, 1\right\}$

$\Rightarrow x \in \left[\dfrac{1}{2}, 2\right)$

20. **(a)** $(x-2)^3(x-3) < 0 \Rightarrow \qquad x \in (2, 3)$

21. **(c)** $(x-2)^4(x-3)^3(x-4)^2(1-x) < 0$ or $x \in \{2, 3, 4, 1\}$

$\Rightarrow (x-3)^3(x-1) > 0$ or $x \in \{1, 2, 3, 4\}$

$\Rightarrow x \in (-\infty, 1] \cup \{2\} \cup [3, \infty)$

22. **(b)** $c < d,\ x^2 + (c+d)x + cd < 0$

$\Rightarrow x(x+c) + d(x+c) < 0$

$\Rightarrow (x+c)(x+d) < 0 \qquad \Rightarrow x \in (-d, -c)$

23. **(a)** $a, b, c > 0$, and $a(1-b) > 1/4$

$\Rightarrow 0 < b < 1$

Similarly $0 < a < 1,\ 0 < c < 1$

$\Rightarrow a,\ 1 - b > 0$

By AM-GM inequality, $\dfrac{a+1-b}{2} \ge \sqrt{a(1-b)} > \sqrt{\dfrac{1}{4}}$

$\Rightarrow a + 1 - b > 1 \qquad \Rightarrow a > b$

Similarly, $b > c$ and $c > a$

$\Rightarrow a > b > c > a \qquad \Rightarrow a > a$, which is impossible

TEXTUAL EXERCISE–4: (SUBJECTIVE)

1. **(a)** $y = x^3 + x$

$\Rightarrow \dfrac{dy}{dx} = 3x^2 + 1 \ge 0 \forall x \in \mathbb{R}$

$\therefore \dfrac{dy}{dx} > 0$ every where, hence, $y = x^3 + x$ increases every where.

(b) $y = x^4 - 2x^2 - 5$

$\Rightarrow \dfrac{dy}{dx} = 4x^3 - 4x$

$\Rightarrow f'(x) = 4x(x^2 - 1) = 4x(x - 1)(x + 1)$

$$\begin{array}{cccc} - & + & - & + \\ & -1 & 0 & 1 \end{array}$$

$\Rightarrow f'(x) \ge 0$, i.e., $f(x) \uparrow$, when $x \in [-1, 0]$ or $[1, \infty)$ and $f'(x) \le 0$, i.e., $f(x) \downarrow$, when $x \in (-\infty, -1]$ or $[0, 1]$.

2. $y = \sin x + \cos x$, $x \in (0, \pi)$

$$\Rightarrow \frac{dy}{dx} = \cos x - \sin x = \begin{cases} +ve; & x \in \left(0, \frac{\pi}{4}\right) \\ -ve; & x \in \left(\frac{\pi}{4}, \pi\right) \end{cases}$$

Hence, $f(x)\uparrow$, when $x \in \left(0, \frac{\pi}{4}\right)$ and $\downarrow$ when $x \in \left[\frac{\pi}{4}, \pi\right)$

For $x \in \left(0, \frac{\pi}{4}\right]$; $f(x)$ is increasing

$\Rightarrow f(27°) > f(23°)$

$\Rightarrow \sin 27° + \cos 27° > \sin 23° + \cos 23°$

$\Rightarrow \sin 53° + \cos 53° > \sin 23° + \cos 23°$

Hence, the given result is false.

3. $y = \dfrac{\sin x}{x}$

$$\Rightarrow \frac{dy}{dx} = \frac{x\cos x - \sin x}{x^2} = \frac{x\cos x\left(1 - \frac{\tan x}{x}\right)}{x^2}$$

$$= \frac{\cos x\left(1 - \frac{\tan x}{x}\right)}{x}$$

For $\left(0, \frac{\pi}{2}\right)$

$\Rightarrow \left(1 - \dfrac{\tan x}{x}\right)$ is –ve, x is +ve and cos x is +ve and

$\dfrac{dy}{dx} = -\dfrac{4}{\pi^2} < 0$

$\Rightarrow \dfrac{dy}{dx}$ is –ve in $\left(0, \frac{\pi}{2}\right]$

From $\left(\frac{\pi}{2}, \pi\right)$

$\Rightarrow 1 - \dfrac{\tan x}{x}$ is +ve, x is +ve and cos x is –ve

$\Rightarrow \dfrac{dy}{dx}$ is –ve over the interval $(0, \pi)$, hence, y is always decreasing

$\Rightarrow 0 < \alpha < \beta < \pi$

$\Rightarrow \dfrac{\sin\alpha}{\alpha} > \dfrac{\sin\beta}{\beta}$, hence, verified.

4. $y = 2x^3 + 3x^2 - 12x + 1$

$\Rightarrow \dfrac{dy}{dx} = 6x^2 + 6x - 12 = 6(x^2 + x - 2) = 6(x + 2)(x - 1)$

$$\begin{array}{ccc} + & - & + \\ & -2 & 1 \end{array}$$

$$\Rightarrow \frac{dy}{dx} = \begin{cases} -ve \text{ when } x \in (-2, 1) \\ +ve \text{ when } x \in (-\infty, -2) \cup (1, \infty) \end{cases}$$

Hence, function decreases in the interval $(-2, 1)$

5. $y = \sqrt{2x - x^2} \Rightarrow \dfrac{dy}{dx} = \dfrac{2 - 2x}{2\sqrt{2x - x^2}} = \dfrac{1 - x}{\sqrt{2x - x^2}} > 0$ for $x - 1 \le 0$ and $(2 - x)x > 0$, i.e., $x \le 1$ and x $(0, 2)$

$\Rightarrow x \in (0, 1]$

$\therefore f(x) \uparrow$ on $(0, 1)$ and $\downarrow$ on $(1, 2)$.

6. $y = \dfrac{x^2 - 1}{x}$

$$\Rightarrow \frac{dy}{dx} = \frac{x(2x) - (x^2 - 1)}{x^2} = \frac{2x^2 - x^2 + 1}{x^2} = \frac{x^2 + 1}{x^2} = 1 + \frac{1}{x^2}$$

$\Rightarrow \dfrac{dy}{dx}$ is always +ve $\forall\, x \in (-\infty, \infty) - \{0\}$, i.e., increasing on every interval not containing 0.

7. **(a)** $y = (x - 2)^5 (2x + 1)^4$

$y' = 5(x - 2)^4 (2x + 1)^4 + 8(2x + 1)^3 (x - 2)^5$

$= (x - 2)^4 (2x + 1)^3 [5(2x + 1) + 8(x - 2)]$

$= (x - 2)^4 (2x + 1)^3 (18x - 11)$

$$\begin{array}{ccccc} + & & - & + & + \\ & -\frac{1}{2} & \frac{11}{18} & 2 & \end{array}$$

$\therefore y \uparrow$ when $x \in \left(-\infty, -\frac{1}{2}\right)$ or $\left(\frac{11}{8}, \infty\right)$ and $y \downarrow$ when $x \in \left[-\frac{1}{2}, \frac{11}{18}\right]$

(b) $y = x - e^x \quad \Rightarrow y' = 1 - e^x$

$\Rightarrow y \uparrow$ when $x \in (-\infty, 0]$ and $y \downarrow$ when $x \in [0, \infty)$

$$\begin{array}{cc} + & - \\ & 0 \end{array}$$

(c) $y = x^2 e^{-x}$

$\Rightarrow y' = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2 - x) = x(2 - x)e^{-x}$

$\because e^{-x}$ is always +ve

$\Rightarrow$ sign of y' depends on $x(2 - x)$

$$\begin{array}{ccc} - & + & - \\ & 0 \quad 2 & \end{array}$$

$\Rightarrow y \uparrow$ when $x \in [0, 2]$ and $y \downarrow$ when $x \in (-\infty, 0]$ or $[2, \infty)$

(d) $y = \dfrac{x}{\ln x} \quad \Rightarrow y' = \dfrac{\ln x - \frac{1}{x} \times x}{(\ln x)^2} = \dfrac{\ln x - 1}{(\ln x)^2}$

$$\begin{array}{ccc} - & - & + \\ 0 & & e \end{array}$$

$\Rightarrow y$ when $x \in [e, \infty)$ and $y \downarrow$ when $x \in (0, e] - \{1\}$

(e) $y = x - 2\sin x$; $x \in [0, 2\pi]$

$$\Rightarrow y' = 1 - 2\cos x = \begin{cases} +ve & \left(\frac{\pi}{3}, \frac{5\pi}{3}\right) \\ -ve & \left(\frac{5\pi}{3}, 2\pi\right) \cup \left(0, \frac{\pi}{3}\right) \end{cases}$$

$\Rightarrow \quad y\uparrow \forall x \in \left[\frac{\pi}{3},\frac{5\pi}{3}\right]$ and $y\downarrow \forall x \in \left[0,\frac{\pi}{3}\right]$ or $\left[\frac{5\pi}{3},2\pi\right]$

(f) $y = x + \cos 2x;\ x \in [0, 2\pi]$

$$\Rightarrow \quad y' = 1 - 2\sin 2x = \begin{cases} \le \text{ for } x \in \left[\frac{\pi}{12},\frac{5\pi}{12}\right] \text{or} \left[\frac{13\pi}{12},\frac{17\pi}{12}\right]; y\downarrow \\ \ge \text{ for } x \in \left[0,\frac{\pi}{12}\right] \text{or} \left[\frac{5\pi}{12},\frac{13\pi}{12}\right] \cup \left[\frac{17\pi}{12},2\pi\right]; y\uparrow \end{cases}$$

8. **(i)** $f(x) = 2x + 3 \quad \Rightarrow f'(x) = 2$

$\Rightarrow$ $f(x)$ is always $\uparrow$

(ii) $f(x) = a^x \quad \Rightarrow \quad a > 1$

$\Rightarrow$ $f'(x) = (\ln a)\, a^x \rightarrow +\text{ve}$

$\Rightarrow$ $f(x)$ is always $\uparrow$

(iii) $f(x) = x^3 \quad \Rightarrow \quad f'(x) = 3x^2 \ge 0$

$\Rightarrow$ $f(x)$ is always $\uparrow$

(iv) $f(x) = \sqrt{x} \quad \Rightarrow \quad f'(x) = \dfrac{1}{2\sqrt{x}} > 0$

$\Rightarrow$ $f(x)$ is always $\uparrow$

(v) $f(x) = \sqrt[3]{x} \Rightarrow f'(x) = \dfrac{1}{3(x)^{\frac{2}{3}}} > 0$. Hence, $f(x)$ is always $\uparrow$

(vi) $f(x) = \log_a x$ and $a > 1 \Rightarrow \quad f'(x) = \dfrac{1}{x\log_e a} > 0$

$\Rightarrow$ $f(x)$ is always $\uparrow$

9. $f(x) = x^3 - 3x^2 + 4x \Rightarrow f'(x) = 3x^2 - 6x + 4$

$D = 36 - 48 = -12 < 0$

Since $D < 0$ and leading coefficient is +ve, and hence, $f'(x)$ is always +ve.

Hence, $f(x)$ is strictly increasing in its domain.

10. **(a)** $f(x) = \sin 3x,\ x \in \left[0,\frac{\pi}{2}\right]$

$\Rightarrow$ $f'(x) = 3\cos 3x \ge 0$ for $x \in \left[0,\frac{\pi}{6}\right]$

(b) $f(x) = (x + 1)^3(x - 3)^3 = (x^2 - 2x - 3)^3$

$\Rightarrow$ $f(x) = 3(2x - 2)(x^2 - 2x - 3)^3$

$\therefore$ $f'(x) \ge 0$ for $(x - 1) \ge 0$ or $x = -1, 3$

$\Rightarrow$ $f(x)\uparrow$ for $x \in [1, \infty)$

(c) $f(x) = 2x^2 - 3x$

$\Rightarrow$ $f'(x) = 4x - 3 \ge 0$ for $x \in \left[\frac{3}{4},\infty\right)$

TEXTUAL EXERCISE–4: (OBJECTIVE)

1. **(b)** **(i)** $f(x) = \ln(x + \sqrt{x^2+1})$

$\Rightarrow \quad f'(x) = \dfrac{1}{\sqrt{x^2+1}} > 0 \forall x \in \mathbb{R}$

$\Rightarrow$ (i)$\rightarrow$(a)

(ii) $f(x) = a^x,\ a > 1 \quad \Rightarrow \quad f'(x) = a^x \ln a > 0\ \forall\ a > 1$

$\Rightarrow$ (ii)$\rightarrow$(a)

(iii) $f(x) = x^3 + x \quad \Rightarrow \quad f'(x) = 3x^2 + 1 > 0$

$\Rightarrow$ (iii)$\rightarrow$(a)

(iv) $f(x) = \tan x \quad \Rightarrow \quad f'(x) = \sec^2 x > 0$

$\Rightarrow$ (iv)$\rightarrow$(a)

(v) $f(x) = x^2 - 1 \quad \Rightarrow \quad f'(x) = 2x$

$\Rightarrow$ $f(x)\uparrow$ for $x > 0$ and $f(x)\downarrow$ for $x < 0$

$\Rightarrow$ Non-monotonic $\quad \Rightarrow$ (v)$\rightarrow$(e)

(vi) $f(x) = \cot x \quad \Rightarrow \quad f'(x) = -\text{cosec}^2\, x < 0$

$\Rightarrow$ $f(x)\downarrow \quad \Rightarrow$ (vi)$\rightarrow$(b)

2. **(b), (c)** $f(x) = x^2(x - 2)^2$

$\Rightarrow$ $f'(x) = 2x(x - 2)^2 + 2x^2(x - 2) = 2x(x - 2)(x - 2 + x)$

$= 4x(x - 2)(x - 1)$

$\Rightarrow$ $f(x)\uparrow$ when $x \in (0, 1) \cup (2, \infty)$ and $f(x)\downarrow$ when $x \in (-\infty, 0) \cup (1, 2)$

– + – +

0 1 2

3. **(b), (c)** $f(x) = \ln(1 + x) - \dfrac{2x}{2 + x}$

$D_f = (-1,\infty)$ and $f'(x) = \dfrac{1}{(1+x)} - \dfrac{[(2+x)(2) - 2x]}{(2+x)^2}$

$= \dfrac{1}{(1+x)} - \dfrac{2}{(x+2)^2} = \dfrac{x^2+2x+2}{(x+1)(x+2)^2} > 0$ on $(-1, \infty)$

$\Rightarrow$ $f(x)$ is M.I. on its domain.

4. **(a)** $x - \sin x = y \quad \Rightarrow \quad \dfrac{dy}{dx} = 1 - \cos x$

$\Rightarrow \quad \dfrac{dy}{dx} \ge 0\ \forall x \in \mathbb{R}$

$\Rightarrow$ $x - \sin x$ is increasing for all real numbers.

5. **(c)** $y = \log \sin x \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{\sin x} \times \cos x = \cot x$

$\Rightarrow \quad \dfrac{dy}{dx}$ is +ve when $x \in \left(0,\frac{\pi}{2}\right)$ and is –ve when $x \in \left(\frac{\pi}{2},\pi\right)$

Therefore, log sin x is increasing in $\left(0,\frac{\pi}{2}\right)$ and decreasing in $\left(\frac{\pi}{2},\pi\right)$

6. **(a), (b)** $y = a^{kx} - a^{-kx}\ \forall\ a > 1$

$\Rightarrow$ $f'(x) = ka^{kx}\ln a + ka^{-kx}\ln a = (k\ln a)(a^{kx} + a^{-kx})$

$\Rightarrow$ $f'(x) \ge 0$ for $k\ln a \ge 0$ for $k \ge 0$ and $f'(x) \le 0$ for $k \le 0$

$\Rightarrow$ $f(x)\uparrow$ for $k > 0$ and $\downarrow$ for $k < 0$

7. **(a)** $y = \sin(\cos x);\ x \in [\pi, 2\pi]$

$\dfrac{dy}{dx} = \cos(\cos x) \times (-\sin x) = -\sin x \cos(\cos x)$

In $[\pi, 2\pi]$, $\sin x < 0$ and $\cos x \in [-1, 1]$

Therefore, $\cos(\cos x) > 0$.

$\Rightarrow \quad \dfrac{dy}{dx} > 0$

Therefore, sin (cos x) is increasing in $[\pi, 2\pi]$

8. **(b)** $f(x) = \cos(\sin x)$

$\Rightarrow$ $f'(x) = -\cos x.\sin(\sin x) < 0$ for $x\ (0, /2)$

9. **(d)** (a) $\because f(x) = a^x$ for $a > 1$ is an increasing function

$\Rightarrow$ $x^1 < x^2 < x^3 < x^4 < \ldots$ for $x \in (1, \infty)$

$\Rightarrow$ correct order given.

(b) $\because$ $f(x) = a^x$ for $0 < a < 1$ is a decreasing function.

$\Rightarrow$ $x > x^2 > x^3 > x^4 > \ldots\ldots$ for $x \in (0, 1)$

$\Rightarrow$ correct order given

(c) $\because$ for $x \in (0, 1)$

$\Rightarrow$ $x > x^2 > x^3 > x^4 > x^5 > \ldots\ldots$

$\Rightarrow$ $x > x^3 > x^5 > \ldots\ldots$

$\Rightarrow$ $-x < -x^3 < -x^5 < \ldots\ldots$

$\Rightarrow$ $y < y^3 < y^5 \ldots.$ for $y \in (-1, 0)$; $y = -x$

Also, for $x \in (0, 1)$, $x^2 > x^4 > x^6 > \ldots\ldots$

$\Rightarrow$ $(-x)^2 > (-x)^4 > (-x)^6 > \ldots\ldots$

$\Rightarrow$ $y^2 > y^4 > y^6 > \ldots$ for $y \in (-1, 0)$

$\Rightarrow$ correct order given

(d) $\because$ for $x \in (1, \infty)$

$\Rightarrow$ $x < x^2 < x^3 < x^4 < \ldots\ldots$

$\Rightarrow$ $x < x^3 < x^5 < \ldots\ldots$

$\Rightarrow$ $-x > -x^3 > -x^5 > \ldots\ldots$

$\Rightarrow$ $y > y^3 > y^5 > \ldots.$ for $y \in (-\infty, -1)$

Also, for $x \in (1, \infty)$, $x^2 < x^4 < x^6 < \ldots\ldots$

$\Rightarrow$ $(-x)^2 < (-x)^4 < (-x)^6 < \ldots\ldots$

$\Rightarrow$ $y^2 < y^4 < y^6 < \ldots$ for $y \in (-\infty, -1)$

$\Rightarrow$ Incorrect order given

10. **(c)** (a) For a $\in (0, 1)$, $a^x \downarrow$

$\Rightarrow$ $x > x^2 > x^3 > \ldots..$ for $x \in (0, 1)$

$$\Rightarrow \left(\frac{1}{x}\right) > \left(\frac{1}{x}\right)^2 > \left(\frac{1}{x}\right)^3 > \ldots\ldots. \text{ for } x \in (1, \infty)$$

$\Rightarrow$ Correct sequence.

(b) For $a \in (1, \infty)$, $a^x \uparrow$

$\Rightarrow$ $x < x^2 < x^3 < \ldots.$ for $x \in (0, 1)$

$$\Rightarrow \left(\frac{1}{x}\right) < \left(\frac{1}{x}\right)^2 < \left(\frac{1}{x}\right)^3 < \ldots. \text{ for } x \in (0, 1)$$

$\Rightarrow$ Correct sequence.

(c) For $x \in (0, 1)$

$$\frac{1}{x} < \frac{1}{x^2} < \frac{1}{x^3} < \ldots.\text{(By part(b))}$$

$$\Rightarrow \frac{1}{x^2} < \frac{1}{x^4} < \frac{1}{x^6} < \ldots. \text{ for } x \in (0, 1)$$

$$\Rightarrow \left(\frac{-1}{x}\right)^2 < \left(\frac{-1}{x}\right)^4 < \left(\frac{-1}{x}\right)^6 < \ldots. \text{ for } x \in (-1, 0)$$

$$\Rightarrow \frac{1}{x^2} < \frac{1}{x^4} < \frac{1}{x^6} < \ldots. \text{ for } x \in (-1, 0)$$

Also $\frac{1}{x} < \frac{1}{x^3} < \frac{1}{x^5} < \ldots$ for $x \in (0, 1)$

$$\Rightarrow \left(\frac{-1}{x}\right) < \left(\frac{-1}{x^3}\right) < \left(\frac{-1}{x^5}\right) < \ldots \text{ for } x \in (-1, 0)$$

$$\Rightarrow \frac{1}{x} > \frac{1}{x^3} > \frac{1}{x^5} > \ldots.. \text{ for } x \in (-1, 0)$$

$\Rightarrow$ In correct sequence.

(d) for $x \in (1, \infty)$,

$$\frac{1}{x} > \frac{1}{x^2} > \frac{1}{x^3} > \ldots\ldots \text{ (By) part(a))}$$

$$\Rightarrow \frac{1}{x^2} > \frac{1}{x^4} > \frac{1}{x^6} > \ldots.. \text{ for } x \in (1, \infty)$$

$$\Rightarrow \left(\frac{1}{-x}\right)^2 > \left(\frac{1}{-x}\right)^4 > \left(\frac{1}{-x}\right)^6 > \ldots. \text{ for } x \in (-, -1)$$

$$\Rightarrow \frac{1}{x^2} > \frac{1}{x^4} > \frac{1}{x^6} > \ldots.. \text{ for } x \in (-\infty, -1)$$

Also $\frac{1}{x} > \frac{1}{x^3} > \frac{1}{x^5} > \ldots$ for $x \in (1, \infty)$

$$\Rightarrow \frac{1}{-x} > \frac{1}{(-x)^3} > \frac{1}{(-x)^5} > \ldots.. \text{ for } x \in (-\infty, -1)$$

$$\Rightarrow \frac{1}{x} < \frac{1}{x^3} < \frac{1}{x^5} < \ldots\ldots \text{ for } x \in (-\infty, -1)$$

$\Rightarrow$ Correct sequence

11. **(d)** (a) $\because$ for $a \in (1, \infty)$, $a^x \uparrow$

$\Rightarrow$ $x^{1/2} > x^{1/3} > x^{1/4} \ldots$ for $x \in (1, \infty)$

$\Rightarrow$ Correct sequence.

(b) $\because$ for $a \in (0, 1)$, $a^x \downarrow$

$x^{1/2} < x^{1/3} < x^{1/4} < \ldots$ for $x \in (0, 1)$

$\Rightarrow$ Correct sequence.

(c) By part (b), $x^{1/2} < x^{1/3}$, $x^{1/4} < x^{1/5} < \ldots$ for $x \in (0, 1)$

$\Rightarrow$ $x^{1/3} < x^{1/5} < x^{1/7} < \ldots..$ for $x \in (0, 1)$

$\Rightarrow$ $(-x)^{1/3} < (-x)^{1/5} < (-x)^{1/7} < \ldots$ for $x \in (-1, 0)$

$\Rightarrow$ $x^{1/3} > x^{1/5} > x^{1/7} > \ldots$ for $x \in (-1, 0)$

$\Rightarrow$ Correct sequence.

(d) By part(a), $x^{1/2} > x^{1/3} > x^{1/4} > \ldots.$ for $x \in (1, \infty)$

$x^{1/3} > x^{1/5} > x^{1/7} > \ldots$ for $x \in (1, \infty)$

$\Rightarrow$ $(-x)^{1/3} > (-x)^{1/5} > (-x)^{1/7} > \ldots.$ for $x \in (-\infty, -1)$

$\Rightarrow$ $x^{1/3} < x^{1/5} < x^{1/7} < \ldots..$ for $x \in (-\infty, -1)$

$\Rightarrow$ In correct sequence.

12. **(b), (c)** $y = \dfrac{e^{2x}-1}{e^{2x}+1}$

$$\Rightarrow \frac{dy}{dx} = \frac{2(e^{2x}+1)e^{2x} - 2e^{2x}(e^{2x}-1)}{(e^{2x}+1)^2}$$

$$= \frac{4e^{2x}}{(e^{2x}+1)^2} > 0\ \forall x \in \mathbb{R}$$

$$\Rightarrow \frac{e^{2x}-1}{e^{2x}+1} \text{ is increasing } \forall\ x \in \mathbb{R}$$

$$\Rightarrow f(x) = \frac{e^{2x}-1}{e^{2x}+1}$$

$$\Rightarrow f(-x) = \frac{e^{-2x}-1}{e^{-2x}+1} = \frac{1-e^{2x}}{1+e^{2x}} = -\left(\frac{e^{2x}-1}{e^{2x}+1}\right) = -f(x)$$

Hence, $\dfrac{e^{2x}-1}{e^{2x}+1}$ is odd.

TEXTUAL EXERCISE–5: (SUBJECTIVE)

1. **(a)** $|x| = 5$ $\Rightarrow$ $x = 5$ or $x = -5$. Hence, $x \in \{5, -5\}$

(b) $|x| = -2$ never possible as modulus of a number cannot be –ve. Hence, $x \in \{\}$.

(c) $x + |x| = 0$ $\Rightarrow$ $|x| = -x$

As modulus of a number is always non-negative, hence, x is always –non positive, hence, $x \in (-\infty, 0]$.

(d) $x + |x| = 2x \Rightarrow |x| = x$
$\Rightarrow x \in [0, \infty)$

(e) $\dfrac{x}{|x|} = -1 \Rightarrow x \neq 0$
$\Rightarrow x = -|x| \Rightarrow |x| = -x$
$\Rightarrow x \in (-\infty, 0]$ and $x \neq 0$
$\Rightarrow x \in (-\infty, 0)$

(f) $3x\sqrt{2} = \sqrt{18x^2} = 3\sqrt{2}\,|x|$
$\Rightarrow x = |x| \Rightarrow x \in [0, \infty)$

(g) $x|x| - x^2 = 0 \Rightarrow x(|x| - x) = 0$
$\Rightarrow x = 0$ or $|x| = x \Rightarrow x \in [0, \infty)$

(h) $|x| < -x$, never possible
$\Rightarrow x \in \phi$

2. (a) $|x| + x^2 + 1 = 0$; $\because |x| \geq 0, x^2 \geq 0$
$\Rightarrow x^2 + |x| \geq 0$
$\Rightarrow x^2 + |x| \neq -1$, hence, $x \in \{\}$

(b) $|5x^2 - 3| = 2 \Rightarrow 5x^2 - 3 = 2$ or $5x^2 - 3 = -2$
$\Rightarrow 5x^2 = 5$ or $5x^2 = 1 \Rightarrow x^2 = 1$ or $x^2 = \dfrac{1}{5}$
$\Rightarrow x = \pm 1$ or $x = \pm\dfrac{1}{\sqrt{5}} \Rightarrow x \in \left\{\dfrac{1}{\sqrt{5}}, \dfrac{-1}{\sqrt{5}}, 1, -1\right\}$

(c) $\left|\dfrac{x+4}{x+2}\right| = 3 \Rightarrow \dfrac{x+4}{x+2} = 3$ or $\dfrac{x+4}{x+2} = -3$
$\Rightarrow \dfrac{x+4-3x-6}{x+2} = 0$ or $\dfrac{x+4+3x+6}{x+2} = 0$
$\Rightarrow \dfrac{-2x-2}{x+2} = 0$ or $\dfrac{4x+10}{x+2} = 0$
$\Rightarrow x = -1$ or $x = -\dfrac{5}{2}$
$\Rightarrow x \in \left\{-1, -\dfrac{5}{2}\right\}$

(d) $|x^2 - 4x| = 5 \Rightarrow x^2 - 4x = 5$ or $x^2 - 4x = -5$
$\Rightarrow x^2 - 4x - 5 = 0$ or $x^2 - 4x + 5 = 0$
$\Rightarrow (x - 5)(x + 1) = 0$ or $x^2 - 4x + 5 = 0$
$\Rightarrow x = 5$ or $x = -1$ or $x^2 - 4x + 5 = 0$
$\Rightarrow D = 16 - 20 = -4$.
Hence, no real roots, $x \in \{5, -1\}$

(e) $|x + 1| + 2 = 2 \Rightarrow |x + 1| = 0$
$\Rightarrow x = -1$

(f) $|3x - 4| = \dfrac{1}{2} \Rightarrow 3x - 4 = \dfrac{1}{2}$ or $3x - 4 = -\dfrac{1}{2}$
$\Rightarrow x = \dfrac{3}{2}$ or $x = \dfrac{7}{6} \Rightarrow x \in \left\{\dfrac{3}{2}, \dfrac{7}{6}\right\}$

(g) $|x + 2| = 2(3 - x)$ and $3 - x > 0$
$\Rightarrow x < 3$
$\Rightarrow x + 2 = 2(3 - x)$ or $-x - 2 = 2(3 - x)$
$\Rightarrow x + 2 = 6 - 2x$ or $-x - 2 = 6 - 2x$
$\Rightarrow x = \dfrac{4}{3}$ or $\Rightarrow x = 8$ but $x < 3$
$\Rightarrow x = \dfrac{4}{3}$

(h) $|x| = -3x - 5 \Rightarrow -3x - 5 \geq 0$
$\Rightarrow -5 \geq 3x$
$\Rightarrow -\dfrac{5}{3} \geq x \Rightarrow x \in \left(-\infty, -\dfrac{5}{3}\right]$
$\Rightarrow |x| = -x$ $(\because x < 0)$
$\Rightarrow -x = -3x - 5 \Rightarrow 2x = -5$
$\Rightarrow x = -\dfrac{5}{2}$

(i) $x^2 + |x - 1| = 1$; $|x - 1| = 1 - x^2$; $1 - x^2 \geq 0$
$\Rightarrow x \in [-1, 1] \Rightarrow x - 1 \leq 0$
$\Rightarrow -x + 1 = 1 - x^2 \Rightarrow x^2 - x = 0$
$\Rightarrow x(x - 1) = 0 \Rightarrow x = 0$ or $x = 1$
$\Rightarrow x \in \{0, 1\}$

3. $\sqrt{x^2 - 2x + 1} - \sqrt{x^2 + 2x + 1} = \sqrt{(x-1)^2} - \sqrt{(x+1)^2}$
$= |x - 1| - |x + 1|$

Case (i): $x \leq -1 \Rightarrow x - 1 \leq -2$ and $x + 1 \leq 0$
$\Rightarrow |x - 1| = -(x - 1)$ and $|x + 1| = -(x + 1)$
$\Rightarrow \sqrt{x^2 - 2x + 1} - \sqrt{x^2 + 2x + 1} = |x - 1| - |x + 1| = -x + 1 + x + 1 = 2$

Case (ii): $x \in (-1, 1)$; $x - 1 \in (-2, 0)$ and $x + 1 \in (0, 2)$
$\Rightarrow |x - 1| = 1 - x$ and $|x + 1| = x + 1$
$\Rightarrow \sqrt{x^2 - 2x + 1} - \sqrt{x^2 + 2x + 1} = |x - 1| - |x + 1| = 1 - x - x - 1 = -2x$

Case (iii): $x \geq 1 \Rightarrow x - 1 \geq 0$ and $x + 1 \geq 2$
$\Rightarrow |x - 1| = x - 1$ and $|x + 1| = x + 1$
$\Rightarrow \sqrt{x^2 - 2x + 1} - \sqrt{x^2 + 2x + 1} = x - 1 - (x + 1) = -2$

4. (a) $|x| > 2 \Rightarrow x > 2$ or $-x > 2$
$\Rightarrow x \in (2, \infty)$ or $x \in (-\infty, -2)$
$\Rightarrow x \in (-\infty, -2) \cup (2, \infty)$

(b) $|x - 1| > 3 \Rightarrow x - 1 > 3$ or $-x + 1 > 3$
$\Rightarrow x > 4$ or $-2 > x \Rightarrow x \in (-\infty, -2) \cup (4, \infty)$

(c) $|x - 2| < 1 \Rightarrow 0 \leq x - 2 < 1$ or $0 \leq -x + 2 < 1$
$\Rightarrow x \in [2, 3)$ or $x \in (1, 2]$
$\Rightarrow x \in (1, 3)$

(d) $|x + 1| \geq 2 \Rightarrow x + 1 \geq 2$ or $-x - 1 \geq 2$
$\Rightarrow x \geq 1$ or $x \leq -3 \Rightarrow x \in (-\infty, -3] \cup [1, \infty)$

(e) $|x - 1| < 5 \Rightarrow 0 \leq x - 1 < 5$ or $0 \leq 1 - x < 5$
$\Rightarrow x \in [1, 6)$ or $x \in (-4, 1]$
$\Rightarrow x \in (-4, 6)$

5. (a) $|x - 1| + |x - 3| = 2$

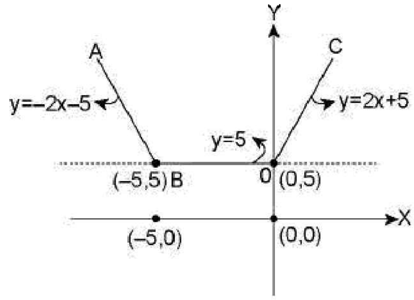
Case: (i) Let $x \in (-\infty, 1]$, then $1 - x + 3 - x = 2$
$\Rightarrow 2 = 2x \Rightarrow x = 1$

Case: (ii) Let $x \in (1, 3]$, then $x - 1 + 3 - x = 2$
$\Rightarrow 2 = 2$ true for all $x \in (1, 3]$

Case: (iii) Let $x \in (3, \infty)$, then $x - 1 + x - 3 = 2$
$\Rightarrow x = 3$, but $x \in (3, \infty)$, hence, no $x \in (3, \infty)$ satisfies the equation.

Hence, solution of equation is $x \in [1, 3]$

(b) $|x| + |x + 5| = 5$. Let $y = |x| + |x + 5| = f(x)$, then $ABOC$ represents the graph of $y = f(x)$ as shown below:



$\therefore$ $y = 5$ holds from B to 0

$\Rightarrow$ $x \in [-5, 0]$

Hence, solution of equation is $x \in [-5, 0]$

(c) $|x-1| + |x-4| = 2$

Case: (i) Let $x \in (-\infty, 1]$

$\Rightarrow$ $|x-1| + |x-4| = 1 - x + 4 - x = 5 - 2x$

$\Rightarrow$ $5 - 2x = 2 \quad \Rightarrow \quad \frac{3}{2} = x$, but $\frac{3}{2} \notin (-\infty, 1]$

$\Rightarrow$ No solution in this case.

Case: (ii) Let $x \in (1, 4]$

$\Rightarrow$ $|x-1| + |x-4| = x - 1 + 4 - x = 3 \neq 2$

Hence, no $x \in (1, 4]$ satisfies the equation.

Case: (iii) Let $x \in [4, \infty)$

$\Rightarrow$ $|x-1| + |x-4| = 2x - 5$

$\Rightarrow$ $2x - 5 = 2 \quad \Rightarrow \quad x = \frac{7}{2}$, but $\frac{7}{2} \notin [4, \infty)$

$\Rightarrow$ No solution in this case.

Hence, there is no solution to the equation.

$\Rightarrow$ $x \in \phi$

(d) $|x^2 - 2x| + |x-4| = |x^2 - 3x + 4|$

$\Rightarrow$ $|x(x-2)| + |x-4| = x^2 - 3x + 4$

($|x^2 - 3x + 4| = x^2 - 3x + 4$ as leading coefficient is +ve and $D < 0$, so, the value of quadratic will always be +ve)

Case: (i) Let $x \in (-\infty, 0]$

$\Rightarrow$ $x^2 - 2x + 4 - x = x^2 - 3x + 4$

$\Rightarrow$ $4 = 4$

Hence, true $\forall\, x \in (-\infty, 0]$

Case: (ii) Let $x \in (0, 2]$

$\Rightarrow$ $2x - x^2 + 4 - x = x^2 - 3x + 4$

$\Rightarrow$ $4x = 2x^2$

$\Rightarrow$ $x = 0$ or $x = 2$

$\Rightarrow$ $x = 2$ (By case condition)

Case: (iii) Let $x \in (2, 4]$

$\Rightarrow$ $x^2 - 2x + 4 - x = x^2 - 3x + 4$

$\Rightarrow$ $4 = 4$

$\Rightarrow$ True for all $x \in (2, 4]$.

Case: (iv) Let $x \in (4, \infty)$

$\Rightarrow$ $x^2 - 2x + x - 4 = x^2 - 3x + 4$

$\Rightarrow$ $x = 4 \notin (4, \infty)$

Hence, solution of the equation is $x \in (-\infty, 0] \cup [2, 4]$

(e) $|x^2 - 2x| + |x-4| < |(x^2 - 2x) - (x-4)|$

$$\begin{bmatrix} \because & |a \pm b| \le |a| + |b| \\ \Rightarrow & |a| + |b| \not< |a \pm b| \end{bmatrix}$$

6. $\sqrt{9 - 6a + a^2} + \sqrt{9 + 6a + a^2} = \sqrt{(a-3)^2} + \sqrt{(a+3)^2}$

$= |a-3| + |a+3|$; $a < -3$

$\Rightarrow$ $|a-3| = 3 - a$ and $|a+3| = -3 - a$

$\Rightarrow$ $\sqrt{9 - 6a + a^2} + \sqrt{9 + 6a + a^2} = |a-3| + |a+3| = 3 - a - 3 - a = -2a$

7. **(a)** $f(x) = \operatorname{cosec} x$; $D_f = \mathbb{R} - \{n\pi;\, n \in \mathbb{Z}\}$; $R_f = (-\infty, -1] \cup [1, \infty)$

$g(x) = \frac{1}{\sin x}$; $D_f = \mathbb{R} - \{n\pi;\, n \in \mathbb{Z}\}$; $R_f = (-\infty, -1] \cup [1, \infty)$

$\Rightarrow$ $f(x)$ and $g(x)$ are identical

(b) $f(x) = \tan x$; $D_f = \mathbb{R} - \{2n + 1)\,\frac{\pi}{2}\}$; $R_f = (-\infty, \infty)$

$g(x) = \frac{1}{\cot x}$; $D_f = \mathbb{R} - \{n\pi,\ (2n+1)\,\frac{\pi}{2};\ n \in \mathbb{Z}\}$; $R_f = (-\infty, \infty)$

$\Rightarrow$ $f(x)$ and $g(x)$ are non-identical.

(c) $f(x) = \ln e^x$; $D_f = \mathbb{R}$ and $R_f = (-\infty, \infty)$; $g(x) = x$; $D_f = \mathbb{R}$ and $R_f = (-\infty, \infty)$ and $f(x) = g(x)$

$\Rightarrow$ Identical.

(d) $f(x) = \sec x$

$= \frac{1}{\cos x} = g(x)$ and $D_f = D_g = 1$

$= \mathbb{R} - \{(2n+1)\,\frac{\pi}{2};\ n \in \mathbb{Z}\}$ and $R_f = R_g = \mathbb{R} - (-1, 1)$

8. **(a)** $f(x) = \ln x$ and $g(x) = \dfrac{1}{\log_x e}$

$$\Rightarrow \quad f(x) = \begin{cases} \ln x & \forall\, x \in (0, \infty) \\ \text{not deifined} & \forall x \in (-\infty, 0] \end{cases}$$

$$\Rightarrow \quad g(x) = \begin{cases} \ln x & x \in (0, \infty) \sim (1) \\ \text{not defined} & \forall x \in (-\infty, 0] \cup \{1\} \end{cases}$$

Hence, $f(x)$ and $g(x)$ are not identical as they don't have common natural domain.

(b) $f(x) = \sqrt{x^2 - 1}$ and $g(x) = \sqrt{x-1}\sqrt{x+1}$

$$\Rightarrow \quad f(x) = \begin{cases} \sqrt{x^2 - 1} & \text{if } x \in (-\infty, -1] \cup [1, \infty) \\ \text{not defined} & \text{if } x \in (-1, 1) \end{cases}$$

$$\Rightarrow \quad g(x) = \begin{cases} \sqrt{x-1}\,\sqrt{x+1} & \text{if } x \in [1, \infty) \\ \text{not defined} & \text{if } x \in (-\infty, 1) \end{cases}$$

Hence, $f(x)$ and $g(x)$ are not identical as they don't have common natural domain.

(c) $f(x) = \sqrt{1 - x^2}$ and $g(x) = \sqrt{1-x}\sqrt{1+x}$

$$\Rightarrow \quad f(x) = \begin{cases} \sqrt{1 - x^2} & \text{if } x \in [-1, 1] \\ \text{not defined} & \text{if } x \in (-\infty, -1) \cup (1, \infty) \end{cases}$$

$$\Rightarrow \quad g(x) = \begin{cases} \sqrt{1-x}\sqrt{1+x} & \text{if } x \in [-1, 1] \\ \text{not defined} & \text{if } x \in (-\infty, -1) \cup (1, \infty) \end{cases}$$

$\Rightarrow$ $f(x)$ and $g(x)$ have same analytical formula and hence, same natural domain $[-1, 1]$, hence, are identical.

9. **(i)** $\operatorname{sgn}(x^2 + 2) = 1 \quad \Rightarrow \quad x^2 + 2 > 0$, true $\forall\, x \in \mathbb{R}$

(ii) $\operatorname{sgn}(x^2 + 2) = -1$

$\Rightarrow$ $x^2 + 2 < 0$, not true for any $x \in \mathbb{R}$

Hence, $x \in \phi$

(iii) $\text{sgn}(x^2 - 9) + \text{sgn}(x + 3) = 0$

Case: (a) $x^2 - 9 = 0$ and $x + 3 = 0$

$\Rightarrow$ $(x - 3)(x + 3) = 0$ and $x = -3$

$\Rightarrow$ $x = 3, -3$ and $x = -3$

$\Rightarrow$ $x = -3$

Case: (b) $x^2 - 9 > 0$ and $x + 3 < 0$

$\Rightarrow$ $x \in (-\infty, -3) \cup (3, \infty)$ and $x \in (-\infty, -3)$

$\Rightarrow$ $x \in (-\infty, -3)$

Case: (c) $x^2 - 9 < 0$ and $x + 3 > 0$

$\Rightarrow$ $x \in (-3, 3)$ and $x \in (-3, \infty)$

$\Rightarrow$ $x \in (-3, 3)$

Therefore, all $x \in (-\infty, 3)$ satisfies above equation.

(iv) $|\text{sgn}(x - 1)| = 1$

Case: (a) $\text{sgn}(x - 1) = 1$

$\Rightarrow$ $x - 1 > 0 \Rightarrow x \in (1, \infty)$

Case: (b) $\text{sgn}(x - 1) = -1$

$\Rightarrow$ $x - 1 < 0$ $\quad\Rightarrow$ $x \in (-\infty, 1)$

$\Rightarrow$ $x \in \mathbb{R} \sim \{1\}$

10. **(i)** $|x - 5| \geq 2 \Rightarrow x - 5 \leq -2$ or $x - 5 \geq 2$

$\Rightarrow$ $x \leq 3$ or $x \geq 7$ $\quad\Rightarrow$ $x \in (-\infty, 3] \cup [7, \infty)$

(ii) $|\text{sgn}(x - 5)| > 0$ $\quad\Rightarrow$ $(x - 5) \neq 0$

$\Rightarrow$ $x \neq 5$ $\quad\Rightarrow$ $x \in \mathbb{R} - \{5\}$

(iii) $\text{sgn}[\tan^{-1} x] = 1$ $\quad\Rightarrow$ $\tan^{-1} x > 0$

$\Rightarrow$ $x \in (0, \infty)$

TEXTUAL EXERCISE–5: (OBJECTIVE)

1. **(a)** $\sqrt{2-|x|}+\sqrt{1+|x|}$ is defined when $2 - |x| > 0$ and $1 + |x| > 0$

$\Rightarrow$ $|x| \leq 2$ and $|x| \geq -1$ $\quad\Rightarrow$ $x \in [-2, 2]$ and $x \in \mathbb{R}$

$\Rightarrow$ $x \in [-2, 2]$

2. **(a)** $f(x)=\sqrt{x^2-|x|-2}$ is defined when $|x|^2 - |x| - 2 \geq 0$

$(|x|^2 - |x| - 2) \geq 0$ $\quad\Rightarrow$ $(|x| - 2)(|x| + 1) \geq 0$

$\Rightarrow$ $|x| \in (-\infty, -1] \cup [2, \infty)$

But $|x| \geq 0$

$\Rightarrow$ $|x| \in [2, \infty)$ $\quad\Rightarrow$ $x \in (-\infty, -2] \cup [2, \infty)$

$\Rightarrow$ $x \in \mathbb{R} \sim (-2, 2)$

3. **(b)** $f(x) = |x - 1| + |x - 2|;\ -1 \leq x \leq 3$

Case: (i) when $x \in [-1, 1]$, then $f(x) = 1 - x + 2 - x = 3 - 2x$

$\Rightarrow$ $x \in [-1, 1]$ $\quad\Rightarrow$ $-2x \in [-2, 2]$

$\Rightarrow$ $3 - 2x \in [1, 5]$

Case: (ii) when $x \in (1, 2]$, then $f(x) = x - 1 + 2 - x = 1$

Case: (iii) when $x \in (2, 3]$, then $f(x) = x - 1 + x - 2 = 2x - 3$

$\Rightarrow$ $x \in (2, 3]$

$\Rightarrow$ $2x - 3 \in (1, 3]$

$\Rightarrow$ Range of $f(x)$ when $x \in [-1, 3]$ is $[1, 5]$

4. **(d)** (a) $x \,\text{sgn}\, x = \begin{cases} -x \text{ for } x<0 \\ x \text{ for } x\geq 0 \end{cases} = |x|$

(b) $|x|\ \text{sgn} = \begin{cases} -|x| \text{ for } x<0 \\ |x| \text{ for } x\geq 0 \end{cases} = \{x \forall x \in \mathbb{R}$

(c) $x(\text{sgn})(\text{sgn}) = \begin{cases} x \text{ for } x<0 \\ x \text{ for } x\geq 0 \end{cases} = x \forall x \in \mathbb{R}$

(d) $|x|(\text{sgn}\, x)^3 = \begin{cases} -|x| \text{ for } x<0 \\ |x| \text{ for } x\geq 0 \end{cases} = x \neq |x|$

5. **(a)** $\{(\text{sgn}\, x)^{\text{sgn}\, x}\}^n = \begin{cases} \left((-1)^{-1}\right)^n \text{ for } x<0 \\ \text{Not defined at } x=0 \\ \left((1)^1\right)^n \text{ for } x>0 \end{cases}$

$= \begin{cases} -1 \text{ for } x<0 \\ \text{Not defined at } x=0 \\ 1 \text{ for } x>0 \end{cases}$

$\Rightarrow$ $f(x)$ is an odd function

6. **(d)** $f(x)=\left(\dfrac{x}{1-|x|}\right)^{\frac{1}{2001}}$ $\quad\Rightarrow\quad |x| \neq 1$

$\Rightarrow$ $x \neq 1, -1$ $\quad\Rightarrow$ $x \in \mathbb{R} \sim \{1, -1\}$

7. **(c)** $f(x)=\left(\dfrac{x}{1-|x|}\right)^{\frac{1}{2002}}$ $\quad\Rightarrow\quad |x| \neq 1$

and $\dfrac{x}{1-|x|} \geq 0$

Case (i): $x \geq 0$

$\Rightarrow$ $\dfrac{x}{1-x} \geq 0$

$\Rightarrow$ $x(1 - x) \geq 0;\ x \neq 1$

$\Rightarrow$ $x \in [0, 1]$

Case (ii): $x < 0$

$\Rightarrow$ $\dfrac{x}{1+x} \geq 0$ $\quad\Rightarrow$ $x(1 + x) > 0$

$\Rightarrow$ $x \in (-\infty, -1)$

$\therefore$ $D_f = (-\infty, -1) \cup [0, 1)$

8. **(c)** $f(x) = \dfrac{\sqrt{|\tan x|+\tan x}}{\sqrt{3x}} \Rightarrow x>0, x \neq (2n+1)\dfrac{\pi}{2}$ and $|\tan x| + \tan x \geq 0$

$\Rightarrow$ $x > 0, x \neq (2n + 1)\dfrac{\pi}{2}$; $|\tan x| \geq -\tan x$

$\Rightarrow$ $D_f = \mathbb{R}^+ - \left\{n\pi-\dfrac{\pi}{2}; n\in\mathbb{N}\right\} = \mathbb{R}^+ - \left\{n\pi-\dfrac{\pi}{2}; n \in W\right\}$

9. **(d)** (a) $f(x) = \sqrt{x^2} = |x|;\ g(x) = \left(\sqrt{x}\right)^2 = x$

$\Rightarrow$ $f(x) \neq g(x)$, i.e., not identical.

(b) $f(x) = \dfrac{1}{\sqrt{x^2}} = \dfrac{1}{|x|}$ and $x \in \mathbb{R} - \{0\}$ and $g(x) = \dfrac{x}{x^2} = \dfrac{1}{x}$ for $x \in \mathbb{R} - \{0\}$

$\Rightarrow$ $f(x)$ and $g(x)$ are not identical.

(c) $f(x) = \log(x - 1) + \log(x - 2);\ x > 2$

$= \log(x - 1)(x - 2);\ x \in (2, \infty)$

$g(x) = \log(x - 1)(x - 2))\ ;\ x \in (-\infty, 1) \cup (2, \infty)$

$\Rightarrow$ $f(x)$ and $g(x)$ are not identical.

(d) $f(x) = \sin^2 x + \cos^2 x = 1\ \forall\ x \in \mathbb{R}$ and $g(x) = 1\ \forall\ x \in \mathbb{R}$.

$\Rightarrow$ $f(x)$ and $g(x)$ are identical.

TEXTUAL EXERCISE–6: (SUBJECTIVE)

1. **(a)** $\dfrac{(15)^3.(21)^3}{(35)^2.(3)^4} = \dfrac{3^6 \times 5^3 \times 7^3}{5^2 \times 7^2 \times 3^4} = 9 \times 5 \times 7 = 315$

(b) $(-1.4)^3\left(\dfrac{25}{7}\right)^3 = -\dfrac{7^3}{5^3} \times \dfrac{5^6}{7^3} = -125$

(c) $\dfrac{5.2^{k-2}+10.2^{k-1}}{10^{k+2}} = \dfrac{5.2^{k-2}+5.2^k}{10^{k+2}}$

$= \dfrac{5^2.2^{k-2}}{5^{k+2}.2^{k+2}} = \dfrac{1}{5^k.16} = \dfrac{5^{-k}}{16}$

(d) $\left(\dfrac{\sqrt[4]{ab}-\sqrt{b}}{\sqrt{a}-\sqrt[4]{ab}}\right)^{-4} = \left(\dfrac{\sqrt{a}-\sqrt[4]{ab}}{\sqrt[4]{ab}-\sqrt{b}}\right)^4$

$= \left(\dfrac{\sqrt[4]{a}}{\sqrt[4]{b}}\left(\dfrac{\sqrt[4]{a}-\sqrt[4]{b}}{\sqrt[4]{a}-\sqrt[4]{b}}\right)\right)^4 = \dfrac{a}{b}$

2. **(a)** 2^{300} or 3^{200}

Let $2^{300} > 3^{200}$; taking log on both sides 300.

$\ln 2 > 200 \ln 3$

$\Rightarrow$ $\ln 8 > \ln 9$; which is false.

Hence, 3^{200} is greater.

(b) 54^4 or 21^{12}

Let $54^4 > 21^{12}$

$\Rightarrow$ $3^{12}.\ 2^4 > 3^{12}.\ 7^{12}$

$\Rightarrow$ $3^{12}\ (2^4 - 7^{12}) > 0$; which is false.

$\Rightarrow$ Hence, 21^{12} is greater.

(c) $(0.4)^4$ or $(0.8)^3$ $\Rightarrow$ $(0.\ 8)^3 > (0.\ 8)^4$ and $(0.\ 8)^4 > (0.\ 4)^4$ $[\because a^x \downarrow$ for $a \in (0, 1)]$

$\Rightarrow$ $(0.\ 8)^3 > (0.\ 4)^4$

(d) 10^{20} or 40^{10}

$10^{20} = (10^2)^{10} = (100)^{10} > (40)^{10}$

$\Rightarrow$ 10^{20} is greater

3. **(a)** $\left(\left(\dfrac{x^{\frac{2}{3}}}{y}\right)^2\left(\dfrac{x^2}{y}\right)^{\frac{1}{2}}\left(y^{\frac{2}{3}}\right)^{\frac{3}{2}}\right)^6 = \left(\dfrac{x^{\frac{4}{3}}}{y^2}\dfrac{x}{y^{\frac{1}{2}}}y\right)^6$

$= \left(\dfrac{x^{\frac{7}{3}}}{y^{\frac{3}{2}}}\right)^6 = \dfrac{x^{14}}{y^9} = x^{14}y^{-9}$

(b) $\sqrt{\dfrac{a}{b}\sqrt{\dfrac{a}{b}\times\dfrac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}}}}.a^{-\frac{1}{3}}.b^{\frac{1}{3}} = \sqrt{\dfrac{a}{b}.\dfrac{a^{\frac{2}{3}}}{b^{\frac{2}{3}}}}\ a^{-\frac{1}{3}}.b^{\frac{1}{3}}$

$= \dfrac{a^{\frac{5}{6}}}{b^{\frac{5}{6}}}a^{-\frac{1}{3}}.b^{\frac{1}{3}} = a^{\frac{3}{6}}b^{-\frac{1}{2}} = \left(\dfrac{a}{b}\right)^{1/2}$

4. **(a)** $4^{2x^2-1} = 2 \Rightarrow 2^{4x^2-2} = 2$

$\Rightarrow$ $4x^2 - 2 = 1$ $\Rightarrow$ $x^2 = \dfrac{3}{4}$

$\Rightarrow$ $= \pm\dfrac{\sqrt{\ }}{\ }$

(b) $5^{x-1} = 5^x 2^x.\ 2^{-x}.5^{x+1}$

$\Rightarrow$ $5^{x-1} = 5^{2x+1}$ $\Rightarrow$ $x - 1 = 2x + 1$

$\Rightarrow$ $-2 = x$

(c) $5^{2x-1} + 5^{x+1} = 250$ $\Rightarrow$ $5^{2x-1} + 5^{x+1} = 2.5^3$

$\Rightarrow$ $5.(5^{x-1})^2 + 25.(5^{x-1}) - 250 = 0$

$\Rightarrow$ $(5^{x-1})^2 + 5(5^{x-1}) - 50 = 0$

$\Rightarrow$ $(5^{x-1} + 10)\ (5^{x-1} - 5) = 0$

$\Rightarrow$ $5^{x-1} = 5$ or $5^{x-1} = -10$ (not possible)

$\Rightarrow$ $5^{x-1} = 5$ $\Rightarrow$ $x - 1 = 1$

$\Rightarrow$ $x = 2$

(d) $9^x + 6^x = 2.4^x$

$\Rightarrow$ $3^{2x} + 2^x\ 3^x = 2^{2x+1}$ $\Rightarrow$ $(3^x)^2 + 2^x\ 3^x - 2\ (2^x)^2 = 0$

$\Rightarrow$ $(3^x - 2^x)(3^x + 2.2^x) = 0$

$\Rightarrow$ $3^x = 2^x$ $\Rightarrow$ $x = 0$

(e) $2^{x+1}.\ 5^x = 200$ $\Rightarrow$ $10^x = 100$

$\Rightarrow$ $x = 2$

(f) $6^x + 6^{x+1} = 2^x + 2^{x+1} + 2^{x+2}$

$\Rightarrow$ $7.6^x = 2^x.\ 7$ $\Rightarrow$ $6^x = 2^x$

$\Rightarrow$ $x = 0$

5. **(a)** $2^{3-8x} > 1$ $\Rightarrow$ $2^{3-8x} > 2^0$

$\Rightarrow$ $3 - 8x > 0$ $\Rightarrow$ $3 > 8x$

$\Rightarrow$ $x \in \left(-\infty, \dfrac{3}{8}\right)$

(b) $16^x > 0.125$ $\Rightarrow$ $2^{4x} > 2^{-3}$

$\Rightarrow$ $4x > -3$ $\Rightarrow$ $x \in \left(-\dfrac{3}{4}, \infty\right)$

(c) $(0.3)^{2x^2-3x+6} < 0.00243$

$\Rightarrow$ $(0.3)^{2x^2-3x+6} < (0.3)^5$

$\Rightarrow$ $2x^2 - 3x + 6 > 5$ $\Rightarrow$ $2x^2 - 3x + 1 > 0$

$\Rightarrow$ $2x^2 - 2x - x + 1 > 0$

$\Rightarrow$ $(2x - 1)\ (x - 1) > 0$ $\Rightarrow$ $x \in \mathbb{R} - \left(\dfrac{1}{2}, 1\right)$

$\Rightarrow$ $x \in \left(-\infty, \dfrac{1}{2}\right) \cup (1, \infty)$

(d) $\left(\dfrac{1}{3}\right)^{\sqrt{x+2}} > 3^{-x}$ $\Rightarrow$ $3^{-\sqrt{x+2}} > 3^{-x}$

$\Rightarrow$ $-\sqrt{x+2} > -x$

$\Rightarrow$ $\sqrt{x+2} < x$

$\Rightarrow$ $x > 0$ as $\sqrt{x+2}$ is always +ve

$\Rightarrow$ $x + 2 < x^2$ $\Rightarrow$ $x^2 - x - 2 > 0$

$\Rightarrow$ $(x - 2)\ (x + 1) > 0$

$\Rightarrow$ $x \in (-\infty, -1) \cup (2, \infty)$ but $x > 0$

$\Rightarrow$ $x \in (2, \infty)$

(e) $\sqrt[x+1]{3} > 9$ $\Rightarrow$ $(3)^{1/x+1} > (3)^2$

$\Rightarrow$ $\dfrac{1}{x+1} > 2$

$\Rightarrow\ x + 1 < \frac{1}{2}$ ($\because x + 1 \geq 2$ and $x + 2\ \mathbb{N}$)

$\Rightarrow\ x < \frac{1}{2}$ but $x \geq 1 \quad \Rightarrow\ x \in \phi$

(f) $8^{\sqrt{8^x}} > 4096 \quad \Rightarrow\ 2^{3.2^{\frac{3x}{2}}} > 2^{12}$

$\Rightarrow\ 3.2^{\frac{3x}{2}} > 12 \quad \Rightarrow\ 2^{\frac{3x}{2}} > 4$

$\Rightarrow\ \frac{3x}{2} > 2 \quad \Rightarrow\ x > \frac{4}{3}$

$\Rightarrow\ x \in \left(\frac{4}{3}, \infty\right)$

(g) $\left(\frac{2}{5}\right)^{\frac{6-5x}{2+5x}} < \left(\frac{2}{5}\right)^{-2} \quad \Rightarrow\ \frac{6-5x}{2+5x} > -2$

$\Rightarrow\ \frac{6-5x+4+10x}{2+5x} > 0$

$\Rightarrow\ \frac{5(x+2)}{5x+2} > 0 \quad \Rightarrow\ (x + 2)(5x + 2) > 0$

$\Rightarrow\ x \in (-\infty, -2) \cup \left(-\frac{2}{5}, \infty\right)$

(h) $2^x + 2^{-x+1} - 3 < 0 \quad \Rightarrow\ 2^x + \frac{2}{2^x} - 3 < 0$

$\Rightarrow\ \frac{2^{2x} - 3.2^x + 2}{2^x} < 0$

$\Rightarrow\ (2^x)^2 - 2^x - 2.2^x + 2 < 0$

$\Rightarrow\ (2^x - 2)(2^x - 1) < 0$

$\Rightarrow\ 2^x - 2 > 0$ and $2^x - 1 < 0$ or $2^x - 2 < 0$ and $2^x - 1 > 0$

$\Rightarrow\ 2^x > 2$ and $2^x < 1$ or $2^x < 2$ and $2^x > 1$

$\Rightarrow\ x > 1$ and $x < 0$ or $x < 1$ and $x > 0$

$\Rightarrow\ x \in \phi$ or $x \leftarrow (0, 1)$

$\Rightarrow\ x \in (0, 1)$

6. (a) $32^{\frac{1}{3}} = 4x \ \Rightarrow\ 2^{\frac{5}{3}} = 4x$

Taking $\log_2$ on both sides, we get $\frac{5}{3} = \log_2 x + 2$

$\Rightarrow\ \log_2 x = -\frac{1}{3}$

(b) $64^{\frac{1}{4}} = 8x \quad \Rightarrow\ 2^{\frac{3}{2}} = 8x$

Taking $\log_2$ on both sides, we get $\frac{3}{2} = 3 + \log_2 x$

$\Rightarrow\ \log_2 x = -\frac{3}{2}$

7. $\frac{\log_a N}{\log_{ab} N} = 1 + \log_a b$

$\Rightarrow\ \frac{\log_a N}{\log_{ab} N} = \frac{\frac{1}{\log_N a}}{\frac{1}{\log_N ab}} = \frac{\log_N ab}{\log_N a}$

$= \log_a ab = 1 + \log_a b$

$\Rightarrow\ a > 0$ and $a \neq 1$; $b > 0$

$\Rightarrow\ ab \neq 1$ and $ab > 0$; $N > 0$ and $N \neq 1$

8. (a) $f(x) = \log_x 3$; $D_f = (0, \infty) - \{1\}$

(b) $f(x) = \log_2(x - 1)$; $D_f = (1, \infty)$

(c) $f(x) = \log_{2x-3}(x - 1)(x - 4)$; $2x - 3 > 0, 1$ and $(x - 1)(x - 4) > 0$

$\Rightarrow\ x > \frac{3}{2}$, $x \neq 2$ and $x \in (-\infty, 1) \cup (4, \infty)$

$\Rightarrow\ x \in (4, \infty)$

(d) $f(x) = \log_2 \left[\frac{(x-4)(x-5)}{(x-6)}\right]$

$\Rightarrow\ (x - 4)(x - 5)(x - 6) > 0$

$\Rightarrow\ x \in (4, 5) \cup (6, \infty)$

9. $f(x) = \ell n\left(\frac{x^3 - 5x^2 - 6x + 54}{x+3}\right)$

$= \ell n\left[\frac{(x^2 - 8x + 18)(x+3)}{(x+3)}\right]$

$\because\ x^2 - 8x + 18 > 0\ \forall x \in \mathbb{R}$.

$\Rightarrow\ x + 3 \in 0$

$\therefore\ D_f = \mathbb{R} - \{-3\}$.

$\Rightarrow\ f(x) = \ell n(x^2 - 8x + 18)$; $x \in \mathbb{R} - \{3\} = \ell n[(x - 4)^2 + 2]$; $x \in \mathbb{R} - \{3\}$

$\Rightarrow\ f(x) \in [\ell n\, 2, \infty)$

$\Rightarrow$ Smallest possible value $= \ell n\, 2$ at $x = 4$

10. (i) $\log_{(x+3)} (x + 4) = 2$

For log to be defined $x + 4 > 0$

$\Rightarrow\ x \in (-4, \infty)$

$\Rightarrow$ And $x + 3 > 0$ and $x + 3 \neq 1$

$\Rightarrow\ x > -3$ and $x \neq -2$

$\Rightarrow\ x \in (-3, -2) \cup (-2, \infty)$

Taking anti log on both sides, we get $x + 4 = (x + 3)^2$

$\Rightarrow\ x + 4 = x^2 + 6x + 9$

$\Rightarrow\ x^2 + 5x + 5 = 0$

$\Rightarrow\ x = \frac{-5 \pm \sqrt{25 - 20}}{2} = \frac{-5 \pm \sqrt{5}}{2}$

$\Rightarrow\ x = \frac{-5 + \sqrt{5}}{2}$ by applying domain restriction

(ii) $2^{\log_4(x+5)} = x + 5$

$\Rightarrow\ x + 5 > 0 \quad \Rightarrow\ x \in (-5, \infty)$

$\Rightarrow\ 2^{\frac{1}{2}\log_2(x+5)} = x + 5 \quad \Rightarrow\ (x+5)^{\frac{1}{2}} = x + 5$

$\Rightarrow\ x^2 + 10x + 25 = x + 5$

$\Rightarrow\ x^2 + 9x + 20 = 0 \quad \Rightarrow\ (x + 4)(x + 5) = 0$

$\Rightarrow\ x = -4$ or $x = -5$

Applying domain restrictions $x = -4$

11. $x + 4 > 0$; $x + 4 \neq 1$

$\Rightarrow\ x > -4$; $x \neq -3$

$\Rightarrow\ x \in (-4, -3) \cup (-3, \infty)$

Also $(x + 4)^{y+2} = 16$ as, $x, y \in \mathbb{Z}$, $x + 4$, $y + 2 \in \mathbb{Z}$

$\Rightarrow\ (x + 4)^{y+2} = (-4)^2, (-2)^4, (2)^4$, and $(4)^2, (16)^1$

$\Rightarrow\ (x, y) \in \{(-8, 0), (-6, 2), (-2, 2), (0, 0), (12, -1)\}$

But, by domain restrictions $(x, y) \in \{(-2, 2), (0, 0), (12, -1)\}$

12. $\sqrt[4]{|x-3|^{x+1}} = \sqrt[3]{|x-3|^{x-2}}$

$\Rightarrow$ $(|x-3|^{(x+1)})^3 = (|x-3|^{(x-2)})^4$

$\Rightarrow$ $|x-3|^{(3x+3)} = |x-3|^{(4x-8)}$

$\Rightarrow$ $|x-3|^{x-11} = 1$ $\quad\Rightarrow$ $x - 11 = 0$ or $|x-3| = 1$

$\Rightarrow$ $x = 11$ or $x = 4$ or 2

Also $x = 3$ satisfies the equation

$\Rightarrow$ $x \in \{2, 3, 4, 11\}$

13. $\log_{\sin x} 5 . \log_{\sin^2 x} a + 1 = 0$; $a > 0$ and no solution for $a = 1$

$\Rightarrow$ $a > 0, \neq 1$

$\Rightarrow$ $\dfrac{\log_a 5}{\log_a \sin x} . \dfrac{1}{\log_a \sin^2 x} = -1$

$\Rightarrow$ $\log_a 5 = -2(\log_a \sin x)^2$

$\Rightarrow$ $(\log_a \sin x) = \pm\sqrt{\dfrac{\log_a 5}{-2}}$

Case (i): $0 < a < 1$; L.H.S > 0

$\Rightarrow$ $\log_a \sin x = \sqrt{-\log_a \sqrt{5}} \Rightarrow \sin x = a^{\sqrt{-\log_a \sqrt{5}}}$

$\Rightarrow$ $x = n\pi + (-1)^n . \sin^{-1}\left(a^{\sqrt{-\log_a \sqrt{5}}}\right)$

Case (ii): $a > 1$; L.H.S < 0

$\Rightarrow$ $\log_a \sin x = -\sqrt{-\log_a \sqrt{5}} \Rightarrow \sin x = a^{-\sqrt{-\log_a \sqrt{5}}}$

$\Rightarrow$ $x = n\pi + (-1)^n . \sin^{-1}\left(a^{-\sqrt{-\log_a \sqrt{5}}}\right)$

14. $\log_{x-2} + \log_{x+2} 8 > \log_{x-2} 8 . \log_{x+2} 8$

Here $x - 2 > 0, \neq 1$ and $x + 2 > 0, \neq 1$

$\Rightarrow$ $x > 2$, 3, $x > -2 \neq -1$

$\Rightarrow$ $x \in (2, \infty) - \{3\}$

Case (i): $x > 3$; $\log_{x-2} 8 . \log_{x+2} 8 > 0$

$\Rightarrow$ $\dfrac{1}{\log_{x+2} 8} + \dfrac{1}{\log_{x-2} 8} > 0$

$\Rightarrow$ $\log_8 (x+2)(x-2) > 0$

$\Rightarrow$ $(x+2)(x-2) > 0$ $\quad\Rightarrow$ $x^2 - 5 > 0$

$\Rightarrow$ $x \in (-\infty, -\sqrt{5}) \cup (\sqrt{5}, \infty)$

Due to case restriction, $x \in (3, \infty)$

Case (ii): $2 < x < 3$; $\log_{x-2} 8 . \log_{x+2} 8 < 0$

$\Rightarrow$ $\log_8 (x+2)(x-2) < 0$

$\Rightarrow$ $0 < x^2 - 4 < 1$ $\quad\Rightarrow$ $4 < x^2 < 5$

$\Rightarrow$ $x \in (2, \sqrt{5})$ [Due to base restriction]

$\therefore$ $x\ (2, \sqrt{5}) \cup (3, \infty)$

15. $x > 0, y > 0, z > 0,$

L.H.S. $= x^{\ell n\left(\frac{y}{z}\right)} + y^{\ell n\left(\frac{z}{x}\right)} + z^{\ell n\left(\frac{x}{y}\right)}$

By A.M $\geq$ G.M for non-negative reals.

L.H.S $\geq 3\sqrt[3]{x^{\ell n\frac{y}{z}} . y^{\ell n\frac{z}{x}} . z^{\ell n\frac{x}{y}}} = 3\sqrt[3]{\left(\dfrac{y}{z}\right)^{\ell nx} . \left(\dfrac{z}{x}\right)^{\ell ny} . \left(\dfrac{x}{y}\right)^{\ell nz}}$

$= 3\sqrt[3]{\dfrac{y^{\ell nx}}{z^{\ell nx}} . \dfrac{z^{\ell ny}}{x^{\ell ny}} . \dfrac{x^{\ell nz}}{y^{\ell nz}}} = 3\sqrt[3]{\dfrac{x^{\ell ny} y^{\ell nz} . z^{\ell nx}}{z^{\ell nx} . x^{\ell ny} . y^{\ell nz}}}$

$= 3\ \left[\because a^{\log_N b} = b^{\log_N a}\right]$

$\therefore$ L.H.S 3

16. $\log_a\left(\dfrac{x^2 - x + 2}{-x^2 + 2x + 3}\right) > 0$. At $x = \dfrac{1}{2}, \dfrac{x^2 - x + 2}{-x^2 + 2x + 3} = \dfrac{\frac{1}{4} - \frac{1}{2} + 2}{-\frac{1}{2} + 1 + 3}$

$= \dfrac{\frac{3}{2} + \frac{1}{4}}{\frac{7}{2}} = \dfrac{\left(\frac{6+1}{4}\right)}{\left(\frac{7}{2}\right)} = \dfrac{7}{4} \times \dfrac{2}{7} = \dfrac{1}{2} < 1$

$\Rightarrow$ $0 < a < 1$ $\quad\Rightarrow$ $0 < \dfrac{x^2 - x + 2}{-x^2 + 2x + 3} < 1$

$\Rightarrow$ $-x^2 + 2x + 3 > 0$; $\dfrac{x^2 - x + 2 + x^2 - 2x - 3}{-x^2 + 2x + 3} < 0$

$\Rightarrow$ $-x^2 + 2x + 3 > 0$; $\dfrac{2x^2 - 3x - 1}{-x^2 + 2x + 3} < 0$

$\Rightarrow$ $(2x^2 - 3x - 1) < 0$

$\Rightarrow$ $\left(x - \left(\dfrac{3 - \sqrt{17}}{4}\right)\right) . \left(x + \left(\dfrac{3 + \sqrt{17}}{4}\right)\right) < 0$

$\Rightarrow$ $x \in \left(\dfrac{3 - \sqrt{17}}{4}, \dfrac{3 + \sqrt{17}}{4}\right)$

TEXTUAL EXERCISE–6: (OBJECTIVE)

1. **(a)** $4^{2\log_9 3} = 4^{\frac{2}{2}\log_3 3} = 4^1 = 4$

2. **(b)** $\log_7 2 = n$

$\Rightarrow$ $\log_{49} 28 = \dfrac{1}{2}\log_7 28 = \dfrac{1}{2}[2\log_7 2 + 1] = \eta + \dfrac{1}{2} = \dfrac{2\eta + 1}{2}$

3. **(c)** $a^2 + 4b^2 = 12ab$ $\quad\Rightarrow$ $\log(a + 2ab)$

$\Rightarrow$ $a^2 + 4b^2 + 4ab = 16ab$

$\Rightarrow$ $(a + 2b)^2 = 16ab$

Taking log on both sides, we get $2\log(a + 2b) = \log 16 + \log a + \log b$

$\Rightarrow$ $\log(a + 2b) = \dfrac{1}{2}(4\log 2 + \log a + \log b)$

4. **(c)** $\log_a ab = x$ $\quad\Rightarrow$ $1 + \log_a b = x$

$\Rightarrow$ $x - 1 = \log_a b$

$\Rightarrow$ $\dfrac{1}{x-1} = \log_b a$ $\quad\Rightarrow$ $1 + \dfrac{1}{x-1} = 1 + \log_b a$

$\Rightarrow$ $\dfrac{x}{x-1} = \log_b ab$

5. **(c)** Let $\dfrac{\ln a}{y - z} = \dfrac{\ln b}{z - x} = \dfrac{\ln c}{x - y} = m$

Let $a^{y^2 + yz + z^2} . b^{z^2 + zx + x^2} . c^{x^2 + xy + y^2} = t$

Taking log on both sides, we get $(y^2 + yz + z^2)\ln a + (z^2 + zx + x^2)\ln b + (x^2 + xy + y^2)\ln c = \ln t$

$\Rightarrow$ $\dfrac{(y^3 - z^3)}{(y - z)}\ln a + \dfrac{(z^3 - x^3)}{(z - x)}\ln b + \dfrac{\ln c}{x - y}(x^3 - y^3) = \ln t$

$\Rightarrow$ $m(y^3 - z^3 + z^3 - x^3 + x^3 - y^3) = \ln t$

$\Rightarrow$ $m(0) = \ln t$ $\quad\Rightarrow$ $0 = \ln t$

$\Rightarrow$ $t = 1$

6. (d) $81^{\log_3 5} + 27^{\frac{1}{2}\log_3 36} + 3^{4\times\frac{1}{2}\log_3 7} = 5^4 + 6^3 + 7^2 = 625 + 216 + 49 = 890$

7. (d) $f(x) = \log\left(\frac{1+x}{1-x}\right)$

(a) $f(x_1).f(x_2) = \log\left(\frac{1+x_1}{1-x_1}\right)\log\left(\frac{1+x_2}{1-x_2}\right) \neq \log\left(\frac{1+x_1+x_2}{1-x_1-x_2}\right)$,

(b) $f(x+2) - 2f(x+1) + f(x)$

$= \log\left(\frac{3+x}{-x-1}\right) - 2\log\left(\frac{2+x}{-x}\right) + \log\left(\frac{1+x}{1-x}\right)$

$\Rightarrow \log\left(\frac{3+x}{-x-1}\right)\left(\frac{1+x}{1-x}\right)\left(\frac{x^2}{(x+2)^2}\right)$

$= \log\frac{(x+3)x^2}{(x-1)(x+2)^2} \neq 0$

(c) $f(x) = \log\left(\frac{1+x}{1-x}\right)$ and $f(x+1) = \log\left(\frac{2+x}{-x}\right)$

$\Rightarrow f(x) \neq f(x+1)$

(d) $f(x_1) + f(x_2) = \log\left(\frac{1+x_1}{1-x_1}\right) + \log\left(\frac{1+x_2}{1-x_2}\right)$

$= \log\left(\frac{1+x_1+x_2+x_1x_2}{1-x_1-x_2+x_1x_2}\right)$

$= \log\left(\frac{1+\frac{x_1+x_2}{1+x_1x_2}}{1-\frac{(x_1+x_2)}{1+x_1x_2}}\right) = f\left(\frac{x_1+x_2}{1+x_1x_2}\right)$

8. (c) $y = 2^{\frac{1}{\log_x 4}} = 2^{\frac{1}{2}\log_2 x} = \sqrt{x}$

$\Rightarrow x = y^2$

9. (b) $\frac{1}{\log_a x} + \frac{1}{\log_c x} = \frac{2}{\log_b x}$

$\Rightarrow \log_x ac = \log_x b^2 \quad \Rightarrow b^2 = ac$

$\Rightarrow$ a, b, c are in G.P.

10. (c) $\log_{10}(x^2 - x - 6) - x = \log_{10}(x+2) - 4$

$\Rightarrow \log_{10}\left(\frac{x^2-x-6}{x+2}\right) = x - 4$

$\Rightarrow \log_{10}(x-3) = x - 4$

$\Rightarrow x - 3 = 10^{x-4}$

if $x = 5$, $\Rightarrow 2 = 10$ which is not true

Hence, $x \neq 5$

If $x = 3$, $\Rightarrow 0 = 10^{-1}$ which is not true

$\Rightarrow x \neq 3$

If $x = 4$, $\Rightarrow 1 = 10^0$

$\Rightarrow$ true

Hence, $x = 4$

11. (b) $(\log_{16}x)^2 - (\log_{16} x) + \log_{16} k = 0$; $k > 0$; $x > 0$

For exactly one solution, $D = 0$

$\Rightarrow 1 - 4\log_{16} k = 0 \quad \Rightarrow 1 - \frac{4}{4}\log_2 k = 0$

$\Rightarrow 1 = \log_2 k \quad \Rightarrow k = 2$

So, number of values of k is 1.

12. (c) $x + \log_{10}(1 + 2^x) = x\log_{10} 5 + \log_{10} 6$

$\Rightarrow 10^x = \frac{6.5^x}{1+2^x} \quad \Rightarrow (2^x)^2 + 2^x = 6$

$\Rightarrow (2^x)^2 + 2^x - 6 = 0$

$\Rightarrow (2^x + 3)(2^x - 2) = 0 \quad \Rightarrow 2^x = 2$

$\Rightarrow x = 1$

13. (a) $\log_{1/\sqrt{2}} \sin x \geq 0 \quad \Rightarrow \quad 0 < \sin x \leq 1$

$\Rightarrow x \in (0, \pi)$ and $(2\pi, 3\pi)$

$\Rightarrow$ Integer multiples of $\pi/4 \in \left\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{5\pi}{2}, \frac{11\pi}{4}\right\}$

That is, 6 solutions.

14. (b) $\log_{1/2}(x+1) \leq \log_2(2-x)$

$\Rightarrow 2 - x > 0$ and $x + 1 > 0$

$\Rightarrow x \in (-1, \infty)$ and $x < 2$

$\Rightarrow x \in (-1, 2) - \log_2(x+1) \leq \log_2(2-x)$

$\Rightarrow \log_2(2-x)(x+1) \geq 0$

$\Rightarrow (2-x)(x+1) \geq 1$

$\Rightarrow x^2 - x - 1 \leq 0 \quad \Rightarrow x \in \left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right]$

15. (a) $\log_{0.3}(x-1) < \log_{0.09}(x-1)$

$\Rightarrow (x-1) > 0$ and $\log_{0.3}(x-1) < \frac{1}{2}\log_{0.3}(x-1)$

$\Rightarrow (x-1) > \sqrt{x-1}$; $x > 1$

$\Rightarrow (x-1)^2 > (x-1)$; $x > 1$

$\Rightarrow x^2 - 3x + 2 > 0$; $x > 1$

$\Rightarrow (x-1)(x-2) > 0$; $x > 1$

$\Rightarrow x < 1$ or $x > 2$ and $x > 1$

$\Rightarrow x \in (2, \infty)$

16. (a) $\log_{|\sin x|}(1 + \cos x) = 2$; $x \in [0, n\pi]$; $n \in \mathbb{Z}$

$\Rightarrow 1 + \cos x > 0 \quad \Rightarrow \cos x > -1$

$\Rightarrow x \neq (\text{odd})\pi$

$\Rightarrow$ Also $|\sin x| \in (0, 1) \quad \Rightarrow x \neq \frac{n\pi}{2}$

$\therefore$ Domain of equation $[0, n\pi] - \left\{\frac{n\pi}{2}; n \in \mathbb{Z}\right\}$

$\Rightarrow$ By given equation, $(1 + \cos x) = |\sin x|^2$

$\Rightarrow 1 + \cos x = \sin^2 x$

$\Rightarrow \cos^2 x + \cos x = 0$

$\Rightarrow \cos x(1 + \cos x) = 0$ but $1 + \cos x > 0$

$\Rightarrow \cos x = 0$

$\Rightarrow$ But $x \neq \frac{n\pi}{2} \quad \Rightarrow \cos x \neq 0$

$\Rightarrow$ No real $x \in [0, n\pi]$ satisfy, i.e., $x \in \{\}$

17. (d) $\log_{\cos x} \sin x \geq 0$; $x \in [0, 3\pi]$

$\Rightarrow$ Here $\sin x > 0$, $\cos x > 0, \neq 1$

$\Rightarrow x \in \left(0, \frac{\pi}{2}\right) \cup \left(2\pi, \frac{5\pi}{2}\right)$

$\Rightarrow$ Also by given inequality, $0 < \sin x \leq 1$ which holds $\forall\, x \in \left(0, \frac{\pi}{2}\right) \cup \left(2\pi, \frac{5\pi}{2}\right)$

18. **(d)** $\log_{\sqrt{3}}\left(\sin x+2\sqrt{2}\cos x\right)\ge 2;\ -2\pi < x \le 2\pi$

$\Rightarrow$ Here $\sin x + 2\sqrt{2}\cos x \ge 3$ ....(1)

$\Rightarrow$ Also $-3 \le \sin x + 2\sqrt{2}\cos x \le 3$ ....(2)

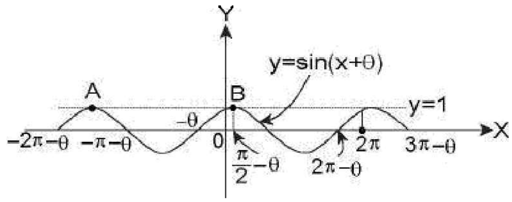
$\Rightarrow$ From (1) and (2), we get $1\sin x + 2\sqrt{2}\cos x = 3$

Let 1 = r cos

$\Rightarrow$ $r = 3$; $\tan\theta = 2\sqrt{2}$

$\Rightarrow$ $3\sin x\cos\theta + 3\sin\theta\cos x = 3$

$\Rightarrow$ $\sin(x+\theta) = 1;\ \theta = \tan^{-1}\left(2\sqrt{2}\right) = \cos^{-1}(1/3) \in \left(\frac{\pi}{3},\frac{\pi}{2}\right)$



$\Rightarrow$ There will be exactly 2 solutions.

TEXTUAL EXERCISE–7: (SUBJECTIVE)

1. L.H.S. = [sin x + [sin x + [sin x + [sin x]]]

= [sin x] + [sin x + [sin x]]

($\because$ $[x + m] = m + [x]$; $m \in \mathbb{Z}$)

= [sin x] + [sin x] + [sin x] = 3[sin x] = R. H.S.

2. $f(x)$ = [sin x] + [tan x + [cos x + [sin x]]]

= [sin x] + [tan x] + [cos x + [sin x]]

= [sin x] + [tan x] + [sin x] + [cos x]

$\because$ $x \in (0, \pi/4)$

$\Rightarrow$ $\sin x \in \left(0,\frac{1}{\sqrt{2}}\right)$

$\tan x \in (0, 1)$, $\cos x \in \left(\frac{1}{\sqrt{2}},1\right)$

$\Rightarrow$ [sin x] = 0, [tan x] = 0, [cos x] = 0

$\therefore$ $f(x) = 0 + 0 + 0 = 0$

3. $y=\frac{1}{3}[\sin x+[\sin x+[\sin x]]]\ =\ \frac{1}{3}[3[\sin x]]=[\sin x]$

$\Rightarrow$ $[y + [y]] = 2\cos x$ $\Rightarrow$ $2[y] = 2\cos x$

$\Rightarrow$ $[y] = \cos x$ $\Rightarrow$ $[[\sin x]] = \cos x$

$\Rightarrow$ $[\sin x] = \cos x$ $\Rightarrow$ $\cos x$ has to be integer

$\Rightarrow$ $\cos x \in \{1, 0, -1\}$

If $\cos x = 1$, then $\sin x = 0$

$\Rightarrow$ $[\sin x] = 0, \ne 1$, hence, not possible

If $\cos x = 0$, then $\sin x = 1$

$\Rightarrow$ $[\sin x] = 1, \ne 0$, hence, not possible

If $\cos x = -1$, then $\sin x = -1$

$\Rightarrow$ $[\sin x] = 0, \ne 1$, hence, not possible

$\therefore$ No solution

4. **(a)** $4\{x\} = x + [x]$ $\Rightarrow$ $4\{x\} = \{x\} + 2[x]$

$\Rightarrow$ $\frac{3}{2}\{x\}=[x]$

But, $\{x\} \in [0, 1)$ $\Rightarrow$ $\frac{3}{2}\{x\}\in\left[0,\frac{3}{2}\right)$

$\Rightarrow$ $[x] = 0$ or 1

If $[x] = 0$ $\Rightarrow$ $\{x\} = 0$

$\Rightarrow$ $x = [x] + \{x\} = 0$

If $[x] = 1$ $\Rightarrow$ $\{x\} = \frac{2}{3}$

$\Rightarrow$ $x=[x]+\{x\}=1+\frac{2}{3}=\frac{5}{3}$

$\therefore$ 0, 5/3

(b) $[x] - 2x| = 4$

$\Rightarrow$ $[x] - 2x = 4$ or $[x] - 2x = -4$

$\Rightarrow$ $-[x] - 2\{x\} = 4$ or $-[x] - 2\{x\} = -4$

$\Rightarrow$ $2\{x\} = -[x] - 4$ or $2\{x\} = 4 - [x]$

$\{x\} \in [0, 1)$

$\Rightarrow$ $2\{x\} \in [0, 2)$ $\quad -[x] - 4,\ 4 - [x] \in \{0, 1\}$

Case (i): $-[x] - 4 = 0$

$\Rightarrow$ $[x] = -4$ $\Rightarrow$ $\{x\} = 0$

$\Rightarrow$ $x = -4 + 0 = -4$

Case (ii): $\quad -[x] - 4 = 1$

$\Rightarrow$ $[x] = -5$ $\Rightarrow$ $\{x\} = 1/2$

$\Rightarrow$ $x = 5 + \frac{1}{2} = -\frac{9}{2}$

Case (iii): $4 - [x] = 1$ $\Rightarrow$ $[x] = 3$

$\Rightarrow$ $\{x\} = 1/2$

$\Rightarrow$ $x = 3 + 1/2 = 7/2$

Case (iv): $4 - [x] = 0$ $\Rightarrow$ $[x] = 4$

$\Rightarrow$ $\{x\} = 0$

$\Rightarrow$ $x = 4 + 0 = 4$ $\Rightarrow$ $\left\{\pm 4,\frac{7}{2},\frac{-9}{2}\right\}$

5. Follow the steps given Illustration No. 93

TEXTUAL EXERCISE–7: (OBJECTIVE)

1. **(b)** $\log_{[x]} x^2$ $\Rightarrow$ $[x] > 0$ and $[x] \ne 1$

$\Rightarrow$ $[x] > 1$ $\Rightarrow$ $[x] \ge 2$

$\Rightarrow$ $x \in [2, \infty)$

2. **(b)** $[x]\sin\left(\frac{\pi}{[x+1]}\right)$ $\Rightarrow$ $[x + 1] \ne 0$

$\Rightarrow$ $x + 1 \notin [0, 1)$ $\Rightarrow$ $x \notin [-1, 0)$

$\Rightarrow$ $x \in \mathbb{R} - [-1, 0)$

3. **(a)** $\sqrt{\frac{2-[x]}{[x]-3}}$ $\Rightarrow$ $\frac{2-[x]}{[x]-3} > 0$ or $2 - [x] = 0$ and $[x] \in \mathbb{Z}$

$\Rightarrow$ $[x] \in (2, 3)$ and $[x] \in \mathbb{Z}$ or $[x] = 2$

$\Rightarrow$ $x \in [2, 3)$

4. **(c)** $\sin^{-1}[2 - 3x^2]$

$[2 - 3x^2] \in [-1, 1]$

$\Rightarrow$ $(2 - 3x^2) \in [-1, 2)$ $\Rightarrow$ $-3x^2 \in [-3, 0)$

$\Rightarrow$ $x^2 \in (0, 1]$

$\Rightarrow$ $x \in [-1, 0) \cup (0, 1]$

5. **(b)** $f(x) = \log\left\{\frac{}{([\cos x]\quad[\sin x])}\right\}$; $[\cos x] - [\sin x] > 0$

$\Rightarrow$ $[\cos x] > [\sin x]$

$\because$ [cos x] – [sin x] is periodic with period 2π, we shall work only for $x \in [0, 2\pi]$.

$\therefore$ [cos x] > [sin x} at $x = 0$ and $x \in \left[\frac{3\pi}{2}, 2\pi\right]$

$\Rightarrow x \in \left[2n\pi + \frac{3\pi}{2}, 2n\pi + 2\pi\right]$, i.e., x

$\in \left[(2n+1)\pi + \frac{\pi}{2}, (2n+2)\pi\right]$

6. **(c)** $\sqrt{[x]-1} + \sqrt{4-[x]}$

$\Rightarrow$ $[x] - 1 \geq 0$ and $4 - [x] \geq 0$

$\Rightarrow$ $[x] \geq 1$ and $4 \geq [x]$

$\Rightarrow$ $x \in [1, \infty)$ and $x \in (-\infty, 5)$

$\Rightarrow$ $x \in [1, 5)$

7. **(a)** $e^x + \sin^{-1}\left[\left(\frac{x}{2}-1\right)\right] + \log\sqrt{x-[x]}$

$= e^x + \sin^{-1}\left[\left(\frac{x}{2}\right)-1\right] + \log\sqrt{\{x\}}$

$\left[\frac{x}{2}-1\right] \in [-1, 1] \quad \Rightarrow \quad \frac{x}{2} - 1 \in [-1, 2)$ and $\{x\} \neq 0$

$\Rightarrow$ $x \in [0, 6)$ and $x \notin \mathbb{Z}$ $\Rightarrow$ $x \in (0, 6) - \{1, 2, 3, 4, 5\}$

8. **(b)** $f : (2, 3) \to (1, 3)$; $f(x) = x - \{-x\} = x - (1 - \{x\})$

$= x + \{x\} - 1 = [x] + 2\{x\} - 1$

$\Rightarrow f(x) = \begin{cases} = 2 + 2(x-2) - 1 \\ = 2x - 3 \end{cases}$

$\Rightarrow y = 2x - 3 \quad \Rightarrow x = \frac{y+3}{2}$

$\Rightarrow f^{-1}(x) = \frac{x+3}{2}$

9. **(c)** $f(x) = \begin{cases} x - [x]; I \leq x \leq I + 0.5 \\ [x]; I + 0.5 \leq I + 1 \end{cases} = \begin{cases} x - I; I \leq x \leq I + 0.5 \\ I; I + 0.5 \leq x < I + 1 \end{cases}$

$f(g(x)) = f(\sin^4 x + \cos^4 x)$

$= f\left[1 - \frac{1}{2}\sin^2 2x\right]; k = 1 - \frac{1}{2}\sin^2 2x$

$= f\left[\frac{1}{2} \leq k \leq 1\right] = \left\{\text{for} \frac{1}{2} \leq k \leq 1\right.$

10. **(a)** $f(x) = \frac{x-[x]}{1+x-[x]} = \frac{\{x\}}{1+\{x\}} = 1 - \frac{1}{1+\{x\}}$

$\Rightarrow$ $\{x\} \in [0, 1) \quad \Rightarrow \{x\} + 1 \in [1, 2)$

$-\frac{1}{\{x\}+1} \in \left[-1, -\frac{1}{2}\right) \quad \Rightarrow \quad 1 - \frac{1}{\{x\}+1} \in \left[0, \frac{1}{2}\right)$

$\Rightarrow R_f \to \left[0, \frac{1}{2}\right)$

11. **(c)** $f(x) = \frac{e^x}{1+[x]}; x \geq 0 = \begin{cases} e^x \text{ for } 0 \leq x < 1 \\ \frac{e^x}{2} \text{ for } 1 \leq x < 2 \\ \frac{e^x}{3} \text{ for } 2 \leq x < 3 \end{cases}$ and so on.

$\Rightarrow$ $f(x) \in [1, e)$ for $x \in [0, 1)$

$\Rightarrow$ $f(x) \in \left[\frac{e}{2}, \frac{e^2}{2}\right)$ for $x \in [1, 2)$

$\Rightarrow$ $f(x) \in \left[\frac{e^2}{3}, \frac{e^3}{3}\right)$ for $x \in [2, 3)$ and so on.

$\Rightarrow$ $f(x) \in [1, \infty)$

12 **(b)** $x^3 - [x] = 3 \quad \Rightarrow x^3 - 3 = [x]$

$\Rightarrow$ x^3 should be integer

$\Rightarrow$ only $4^{\frac{1}{3}}$ satisfy the equation $(4^{\frac{1}{3}})^3 - 3 = 1 = [4^{\frac{1}{3}}]$

13. **(c)** $(x)^2 + (x + 1)^2 = 25$

$\Rightarrow$ $(x)^2 + (x)^2 + 2(x) + 1 = 25$

$\Rightarrow$ $(x)^2 + (x) - 12 = 0$

$\Rightarrow$ $((x) + 4)((x) - 3) = 0$

$\Rightarrow$ $(x) = -4$ or $(x) = 3$

$\Rightarrow$ $x \in (-5, -4]$ or $x \in (2, 3]$

$\Rightarrow$ $x \in (-5, -4] \cup (2, 3]$

14. **(a)** **Case (i):** $x \notin \mathbb{Z}$

$[x]^2 + (x)^2 > 25$

$[x]^2 + ([x] + 1)^2 > 25$

$2[x]^2 + 2[x] + 1 - 25 > 0$

$\Rightarrow$ $[x]^2 + [x] - 12 > 0$

$\Rightarrow$ $([x] + 4)([x] - 3) > 0$

$\Rightarrow$ $[x] < -4$ or $[x] > 3$

$\Rightarrow$ $[x] \leq -5$ or $[x] \geq 4$

$\Rightarrow$ $x < -4$ or $x \geq 4$

$\Rightarrow$ $x \in (-\infty, -4) \cup [4, \infty)$ ……(i)

Case (ii): $x \in \mathbb{Z}$

$\Rightarrow$ $[x] = (x) = x$

$\Rightarrow$ $2x^2 > 25$

$\Rightarrow$ $x^2 > 12.5$

$\Rightarrow$ $x > \sqrt{12.5}$ or $x < -\sqrt{12.5}$

$\Rightarrow$ $x \in [4, \infty)$ or $x \in (-\infty, -4]$ …..(ii)

$\therefore$ Combing (i) and (ii), we get $x \in (-\infty, -4] \cup [4, \infty)$

TEXTUAL EXERCISE–8: (SUBJECTIVE)

1. **(i)** $a * b = \text{HCF}(a, b)$ is a binary operation as HCF of two natural number is always another natural number.

(ii) $\because$ LCM$(a, b) \in \mathbb{N}$ for $a, b \in \mathbb{N}$

$\Rightarrow$ Binary operation

(iii) $a + b = ab \in \mathbb{Q}\ \forall\ a, b \in \mathbb{Q}$

$\Rightarrow$ $a + b + ab \in \mathbb{Q} \quad \Rightarrow$ Binary operation

(iv) $a * b = \sqrt{ab}$ is not a binary operation on $\mathbb{R}$.

e.g., $a = 2, b = -2 \quad \Rightarrow \quad \sqrt{-4} \notin \mathbb{R}$

(v) $a * b = \sqrt{a+b}$ is not a binary operation on $\mathbb{R}$.

e.g., $a = 2$ $b = -3 \quad \Rightarrow \quad a + b = -1$

$\Rightarrow$ $\sqrt{a+b} \notin \mathbb{R}$

Hence, not a binary operation.

(vi) $\because$ For $b, a \in \mathbb{R}$, Disc. $= (-3b)^2 - 4(1)(4b)^2 = -7b^2 < 0$

$\Rightarrow$ $a^2 - 3ab + 4b^2 > 0\ \forall\ a, b \in \mathbb{R}$

$\Rightarrow$ $\sqrt{a^2 - 3ab + 4b^2} \in \mathbb{R}$ Binary operation

(vii) $a * b = (a - b)$ on $\mathbb{Z}^+$
Put $a = 2, b = 3$ $\Rightarrow$ $a - b = -1 \notin \mathbb{Z}^+$
Not a binary operation.
(viii) If $a = b$, then $a * b = |a - b| = 0 \in \mathbb{Z}^+$
$\Rightarrow$ Not a binary operation
(ix) $\because$ $a, b \in \mathbb{Z}^+$ $\Rightarrow$ $a.b \in \mathbb{Z}^+$
$\Rightarrow$ Binary operation
$\Rightarrow$ (x) $a, b \in \mathbb{R}$ $\Rightarrow$ $a, b^2 \in \mathbb{R}$
$\Rightarrow$ $a.\, b^2 \in \mathbb{R}$ $\Rightarrow$ Binary operation.
(xi) $\because a, b \in \mathbb{Z}^+$ $\Rightarrow$ $a * b = a \in \mathbb{Z}^+$
$\Rightarrow$ Binary operation

2. **(i)** $a * b = a - b$
Neither commutative nor associative as subtraction of real numbers is so.
(ii) $a * b = a^3 + b^3$
$\Rightarrow$ Commutative as $a * b = b * a$, i.e., $a^3 + b^3 = b^3 + a^3$
Not associative as $a *(b * c) = a *(b^3 + c^3) = a^3 + (b^3 + c^3)^3$ and $(a * b) * c = (a^3 + b^3) * c = (a^3 + b^3) + c^3$
(iii) $a * b = a + 2ab$
$\Rightarrow$ Not commutative, as $b * a = b + 2ab$
$\Rightarrow$ Not associative, as $(a * b)*c = (a + 2ab)* c = (a + 2ab) + 2(a + 2ab)c$ and $a*(b * c) = a + 2a(b * c) = a + 2a\,(b + 2bc)$
(iv) $a * b = (a - b)^4$
$\Rightarrow$ Commutative, $a * b = b * a$ as $(a - b)^4 = (b - a)^4$
$\Rightarrow$ Not associative, as a $*(b * c) = a *(b - c)^4 = [a - (b - c)^4]^4$ and $(a * b) * c = (a - b)^4 * c = [(a - b)^4 - c]^4$
(v) $a * b = ab/8$ $\because$ $ab/8 = \mathrm{ba}/8$
$\Rightarrow$ Commutative as well as associative as $(a * b) * c$ $= \left(\frac{ab}{8}\right) * c = \frac{abc}{64}$ and (a $*(b * c) =$ $a * \left(\frac{bc}{8}\right) = \frac{abc}{64}$

(vi) $a * b = 3ab^2$ and $(b * a) = 3ba^2$
$\Rightarrow$ Not commutative
Also, $(a * b) * c = 3ab^2 * c = 3(3ab^2)c^2 = 9ab^2c^2$
and $a * (b * c) = a * (3bc^2) = 3a(3bc^2)^2 = 27ab^2c^4$
$\Rightarrow$ Not associative

3. LCM of $(a, b) = a * b$ is not a binary operation on A as $3 * 5 = 15 \notin A$ and $5 * 7 = 35 \notin A$

4. Yes, $a * b$ is a binary operation of N
HCF (a, b) = HCF (b, a)
$\Rightarrow$ * is commutative
Also * *is* associative.
No identity element exists for * as $\exists$ no $e \in \mathbb{N}$ such that HCF $(a, e) = a$

5. **(i)** $6 * 7 = 42$
$15 * 24 = 120$
(ii) $(a * b) * c = \mathrm{a} * (b * c)$
$\Rightarrow$ * is associative
(iii) LCM (a, b) = LCM (b, a)
$\Rightarrow$ * is a commutative
(iv) LCM $(1, a) = a$
$\Rightarrow$ 1 is the identity elements of *
(v) 1 is the only inventible element

6. $(a, b) * (c, d) = (a + c, b + d) = (c + a, d + b) = (c, d)* (a, b)$
$\Rightarrow$ * is commutative
Let $(a, b), (c, d), (e, f) \in \mathbb{N} \times \mathbb{N}$, then $(a, b) * [(c, d)* (e, f)]$
$= (a, b)* [(c + e, d + f)] = (a + (c + e), b + (d + f)) = ((a + c), e, (b + d) + f) = ((a + c), (b + d)) * (e, f) = ((a, b) * (c, d)) * (e, f)$
$\Rightarrow$ * is associative on $\mathbb{N} \times \mathbb{N}$.

7. **(i)** $(3 * 4) * 5 = (1 * 5) = 1$
(ii) Yes it is commutative
(iii) $(3 * 4) * (4 * 5) = (1) * (1) = 1$

8. See the Answer.

9. **(a)** $a * b = a + 2b$
$(a * b) * c = (a + 2b) * c = a + 2b + 2c$
$a * (b * c) = \mathrm{a} * (b + 2c) = a + 2b + 4c$
$\Rightarrow$ Not associative
(b) $a * b = a + 2b$
$b * a = b + 2a$
$\Rightarrow$ $a * b \neq b * a$ $\Rightarrow$ Not commutative

10. $\because$ for $a \in \mathbb{N}, -a \notin \mathbb{N}$
$\Rightarrow$ $-a$ can't be inverse of 'a' moreover the identity element '0' does not exist in $\mathbb{N}$.
Also for $a \in \mathbb{N} - \{1\}$, $\frac{1}{a} \in \mathbb{N}$, and hence, $\frac{1}{a}$ can't be the multiplicative inverse of a on $\mathbb{N}$.

11. **(i)** $(a * b) = a + 4b^2$ ($\because$ Product and sum of two real numbers is real)
$\Rightarrow$ $a + 4b^2 \in \mathbb{R}$ $\Rightarrow$ $a * b$ is a binary operation.
(ii) $\Delta(a, b) = \min\{a, b\}$.
If a, b are reals.
$\Rightarrow$ $a * b = \min\{a, b\}$ is also real.
Similarly $a * b = \max\{a, b\}$ is also binary operations.
(iii) $\cup$: $\mathrm{P} \times P \to P$
$\cup (A, B) = A \cup B$
If $^A \in \mathrm{P}, B \in P$ $\Rightarrow$ $A \cup B \in P$
$\Rightarrow$ $\cup$ is binary operation on P
Also, $\cap : P * P \to P$, defined by $\cap (A, B) = (A \cap \mathrm{B})$
Similar as explained above if $A \in P$
$\Rightarrow$ $B \in P$ $\Rightarrow$ $A \cap B \in P$
$\Rightarrow$ $\cap$ is a binary operation on P

12. **(a)** $a * b = 4$ is commutative as $a * b = b * a = 4$
Also a $* (b * c) = \mathrm{a} * 4 = 4$ and $(a * b) * c = 4 * c = 4$
$\Rightarrow$ * is associative.
(b) $a * b = \frac{a+b}{4} = b * a$
$\Rightarrow$ * is commutative
Also, $(a * b) * c = \left(\frac{a+b}{4}\right) * c = \frac{\frac{a+b}{4} + c}{4} = \frac{a+b+4c}{16}$

$$a * (b * c) = a * \left(\frac{b+c}{4}\right) = \frac{a + \frac{b+c}{4}}{4} = \frac{4a+b+c}{16}$$

$\Rightarrow$ * is not associative

13. $A * B = A \cap B \ \forall A, B \in P(W)$

$\therefore$ $W \cap$ (any element $P(W)$) = that element of $P(W)$, as $P(W)$ is the collection of all subsets of W

$\Rightarrow$ W is the identity element with respect to *.

$\because$ $A * B = W$ $\Rightarrow$ $A \cap B = W \Leftrightarrow A = B = W$

$\Rightarrow$ W is the only invertible element and self invertible.

14. $a * b = |a - b|$

$a \text{ o } b = a$

$a * b = |a - b| = b * a = |b - a|$

$\Rightarrow$ * is commutative.

Next, $(a * b) * c = |a - b| * c = ||a - b| - c|$ and $a * (b * c) = a * |b - c| = |a - |b - c||$

$\Rightarrow$ * is not associative.

Now, $aob = a$ but $boa = b$

$\Rightarrow$ o is not commutative.

Next, $ao(boc) = aob = a$

Also, $(aob)oc = aoc = a$

$\Rightarrow$ o is associative .

We have a * $(boc) = |a - b|$ and $(a * b)o(a * b) = |a - b|o|a - b| = |a - b|$

$\Rightarrow$ Yes * distributes over o

Next, $ao(b * c) = ao(|b - c|) = a$ and $(aob) * (aoc) = a * a = |a - a| = 0$

$\Rightarrow$ No 'o' does not distribute over *

TEXTUAL EXERCISE–8: (OBJECTIVE)

1. **(b), (c)** An operation * is associative on set A

$\Leftrightarrow$ $(x * y) * z = x * (y * z) \ \forall x, y, z \in A$

(a) $(a * b) * c = (a - b) - c = a - b - c$ and $a * (b * c) = a * (b - c) = a - b + c$

$\Rightarrow$ * is not associative

(b) $(a * b) * c = \left(\dfrac{ab}{4}\right) * c = \dfrac{abc}{16}$ and a * $(b * c)$

$= a * \left(\dfrac{bc}{4}\right) = \dfrac{abc}{16}$

$\Rightarrow$ * is associative

(c) $(a * b) * c = \left(\sqrt{a^2b^2}\right) * c = \sqrt{a^2b^2c^2} = abc$

$\Rightarrow$ $a * (b * c) = a + \left(\sqrt{b^2c^2}\right) = \sqrt{a^2b^2c^2} = abc$

$\Rightarrow$ * is associative

(d) $(a * b) * c = (a^2 - b^2) * c$

$= (a^2 - b^2) - c^2$

$= (a^2 - b^2 + c)(a^2 - b^2 - c)$

Also, $a * (b * c) = a * (b^2 - c^2)$

$= a^2 - (b^2 - c^2)^2$

$= (a - b^2 + c^2)(a + b^2 - c^2)$

$\Rightarrow$ * is not associative

2. **(a), (b)** Precisely, a binary operation * on a sets is a map which sends elements of the cartesian products $S \times S$ to S.

$* \ S \times S \to S$

(a) $a * b = \sqrt{a^2b^2 + b^2 + 2ab^2}$; where $a, b \in \mathbb{Z}$

$\Rightarrow$ $a * b = \sqrt{(ab + b)^2} = |b(a + 1)| \in \mathbb{Z}$

$\Rightarrow$ * is a binary operation

(b) $a * b = \gcd(a, b) \in \mathbb{Z}$

$\Rightarrow$ * is a binary operation

(c) $a * b = \dfrac{a - b}{2}$ need not be an integer, e.g., if $a = 7$, $b = 2$, then $a * b = \dfrac{7 - 2}{2} = \dfrac{5}{2} \notin \mathbb{Z}$

$\Rightarrow$ * is not a binary operation on $\mathbb{Z}$.

(d) $a * b = \dfrac{a^2 - b^2}{4}$ need not be an integer, e.g., if $a = 4$, $b = 3$, then $a * b = \dfrac{16 - 9}{4} = \dfrac{7}{4} \notin \mathbb{Z}$

$\Rightarrow$ * is not a binary operation on $\mathbb{Z}$.

3. **(a)** A binary operation f: $A \times A \to B$ is commutative it $f(x, y) = f(y, x) \ \forall x, y \in A$

(a) is commutative

$a * b = \sqrt{a^2 + b^2 - 2ab}$

$b * a = \sqrt{b^2 + a^2 - 2ab}$

(b) $a * b = a - b$

$b * a = b - a$

$\Rightarrow$ Not commutative.

(c) $a * b = \begin{cases} (a)^b & \text{if } b \geq 0 \\ ab & \text{if } b < 0 \end{cases}$

Let a = 2, $b = 3$

$\Rightarrow$ $a * b = (2)^3 = 8$ and $b * a = (3)^2 = 9$

$\Rightarrow$ * is not commutative.

if $a = 2$, $b = -4$

$\Rightarrow$ $a * b = a.\ b = -8$ and $b * a = (-4)^2 = 16$

$\Rightarrow$ $a * b \neq b * a$

$\Rightarrow$ * is not commutative.

(d) $a * b = a^2\, b$ and $b * a = b^2$ a

$\Rightarrow$ * is not commutative

4. **(c)** $A = \{1, \omega, \omega^2\}$; = ω cube root of unity

$a * b = ab$

$\therefore$ $1.\ 1 = 1$, $1.\ \omega = \omega$ and $1.\ \omega^2 = \omega^2$

$\Rightarrow$ 1 is an identity element in A.

5. **(c), (d)** $A = \{1, \omega, \omega^2\}$

$\therefore$ $a*b = ab$

$\therefore$ $\omega\omega^2 = \omega^3 = 1$

$\Rightarrow$ Inverse of ω^2 is ω and inverse of ω is ω^2

6. **(c)** $* : A \times A \to A$

$n(A \times A) = 9$ $\because$ A $\times$ A has 9 elements and each element has 3 choices.

$\Rightarrow$ Total number of binary operations defined on set $A = (3)^9$

7. **(a), (b), (c), (d)**

(a) $a * b$ = least non- negative integer obtained when $(a.b)$ is divided by 6

$\Rightarrow$ 1 is the identity elements in A

Clearly, 1 * 3 = 3 and 1 * 5 = 5

$\Rightarrow$ (a) is true

(b) 3 * a $\Rightarrow$ 3 * 1 = 3

$\Rightarrow$ $3 * 5 = 3$ $\Rightarrow$ $3 * 3 = 3$

$\Rightarrow$ (b) is true

(c) No elements of A can give $3 * a = 1$

$\Rightarrow$ (c) is true

(d) Clearly, $5 * 5 = 1$ $\Rightarrow$ (d) is true

8. (a), (b), (c)

$A = \{2, 4, 6, 8, 10\}$

Option (a) is correct as $\forall\ a \in A$

$6 * a = a$

(b) $*$ is commutative as $\forall\ a, b \in A$

$a * b = b * a$ by the table

(c) Since $8 * 10 = 6$, $10 * 8 = 6$

Where 6 is the identity elements of $*$ as proved above

$\Rightarrow$ Inverse of 8 is 10

(d) $2 * (4 * 8) = 2 * 10 = 8$ and $(2 * 4) * 8 = (6 * 8) = 8$

$\Rightarrow$ (d) is false

TEXTUAL EXERCISE–9: (SUBJECTIVE)

1. (a) $f(x) = \dfrac{\sqrt{x-1}+\sqrt{6-x}}{\sqrt{1-x}+\sqrt{x-6}}$

$x - 1 \geq 0$ and $6 - x \geq 0$ and $1 - x \geq 0$ and $x - 6 \geq 0$ and $\sqrt{1-x}+\sqrt{x-6} \neq 0$

$\Rightarrow$ $x \geq 1$ and $x \leq 6$ and $x \leq 1$ and $x \geq 6$ and $1 - x \neq x - 6$ (never possible simultaneously)

$x \in \phi$

(b) $f(x) = \dfrac{x^2+1}{x-\sqrt{x+2}}$

$\Rightarrow$ $x-\sqrt{x+2} \neq 0$ and $x + 2 \geq 0$

$\Rightarrow$ $x^2 - x - 2 \neq 0$ and $x \in [-2, \infty)$

$\Rightarrow$ $x \neq 2, -1$ and $x \in [-2, \infty)$

$\Rightarrow$ $x \in [-2, \infty) - \{2, -1\}$

(c) $f(x) = \sqrt{\dfrac{x-2}{x+2}} + \sqrt{\dfrac{1-x}{1+x}}$

$\Rightarrow$ $\dfrac{x-2}{x+2} > 0$ or $x = 2$ and $\dfrac{1-x}{1+x} > 0$ or $x = 1$

$\Rightarrow$ $x \in (-\infty, -2) \cup (2, \infty)$ or $x = 2$ and $x \in (-1, 1)$ or $x = 1$

$\Rightarrow$ $x \in (-\infty, -2)\ \ [2, \infty)$ and $x \in (-1, 1]$

$\Rightarrow$ $x \in \phi$

2. (a) $f(x) = \sqrt{\dfrac{1-5^x}{7^{-x}-7}} = \sqrt{7^x\left(\dfrac{1-5^x}{1-7^{x+1}}\right)}$

$\Rightarrow$ $7^x > 0$ $\Rightarrow$ $\dfrac{1-5^x}{1-7^{x+1}} \geq 0$

$\Rightarrow$ $\dfrac{5^x-1}{7^{x+1}-1} > 0$ or $x = 0$

$5^x - 1 > 0$ and $7^{x+1} - 1 > 0$ or $5^x - 1 < 0$ and $7^{x+1} - 1 < 0$ or $x = 0$

$\Rightarrow$ $x > 0$ and $x + 1 > 0$ or $x < 0$ and $x + 1 < 0$ or $x = 0$

$\Rightarrow$ $x \in (0, \infty)$ or $x \in (-\infty, -1)$ or $x = 0$

$x \in (-\infty, -1) \cup [0, \infty)$

(b) $f(x) = \dfrac{\sqrt{5-x}+\sqrt{x-2}}{\sqrt{3-x}-\sqrt{x-1}}$

$x \leq 5$ and $x \geq 2$ and $x \leq 3$ and $x \geq 1$ and $3 - x \neq x - 1$

$x \in [2, 3]$ and $x \neq 2$ $\Rightarrow$ $x \in (2, 3]$

3. (a) $f(x) = \log_{2\{x\}-3}(x^2 - 5x + 13)$

$x^2 - 5x + 13 > 0$ and $2\{x\} - 3 > 0$ and $2\{x\} - 3 \neq 1$

$\Rightarrow$ $x^2 - 5x + 13 > 0$ is true for all $x \in \mathbb{R}$ as $D < 0$ and leading coefficient is +ve and

$\{x\} > \dfrac{3}{2}$ is not possible for any value of x and $\{x\} \neq$ is 2 true for all x.

Hence, $x \in \phi$.

(b) $f(x) = \log_{10}\left(\dfrac{9-x^2}{x^2-4}\right)$

$\Rightarrow$ $\dfrac{9-x^2}{x^2-4} > 0$

$\Rightarrow$ $(3 - x)(3 + x)(x - 2)(x + 2) > 0$

$\Rightarrow$ $x \in (-3, -2) \cup (2, 3)$

(c) $f(x) = \sqrt{\log_2\left(\dfrac{5x-x^2}{4}\right)}$

$\Rightarrow$ $\dfrac{5x-x^2}{4} > 0$ and $\log_2\left(\dfrac{5x-x^2}{4}\right) \geq 0$

$\Rightarrow$ $x(5 - x) > 0$ and $\dfrac{x(5-x)}{4} \geq 1$

$\Rightarrow$ $x \in (0, 5)$ and $x^2 - 5x + 4 \leq 0$

$\Rightarrow$ $x \in (0, 5)$ and $x \in [1, 4]$

$\Rightarrow$ $x \in [1, 4]$

(d) $f(x) = \sqrt{\log_{\frac{1}{2}}\left(\dfrac{5x-x^2}{4}\right)}$

$\dfrac{x(5-x)}{4} > 0$ and $\log_{\frac{1}{2}}\left(\dfrac{5x-x^2}{4}\right) \geq 0$

$x \in (0, 5)$ and $x(5 - x) \leq 4$

$\Rightarrow$ $x \in (0, 5)$ and $x \in (-\infty, 1]\ \ [4, \infty)$

$\Rightarrow$ $x \in (0, 1] \cup [4, 5)$

4. (a) $f(x) = \dfrac{1}{\sqrt{x-|x|}}$ $\Rightarrow$ $x - |x| > 0$

$\Rightarrow$ $x > |x|$ $\Rightarrow$ $x \in \phi$

(b) $f(x) = \dfrac{1}{\sqrt{|x|-x}}$ $\Rightarrow$ $|x| > x$

$\Rightarrow$ $x \in (-\infty, 0)$

(c) $f(x) = \sin^{-1}\left(\dfrac{3-|x|}{2}\right)$

$\Rightarrow$ $\dfrac{3-|x|}{2} \in [-1, 1]$ $\Rightarrow$ $3 - |x| \in [-2, 2]$

$\Rightarrow$ $-|x| \in [-5, -1]$

$\Rightarrow$ $|x| \in [1, 5]$ $\Rightarrow$ $x\ \ [-5, -1] \cup [1, 5]$

(d) $f(x) = \sqrt{x^2 - 3|x| + 2}$

$\Rightarrow$ $x^2 - 3|x| + 2 \geq 0$ $\Rightarrow$ $|x|^2 - 3|x| + 2 \geq 0$

$\Rightarrow$ $(|x| - 1)(|x| - 2) \geq 0$

$\Rightarrow$ $|x| \in (-\infty, 1] \cup [2, \infty)$ but $|x| \geq 0$

$\Rightarrow$ $|x| \in [0, 1] \cup [2, \infty)$

$\Rightarrow$ $x \in (-\infty, -2] \cup [-1, 1] \cup [2, \infty)$

5. **(a)** $f(x) = \sqrt{\sin x} + \sqrt{16 - x^2}$

$\sin x \geq 0$ and $16 - x^2 \geq 0$

$\Rightarrow$ $x \in \bigcup\limits_{n \in \mathbb{Z}} [2n\pi, (2n+1)\pi]$ and $x \in [-4, 4]$

$\Rightarrow$ $x \in [-4, -\pi] \cup [0, \pi]$

(b) $f(x) = \sqrt{\cos(\sin x)} + \sqrt{\sin(\cos x)}$

$\Rightarrow$ $\cos(\sin x) \geq 0$ and $\sin(\cos x) \geq 0$

$\Rightarrow$ $x \in \mathbb{R}$ and $\cos x \in [0, 1]$

$\Rightarrow$ $x \in \bigcup\limits_{n \in \mathbb{Z}} \left[(4n-1)\dfrac{\pi}{2}, (4n+1)\dfrac{\pi}{2}\right]$

6. **(a)** $f(x) = \cos^{-1}[x]$ $\Rightarrow$ $[x] \in [-1, 1]$

$\Rightarrow$ $x \in [-1, 2)$

$\Rightarrow$ $R_f = \{\cos^{-1}(-1), \cos^{-1}(0), \cos^{-1}(1)\} = \{\pi, \pi/2, 0\}$

(b) $f(x) = \sin^{-1}(x^2 + 1)$ $\Rightarrow$ $x^2 + 1 \in [-1, 1]$

$\Rightarrow$ $x^2 \in [-2, 0]$ but $x^2 \geq 0$

$\Rightarrow$ $x^2 = 0$

$\Rightarrow$ $x = 0$; $x \in \{0\}$ and $R_f = \{\sin^{-1}\} = \{\pi/2\}$

7. **(a)** $f(x) = \cos^{-1}\left(\dfrac{2x}{1+x^2}\right)$

$\Rightarrow$ $\dfrac{2x}{1+x^2} \in [-1, 1]$ $\Rightarrow$ $-1 \leq \dfrac{2x}{1+x^2} \leq 1$

$\Rightarrow$ $\dfrac{(x+1)^2}{1+x^2} \geq 0$ and $\dfrac{(x-1)^2}{1+x^2} \geq 0$

$x = -1$ or $\dfrac{(x+1)^2}{1+x^2} > 0$ true $\forall\ x \in \mathbb{R}$ and $x = 1$ or $\dfrac{(x-1)^2}{1+x^2} > 0$ true $\forall\ x \in \mathbb{R}$

Hence, $x \in \mathbb{R}$

(b) $f(x) = \cos^{-1}\left(\dfrac{1+x^2}{2x}\right)$

$\Rightarrow$ $\dfrac{1+x^2}{2x} \in [-1, 1]$ $\Rightarrow$ $-1 \leq \dfrac{1+x^2}{2x} \leq 1$

$\Rightarrow$ $\dfrac{(x+1)^2}{2x} \geq 0$ and $\dfrac{(x-1)^2}{2x} \leq 0$

$\Rightarrow$ $x = -1$ or $x > 0$ and $x = 1$ or $x < 0$

$\Rightarrow$ $x \in (0, \infty) \cup \{-1\}$ and $x \in (-\infty, 0) \cup \{1\}$

$\Rightarrow$ $x \in \{-1\} \cup \{1\}$ $\Rightarrow$ $x \in \{-1, 1\}$

(c) $f(x) = \tan^{-1}(1 + x^2)$ $\Rightarrow$ $1 + x^2 \in (-\infty, \infty)$

$\Rightarrow$ $x^2 \in (-\infty, \infty)$ but $x^2 \geq 0$

$\Rightarrow$ $x^2 \in [0, \infty)$ $\Rightarrow$ $x \in (-\infty, \infty)$

$\Rightarrow$ $x \in \mathbb{R}$

(d) $f(x) = \sin^{-1}\left(\dfrac{x^2-1}{2(x^2+1)}\right)$

$\Rightarrow$ $-1 \leq \dfrac{x^2-1}{2(x^2+1)} \leq 1$

$\dfrac{x^2 - 1 + 2x^2 + 2}{2(x^2+1)} \geq 0$ and $\dfrac{x^2 - 1 - 2x^2 - 2}{2(x^2+1)} \leq 0$

$\Rightarrow$ $\dfrac{3x^2+1}{2(x^2+1)} \geq 0$ and $\dfrac{x^2+3}{2(x^2+1)} \leq 0$

$\Rightarrow$ $x \in \mathbb{R}$

8. **(a)** $f(x) = \log_2(3 + 5\sin x)$

$\Rightarrow$ $3 + 5\sin x > 0$

$\Rightarrow$ $\sin x > -\dfrac{3}{5}$ and $\sin x \in [-1, 1]$

$\Rightarrow$ $\sin x \in \left(-\dfrac{3}{5}, 1\right]$

$\Rightarrow$ $x \in \bigcup\limits_{n \in \mathbb{Z}} \left(2n\pi - \sin^{-1}\dfrac{3}{5}, (2n+1)\pi + \sin^{-1}\dfrac{3}{5}\right)$

(b) $f(x) = \log_2(3 + 2\sin x)$ $\Rightarrow 3 + 2\sin x \geq 0$

$\Rightarrow$ $\sin x \geq -\dfrac{3}{2}$ $\Rightarrow$ $x \in \mathbb{R}$

9. **(a)** $f(x) = \sqrt{\log_x(\cos 2\pi x)}$

$\Rightarrow$ $x > 0, \neq 1$, $\log_x(\cos 2\pi x) \geq 0$, $\cos 2\pi x > 0$

Case (i): $0 < x < 1$

$\Rightarrow$ $0 < \cos 2\pi x \leq 1$

For $0 < x < 1$, $0 < 2\pi x < 2$

$\Rightarrow$ $2\pi x \in \left(0, \dfrac{\pi}{2}\right)$ or $\left(\dfrac{3\pi}{2}, 2\pi\right)$

$\Rightarrow$ $x \in \left(0, \dfrac{1}{4}\right) \cup \left(\dfrac{3}{4}, 1\right)$

Case (ii): $x > 1$; $\cos 2\pi x \ \ 1$

$\Rightarrow$ $\cos 2\pi x = 1$

$\Rightarrow$ $x \in \{2, 3, 4, \ldots\}$

$\therefore$ $D_f = \left(0, \dfrac{1}{4}\right) \cup \left(\dfrac{3}{4}, 1\right) \cup \{2, 3, 4, \ldots\ldots \infty\}$

(b) $f(x) = \sqrt{(x^2 - 3x - 10)\ln^2(x-3)}$

$\Rightarrow$ $x^2 - 3x - 10 \geq 0$ or $x - 3 = 1$ and $x - 3 > 0$

$\Rightarrow$ $x \in (-\infty, -2]\ \ [5, \infty)$ or $x > 3$ and $x = 4$

$\Rightarrow$ $x \in \{4\} \cup [5, \infty)$

(c) $f(x) = \cos^{-1}\left(\dfrac{2 - |x|}{4}\right) + (\ln(3-x))^{-1}$

$\Rightarrow$ $\dfrac{2 - |x|}{4} \in [-1, 1]$ and $3 - x \neq 1$ and $3 - x > 0$

$\Rightarrow$ $-|x| \in [-6, 2]$ and $x \neq 2$ and $x < 3$

$\Rightarrow$ $|x| \in [-2, 6]$ and $x \neq 2$ and $x < 3$

$|x| \in [0, 6]$ and $x \in (-\infty, 3) - \{2\}$

$\Rightarrow$ $x \in [-6, 6]$ and $x \in (-\infty, 3) - \{2\}$

$\Rightarrow$ $x \in [-6, 3) - \{2\}$

10. **(a)** $f(x) = \dfrac{\sqrt{\cos x - \dfrac{1}{2}}}{\sqrt{6 + 35x - 6x^2}}$

$\Rightarrow$ $\cos x - \dfrac{1}{2} \geq 0$ and $6 + 35x - 6x^2 > 0$

$\Rightarrow$ $\cos x \geq \dfrac{1}{2}$ and $6x^2 - 35x - 6 < 0$

$\Rightarrow\ \cos x \in \left[\frac{1}{2},1\right]$ and $(6x + 1)(x - 6) < 0$

$\Rightarrow\ x \in \bigcup_{n\in\mathbb{Z}}\left[2n\pi-\frac{\pi}{3},2n\pi+\frac{\pi}{3}\right]$ and $x \in \left(-\frac{1}{6},6\right)$

$\Rightarrow\ x \in \bigcup\left[-\frac{\pi}{3},\frac{\pi}{3}\right]\cup\left[\frac{5\pi}{3},\frac{7\pi}{3}\right]\cup....$ and $x \in \left(-\frac{1}{6},6\right)$

$\Rightarrow\ x \in \left(-\frac{1}{6},\frac{\pi}{3}\right]\cup\left[\frac{5\pi}{3},6\right)$

(b) $f(x)=\left(\log_{\left(\frac{x-2}{x+3}\right)}2\right)+\sqrt{9-x^2}$

$\Rightarrow\ 9 - x^2 \geq 0$ and $\frac{x-2}{x+3}\neq 1$ (always hold) and $\frac{x-2}{x+3}>0$

$\Rightarrow\ x \in [-3, 3]$ and $(x - 2)(x + 3) > 0$

$\Rightarrow\ x \in [-3, 3]$ and $x \in (-\infty, -3) \cup (2, \infty)$

$\Rightarrow\ x \in (2, 3]$

11. **(a)** $f(x)=\frac{1}{[x]}+\log_{1-\{x\}}(x^2-3x+10)$

$+\frac{1}{\sqrt{2-|x|}}+\frac{1}{\sqrt{\sec(\sin x)}}$

$\Rightarrow\ [x] \neq 0$ and $1 - \{x\} > 0$ and $1 - \{x\} \neq 1$ and $x^2 - 3x + 10 > 0$ and $2 - |x| > 0$ and $\sec(\sin x) > 0$

$\Rightarrow\ x \notin [0, 1)$ and $x \in \mathbb{R}$ and $x \notin \mathbb{Z}$ and $x \in \mathbb{R}$ and $x \in (-2, 2)$ and $x \in \mathbb{R}$

$\Rightarrow\ x \in (-2, -1) \cup (-1, 0) \cup (1, 2)$

(b) $f(x)=\frac{1}{\sqrt{\sin(\cos x)}}+\sin^{-1}\left(\frac{2x}{\pi}\right)$

$+\frac{1}{\{-x\}}+\frac{1}{\ln\left(\left(1-\left[\tan\frac{x}{2}\right]\right)-\left[-\tan\frac{x}{2}\right]\right)}$

$\sin(\cos x) > 0$ and $\frac{2x}{\pi}\in(-1,1)$ and $x \notin \mathbb{Z}$ and

$\ln\left(1-\left[\tan\frac{x}{2}\right]\right)-\left[-\tan\frac{x}{2}\right]\neq 0$ and $1-\left[\tan\frac{x}{2}\right]>0$

$\cos x \in (0, 1]$ and $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and $x \notin \mathbb{Z}$ and

$\ln\left(1-\left[\tan\frac{x}{2}\right]\right)-\left[-\tan\frac{x}{2}\right]\neq 0$ and $\tan\frac{x}{2}\in(-\infty,1)$

$\Rightarrow\ x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and $x \notin \mathbb{Z}$ and

$\ln\left(1-\left[\tan\frac{x}{2}\right]\right)+\left[-\tan\frac{x}{2}\right]\neq 0,\ \tan\frac{x}{2}\in(-\infty,1)$

$\Rightarrow\ x\in\left(\frac{-\pi}{2},-1\right)\cup(-1,0)\cup(0,1)\cup\left(1,\frac{\pi}{2}\right)$ and ln

$\left(1-\left[\tan\frac{x}{2}\right]\right)+\left[-\tan\frac{x}{2}\right]\neq 0$

$\because$ If $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\quad\Rightarrow\quad\frac{x}{2}\in\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$

$\Rightarrow\ \tan\frac{x}{2}\in(-1,1)$

$\because\ \tan\frac{x}{2}\in(-1,1)\Rightarrow\left[\tan\frac{x}{2}\right]\in\{-1,0\}$

Case (i): $\frac{-\pi}{2}<x<-1$

$\Rightarrow\ \frac{-\pi}{4}<\frac{x}{2}<\frac{-1}{2}$

$\Rightarrow\ -1<\tan\frac{x}{2}<\tan\left(\frac{-1}{2}\right)$

$\Rightarrow\ \left[\tan\frac{x}{2}\right]=-1$ and $\left[-\tan\frac{x}{2}\right]=0$

$\Rightarrow\ \ln\left(1-\left[\tan\frac{x}{2}\right]\right)+\left[-\tan\frac{x}{2}\right]=\ln 2\neq 0$

Case (ii): $-1 < x < 0$

$\Rightarrow\ \ln\left(1-\left[\tan\frac{x}{2}\right]\right)+\left[-\tan\frac{x}{2}\right]=\ln 2$

Case (iii): $0 < x < 1$

$\Rightarrow\ \ln\left(1-\left[\tan\frac{x}{2}\right]\right)+\left[-\tan\frac{x}{2}\right]=-1\neq 0$

Case (iv): $1 < x < \pi/2$

$\Rightarrow\ \ln\left(1-\left[\tan\frac{x}{2}\right]\right)+\left[-\tan\frac{x}{2}\right]=-1\neq 0$

$D_f=\left(\frac{-\pi}{2},-1\right)\cup(-1,0)\cup(0,1)\cup\left(1,\frac{\pi}{2}\right)$

TEXTUAL EXERCISE–9: (OBJECTIVE)

1. **(c)** $\sqrt{x(x-1)}\quad\Rightarrow\ x(x-1)\geq 0$

$\Rightarrow\ x \in (-\infty, 0] \cup [1, \infty)$

2. **(a)** $x^{\frac{1}{\ln x}}\quad\Rightarrow\ x > 0$ and $x \neq 1$

$\Rightarrow\ x \in (0, \infty) \sim \{1\}$

3. **(a)** $\frac{1}{\sqrt{x-|x|}}\quad\Rightarrow\ x - |x| > 0$

$\Rightarrow\ x > |x|\quad\Rightarrow\ x \in \phi$

4. **(b)** $\frac{1}{\sqrt{|x|-x}}\quad\Rightarrow\ |x| - x > 0$

$\Rightarrow\ |x| > x\quad\Rightarrow\ x \in (-\infty, 0)$

5. **(b)** $\frac{1}{2-\cos 3x}\quad\Rightarrow\ 2 - \cos 3x \neq 0$

$\Rightarrow\ \cos 3x \neq 2\quad\Rightarrow\ x \in \mathbb{R}$

6. **(a)** $\log_{10}(1 + x^3)\quad\Rightarrow\ 1 + x^3 > 0$

$\Rightarrow\ x^3 > -1\quad\Rightarrow\ x \in (-1, \infty)$

7. **(c)** $\frac{1}{\sqrt{x(x-2)(x-3)}}\quad\Rightarrow\ x(x-2)(x-3)>0$

$\Rightarrow\ x \in (0, 2) \cup (3, \infty)$

8. **(b)** $\dfrac{1}{\sqrt[3]{(x-1)(x-2)(x-4)}}$

$\Rightarrow$ $(x - 1)(x - 2)(x - 4) \neq 0$

$\Rightarrow$ $x \neq 1, 2, 4$ $\Rightarrow$ $x \in \mathbb{R} \sim \{(1, 2, 4)\}$

9. **(d)** $2^x + 2^y = 2$ $\Rightarrow$ $2^y = 2 - 2^x$

$\Rightarrow$ $2^y > 0$ $\Rightarrow$ $2 - 2^x > 0$

$\Rightarrow$ $2 > 2^x$ $\Rightarrow$ $x < 1$

$\Rightarrow$ $x \in (-\infty, 1)$

10. **(a)** $f(x) = \sqrt{3-x} + \cos^{-1}\left(\dfrac{3-2x}{5}\right)$

$\Rightarrow$ $3 - x \geq 0$ and $\dfrac{3-2x}{5} \in [-1,1]$

$\Rightarrow$ $x \in (-\infty, 3]$ and $x \in [-1, 4]$

$\Rightarrow$ $x \in [-1, 3]$

11. **(c)** $f(x) = \sin^{-1}\left(\dfrac{x-3}{2}\right) - \log_{10}(4-x)$

$\Rightarrow$ $\dfrac{x-3}{2} \in [-1,1]$ and $4 - x > 0$

$\Rightarrow$ $x \in [1, 5]$ and $x < 4$ $\Rightarrow$ $x \in [1, 4)$

12. **(d)** $y = \dfrac{1}{\log_{10}(3-x)} + \sqrt{x+2}$

$\Rightarrow$ $x > -2$ and $3 - x \neq 1$ and $3 - x > 0$

$\Rightarrow$ $x > -2$ and $x \neq 2$ and $x < 3$

$\Rightarrow$ $x \in (-2, 3) - \{2\}$

13. **(a)** $f(x) = \sqrt{\dfrac{(x+1)(x-3)}{x-2}}$

$\Rightarrow$ $\dfrac{(x+1)(x-3)}{(x-2)} > 0$ or $x = -1, 3$

$\Rightarrow$ $x \in (-1, 2) \cup (3, \infty)$ or $x = -1, 3$

$\Rightarrow$ $x \in [-1, 2) \cup [3, \infty)$

14. **(c)** $f(x) = \sqrt{\dfrac{1-|x|}{2-|x|}}$

$\Rightarrow$ $1 - |x| = 0$ or $\dfrac{1-|x|}{2-|x|} > 0$

$\Rightarrow$ $x = -1, 1$ or $|x| \in (-\infty, 1) \cup (2, \infty)$ and $|x| \geq 0$

$\Rightarrow$ $x = -1, 1$ or $|x| \in [0, 1) \cup (2, \infty)$

$\Rightarrow$ $x = -1, 1$ or $x \in (-\infty, -2) \cup (-1, 1) \cup (2, \infty)$

$\Rightarrow$ $x \in (-\infty, -2) \cup [-1, 1] \cup (2, \infty)$

15. **(d)** $f(x) = \cos^{-1}\left(\dfrac{3}{4+2\sin x}\right)$

$\Rightarrow$ $\dfrac{3}{4+2\sin x} \in [-1, 1]$ $\Rightarrow$ $\dfrac{1}{4+2\sin x} \in \left[-\dfrac{1}{3}, \dfrac{1}{3}\right]$

$\Rightarrow$ $4 + 2\sin x \in (-\infty, -3] \cup [3, \infty)$

$\Rightarrow$ $2\sin x \in (-\infty, -7] \cup [-1, \infty)$

$\Rightarrow$ $\sin x \in \left(-\infty, -\dfrac{7}{2}\right] \cup \left[-\dfrac{1}{2}, \infty\right)$, but $\sin x \in [-1, 1]$

$\Rightarrow$ $\sin x \in \left[-\dfrac{1}{2}, 1\right]$

$\Rightarrow$ $x \in \bigcup_{n\in\mathbb{Z}} \left[2n\pi - \dfrac{\pi}{6}, (2n+1)\pi + \dfrac{\pi}{6}\right]$

16. **(a)** $f(x) = \dfrac{1}{\sqrt{\sin x}} + \sqrt[3]{\sin x}$

$\Rightarrow$ $\sin x > 0$ $\Rightarrow$ $x \in \bigcup_{k\in\mathbb{Z}} (2k\pi, (2k + 1)\pi)$

17. **(d)** $f(x) = \dfrac{1}{\sqrt{|\cos x| + \cos x}}$

$\Rightarrow$ $|\cos x| + \cos x > 0$

$\Rightarrow$ $|\cos x| > -\cos x$ $\Rightarrow$ $\cos x > 0$

$\Rightarrow$ $x \in \left(2n\pi - \dfrac{\pi}{2}, 2n\pi + \dfrac{\pi}{2}\right)$

$\Rightarrow$ $x \in \bigcup_{n\in\mathbb{Z}} \left((4n-1)\dfrac{\pi}{2}, (4n+1)\dfrac{\pi}{2}\right)$

TEXTUAL EXERCISE–10: (SUBJECTIVE)

1. **(a)** $\sqrt{x^2+4}$ $\Rightarrow$ $x^2 + 4 \geq 0$ true $\forall\, x \in \mathbb{R}$

$\Rightarrow$ $x \in (-\infty, \infty)$ $\Rightarrow$ $x^2 \in [0, \infty)$

$\Rightarrow$ $x^2 + 4 \in [4, \infty)$

(b) $\dfrac{1}{(x-1)(x-5)}$ $\Rightarrow$ $x \neq 1, 5$

$\Rightarrow$ $x \in \mathbb{R} \sim \{1, 5\}$ is the domain

$\dfrac{1}{(x-1)(x-5)} = \dfrac{1}{x^2-6x+5} = \dfrac{1}{(x-3)^2-4}$

$x \in (-\infty, 1) \cup (1, 5) \cup (5, \infty)$

$x - 3 \in (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

$\Rightarrow$ $(x - 3)^2 \in [0, 4) \cup (4, \infty)$

$\Rightarrow$ $(x - 3)^2 - 4 \in [-4, 0)\ (0, \infty)$

$\Rightarrow$ $\dfrac{1}{(x-3)^2-4} \in \left(-\infty, -\dfrac{1}{4}\right] \cup (0, \infty)$

(c) $\sqrt[3]{x+8}$; $x \in \mathbb{R}$

$x + 8 \in \mathbb{R}$ $\Rightarrow$ $x + 8\ (-\infty, \infty)$

$\sqrt[3]{x+8} \in (-\infty, \infty)$ $\therefore$ Range is $\mathbb{R}$

(d) $y = \dfrac{1}{x^2+4}$; $x^2 + 4 \neq 0$ true $\forall\, x \in \mathbb{R}$

Now as $x \in (-\infty, \infty)$ $\Rightarrow$ $x^2 \in [0, \infty)$

$\Rightarrow$ $x^2 + 4 \in [4, \infty)$ $\Rightarrow$ $\dfrac{1}{x^2+4} \in \left(0, \dfrac{1}{4}\right]$

$\therefore$ Range is $\left(0, \dfrac{1}{4}\right]$

2. $f(x) = {}^{x+1}C_{2x-8}$; $g(x) = {}^{2x-8}C_{x+1}$

$\Rightarrow$ $h(x) = f(x).g(x) = {}^{x+1}C_{2x-8}.{}^{2x-8}C_{x+1}$

$\Rightarrow$ $x + 1 > 0$ and $2x - 8 > 0$ and $x + 1 \geq 2x - 8$ and $2x - 8 \geq x + 1$

$\Rightarrow$ $x \in (4, \infty)$ and $x = 9$ $\Rightarrow$ $x = 9$

$\therefore$ $D_f(h(x)) = \{9\}$; $R_f = \{1\}$

3. (a) $f(x) = 4\tan x\cos x = 4\dfrac{\sin x}{\cos x}\cos x$

$= \begin{cases} 4\sin x & x \in \mathbb{R} \sim \bigcup\limits_{n\in\mathbb{Z}} (2n+1)\dfrac{\pi}{2} \\ \text{undefined} & \text{otherwise} \end{cases}$

Hence, $D_f = \mathbb{R} \sim \bigcup\limits_{n\in\mathbb{Z}} \left\{(2n+1)\dfrac{\pi}{2}\right\}$

$\Rightarrow\ x \in \mathbb{R} \sim \bigcup\limits_{n\in\mathbb{Z}} \left\{(2n+1)\dfrac{\pi}{2}\right\}$

$\Rightarrow$ $\sin x \in (-1, 1)$

$\Rightarrow$ $4\sin x \in (-4, 4)$

$\Rightarrow$ $R_f = (-4, 4)$

(b) $\cos(2\sin x)$

$\sin x \in [-1, 1]$

$\Rightarrow$ $2\sin x \in [-2, 2]$

$2\sin x \in [-2, 0]$ or $2\sin x \in [0, 2]$

$\cos(2\sin x) \in [\cos(-2), 1]$ or $\cos(2\sin x \in [\cos 2, 1])$

$\cos(2\sin x) \in [\cos 2, 1] \cup [\cos 2, 1]$

$\cos(2\sin x) \in [\cos 2, 1]$

(c) $\sin(\log_2 x)$ for $\log_2 x$ to be defined, $x > 0$

$\Rightarrow$ $x \in (0, \infty)$ $\quad\Rightarrow$ $\log_2 x \in (-\infty, \infty)$

$\Rightarrow$ $\sin(\log_2 x) \in [-1, 1]$

4. (a) $f(x) = [\{x\}]$ or $\{[x]\}$

$\{x\} \in [0, 1)$ $\quad\Rightarrow$ $[\{x\}] \in \{0\}$

Also, $[x] \in \mathbb{Z}$ $\quad\Rightarrow$ $\{[x]\} \in \{0\}$

(b) $g(x) = \dfrac{\tan(\pi[x-\pi])}{x^2 - 3x + 4}$

$x^2 - 3x + 4 \neq 0$ true $\forall\ x \in \mathbb{R}$ as $D < 0$, hence, no real roots

$\tan(\pi[x-\pi])$

$\Rightarrow$ The numerator will always be the tangent of multiple of π, hence, it is 0.

$\Rightarrow$ $R_f \in \{0\}$

(c) $f(x) = \cos 2x - \sin 2x = \sqrt{2}\left(\dfrac{1}{\sqrt{2}}\cos 2x - \dfrac{1}{\sqrt{2}}\sin 2x\right)$

$= \sqrt{2}\left(\cos\left(2x + \dfrac{\pi}{4}\right)\right)$

$\cos\left(2x + \dfrac{\pi}{4}\right) \in [-1, 1]$

$\Rightarrow$ $\sqrt{2}\cos\left(2x + \dfrac{\pi}{4}\right) \in [-\sqrt{2}, \sqrt{2}]$

(d) $f(x) = \cot^2\left(x - \dfrac{\pi}{4}\right)$; $x - \dfrac{\pi}{4} \in (-\infty, \infty)$

$\cot\left(x - \dfrac{\pi}{4}\right) \in (-\infty, \infty)$

$\Rightarrow$ $\cot^2\left(x - \dfrac{\pi}{4}\right) \in [0, \infty)$

5. (a) $f(x) = \sin^{-1}\left(\dfrac{x^2 - 1}{2(x^2+1)}\right)$

$\dfrac{x^2-1}{2(x^2+1)} = \dfrac{1}{2} - \dfrac{1}{x^2+1}$

$-1 \le \dfrac{1}{2} - \dfrac{1}{x^2+1} \le 1$

$\Rightarrow$ $-\dfrac{3}{2} \le -\dfrac{1}{x^2+1} \le \dfrac{1}{2}$

$\Rightarrow$ $-\dfrac{1}{2} \le \dfrac{1}{x^2+1} \le \dfrac{3}{2}$, but $\dfrac{1}{x^2+1} > 0$

$\Rightarrow$ $0 < \dfrac{1}{x^2+1} \le \dfrac{3}{2}$ $\quad\Rightarrow$ $\dfrac{2}{3} \le x^2 + 1 < \infty$

$\Rightarrow$ $-\dfrac{1}{3} \le x^2 < \infty$ true $\forall\ x \in \mathbb{R}$

Now as $x \in (-\infty, \infty)$ $\Rightarrow$ $x^2 + 1 \in [1, \infty)$

$\dfrac{1}{x^2+1} \in (0, 1]$ $\quad\Rightarrow$ $-\dfrac{1}{x^2+1} \in [-1, 0)$

$\Rightarrow$ $\dfrac{1}{2} - \dfrac{1}{x^2+1} \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right)$

$\Rightarrow$ $\sin^{-1}\left(\dfrac{x^2-1}{2(x^2+1)}\right) \in \left[-\dfrac{\pi}{6}, \dfrac{\pi}{6}\right)$

(b) $f(x) = \cos^{-1}\left(\dfrac{2x}{1+x^2}\right)$

$-1 \le \dfrac{2x}{1+x^2} \le 1$ $\quad\Rightarrow$ $\dfrac{(x+1)^2}{x^2+1} \ge 0$ or $\dfrac{(x-1)^2}{(x^2+1)} \ge 0$

$\Rightarrow$ $x \in \mathbb{R}$

$\dfrac{2x}{1+x^2} = \dfrac{2}{x + \dfrac{1}{x}}$ if $x \in \mathbb{R} \sim \{0\}$

$\Rightarrow$ $x + \dfrac{1}{x} \in (-\infty, -2] \cup [2, \infty)$

$\Rightarrow$ $\dfrac{1}{x + \dfrac{1}{x}} \in \left[-\dfrac{1}{2}, 0\right) \cup \left(0, \dfrac{1}{2}\right]$

$\dfrac{2}{x + \dfrac{1}{x}} \in [-1, 0) \cup (0, 1]$

$\Rightarrow$ $\cos^{-1}\left(\dfrac{2x}{1+x^2}\right) \in [0, \pi] \sim \left\{\dfrac{\pi}{2}\right\}$

If $x = 0$, then $\dfrac{2x}{1+x^2} = 0$

$\Rightarrow$ $\cos^{-1} 0 = \dfrac{\pi}{2}$ $\quad\Rightarrow$ $R_f \to [0, \pi]$

(c) $f(x) = \cos^{-1}\left(\dfrac{1+x^2}{2x}\right)$

$-1 \le \dfrac{1+x^2}{2x} \le 1$

$-1 \le \dfrac{1+x^2}{2x}$

$$0 \le \frac{x^2+2x+1}{2x}$$

$\Rightarrow \dfrac{(x+1)^2}{2x} \ge 0 \qquad \Rightarrow x > 0$ or $x = -1$

Also, $\dfrac{1+x^2}{2x} - 1 \le 0$

$\Rightarrow \dfrac{x^2+1-2x}{2x} \le 0 \qquad \Rightarrow \dfrac{(x-1)^2}{2x} \le 0$

$\Rightarrow x < 0$ or $x = 1$

$\Rightarrow x \in \{-1, 1\}$

$\Rightarrow f(x) = \cos^{-1}(-1)$ or $\cos^{-1}(1)$, i.e., π or 0

$\Rightarrow R_f = \{0, \}$

(d) $f(x) = \tan^{-1}(1 + x^2)$

$x^2 + 1 \in [1, \infty) \qquad \Rightarrow \tan^{-1}(x^2+1) \in \left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

(e) $f(x) = \log_2(3 + 5\sin x)$

Now as $3 + 5\sin x > 0$

$\Rightarrow \sin x > -\dfrac{3}{5}$ and $\sin x \in [-1, 1]$

$\Rightarrow \sin x \in \left(-\dfrac{3}{5}, 1\right] \qquad \Rightarrow 5\sin x \in (-3, 5]$

$\Rightarrow 3 + 5\sin x \in (0, 8]$

$\Rightarrow \log_2(3 + 5\sin x)\ (-\infty, 3]$

(f) $f(x) = \tan^{-1}\dfrac{2x}{1+x^2}$

If $x \ne 0$, then $\dfrac{2x}{1+x^2} = \dfrac{2}{x+\dfrac{1}{x}}$

$\Rightarrow x + \dfrac{1}{x} \in (-\infty, -2] \cup [2, \infty)$

$\dfrac{2}{x+\dfrac{1}{x}} \in [-1, 0) \cup (0, 1]$

If $x = 0$, then $\dfrac{2x}{1+x^2} = 0$

$\Rightarrow \dfrac{2x}{1+x^2} \in [-1, 1]$

$\Rightarrow \tan^{-1}\dfrac{2x}{1+x^2} \in \left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$

6. **(a)** $5x^2 - 8x + 4 = 5\left[\left(x^2 - \dfrac{8}{5}x\right) + \dfrac{4}{5}\right]$

$= 5\left(x^2 - 2\times\dfrac{4}{5}x + \dfrac{16}{25}\right) + 4 - \dfrac{16}{5} = 5\left(x - \dfrac{4}{5}\right)^2 + \dfrac{4}{5}$

Now, $5\left(x - \dfrac{4}{5}\right)^2 \in [0, \infty)$

$\Rightarrow 5\left(x - \dfrac{4}{5}\right)^2 + \dfrac{4}{5} \in \left[\dfrac{4}{5}, \infty\right)$

$\Rightarrow \ln(5x^2 - 8x + 4) \in \left[\ln\dfrac{4}{5}, \infty\right)$

(b) $\cot^{-1}\left[\log_{\frac{4}{5}}(5x^2 - 8x + 4)\right)$

$5x^2 - 8x + 4 \in \left[\dfrac{4}{5}, \infty\right)$

$\log_{4/5}(5x^2 - 8x + 4) \in (0, 1]$

$\cot^{-1}(\log_{4/5}(5x^2 - 8x + 4)) \in \left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

(c) $5x^2 - 8x + 4 \in \left[\dfrac{4}{5}, \infty\right)$

$\log_{5/4}(5x^2 - 8x + 4) \in [-1, \infty)$

$\cot^{-1}\log_{\frac{5}{4}}(5x^2 - 8x + 4) \in \left(0, \dfrac{3\pi}{4}\right]$

7. **(a)** $f(x) = \dfrac{x^2 - x + 1}{x^2 + x + 1} = 1 - \dfrac{2x}{x^2 + x + 1}$

If $x \ne 0$, then $\dfrac{2x}{x^2+x+1} = \dfrac{2}{x + \dfrac{1}{x} + 1}$;

$x + \dfrac{1}{x} \in (-\infty, -2] \cup [2, \infty)$

$x + \dfrac{1}{x} + 1 \in (-\infty, -1] \cup [3, \infty)$

$\Rightarrow \dfrac{1}{x + \dfrac{1}{x} + 1} \in \left(0, \dfrac{1}{3}\right] \cup [-1, 0)$

$\Rightarrow \dfrac{2}{x + \dfrac{1}{x} + 1} \in \left(0, \dfrac{2}{3}\right] \cup [-2, 0)$

$\Rightarrow 1 + \left(\dfrac{-2}{x + \dfrac{1}{x} + 1}\right) \in \left[\dfrac{1}{3}, 1\right) \cup (1, 3]$

If $x = 0$, then $\dfrac{x^2 - x + 1}{x^2 + x + 1} = 1$

$\Rightarrow \dfrac{x^2 - x + 1}{x^2 + x + 1} \in \left[\dfrac{1}{3}, 3\right]$

(b) $f(x) = \dfrac{x}{\ln x}$; $x \in (0, \infty) \sim \{1\}$

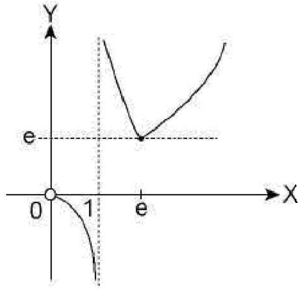
$f'(x) = \dfrac{\ln x - 1}{(\ln x)^2}$; $f'(x) = 0$ when $\ln x = 1$

$\Rightarrow x = e$

If $x > e$, then $f'(x) > 0$ and if $x < e$, then $f'(x) < 0$

$\therefore$ On $(0, 1)$, $f'(x) < 0$

$\Rightarrow \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0} \dfrac{x}{\ln x} = 0$



Also, $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{h\to 0} \dfrac{1-h}{\ln(1-h)} = -\infty$

On $(1, e)$; $f'(x) < 0$ and $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{h\to 0} \dfrac{1+h}{\ln(1+h)} = \infty$

$f(e) = e$

On (e, ∞); $f'(x) > 0$; $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{x}{\ln x}$

Applying L-H rule $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{x}{\ln x} = \lim\limits_{x\to\infty} \dfrac{1}{\frac{1}{x}} = \lim\limits_{x\to\infty} x = \infty$

$\therefore$ Range is $(-\infty, 0) \cup [e, \infty)$

8. $f(x) = \sin^{-1} x^2 + [\{\ln\sqrt{x-[x]}\}] + \cot^{-1}\left(\dfrac{1}{1+\sqrt{2x^2}}\right)$

$= \sin^{-1} x^2 + [\{\ln\sqrt{\{x\}}\}] + \cot^{-1}\left(\dfrac{1}{1+\sqrt{2x^2}}\right)$

If $x \notin \mathbb{Z}$, then $f(x) = \sin^{-1} x^2 + 0 + \cot^{-1}\left(\dfrac{1}{1+\sqrt{2x^2}}\right)$

$= \sin^{-1} x^2 + \cot^{-1}\left(\dfrac{1}{1+\sqrt{2x^2}}\right)$

$\Rightarrow x \notin \mathbb{Z}$ and $x^2 \in [-1, 1]$ and $x^2 \in [0, \infty)$ and

$\dfrac{1}{1+\sqrt{2x^2}} \in (-\infty, \infty)$

$\Rightarrow$ $x \notin \mathbb{Z}$ and $x^2 \in [0, 1]$ and $1+\sqrt{2}x^2 \in (-\infty, 0) \cup (0, \infty)$

$\Rightarrow$ $x \notin \mathbb{Z}$ and $x \in [-1, 1]$ and $x^2 \in \left(-\infty, -\dfrac{1}{\sqrt{2}}\right) \cup \left(-\dfrac{1}{\sqrt{2}}, \infty\right)$

but $x^2 \geq 0$

$\Rightarrow$ $x \in [-1, 1]$ and $x^2 \in [0, \infty)$ and $x \notin \mathbb{Z}$

$\Rightarrow$ $x \in [-1, 1]$ and $x \in \mathbb{R}$ and $x \notin \mathbb{Z}$

$\Rightarrow$ $x \in [-1, 1]$ and $x \notin \mathbb{Z}$

$\Rightarrow$ $x \in (-1, 1) \sim \{0\}$

If $x \in \mathbb{Z}$, then function $f(x)$ will be undefined as ln 0 is not defined. For the respective domain $(-1, 1) \sim \{0\}$

$f(x) = \sin^{-1} x^2 + [\{\ln\sqrt{x-\{x\}}\}] + \cot^{-1}\left(\dfrac{1}{1+\sqrt{2x^2}}\right)$

$= \sin^{-1} x^2 + \cot^{-1}\left(\dfrac{1}{1+\sqrt{2x^2}}\right)$

$\Rightarrow$ Range of $f(x) = \left(\dfrac{\pi}{4}, \dfrac{7\pi}{8}\right)$

9. **(i)** For $\sqrt{x-1}$ to be defined $x - 1 \geq 0$

$\Rightarrow$ $x \geq 1$ $\quad\Rightarrow$ $x \in [1, \infty)$

$\Rightarrow$ $x - 1 \in [0, \infty)$ $\quad\Rightarrow$ $\sqrt{x-1} \in [0, \infty)$

(ii) $\sqrt[3]{x-2}$; $x \in \mathbb{R}$ $\quad\Rightarrow$ $x - 2 \in (-\infty, \infty)$

$\Rightarrow$ $\sqrt[3]{x-2} \in (-\infty, \infty)$ $\quad\Rightarrow$ $\sqrt[3]{x-2} \in \mathbb{R}$

(iii) $\sqrt{(x-1)(x-4)}$ for square root to be defined $(x - 1)(x - 4) \geq 0$

$\Rightarrow$ $x \in (-\infty, 1] \cup [4, \infty)$

Now, $(x - 1)(x - 4) = x^2 - 5x + 4 = \left(x - \dfrac{5}{2}\right)^2 - \dfrac{9}{4}$

$\Rightarrow$ $x \in (-\infty, 1] \cup [4, \infty)$

$\Rightarrow$ $\left(x - \dfrac{5}{2}\right) \in \left(-\infty, -\dfrac{3}{2}\right] \cup \left[\dfrac{3}{2}, \infty\right)$

Now, $\left(x - \dfrac{5}{2}\right)^2 - \dfrac{9}{4} \in [0, \infty)$

$\Rightarrow$ $\sqrt{(x-1)(x-4)} \in [0, \infty)$

(iv) $\dfrac{1}{(x-1)(x-2)(x-3)}$; $x \in \mathbb{R} \sim \{1, 2, 3\}$

The expression is the denominator is cubic in x it will take all values from $(-\infty, \infty)$ and its reciprocal will take all values except 0.

Hence, $R_f = (-\infty, \infty) \sim \{0\}$

(v) $\dfrac{x-1}{x^2-4x+3} = \dfrac{(x-1)}{(x-1)(x-3)}$; $D_f \to \mathbb{R} \sim \{1, 3\}$

If $x = 1$, then $\dfrac{x-1}{x^2-4x+3}$ is not defined

If $x \neq 1$, then $\dfrac{1}{x\ \ -4x+3}$ $\quad \dfrac{1}{x-3}$

$\Rightarrow$ $x - 3 \in (-\infty, \infty) \sim \{-2\}$

$\Rightarrow$ $x - 3 \in (-\infty, -2) \cup (-2, \infty)$

$\Rightarrow$ $\dfrac{1}{x-3} \in \left(-\dfrac{1}{2}, 0\right) \cup \left(-\infty, -\dfrac{1}{2}\right) \cup (0, \infty)$

$\Rightarrow$ $R_f \to \mathbb{R} \sim \left\{0, -\dfrac{1}{2}\right\}$

(vi) $\dfrac{x^2-1}{x^2+5x+4} = \dfrac{(x-1)(x+1)}{(x+4)(x+1)}$; $x \in \mathbb{R} \sim \{-1, -4\}$

If $x = -1, -4$, then $\dfrac{x^2-1}{x^2+5x+4}$ is not defined

If $x \neq -1, -4$, then $\dfrac{x^2-1}{x^2+5x+4} = \dfrac{x-1}{x+4} = 1 - \dfrac{5}{x+4}$

$x \in (-\infty, -4) \cup (-4, -1) \cup (-1, \infty)$

$x + 4 \in (-\infty, 0) \cup (0, 3) \cup (3, \infty)$

$\dfrac{1}{x+4} \in (-\infty, 0) \cup \left(0, \dfrac{1}{3}\right) \cup \left(\dfrac{1}{3}, \infty\right)$

$1 - \dfrac{5}{x+4} \in \left(-\infty, -\dfrac{2}{3}\right) \cup \left(-\dfrac{2}{3}, 1\right) \cup (1, \infty)$

$\Rightarrow \quad R_f \to \mathbb{R} \sim \left\{1, -\frac{2}{3}\right\}$

(vii) $\dfrac{x^2+2x-3}{x^2-1} = \dfrac{(x+3)(x-1)}{(x-1)(x+1)}; x \in \mathbb{R} \sim \{1, -1\}$

If $x \neq 1, -1$, then $\dfrac{x^2+2x-3}{x^2-1} = \dfrac{x+3}{x+1} = 1 + \dfrac{2}{x+1}$

$x \in (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

$\Rightarrow \quad x + 1 \in (-\infty, 0) \cup (0, 2)\ (2, \infty)$

$\Rightarrow \quad \dfrac{1}{x+1} \in (-\infty, 0) \cup \left(0, \dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, \infty\right)$

$\Rightarrow \quad 1 + \dfrac{2}{x+1} \in (-\infty, 1) \cup (1, 2) \cup (2, \infty)$

$\Rightarrow \quad R_f \to \mathbb{R} \sim \{1, 2\}$

(viii) $\dfrac{x^2+2x+3}{x}; D_f \to \mathbb{R} \sim \{0\}$

If $x \neq 0$, then $\dfrac{x^2+2x+3}{x} = x + \dfrac{3}{x} + 2$

$\Rightarrow \quad x + \dfrac{3}{x} \in (-\infty, -2\sqrt{3}] \cup [2\sqrt{3}, \infty)$

$x + \dfrac{3}{x} + 2 \in (-\infty, 2 - 2\sqrt{3}] \cup [2 + 2\sqrt{3}, \infty)$

(ix) $f(x) = \dfrac{x+1}{x^2-2x+3} = \dfrac{x+1}{(x+3)(x-1)} = \dfrac{A}{x+3} + \dfrac{B}{x-1}$

$\Rightarrow \quad B = 1/2, A = 1/2$

Let $y = \dfrac{x+1}{x^2-2x+3} \quad \Rightarrow x^2y - 2xy + 3y = x + 1$

$\Rightarrow \quad x^2y - (2y + 1)x + 3y - 1 = 0$

For $y \in \mathbb{R}$ and $y \neq 0$, i.e., $x \neq -1$

Disc. $\geq 0 \quad \Rightarrow \quad (2y + 1)^2 - 4y(3y - 1) \geq 0$

$\Rightarrow \quad 4y^2 + 4y + 1 - 12y^2 + 4y \geq 0$

$\Rightarrow \quad -8y^2 + 8y + 1 \geq 0$

$\Rightarrow \quad 8y^2 - 8y + 1 \geq 0$

$\Rightarrow \quad y \in \left[\dfrac{1}{2} - \dfrac{\sqrt{6}}{4}, \dfrac{1}{2} + \dfrac{\sqrt{6}}{4}\right]$

For $x = -1, y = 0$

$\Rightarrow \quad R_f = \left[\dfrac{1}{2} - \dfrac{\sqrt{6}}{4}, \dfrac{1}{2} + \dfrac{\sqrt{6}}{4}\right]$

(x) $y = \sqrt{x-1} + \sqrt{5-x}; x \in [1, 5]$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{\sqrt{5-x} - \sqrt{x-1}}{2\sqrt{x-1}\sqrt{5-x}}$ if $\sqrt{5-x} > \sqrt{x-1}$

$\Rightarrow \quad \dfrac{dy}{dx} > 0$

If $x < 3$, then $\dfrac{dy}{dx} > 0$

If $x > 3$, then $\dfrac{dy}{dx} < 0$

If $x = 3$, then $\dfrac{dy}{dx} = 0$

For $x \in [1, 3] \quad \Rightarrow f(x) \in [f(1), f(3)]$

$\Rightarrow \quad f(x) \in [2, 2\sqrt{2}]$

For $x \in [3, 5] \quad \Rightarrow f(x) \in [f(5), f(3)]$

$\Rightarrow \quad f(x) \in [2, 2\sqrt{2}]$

$\Rightarrow \quad R_f \to [2, 2\sqrt{2}]$

(xi) $\sin^2 x - 5\sin x - 6 = \left(\sin x - \dfrac{5}{2}\right)^2 - \dfrac{49}{4}$

$\sin x \in [-1, 1]; \left(\sin x - \dfrac{5}{2}\right)^2 \in \left[\dfrac{9}{4}, \dfrac{49}{4}\right]$

$\left(\sin x - \dfrac{5}{2}\right)^2 - \dfrac{49}{4} \in [-10, 0]$

(xii) ${}^{7-x}P_{x-3}$

$\Rightarrow \quad x \in \mathbb{Z}$ and $7 - x \geq 0$ and $x - 3 \geq 0$ and $7 - x \geq x - 3$

$\Rightarrow \quad x \in \mathbb{Z}$ and $x \in [3, 7]$ and $x \in (-\infty, 5]$

$\Rightarrow \quad x \in \mathbb{Z}$ and $x \in [3, 5] \quad \Rightarrow \quad x \in \{3, 4, 5\}$

$f(3) = {}^{7-3}P_{3-3} = {}^4P_0 = 24; f(4) = {}^{7-4}P_{4-3} = {}^3P_1 = 6$

$f(5) = {}^{7-5}P_{5-3} = {}^2P_2 = 1$

$R_f = \{1, 24, 6\}$

10. (i) $\log_e(3x^2 - 4x + 5) = \ln(3x^2 - 4x + 5) = 3\left[\left(x^2 - \dfrac{4}{3}x\right) + \dfrac{5}{3}\right]$

$= 3\left[\left(x^2 - 2.\left(\dfrac{2}{3}\right).x + \dfrac{4}{9}\right) - \dfrac{4}{9} + \dfrac{15}{9}\right]$

$\Rightarrow \quad 3\left[\left(x - \dfrac{2}{3}\right)^2 + \dfrac{11}{9}\right]$

$\Rightarrow \quad 3(x - 2/3)^2 \in (0, \infty)$

$\Rightarrow \quad 3(x - 2/3)^2 + 11/3 \in [11/3, \infty)$

$\Rightarrow \quad \ln(3x^2 - 4x + 5) \in [\ln(11/3), \infty)$

(ii) $f(x) = \cot^{-1}(x^2 - 4x + 5); D_f \in \mathbb{R}$

$(x^2 - 4x + 5) = (x^2 - 4x + 4) + 1$

$= (x - 2)^2 + 1 \in [1, \infty)$

$\Rightarrow \quad R_f = (0, /4]$

(iii) $\cot^{-1}(-x^2 + 2x) = \cot^{-1}(-x(x - 2))$

Domain of $\cot^{-1}(x) = \mathbb{R}$ and $-x(x - 2) \in (-\infty, 1)$

$\Rightarrow \quad \cot^{-1}(-x(x - 2))\ [\cot^{-1}(1), \cot^{-1}(\infty)]$

$\Rightarrow \quad R_f = [\pi/4, \pi)$

(iv) $f(x) = \dfrac{x}{1 + |x|}$

For $x > 0; y = \dfrac{x}{1+x}$

$\Rightarrow \quad xy + y = x \quad \Rightarrow \quad x(y - 1) = -y$

$\Rightarrow \quad x = \dfrac{-y}{y-1} > 0$

$\Rightarrow \quad y(y - 1) < 0 \quad \Rightarrow \quad y \in (0, 1)$

For $x < 0; y = \dfrac{x}{1-x}$

$\Rightarrow \quad y - xy = x \quad \Rightarrow \quad x(1 + y) = y$

$\Rightarrow \quad x = \dfrac{y}{1+y} < 0 \quad \Rightarrow \quad y \in (-1, 0)$

For $x = 0$; $y = 0 \quad \Rightarrow \quad y \in (-1, 1) = R_f$

(v) $\cos([x]\,\pi)$

$[x]$ is always an integer, i.e., positive or negative

$\Rightarrow$ cos[[x] π] = cosine of a negative integer or a positive integer.

$\Rightarrow$ cos [[x]π] = {–1, 1}

(vi) $f(x) = 4^x - 2^x + 1$

Let $2^x = y$

$f(y) = y^2 - y + 1 \quad \Rightarrow \quad y^2 - 2\left(\frac{1}{2}\right)y + \frac{1}{4} + \frac{3}{4}$

$\Rightarrow \left(y - \frac{1}{2}\right)^2 + \frac{3}{4} \quad \Rightarrow \quad f(y) \in [3/4, \infty)$

Hence, range of $f(x)$ as $2^x \in (0, \infty)$

$\Rightarrow f(x) \in [3/4, \infty)$

11. **(i)** $\dfrac{1}{(|x|-1)(\cos^{-1}(2x+1))\tan 3x}$

$|x| - 1 \neq 0$ and $2x + 1 \in [-1, 1]$ and $\cos^{-1}(2x + 1) \neq 0$

and $3x \in \mathbb{R} \sim \{0\}$ and $3x \neq (2n+1)\dfrac{\pi}{2}, n \in \mathbb{Z}$

$\Rightarrow x \neq 1, -1$ and $x \in [-1, 0]$ and $x \neq 0$ and $x \in \mathbb{R} \sim \{0\}$

and $3x \neq (2n+1)\dfrac{\pi}{2}; n \in \mathbb{Z}$

$\Rightarrow x \in (-1, 0)$ and $x \neq -\dfrac{\pi}{6}$

$\Rightarrow x \in (-1, 0) - \left\{-\dfrac{\pi}{6}\right\}$

(iii) $\log_2 \log_{1/3} \log_3 (x^2 - 4x + 3)$

$x^2 - 4x + 3 > 0$ and $\log_{1/3} \log_3 (x^2 - 4x + 3) > 0$

$\Rightarrow x \in (-\infty, 1) \cup (3, \infty)$ and $x^2 - 4x + 3 < 3$

$\Rightarrow x \in (-\infty, 1) \cup (3, \infty)$ and $x \in (0, 4)$

$\Rightarrow x \in (0, 1) \cup (3, 4)$

(iv) $\log_2 \log_3 \log_{4/\pi} (\tan^{-1} x)^{-1}$

$\Rightarrow (\tan^{-1} x)^{-1} > 0$ and $\log_3 \log_{4/\pi} (\tan^{-1} x)^{-1} > 0$

$\Rightarrow (\tan^{-1} x)^{-1} > \dfrac{4}{\pi}$ and $(\tan^{-1} x)^{-1} > 0$

$\Rightarrow \tan^{-1} x < \dfrac{\pi}{4}$ and $(0, \infty)$

$\Rightarrow x \in (0, 1)$

(v) $f(x) = \left(\dfrac{x}{1-|x|}\right)^{1/2002}$ for $x \geq 0$

$\Rightarrow f(x) = \left(\dfrac{x}{1-x}\right)^{1/2002} \quad \Rightarrow \quad f(x) = \left(\dfrac{-1}{1-x}+1\right)^{1/2002}$

Similarly for $x < 0$

$\Rightarrow f(x) = \left(\dfrac{1}{x+1}-1\right)^{1/2002}$

$\Rightarrow R_f \in (-\infty, -1) \cup [0, 1)$

12. **(i)** $\dfrac{2x}{x^2+1}$

This can be written as $\dfrac{2}{x+\dfrac{1}{x}}$

$\therefore \quad x + 1/x \in (-\infty, -2] \cup [2, \infty)$

$\Rightarrow \dfrac{1}{\dfrac{1}{x}+x} \in [-1/2, 0) \cup (0, 1/2]$

$\Rightarrow \dfrac{2}{x+\dfrac{1}{x}} \in [-1, 0)\ (0, 1]$

$\Rightarrow$ Minimum = – 1

$\Rightarrow$ Maximum = 1

(ii) $f(x) = \dfrac{x^2-1}{x^2+1} = \left(1 - \dfrac{2}{x^2+1}\right)$

$\Rightarrow x^2 \in [0, \infty)$

$\Rightarrow x^2 + 1 \in [1, \infty)$

$\Rightarrow 1/(x^2 + 1) \in (0, 1]$

$\Rightarrow 2/(x^2 + 1) \in (0, 2] \quad \Rightarrow \quad f(x) \in [-1, 1)$

$\Rightarrow$ Min. $f(x) = -1$, Max $f(x)$ does not exist, but lub = 1

(iii) $\dfrac{x^2-3x+4}{x^2+3x+4} = y$

$\Rightarrow y(x^2 + 3x + 4) = x^2 - 3x + 4$

$\Rightarrow (y - 1)x^2 + 3(y + 1)x + 4(y - 1) = 0$

$D \geq 0$

$\Rightarrow 9(y + 1)^2 - 16(y - 1)^2 \geq 0$

$\Rightarrow (7y - 1)(7 - y) \geq 0$

$\Rightarrow y \in [1/7, 7] \quad \Rightarrow \quad y_{\min} = \dfrac{1}{7}, y_{\max} = 7$

(iv) $y = \dfrac{1}{2+x^2}$

$\Rightarrow x^2 \in [0, 36]$

$\Rightarrow x^2 + 2 \in [2, 38]$

$\Rightarrow \dfrac{1}{x^2+2} \in \left[\dfrac{1}{38}, \dfrac{1}{2}\right]$

$\Rightarrow$ Lower bound = 1/38

$\Rightarrow$ Upper bound = 1/2

(v) $S = \dfrac{3-x}{1-x} = \left(\dfrac{2}{1-x}+1\right); x > 0, x \neq 1$

$\Rightarrow -x < 0; -x \neq -1$

$\Rightarrow 1 - x < 1; 1 - x \neq 0 \quad \Rightarrow \quad \dfrac{1}{1-x} \in (-\infty, 0) \cup (1, \infty)$

$\Rightarrow \dfrac{2}{1-x} + 1 \in (-\infty, 1) \cup (3, \infty)$

$\Rightarrow$ Upper and lower does not exist.

(vi) $|x| \leq \sqrt{\dfrac{3}{2}} \Rightarrow -\sqrt{\dfrac{3}{2}} \leq x \leq \sqrt{\dfrac{3}{2}}$

$\Rightarrow 0 \leq x^2 \leq 3/2 \quad \Rightarrow \quad -x^2 \geq -\dfrac{3}{2}$

$\Rightarrow 1 \geq 1 - x^2 \geq -\dfrac{1}{2}$

$\Rightarrow 1 \geq \sqrt{1-x^2} \geq 0$

$\Rightarrow$ *glb* = 0 and least upper bond = 1

(vii) $|x - 1| \leq 2$

$\Rightarrow -2 \leq x - 1 \leq 2 \quad \Rightarrow \quad -1 \leq x \leq 3$

$\Rightarrow -2 \leq 2x \leq 6$

$\Rightarrow 1 \leq 2x + 3 \leq 9 \quad \Rightarrow \quad \dfrac{1}{9} \leq \dfrac{1}{2x+3} \leq 1$

$y = \dfrac{3x+1}{2x+3} = \dfrac{3}{2} + \left(\dfrac{-7/2}{2x+3}\right)$

$\Rightarrow -\dfrac{7}{18} \geq \dfrac{-7/2}{2x+3} \geq -\dfrac{7}{2}$

$\Rightarrow \dfrac{3}{2} - \dfrac{7}{18} \geq \dfrac{3}{2} + \left(\dfrac{-7/2}{2x+3}\right) \geq \dfrac{3}{2} - \dfrac{7}{2}$

$\Rightarrow \dfrac{10}{9} \geq y \geq -2$ $\Rightarrow$ glb = –2 and lub = 10/9

TEXTUAL EXERCISE–10: (OBJECTIVE)

1. (c) $y = [x]$ $\Rightarrow R_f = \mathbb{Z}$

2. (b) $\dfrac{2x-1}{2x+1}$ $\Rightarrow x \in \mathbb{R} - \left\{-\dfrac{1}{2}\right\}$

$\dfrac{2x-1}{2x+1} = 1 - \dfrac{2}{2x+1}$; $x \in \left(-\infty, -\dfrac{1}{2}\right) \cup \left(-\dfrac{1}{2}, \infty\right)$

$\Rightarrow 2x + 1 \in (-\infty, 0) \cup (0, \infty)$

$\Rightarrow \dfrac{1}{2x+1} \in (-\infty, 0) \cup (0, \infty)$

$\Rightarrow 1 - \dfrac{2}{2x+1} \in (-\infty, 1) \cup (1, \infty)$

$\Rightarrow R_f \to \mathbb{R} \sim \{1\}$

3. (a) $\dfrac{x^2-1}{x^2+1} = 1 - \dfrac{2}{x^2+1}; x^2 \geq 0$

$\Rightarrow 1 + x^2 \geq 1$ $\Rightarrow \dfrac{1}{x^2+1} \in (0, 1]$

$\Rightarrow 1 - \dfrac{2}{x^2+1} \in [-1, 1)$

4. (b) $\dfrac{2x}{1+x^2} = y$

$\Rightarrow yx^2 + y - 2x = 0$

For real x; $D \geq 0$ $\Rightarrow 4 - 4y^2 \geq 0 = 1 - y^2 \geq 0$

$\Rightarrow (1 + y)(1 - y) \geq 0$ $\Rightarrow (y + 1)(y - 1) \leq 0$

$\Rightarrow y \in [-1, 1]$

5. (c) $y = \dfrac{x^2 - x + 1}{x^2 + x + 1}$

$\Rightarrow yx^2 + yx + y = x^2 - x + 1$

$\Rightarrow x^2(y - 1) + x(y + 1) + y - 1 = 0$

For real roots $= (y + 1)^2 - 4(y - 1)^2 \geq 0$

$\Rightarrow y^2 + 1 + 2y - 4(y^2 + 1 - 2y) \geq 0$

$\Rightarrow -3y^2 + 10y - 3 \geq 0$

$\Rightarrow -3y^2 + 9y + y - 3 \geq 0$

$\Rightarrow -3y(y - 3) + (y - 3) > 0$

$\Rightarrow (y - 3)(-3y + 1) \geq 0$

$\Rightarrow (y - 3)(y - 1/3) \leq 0$

$\Rightarrow y \in [1/3, 3]$

6. (c) $y = \dfrac{1}{(x-1)(x-2)}$

$\Rightarrow y = \dfrac{1}{x^2 - 3x + 2}$; $y \neq 0, x \neq 1, 2$

$\Rightarrow yx^2 - 3yx + 2y - 1 = 0$

$\Rightarrow 9y^2 - 4y(2y - 1) \geq 0$ and $y \neq 0$

$\Rightarrow 9y^2 - 8y^2 + 4y \geq 0$ $\Rightarrow y^2 + 4y \geq 0, y \neq 0$

$\Rightarrow y(y + 4) \geq 0, y \neq 0$ $\Rightarrow y \in (-\infty, -4] \cup (0, \infty)$

7. (a) $y = A \sin x + B \cos x$

Which can be written as

$y = \left(\dfrac{A \sin x}{\sqrt{A^2+B^2}} + \dfrac{B \cos x}{\sqrt{A^2+B^2}}\right)\sqrt{A^2+B^2}$

Let $\cos u = \dfrac{A}{\sqrt{A^2+B^2}}$

$\Rightarrow y = (\cos u \sin x + \sin u . \cos x) \sqrt{A^2+B^2}$

$\Rightarrow y = \sin(x + u)\sqrt{A^2+B^2}$

$\Rightarrow y \in \left[-\sqrt{A^2+B^2}, \sqrt{A^2+B^2}\right]$

8. (a) $y = \dfrac{x^2}{1+x^2}; y \neq 1$

$\Rightarrow y + yx^2 = x^2$ $\Rightarrow yx^2 - x^2 + y = 0$

$\Rightarrow x^2(y - 1) + y = 0, y \neq 1$

$\Rightarrow -4y(y - 1) \geq 0, y\ 1$ $\Rightarrow y(y - 1) \leq 0, y \neq 1$

$\Rightarrow y \in [0, 1)$

9. (c) $y = \dfrac{1}{2 - \sin 3x}$; $\sin 3x \in [-1, 1]$; $2 - \sin 3x \in [1, 3]$

$\Rightarrow \dfrac{1}{2 - \sin 3x} \in \left[\dfrac{1}{3}, 1\right]$

10. (a) $x^2 + x + 1 = \left(x + \dfrac{1}{2}\right)^2 + \dfrac{3}{4}$ $\Rightarrow x^2 + x + 1 \geq \dfrac{3}{4}$

Hence, min $= \dfrac{3}{4}$; max $\to \infty$

11. (c) $x^2 - 4x + 5 = (x - 2)^2 + 1$

$x^2 - 4x + 5 \geq 1$ $\Rightarrow$ min = 1; max $\to \infty$

12. (a) $ax^2 + bx + c = a\left(x^2 + 2 \times \dfrac{b}{2a}x\right) + c$; $a > 0$

$= a\left(x + \dfrac{b}{2a}\right)^2 + c - \dfrac{b^2}{4a} = a\left(x + \dfrac{b}{2a}\right)^2 + \dfrac{4ac - b^2}{4a}$

$\Rightarrow ax^2 + bx + c \in \left[\dfrac{4ac - b^2}{4a}, \infty\right)$

13. (b) $2 - 3x - x^2 = -(x^2 + 3x) + 2 = -\left(x^2 + 3x + \dfrac{9}{4}\right) + \dfrac{9}{4} + 2$

$= -\left(x + \dfrac{3}{2}\right)^2 + \dfrac{17}{4}$

$\because \left(x + \dfrac{3}{2}\right)^2 \leq 0$ $\Rightarrow -\left(x + \dfrac{3}{2}\right)^2 + \dfrac{17}{4} \leq \dfrac{17}{4}$

$\Rightarrow$ min $\to -\infty$ and max $= \dfrac{17}{4}$

14. (a) $\dfrac{x^2-1}{x^2+1} = 1 - \dfrac{2}{x^2+1}$

Now as $x^2 + 1 \geq 1$ $\Rightarrow 0 \geq -\dfrac{1}{x^2+1} \geq -1$

$\Rightarrow 1 \geq 1 - \dfrac{2}{x^2+1} \geq -1$

$\Rightarrow \dfrac{x^2-1}{x^2+1} \in [-1, 1]$

15. **(b)** $\dfrac{x^2-3x+4}{x^2+3x+4}=1-\dfrac{6x}{x^2+3x+4}$

If $x = 0$, then $\dfrac{x^2-3x+4}{x^2+3x+4}=1$

If $x \neq 0$, then $\dfrac{x^2-3x+4}{x^2+3x+4}=1-\dfrac{6x}{x^2+3x+4}=1-\dfrac{6}{x+\dfrac{4}{x}+3}$

$\Rightarrow\ x+\dfrac{4}{x}\in(-\infty,-4]\cup[4,\infty)$

$\Rightarrow\ x+\dfrac{4}{x}+3\in(-\infty,-1]\cup[7,\infty)$

$\Rightarrow\ \dfrac{1}{x+\dfrac{4}{x}+3}\in[-1,0)\cup\left(0,\dfrac{1}{7}\right]$

$\Rightarrow\ 1-\dfrac{6}{x+\dfrac{4}{x}+3}\in\left[\dfrac{1}{7},1\right)\cup(1,7]$

$\Rightarrow\ \dfrac{x^2-3x+4}{x^2+3x+4}\in\left[\dfrac{1}{7},7\right]$

16. **(b)** $y=\dfrac{1}{2+x^2};-6\le x\le 4$

$\Rightarrow\ x^2 \in [0, 36]$ $\qquad\Rightarrow\ x^2 + 2 \in [2, 38]$

$\Rightarrow\ \dfrac{1}{x^2+2}\in\left[\dfrac{1}{38},\dfrac{1}{2}\right]$

$\therefore\ \ g.l.b.=\dfrac{1}{38};\ l.u.b.=\dfrac{1}{2}$

17. **(c)** $y=\dfrac{3-x}{1-x};x>0$

$\Rightarrow\ y - yx = 3 - x$ $\qquad\Rightarrow\ x(1 - y) = 3 - y$

$\Rightarrow\ x=\dfrac{3-y}{1-y};\ x>0$

$\Rightarrow\ (3 - y) (1 - y) > 0$ $\qquad\Rightarrow\ (y - 1) (y - 3) > 0$

$\Rightarrow\ y \in (-\infty, 1) \cup (3, \infty)$

$\Rightarrow$ *glb*, *lub* does not exist.

18. **(a)** $|x - 5| < 1$ $\qquad\Rightarrow\ x \in (4, 6)$

Now as $\dfrac{x}{x+10}=\dfrac{1}{1+\dfrac{10}{x}}$

$\Rightarrow\ \dfrac{1}{x}\in\left(\dfrac{1}{6},\dfrac{1}{4}\right)$ $\qquad\Rightarrow\ \dfrac{10}{x}+1\in\left(\dfrac{16}{6},\dfrac{14}{4}\right)$

$\Rightarrow\ \dfrac{1}{\dfrac{10}{x}+1}\in\left(\dfrac{2}{7},\dfrac{3}{8}\right)$

19. **(c)** $e^x=y+\sqrt{1+y^2}$ $\qquad\Rightarrow\ (e^x - y)^2 = 1 + y^2$

$\Rightarrow\ e^{2x} + y^2 - 2e^xy = 1 + y^2$

$\Rightarrow\ \dfrac{e^{2x}-1}{2e^x}=y$ $\qquad\Rightarrow\ y=\dfrac{e^x-e^{-x}}{2}$

20. **(c)** $f(x)=\begin{cases}x & x\in\mathbb{Q}\\ 1-x & x\notin\mathbb{Q}\end{cases}\ \forall\, x \in [0, 1]$

If $x \in \mathbb{Q}$

$f(x) = x$ $\qquad\Rightarrow\ f(f(x)) = x;\ x \in \mathbb{Q}$

If $x \notin \mathbb{Q}$ $\qquad\Rightarrow\ f(x) = 1 - x$

Since $1 - x$ is also irrational.

$\Rightarrow\ f(f(x)) = (1 - (1 - x)) = x$

21. **(b)** $y=\dfrac{x}{1+x^2}$ $\qquad\Rightarrow\ y = 0$ for $x = 0$

$\Rightarrow\ y + yx^2 = x$ $\qquad\Rightarrow\ yx^2 - x + y = 0$

$\Rightarrow\ 1 - 4y^2 \ge 0$ for $y \neq 0$

$\Rightarrow\ (1 - 2y) (1 + 2y) \ge 0;\ y \neq 0$

$\Rightarrow\ (y - 1/2) (y + 1/2) \le 0$

$\Rightarrow\ y\in\left[-\dfrac{1}{2},\dfrac{1}{2}\right]-\{0\}$

But y = 0 at $x = 0$ $\qquad\Rightarrow\ R_f=\left[-\dfrac{1}{2},\dfrac{1}{2}\right]$

22. **(c)** $y=\dfrac{1}{x(x-1)(x-2)}$

$\Rightarrow\ y=\dfrac{1}{x(x^2-3x+2)}$ $\qquad\Rightarrow\ y=\dfrac{1}{x^3-3x^2+2x}$

Consider $f(x) = x^3 - 3x^2 +2 x$

$\Rightarrow\ f'(x) = 3x^2 - 6x + 2$

$\Rightarrow\ x=\dfrac{6\pm\sqrt{36-24}}{6}=1\pm\dfrac{\sqrt{12}}{6}=1\pm\dfrac{\sqrt{3}}{3}$

$\Rightarrow\ x=1\pm\dfrac{1}{\sqrt{3}}$

$\Rightarrow$ Range of function $\mathbb{R} - \{0\}$

23. **(c)** Consider $e^x - x = g(x)$

$\Rightarrow\ g'(x) = e^x - 1$ $\qquad\Rightarrow\ g'(x) = 0$ at $x = 0$

$\Rightarrow\ g'(x) > 0$ $\qquad\Rightarrow\ x > 0$

$\Rightarrow\ g'(x) < 0$ $\qquad\Rightarrow\ x < 0$

$\Rightarrow$ Range of function $= [0, \infty)$

24. **(b)** $y=\dfrac{x^2}{1+x^2}\ y + yx^2 - x^2 = 0$

$\Rightarrow\ x^2 (y - 1) + y = 0$

$\Rightarrow\ -4(y) (y - 1) \ge 0$ $\qquad\Rightarrow\ y(y - 1) \le 0$

$\Rightarrow\ y \in [0, 1)$

25. **(a), (b), (d)**

Functions (a), (b), (d) are defined for all x.

While tan(log x) is not defined for x. It has the restriction $x > 0$

26. **(a)** $y = \sin^2 x - 5 \sin x - 6$

Replace $\sin x = k$ $\qquad\Rightarrow\ y = k^2 - 5k - 6$

$\Rightarrow\ y' = 2k - 5$

$\Rightarrow\ y = 0$

$\Rightarrow\ k = 5/2 = 2.5$ which is not position.

$\Rightarrow\ y < 0\ \forall\ k < 5/2$, which is always possible. Hence, y is decreasing.

$\Rightarrow$ Range $=[f(k=1),f(k=-1)]=[-10,0]$

TEXTUAL EXERCISE–11: (SUBJECTIVE)

1. **(i)** $f(x) = 5x - 4$

$\Rightarrow f'(x) = 5 > 0 \quad \Rightarrow f(x)$ is injective

(ii) $f(x) = x - (1/x) \quad \Rightarrow f'(x) = 1 + \dfrac{1}{x^2} > 0 \ \forall\ x\ 0$

Also $\lim\limits_{x\to-\infty} f(x) = -\infty,\ \lim\limits_{x\to0^-} f(x) = \infty,\ \lim\limits_{x\to0^+} = -\infty,\ \lim\limits_{x\to\infty} f(x) = \infty$

$\Rightarrow f(x)$ is many–one, i.e., non-injective.

(iii) $f(x) = x^2 - 6x + 15;\ (x < 3)$

$f(x)$ is injective when defined on subset of $(-\infty, 3]$ or on subset of $[3, \infty)$

$\Rightarrow f(x)$ is injective

(iv) $f(x) = \dfrac{(x^2+1)}{2x};\ (x > 0) = \dfrac{1}{2}\left(x + \dfrac{1}{x}\right)$

$\Rightarrow f'(x) = \dfrac{1}{2}\left(1 - \dfrac{1}{x^2}\right) = \dfrac{1}{2}\left(\dfrac{x^2-1}{x^2}\right)$

$\Rightarrow f(x) \downarrow$ on $(0, 1]$ and $\uparrow$ on $[1, \infty)$

$\Rightarrow f(x)$ is non-injective.

(v) $f(x) = 2x^5 + 40\,x^2 + 2$; for $x \in [0, \infty)$

$\Rightarrow f'(x) = 10x^4 + 80x\ \forall\ x \in [0, \infty)$

$\Rightarrow f(x) \uparrow \forall\ x \in [0, \infty)$ and being polynomial is continuous, and hence, it is injective.

(vi) $f(x) = \dfrac{x-2}{x+2};\ x \in \mathbb{R} - \{-2\}$

$\Rightarrow f(x) = 1 - \dfrac{4}{x+2}$

$\Rightarrow f'(x) = \dfrac{4}{(x+2)^2} > 0\ \forall\ x \in \mathbb{R} - \{-2\}$

$\lim\limits_{x\to-2^-} f(x) = \infty,\ \lim\limits_{x\to-2^+} f(x) = -\infty,$

$\lim\limits_{x\to-\infty} f(x) = 1^+;\ \lim\limits_{x\to\infty} f(x) = 1^-$, i.e., $f(x) \in (1, \infty)$ for $x < -2$ and $f(x) \in (-\infty, 1)$ for $x > -2$

$\Rightarrow f(x)$ is injective.

(vii) $f(x) = x^4 + x^3 + 1$; domain $= \mathbb{R}$

$\Rightarrow f'(x) = 4x^3 + 3x^2 = x^2(4x + 3)$

$\Rightarrow f'(x) < 0$ for $x < -3/4$ and $f'(x) \geq 0$ for $x > -3/4$ and $f(x)$ being polynomial is continuous.

$\Rightarrow f(x)$ is non-injective.

(viii) $f(x) = x^6 + x^4 + x^2 + 1$

$\Rightarrow f(x)$ is an even function domain $\mathbb{R}$ (symmetric about origin)

$\Rightarrow f(x)$ is non-injective.

2. **(a)** False, it will be true if $f(x)$ is continuous.

(b) It is true as $f(x)$ is continuous, hence, $f(x) \in (k, k + 1)$ for some $k \in \mathbb{Z}$ and for some set A.

$\Rightarrow [f(x)] = k\ \forall\ x \in A \quad \Rightarrow [f(x)]$ is non-injective.

(c) False, as if $f(x)$ is strictly increasing with range $(k, k + 1)$ for some $k \in \mathbb{Z}$, then $\{f(x)\} = x - k\ \forall\ x \in D_f$

$\Rightarrow \{f(x)\}$ is injective .

(d) True, as for such function $\{f(x)\} = f(x) - [f(x)] = f(x) - 0 = f(x)$, which is continuous and strictly increasing, hence, $\{f(x)\}$ is injective.

(e) Since range of $f(x) \supseteq [0, 1]$

$\Rightarrow \{f(x)\} = \begin{cases} f(x)+1 \text{ for } f(x) \in [-1,0) \\ f(x) \text{for } f(x) \in [0,1) \\ f(x)-1 \text{ for } f(x) \in [1,2) \end{cases}$

$\Rightarrow \{f(x)\} \in [0, 1)$ for $f(x)\ [-1, 0)$

Again $\in [0, 1)$ for $f(x)\ [0, 1)$ and again $\in [0, 1)$ for $f(x) \in [0, 1)$

$\Rightarrow \{(f(x)\}$ is many–one function

3. $A = \{1, 2, 3, 4, 5, 6\}$

Let $a + f(a) = k^2$; then $a, f(a)$

$\Rightarrow a + f(a) \in \{2, 3, 4, \ldots, 12\}$

$\Rightarrow k^2 \in \{4, 9\}$

$\Rightarrow a + f(a) = 4$ or $9\ \forall\ a \in A$

$\Rightarrow f(1) = 3, f(2) = 2, f(3) = 1$or $6, f(4) = 5, f(5) = 4, f(6) = 3$

$\Rightarrow f(x)$ is non-injective.

Now, let $B = \{2, 3, 4, 5, 6\}$ and $f: B \to B;\ a, f(a) \in B$

$\Rightarrow a + f(a) \in \{4, 5, 6, 12\}$

$\Rightarrow k^2 \in \{4, 5, 6, \ldots, 12\} \Rightarrow k^2 \in \{4, 9\}$

$\Rightarrow f(2) = 2, f(3) = 6, f(4) = 5, f(5) = 4, f(6) = 3$

$\Rightarrow f(x)$ is injective. Thus, when A is replaced by B, then the statement is no more holds good.

4. $f: \mathbb{R} \to \mathbb{R};\ f(x) = \dfrac{x^2+4x+7}{x^2+x+1}$

$\Rightarrow f(x) = \dfrac{(x^2+x+1)+(3x+6)}{(x^2+x+1)}$

$\Rightarrow f(x) = 1 + \dfrac{3(x+2)}{(x^2+x+1)}$

$\Rightarrow f'(x) = \ldots$, which is not having a unique sign as discriminant of $(x^2 + 4x + 1)$ is possible.

Also $x^2 + x + 1 > 0\ \forall\ x \in \mathbb{R}$, implies $f(x)$ is continuous; hence, $f(x)$ is non-injective.

5. $f(x) = 2\sin x - kx + 3, f'(x) = 2\cos x - k$

Clearly $f(x)$ is continuous and will be injective if either $f'(x) \geq 0\ \forall\ x \in \mathbb{R}$ or $f'(x) \leq 0\ \forall\ x \in \mathbb{R}$.

Now $f'(x) \geq 0\ \forall\ x \in \mathbb{R}$

$\Rightarrow k \leq 2\cos x\ \forall\ x \in \mathbb{R} \Rightarrow k \leq -2$

Also for $f'(x) \leq 0\ \forall\ x \in \mathbb{R}$

$\Rightarrow k \geq 2\cos x\ \forall\ x \in \mathbb{R} \Rightarrow k \geq 2$

Thus, $k \in (-\infty, -2]\ [2, \infty)$

6. $f: D \to \mathbb{R};\ f(x) = \dfrac{5x - x^2 - 4}{5x - 4x^2 - 1} = \dfrac{(x-1)(x-4)}{(4x-1)(x-1)}$

$\Rightarrow D = \mathbb{R} - \left\{\dfrac{1}{4}, 1\right\}$

$\Rightarrow f(x) = \dfrac{(x-4)}{(4x-1)};\ x \in \mathbb{R} - \left\{\dfrac{1}{4}, 1\right\} = D$

$\Rightarrow f'(x) = \dfrac{15}{(4x-1)^2} > 0\ \forall\ x \in D$

$\lim\limits_{x\to-\infty} f(x) = \left(\dfrac{1}{4}\right)^+,\ \lim\limits_{x\to\left(\frac{1}{4}\right)^-} f(x) = \infty,$

$$\lim_{x\to\left(\frac{1}{4}\right)} f(x) = -\infty, \lim_{x\to 1^-} = -1,\ \lim_{x\to 1^+} f(x) = -1, \lim_{x\to\infty} f(x) = \left(\frac{1}{4}\right)^-$$

$\therefore\ f(x) \in \left(-\infty, \frac{1}{4}\right) - \{-1\}$ for $x \in \left(\frac{1}{4}, \infty\right)$ and $f(x)\ \left(\frac{1}{4}, \infty\right)$

for $x \in \left(-\infty, \frac{1}{4}\right)$ and $f(x) \uparrow$ on D

$\Rightarrow$ $f(x)$ is injective.

7. f: $\mathbb{R} \to \mathbb{R}$; $f(x) = \dfrac{x^2+3x+c}{x^2+x+1} = \dfrac{-2x^2+2(1-c)x+(3-c)}{(x^2+x+1)^2}$

Disc. $-2x^2 + 2(1-c)x + (3-c)$

$= 4(1-c)^2 + 8(3-c) = 4[c^2 + 1 - 2c + 6 - 2c] = 4[c^2 - 4c + 7] = 4[c^2 - 4c + 4 + 3] = 4[(c-2)^2 + 3] \geq 12\ \forall\ c\ \mathbb{R}$

$\Rightarrow$ $-2x^2 + 2(1-c)x + (3-c)$ changes its sign $\forall\ c \in \mathbb{R}$

$\Rightarrow$ $f'(x)$ changes its sign for each $c \in \mathbb{R}$ and $x^2 + x + 1 > 0\ \forall\ x \in \mathbb{R}$

$\Rightarrow$ $f(x)$ is continuous. Hence, $f(x)$ is many–one $\forall\ c \in \mathbb{R}$.

TEXTUAL EXERCISE–11: (OBJECTIVE)

1. (a), (b), (c), (d) $f(x) = x^3 - 12x + 1$

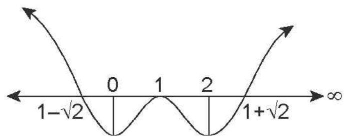
$\Rightarrow$ $f'(x) = 3x^2 - 12 = 3(x^2 - 4) > 0$ for $x \in (-\infty, -2) \cup (2, \infty)$ and ≤ 0 for $x \in [-2, 2]$

$\Rightarrow$ $f(x)$ is non-injective on $\mathbb{R}$.

2. (b) $f(x) = x^4 - 4x^3 + 4x^2 - 1$

$\Rightarrow$ $f(x) = (x-1)^2(x^2 - 2x - 1)$

$f(x)$ will be of the form



Also $f'(x) = 0$ $\quad\Rightarrow\quad$ $x = 1, 0, 2$

$\Rightarrow$ $f(x)$ is non-injective on $(1-k, 1+k)$ for each $k > 0$, injective on $[2, \infty)$ and on $(-\infty, 0)$.

3. (a), (c) $f(x) = x^3 + x^2 + 3x + \sin x$

$\Rightarrow$ $f'(x) = 3x^2 + 2x + 3 + \cos x$

Now, $3x^2 + 2x + 3 \geq \dfrac{8}{3}\left(\because\ \geq \dfrac{-D}{4a}\right)$

$\Rightarrow$ $f'(x) \geq \dfrac{8}{3} - 1 = \dfrac{5}{3} > 0$

$\Rightarrow$ $f(x)$ is increasing and being continuous (polynomial) is injective function.

4. (b) $f(x) = 5 + e^{x^2-2}$

$\Rightarrow$ $f'(x) = \left(e^{x^2-2}\right)(2x) = 2x \,.\, e^{x^2-2} \geq 0\ \forall\ x \in [0, \infty)$ and $\leq 0\ \forall\ x \in (-\infty, 0)$

$\Rightarrow$ $f(x)$ is injective on $[0, \infty)$, $(-\infty, 0)$ and non-injective on $[-k, k]\ \forall\ k > 0$

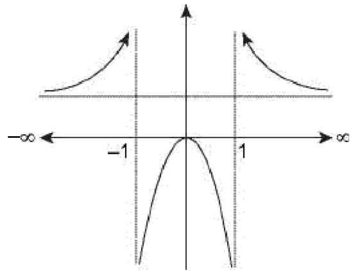
5. (a), (b), (c)

$f(x) = \dfrac{x^2}{x^2-1} = 1 + \dfrac{1}{x^2-1}$

$\Rightarrow$ $f'(x) = \dfrac{-2x}{\left(x^2-1\right)^2} \geq 0$ for $x \leq 0, -1$ and $\leq 0\ \forall\ x \geq 0, 1$

$f(-\infty) = 1^+, f(-1^-) = \infty, f(-1^+) = -\infty, f(-1^-) = -\infty, f(1^+) = \infty, f(\infty) = 1^+$

Graph of $f(x)$ would be as shown below:



$\Rightarrow$ $f(x)$ is injective in $(-\infty, -1)$, non-injective in $(-1, 1)$, non-injective on $\mathbb{R}$

6. (b), (c), (d) $f(x) = \ln(1 - \ln x)$

$D_f = \{x\colon 1 - \ln x > 0;\ x > 0\} = (0, e)$

$f'(x) = \dfrac{1}{1-\ell nx}.\left(\dfrac{-1}{x}\right) = \dfrac{-1}{x(1-\ell nx)}$

$\Rightarrow$ $f'(x) < 0$ as $x > 0$, $1 - \ln x > 0$

$\Rightarrow$ $f(x)$ is a decreasing function with domain $(0, e)$ on $(0, e)$, $\ln x \in (-\infty, 1)$

$\Rightarrow$ $-\ln x \in (-1, \infty)$ $\qquad\Rightarrow\quad 1 - \ln x \in (0, \infty)$

$\Rightarrow$ $\ln(1 - \ln x) \in (-\infty, \infty) = \mathbb{R}$

7. (b), (c) $f(x) = \dfrac{x^3 - 2x^2 - 9x + 18}{x^2-4} = \dfrac{(x-2)\left(x^2-9\right)}{\left(x^2-4\right)}$

$D_f = \mathbb{R} - \{\pm 2\}$

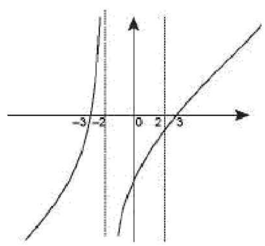
$\Rightarrow$ $f(x) = \dfrac{x^2-9}{x+2}$; $x \in \mathbb{R} - \{\pm 2\}$

$\Rightarrow$ $f'(x) = \dfrac{x^2+4x+9}{(x+2)^2} > 0\ \forall\ x - 2$

$\Rightarrow$ $f(x)$ is an increasing function, but discontinuous.

$f(-\infty) = -\infty, f(-3) = 0, f(-2^-) = \infty, f(-2^+) = -\infty, f(2^-) = \dfrac{-5}{4}, f(2^+) = \dfrac{-5}{4}, f(3) = 0, f(\infty) = \infty$

The graph of $f(x)$ will be as shown below:



Clearly, $f(x)$ is non-injective on its domain, also $f(x)$ is increasing at every point of its domain.

8. (a), (d) $y = [x] + \sqrt{\{x\}}$ for $x \in (k, k+1)$; $k \in \mathbb{Z}$

$y = k + \sqrt{x-k}\ < k + 1$

$\Rightarrow\ f'(x) = \dfrac{1}{2\sqrt{x-k}} > 0\ \forall\ x \in (k, k+1)$

$\Rightarrow\ f(k) = k, f(k+1) = k+1$

Thus, $f(k) = k, f(x) = k + \sqrt{x-k} \in (k, k+1)$

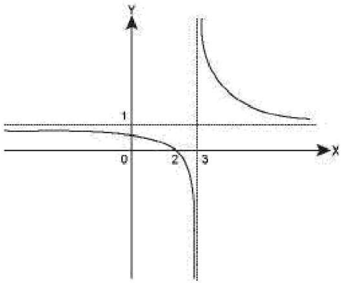
$f(k+1) = k+1$ and $f'(x) > 0\ \forall\ x\ (k, k+1)$

$\Rightarrow\ f(x)$ is continuous and increasing on $\mathbb{R}$.

9. (b), (d) $A = \mathbb{R} - \{3\}, B = \mathbb{R} - \{1\}$ and $f: A \to B; f(x) = \left(\dfrac{x-2}{x-3}\right)$

$\Rightarrow\ f(x)$ is decreasing at every point of its domain, but discontinuous at $x = 3$.

$f(-\infty) = 1^-, f(3^-) = -\infty, f(3^+) = \infty, f(2) = 0, f(\infty) = 1^+$. The graph of $f(x)$ is as shown below.



Thus, $f(x) < 1$ for $x \in (-\infty, 3)$ and $f(x) > 1$ for $x \in (3, \infty)$ and $f(x)$ is decreasing in each of these intervals.

Thus, $f(x)$ is injective function.

TEXTUAL EXERCISE–12: (SUBJECTIVE)

1. (a) $y = 5x + 8$

Injectivity: $f(x_1) = f(x_2)\ \Rightarrow\ 5x_1 + 8 = 5x_2 + 8$

$\Rightarrow\ x_1 = x_2$

Subjectivity: $f(x) = y\ \Rightarrow\ x = \dfrac{y-8}{5} \Rightarrow f\left(\dfrac{y-8}{5}\right) = y$

$\Rightarrow\ f(x)$ is onto

(b) $y = \dfrac{x^2-1}{x^2+1}$

Intectivity: $f(x_1) = f(x_2)$

$\Rightarrow\ \dfrac{x_1^2-1}{x_1^2+1} = \dfrac{x_2^2-1}{x_2^2+1}$

$\Rightarrow\ x_1^2 + x_2^2 + x_1^2 - x_2^2 - 1$

$\Rightarrow\ x_1^2 x_2^2 - x_1^2 + x_2^2 - 1$

$\Rightarrow\ x_1^2 = x_2^2 \nRightarrow x_1 = x_2$

$\Rightarrow\ f(x)$ is many–one.

Subjectivity: Also $x^2 y + y = x^2 - 1$

$\Rightarrow\ x^2(1-y) = 1 + y\qquad \Rightarrow x = \pm\sqrt{\dfrac{1+y}{1-y}}$

$\Rightarrow\ y \in [-1, 1)\qquad \Rightarrow\ f(x)$ is into

(c) $y = \dfrac{ax+b}{ax-b}$

Injectivity: $f(x_1) = f(x_2)\ \Rightarrow\ \dfrac{ax_1+b}{ax_1-b} = \dfrac{ax_2+b}{ax_2-b}$

$\Rightarrow\ a^2 x_1 x_2 - abx_1 + abx_2 - b^2$

$= a^2 x_1 x_2 + abx_1 - abx_2 - b^2$

$\Rightarrow\ 2ab\,x_1 = 2\,ab\,x_2\qquad \Rightarrow\ x_1 = x_2$

Subjectivity: $axy - by = ax + b$

$\Rightarrow\ x(ay - a) = b(1+y)\ \Rightarrow\ x = \dfrac{b(y+1)}{a(y-1)}$

$\Rightarrow\ y\ 1\qquad \Rightarrow\ f(x)$ is into

(d) $f(x) = e^x + e^{-x}$, clearly $f(x)$ is continuous

$f'(x) = e^x - e^{-x} = \dfrac{e^{2x}-1}{e^x} > 0$

$\Rightarrow\ e^{2x} > 1\qquad \Rightarrow\ 2x > 0$

$\Rightarrow\ x > 0$ and $f'(x) \le 0\qquad \Rightarrow\ x \le 0$

$\Rightarrow\ f(x)$ decrease for $x \le 0$ and increase for $x > 0$

$\Rightarrow\ f(x)$ is many–one function.

Also $f(x) = e^x + \dfrac{1}{e^x} \ge 2\ \forall\ x \in \mathbb{R}$

$\Rightarrow\ f(x)$ is into

(e) $y = e^x - e^{-x}$

$\Rightarrow\ f'(x) = e^x + e^{-x} > 0\ \forall\ x \in \mathbb{R}$

$\Rightarrow\ f(x)$ is increasing function.

$\Rightarrow\ f(x)$ is one–one. Clearly, $f(x)$ is continuous function, $f(-\infty) = 0 - \infty$ and $f(\infty) = \infty - 0 = \infty$

$\Rightarrow\ f(x)$ has its range $= (-\infty,)$

$\Rightarrow\ f(x)$ is onto

(f) $y = \log\left(\dfrac{x-1}{x+1}\right)$

$D_f = (-\infty, -1) \cup (1, \infty)$

Let $y = \dfrac{x-1}{x+1}\qquad \Rightarrow\ x = \dfrac{-(z+1)}{(z-1)}$

$\Rightarrow$ As $x \in (-\infty, -1) \cup (1, \infty)$

$\Rightarrow\ \dfrac{(z+1)}{(z-1)} \in (-\infty, -1) \cup (1, \infty)$

$\Rightarrow\ \dfrac{z+1}{z-1} < -1$ or $\dfrac{z+1}{z-1} > 1$

$\Rightarrow\ \dfrac{2z}{(z-1)} < 0$ or $\dfrac{2}{z-1} > 0$

$\Rightarrow\ z \in (0, 1)$ or $z > 1$

$\Rightarrow\ z \in (0, \infty) - \{1\}$

$\therefore\ y = \log(z);\ z \in (0,) - \{1\}$

$\Rightarrow\ y \in (-\infty, \infty) - \{0\}$ and $y = \log(z)$ in increasing on $(0, \infty) - \{1\}$

$\Rightarrow\ f(x)$ is one–one and into.

(g) $y = \log\left(x + \sqrt{x^2+1}\right)$

$f'(x) = \dfrac{1}{\left(x+\sqrt{x^2+1}\right)}\left(\dfrac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right)$

$\Rightarrow\ f'(x) = \dfrac{1}{\sqrt{x^2+1}} > 0\ \forall\ x \in D_f$

$\Rightarrow\ f(x)$ is injective (one–one) and also $\left(x+\sqrt{x^2+1}\right) > 0\ \forall\ x \in \mathbb{R}$.

$\Rightarrow\ y = \log\left(x+\sqrt{x^2+1}\right) \in (-\infty, \infty)$

$\Rightarrow\ f(x)$ is onto.

(h) $f(x) = x + \dfrac{1}{x}$

$f'(x) = 1 - \dfrac{1}{x^2} = \left(\dfrac{x^2-1}{x_2}\right)$

$\therefore\ f'(x) > 0 \quad \Rightarrow \dfrac{x^2-1}{x^2} > 0$

$\Rightarrow x^2 - 1 > 0$

$\Rightarrow x \in (-\infty, -1) \cup (1,)$ and $f'(x) < 0$

$\Rightarrow \dfrac{x^2-1}{x^2} < 0 \quad \Rightarrow x^2 - 1 < 0$

$\Rightarrow x \in (-1, 1) - \{0\}$

Thus, $f(x)$ is many–one, also $f(x) \in (-\infty, -2] \cup [2,)$

$\Rightarrow f(x)$ is into.

2. $A = \{1, 2, 3, 4\}$; $B = \{a, b\}$

Each element of A has two choice a or b.

$\therefore$ Total number of onto functions $= (2)^4 - (2) = 14$

3. $f: X \to Y; f(x) = \sqrt{1-2x} + x$

$A = \{x: 1 - 2x \geq 0\} = \left(-\infty, \dfrac{1}{2}\right]$. Let $y = \sqrt{1+2x} + x$

$\Rightarrow y - x = \sqrt{1-2x} \quad \Rightarrow x^2 - 2x(y-1) + y^2 - 1 = 0$

$\therefore$ For $x \in \mathbb{R}$; Disc. 0 $\quad \Rightarrow (y-1)^2 - 4(y^2-1) \geq 0$

$\Rightarrow 3y^2 + 2y - 5 \leq 0$

$\Rightarrow (3y+5)(y-1) \leq 0 \quad \Rightarrow \dfrac{-3}{5} \leq y \leq 1 \Rightarrow y \in \left[\dfrac{-5}{3}, 1\right]$

$\Rightarrow B = \left[\dfrac{-5}{3}, 1\right]$

4. $f: \mathbb{R} \to A$

(a) $f(x) = \dfrac{e^x - e^{-|x|}}{e^x + e^{-|x|}} = \begin{cases} \dfrac{e^x - e^{-x}}{e^x + e^{-|x|}}; x \geq 0 \\ 0 \quad ; x < 0 \end{cases}; x \geq 0$

Let $y = \dfrac{e^{2x}-1}{e^{2x}+1}$ for $x \geq 0$

$\Rightarrow y' = \dfrac{4e^{2x}}{(e^{2x}+1)} > 0\ \forall x \in \mathbb{R}$

$\Rightarrow f(x)$ is an increasing function, also $f(x)$ is continuous.

$\Rightarrow$ A $= [f(0), f(\infty) \cup \{0\}] = [0, 1) \cup \{0\} = [0, 1)$

(b) $f(x) = \dfrac{e^x - e^{-|x|}}{e^x + e^{|x|}} = \begin{cases} 0 \text{ if } x \leq 0 \\ \dfrac{e^x - e^{-x}}{2e^x}; x \geq 0 \end{cases}$

Let $y = \dfrac{e^x - e^{-x}}{2e^x}; x \geq 0$

$\Rightarrow y' = \dfrac{1}{e^{2x}} > 0\ \forall x \geq 0$

$\Rightarrow f(x)$ has range $=$ A $= \left[0, \dfrac{1}{2}\right)$

(c) $f(x) = \dfrac{e^x - e^{|x|}}{e^x + e^{-|x|}} = \begin{cases} 0 \text{ if } x \geq 0 \\ \dfrac{e^x - e^{-x}}{2e^x} \text{ if } x \leq 0 \end{cases}$

Let $y = \dfrac{e^x - e^{-x}}{2e^x} = \dfrac{1}{2} - \dfrac{1}{2}e^{-2x}$

$\Rightarrow y' = e^{-2x} > 0\ x \leq 0$

$\Rightarrow$ Range $f(x) = $ A $= (f(-\infty), f(0)] = (-\infty, 0]$

5. $f: \mathbb{N} \to \mathbb{N}; f(x) = 3x + 5$

Let $y \in \mathbb{N}$ and $f(x) = y$ (suppose)

$\Rightarrow 3x + 5 = y \quad \Rightarrow x = \dfrac{y-5}{3}$

$\therefore$ For $x \in \mathbb{N}$, $3|(y-5)\ \forall y\ \mathbb{N}$ and $y \geq 6$

$\Rightarrow$ Range of $f(x) = \{8, 11, 14, 17, \ldots\}$

$\Rightarrow f(x)$ is not subjective from $\mathbb{N}$ to $\mathbb{N}$.

6. $f: X \to$ Y; $X = \{x, y, z$ and $Y = \{a, b\}$

(a) Total number of relations from X to $Y = 2^{n(X) \times n(Y)} = 2^6 = 64$

(b) Total number of functions from X to $Y = (2)^3 = 8$

(c) Number of one–one functions from X to $Y = 0$ as $n(Y) < n(X)$

(d) All functions are many–one

$\Rightarrow$ 8 in numbers.

(e) Number of onto functions $= (2)^3 - {}^2C_1\,(1)^3 = 8 - 2 = 6$

7. $f(x) = ax^3 + bx^2 + cx + d \sin x$

$f'(x) = 3ax^2 + 2bx + c + d\cos x$ for $f(x)$ to be injective, $f'(x) \geq 0$

$\Rightarrow 3ax^2 + 2bx + c + d\cos x \geq 0$

$\Rightarrow a > 0,\ 4b^2 - 4(3a)(c + d\cos x) \leq 0\ \forall \cos x$

$\Rightarrow a > 0,\ b^2 - 3a(c + d\cos x) \leq 0$

$\Rightarrow b^2 \leq 3a(c + d\cos x)$

$\Rightarrow b^2 \leq 3a(c - d)$ if $d \geq 0$ or $b^2 \leq 3a(c + d)$ if $d \leq 0$

8. $f: \mathbb{R} \to$ B; $f(x) = \sin x + \cos x - 4$

Range $f(x) = \left[-\sqrt{2} - 4, \sqrt{2} - 4\right]$

9. $f: [-1, 1] \to [0, 3]$ such that $f(x) = ax + b$

$\Rightarrow f'(x) = a$

Case: (i) if $a > 0, f(x) \uparrow \quad \Rightarrow f(-1) = 0, f(1) = 3$

$\Rightarrow -a + b = 0,\ a + b = 3$

$\Rightarrow a = b = 3/2 \quad \Rightarrow f_1(x) = \dfrac{3}{2}x + \dfrac{3}{2}$

Case: (ii) If $a < 0, f(x) \quad \Rightarrow f(-1) = 3, f(1) = 0$

$\Rightarrow -a + b = 3,\ a + b = 0 \Rightarrow b = \dfrac{3}{2} = -a$

$\Rightarrow f_2(x) = \dfrac{-3}{2}x + \dfrac{3}{2}$

$\Rightarrow$ Point of intersection of $f_1(x)$ and $f_2(x)$ is $\left(0, \dfrac{3}{2}\right)$

10. $f: \mathbb{R} \to \left[\dfrac{\pi}{6}, \dfrac{\pi}{2}\right); f(x) = \sin^{-1}\left(\dfrac{x^2-a}{x^2+1}\right)$

For onto function, $\dfrac{\pi}{6} \leq \sin^{-1}\left(\dfrac{x^2-a}{x^2+1}\right) < \dfrac{\pi}{2}$

$\Rightarrow \dfrac{1}{2} \leq \dfrac{x^2-a}{x^2+1} < 1$

$\Rightarrow x^2 + 1 \leq 2x^2 - 2a < 2x^2 + 2$

$\Rightarrow x^2 \geq 2a + 1;\ a > -1$

$\Rightarrow 2a + 1 \leq 0;\ a > -1 \quad \Rightarrow a \in \left(-1, \dfrac{-1}{2}\right]$

Now $f(x) = \dfrac{x^2-a}{x^2+1} = 1 - \dfrac{(a+1)}{x^2+1}$

As $a > -1$ $\Rightarrow$ $a + 1 > 0$

$\Rightarrow$ $f(x) < 1$ and $f(x) \downarrow$ for $x < 0$ and $\uparrow$ for $x > 0$

$\therefore$ $f(0) = \dfrac{1}{2}$ for onto functions

$\Rightarrow$ $-a = \dfrac{1}{2} \Rightarrow a = \dfrac{-1}{2}$

11. f: $\mathbb{R} \to \left[0, \dfrac{\pi}{2}\right)$

$f(x) = \tan^{-1}(x^2 + x + a)$

For onto function, $0 \le \tan^{-1}(x^2 + x + \text{a}) < \dfrac{\pi}{2}$

$\Rightarrow$ $0 \le (x^2 + x + a) < \infty$

$\Rightarrow$ $x^2 + x + a \ge 0$

$\Rightarrow$ $1 - 4a \le 0$ $\Rightarrow$ $a \ge \dfrac{1}{4}$

Let $f(x) = x^2 + x + a = \left(x + \dfrac{1}{2}\right)^2 + \left(a - \dfrac{1}{4}\right)$ and $f(x) \in [0, \infty)$

$\Rightarrow$ $f(x) = 0$, at $x = \dfrac{-1}{2}, a = \dfrac{1}{4}$

$\therefore$ For onto function $a = \dfrac{1}{4}$

12. $f(x)$: $D_f \to Y$

(a) $f(x) = \sin^6 x + \cos^6 x$

$= (\sin^2 x + \cos^2 x)^3 - 3 \sin^2 x \cos^2 x$

$= 1 - \dfrac{3}{4} \sin^2 2x \in \left[\dfrac{1}{4}, 1\right] = Y$

(b) $f(x) = \ln(|\sin x|^4 + |\cos x|^4)$

$(\sin x)^4 + (\cos x)^4 + 1 - 2 \sin^2 x \cos^2 x$

$= 1 - \dfrac{1}{2}\sin^2 2x \in \left[\dfrac{1}{2}, 1\right]$

$\Rightarrow$ $f(x) \in \left[\ln\dfrac{1}{2}, \ln 1\right] = [-\ln 2, 0] = Y$

(c) $f(x) = \sqrt{\ln|\text{cosec}\, x|}$ $\because$ $|\text{cosec}\, x| \in [1, \infty)$

$\Rightarrow$ $\ln|\text{cosec}\, x| \in [0, \infty)$

$\Rightarrow$ $f(x) \in [0, \infty) = Y$

(d) $f(x) = \log_{\sqrt{2}}\left(\dfrac{\cos x + \sin x + 2\sqrt{2}}{\sqrt{2}}\right)$

$\because$ $(\cos x + \sin x + 2\sqrt{2}) \in [\sqrt{2}, 3\sqrt{2}]$

$\Rightarrow$ $\left(\dfrac{\cos x + \sin x + 2\sqrt{2}}{\sqrt{2}}\right) \in [1, 3]$

$\Rightarrow$ $f(x) \in [\log_{\sqrt{2}} 1, \log_{\sqrt{2}} 3] = [0, 2\log_2 3] = Y$

13. $A = \{x: -k \le x \le k, k \in \mathbb{N}\}$ and $B_1 = \left\{\pm\dfrac{m^2}{4}; 0 \le m \le k; n \in \mathbb{Z}\right\}$

$B_2 = A$, $B_3 = \{\pm m^4;\ 0 \le m \le k;\ m \in \mathbb{R}\}$

$B_4 = \{m^2;\ m \in \mathbb{Z};\ 0 \le m \le k\}$; $B_5 = A$

(a) $f(x) = \dfrac{x|x|}{4}$; f: $AB = \begin{cases} \dfrac{-x^2}{4}; x \in [-k, 0); k \in \mathbb{N} \\ \dfrac{x^2}{4}; x \in [0, k]; k \in \mathbb{N} \end{cases}$

$\Rightarrow$ $f'(x) = \begin{cases} \dfrac{-x}{2}; x \in (-k, 0); k \in \mathbb{N} \\ \dfrac{x}{2}; x \in (0, k); k \in \mathbb{N} \end{cases}$

$\Rightarrow$ $f'(x) \ge 0\ \forall\ x \in (-k, 0);\ k \in \mathbb{N}$

$\Rightarrow$ $f(x) \uparrow \forall\ x \in [-k, k]$

$\Rightarrow$ $f(x)$ is injective

Let $y \in B_1$ $\Rightarrow$ $y = \dfrac{m^2}{4}$ or $\dfrac{-m^2}{4}$; $m \in \mathbb{Z}$

$\Rightarrow$ $y = f(m) = \dfrac{m^2}{4}$; $m \in \mathbb{N}$ and $m \in [0, k]$ and

$y = f(-m) = \dfrac{-m^2}{4}$, $-m \in [-k, 0)$

$\Rightarrow$ $f(x)$ is surjective $\Rightarrow$ $f(x)$ is bijective.

(b) $f(x) = \sqrt{x^2}$; $g: A \to B_2$

$\Rightarrow$ $f(x) = |x|$; g: $[-k, k] \to [-k, k]$; $k \in \mathbb{N}$

Clearly, $f(x)$ is many–one as $f(t) = |t|$

$\Rightarrow$ $f(x)$ is not injective.

Thus, $f(x)$ is neither injective nor subjective

(c) $h(x) = x^3 |x|$; $h\ [-k, k] \to \{\pm m^4, 0 \le m \le k; m \le \in \mathbb{R}\}$

$= \begin{cases} x^4; x \ge 0, \text{ i.e., } x \in [0, k] \\ -x^4; x < 0, \text{ i.e., } x \in [-k, 0] \end{cases}$

$\Rightarrow$ $h'(x) = \begin{cases} 4x^3; x \in (0, k) \\ -4x^3; x \in (-k, 0) \end{cases}$

$\Rightarrow$ $h'(x) \ge 0\ \forall\ x \in [-k, k]$

$\Rightarrow$ $h(x)$ is injective

Also, $m^4 = \text{h}(m)$; $m \in [0, k]$ and $-m^4 = h(-m)$;

$-m \in [-k, 0]$

$\Rightarrow$ $h(x)$ is bijetive.

(d) $k(x) = x^2$: $[-k, k]$; $k \in \mathbb{N} \to \{m^2;\ m \in 0 \le m \le k\}$

Clearly, $k(x)$ as many–one as $f(-t) = f(t)\ \forall\ \text{t} \in [-k, k]$

Also range of $k(x) = [0, k^2] \ne \{m^2;\ m \in \mathbb{Z};\ 0 \le m \le k\}$

$\Rightarrow$ $k(x)$ is non-surjective, non-bijective

(e) $\phi(x) = x\cos(2n + 1)\pi x$; $:[-k, k] \to [-k, k]$; $\pi x\ [-k\pi, k]$

$\Rightarrow$ $(2n + 1)\,\pi x\ [-(2n + 1)k\pi, (2n + 1)\,k\pi]$

14. f: $\mathbb{R} \to \mathbb{R}$; $f(x) = x^3 + x^2 + 3x + \sin x$

$f'(x) = 3x^2 + 2x + 3 + (\cos x)$

Now Discrioninant of $f'(x) = (2)^2 - 4(3)(3 + \cos x)$

$= 4[1 - 9 - 3\cos x] = 4[-8 - 3\cos x]$

$\because$ $-3 \le -3\cos x \le 3$

$\Rightarrow$ $4(-8 - 3\cos x)\ [-44, -20]$

$\Rightarrow$ $D < 0$ $\Rightarrow$ $f'(x) > 0\ \forall\ x\ \mathbb{R}$

$\Rightarrow$ $f(x)$ is an increasing function

$\Rightarrow$ $f(x)$ is injective.

Also $(x^3 + x^2 + 3x)\ (-\infty, \infty)$ and $\sin x\ [-1, 1]$

$\Rightarrow$ $f(x)\ (-\infty, \infty)\ f(x)$ is surjective .

15. f: $\mathbb{Z} \to \mathbb{Z}$

(a) $f(x) = x + 3$, which being an increasing function is injective. Also $f(y - 3) = y\ \forall\ \mathbb{Z}$

$\Rightarrow$ $f(x)$ is surjective. Hence, a bijection.

(b) $f(x) = x^5$

$\Rightarrow$ $f'(x) = 5x^4 \ge 0$ $\Rightarrow$ $f(x)$ is increasing function

$\Rightarrow$ Injective.

Also $f(x) = y$ $\Rightarrow$ $y = x^5$

$\Rightarrow$ $x = (y)^{1/5} \notin \mathbb{Z}$ for every y $\mathbb{Z}$, e.g., if $y = 6$, then $f((6)^{1/5}) = 6$ but $(6)^{1/5} \notin \mathbb{Z}$

$\Rightarrow$ $f(x)$ is not surjective.

(c) $f(x) = 3x + 2$

$f'(x) = 3 > 0$ $\Rightarrow$ $f(x)$ is injective .

If y $\mathbb{Z}$, then $y = f(x)$

$\Rightarrow$ $y = 3x + 2$

$\Rightarrow$ $x = \dfrac{y-2}{3}$, which need not necessary an integer

$\Rightarrow$ $f(x)$ is not surjective, and hence, not bijection.

(d) $f(x) = x^2 + x$

$\Rightarrow$ $f'(x) = 2x + 1 > 0$ for $x > -\dfrac{1}{2}$ and < 0 for $x < -\dfrac{1}{2}$

$\Rightarrow$ $f(x)$ is many–one, i.e., not injective and hence, not bijection.

16. f: $\mathbb{R} \to \mathbb{R}$

$\Rightarrow$ $f(x) = \begin{cases} 2x; \text{if } x \geq 0 \\ 2x-3; \text{if } x < 0 \end{cases}$

$\Rightarrow$ $f'(x) = \{2\ \forall\ x\ 0$ and $f(0) = -3$ and $f(0^+) = 0$

$\Rightarrow$ $f(x)$ is injective. Also range $f(x) = (-\infty, -3) \cup [0, \infty)$ $\mathbb{R}$

$\Rightarrow$ $f(x)$ is not onto.

17. f: $\mathbb{R} - \{2\} \to \mathbb{R} - \{1\}$

$f(x) =$ ——

$D_f = \mathbb{R} - \{2\}$

Let $f(x_1) = f(x_2)$ $\Rightarrow$ $\dfrac{x_1-5}{x_1-2} = \dfrac{x_2-5}{x_2-2}$

$\Rightarrow$ $x_1x_2 - 2x_1 - 5x_2 + 10 = x_1x_2 - 5x_1 - 2x_2 + 10$

$\Rightarrow$ $3x_1 = 3x_2$ $\Rightarrow$ $x_1 = x_2$

$\Rightarrow$ $f(x)$ is injective.

Let $y = \dfrac{x-5}{x-2}$ $\Rightarrow$ $xy - 2y = x - 5$

$\Rightarrow$ $x(y - 1) = 2y - 5$ $\Rightarrow$ $x = \dfrac{2y-5}{y-1}$

$\Rightarrow$ $y \neq 1$

$\Rightarrow$ $R_f = \mathbb{R} - \{1\}$ $\Rightarrow$ $f(x)$ is surjective

$\therefore$ $f(x)$ is bijective

18. A: $\mathbb{R} - \{3\} \to \mathbb{R} - \{1\}$

$f(x) = \dfrac{x-2}{x-3}$ $\Rightarrow$ $f'(x) = \dfrac{-1}{(x-3)^2} < 0\ \forall\ x\ 3$

$\Rightarrow$ $f(x)$ is injective function. Also $y = \dfrac{x-2}{x-3}$

$\Rightarrow$ $xy - 3y = x - 2$

$\Rightarrow$ $x(y - 1) = 3y - 2$ $\Rightarrow$ $x = \dfrac{3y-2}{y-1}$

$\Rightarrow$ $y \neq 1$

$\Rightarrow$ $R_f = \mathbb{R} - \{1\}$ $\Rightarrow$ Surjective

Thus, $f(x)$ is bijective function.

19. f: $[1, 7] \to [2, 27]$;

$f(x) = x^2 - 4x + 6$

$f'(x) = 2x - 4 < 0$ for x $[1, 2)$ and $f'(x) \geq 0$ for x $[2, 7]$

$\Rightarrow$ $f(x)$ is not injective.

Range of $f(x) = \left[\dfrac{-(16-24)}{4}, \infty\right) = [2, \infty)$

$\therefore$ $f(x)$: $[1, 7] \to [2, 27]$ is not surjective.

20. **(a)** $f(x) = 2 \cot x$; f: $(\pi, 2\pi) \to \mathbb{R}$

$f'(x) = -2\,\text{cosec}^2\, x < 0\ \forall\ x\ (\pi, 2)$

$\Rightarrow$ $f(x)$ is injective function. Also range $f(x) = (2 \cot (2\pi^-)$, $2 \cot (\pi^+) = (-\infty, \infty) = \mathbb{R}$

$\Rightarrow$ $f(x)$ is surjective (onto).

(b) $f(x) = \dfrac{4}{1+x^2}$; f: $\mathbb{R} \to \mathbb{R}$

$\Rightarrow$ $f'(x) = \dfrac{-8}{1+x^2} \leq 0$ for $x \geq 0$ and > 0 for $x < 0$

$\Rightarrow$ $f(x) \downarrow$ for $x \geq 0$ and $\uparrow$ for $x < 0$ and continuous.

$\Rightarrow$ $f(x)$ is not surjective and range $= (f(-\infty), f(0))$ as $f(x)$ is even

$\Rightarrow$ $R_f \equiv (0, 4)$ $\mathbb{R}$ $\Rightarrow$ $f(x)$ is not surjective.

(c) $f(x) = x^4 + \ln x$; f: $(0,) \to \mathbb{R}$

$\Rightarrow$ $f'(x) = 4x^3 + \dfrac{1}{x} < 0$ for $x < 0$ and > 0 for $x > 0$.

$\Rightarrow$ $f(x) \uparrow$ and continuous for x $(0, \infty)$

$\Rightarrow$ $f(x)$ is injectivite

$\Rightarrow$ Range of $f(x) = (f(0), f(\infty)) \equiv (-\infty, \infty) = \mathbb{R}$

$\Rightarrow$ $f(x)$ is surjective.

21. **(a)** $f(x) = \dfrac{x^2+4x+30}{x^2-8x+18}$

$\Rightarrow$ $f'(x) = \dfrac{-12(x^2+2x-26)}{(x^2-8x+18)^2}$

$\because$ Disc. of $x^2 + 2x - 26 = 108 > 0$

$\Rightarrow$ $f'(x) < 0$ for x $(-\infty, \alpha) \cup (\beta, \infty)$ and > 0 for x (α, β) where α, β are roots of $x^2 + 2x - 26 = 0$

$\Rightarrow$ $f(x)$ is not injective.

Let $y = \dfrac{x^2+4x+30}{x^2-8x+18}$

$\Rightarrow$ $x^2(1 - y) + (4 + 8y)x + (30 - 18y) = 0$

$\Rightarrow$ If $y = 1$, then $x = -1$ and if y 1, then for x $\mathbb{R}$, Disc. ≥ 0

$\Rightarrow$ $y^2 - 32y + 13 \leq 0$

$\Rightarrow$ $R_f = \left[16-\sqrt{243}, 16+\sqrt{243}\right]$

$\Rightarrow$ $f(x)$ is not surjective.

(b) $f(x) = x^3 - 6x^2 + 11x - 6$

$\Rightarrow$ $f'(x) = 3x^2 - 12x + 11$ having disc. $= 12$

$\Rightarrow$ $f'(x)$ would change its sign.

$\Rightarrow$ $f(x)$ is not injective, however, being a cubic polynomial its range $= (-\infty, \infty)$ $\Rightarrow$ $f(x)$ is surjective.

(c) $f(x) = (x^2 + x + 5)(x^2 + x + 3) = x^4 + 2x^3 + 9x^2 + 8x + 15$

$\Rightarrow$ $f'(x) = 4x^3 + 6x^2 + 18x + 8$

$\Rightarrow$ $f'(x)$ $(-\infty, \infty)$ $\Rightarrow$ $f'(x)$ changes its sign

$\Rightarrow$ $f(x)$ is not injective.

Also $x^2 + x + 5 > 0$ as disc. $= -19 < 0$ and $x^2 + x + 3 > 0$ as disc. $= -11 < 0$

$\Rightarrow$ $f(x) > 0$ x $\mathbb{R}$ $\Rightarrow$ $f(x)$ is not surjective.

22. $f(x) = k_x + 3 \sin x + 5$ $\Rightarrow$ $f'(x) = k + 3 \cos x$

$\Rightarrow$ $k - 3 \leq f'(x) \leq k + 3$

$\Rightarrow$ $f'(x) \leq 0$ for $k \leq -3$ and $f'(x) \geq 0$ for $k \geq +3$

Also $f(x)$ being a linear function for k 0 would have its range $(-\infty, \infty)$.

$\therefore$ For one–one and onto function k $(-\infty, -3] \cup [3, \infty)$

23. $f(x) = \dfrac{x-1}{m-x^2+1}; x\ \mathbb{R}$

By given condition, $f(x) \le -1$ or $f(x) \ge \dfrac{-1}{3}$ or

$m \in \left(\mathbb{R} - \left\{m : f(x) \in \left(-1, \dfrac{-1}{3}\right)\right\}\right)$

$\Rightarrow \dfrac{(x-1)}{m-x^2+1} > -1$ and $\dfrac{(x-1)}{m-x^2+1} < -\dfrac{1}{3}\ \forall\ x\ \mathbb{R}$

$\Rightarrow \dfrac{(x-1)+m-x^2+1}{m-x^2+1} > 0$ and $\dfrac{3x-3+m-x^2+1}{m-x^2+1} < 0$

$\Rightarrow \dfrac{-x^2+x+m}{m-x^2+1} > 0$ and $\dfrac{-x^2+3x-2+m}{m-x^2+1} < 0$

$\Rightarrow (x^2-x-m)(x^2-m-1) > 0$ and $(x^2-3x+2-m)$ $(x^2-m-1) < 0\ \forall\ x\ \mathbb{R}$; which is impossible

So, either $f(x) > -1\ \forall\ x\ \mathbb{R}$ or $f(x) < -\dfrac{1}{3}\ \forall\ x\ \mathbb{R}$

24. $f(x) = \dfrac{x^2-8x+18}{x^2+4x+30}$

$\Rightarrow f'(x) = \dfrac{12(x^2+2x-26)}{(x^2+4x+30)^2}$

$\Rightarrow$ Disc. of $x^2+2x-26$ is $108 > 0$

$\Rightarrow f'(x)$ changes its sign and $x^2+4x+30 > 0\ \forall\ x\ \mathbb{R}$

Thus, $f(x)$ is non-monotomic and continuous.

$\Rightarrow f(x)$ is many–one, function, i.e., non-injective.

25. $f(x) = \dfrac{x}{1+|x|}$; $x\ \mathbb{R}$

$\Rightarrow f(x) = \begin{cases} \dfrac{x}{1+x}; x \ge 0 \\ \dfrac{x}{1-x}; x < 0 \end{cases} \Rightarrow f'(x) = \begin{cases} \dfrac{1}{(1+x)^2}; x > 0 \\ \dfrac{1}{(1-x)^2}; x < 0 \end{cases}$

$\Rightarrow f'(x) > 0\ \forall\ x\ \mathbb{R} - \{1, 1\}$

$\Rightarrow f(x)$ is continuous and increasing function

$\Rightarrow f(x)$ is injective

Also $f(-\infty) = -1$ and $f(\infty) = 1$

$\Rightarrow$ Range of $f(x) = (-1, 1)$

$\Rightarrow f(x)$ is injective, not surjective.

26. $f(-1, 1) \to \mathbb{R}$

$\Rightarrow f(x) = \begin{cases} \dfrac{x}{1+x}, -1 < x \le 0 \\ \dfrac{x}{1-x}; 0 < x < 1 \end{cases}$

$\Rightarrow f'(x) = \begin{cases} \dfrac{1}{(1+x)^2}, -1 < x \le 0 \\ \dfrac{1}{(1-x)^2}; 0 < x < 1 \end{cases}$

$\Rightarrow f'(x) > 0\ \forall\ x\ \mathbb{R}$ and $f(x)$ is continuous on $(-1, 1)$

$\Rightarrow f(-1^+) = -\infty, f(1^-) = \infty$

$\Rightarrow f(x)$ is injective and surjective, i.e., bijective.

27. f: $[2, \infty) \to X$; $f(x) = 5 - 4x + x^2$

Principal domain of $f(x) = [2, \infty)$

$\therefore$ For bijective function, $X = [f(2), f(\infty)) \equiv [1, \infty)$

28. f: $\mathbb{N} \to \mathbb{N}$; $f(x)$ = highest prime factor of n

$\because$ $f(3) = 3 = f(6)$

$\Rightarrow f(x)$ is many–one, i.e., not injective.

Also $\exists$ no n $\mathbb{N} - \{1\}$ for which $f(n) = 1$

$\Rightarrow f(x)$ is not onto.

29. $f(x) = 1/(1-x)$

$f_2(x) = f\{f(x)\}; f_3(x) = f\{f\{f(x)\}\ldots\ldots,$

$f_{3n}(x) = ?$

$\Rightarrow f_2(x) = f\left(\dfrac{1}{1-x}\right) = \dfrac{1}{1-\left(\dfrac{1}{1-x}\right)} = \dfrac{1-x}{-x}$

$\Rightarrow f_3(x) = \dfrac{1}{1-\left(\dfrac{x-1}{x}\right)} = \dfrac{x}{1} = x$

$\Rightarrow f_4(x) = f(f_3(x) = f(x)$

$\Rightarrow f_3(x) = f_6(x) = f_9(x) = \ldots = f_{3n}(x) = x$

30. $f(x) = \dfrac{ax+b}{cx+d}, x \ne \dfrac{-d}{c}$ for $d = -a$, $f(x) = \dfrac{ax+b}{cx-a}$

$f(f(x)) = \dfrac{a\left(\dfrac{ax+b}{cx-a}\right)+b}{c\left(\dfrac{ax+b}{cx-a}\right)-a} = \dfrac{a^2x+bcx}{cb+a^2} = x$

$\Rightarrow$ fof is an identity function.

31. f: $\mathbb{R} \to [-1, 1]$; $f(x) = \dfrac{\sin x}{x^2+1}$, clearly $f(x)$ is continuous

Let $f(x) = 1$ $\quad\Rightarrow \dfrac{\sin x}{x^2+1} = 1$

$\Rightarrow \sin x = x^2 + 1$ but $\sin x \le 1$ and $x^2 + 1 \ge 1$

$\Rightarrow \sin x = x^2 + 1 = 1$

$\Rightarrow \sin x = 1$, $x^2 = 0$, i.e., $x = 0$, which is impossible.

$\therefore$ $f(x)$ is into.

32. f: $\mathbb{R} - \{0\} \to \mathbb{R} - \{0\}$; $f(x) = \dfrac{1}{x}$

$\Rightarrow f'(x) = \dfrac{-1}{x^2} < 0\ \forall\ x\ \mathbb{R} - \{0\}$ and $f(x) < 0$ for $x < 0$, $f(x) >$ 0 for $x > 0$

$\Rightarrow f(x)$ is an injective function.

Also for $y\ \mathbb{R} - \{0\}$, $f\left(\dfrac{1}{y}\right) = y$ and $\dfrac{1}{y} \in \mathbb{R} - \{0\}$

$\Rightarrow f(x)$ is onto.

If domain $\mathbb{R} - \{0\}$ is replaced by $\mathbb{N}$ keeping the co-domain same, i.e., f: $\mathbb{N} \to \mathbb{R} - \{0\}$, $f(x) = -$

$f(x) = \dfrac{1}{x}$ being a decreasing function and continuous for $x > 0$ is injective.

However, 2 $\mathbb{R} - \{0\}$ and $f\left(\dfrac{1}{2}\right) = 2$ but $\dfrac{1}{2} \notin \mathbb{N}$

$\Rightarrow f(x)$ is not onto.

33. (i) f: $\mathbb{N} \to \mathbb{N}$; $f(x) = x^2$

$f'(x) = 2x > 0\ \forall\ x > 0$

$\Rightarrow f(x)$ is injective on $\mathbb{N}$

Clearly, $f(x)$ is not surjective, e.g., $f\left(\sqrt{2}\right) = 2$ but $\sqrt{2} \notin \mathbb{N}$

(ii) $f: \mathbb{Z} \to \mathbb{Z}$; for $f(x) = x^2$

$\because$ $f(-k) = f(k) = k^2 \ \forall\ k\ \mathbb{Z}$

$\Rightarrow$ $f(x)$ is not injective. Also $f(x)$ is not surjective, e.g., $f\left(\sqrt{3}\right) = 3\ \mathbb{Z}$ but $\sqrt{3} \notin \mathbb{Z}$

(iii) $f: \mathbb{R} \to \mathbb{R}$; $f(x) = x^2$

Clearly $f(-k) = f(k) = k^2 \ \forall\ k\ \mathbb{R}$

$\Rightarrow$ $f(x)$ is not injective and $f(x) \geq 0\ \forall\ x\ \mathbb{R}$

$\Rightarrow$ $f(x)$ is not surjective.

(iv) $f: \mathbb{N} \to \mathbb{N}$; $f(x) = x^3$

$f'(x) = 3x^2 > 0\ \forall\ x > 0$

$\Rightarrow$ $f(x)$ is increasing, and hence, injective function on $\mathbb{N}$. However, $f(x)$ is not surjective e.g., $f\left(3\sqrt{5}\right) = 5\ \mathbb{N}$ but $3\sqrt{5} \notin \mathbb{N}$.

(v) $f: \mathbb{Z} \to \mathbb{Z}$; $f(x) = x^3$

Let $f(x_1) = f(x_2)$

$\Rightarrow$ $x_1^3 - x_2^3 = 0$ $\Rightarrow$ $(x_1 - x_2)(x_1^2 + x_1 x_2 + x_2^2) = 0$

$\Rightarrow$ $x_1 = x_2$ or $x_1^2 + x_1 x_2 + x_2^2 = 0$

But $x_1^2 + x_1 x_2 + x_2^2 = 0$ gives imaginary roots x_1 for each real x_2.

$\Rightarrow$ $x_1 = x_2$ $\Rightarrow$ $f(x)$ is injective .

Also clearly $f(x)$ is not subjective as $f\left(3\sqrt{5}\right) = 5\ \mathbb{Z}$ but $3\sqrt{5} \notin \mathbb{Z}$.

TEXTUAL EXERCISE–12: (OBJECTIVE)

1. (d) $f: (-\infty, -1) \to (0, e^5)$; $f(x) = e^{x^3-3x+2}$

$\Rightarrow$ $f'(x) = (3x^2 - 3)\,.\,e^{x^3-3x+2} = 3(x^2 - 1).\ e^{x^3-3x+2}$

$\Rightarrow$ $f'(x) > 0$ for $x\ (-\infty, -1) \cup (1, \infty)$ and $< 0\ \forall\ x\ (-1, 1)$

$\Rightarrow$ $f(x) \uparrow$ for $x\ (-\infty, -1)$ $\Rightarrow$ $f(x)$ is injective.

$\Rightarrow$ Range of $f(x) \equiv (f(-\infty), f(-1)) \equiv (0, e^4) \subset (0, e^5)$

$\Rightarrow$ $f(x)$ is not subjective.

2. (a) (i) $f(x) = e^x - e^{-x}$

$\Rightarrow$ $f'(x) = e^x + e^{-x} > 0\ \forall\ x\ \mathbb{R}$, also being continuous and $f(-\infty) = -\infty, f(\infty) = \infty$

$\therefore$ (i) $\to$ (p)

(ii) $f(x) = x^3 - 1$

$\Rightarrow$ $f'(x) = 3x^2 > 0\ \forall\ x\ \mathbb{R}$ and $f(x)$ is continuous

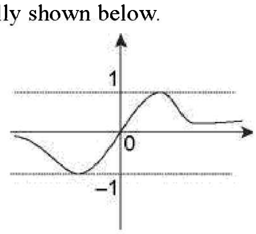
$\Rightarrow$ one–one

Also Range = $(-\infty,)$ $\Rightarrow$ onto

$\therefore$ (ii) $\to$ (p)

(iii) $f(x) = \dfrac{2x}{1+x^2} \Rightarrow f'(x) = \dfrac{2(1-x^2)}{\left(1+x^2\right)^2}$

$\Rightarrow$ $f'(x) < 0$ for $x\ (-\infty, -1) \cup (1, \infty)$ and $f'(x) > 0$ for $x\ (-1, 1)$, graphically shown below.



$\Rightarrow$ $f(x)$ is many–one and having range $[-1, 1]$ is into

$\therefore$ (iii) $\to$ (s)

(iv) $f(x) = a^{x^3}$ $\because$ $f'(x) = \left(a^{x^3} \ell n a\right)\left(3x^2\right)$

$\Rightarrow$ $f'(x)$: $\begin{cases} > 0 \forall x \in \mathbb{R}, a > 1 \\ < 0 \forall x \in \mathbb{R}, a \in (0,1) \end{cases}$

$\Rightarrow$ $f(x) \uparrow$ for $a > 1$ and $\downarrow$ for $a\ (0, 1)$

$\Rightarrow$ $f(x)$ is injective function as continuous. Also $f(x) > 0\ \forall\ x\ \mathbb{R}$ and $\forall$ a > 0

$\Rightarrow$ $R_f\ \mathbb{R}$ $\Rightarrow$ $f(x)$ is into

$\therefore$ (iv) $\to$ (q)

(v) $f(x) = x^3\,|x| = \begin{cases} x^4; x \geq 0 \\ -x^4; x < 0 \end{cases}$

$\Rightarrow$ $f'(x) = \begin{cases} 4x^3; x > 0 \\ -4x^3; x < 0 \end{cases}$

$\Rightarrow$ $f(x) \uparrow\ \forall\ x \in \mathbb{R}$ and is continuous. Also $f(-\infty) = -\infty$ and $f(\infty) = \infty$

$\therefore$ (v) $\to$ (p)

(vi) $f(x) = \dfrac{x^2 - x + 1}{x^2 + x + 1}$

$\Rightarrow$ $f'(x) = \dfrac{2\left(x^2-1\right)}{\left(x^2+x+1\right)} < 0$ for $x\ (-1, 1) > 0$ for $x < -1$ or $x > 1$

Also $x^2 + x + 1\ 0\ \forall\ x\ \mathbb{R}$

$\Rightarrow$ $f(x)$ is continuous function.

$\Rightarrow$ $f(x)$ is many–one function and $x^2 - x + 1 > 0$, $x^2 + x + 1 > 0\ \forall\ x\ \mathbb{R}$

$\Rightarrow$ $f(x) > 0\ \forall\ x\ \mathbb{R}$ $\Rightarrow$ $f(x)$ is into

$\therefore$ (vi) $\to$ (s)

(vii) $f(x) = \sin^3 x$

$\Rightarrow$ $f'(x) = 3\sin^2 x \cos x \geq 0\ \forall\ x\ \left[2n\pi - \dfrac{\pi}{2}, 2n\pi + \dfrac{\pi}{2}\right]$

$\Rightarrow$ $f(x)$ is not injective.

Also $f(x)\ [-1, 1]$ $\Rightarrow$ $f(x)$ is into

$\therefore$ (viii)$\to$ (s)

3. (i) (c) $f(x) = \sqrt{\ell n \dfrac{1}{|\sec x|}}$

For domain, $\ell n \dfrac{1}{|\sec x|} \geq 0$

$\Rightarrow$ $\dfrac{1}{|\sec x|} \geq 1$ $\Rightarrow$ $|\sec x| \leq 1$

$\Rightarrow$ $|\sec x| = 1$ $\Rightarrow$ $R_f = \{0\}$

(ii) (a) $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{e^{2x} - 1}{e^{2x} + 1}$

$\Rightarrow$ -1 for $x \to -\infty$ and $\to 1$ for $x \to \infty$

Also $f'(x) = \dfrac{4}{\left(e^x + e^{-x}\right)^2} > 0\ \forall\ x \in \mathbb{R}$

$\Rightarrow$ $f(x)$ is increasing, and hence, injective with range $(-1, 1)$

(iii) (c) $f(x) = \dfrac{x^2}{1+x^4} = \dfrac{1}{\left(x^2 + \dfrac{1}{x_2}\right)}$; $x\ 0$

$\because\ x^2 + \dfrac{1}{x^2}\ [2, \infty)$ for $x\ 0$

$\Rightarrow f(x) \left(0, \dfrac{1}{2}\right]$ for $x\ 0$

Also $f(0) = 0 \quad \Rightarrow R_f = \left[0, \dfrac{1}{2}\right]$

(iv) (a) $f(x) = \sin^4 x + \cos^2 x = \sin^4 x + 1 - \sin^2 x = \sin^4 x - \sin^2$

$x + \dfrac{1}{4} + \dfrac{3}{4} = \left(\sin^2 x - \dfrac{1}{2}\right)^2 + \dfrac{3}{4}$

$\because\ 0 \le \left(\sin^2 x - \dfrac{1}{2}\right)^2 \le \dfrac{1}{4}$

$\Rightarrow f(x) \in \left[\dfrac{3}{4}, 1\right]$

4. **(d)** $f: \mathbb{R} \to \mathbb{R}; f(x) = x^3 - 6x^2 + 11x - 6$
$\Rightarrow f'(x) = 3x^2 - 12x + 11$, Disc. $= 12 > 0$
$\Rightarrow f'(x)$ would change its sign and $f(x)$ is continuous
$\Rightarrow f(x)$ is many–one.
Also being a cubic polynomial range $= (-\infty, \infty)$.
$\Rightarrow f(x)$ is many–one onto.

5. **(a)** $f(x) = {}^{7-x}P_{x-3}$
$\Rightarrow 7 - x \ge x - 3 \quad \Rightarrow x \le 5$ and $x - 3 \ge 0$
$\Rightarrow x\ [3, 5]$ but $x\ \mathbb{Z} \quad \Rightarrow x\ \{3, 4, 5\}$
$\Rightarrow R_f = \{{}^4P_0, {}^3P_1, {}^2P_2\} = \{1, 3, 2\}$

6. **(b), (c)** $f(x) = |x + 1|; x\ [-1, 0)$
$\Rightarrow f(x) = x + 1$ and $f'(x) = 1 > 0$
$\Rightarrow f(x)$ is injective.
$\Rightarrow f(x) = x + \dfrac{1}{x}; x\ (0, \infty)$
$\Rightarrow g'(x) = 1 - \dfrac{1}{x^2}$
$\Rightarrow g'(x) < 0$ for $x\ (0, 1)$ and > 0 for $x\ (1, \infty)$
$\Rightarrow$ non-injective.
$h(x) = x^2 + 4x - 5$, being quadratic polynomial is many–one, i.e., non-injective.
$k(x) = e^{-x}; x\ [0, \infty)$
$\Rightarrow k'(x) = -e^{-x} < 0$ and continuous
$\Rightarrow k(x)$ is injective.

7. **(a), (c)** $f: A \to [-1, 1]; f(x) = \sin 3x$
For one–one and onto function, $3x \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right] \Rightarrow x \in \left[\dfrac{-\pi}{6}, \dfrac{\pi}{6}\right]$

Also for one–one and onto function, $3x \left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right] \Rightarrow x \in \left[\dfrac{\pi}{6}, \dfrac{\pi}{2}\right]$

8. **(a), (b)** $f: \mathbb{N} \to \mathbb{Z}; f(n) = \begin{cases} \dfrac{n-1}{2}; n = \text{odd} \\ -\dfrac{n}{2}; n = \text{even} \end{cases}$

Let $n_1, n_2\ \mathbb{N}$
Case (i): n_1, n_2 odd or n_1, n_2 even, $f(n_1) = f(n_2)$
$\Rightarrow n_1 = n_2$
Case (ii): n_1 = odd, n_2 = even, then $f(n_1) = f(n_2)$
$\Rightarrow \dfrac{n_1 - 1}{2} = \dfrac{-n_2}{2}$
$\Rightarrow n_1 + n_2 = 1$, which is impossible.
$\Rightarrow f(x)$ is injective. Also if $n \in \mathbb{N}$, then

n = odd + ve	$\Rightarrow (2n + 1) = n$,
n = odd – ve	$\Rightarrow f(-2n) = n$,
n = even + ve	$\Rightarrow f(2n + 1) = n$,
n = even – ve	$\Rightarrow f(-2n) = n$ and $f(1) = 0$

$\Rightarrow f(x)$ is surjective.

9. **(d)** $f: \mathbb{R} \to [-1, 1); f(x) = \dfrac{x^2 - 4}{x^2 + 4}$, clearly $f(-x) = f(x)$
$\Rightarrow f(x)$ is not injective. Also $f(x) = 1 - \dfrac{8}{x^2 + 4} \in [-1, 1)$
$\Rightarrow f(x)$ is surjective.

10. **(c)** $f: (-1, 1) \to Y, f(x) = \tan^{-1}\left(\dfrac{2x}{1 - x^2}\right)$

$\because\ \dfrac{d}{dx}\left(\dfrac{2x}{1 - x^2}\right) = \dfrac{1 + 2x^2}{(1 - x^2)^2} > 0\ \forall\ x\ (-1, 1)$

$\Rightarrow f(x)$ is $\uparrow \forall\ x\ (-1, 1)$ and continuous
$\Rightarrow Y = (f(-1), f(1)) = (\tan^{-1}(-\infty), \tan^{-1}(\infty)) = \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$

11. **(b)** $g: \mathbb{R} \to \left(0, \dfrac{\pi}{3}\right]; g(x) = \cos^{-1}\left(\dfrac{x^2 - \alpha}{1 + x^2}\right)$

Here $0 < \cos^{-1}\left(\dfrac{x^2 - \alpha}{1 + x^2}\right) \le \dfrac{\pi}{3} \forall x \in \mathbb{R}$

$\Rightarrow \cos\dfrac{\pi}{3} \le \dfrac{x^2 - \alpha}{1 + x^2} < \cos 0$

$\Rightarrow \dfrac{1}{2} \le \left(\dfrac{x^2 - \alpha}{1 + x^2}\right) < 1\ \forall\ x\ \mathbb{R}$ …(1)

Now $\dfrac{x^2 - \alpha}{1 + x^2} = \dfrac{x^2 + 1 - (\alpha + 1)}{1 + x^2} = 1 - \dfrac{(\alpha + 1)}{(x^2 + 1)}$ …..(2)

From (1), we get $1 + x^2 \le 2x^2 - 2\alpha < 2 + 2x^2\ \forall\ x\ \mathbb{R}$
$\Rightarrow 2\alpha + 1 \le x^2; \alpha > -1\ \forall\ x\ \mathbb{R}$
$\Rightarrow 2\alpha + 1 \le 0; \alpha > -1$
$\Rightarrow \alpha \left(-1, \dfrac{-1}{2}\right]$ …… (3)

From (2) and (3), clearly $\dfrac{x^2 - \alpha}{1 + x^2} < 1\ \forall\ x\ \mathbb{R}$

Let $h(x) = \dfrac{x^2 - \alpha}{1 + x^2} \Rightarrow h'(x) = \dfrac{2(1 + \alpha)x}{1 + x^2}$

$\Rightarrow h(x) \downarrow$ for $x < 0$ and $h(x) \uparrow$ for $x > 0$
$\Rightarrow h(x)$ has its minimum value $= \dfrac{1}{2}$ at $x = 0$
$\Rightarrow \dfrac{0 - \alpha}{1 + 0} = \dfrac{1}{2} \Rightarrow \alpha = \dfrac{-1}{2}$

12. **(c)** $f: \mathbb{R} \to \mathbb{R}; f(x) = \begin{cases} x|x| - 6; x \in \mathbb{Q} \\ x|x| - \sqrt{2}; x \notin \mathbb{Q} \end{cases}$

If $x_1, x_2\ \mathbb{R}$, then $f(x_1) = f(x_2)$ is possible when either both $x_1, x_2\ \mathbb{Q}$ or both $x_1, x_2 \in \mathbb{Q}$

If x_1, x_2 $\mathbb{Q}$, then $x_1 |x_1| - 6 = x_2 |x_2| - 6$

$\Rightarrow$ $x_1|x_1| = x_2 |x_2|$

$\Rightarrow$ $x_1^{\,2} = x_2^{\,2}$ for $x_1, x_2 \geq 0$ and $-x_1^{\,2} = -x_2^{\,2}$ for $x_1, x_2 < 0$

$\Rightarrow$ $(x_1 - x_2)(x_1 + x_2) = 0$

$\Rightarrow$ $x_1 = x_2$ $\quad$ ($\because x_1, x_2 \geq 0$ or $x_1, x_2 < 0$)

$\Rightarrow$ $f(x)$ is injective.

For $y = -\sqrt{2}$

If $x|x| - 6 = -\sqrt{2}$

$\Rightarrow$ $x|x| = 6 - \sqrt{2}$ $\quad \Rightarrow$ $x = \sqrt{6-\sqrt{2}} \notin \mathbb{Q}$

If $x\,|x| - \sqrt{2} = -\sqrt{2}$ $\quad \Rightarrow$ $x\,|x| = 0$

$\Rightarrow$ $x = 0 \in \mathbb{Q}$

$\Rightarrow$ no value of x satisfy $f(x) = -\sqrt{2}$

$\Rightarrow$ $f(x)$ is not surjective.

13. **(c)** $\because \sin [x]\,\pi = 0$

$\Rightarrow$ $f(x) = 0\ \forall\ x\ \mathbb{R}$

$\Rightarrow$ $f(x)$ is many–one function and a constant function.

14. **(d)** $f\colon X \xrightarrow{onto} Y$;

$X = \{a_1, a_2, a_3, \ldots . a_6\}$; $Y = \{b_1, b_2, b_3\}$

Let $f(a_1) = f(a_2) = f(a_3) = b_1$, choose a_i, a_j, a_k in ${}^6C_3$ ways. Each of a_4, a_5, a_6 has two choice b_2 and b_3, but there are two cases when each of a_4, a_5, a_6 is associated with b_2 or b_3 in this case b_3 or b_2 remains unassociated.

Thus, number of such onto functions $= {}^6C_3\,((2)^3 - 2)$

$= 20(6) = 120$

15. **(b)** $f\colon Y \to Y$; $Y = \{4, 5, 6\}$; $f(4)\ 6$, $f(5)\ 4$, $f(6)\ 5$

$\therefore$ $f(4)\ 6$ $\quad \Rightarrow$ $f(4) = 4$ or 5,

$f(5)\ 4$ $\quad \Rightarrow$ $f(5) = 5$ or 6,

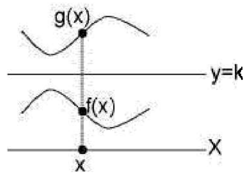
$f(6)\ 5$ $\quad \Rightarrow$ $f(6) = 4$ or 6

$\therefore$ If $f(4) = 4$ $\quad \Rightarrow$ $f(6) = 6, f(5) = 5$

If $f(4) = 5$ $\quad \Rightarrow$ $f(5) = 6, f(6) = 4$

$\Rightarrow$ $f_1 = \{(4, 4), (5, 5), (6, 6)\}$ and $f_2 = \{(4, 5), (5, 6), (6, 4)\}$ are two possible bijective functions.

16. **(d)** By given condition of mirror image about $y = \mathrm{k}$, $f(x) - k = k - f(x)$



$\Rightarrow$ $f(x) + f(x) = 2k$ $\quad \therefore$ $\phi(x) = f(x) + f(x)$

$\Rightarrow$ $\phi(x) = 2k\ \forall\ k\ \mathbb{R}$ $\quad \Rightarrow$ ϕ is a constant function.

$\Rightarrow$ Many–one and into

17. **(b)** $X = \{a_1, a_2, a_3, a_4, a_5\}$; $Y = \{b_1, b_2, b_3, b_4, b_5\}$

$f\colon X \to Y$ such that, $f(x)$ is bijective and $f(a_i)\ b_i$, then number of such functions will be equal to the number of ways of placing 5 letters into five envelops, so that no letter goes to its corresponding addressed envelope.

$= 5!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right]$

$= 5!\left[\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right] = 60 - 20 + 5 - 1 = 44$

TEXTUAL EXERCISE–13: (SUBJECTIVE)

1. $fof(-3) = f(f(-3)) = f(0) = -8$, $fof(7) = f(f(7)) = f(53) = 427$, $fog\,(9) = f(f(9)) = f(85) = 683$, $gof\,(2) = f(f(2)) = f(13) = 173$, $gof\,(0) = f(f\,(0)) = f(-8) = |-8| = 8$, $gof\,(6) = f(f(6)) = f(29) = 845$

2. $f(x) = \dfrac{ax}{x+1}$, $x\ {-1}$, $fof\,(x) = \dfrac{a\,f(x)}{f(x)+1} = \dfrac{a\left(\dfrac{ax}{x+1}\right)}{\dfrac{ax}{x+1}+1}$

$= \dfrac{a^2x}{ax+x+1} = \dfrac{a^2x}{(a+1)x+1} = x$

$\Rightarrow$ $a^2\,x = (a + 1)x^2 + x$ $\quad \Rightarrow$ $(a + 1)x^2 + (1 - a^2)x = 0$

$\Rightarrow$ $(a + 1)\,[x^2 + (1 - a)\,x] = 0$

$\Rightarrow$ $(a + 1)x\,(x + 1 - a) = 0\ \forall\ x\ {-1}$

$\Rightarrow$ $a = -1$

3. $D_f\,[0, 1)$

(i) $f(|[x - 2.5]|)$ $\quad \Rightarrow$ $|[x - 2.5]|\ [0, 1)$

$\Rightarrow$ $|[x - 2.5]|\ (-1, 1)$

$\Rightarrow$ $[x - 2.5] = 0$ $\quad \Rightarrow$ $(x - 2.5)\ [0, 1)$

$\Rightarrow$ $x\ [2.5, 3.5)$

$\Rightarrow$ $x\ \left[\frac{5}{2}, \frac{7}{2}\right)$

(ii) $f(\cos x)$ $\quad \Rightarrow$ $\cos x\ [0, 1)$

$\Rightarrow$ $x \in \bigcup\limits_{n\in\mathbb{Z}} \left\{\left[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right] - \{2n\pi\}\right\}$

4. **(i)** $f(x) = [x]$, $f(x) = \sin x$

$\Rightarrow$ $fof(x) = [\sin x]$

$\Rightarrow$ Domain of fog $(x) = \mathbb{R}$ and range of $fog\,(x) = \{-1, 0, 1\}$; gof $(x) = \sin [x]$

$\Rightarrow$ Domain of gof $(x) = \mathbb{R}$; Range of $gof\,(x) = \{\sin x;\ a\ \mathbb{Z}\}$

(ii) $f(x) = \tan x$, $x\ \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$; $f(x) = \sqrt{1-x^2}$

$\Rightarrow$ $fog\,(x) = \tan \sqrt{1-x^2}$

$\Rightarrow$ $x\ [-1, 1]$ $\quad \Rightarrow$ Domain of $fog\,(x) = [-1, 1]$

$\Rightarrow$ Range of $fog\,(x) = [0, \tan 1]$

$\Rightarrow$ $gof\,(x) = \sqrt{1-\tan^2 x}$

$\Rightarrow$ $\tan x\ [-1, 1]$

$\Rightarrow$ Domain of $gof\,(x) = \left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$ and range of $gof\,(x) = [0, 1]$

5. **(a)** $f(x) = \begin{cases} 1+x;\ 0 \leq x \leq 2 \\ 3-x;\ 2 < x \leq 3 \end{cases}$

$\Rightarrow$ $fof(x) = \begin{cases} 1+f(x); 0 \leq \mathrm{f}(x) \leq 2 \\ 3-f(x); 2 < f(x) \leq 3 \end{cases}$

$= \begin{cases} 1+(1+x); 0 \leq 1+x \leq 2; 0 \leq x \leq 2 \\ 1+(3-x); 0 \leq 3-x \leq 2; 2 < x \leq 3 \\ 3-(1+x); 2 < 1+x \leq 3; 0 \leq x \leq 2 \\ 3-(3-x); 2 < 3-x \leq 3; 2 < x \leq 3 \end{cases}$

$= \begin{cases} 2+x; 0 \leq x \leq 1 \\ 4-x; 2 < x \leq 3 \\ 2-x; 1 < x \leq 2 \end{cases} = \begin{cases} 2+x; 0 \leq x \leq 1 \\ 2-x; 1 < x \leq 2 \\ 4-x; 2 < x \leq 3 \end{cases}$

(b) $f(x) = -1 + |x-2|;\ 0 \le x \le 4$

$$\Rightarrow f(x) = \begin{cases} 1-x; 0 \le x < 2 \\ x-3; 2 \le x \le 4 \end{cases}$$

$$\Rightarrow fof(x) = \begin{cases} 1-f(x); 0 \le f(x) < 2 \\ f(x)-3; 2 \le f(x) \le 4 \end{cases}$$

$$= \begin{cases} 1-(1-x); 0 \le 1-x < 2; 0 \le x < 2 \\ 1-(x-3); 0 \le x-3 < 2; 2 \le x \le 4 \\ (1-x)-3; 2 \le 1-x \le 4; 0 \le x < 2 \\ (x-3)-3; 2 \le x-3 \le 4; 2 \le x \le 4 \end{cases}$$

$$= \begin{cases} x; 0 \le x \le 1 \\ 4-x; 3 \le x \le 4 \end{cases}$$

$$\therefore\ fof(x) = \begin{cases} x; 0 \le x \le 1 \\ \text{Not defined for } 1 < x < 3 \\ 4-x; 3 \le x \le 4 \end{cases}$$

(c) $f(x) = 2 - |x|;\ -1 \le x \le 3$

$$\Rightarrow f(x) = \begin{cases} 2+x; x < 0 \\ 2-x; x \ge 0 \end{cases}$$

$$\Rightarrow f(f(x)) = \begin{cases} 2+f(x); -1 \le f(x) < 0 \\ 2-f(x); 0 \le f(x) \le 3 \end{cases}$$

$$= \begin{cases} 2+(2+x); -1 \le 2+x < 0, x < 0 \\ 2+(2-x); -1 \le 2-x < 0, x \ge 0 \\ 2-(2+x); 0 \le 2+x \le 3, x < 0 \\ 2-(2-x); 0 \le 2-x \le 3, x \ge 0 \end{cases} = \begin{cases} 4-x; 2 < x \le 3 \\ -x; -1 \le x \le 0 \\ x; 0 < x \le 2 \end{cases}$$

(d) $f(x) = \begin{cases} x+1; x \le 1 \\ 5-x^2; x > 1 \end{cases}$

$$\Rightarrow fof(x) = \begin{cases} f(x)+1; f(x) \le 1 \\ 5-(f(x))^2; f(x) > 1 \end{cases}$$

$$= \begin{cases} (x+1)+1; x+1 \le 1; x \le 1 \\ (5-x)^2+1; (5-x^2) \le 1; x > 1 \\ 5-(x+1)^2; (x+1) > 1; x \le 1 \\ 5-(5-x^2)^2; 5-x^2 > 1; x > 1 \end{cases}$$

$$= \begin{cases} x+2; x \le 0 \\ 6-x^2; x \ge 2 \\ 5-(x+1)^2; 0 < x \le 1 \\ 5-(5-x^2)^2; 1 < x < 2 \end{cases}$$

(e) $f(x) = \begin{cases} -x; & x < 0 \\ x; & 0 \le x \le 1 \\ 2-x; & x > 1 \end{cases}$

$$\Rightarrow f(f(x)) = \begin{cases} -f(x); f(x) < 0 \\ f(x); 0 \le f(x) \le 1 \\ 2-f(x); f(x) > 1 \end{cases}$$

$$= \begin{cases} -(-x); -x < 0; x < 0 \\ -(x); x < 0; 0 \le x \le 1 \\ -(2-x); 2-x < 0; x > 1 \\ -x; 0 \le -x \le 1; x < 0 \\ x; 0 \le x \le 1; 0 \le x \le 1 \\ 2-x; 0 \le 2-x \le 1; x > 1 \\ 2-(-x); -x > 1; x < 0 \\ 2-x; x > 1; 0 \le x \le 1 \\ 2-(2-x); 2-x > 1; x > 1 \end{cases} = \begin{cases} 2+x; x < -1 \\ x-2; x > 2 \\ -x;\ -1 \le x < 0 \\ x; 0 \le x \le 1 \\ 2-x; 1 < x \le 2 \end{cases}$$

6. (a) $f(x) = \begin{cases} 1+x^2; x \le 1 \\ x+1; 1 < x \le 2 \end{cases}$ and $f(x) = \{1 - x;\ -2 \le x \le 1$

$$\text{fog}(x) = \begin{cases} 1+(g(x)^2; g(x) \le 1 \\ g(x)+1; 1 < g(x) \le 2 \end{cases}$$

$$= \begin{cases} 1+(1-x)^2; (1-x) \le 1, -2 \le x \le 1 \\ (1-x)+1; 1 < 1-x \le 2; -2 \le x \le 1 \end{cases}$$

$$= \begin{cases} 2-2x+x^2; 0 \le x \le 1 \\ 2-x; -1 \le x < 0 \end{cases}$$

(b) $f(x) = \begin{cases} 1+x; x \le 1 \\ 2x+2; 1 < x \le 2 \end{cases}$ and $f(x) = \begin{cases} x^2; -1 \le x < 2 \\ x+2; 2 \le x \le 3 \end{cases}$

$$\Rightarrow \text{fog}(x) = \begin{cases} 1+g(x); g(x) \le 1 \\ 2.g(x)+2; 1 < g(x) \le 2 \end{cases}$$

$$= \begin{cases} 1+x^2; x^2 \le 1; -1 \le x \le 2 \\ 1+(x+2); x+2 \le 1; 2 \le x \le 3 \\ 2x^2+2; 1 < x^2 \le 2; -1 \le x < 2 \\ 2(x+2)+2; 1 < x+2 \le 2; 2 \le x \le 3 \end{cases}$$

$$= \begin{cases} 1+x\ ; -1 \le x \le 1 \\ 2x\ +2; 1 < x \le \sqrt{2} \end{cases}$$

7. (a) $f(x) = -1 + |x-2|;\ 0 \le x \le 4; f(x) = 2 - |x|;\ -1 \le x \le 3$

$fog(x) = -1 + |g(x) - 2|;\ 0 \le f(x) \le 4;\ -1 \le x \le 3$

$= -1 + |2 - |x| - 2|;\ 0 \le 2 - |x| \le 4;\ -1 \le x \le 3$

$= -1 + |x|;\ 0 \le |x| \le 2,\ -1 \le x \le 3$

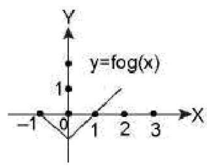
$= -1 + |x|;\ -1 \le x \le 2$ and $gof(x) = 2 - |f(x)|;\ -1 \le f(x) \le 3;\ 0 \le x \le 4$

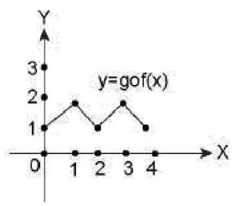
$= 2 - |-1 + |x-2||;\ -1 \le -1 + |x-2| \le 3;\ 0 \le x \le 4$

$$= \begin{cases} 2+(-1+|x-2|); 0 \le |x-2| \le 1; 0 \le x \le 4 \\ 2-(-1+|x-2|); 1 \le |x-2| \le 4; 0 \le x \le 4 \end{cases}$$

$$= \begin{cases} 1+|x-2|; 1 \le x \le 3 \\ 3-|x-2|; 0 \le x \le 1, 3 \le x \le 4 \end{cases}$$

The graphs of fog (x) and gof (x) are as given below.





(b) $f(x) = \begin{cases} x+1; x \le 1 \\ 2x+1; 1 < x \le 2 \end{cases}$ and $f(x) = \begin{cases} x^2; -1 < x \le 2 \\ x+2; 2 < x \le 3 \end{cases}$

fog $(x) = \begin{cases} g(x)1; g(x) \le 1; -1 < x \le 3 \\ 2g(x)+1; 1 < g(x) \le 2; -1 < x \le 3 \end{cases}$

$= \begin{cases} x^2+1; x^2 \le 1; -1 < x \le 2 \\ (x+2)+1; (x+2) \le 1; 2 < x \le 3 \\ 2x^2+1; 1 < x^2 \le 2; -1 < x \le 2 \\ 2(x+2)+1; 1 < (x+2) \le 2; 2 < x \le 3 \end{cases}$

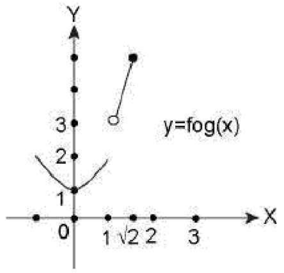
$= \begin{cases} x^2+1; -1 < x \le 1 \\ 2x^2+1; 1 < x \le \sqrt{2} \end{cases}$ and gof (x)

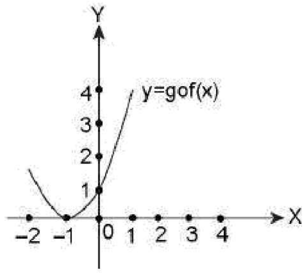
$= \begin{cases} (f(x))^2; -1 < f(x) \le 2, x \le 2 \\ f(x)+2; 2 < f(x) \le 3; x \le 2 \end{cases}$

$= \begin{cases} (x+1)^2; -1 < (x+1) \le 2; x \le 1 \\ (2x+1)^2; -1 < (2x+1) \le 2; 1 < x \le 2 \\ (x+1)+2; 2 < x+1 \le 3; x \le 1 \\ (2x+1)+2; 2 < 2x+1 \le 3; 1 < x \le 2 \end{cases}$

$= \{(x+1)^2; -2 < x \le 1$

The graphs of fof(x) and gof (x) are as given below.





8. **Example1:** $f(x) = 2x + 4$; $f(x) = \dfrac{x-4}{2}$

Clearly $f(x)$ is not onto as gives even images only and gof(x) $= x$ which is onto.

Example 2: $f(x) = x^2 + 3$; $g(x) = \sqrt{x-3}$

$f(x)$ 5 as $x^2 + 3 = 5$ $\Rightarrow$ $x^2 = 2$

$\Rightarrow x = \pm\sqrt{2}$; x ℕ

$\Rightarrow$ $f(x)$ is not onto

Now, gof $(x) = |x| = x$ as x ℕ

$\Rightarrow$ gof (x) is onto.

9. Let $y \in Y$ $\Rightarrow$ fog (y) $= y$

$\Rightarrow$ $f(f(y)) = y$ but $f(y)$ X

$\Rightarrow$ $f(y) = x$ for some $x \in x$

$\Rightarrow$ $f(x) = y$ for some $x \in x$

$\Rightarrow$ $f(x)$ is onto.

10. Let $f(x_1) = f(x_2)$ for some $x_1, x_2 X$

$\Rightarrow$ $f(x_1), f(x_2)$ Y and g is a function from $Y \to X$

$\Rightarrow$ $f(f(x_1)) = f(f(x_2)))$ (unique image)

$\Rightarrow$ $x_1 = x_2$ ($\because$ gof$(x) = x \ \forall\ X$)

$\Rightarrow$ $f(x)$ is one–one.

11. $f(x) = \cot^{-1}(\text{sgn}(x)) + \sin^{-1}(x - \{x\}) = \cot^{-1}(\text{sgn}(x)) + \sin^{-1}[x]$

$\Rightarrow$ $[x]$ $\{-1, 0, 1\}$

$\Rightarrow$ x $[-1, 2)$, which is the largest domain.

Also $f([-1, 2)) = \{\cot^{-1}(-1) + \sin^{-1}(-1), \cot^{-1} 1 + \sin^{-1} 0,$ $\cot^{-1}(0) + \sin^{-1}(0), \cot^{-1}(1) + \sin^{-1} 1\} = \left\{\dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{3\pi}{4}\right\}$

$\because$ $f(x) = \begin{cases} \dfrac{\pi}{4} \text{ for } x \in [-1,0) \cup (0,1) \\ \dfrac{\pi}{2} \text{ for } x = 0 \\ \dfrac{3\pi}{4} \text{ for } x \in [1,2) \end{cases}$

$\Rightarrow$ $f(x)$ is not injective function.

12. $f(x) = \begin{cases} x+2; x \le 1 \\ 2x+1; 1 < x \le 3 \end{cases}$ and $f(x) = \begin{cases} x^2+1; -1 \le x \le 4 \\ x-1; 4 < x \le 6 \end{cases}$

$\therefore$ $f(f(x)) = \begin{cases} g(x)+2; g(x) \le 1; -1 \le x \le 6 \\ 2g(x)+1; 1 < g(x) \le 3; -1 \le x \le 6 \end{cases}$

$= \begin{cases} (x^2+1)+2; (x^2+1) \le 1; -1 \le x \le 4 \\ (x-1)+2; (x-1) \le 1; 4 < x \le 6 \\ 2(x^2+1)+1; 1 < x^2+1 \le 3; -1 \le x \le 4 \\ 2(x-1)+1; 1 < x-1 \le 3; 4 < x \le 6 \end{cases}$

$= \begin{cases} 3; x = 0 \\ 2x^2+3; -1 \le x <, 0 \text{ or } 0 < x \le \sqrt{2} \end{cases}$

$= 2x^2 + 3; x \in [-1, \sqrt{2}]$

$\therefore$ Domain $= [-1, \sqrt{2}]$, Range $= [3, 7]$

$\therefore$ $f(f(x)) = \begin{cases} (f(x))^2+1; -1 \le f(x) \le 4; x \le 3 \\ f(x)-1; 4 < f(x) \le 6 \end{cases}$

$= \begin{cases} (x+2)^2+1; -1 \le x+2 \le 4; x \le 1 \\ (2x+1)^2+1; -1 \le 2x+1 \le 4; 1 < x \le 3 \\ (x+2)-1; 4 < x+2 \le 6; x \le 1 \\ (2x+1)-1; 4 < 2x+1 \le 6; 1 < x \le 3 \end{cases}$

$= \begin{cases} x^2+4x+5; -3 \le x \le 1 \\ 4x^2+4x+2; 1 < x \le \dfrac{3}{2} \\ 2x; \dfrac{3}{2} < x \le \dfrac{5}{2} \end{cases}$

$\Rightarrow$ Domain $= \left[-3, \dfrac{5}{2}\right]$ and range $= [1, 17]$

(a) $f(f(x)) = 5.01$

$\Rightarrow$ $2x^2 + 3 = 5.01$

For x $[-1, 0)$; fog (x) decrease from 5 to 3 and for x $[0, \sqrt{2}]$, fog (x) increase from 3 to 7.

$\Rightarrow$ $f(f(x)) = 5.01$ has only one solution.

(b) $f(f(x)) = 5$ has solution.

(c) $f(f(x)) = 1$ has no solution.

13. $f(x) = \begin{cases} x+a; \text{if } x<0 \\ |x-1|; \text{if } x \geq 0 \end{cases}$ and $f(x) = \begin{cases} (x+1); \text{if } x<0 \\ (x-1)^2 + b \text{ if } x \geq 0 \end{cases}$; $a, b \geq 0$

$\Rightarrow$ (gof)(x) = $\begin{cases} f(x)+1; f(x)<0 \\ (f(x)-1)^2+b; f(x)\geq 0 \end{cases}$

$= \begin{cases} (x+a)+1; (x+a)<0; x<0 \\ |x-1|+1; |x-1|<0; x\geq 0 \\ (x+a-1)^2+b; (x+a)\geq 0; x<0 \\ (|x-1|)^2+b; |x-1|\geq 0; x\geq 0 \end{cases}$

$= \begin{cases} x+a+1; x<-a \\ (x+a-1)^2+b; -a\leq x<0 \\ (|x-1|-1)^2+b; x\geq 0 \end{cases}$

$= \begin{cases} (x+a+1); x<-a \\ (x+a-1)^2+b; -a\leq x<0 \\ (x)^2+b; 0\leq x\leq 1 \\ (x-2)^2+b; x\geq 1 \end{cases}$

$\because$ gof (x) is continuous.

$\Rightarrow \lim\limits_{x\to -a^-} gof(x) = gof(-a) = 1+b$

$\Rightarrow 1 = 1 + b \qquad \Rightarrow \quad b = 0$

Also $\lim\limits_{x\to 0^-} gof(x) = gof(0)$

$\Rightarrow (a-1)^2 + b = b$

$\Rightarrow a = 1$

$\Rightarrow$ (gof)$'$ (x) = $\begin{cases} 1; x<-1 \\ 2(x); -1<x<0 \\ 2x; 0\leq x<1 \\ 2(x-2); x>1 \end{cases}$

$\therefore$ L.H.D of gof (x) at $x = 0$ equals 0 and R.H.D of gof (x) at $x = 0$ equals 0.

$\Rightarrow$ gof(x) is differentiable at $x = 0$.

14. $h'(x) = (fog)'(x) = f'(f(x)).g'(x) \leq 0$ as $f'(x) \geq 0\ \forall\ x\ [0, \infty)$, $g'(x) \leq 0\ \forall\ x\ [0, \infty)$

$\Rightarrow$ $h(x)$ is a decreasing function from $[0, \infty)$ to $[0,)$. Also $h(0) = 0$

$\Rightarrow h(x) = 0\ \forall\ x\ [0, \infty)$

15. f: $[0, 1] \to [1, 2]$; $f(x) = (1 + x)$ and g: $[1, 2] \to [0, 1]$; $f(x) = (2 - x)$ $gof(x) = 2 - f(x)$; $x\ [0, 1] = 2 - (1 + x)$; $x\ [0, 1]$

$\Rightarrow gof(x) = 1 - x$; $x\ [0, 1]$

16. (i) $f(x) = \cos^{-1}\sqrt{\log_{[x]}\dfrac{|x|}{x}}$

For domain, $\sqrt{\log_{|x|}\dfrac{|x|}{x}} \in [0,1]$

$\Rightarrow \log_{|x|}\dfrac{|x|}{x}\ [0, 1]; [x] > 0\ 1$

$\Rightarrow \log_{|x|}\dfrac{|x|}{x}\ [0, 1]; [x] \geq 2$

$\Rightarrow 1 \leq \dfrac{|x|}{x} \leq [x]; x \geq 2$

$\Rightarrow x \leq |x| \leq x\,[x]; x \geq 2 \quad \because \quad x \geq 2$

$\Rightarrow |x| = x$

$\Rightarrow x \leq x\,[x]$

$\Rightarrow$ $1 \leq [x]$; $x \geq 2$; which is true $\forall\ x \geq 2$

$\therefore$ Domain of $f(x) = [2, \infty)$

$f(x) = \cos^{-1}\sqrt{\log_{[x]}1}$ $\quad (\because x > 0 \Rightarrow |x| = x)$

$\Rightarrow f(x) = \cos^{-1}(0) = \dfrac{\pi}{2}$

$\Rightarrow$ Range $= \left\{\dfrac{\pi}{2}\right\}$

(ii) $f(x) = \sqrt{\ell n(\cos(\sin x))}$

For domain ln (cos (sin x)) ≥ 0

$\Rightarrow$ cos (sin x) ≥ 1

$\Rightarrow$ cos (sin x) = 1

$\Rightarrow$ sin $x = 2n\pi$; n $\mathbb{Z}$

$\Rightarrow$ sin $x = 0$

$\Rightarrow$ Domain = $\{n\}$; n $\mathbb{Z}$ and range = $\{0\}$

TEXTUAL EXERCISE–13: (OBJECTIVE)

1. (a), (b), (c) Since the composition of two bijective function is a bijective function, then gof is injective surjective also.

2. (d) $f(f(x)) = x\ \forall\ x\ \mathbb{R}$

$\Rightarrow f(x) = f^{-1}(x)\ \forall\ x\ \mathbb{R}$, i.e., function is self invertible.

$\therefore$ $(\alpha, \beta) f$

$\Rightarrow$ $(\beta, \alpha) f$

$\Rightarrow f(x)$ and $f^{-1}(x)$ intersect along the line, $(y - \beta)$ $= \dfrac{\alpha-\beta}{\beta-\alpha}(x-\alpha)$

$\Rightarrow (y - \beta) = -(x - \alpha)$

$\Rightarrow y = -x + (\alpha - \beta)$

$\Rightarrow$ Such functions are infinitely many.

3. (b), (d) $f(x) = \dfrac{1}{x}$ and $f(x) = \dfrac{1}{\sqrt{x}}$

$\Rightarrow D_f = \mathbb{R} - \{0\}$; $D_g = (0, \infty)$

Now, $f(f(x)) = \dfrac{1}{g(x)} = \sqrt{x}$ with its domain $(0, \infty) = D_g$ and range of $f(x)$.

$f(f(x)) = \dfrac{1}{\sqrt{f(x)}} = \sqrt{x}$ with its domain $(0, \infty) = D_g$ and range = $(0, \infty)$

4. (d) f, g: $\mathbb{R} \to \mathbb{R}$

$f(x) = \dfrac{x+|x|}{2}$ and $f(x) = \begin{cases} x \text{ for } x<0 \\ x^2 \text{ for } x\geq 0 \end{cases}$

fog $(x) = \dfrac{g(x)+|g(x)|}{2}$

$= \begin{cases} 0 \text{ if } g(x) < 0 \\ g(x) \text{ if } g(x) \geq 0 \end{cases} = \begin{cases} 0 \text{ if } x < 0 \\ x^2 \text{ if } x \geq 0 \end{cases}$ and gof (x)

$= \begin{cases} f(x) \text{ for } f(x) < 0 \\ (f(x))^2 \text{ for } f(x) \geq 0 \end{cases} = \begin{cases} x^2 \text{ for } x \geq 0 \\ 0 \text{ for } x < 0 \end{cases}$

$\Rightarrow$ fog (x) = gof (x) $\forall$ x $\mathbb{R}$

5. **(d)** $f(x) = \sin x, f(x) = |\ln x|$

$\Rightarrow$ fog (x) = sin $|\ln x|$ with its domain = $(0, \infty)$ and range = $[-1, 1] = R_1$

And gof $(x) = |\ln (\sin x)|$ with its domain

$= \bigcup_{n\in\mathbb{Z}} (2n\pi, (2n+1)\pi)$ and range = $[0, \infty) = R_2$

7. **(c)** $f(x) = -1 + |x - 1|$; $-1 \leq x \leq 3$ and $f(x) = 2 - |x + 1|$; $-2 \leq x \leq 2$

fog $(3/2) = f(f(3/2)) = f\left(2 - \left|\frac{5}{2}\right|\right) = f\left(-\frac{1}{2}\right)$

$= -1 + \left|-\frac{1}{2} - 1\right| = \frac{1}{2}$

8. **(b), (c)** $f(x) = \begin{cases} x; x \in \mathbb{Q} \\ 1 - x; x \notin \mathbb{Q} \end{cases}$

$\Rightarrow f(f(x)) = \begin{cases} f(x); f(x) \in \mathbb{Q} \\ 1 - f(x); f(x) \notin \mathbb{Q} \end{cases}$

$= \begin{cases} x; x \in \mathbb{Q}; x \in \mathbb{Q} \\ 1 - x; 1 - x \in \mathbb{Q}; x \notin \mathbb{Q} \\ 1 - x; x \notin \mathbb{Q}, x \in \mathbb{Q} \\ 1 - (1 - x); 1 - x \notin \mathbb{Q}, x \in \mathbb{Q} \end{cases}$

$= \begin{cases} x; x \in \mathbb{Q} \\ x; x \notin \mathbb{Q}\ \forall x \in \mathbb{R} \end{cases} \Rightarrow f(f(x)) = x$

$\Rightarrow$ An identity function an odd linear as polynomial.

9. **(c)** Clearly, abscissa of $D = x_0$ and ordinate of D = ordinate of C.

= g (abscissa of B) = g (ordinate of B as $y = x$)

= g (ordinate of A) = $f(f(x_0))$

$\Rightarrow D = (x_0, f(f(x_0)))$

11. **(b)** $f(x) = x^2 + 2x + 1 = (x + 1)^2 \Rightarrow g[f(x)] = |x + 1|$

$\Rightarrow f((x + 1)^2) = |x + 1| = \sqrt{(x+1)^2}$

$\Rightarrow f(x) = \sqrt{x}$

12. **(a), (b)** $f(x) = (ax^3 + b)^5$; $f(f(x)) = f(f(x))$

$\Rightarrow f(x) = (f(x))^{-1} = \left(\frac{x^{1/5} - b}{a}\right)^{1/3}$

Also $f(f(x)) = f(f(x)) = x$

TEXTUAL EXERCISE–14: (SUBJECTIVE)

1. $f(x) = \begin{cases} x; -\infty < x < 1 \\ x^2; 1 \leq x \leq 4 \\ 2^x; 4 < x < \infty \end{cases}$

$\Rightarrow x = \begin{cases} y; -\infty < x < 1; -\infty < y < 1 \\ \sqrt{y}; 1 \leq x \leq 4; 1 \leq y \leq 16 \\ \log_2 y; 4 < x < \infty; 16 < y < \infty \end{cases}$

$\Rightarrow f^{-1}(x) = \begin{cases} x; -\infty < x < 1 \\ \sqrt{x}; 1 \leq x \leq 16 \\ \log_2 x: 16 < x < \infty \end{cases}$

2. **(i)** $f: (-\infty, -1) \to (-\infty, -2)$; $f(x) = -(x + 1)^2 - 2$

$\Rightarrow f'(x) = -2(x + 1) \geq 0$ as $x \leq -1$ and $y + 2 = -(x + 1)^2$

$\Rightarrow (x + 1)^2 = -(y + 2)$

$\Rightarrow x + 1 = \pm\sqrt{-(y+2)}$ but $x + 1 \leq 0$ and $y + 2 \leq 0$

$\Rightarrow x + 1 = -\sqrt{-(y+2)} \Rightarrow f^{-1}(x) = -1 - \sqrt{-(x+2)}$

(ii) $f: \left[\frac{\pi}{6}, \frac{7\pi}{6}\right] \to [-1, 1]$; $f(x) = \sin\left(x + \frac{\pi}{3}\right)$

$\Rightarrow f'(x) = \cos\left(x + \frac{\pi}{3}\right) \leq 0$ for $x \in \left[\frac{\pi}{6}, \frac{7\pi}{6}\right]$

$\Rightarrow f(x)$ is decreasing function and $y = \sin\left(x + \frac{\pi}{3}\right)$

$\Rightarrow \sin^{-1} y = \sin^{-1}\left(\sin\left(x + \frac{\pi}{3}\right)\right)$

$\Rightarrow \sin^{-1} y = \sin^{-1}\left[\sin\left(\pi - \left(x + \frac{\pi}{3}\right)\right)\right]$

$= \sin^{-1}\left[\sin\left(\frac{2\pi}{3} - x\right)\right] \quad \left(\because \frac{-\pi}{2} \leq \pi - \left(x + \frac{\pi}{3}\right) \leq \frac{\pi}{2}\right)$

$\Rightarrow \sin^{-1}(y) = \frac{2\pi}{3} - x$

$\Rightarrow x = \frac{2\pi}{3} - \sin^{-1}(y)$

$\Rightarrow f^{-1}(x) = \frac{2\pi}{3} - \sin^{-1} x; x \in [-1, 1].$

3. $f(x) = \begin{cases} 0, \text{ if } x \text{ is rational} \\ x, \text{if } x \text{ is irrational} \end{cases}$ and $f(x) = \begin{cases} x, \text{ if } x \text{ is rational} \\ 0, \text{if } x \text{ is irrational} \end{cases}$

$\Rightarrow (f - g)(x) = \begin{cases} -x, \text{ if } x \in \mathbb{Q} \\ x, \text{ if } x \in \overline{\mathbb{Q}} \end{cases}$

We claim that $h = (f - g)$ is one–one and onto

Injectivity: Let $h(x_1) = h(x_2)$

$\Rightarrow \begin{cases} -x_1 = -x_2 \text{ if } x_1, x_2 \in \mathbb{Q} \\ x_1 = x_2 \text{ if } x_1, x_2 \in \overline{\mathbb{Q}} \\ -x_1 = x_2 \text{ if } x_1 \in \mathbb{Q}, x_2 \in \overline{\mathbb{Q}} \end{cases}$

$\Rightarrow x_1 = x_2 \quad \Rightarrow$ Injectivity

Subjectivity: Also if y $\mathbb{R}$, then h($-y$) if y $\mathbb{Q}$ and $h(y) = y$ if y $\mathbb{Q}$

$\Rightarrow h(x) = (f - g)(x)$ is subjective.

Hence, $(f - g)(x)$ is bijective, i.e., invertible.

Also $(f + g)(x) = x\ \forall\ \mathbb{R}$, which is clearly bijective, and hence, invertible.

Now $(f.g)(x) = 0\ \forall\ x\ \mathbb{R}$, Which being constant, and hence, many–one.

$\Rightarrow$ Not invertible.

4. $f: \mathbb{R} \to \mathbb{R}; f(x) = \dfrac{e^x - e^{-x}}{2}$

$\Rightarrow f'(x) = \dfrac{e^x + e^{-x}}{2} > 0\ \forall\ x\ \mathbb{R}$

$\Rightarrow f(x)$ is increasing and continuous function. Also $f(-\infty) = \dfrac{e^{-\infty} - e^{\infty}}{2} = -\infty$ and $f(\infty) = \dfrac{e^{\infty} + e^{-\infty}}{2} = \infty$

$\Rightarrow f: \mathbb{R} \to \mathbb{R}$ is invertible. Now, $y = \dfrac{e^x - e^{-x}}{2} = \dfrac{e^{2x} - 1}{2e^x}$

$\Rightarrow (e^x)^2 - 2y\,e^x - 1 = 0$

$\Rightarrow e^x = \dfrac{2y \pm \sqrt{4y^2 + 4}}{2} \quad \Rightarrow \quad e^x = y \pm \sqrt{y^2 + 1}$

$\Rightarrow e^x = y + \sqrt{y^2+1}$ as $e^x > 0\ \forall\ x\ \mathbb{R}$

$\Rightarrow x = \ln\left(y + \sqrt{y^2+1}\right) \quad \Rightarrow \quad f^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)$

5. $y = \log_a\left(x + \sqrt{x^2+1}\right); a > 1$

$\Rightarrow a^y = x + \sqrt{x^2+1} \qquad \Rightarrow a^y - x = \sqrt{x^2+1}$

$\Rightarrow a^{2y} + x^2 - 2a^y.x = x^2 + 1$

$\Rightarrow a^{2y} - 2\,x.a^y - 1 = 0 \quad \Rightarrow \quad x = \dfrac{a^{2y} - 1}{2a^y} = \dfrac{a^y - a^{-y}}{2}$

$\Rightarrow f^{-1}(x) = \dfrac{\left(a^x - a^{-x}\right)}{2}$

6. $f(x) = f^{-1}(x)$ and $f'(x) = \dfrac{1}{1+x^3}$

Let $y = f(x) = f^{-1}(x)$

$\Rightarrow f(y) = x$

$\Rightarrow f'(y) = \dfrac{dx}{dy}$ (Diff. w.r.t y)

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{f'(y)} = 1 + y^3 = 1 + (f(x))^3$

$\Rightarrow \dfrac{d}{dx}(g(x)) = 1 + (g(x))^3$

7. $f(x) = \sin x + \cos x; f(x) = x^2 - 4$

$\Rightarrow f(x) = \sqrt{2}\sin\left(x + \dfrac{\pi}{4}\right); g(x) = x^2 - 4$

$\Rightarrow f(f(x)) = (f(x))^2 - 4 = 2\sin^2\left(x + \dfrac{\pi}{4}\right) - 4 = 1 - \cos\left(2x + \dfrac{\pi}{2}\right) - 4 = -3 + \sin 2x$

$\therefore\ f(x) = \sin 2x - 3$

$\Rightarrow f(x)$ is invertible for $2x \in \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$

$\Rightarrow x \in \left[\dfrac{-\pi}{4}, \dfrac{\pi}{4}\right]$

8. $f(x) = \sqrt{2}\sin\left(x + \dfrac{\pi}{4}\right); f(x) = x^2 + 7$

$\Rightarrow f(x) = x + 7$

$\Rightarrow f(f(x)) = (f(x)) + 7 = \sqrt{2}\sin\left(x + \dfrac{\pi}{4}\right) + 7$, which is invertible for $x + \dfrac{\pi}{4} \in \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$

$\Rightarrow x \in \left[\dfrac{-3\pi}{4}, \dfrac{\pi}{4}\right]$

Also $y = \sqrt{2}\sin\left(x + \dfrac{\pi}{4}\right) + 7$

$\Rightarrow \dfrac{y-7}{\sqrt{2}} = \sin\left(x + \dfrac{\pi}{4}\right)$

$\Rightarrow \sin^{-1}\left(\dfrac{y-7}{\sqrt{2}}\right) = \sin^{-1}\left(\sin\left(x + \dfrac{\pi}{4}\right)\right) = x + \dfrac{\pi}{4}$

For $x + \dfrac{\pi}{4} \in \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$ and $\dfrac{y-7}{\sqrt{2}} \in [-1, 1]$

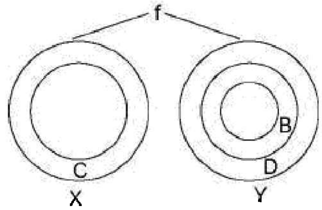
$\Rightarrow x = \sin^{-1}\left(\dfrac{y-7}{\sqrt{2}}\right) - \dfrac{\pi}{4}$

$\Rightarrow f^{-1}(x) = \sin^{-1}\left(\dfrac{x-7}{\sqrt{2}}\right) - \dfrac{\pi}{4}$ with domain $x \in \left[7 - \sqrt{2}, 7 + \sqrt{2}\right]$

9. $f(C) = \{f(x) : x\ C\}$

$F^{-1}(D) = \{x \in x : f(x)\ D\}$

Let $y \in f(f^{-1}(B))$



$\Rightarrow y = f(x)$ for some $x\ f^{-1}(B)$

$\Rightarrow f(x)\ B \qquad \Rightarrow y \in B \subseteq f(X)$

$\Rightarrow f(f^{-1}(B)) \subseteq B$ ....(i)

Conversely, let $y \in B \subseteq f(X)$

As $y \in f(X)$

$\Rightarrow \exists\ x \in X$ such that $y = f(x) \in B$

$\Rightarrow x \in f^{-1}(B) \subseteq X$

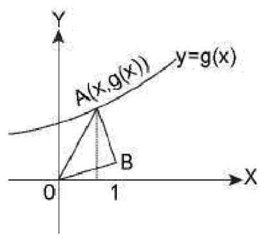
$\Rightarrow f(x) \in f(f^{-1}(B))$

$\Rightarrow y \in f(f^{-1}(B))$

$\Rightarrow B \subseteq f(f^{-1}(B))$ ....(iii)

$\therefore$ From (i) and (ii), $f(f^{-1}(B)) = B$

10. Area of $\Delta = \dfrac{\sqrt{3}}{4}a^2 = \dfrac{\sqrt{3}}{4}[x^2 + (f(x))^2] = \dfrac{\sqrt{3}}{4}$ (given)



$\Rightarrow\ x^2 + (f(x))^2 = 1 \qquad \Rightarrow\ x = \pm\sqrt{1-y^2}\,;\, y \ge 0$

$\Rightarrow\ x = \sqrt{1-y^2} \qquad \Rightarrow\ g^{-1}(x) = \sqrt{1-x^2}\,;\, x \in [0, 1]$

11. f: {1, i, w} → {5, 6, 7}

Case (i) : $f(1) = 5$ (true) , $f(i)$ 5 (false), $f(w)$ 7 (false).

$\Rightarrow\ f(1) = 5, f(i) = 5, f(w) = 7$ but it is many–one.

Case (ii): $f(1) = 5$ (false), $f(i)$ 5

$\Rightarrow\ f(w) = 7, f(1)$ 5, f (i) 5

$\Rightarrow\ f(x)$ is not onto.

Case (iii): $f(1) = 5$ (false), $f(i)$ 5 (false), $f(w)$ 7 (true)

$\Rightarrow\ f(1)$ 5, f (i) = 5, $f(w)$ 7

$\Rightarrow\ f(1) = 7, f(i) = 5, f(w) = 6$

$\Rightarrow\ f^{-1}(6) = w$

$\Rightarrow\ (f^{-1})^{2012}(6) = w^{2012} = (w^3)^{670}.w^2 = w^2$

12. $2x^2 - 5x + 2 = \dfrac{5-\sqrt{9+8x}}{4}; x < \dfrac{5}{4}$ …(i)

Let $y = 2x^2 - 5x + 2$

$\Rightarrow\ 2x^2 - 5x + (2-y) = 0 \Rightarrow\ x = \dfrac{5\pm\sqrt{25-4(2)(2-y)}}{4}$

$\Rightarrow\ x = \dfrac{5\pm\sqrt{9+8y}}{4} \qquad \Rightarrow\ x = \dfrac{5+\sqrt{9+8y}}{4} > \dfrac{5}{4}$

But $x < 5/4$

$\Rightarrow\ x = \dfrac{5-\sqrt{9+8y}}{4}$

$\Rightarrow\ f^{-1}(x) = \dfrac{5-\sqrt{9+8x}}{4}$ …..(ii)

From (i) and (ii), $f(x) = f^{-1}(x)$

$\Rightarrow\ f(x) = x \qquad \Rightarrow\ 2x^2 - 5x + 2 = x$

$\Rightarrow\ 2x^2 - 6x + 2 = 0 \qquad \Rightarrow\ x^2 - 3x + 1 = 0$

$\Rightarrow\ x = \dfrac{3\pm\sqrt{9-4}}{2} = \dfrac{3\pm\sqrt{5}}{2}$ but $x < 5/4$

$\Rightarrow\ x = \dfrac{3-\sqrt{5}}{2}$

TEXTUAL EXERCISE–14: (OBJECTIVE)

1. (a) f: $(-\infty, 1] \to (-\infty, 1]$

$f(x) = x(2-x)$

$\Rightarrow\ y = -x^2 + 2x \qquad \Rightarrow\ x^2 - 2x + y = 0$

$\Rightarrow\ x = \dfrac{2\pm\sqrt{4-4y}}{2} \qquad \Rightarrow\ x = 1\pm\sqrt{1-y}\,;\, x \le 1;\, y \le 1$

$\Rightarrow\ x = 1-\sqrt{1-y} \qquad \Rightarrow\ f^{-1}(x) = 1-\sqrt{1-x}$

2. (b) $f(x) = 3x - 5 \quad \Rightarrow\ f^{-1}(x) = \dfrac{x+5}{3}$

$\Rightarrow\ f(x) = \dfrac{x+5}{3} \qquad \Rightarrow\ f(4) = 3$

3. (a) $f(x) = (x+2)^2$ for $x \le -2$

$\Rightarrow\ y = (x+2)^2 \qquad \Rightarrow\ x = -2\pm\sqrt{y}$

But $x \le -2 \qquad \Rightarrow\ x = -2-\sqrt{y}$

$\Rightarrow\ f(x) = f^{-1}(x) = -2-\sqrt{x}\,;\, x \ge 0$

4. (d) $y = f(x) = \dfrac{x+2}{x+1}$

$\Rightarrow\ xy + y = x + 2 \qquad \Rightarrow\ x(y-1) = 2 - y$

$\Rightarrow\ x = \dfrac{2-y}{y-1} \qquad \Rightarrow\ f^{-1}(x) = \dfrac{2-x}{x-1}$

$\Rightarrow\ f^{-1}(4) = \dfrac{-2}{3}; f^{-1}(0) = -2$, also $f(y) + 1 = \dfrac{y+2}{y+1} + 1 = \dfrac{2y+3}{y+1}$

$\Rightarrow\ x f(y) + 1 \qquad \Rightarrow\ f'(x) = \dfrac{-1}{(x+1)^2}$

$\Rightarrow\ f(x)$ decrease for x –1

5. (c) f: $\mathbb{R} \to \mathbb{R}$; $f(x) = x - [x]$

$\Rightarrow\ f(x) = \{x\}$, which is many–one

$\Rightarrow\ f^{-1}(x)$ is not defined on $\mathbb{R}$.

6. (c) f: $(2, 4) \to (1, 3)$; $f(x) = x - \left[\dfrac{x}{2}\right]$

$\therefore\ \dfrac{x}{2} \in (1,2) \Rightarrow \left[\dfrac{x}{2}\right] = 1$

$\Rightarrow\ f(x) = x - 1 \qquad \Rightarrow\ x = y + 1$

$\Rightarrow\ f^{-1}(x) = x + 1$

7. (b), (c) $f(x)$: $[0, \infty) \to [0,)$

$f(x) = \dfrac{x}{1+x} \qquad \Rightarrow\ y + xy = x$

$\Rightarrow\ x(y-1) = -y \qquad \Rightarrow\ x = \dfrac{y}{1-y}$

$\Rightarrow\ f'(x) = \dfrac{1}{(1+x)^2} \to 0$

$\Rightarrow\ f(0) = 0, f(\infty) = 1, f(x)$ is continuous on $[0, \infty)$

$\Rightarrow\ f(x)$ has its range $[0, 1)$

$\Rightarrow\ f(x)$ is one–one but not onto, and hence, is non-invertible on $[0, \infty)$

8. (a) f: $[1, \infty) \to [2, \infty)$; $f(x) = x + \dfrac{1}{x}$

$\Rightarrow\ xy = x^2 + 1 \qquad \Rightarrow\ x^2 - xy + 1 = 0$

$\Rightarrow\ x = \dfrac{y\pm\sqrt{y^2-4}}{2}; x \ge 1$

$\Rightarrow\ f^{-1}(x) = \dfrac{x+\sqrt{x^2-4}}{2}$

9. (b) f: $[1, \infty) \to [1, \infty)$; $f(x) = 2^{x(x-1)}$

$\Rightarrow\ y = 2^{x(x-1)} \qquad \Rightarrow\ \log_2 y = x^2 - x$

$\Rightarrow\ x = \dfrac{1\pm\sqrt{1+4\log_2 y}}{2}$

But $x \ge 1 \qquad \Rightarrow\ x = \dfrac{1+\sqrt{1+4\log_2 y}}{2}$

$\Rightarrow\ f^{-1}(x) = \dfrac{1+\sqrt{1+4\log_2 x}}{2}$

10. (c) Let $f(x) = f^{-1}(x)$; $f(x) = x^x$

$\Rightarrow\ f(f(x)) = x \qquad \Rightarrow\ f'(f(x)).g'(x) = 1$

$\Rightarrow\ g'(x) = \dfrac{1}{f'(g(x))} = \dfrac{1}{(g(x))^{g(x)}(1+\log g(x))}$

$= \dfrac{1}{x(1+\log_e g(x))} \left(\begin{array}{l}\therefore (g(x))^{g(x)} \\ = f(g(x)) = x\end{array}\right)$

11. **(a)** $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} \Rightarrow y = \dfrac{e^{2x}-1}{e^{2x}+1}$

$\Rightarrow e^{2x}.y + y = e^{2x} - 1$

$\Rightarrow e^{2x}(y-1) = -y-1 \quad \Rightarrow e^{2x} = \dfrac{y+1}{1-y}$

$\Rightarrow 2x = \ell n\left(\dfrac{y+1}{1-y}\right)$

$\Rightarrow x = \dfrac{1}{2}\ell n\left(\dfrac{1+y}{1-y}\right) \quad \Rightarrow f^{-1}(x) = \dfrac{1}{2}\ell n\left(\dfrac{1+x}{1-x}\right)$

12. **(c)** $A_1 = \{1, 2, 3, 4\}; A_2 = \{5, 6, 7, 8\}; A_3 = \{9, 10\};$
$A_4 = \{11, 12, 13, 14\}$
$n_{12} = 4!, n_{13} = 0, n_{14} = 4!\ , n_{21} = 4!, n_{23} = 0, n_{24} = 4!, n_{31} = 0, n_{32} = 0, n_{34} = 0, n_{41} = 4!, n_{42} = 4!, n_{43} = 0$

$\Rightarrow \sum n_{ij} = 6\,(4!) = 6\,(24) = 144$

13. **(b)** $f(x) = \sqrt{3}\sin x - \cos x + 2$

$= 2\left(\dfrac{\sqrt{3}}{2}\sin x - \dfrac{1}{2}\cos x\right) + 2 = 2\sin\left(x - \dfrac{\pi}{6}\right) + 2$

$\Rightarrow \dfrac{y-2}{2} = \sin\left(x - \dfrac{\pi}{6}\right) \Rightarrow x - \dfrac{\pi}{6} = \sin^{-1}\left(\dfrac{y-2}{2}\right)$

$\Rightarrow f^{-1}(x) = \dfrac{\pi}{6} + \sin^{-1}\left(\dfrac{x-2}{2}\right)$

14. **(b)** $f(x) = \sqrt{\sin^{-1}(2x) + \dfrac{\pi}{6}}$ .....(1)

$\Rightarrow y^2 = \sin^{-1}(2x) + \pi/6$

$\Rightarrow \left(y^2 - \dfrac{\pi}{6}\right) = \sin^{-1}(2x)$

$\Rightarrow x = \dfrac{1}{2}\sin\left(y^2 - \dfrac{\pi}{6}\right)$

$\Rightarrow f^{-1}(x) = \dfrac{1}{2}\sin\left(x^2 - \dfrac{\pi}{6}\right) = f(x)$ (say)

From (i), $\sin^{-1}(2x)$

$\in \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right] \Rightarrow \sin^{-1}(2x) + \dfrac{\pi}{6} \in \left[\dfrac{-\pi}{3}, \dfrac{2\pi}{3}\right]$

$\Rightarrow y \in \left[0, \sqrt{\dfrac{2\pi}{3}}\right]$

$\Rightarrow$ Domain of $f^{-1}(x)$ is $\left[0, \sqrt{\dfrac{2\pi}{3}}\right]$

15. **(b)** $f(x) = \sqrt{\cos^{-1}(2x) + \dfrac{\pi}{6}}$

Range of $f(x) = f^{-1}(x)$ means domain of $f(x)$

$\Rightarrow \cos^{-1}(2x) + \dfrac{\pi}{6} \geq 0$

$\Rightarrow \cos^{-1}(2x) \geq \dfrac{-\pi}{6}$, which is true $\forall\ x \in \left[\dfrac{-1}{2}, \dfrac{1}{2}\right]$

$\Rightarrow$ Range $f(x)$ is $\left[\dfrac{-1}{2}, \dfrac{1}{2}\right]$

16. **(a)** $f(x) = (1 + b)\,x^2 + 2bx + 1$

$\Rightarrow m(b) = \dfrac{-\left[(2b)^2 - 4(1+b^2)\right]}{4(1+b^2)} = \dfrac{1}{4(1+b^2)}$

Which has principle domain $[0, \infty)$ and range $(0, 1]$

Let $y = \dfrac{1}{1+b^2} \Rightarrow 1 + b^2 = \dfrac{1}{y} \Rightarrow b^2 = \dfrac{1}{y} - 1$

$\Rightarrow b = \pm\sqrt{\dfrac{1}{y} - 1}$; $b \in [0, \infty)$

$\Rightarrow b = \sqrt{\dfrac{1}{y} - 1} \Rightarrow m^{-1}(x) = \sqrt{\dfrac{1}{x} - 1}$

17. **(c)** $f: (0, \infty) \to (1, \infty)$

$F(x) = \int_0^x f(t)dt$

$\Rightarrow F'(x) = f(x)$ ……(i)

Now $F(x^2) = x^2(1 + x)$

$\Rightarrow F(x) = x(1 + \sqrt{x})$

$\Rightarrow F'(x) = 1 + \dfrac{3}{2}\sqrt{x}$ ….(ii)

$\therefore$ From (i) and (ii), we get $f(x) = 1 + \dfrac{3}{2}\sqrt{x}$

$\Rightarrow x = \left[\dfrac{2}{3}(y-1)\right]^2 \quad \Rightarrow f^{-1}(x) = \dfrac{4}{9}(x-1)^2 = f(x)$

$\Rightarrow g(10) = 36$

18. **(c)** From above, $f(x) = f(x)$

$\Rightarrow f(x) = x$

$\Rightarrow 1 + \dfrac{3}{2}\sqrt{x} = x \quad \Rightarrow \dfrac{3}{2}\sqrt{x} = x - 1$

$\Rightarrow 2x - 3\sqrt{x} - 2 = 0$

$\Rightarrow \left(2\sqrt{x}+1\right)\left(\sqrt{x}-2\right) = 0$

$\Rightarrow \sqrt{x} = 2 \quad \Rightarrow x = 4$

$\Rightarrow$ Required points are (4, 4)

TEXTUAL EXERCISE–15: (SUBJECTIVE)

1. **(i)** Domain of $\tan x$ is symmetric about origin and $\tan(-x) = -\tan x$

$\Rightarrow$ odd function

(ii) Domain of $\cos x$ is symmetric about origin and $\cos(-x) = \cos x$

$\Rightarrow$ even function

(iii) Domain of $\sin(x^2 + 1)$ is $\mathbb{R}$ and $\sin((-x)^2 + 1) = \sin(x^2 + 1)$

$\Rightarrow$ even function

(iv) Domain of $x + x^2$ is $\mathbb{R}$ and $f(-x) = -x + x^2 f(x), -f(x)$

$\Rightarrow f(x)$ is neither even nor odd

(v) Domain of $x - x^3$ is $\mathbb{R}$ and $f(-x) = -f(x)$

$\Rightarrow$ odd function

(vi) $f(x) = x\left(\dfrac{a^x - 1}{a^x + 1}\right)$; $D_f = \mathbb{R}$ and $f(-x) = -x\left(\dfrac{a^{-x} - 1}{a^{-x} + 1}\right) = x\left(\dfrac{a^x - 1}{a^x + 1}\right) = f(x)$

$\Rightarrow$ even function

(vii) $f(x) = \log\left(x+\sqrt{x^2+1}\right)$

$\because \quad \sqrt{x^2+1} > |x| \geq \pm x \Rightarrow x+\sqrt{x^2+1} > 0$

$\Rightarrow D_f = \mathbb{R}$ and $f(-x) = \log\left(-x+\sqrt{x^2+1}\right)$

$= \log\left(\dfrac{-x+\sqrt{x^2+1}}{-x-\sqrt{x^2+1}} \times \dfrac{-x-\sqrt{x^2+1}}{1}\right)$

$= \log\left(\dfrac{x^2-x^2-1}{-x-\sqrt{x^2+1}}\right) = -\log\left(x+\sqrt{x^2+1}\right) = -f(x)$

$\Rightarrow f(x)$ is an odd function

(viii) $f(x) = \sin x + \cos x$

$f(-x) = -\sin x + \cos x\, f(x), -f(x)$

$\Rightarrow f(x)$ is neither odd nor even.

(ix) $f(x) = (x^2-1)\,|x|$

$\Rightarrow f(-x) = (x^2-1)\,|-x| = (x^2-1) = f(x)$

$\Rightarrow f(x)$ is an even function

2. (a) $f(x) = 2^{x^2-x^4} \Rightarrow D_f = \mathbb{R}$ and

$f(-x) = 2^{x^2-x^4} \Rightarrow f(x) = f(-x)$

$\Rightarrow f(x)$ is an even function.

(b) $f(x) = \sin x + \cos x;\ D_f = \mathbb{R}$

$\Rightarrow f(-x) = -\sin x + \cos x\, f(x), -f(x)$

$\Rightarrow f(x)$ is neither even nor odd .

(c) $f(x) = x^2 - |x|$

$\Rightarrow f(-x) = x^2 - |-x| = x^2 - |x| = f(x)$

$\Rightarrow f(x)$ is an even function

(d) $f(x) = \dfrac{x}{2^x-1} + \dfrac{x}{2} + 1$

$\Rightarrow f(-x) = \dfrac{-x}{2^{-x}-1} - \dfrac{x}{2} + 1 = \dfrac{x.2^x}{2^x-1} - \dfrac{x}{2} + 1$

$= x\left(\dfrac{2^x-1+1}{2^x-1}\right) - \dfrac{x}{2} + 1 = x + \dfrac{x}{2^x-1} - \dfrac{x}{2} + 1$

$= \dfrac{x}{2} + \dfrac{x}{2^x-1} + 1 = f(x)$

$\Rightarrow f(x)$ is an even function

(e) $f(x) = x^{10}.\log\left(\dfrac{1+\sin x}{1-\sin x}\right)$

$\Rightarrow f(-x) = x^{10}.\log\left(\dfrac{1-\sin x}{1+\sin x}\right) = -x^{10}\log\left(\dfrac{1+\sin x}{1-\sin x}\right) = -f(x)$

$\Rightarrow f(x)$ is an odd function.

(f) $|f(x)| + 1$; $f(x)$ is and odd function

Let $f(x) = |f(x)| + 1$

$\Rightarrow g(-x) = |f(-x)| + 1 = |-f(x)| + 1 = |f(x)| + 1 = f(x)$

$\Rightarrow$ even function

(g) $f(x) = \dfrac{\left(1+2^x\right)^9}{2^x}$

$\Rightarrow f(-x) = \dfrac{(1+2^{-x})^9}{2^{-x}} = \dfrac{\left(1+2^x\right)^9}{2^{8x}} \in f(x), -f(x)$

$\Rightarrow f(x)$ is neither odd nor even.

(h) $f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2}$

$\Rightarrow f(-x) = \sqrt{1-x+x^2} - \sqrt{1+x+x^2} = -f(x)$

$\Rightarrow$ odd function

(i) $\phi(x) = [f(x) + f(-x)].[f(x) - g(-x)]$

$\Rightarrow \phi(-x) = [(-x) + f(x)].[f(-x) - f(x)] = -\phi(x)$

$\Rightarrow$ odd function

(j) $f(x) = \dfrac{\sec x + x^2 - 9}{x \sin x}$

$f(-x) = \dfrac{\sec x + x^2 - 9}{(-x)(-\sin x)} = f(x)$

$\Rightarrow$ even function

3. $f(x) = (-1)^{[x]}$

$\Rightarrow f(-x) = (-1)^{[-x]} = \begin{cases} (-1)^{-[x]} \text{for } x \in \mathbb{Z} \\ (-1)^{-1-[x]} \text{for } x \notin \mathbb{Z} \end{cases}$

$\Rightarrow f(-x) = \begin{cases} \dfrac{1}{(-1)^{[x]}} \text{for } x \in \mathbb{Z} \\ (-1)^{-1}.(-1)^{-[x]} \text{ for } x \notin \mathbb{Z} \end{cases}$

$\Rightarrow f(x)$ is even function for $x \in \mathbb{Z}$ and odd function for $x \notin \mathbb{Z}$

$\Rightarrow f(x)$ is neither even nor odd

4. $f(x) = \begin{cases} 4; \text{if } x < -1 \\ -4x; \text{if } -1 \le x \le 0 \end{cases}$ .....(i)

$\because$ $f(x)$ is an even function, $f(-x) = f(x)$, $\forall\, x \in \mathbb{R}$ and $f(-x)$

$= \begin{cases} 4 \text{ if } x > 1 \\ -4(-x);\ \text{if } 0 \le x \le 1 \end{cases} = f(x)$ .....(ii)

$\therefore$ From (i) and (ii), we get $f(x) = \begin{cases} 4;\ x < -1 \text{ or } > 1 \\ -4x;\ -1 \le x \le 0 \\ 4x;\ 0 \le x \le 1 \end{cases}$

$\Rightarrow f(x) = \begin{cases} 4x;\ 0 < x \le 1 \\ 4;\ x > 1 \end{cases}$

5. $f(x+y) + f(x-y) = 2f(x).f(y)$

For $x = y = 0$, we have $2f(0) = 2[f(0)]^2$

$\Rightarrow f(0) = 0$ or $f(0) = 1$

For $y = 0$, we have $f(x) + f(x) = 2f(x).f(0)$

$\Rightarrow 2f(x)\,(f(0) - 1) = 0$

$\Rightarrow f(x) = 0\ \forall\, x \in \mathbb{R}$ or $f(0) = 1$ if $f(x) = 0\ \forall\, x \in \mathbb{R}$, then $f(x)$ is an even function.

If $f(0) = 1$; then $f(y) + f(-y) = 2f(y)$

$\Rightarrow f(-y) = f(y)\ \forall\, y \in \mathbb{R}$

$\Rightarrow f(x)$ is an even function. Thus $f(x)$ is an even function.

7. $f(x) = \dfrac{2x(\sin x + \tan x)}{2\left[\dfrac{x+21\pi}{\pi}\right] - 41} = \dfrac{2x(\sin x + \tan x)}{2\left[\dfrac{x}{\pi} + 21\right] - 41}$

$= \dfrac{2x(\sin x + \tan x)}{2\left[\dfrac{x}{\pi}\right] + 1}$ and $f(-x) = \dfrac{2(-x)(-\sin x - \tan x)}{2\left[\dfrac{-x}{\pi}\right] + 1}$

$= \dfrac{2x(\sin x + \tan x)}{2\left(-1 - \left[\dfrac{x}{\pi}\right]\right) + 1}$ if $x\ n\pi;\ x \in \mathbb{Z}$

$= \dfrac{2x(\sin x+\tan x)}{-1-2\left[\dfrac{x}{\pi}\right]} = -\dfrac{2x(\sin x+\tan x)}{2\left[\dfrac{x}{\pi}\right]+1} = -f(x)$ if $x\ n\pi;\ n \in \mathbb{Z}$

If $x = n\pi$; then $f(x) = f(-x) = 0$

$\Rightarrow$ $f(x)$ is an odd function .

8. $f(x) = \begin{cases} \sqrt{\cos x}\ ; x \in \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right] \\ \sqrt{1-\cos x}\ ; x \in \left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right] \end{cases}$

$\because$ Domain of $f(x)$ is not symmetric about origin.

$\Rightarrow$ $f(x)$ is neither even nor odd.

9. $f: [-20, 20] \to \mathbb{R}, f(x) = \left[\dfrac{x^2}{a}\right] \sin x + \cos x$

$\Rightarrow f(-x) = -\left[\dfrac{x^2}{a}\right]\sin x + \cos x$

$\because$ $f(x)$ is an even function.

$\Rightarrow \left[\dfrac{x^2}{a}\right] = 0\ \forall\ x \in [-20, 20]$

$\Rightarrow \left[\dfrac{x^2}{a}\right] = 0\ \forall\ x^2 \in [0, 400]$

$\Rightarrow 0 \le \dfrac{x^2}{a} < 1\ \forall\ x^2 \in [0, 400]$

$\Rightarrow a \in (400, \infty)$

10. $f(x) = \begin{cases} x|x|; 0 \le x < 1 \\ 2x\ ; x \ge 1 \end{cases}$

(i) For $f(x)$ to be even, $f(-x) = f(x)$

$\Rightarrow f(-x) = \begin{cases} -x|x|; 0 \le -x < 1 \\ -2x;\ -x \ge 1 \end{cases}$

$\Rightarrow \begin{cases} -x|x|;\ -1 < x \le 0 \\ -2x\ ;\ x \le -1 \end{cases} \quad \Rightarrow \quad f(x) = \begin{cases} -2x;\ x \le -1 \\ -x|x|; -1 < x \le 0 \\ x|x|;\ 0 \le x < 1 \\ 2x;\ x \ge 1 \end{cases}$

(ii) For $f(x)$ to be odd, $f(-x) = -f(x)$

$\Rightarrow f(x) = \begin{cases} 2x;\ x \le -1 \text{ or } x \ge 1 \\ x|x|;\ -1 < x < 1 \end{cases}$

11. $f(x) = \begin{cases} x^2+1\ ; 0 < x \le 1 \\ x+1\ ;\ 1 < x \le 2 \end{cases}$

$\Rightarrow f(-x) = \begin{cases} x^2+1\ ;\ 0 < -x \le 1 \\ -x+1\ ;\ 1 < -x \le 2 \end{cases} = \begin{cases} x^2+1\ ;\ -1 \le x < 0 \\ -x+1\ ;\ -2 \le x < -1 \end{cases}$

(i) For even extension: $f(-x) = f(x)$

$\Rightarrow f(x) = \begin{cases} -x+1;\ -2 \le x < -1 \\ x^2+1;\ -1 \le x < 0 \\ x^2+1;\ 0 < x \le 1 \\ x+1;\ 1 < x \le 2 \end{cases}$

(ii) For odd Extension: $f(-x) = -f(x)$

$\Rightarrow f(x) = \begin{cases} x-1; -2 \le x < -1 \\ -x^2-1;\ -1 \le x < 0 \\ x^2+1;\ 0 < x \le 1 \\ x+1;\ 1 < x \le 2 \end{cases}$

12. $f(x) = x^2 + x$ for x [0, 3]

$\Rightarrow f(-x) = x^2 - x$ for $-x \in [0, 3]$

$\Rightarrow f(-x) = x^2 - x$ for $x \in [-3, 0]$

(i) For even function: $f(-x) = f(x)$

$\Rightarrow f(x) = \begin{cases} x^2 - x \text{ for } x \in [-3,0] \\ x^2 + x \text{ for } x \in [0,3] \end{cases}$

(ii) For odd function: $f(-x) = -f(x)$

$\Rightarrow f(x) = \begin{cases} -x^2 + x \text{ for } x \in [-3,0] \\ x^2 + x \text{ for } x \in [0,3] \end{cases}$

13. $f(x) = \begin{cases} x^2 + \sin x;\ 0 \le x < 1 \\ -x + e^{-x};\ x \ge 1 \end{cases}$

$\Rightarrow f(-x) = \begin{cases} x^2 - \sin x\ ; 0 \le -x < 1 \\ -x + e^x\ ;\ -x \ge 1 \end{cases}$

$\Rightarrow f(-x) = \begin{cases} x^2 - \sin x; -1 < x \le 0 \\ -x + e^x\ ;\ x \le -1 \end{cases}$

(i) For even function, $f(-x) = f(x)$

$\Rightarrow f(x) = \begin{cases} -x + e^x\ ; x \le -1 \\ x^2 - \sin x\ ;\ -1 < x \le 0 \\ x^2 + \sin x;\ 0 \le x < 1 \\ x + e^{-x}\ ;\ x \ge 1 \end{cases}$

(ii) For odd function, $f(-x) = -f(x)$

$\Rightarrow f(x) = \begin{cases} x - e^x\ ; x \le -1 \\ -x^2 + \sin x : -1 < x \le 0 \\ x^2 + \sin x\ ; 0 \le x < 1 \\ x + e^{-x}\ ;\ x \ge 1 \end{cases}$

TEXTUAL EXERCISE–15: (OBJECTIVE)

1. (i) $f(x) = \dfrac{1-x}{1+x} \Rightarrow f(-x) = \dfrac{1+x}{1-x} \ne f(x), -f(x)$

$\Rightarrow$ Neither odd nor even

(ii) $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$\Rightarrow f(-x) = \dfrac{e^{-x} - e^x}{e^{-x} + e^x} = -\left(\dfrac{e^x - e^{-x}}{e^x + e^{-x}}\right) = -f(x)$

$\Rightarrow$ $f(x)$ is an odd function.

(iii) $f(x) = \sin(\sin x)$

$\Rightarrow f(-x) = \sin(\sin(-x)) = \sin(-\sin x) = -\sin(\sin x) = -f(x)$

$\Rightarrow$ $f(x)$ is an odd function.

(iv) $f(x) = \sin(\cos x) \quad \Rightarrow f(-x) = \sin(\cos(-x)) = f(x)$

$\Rightarrow$ $f(x)$ is an even function.

(v) $f(x) = \cos(\cos x)$

$\Rightarrow f(-x) = \cos(\cos(-x)) = \cos(\cos x) = f(x)$

$\Rightarrow f(x)$ is an even function.

(vi) $f(x) = \dfrac{\sin^4 x + \cos^4 x}{x + x^2 \tan x}$

$\Rightarrow f(-x) = \dfrac{\sin^4 x + \cos^4 x}{-x - x^2 \tan x} = -f(x)$

$\Rightarrow f(x)$ is an odd function.

(vii) $f(x) = x^{17} = \left(\dfrac{a^{\sin x} - 1}{a^{\sin x} + 1}\right)$

$\Rightarrow f(-x) = -x^{17}\left(\dfrac{a^{-\sin x} - 1}{a^{-\sin x} + 1}\right) = x^{17}\left(\dfrac{a^{\sin x} - 1}{a^{\sin x} + 1}\right) = f(x)$

$\Rightarrow f(x)$ is an even function.

2. (i) $f(x) = x^2 \sin\left\{\dfrac{1}{\sin\left(\dfrac{1}{x}\right)}\right\}$

$\Rightarrow f(-x) = x^2 \sin\left(\dfrac{1}{-\sin\left(\dfrac{1}{x}\right)}\right)$

$\Rightarrow f(-x) = -x^2 \sin\left\{\dfrac{1}{\sin\left(\dfrac{1}{x}\right)}\right\} = -f(x)$

$\Rightarrow f(x)$ is an odd function.

(ii) $f(x) = \cos x.\log\left(\dfrac{1 + \tan x}{1 - \tan x}\right)$

$\Rightarrow f(-x) = \cos x.\log\left(\dfrac{1 - \tan x}{1 + \tan x}\right) = -\cos x.\log\left(\dfrac{1 + \tan x}{1 - \tan x}\right)$

$= -f(x)$

$\Rightarrow$ odd function.

(iii) $f(x) = \left(\dfrac{a^{2x} + 1}{a^x}\right) \Rightarrow f(-x) = \dfrac{a^{-2x} + 1}{a^{-x}} = \dfrac{1 + a^{2x}}{a^x} = f(x)$

$\Rightarrow$ even function.

(iv) $f(x) = \dfrac{(1 + e^x)^2}{e^x}$

$\Rightarrow f(-x) = \dfrac{(1 + e^{-x})^2}{e^{-x}} = \dfrac{(1 + e^x)^2}{e^x} = f(x)$

$\Rightarrow$ even function.

(v) $f(x) = \sqrt[3]{1 + x + x^2} + \sqrt[3]{1 - x + x^2}$

$\Rightarrow f(-x) = \sqrt[3]{1 - x + x^2} + \sqrt[3]{1 + x + x^2} = f(x)$

$\Rightarrow$ even function.

(vi) $f(x) = \sqrt{1 + 2x + 3x^2} - \sqrt{1 - 2x + 3x^2}$

$\Rightarrow f(-x) = \sqrt{1 - 2x + 3x^2} - \sqrt{1 + 2x + 3x^2} = -f(x)$

$\Rightarrow$ odd function

3. (a), (c) (a) $f(x) = \ln|x|$

$\Rightarrow f(-x) = \ln|-x| = \ln|x| = f(x)$

$\Rightarrow$ even function.

(b) $f(x) = \sin^{-1}\left(\dfrac{2x}{1 + x^2}\right) \Rightarrow f(-x) = \sin^{-1}\left(\dfrac{-2x}{1 + x^2}\right) = -\sin^{-1}\left(\dfrac{2x}{1 + x^2}\right)$

$\Rightarrow$ odd function

(c) $f(x) = x\left(\dfrac{a^x - 1}{a^x + 1}\right)$

$\Rightarrow f(-x) = -x\left(\dfrac{a^{-x} - 1}{a^{-x} + 1}\right) = x\left(\dfrac{a^x - 1}{a^x + 1}\right)$

$\Rightarrow$ even function.

(d) $f(x) = \sin\left[\log\left(x + \sqrt{x^2 + 1}\right)\right]$

$\Rightarrow f(-x) = \sin\left[\log\left(-x + \sqrt{x\ \ + 1}\right)\right]$

$= \sin\left[\log\left(\dfrac{-1}{-x - \sqrt{x^2 + 1}}\right)\right] = -\sin\left[\log\left(x + \sqrt{x^2 + 1}\right)\right]$

$= -f(x)$

$\Rightarrow$ odd function.

4. (d) (a) $f(x) = \operatorname{sgn} x + x^{2000}$

$\Rightarrow f(-x) = \operatorname{sgn}(-x) + (-x)^{2000} = -\operatorname{sgn} x + x^{2000}$

$\Rightarrow$ Neither even nor odd.

(b) $f(x) = |x| - \tan x \quad \Rightarrow f(-x) = |x| + \tan x\, f(x), -f(x)$

$\Rightarrow$ Neither even nor odd.

(c) $f(x) = x^3 \cot x \quad \Rightarrow f(-x) = x^3 \cot x = f(x)$

$\Rightarrow$ even function

(d) $f(x) = \operatorname{cosec} x^{55}$

$\Rightarrow f(-x) = \operatorname{cosec}(-x)^{55} = -\operatorname{cosec} x^{55}$

$\Rightarrow$ odd function.

5. (b) $f(x) = \sec\left[\log\left(x + \sqrt{1 + x^2}\right)\right]$

$\Rightarrow f(-x) = \sec\left[\log\left(-x + \sqrt{1 + x^2}\right)\right]$

$= \sec\left[\log\left(\dfrac{-1}{-x - \sqrt{1 + x^2}}\right)\right]$

$= \sec\left[\log\left(x + \sqrt{1 + x^2}\right)\right] = f(x)$

$\Rightarrow$ even function.

6. (b) $f(x) = \sin\left(\tan\left(\log\left(x + \sqrt{x^2 + 1}\right)\right)\right)$

$\Rightarrow f(-x) = \sin\left(\tan\left(\log\left(-x + \sqrt{x^2 + 1}\right)\right)\right)$

$\Rightarrow f(-x) = \sin\left(\tan\left(\log\left(x + \sqrt{x^2 + 1}\right)^{-1}\right)\right) = -\sin$

$\left(\tan\left(\log\left(x + \sqrt{x^2 + 1}\right)\right)\right) = -f(x)\, f(x)$ is an odd function

7. **(a)** $f(x) = \begin{cases} 0 \; ; \text{for } x = 0 \\ x^2 \sin\left(\dfrac{\pi}{x}\right); \text{for} -1 < x < 1 (x \neq 0) \\ x|x|; \text{for } x > 1 \text{ or } x < -1 \end{cases}$

For $x < -1$; $f(x) = x|x| = -x^2$, $f(-x) = (-x)|-x| = (-x)(-x) = x^2$

For $-1 < x < 1$; $f(x) = x^2 \sin\left(\dfrac{\pi}{x}\right)$, $f(-x) = x^2 \sin\left(\dfrac{-\pi}{x}\right) = -x^2 \sin \dfrac{\pi}{x}$

For $x > 1$; $f(x) = x|x| = x^2$, $f(-x) = (-x)|-x| = (-x)(x) = -x^2$

Clearly, $f(-x) = -f(x) \; \forall \; x \in \mathbb{R}$.

$\Rightarrow$ odd function.

8. **(b)** $f(x) = 2x^6 + 3x^4 + 4x^2 + 7$, $f'(x) = 12x^5 + 12x^3 + 8x$

$\Rightarrow f'(-x) = 12(-x)^5 + 12(-x)^3 + 8(-x)$

$\Rightarrow -(12x^5 + 12x^3 + 8x) = -f'(x)$

$\Rightarrow f'(x)$ is an odd function.

9. **(b), (c)** $f(x) = \begin{cases} x \; ; \text{if } x \in \mathbb{Q} \\ 1 - x \; ; \text{if } x \notin \mathbb{Q} \end{cases}$

$\Rightarrow f(f(x)) = \begin{cases} f(x) \text{if } f(x) \in \mathbb{Q} \\ 1 - f(x) \text{if } f(x) \notin \mathbb{Q} \end{cases} = \begin{cases} x; \text{if } x \in \mathbb{Q} \\ 1 - (1 - x) \; ; \text{if } x \notin \mathbb{Q} \end{cases}$

$\Rightarrow f(f(x)) = x \; \forall \; x \in \mathbb{R}$

$\Rightarrow f(f(x))$ is an odd function and is an identity function and clearly a linear polynomial.

10. **(c)** $f(x) = \begin{cases} 0 \; ; \text{if } x \text{ is rational} \\ x; \text{if } x \text{ is irrational} \end{cases}$ and $f(x) = \begin{cases} 0 \; ; \text{ if } x \text{ is irrational} \\ x \; ; \text{if } x \text{ is rational} \end{cases}$

$\Rightarrow (f - g)(x) = f(x) - f(x) = \begin{cases} 0 - x \text{ if } x \in \mathbb{Q} \\ x - 0 \text{ if } x \notin \mathbb{Q} \end{cases} = \begin{cases} -x \text{ if } x \in \mathbb{Q} \\ x \text{ if } x \notin \mathbb{Q} \end{cases}$

$\because$ If $x \in \mathbb{Q}$, then $-x \in \mathbb{Q}$

$\Rightarrow f(-x) = -(-x) = x = f(x)$ and if $x \notin \mathbb{Q}$, then $-x \notin \mathbb{Q}$

$\Rightarrow f(-x) = -x = -f(x)$

$\Rightarrow (f - g)(x)$ is neither even nor odd.

11. **(a), (c)** **(a)** Let $f(x) = \dfrac{f(-x) + f(x)}{2} = \dfrac{-f(x) + f(x)}{2} = 0$ as $f(x)$ is an function

$\Rightarrow f(x)$ is an even function

(b) $f(x) = \dfrac{f(x) - f(-x)}{2} = \dfrac{f(x) + f(x)}{2}$

$\Rightarrow f(x) = f(x) \qquad \Rightarrow$ an odd function.

(c) $h(x) = [|f(x)| + 1]$

$\Rightarrow h(-x) = [|f(-x)| + 1] = [|-f(x)| + 1] = [|f(x)| + 1] = h(x)$

$\Rightarrow$ an even function

12. **(a)** $f(x) = a^{\sin x^3 \operatorname{sgn} x^9} (\tan^3 x)$; a < 1

$\Rightarrow f(-x) = a^{(-\sin x^3)(-\operatorname{sgn} x^9)} \cdot \tan^3 x$

$\Rightarrow f(-x) = -a^{(\sin x^3)(\operatorname{sgn} x^9)} \cdot \tan^3 x = -f(x)$

$\Rightarrow f(x)$ is an odd function.

13. **(b)** $f(x) = (a - x^n)^{1/n}$; $n \in \mathbb{N}$,

$f(f(x)) = (a - (f(x))^n)^{1/n}$; $n \in \mathbb{N}$

$= (a - (a - x^n))^{1/n}$; $n \in \mathbb{N}$

$= x \; \forall \; x \in \mathbb{R}, n \in \mathbb{N}$, which is an odd function.

14. **(b)** $f(-x) = f(x) \qquad (\because f(x)$ is even)

$\Rightarrow \; -f'(-x) = f'(x)$

$\Rightarrow f'(x)$ is an odd function.

$\Rightarrow f'(e) + f'(-e) = f'(e) = -f'(e) = 0.$

15. **(b)** $f(x)$ is not identically zero and $f(x + y) = f(x) + f(y)$ (given)

Replacing y by 0, we get $f(x) = f(x) + f(0)$

$\Rightarrow f(0) = 0$

Again, replacing y by $-x$, we get $f(x - x) = f(x) + f(-x)$

$\Rightarrow \; 0 = f(x) + f(-x)$

$\Rightarrow f(-x) = -f(x) \qquad \Rightarrow f(x)$ and odd function.

$\therefore \; f(-x) = a^{f(-x)} + a^{-f(-x)} = a^{-f(x)} + a^{f(x)} = f(x)$; $a > 1$

$\Rightarrow f(x)$ is an even function.

16. **(c)** $f(f(x)) = |\sin x|$; $f(f(x)) = \left(\sin\sqrt{x}\right)^2$

Clearly, $f(x) = \sqrt{x}$, $f(x) = \sin^2 x$

Domain of $f(x)$ is $[0, \infty)$, which is not symmetric about origin.

$\Rightarrow f(x)$ is neither even nor odd and $f(-x) = f(x) = \sin^2 x$

$\Rightarrow f(x)$ is even.

17. **(c)** $f(x) = x^3 - x^2 + \sin^{-1}\left(\dfrac{2 - |x|}{3}\right) + \log(|x| + 1)$, $0 \le x \le 2$

$f(-x) = -x^3 - x^2 + \sin^{-1}\left(\dfrac{2 - |x|}{3}\right) + \log(|x| + 1)$; $0 \le -x \le 2$

$\Rightarrow f(-x) = -x^3 - x^2 + \sin^{-1}\left(\dfrac{2 - |x|}{3}\right) + \log(|x| + 1)$; $-2 \le x \le 0$\

For odd extension $f(-x) = -f(x)$, i.e., $f(x) = -f(-x)$

$$\therefore \quad f(x) = \begin{cases} x^3 + x^2 - \sin^{-1}\left(\dfrac{2 - |x|}{3}\right) + \log(|x| + 1); -2 \le x \le 0 \\ x^3 - x^2 + \sin^{-1}\left(\dfrac{2 - |x|}{3}\right) + \log(|x| + 1); 0 \le x \le 2 \end{cases}$$

18. **(a)** $f(x) = \sin^2 x + \sin^2\left(x + \dfrac{\pi}{3}\right) + \cos x \cdot \cos\left(x + \dfrac{\pi}{3}\right)$

$$= \frac{1 - \cos 2x}{2} + \frac{1 - \cos\left(2x + \dfrac{2\pi}{3}\right)}{2} + \frac{1}{2}\left[\cos\left(2x + \frac{\pi}{3}\right) + \cos\frac{\pi}{3}\right]$$

$$= \frac{5}{4} - \frac{1}{2}\cos 2x - \frac{1}{2}\cos\left(2x + \frac{2\pi}{3}\right) + \frac{1}{2}\cos\left(2x + \frac{\pi}{3}\right)$$

$$= \frac{5}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\cos 2x + (\sin 2x)\frac{\sqrt{3}}{4}$$

$$+ \frac{1}{4}\cos 2x - \frac{\sqrt{3}}{4}\sin 2x = \frac{5}{4}$$

$\Rightarrow f(x) = \dfrac{5}{4} \; \forall \; x \in \mathbb{R} \qquad \Rightarrow$ even function

TEXTUAL EXERCISE-16: (SUBJECTIVE)

1. **(i)** $f(x) = 2 + 3\cos(x - 2)$

Fundamental period $= 2\pi$ as it is same for $f(x)$ and $(x \pm k)$ and $\alpha \pm f(x \pm k)$; $\alpha, k \in \mathbb{R}$.

(ii) $f(x) = \cos\frac{3}{5}x - \sin\frac{2}{7}x$

Fundamental period of $\cos\frac{3}{5}x = \frac{2\pi}{\frac{3}{5}} = \frac{10\pi}{3}$

Fundamental period of $\sin\frac{2}{7}x = \frac{2\pi}{\frac{2}{7}} = 7\pi$

$\therefore$ Period $\left(\cos\frac{3}{5}x - \sin\frac{2}{7}x\right)$ = L.C.M $\left(\frac{10\pi}{3}, 7\pi\right)$ = L.C.M $\left(\frac{10}{3}, 7\right)\pi = \frac{70}{1}\pi$

= 70 π which is also fundamental period.

(iii) $f(x) = \sin\frac{\pi x}{4} + \sin\frac{\pi x}{3}$

Fundamental period of $\sin\frac{\pi x}{4} = \frac{2\pi}{\frac{\pi}{4}} = 8$

Fundamental period of $\sin\left(\frac{\pi x}{3}\right) = \frac{2\pi}{\frac{\pi}{3}} = 6$

Fundamental period of $\sin\left(\frac{\pi x}{4}\right) + \sin\left(\frac{\pi x}{3}\right)$ = L.C.M (8, 6) = 24

(iv) $f(x) = \frac{1}{1-\cos x}$

$\Rightarrow f(x) = \frac{1}{2\sin^2\frac{x}{2}} = \frac{1}{2}\text{cosec}^2 x\frac{x}{2} = \frac{1}{2}\left(1+\cot^2\frac{x}{2}\right)$

Which has its fundamental period = $\frac{\pi}{\frac{1}{2}} = 2\pi$

(v) $f(x) = \sin(2x + \cos x)$

$\because$ $f(2\pi + x) = \sin[2(2\pi + x)) + \cos(2\pi + x)]$

$= \sin(4\pi + 2x + \cos x) = \sin(2x + \cos x)$

$\Rightarrow$ $f(x)$ has fundamental period 2π.

2. **(i)** $f(x) = \sin 3x + \cos x + |\tan x|$

Fundamental period = L.C.M$\left(\frac{2\pi}{3}, \pi, \pi\right) = 2\pi$

(ii) $f(x) = [\sin 3x] + |\cos 6x|$

Fundamental period of $[\sin 3x]$ = $\frac{2\pi}{3}$ and fundamental period of $|\cos 6x| = \pi/6$

Fundamental period of $[\sin 3x] + |\cos 6x|$

= L.C.M $\left(\frac{2\pi}{3}, \pi\right) = 2\pi$

(iii) $f(x) = \frac{\sin 12x}{1+\cos^2 6x} = \frac{\sin 12x}{1+\frac{1}{2}(1+\cos 12x)}$

$\Rightarrow f(x) = \frac{2\sin 12x}{3+\cos 12x}$ having fundamental period $\frac{2\pi}{12} = \frac{\pi}{6}$

(iv) $f(x) = \sec^3 x + \text{cosec}^3 x$

which has fundamental period 2π.

3. **(i)** $f(x) = \frac{\sin x + \sin 3x}{\cos x + \cos 3x} = \frac{2\sin 2x\cos x}{2\cos 2x\cos x} = \tan 2x$ having fundamental period $\frac{\pi}{2}$.

(ii) $f(x) = 1 - \frac{\sin^2 x}{1+\cot x} - \frac{\cos^2 x}{1+\tan x}$

$= 1 - \frac{\sin^3 x}{\sin x + \cos x} - \frac{\cos^3 x}{\cos x + \sin x}$

$= 1 - \frac{(\sin x + \cos x)(1 + \sin x\cos x)}{(\sin x + \cos x)}$

$= 1 - 1 - \sin x \cos x = -\frac{1}{2}\sin 2x$ having fundamental period = $\frac{2\pi}{2} = \pi$

(iii) $f(x) = \tan\frac{\pi}{2}[x]$ $\quad\because$ $f(x+2) = \tan\frac{\pi}{2}[x+2]$

$= \tan\frac{\pi}{2}(2+[x]) = \tan\left(\pi + \frac{\pi}{2}[x]\right) = \tan\frac{\pi}{2}[x] = f(x)$

$\Rightarrow$ $f(x)$ is periodic with period 2.

(iv) $f(x) = \ln(2 + \cos 3x) = \ln\left(1 + 2\cos^2\frac{3}{2}x\right)$

which is periodic with period $\frac{2\pi}{3}$

(v) $f(x) = e^{\ln \sin x} + \tan^3 x - \text{cosec}(3x - 5)$

$= \sin x + \tan^3 x - \text{cosec}(3x - 5)$

$\Rightarrow$ Period of $f(x)$ = L.C.M $\left(2\pi, \pi, \frac{2\pi}{3}\right) = 2\pi$

(vi) $f(x) = \sin x + \tan\frac{x}{2} + \sin\frac{x}{2^2} + \frac{x}{2^3} + \ldots\ldots$

$+ \sin\frac{x}{2^{n-1}} + \tan\frac{x}{2^n}$

Period of $f(x)$ = L.C.M. $\{2\pi, 2\pi, 8\pi, 8\pi, \ldots\ldots, 2^n\pi, 2^n\pi$

= L.C.M. $\{(2, 2^3, \ldots .2^n)\}.\pi = 2^n\pi$

4. **(i)** $f(x) = \cos\frac{4x}{5} + \sin\frac{4x}{3} + \tan\frac{8x}{3}$

$\Rightarrow$ Period = L.C.M $\left(\frac{2\pi}{\frac{4}{5}}, \frac{2\pi}{\frac{4}{3}}, \frac{\pi}{\frac{8}{3}}\right)$

$\Rightarrow$ L.C.M. $\left(\frac{5\pi}{2}, \frac{3\pi}{2}, \frac{3\pi}{8}\right)$

$\Rightarrow$ L.C.M. $\left(\frac{5}{2}, \frac{3}{2}, \frac{3}{8}\right)\pi = \frac{15}{2}\pi$

(ii) $f(x) = \frac{(1+\sin x)(1+\sec x)}{(1+\cos x)(1+\text{cosec} x)} = \frac{\sin x}{\cos x}$ for $\sin x \neq -1, 0$ and $\cos x \neq -1, 0$

$= \tan x,\ x \neq \frac{n\pi}{2}, n \in \mathbb{Z}$

⇒ Period of $f(x)$ = L.C.M. $\left(\frac{\pi}{2}, \pi\right) = \pi$. Hence, $\pi/2$ is the period of discontinuity.

(iii) $f(x) = \sin\left(2\pi x + \frac{\pi}{3}\right) + 2\sin\left(3\pi x + \frac{\pi}{4}\right) + 3\sin(5\pi x)$

Period of $f(x)$ = L.C.M $\left(\frac{2\pi}{2\pi}, \frac{2\pi}{3\pi}, \frac{2\pi}{5\pi}\right)$

= L.C.M $\left(1, \frac{2}{3}, \frac{2}{5}\right)$

$= \frac{\text{L.C.M.}(1,2,2)}{\text{L.C.F.}(1,3,5)} = \frac{2}{1} = 2$

5. (i) $f(x) = \left|\cos^5\left(\frac{2x}{3}\right)\right|$ ∵ Period of $|\cos^5 x| = \pi$

⇒ Period of $\left|\cos^5\left(\frac{2x}{3}\right)\right| = \frac{\pi}{\frac{2}{3}} = \frac{3\pi}{2}$

(ii) $f(x) = \sin(\pi x) + [x] = 2 - x$

$= \sin(\pi x) + x - \{x\} + 2 - x$

$= \sin(\pi x) + 2 - \{x\}$

⇒ Period of $f(x)$ = L.C.M. $\left(\frac{2\pi}{\pi}, 1\right)$ = L.C.M (2, 1) = 2

(iii) $f(x) = x - [x - b] = x - (x - b - \{x - b\}) = b + \{x - b\}$

⇒ Period of $f(x) = 1$

6. $f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational} \end{cases}$

Let if possible $T > 0$ be the fundamental period $f(x)$

⇒ $f(x + T) = f(x)\ \forall\ x \in \mathbb{R}$

If $x \in \mathbb{Q}$ and $T \in \mathbb{Q}$, then $f(x + T) = f(x) = 1$

Also $T/2 \in \mathbb{Q}$

⇒ $f(x + T/2) = f(x) = 1$

Similarly $f(x + T/2^n) = f(x) = 1\ \forall\ x \in \mathbb{N}$

⇒ There is no fundamental period of $f(x)$. Hence, $f(x)$ is a periodic function period.

7. (i) $f(x) = \frac{1}{2}\left\{\frac{|\sin x|}{\cos x} + \frac{|\cos x|}{\sin x}\right\}$

$= \frac{1}{2}\left\{\frac{\sin x|\sin x| + \cos x|\cos x|}{\sin x \cos x}\right\}$

$= \frac{1}{2}\begin{cases} \sec x . \text{cosec}\, x;\ x \in \left[0, \frac{\pi}{2}\right) \\ \frac{\sin^2 x - \cos^2 x}{\sin x . \cos x};\ x \in \left(\frac{\pi}{2}, \pi\right) \\ -\sec x . \text{cosec}\, x;\ x \in \left(\pi, \frac{3\pi}{2}\right) \\ \frac{\cos^2 x - \sin^2 x}{\sin x . \cos x};\ x \in \left(\frac{3\pi}{2}, 2\pi\right) \end{cases}$

$= \begin{cases} \text{cosec}\, 2x;\ x \in \left(0, \frac{\pi}{2}\right) \\ -\cot 2x;\ x \in \left(\frac{\pi}{2}, \pi\right) \\ -\text{cosec}\, 2x;\ x \in \left(\pi, \frac{3\pi}{2}\right) \\ \cot 2x;\ x \in \left(\frac{3\pi}{2}, 2\pi\right) \end{cases}$

⇒ Period of $f(x)$ = L.C.M. $\left\{\pi, \frac{\pi}{2}, 2\pi\right\} = 2\pi$

(ii) $f(x) = \sin\frac{\pi x}{n!} - \cos\frac{\pi x}{(n+1)!}$

Period of $f(x)$ = L. C. $M\left\{\frac{2\pi}{\frac{\pi}{n!}}, \frac{2\pi}{\frac{\pi}{(n+1)!}}\right\}$

= L.C.M $\{2n!, 2(n + 1)!\} = 2((n + 1)!)$

8. (a) $f(x + \lambda) = -f(x)$

⇒ $f(x) = -f(x + \lambda) = -[-f(x + 2\lambda)]$

⇒ $f(x) = f(x + 2\lambda)$

⇒ $f(x)$ is periodic with period 2λ.

(b) $f(x + \lambda) = \pm\frac{1}{f(x)}$

⇒ $f(x) = \pm\frac{1}{f(x+\lambda)} = \frac{\pm 1}{\left[\pm\frac{1}{f(x+2\lambda)}\right]}$

⇒ $f(x) = f(x + 2\lambda)$

⇒ $f(x)$ is periodic with period 2λ.

9. Given, $f(x + a) = \frac{1+f(x)}{1-f(x)}\ \forall x \in \mathbb{R}$

⇒ $\frac{f(x+a)+1}{f(x+a)-1} = \frac{2}{2f(x)}$

⇒ $f(x) = \frac{f(x+a)-1}{f(x+a)+1}$

⇒ $f(x) = \frac{\frac{f(x+2a)-1}{f(x+2a)+1} - 1}{\frac{f(x+2a)-1}{f(x+2a)+1} + 1} = \frac{-2}{2f(x+2a)} = \frac{-1}{f(x+2a)}$

⇒ $f(x) = \frac{-1}{\left(\frac{-1}{f(x+4a)}\right)} = f(x + 4a)$

⇒ $f(x)$ is periodic with period '$4a$'.

10. $f(x) = \sin x + \cos\left(\sqrt{4-a^2}\right)x$, Period of $\sin x = 2\pi$

Period of $\cos\left(\sqrt{4-a^2}\right)x = \frac{2\pi}{\sqrt{4-a^2}}$

$\Rightarrow$ Period of $f(x)$ = L.C.M. $\left(2\pi, \dfrac{2\pi}{\sqrt{4-a^2}}\right) = 2\pi$

$\Rightarrow$ L.C.M $\left(1, \dfrac{1}{\sqrt{4-a^2}}\right)$, which exists if $\sqrt{4-a^2}$ is equal

$\Rightarrow$ $4 - a^2$ is a perfect square

$\Rightarrow$ $4 - a^2 = k^2$; $k \in \mathbb{Q}$

$\Rightarrow$ $a = \pm\sqrt{4-k^2}$ for a $\in \mathbb{Z}$, $4 - k^2 \geq 0$ and must be a perfect square.

$\Rightarrow$ $k^2 = 0$ or 4 $\quad\Rightarrow$ $k = 0, \pm 2$

$\Rightarrow$ $a = 0, \pm 2$ $\quad\Rightarrow$ $a \in \{-2, 0, 2\}$

TEXTUAL EXERCISE–16: (OBJECTIVE)

1. **(d)** $f(x) = \sin(2\pi x) + \left(\dfrac{\pi x}{3}\right) + \sin\left(\dfrac{\pi x}{5}\right)$

$\Rightarrow$ Period of $f(x)$ = L.C.M. $\left\{1, \dfrac{2\pi}{\frac{\pi}{3}}, \dfrac{2\pi}{\frac{\pi}{5}}\right\}$

= L.C.M $\{1, 6, 10\} = 30$

2. **(a), (b), (c)** $f(x) = \text{sgn}\, x.\sin x$

$$= \begin{cases} -\sin x & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ \sin x & \text{for } x > 0 \end{cases}$$

$\Rightarrow$ $f(x + 2\pi) = -f(x)$ for $x \in (-\infty, -2\pi]$ and for $x \in [0, \infty)$ and $f(x + 2\pi) = -f(x)$ for $x \in (-2\pi, 0)$

$\Rightarrow$ $f(x)$ is non-periodic but clearly $f(x)$ is a continuous function.

Also $f'(x) = \begin{cases} -\cos x & \text{for } x < 0 \\ \cos x & \text{for } x > 0 \end{cases}$

$\Rightarrow$ $f'(0^-) = -1, f'(0^+) = 1$

$\Rightarrow$ $f(x)$ is non-differentiable at $x = 0$.

Also $f(-x) = -\sin(-x)$ for $x > 0$

$= \sin x = f(x)$

And $f(-x) = \sin(-x)$ for $x < 0$

$= -\sin x = f(x)$

And $f(-0) = f(0) = 0$

$\Rightarrow$ $f(x)$ is an even function.

3. **(a), (c), (d)**

(a) Period of $f(x) = |\sin x|$ is π

(b) $f(x) = [x + \pi]$; which being an increasing function is non-periodic

(c) $f(x) = \cos(\sin x)$

$\Rightarrow$ $f(x + \pi) = \cos(\sin(x + \pi)) = \cos(-\sin x) = \cos(\sin x)$

$\Rightarrow$ $f(x)$ is periodic with period π.

(d) $f(x) = \cos^2 x$ is periodic with period π.

4. **(b)** $f(x) = (nx + n) - [nx + n]$

$= nx + n - [nx] - n$

$= nx - [nx] = \{nx\}$

which is periodic with period $\dfrac{1}{n}$

5. **(c)** $f(x) = \sin(x + 3 - [x + 3]) = \sin(x + 3 - [x] - 3) = \sin(\{x\})$

$\Rightarrow$ $f(x + 1) = \sin(\{x + 1\}) = \sin(\{x\}) = f(x)$

$\Rightarrow$ $f(x)$ is periodic with period 1

6. **(c)** $f(x) = |\sin^3 2x| + |\cos^3 2x| = |\sin 2x|^3 + |\cos 2x|^3$

Fundamental period of $|\sin x|^3 + |\cos x|^3$ is $\dfrac{\pi}{2}$

Fundamental period of $|\sin 2x|^3 + |\cos 2x|^3$ is $\dfrac{\pi}{4}$

Verification:- f

$$f\left(x + \frac{\pi}{4}\right) = \left|\sin 2\left(x + \frac{\pi}{4}\right)\right|^3 + \left|\cos 2\left(x + \frac{\pi}{4}\right)\right|^3$$

$$= \left|\sin\left(2x + \frac{\pi}{2}\right)\right|^3 + \left|\cos\left(2x + \frac{\pi}{2}\right)\right|^3$$

$$= |\cos 2x|^3 + |\sin 2x|^3 = f(x)$$

7. **(c)** $f(x) = \dfrac{|\sin x| + |\cos x|}{|\sin x - \cos x|}$

Clearly, $f\left(x + \dfrac{\pi}{2}\right) = \dfrac{|\cos x| + |\sin x|}{|\cos x + \sin x|} \neq f(x)$ and $f(x + \pi)$

$= \dfrac{|-\sin x| + |-\cos x|}{|-\sin x + \cos x|} = \dfrac{|\sin x| + |\cos x|}{|\sin x - \cos x|} = f(x)$

$\Rightarrow$ $f(x)$ is periodic with period π

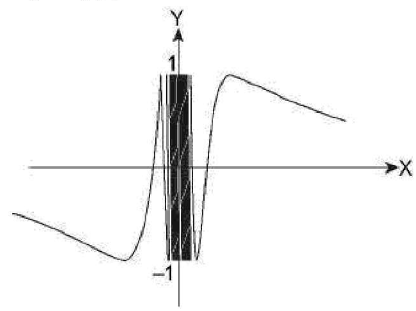
8. **(b)** Functions given in options (a), (c) and (d) being the sum of periodic functions are periodic .

Also $\cos\sqrt{x}$ being non-periodic function, $f(x) = \cos\sqrt{x} + \cos^2 x$ is non-periodic.

9. **(a), (d)** **(a)** $f(x) = x - [x] = \{x\}$ which is periodic with period 1.

(b) $f(x) = \begin{cases} \sin\dfrac{1}{x}; & x \neq 0 \\ 0\ ; & x = 0 \end{cases}$

Graph of $f(x)$ is shown below.



Clearly $f(x)$ is non-periodic as $x \to 0$, $\dfrac{1}{x} \to \infty$ and $\sin\dfrac{1}{x}$ oscillates in between -1 and 1 with increasing frequency.

(c) $h(x) = x\cos x$ is non-periodic.

(d) $w(x) = \sin^{-1}(\sin x)$ is periodic with period 2π.

10. **(c)** If $f(x) = e^{\sin (x-[x]\cos \pi x}$

$\Rightarrow f(x) = e^{\sin\{x\}.\cos\pi x}$

sin $\{x\}$ has period 1 and cos πx has period 2

$\Rightarrow$ Period of $f(x)$ = L.C.M.(1, 2) = 2

Also $f(x+2) = e^{\sin\{x+2\}.\cos\pi(x+2)} = e^{\sin\{x\}.\cos\pi x} = f(x)$

11. **(a)** $f(x) = 2\tan 3x + 5\sqrt{1-\cos 6x}$

$\Rightarrow f(x) = 2\tan 3x + 5\sqrt{2\sin^2 3x}$

$\Rightarrow f(x) = 2\tan 3x + 5\sqrt{2}|\sin 3x|$

$\Rightarrow$ Period of $f(x)$ = L.C.M $\left(\frac{\pi}{3},\frac{\pi}{3}\right) = \frac{\pi}{3}$

(a) $f(x) = (\sec^2 3x + \text{cosec}^2 3x)\tan^2 3x$

$= \left(\frac{1}{\cos^2 3x\sin^2 3x}\right).\tan^2 3x$

$= \frac{1}{\cos^4 3x} = \sec^4 3x$ having period $\frac{\pi}{3}$

(b) $h(x) = 2\sin 3x + 3\cos 3x$

$\Rightarrow$ Period of $h(x) = \frac{2\pi}{3}$

(c) $k(x) = 2\sqrt{1-\cos^2 3x} + \text{cosec } 3x$

$= 2\sqrt{\sin^2 3x} + \text{cosec } 3x$

$= 2|\sin 3x| + \text{cosec } 3x$

$\Rightarrow$ Period of $k(x)$ = L.C.M $\left(\frac{\pi}{3},\frac{2\pi}{3}\right) = \frac{2\pi}{3}$

(d) $p(x) = 3\text{ cosec } 3x + 2\tan 3x$

$\Rightarrow$ Period of $p(x)$ = L.C.M. $\left(\frac{2\pi}{3},\frac{\pi}{3}\right) = \frac{2\pi}{3}$

12. **(c)** $n \in \mathbb{N}, f(x) = \frac{\cos nx}{\sin\left(\frac{x}{n}\right)}$

$\Rightarrow$ Period of $f(x)$ = L.C.M. $\left(\text{—}, 2\quad\right) = 2\pi$

L.C.M. $\left(\frac{1}{n}, n\right) = 2n\pi = 4\pi$ (given)

$\Rightarrow n = 2$

13. **(b)** **(a)** Let $f(x)$ be periodic and differentiable with period T.

$\Rightarrow f(x+T) = f(x)\ \forall x \in \mathbb{R}$

$\Rightarrow f'(x+T) = f'(x) = \forall x \in \mathbb{R}$

$\Rightarrow f'(x)$ is also periodic with period T

(b) Let $f(x) = (x^2+1)$ and $f(x) = \frac{2}{(x^2+1)}$

Clearly, $f(x)$ and $f(x)$ are defined on entire number line and are non-periodic.

However, $f(x).f(x) = 2$, which is periodic function.

(c) Let $f(x)$ is an even differential function, then $f(-x) = f(x)$

$\Rightarrow -f'(-x) = f'(x)$

$\Rightarrow f'(-x) = -f'(x)$

$\Rightarrow f'(x)$ is an odd function.

Similarly derivative of an odd differentiable function is even.

(d) $f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\textit{even function}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\textit{odd function}}$

14. **(a)** $f(-x) = -f(x)\ \forall x \in \mathbb{R}$ and $f(x+2) = f(x)\ \forall x \in \mathbb{R}$

$f(4) = -f(-4)$ ....(1)

But $f(-4) = f(-4+2) = f(-2+2) = f(0+2)$

$= f(2+2) = f(4)$ ....(2)

From (1) and (2), we get $f(4) = 0$

15. **(c)** $f(x) = \frac{\sin 8x\cos x - \sin 6x\cos 3x}{\cos 2x\cos x - \sin 3x\sin 4x}$

$= \frac{(\sin 9x+\sin 7x)-(\sin 9x+\sin 3x)}{(\cos 3x+\cos x)-(\cos x-\cos 7x)}$

$= \frac{\sin 7x - \sin 3x}{\cos 3x + \cos 7x}$

$= \frac{2\cos 5x\sin 2x}{2\cos 5x\cos 2x} = \tan 2x$

which is periodic with period $\frac{\pi}{2}$

16. **(b)** $f\left(x+\frac{1}{2}\right) = f(x); f(2) = 5; f\left(\frac{9}{4}\right) = 2, f(-3) - f\left(\frac{1}{4}\right) = ?$

$\because f(x)$ is periodic with period $\frac{1}{2}$

$\Rightarrow f(x) = f\left(x+\frac{n}{2}\right) \forall x \in \mathbb{N}$

$\Rightarrow f(-3) = f\left(-3+\frac{10}{2}\right) = f(2) = 5$ and

$f\left(\frac{1}{4}\right) = f\left(\frac{1}{4}+4\left(\frac{1}{2}\right)\right) = f\left(\frac{9}{4}\right) = 2$

$\therefore f(-3) - f\left(\frac{1}{4}\right) = 5 - 2 = 3$

SECTION–III (SINGLE ANSWER CORRECT)

1. **(b)** If $2f(x^2) + 3f\left(\frac{1}{x^2}\right) = x^2 - 1\quad \forall x \in \mathbb{R} \sim \{0\}$. ....(i)

Replacing x^2 by $\frac{1}{x^2}$,

we get $2f\left(\frac{1}{x^2}\right) + 3f(x^2) = \frac{1}{x^2} - 1$ ....(ii)

Solving (i) and (ii), we get $-5f(x^2) = 2x^2 - \frac{3}{x^2} + 1$

$\Rightarrow f(x^2) = \frac{-2}{5}x^2 + \frac{3}{5x^2} - \frac{1}{5} = \frac{-2x^4+3-x^2}{5x^2}$

$= \frac{(1-x^2)(2x^2+3)}{5x^2}$

2. **(a)** $f(x) = \sqrt{\frac{2-[x]}{[x]-3}}$; for domain of function, $\frac{2-[x]}{[x]-3} \geq 0$

$\Rightarrow (2-[x])([x]-3) \geq 0;\ [x]-3 \neq 0$

$\Rightarrow$ $([x]-2)([x]-3) \le 0$; $[x] \ne 3$

$\Rightarrow$ $[x] \in [2, 3]$; $[x] \ne 3$

$\Rightarrow$ $x \in [2, 4)$; $[x] \ne 3$

$\Rightarrow$ $x \in [2, 3)$

3. (c) $f(x) = \sin^{-1}[2-3x^2]$; for domain $[2-3x^2] \in [-1, 1]$

$\Rightarrow$ $2-3x^2 \in [-1, 2)$; $-2 < 3x^2 - 2 \le 1$

$\Rightarrow$ $3x^2 > 0$ and $x^2 \le 1$

$\Rightarrow$ $x \ne 0$ and $x \in [-1, 1]$

$\therefore$ Domain $= [-1, 0] \cup (0, 1]$

4. (c) $f(x) = \cos^{-1}\sqrt{x^2+3x+1} + \cos^{-1}\sqrt{x^2+3x}$; for domain $0 \le \sqrt{x^2+3x+1} \le 1$ and $0 \le \sqrt{x^2+3x} \le 1$

$\Rightarrow$ $0 \le x^2 + 3x + 1 \le 1$ and $0 \le x^2 + 3x \le 1$

$\Rightarrow$ $x^2 + 3x + 1 \ge 0$; $x^2 + 3x \le 0$ and $x^2 + 3x \ge 0$; $x^2 + 3x - 1 \le 0$

$\Rightarrow$ $x^2 + 3x = 0$ $\quad\Rightarrow$ $x(x+3) = 0$

$\Rightarrow$ $x = 0$ or $x = -3$

$\therefore$ Domain $= \{0, -3\}$

5. (c) $f(x) = \sin^{-1}x + \sqrt{2x - x^2} - \dfrac{1}{\sqrt{8x-4x^2-3}}$; for domain $x \in [-1, 1]$

$\Rightarrow$ $2x - x^2 \ge 0$ and $-4x^2 + 8x - 3 > 0$, i.e., $x \in [-1, 1]$; $x \in [0, 2]$ and $x \in \left(\dfrac{1}{2}, \dfrac{3}{2}\right)$ $x \in \left(\dfrac{1}{2}, 1\right]$

6. (b) $f(x) = \dfrac{1}{\sqrt{1-\sqrt{1-\sqrt{1-x^2}}}}$; for domain $1-\sqrt{1-\sqrt{1-x^2}} > 0$

$\Rightarrow$ $\sqrt{1-\sqrt{1-x^2}} \in [0, 1)$ $\Rightarrow$ $0 \le 1-\sqrt{1-x^2} < 1$

$\Rightarrow$ $0 < \sqrt{1-x^2} \le 1$

$\Rightarrow$ $0 < 1 - x^2 \le 1$ $\quad\Rightarrow$ $0 \le x^2 < 1$

$\Rightarrow$ $x \in (-1, 1)$

7. **(c)** $f(x) = \sqrt{3-2^x-2^{1-x}-2^{1-x}} + \sqrt{\sin^{-1}x}$; for domain $3 - 2^x - 2^{1-x} \ge 0$; $\sin^{-1}x \in \left[0, \dfrac{\pi}{2}\right]$

$\Rightarrow$ $3 - 2^x - \dfrac{2}{2^x} \ge 0$; $x \in [0, 1]$

$-(2^x)^2 + 3(2^x) - 2 \ge 0$; $x \in [0, 1]$

$\Rightarrow$ $(2^x)^2 - 3(2^x) + 2 \le 0$; $x \in [0, 1]$

$\Rightarrow$ $(2^x - 1)(2^x - 2) \le 0$; $x \in [0, 1]$

$\Rightarrow$ $1 \le 2^x \le 2$ and $x \in [0, 1]$

$\Rightarrow$ $x \in [0, 1]$

8. (c) $f(x) = \cos^{-1}x + \sqrt{1-\log_3(2x^2+6x-5)}$; for domain $x \in [-1, 1]$ and $\log_3(2x^2 + 6x - 5) \le 1$

$\Rightarrow$ $x \in [-1, 1]$ and $2x^2 + 6x - 5 \in (0, 3]$

$\Rightarrow$ $x \in [-1, 1]$ and $2x^2 + 6x - 5 > 0$ and $2x^2 + 6x - 8 \le 0$

$\Rightarrow$ $x \in [-1, 1]$ and $x \in \left(-\infty, \dfrac{-6-\sqrt{76}}{4}\right) \cup \left(\dfrac{-6+\sqrt{76}}{4}, \infty\right)$ and $x \in [-4, 1]$

$\Rightarrow$ $x \in \left(\dfrac{-3+\sqrt{19}}{2}, 1\right]$

9. (a) $f(x) = \cos^{-1}\left(\dfrac{3}{4+2\sin x}\right)$; for domain $-1 \le \dfrac{3}{4+2\sin x} \le 1$

$\Rightarrow$ $-4 - 2\sin x \le 3 \le 4 + 2\sin x$

$\Rightarrow$ $2\sin x \ge -7$ and $2\sin x \ge -1$

$\Rightarrow$ $\sin x \ge \dfrac{-1}{2}$

$\Rightarrow$ $x \in \mathbb{R} - \bigcup\limits_{n\in\mathbb{Z}}\left((2n+1)\pi + \dfrac{\pi}{6}, (2n+2)\pi - \dfrac{\pi}{6}\right)$

10. (c) $f(x) = \dfrac{1}{\sqrt{x^2-x-2}} + \dfrac{\sin^{-1}\left(\log_2\left(\dfrac{x^2}{2}\right)\right)}{\sqrt{5|x|-x^2-6}}$; for domain $x^2 - x - 2 > 0$; $\log_2\left(\dfrac{x^2}{2}\right) \in [-1, 1]$ and $x^2 - 5|x| + 6 < 0$

$\Rightarrow$ $x \in (-\infty, -1) \cup (2, \infty)$ and $x \in [-2, -1] \cup [1, 2]$ and $2 < |x| < 3$ (i.e., $x \in (-3, -2)$ $(2, 3)$)

$\Rightarrow$ $x \in \phi$

11. (a) $f(x) = e^x + \sin^{-1}\left[\left(\dfrac{x}{2}\right) - 1\right] + \log\sqrt{x-[x]}$; for domain $\left[\dfrac{x}{2} - 1\right] \in [-1, 1]$ and $x - [x] > 0$

$\Rightarrow$ $x \in (0, 6)$ and $\{x\} > 0$

$\Rightarrow$ $x \in (0, 6)$ and $x \notin \mathbb{Z}$

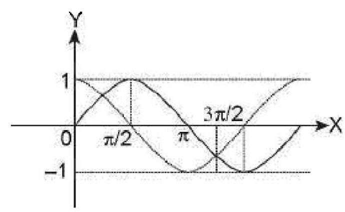
$\Rightarrow$ $x \in (0, 6) - \{1, 2, 3, 4, 5\}$

12. (b) $f(x) = \log\left\{\dfrac{1}{([\cos x]-[\sin x])}\right\}$; for domain $[\cos x] - [\sin x] > 0$

$\cos x$ and $\sin x$ are periodic with period 2π.

$\Rightarrow$ $x = 0, \left[\dfrac{3\pi}{2}, 2\pi\right]$

$\Rightarrow$ $x \in \left[2n\pi - \dfrac{\pi}{2}, 2n\pi\right]; n \in \mathbb{Z}$



Equivalently, $x \in \left[(2n+1)\pi + \dfrac{\pi}{2}, (2n+1)\pi\right]$

13. (b) $\log_{2004}(\log_4(\log_{2002}(\log_{2001}x)))$

$\Rightarrow$ $\log_4(\log_{2002}(\log_{2001}x)) > 0$

$\Rightarrow$ $\log_{2002}(\log_{2001}x) > 1$

$\Rightarrow$ $\log_{2001}(x) > 2002$

$\Rightarrow$ $x > (2001)^{2002}$

14. **(b)** $f(x) = \sqrt{\sin^{-1}(\log_2 x)} + \sqrt{\cos(\sin x)} + \sin^{-1}\left(\frac{1+x^2}{2x}\right)$; for domain $\sin^{-1}(\log_2 x) \in \left[0, \frac{\pi}{2}\right]$ and $\cos(\sin x) \geq 0$; and $\left(\frac{1+x^2}{2x}\right) \in [-1, 1]$

$\Rightarrow$ $\log_2 x \in [0, 1]$ and $\frac{1+x^2}{2x} = -1, 1$

$\Rightarrow$ $x \in [1, 2]$ and $x = \pm 1 \Rightarrow$ $x \in \{1\}$

15. **(b)** $f(x) = \sin^{-1}\left(\frac{2-|x|}{4}\right) + \cos^{-1}\left(\frac{2-|x|}{4}\right) + \tan^{-1}\left(\frac{2-|x|}{4}\right)$;

for domain $\left(\frac{2-|x|}{4}\right) \in [-1, 1]$

$\Rightarrow$ $2 - |x| \in [-4, 4]$ $\Rightarrow$ $-|x| \in [-6, 2]$

$\Rightarrow$ $|x| \in [-2, 6]$ $\Rightarrow$ $x \in [-6, 6]$

16. **(d)** $f(x) = \frac{\sqrt{x^2-4}}{\sin^{-1}(2-x)}$

Here $x^2 - 4 \geq 0$ and $(2 - x) \in [-1, 1] - \{0\}$

$\Rightarrow$ $x \in (-\infty, -2] \cup [2, \infty)$ and $x \in (1, 3)$

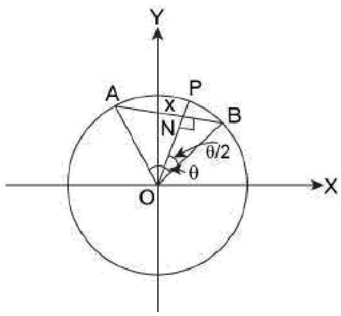
$\Rightarrow$ $x \in [2, 3) - \{x\}$, i.e., $x \in (2, 3)$

$\therefore$ Least value of x does not exist.

17. **(c)** $PN = x$ $\Rightarrow$ $ON = (a - x)$

Since distance PN is lying along OP.

$\Rightarrow$ $ON \perp AB$ and $AN = BN$



$\Rightarrow$ ONP is $\perp$ bisector of chord AB

$\Rightarrow$ In ΔOBN, $\angle BON = \frac{\theta}{2}$

$\therefore$ $\cos\frac{\theta}{2} = \frac{ON}{OB} = \frac{|a-x|}{a}$

$\Rightarrow$ $\theta = 2\cos^{-1}\left(\frac{|a-x|}{a}\right)$; $0 < x < 2a$

18. **(b)** $\because$ There exist more than one Δ's of given area

$\Rightarrow$ f: S $\to \mathbb{R}^+$ *is* many–one, and hence, not injective .

Let $k \in \mathbb{R}^+$, then $f(\Delta) = k = \frac{1}{2}(a) \times (b)$

$\Rightarrow$ $b = \frac{2k}{a}$

$\therefore$ If $k \in \mathbb{R}^+$ and $a \in \mathbb{R}^+$, then we can find $b = \frac{2k}{a}$; such that $k = \frac{1}{2}(a)(b)$, e.g., $k = 4$, $a = 3$, $b = \frac{8}{3}$

$\therefore$ We can form a Δ having base length 3 and altitude $\frac{8}{3}$

$\therefore$ f is subjective.

19. **(b)** $4^{f(x)} + 4^{1-f(x)} = 4^x$

$\Rightarrow$ $4^{f(x)} + \frac{4}{4^{f(x)}} = 4^x$

$\Rightarrow$ $(4^{f(x)})^2 - (4^x)(4^{f(x)}) + 4 = 0$

$\Rightarrow$ Disc. ≥ 0 $\Rightarrow$ $16^x \geq 16$

$\Rightarrow$ $x \geq 1$ $\Rightarrow$ $x \in [1, \infty)$

20. **(d)** $f(x) = (x^{12} - x^9 + x^4 - x + 1)^{-1/2}$

For $x < 0$, $x^{12} - x^9 + x^4 - x + 1 > 0$,

For $x = 0$, $x^{12} - x^9 + x^4 - x + 1 = 1$,

For $x \in (0, 1)$; $x^{12} > 0$, $x^4 > x^9$, $1 > x$,

$\Rightarrow$ $x^{12} - x^9 + x^4 - x + 1 > 0$

For $x \in (1, \infty)$, $x^{12} > x^9$ and $x^4 > x$

$\Rightarrow$ $x^{12} - x^9 + x^4 - x + 1 > 0$

$\therefore$ $x^{12} - x^9 + x^4 - x + 1 > 0$ $\forall x \in \mathbb{R}$

$\Rightarrow$ $(x^{12} - x^9 + x^4 - x + 1)^{-1/2} \in \mathbb{R}$ $\forall x \in \mathbb{R}$

$\therefore$ Domain = $(-\infty, \infty)$ or $\mathbb{R}$

21. **(c)** $f(x) = \sin^{-1}$ [sec x], [.] = gint function); for domain [sec x] $\in [-1, 1]$

$\Rightarrow$ $\sec x \in [-1, 2)$ $\Rightarrow$ $\sec x = -1$ or $\sec x \in [1, 2)$

$\Rightarrow$ $x = (2n + 1)\pi$; $n \in \mathbb{Z}$ and $\sec x \in [1, 2)$

$\Rightarrow$ $x \in \left(2m\pi - \frac{\pi}{3}, 2m\pi + \frac{\pi}{3}\right)$; $m \in \mathbb{Z}$

$\therefore$ Domain = $\{(2n + 1)\pi;\ n \in \mathbb{Z}\}$

$\cup \left\{\left\{\left(2m\pi - \frac{\pi}{3}, 2m\pi + \frac{\pi}{3}\right); m \in \mathbb{Z}\right\}\right.$

22. **(d)** f : $\mathbb{R} \to \mathbb{R}$; $f(x) = 2x^3 - 50000x^2 + 0.1\,x + \sin x$

$\therefore$ $f(x) = 2x^3 - 50000\,x^2 + 0.1x$ being a cubic polynomial has its range $(-\infty, \infty)$.

Also $h(x) = \sin x$ has its range $[-1, 1]$. Thus $f(x)$ has its range $(-\infty, \infty)$

23. **(d)** $f(x) = \log_{\sqrt{2}}\left(2 - \log_2\left(16\sin^2 x + 1\right)\right)$; for domain $2 - \log_2(16\sin^2 x + 1) > 0$

$\Rightarrow$ $\log_2(16\sin^2 x + 1) < 2$

$\Rightarrow$ $16\sin^2 x + 1 < 4$

Also $16\sin^2 x + 1 \geq 1$ $\therefore$ $1 \leq 16\sin^2 x + 1 < 4$

$\Rightarrow$ $\log_2 1 \leq \log_2(16\sin^2 x + 1) < \log_2 4$

$\Rightarrow$ $0 \leq \log_2(16\sin^2 x + 1) < 2$

$\Rightarrow$ $-2 < -\log_2(16\sin^2 x + 1) \leq 2$

$\Rightarrow$ $0 < 2 - \log_2(16\sin^2 x + 1) \leq 2$

$\Rightarrow$ $-\infty < \log_{\sqrt{2}}(2 - \log_2(16\sin^2 x + 1)) \leq 2$

$\Rightarrow$ Range = $(-\infty, 2]$

24. **(a)** $f(x) = \frac{1}{\sqrt{\cos(\sin x)}}$ $(\because \sin x \in [-1, 1])$

$\Rightarrow$ $\cos(\sin x) \in [\cos 1, 1]$

$\Rightarrow$ $\sqrt{\cos(\sin x)} \in \left[\sqrt{\cos 1}, 1\right]$ $(\because f(x) = (x)^{1/2}$ is increasing)

$\Rightarrow \quad \dfrac{1}{\sqrt{\cos(\sin x)}} \in \left[1, \dfrac{1}{\sqrt{\cos 1}}\right]$

$\Rightarrow$ Range $= \left[1, \sqrt{\sec 1}\right]$

25. (a) $f(x) = [9^x - 3^x + 1]\ x \ \forall \in (-\infty, 1)$

Let $3^x = t$

$\therefore \quad 9^x - 3^x + 1 = t^2 - t + 1 = \left(t - \dfrac{1}{2}\right)^2 + \dfrac{3}{4}$ for $x\ (-\infty, 1)$,

$3^x \in (0, 3)\ t \in (0, 3)$

$\Rightarrow \quad \left(t - \dfrac{1}{2}\right)^2 + \dfrac{3}{4} \in \left[\dfrac{3}{4}, 7\right)$

$\Rightarrow \quad 9^x - 3^x + 1 \in \left[\dfrac{3}{4}, 7\right)$

$\Rightarrow \quad f(x) = [9^x - 3^x + 1] \in \{0, 1, 2, 3, 4, 5, 6\}$

26. (d) $f(x) = 6^x + 3^x + 6^{-x} + 3^{-x} + 2$

$= \left(6^x + \dfrac{1}{6^x}\right) + \left(3^x + \dfrac{1}{3^x}\right) + 2$

$\because \quad 6^x, 3^x > 0\ \forall\ x \in \mathbb{R}$

$\therefore \quad$ By A.M. $\geq$ G.M. $6^x + \dfrac{1}{6^x} \in [2, \infty)$; $3^x + \dfrac{1}{3^x} [2, \infty)$

Also for $x < 0$; both $6^x + \dfrac{1}{6^x}$ and $3^x + \dfrac{1}{3^x}$ are decreasing functions and continuous functions having their range $[2, \infty)$ and both are even functions.

Thus range of $f(x) = [2 + 2 + 2, \infty) = [6, \infty)$

27. (c) $f: \mathbb{R} \to \mathbb{R}; f(x) = 2x^3 - 6x$

$f'(x) = 6x^2 - 6$

$f'(x) > 0$

$\Rightarrow \quad x \in (-\infty, -1) \cup (1, \infty)$ and $f'(x) < 0$

$\Rightarrow \quad x \in (-1, 1)$

$\therefore \quad f(x)$ decrease from $[-1, 1]$ and $f(x)$ increase from $[1, 2]$

$\Rightarrow$ Range of $f(x) = [f(1), f(-1)] \cup [f(1), f(2)] = [-4, 4] \cup [-4, 4] = [-4, 4]$

28. (a) If $2 < x^2 < 3 \quad \Rightarrow \quad \{x^2\} = x^2 - 2$ .....(i)

$\because 2 < x^2 < 3$

$\Rightarrow \quad x \in \left(-\sqrt{3}, -\sqrt{2}\right) \cup \left(\sqrt{2}, \sqrt{3}\right)$

$\Rightarrow \quad \dfrac{1}{x} \in \left(\dfrac{-1}{\sqrt{2}}, \dfrac{-1}{\sqrt{3}}\right) \cup \left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{2}}\right)$

$\Rightarrow \quad \left\{\dfrac{1}{x}\right\} = \dfrac{1}{x} - (-1)$ for $x \in \left(-\sqrt{3}, -\sqrt{2}\right)$ and

$\left\{\dfrac{1}{x}\right\} = \dfrac{1}{x} - (0)$ for $x \in \left(\sqrt{2}, \sqrt{3}\right)$

For positive roots, $x \in \left(\sqrt{2}, \sqrt{3}\right)$

Given equation becomes, $x^2 - 2 = \dfrac{1}{x}$

$\Rightarrow \quad x^3 - 2x = 0 \qquad \Rightarrow \quad x(x^2 - 2) = 0$

$\Rightarrow \quad x = 0$ or $x = \sqrt{2}$, but $x \in \left(\sqrt{2}, \sqrt{3}\right)$

$\Rightarrow$ There is no positive root.

29. (b) $f(x) = \dfrac{\sin x}{\sqrt{1 - \cos^2 x}} + \dfrac{\cos x}{\sqrt{1 - \sin^2 x}} + \dfrac{\tan x}{\sqrt{\sec^2 x - 1}} + \dfrac{\cot x}{\sqrt{\operatorname{cosec}^2 x - 1}}$

$\therefore \quad \sqrt{1 - \cos^2 x} = \pm \sin x$, $\sqrt{1 - \sin^2 x} = \pm \cos x$,

$\sqrt{\sec^2 x - 1} = \pm \tan x$ and $\sqrt{\operatorname{cosec}^2 x - 1} = \pm \cot x$

In I[st] quadrant: $f(x) = 1 + 1 + 1 + 1 = 4$

In II[nd] quadrant: $f(x) = 1 - 1 - 1 - 1 = -2$

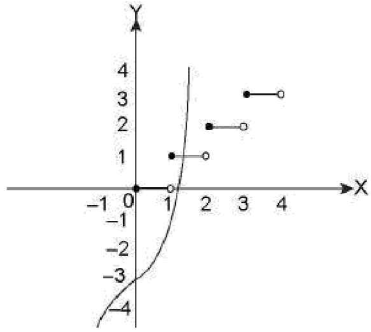
In III[rd] quadrant: $f(x) = -1 - 1 + 1 + 1 = 0$

In IV[th] quadrant: $f(x) = -1 + 1 - 1 - 1 = -2$

$\therefore \quad$ Minimum value of $f(x) = -2$

30. (b) $x^3 - [x] = 3$

$\Rightarrow \quad x^3 - 3 = [x]$ .....(i)



Clearly, $y = [x]$ and $y = x^3 - 3$ intersect each other, where $[x] = 1$

$\Rightarrow \quad x^3 - 3 = 1 \qquad \Rightarrow \quad x^3 = 4$

$\Rightarrow \quad x = (4)^{1/3}$

31. (c) $(x)^2 + (x + 1)^2 = 25$ .....(i)

Let $k < x \leq (k + 1) \quad \Rightarrow \quad (x) = (k + 1); k \in \mathbb{Z}$

$\Rightarrow \quad k + 1 < (x + 1) \leq k + 2$

$\Rightarrow \quad (x + 1) = (k + 2)$

$\therefore \quad$ (i) becomes, $(k + 1)^2 + (k + 2)^2 = 25$

$\Rightarrow \quad 2k^2 + 6k + 5 - 25 = 0 \Rightarrow \quad 2k^2 + 6k - 20 = 0$

$\Rightarrow \quad k^2 + 3k - 10 = 0$

$\Rightarrow \quad (k + 5)(k - 2) = 0 \qquad \Rightarrow \quad k = -5$ or $k = 2$

$\Rightarrow \quad x \in (-5, -4] \cup (2, 3]$

32. (a) Let $k < x < (k + 1); k \in \mathbb{Z}$

$\Rightarrow \quad [x] = k; (x) = (k + 1)$

$\therefore \quad [x]^2 + (x)^2 > 25 \qquad \Rightarrow \quad k^2 + (k + 1)^2 > 25$

$\Rightarrow \quad 2k^2 + 2k - 24 > 0$

$\Rightarrow \quad k^2 + k - 12 > 0 \qquad \Rightarrow \quad (k + 4)(k - 3) > 0$

$\Rightarrow \quad k \in (-\infty, -4) \cup (3, \infty)$

$\Rightarrow \quad k \in (-\infty, -3] \cup [4, \infty)$ .....(i)

If $x = k \in \mathbb{Z}$, then $[x] = (x) = k$

$\therefore \quad [x]^2 + (x)^2 > 25 \qquad \Rightarrow \quad 2k^2 > 25$

$\Rightarrow \quad k \in \left(-\infty, \dfrac{-5}{\sqrt{2}}\right) \cup \left(\dfrac{5}{\sqrt{2}}, \infty\right)$

$\Rightarrow \quad k \in \{\ldots\ldots, -6, -5, -4, 4, 5, 6, \ldots.\}$ .....(ii)

Combing (i) and (ii), we observe that $x\ (-\infty, -4] \cup [4, \infty)$

33. **(a)** $\because f(x)$ is an even function, $f(-x) = f(x)$

$\Rightarrow$ $[3.5 - b\sin x] = [3.5 + b\sin x]\ \forall\, x \in \mathbb{R}$

$\Rightarrow$ $[3.5 - k] = [3.5 + k]$; $k = b\sin x$

$\Rightarrow$ $\dfrac{-1}{2} < k < \dfrac{1}{2} \Rightarrow \dfrac{-1}{2} < b\sin x < \dfrac{1}{2}$

$\therefore$ $-1 < \sin x < 1$; for $b > 0$, $-b\sin x < b$ and for $b < 0$; $-b > b\sin x > b$

$\Rightarrow$ $b \in \left(\dfrac{-1}{2}, \dfrac{1}{2}\right)$

34. **(b)** $f(x) = \log\sqrt{10.3^{x-2} - 9^{x-1} - 1} + \sqrt{\cos^{-1}(2-x)}$; for domain $\dfrac{10.(3)^x}{9} - \dfrac{(3^x)^2}{9} - 1 > 0$ and $(2 - x) \in [-1, 1]$

$\Rightarrow$ $10\,(3)^x - (3^x)^2 - 9 > 0$ and $x \in [1, 3]$

$\Rightarrow$ $-10\,(3)^x + (3^x)^2 + 9 < 0$ and $x \in [1, 3]$

$\Rightarrow$ $(3^x)^2 - (3^x) - 9\,(3^x) + 9 < 0$ and $x\,[1, 3]$

$\Rightarrow$ $(3^x)\,(3^x - 1) - 9\,(3^x - 1) < 0$ and $x \in [1, 3]$

$\Rightarrow$ $(3^x - 1)\,(3^x - 9) < 0$ and $x \in [1, 3]$

$\Rightarrow$ $1 < 3^x < 9$ and $x \in [1, 3]$

$\Rightarrow$ $x \in (0, 2)$ and $x \in [1, 3]$

$\Rightarrow$ $x \in [1, 2)$

35. **(b)** $f(x) = \sqrt{\sqrt{\cot(5+3x)\,(\cot 5 + \cot 3x)} - \sqrt{\cot 3x + 1}}$; for domain $\sqrt{\cot(5+3x)(\cot 5 + \cot 3x)} \geq \sqrt{\cot 3x + 1}$

$\Rightarrow$ $\cot(5 + 3x)\,(\cot 5 + \cot 3x) \geq \cot 3x + 1 \geq 0$

$\Rightarrow$ $\cot(5+3x)\left[\dfrac{\sin 3x\cos 5 + \cos 3x.\sin 5}{\sin 5.\sin 3x}\right] \geq \cot(3x) + 1 \geq 0$

$\Rightarrow$ $\cot(5 + 3x)\,.\left[\dfrac{\sin(3x+5)}{\sin 5.\sin 3x}\right] \geq \cot 3x + 1 \geq 0$

$\Rightarrow$ $\dfrac{\cos(5+3x)}{\sin 5.\sin 3x} \geq \cot 3x + 1 \geq 0$

$\Rightarrow$ $\dfrac{\cos 5.\cos 3x - \sin 5.\sin 3x}{\sin 5\sin 3x} \geq \cot 3x + 1 \geq 0$

$\Rightarrow$ $(\cot 5).\cot 3x - 1 \geq \cot 3x + 1 \geq 0$

$\Rightarrow$ $(\cot(5) - 1)\cot 3x \geq 2$; $\cot 3x \geq -1$

$\because$ $5 \in \left(\dfrac{3\pi}{2}, 2\pi\right) \Rightarrow \cot 5 < 0$

$\Rightarrow$ $(\cot 5 - 1) < 0$

$\Rightarrow$ $\cot 3x \leq \dfrac{2}{(\cot 5 - 1)}$ and $\cot 3x \geq -1$

$\Rightarrow$ $-1 \leq \cot 3x \leq \dfrac{2}{(\cot 5 - 1)}$

$\Rightarrow$ $n\pi + \cot^{-1}\left(\dfrac{2}{\cot 5 - 1}\right) \leq 3x \leq (n+1)\pi - \dfrac{\pi}{4}$

$\Rightarrow$ $\dfrac{n\pi}{3} + \dfrac{1}{3}\cot^{-1}\left(\dfrac{2}{\cot 5 - 1}\right) \leq x \leq \dfrac{1}{3}\left(n + \dfrac{3}{4}\right)\pi; x \in \mathbb{Z}$

$\Rightarrow$ Domain $= \bigcup\limits_{x\in\mathbb{Z}}\left[\dfrac{n\pi}{3} + \dfrac{1}{3}\cot^{-1}\left(\dfrac{2}{\cot 5 - 1}\right), \dfrac{1}{3}\left(n + \dfrac{3}{4}\right)\pi\right]$

36. **(c)** $f(x) = \log_{100x}\left(\dfrac{2\log_{10} x + 2}{-x}\right)$; $g(x) = \{x\}$;

For fog (x) to exist

Range of $f(x)$ must be subset of domain of $f(x)$.

Now, range $f(x) = [0, 1)$

For domain of $f(x)$: $100\,x > 0, \neq 1$ and $\dfrac{2\log_{10} x + 2}{-x} > 0$

$\Rightarrow$ $x > 0$; $x \neq \dfrac{1}{100}$ and $\dfrac{\log_{10} x + 1}{x} < 0$

$\Rightarrow$ $x > 0$; $x \neq \dfrac{1}{100}\log_{10} x < -1$

$\Rightarrow$ $x > 0$; $x \neq \dfrac{1}{100}; x < \dfrac{1}{10}$

$\Rightarrow$ $x \in \left(0, \dfrac{1}{10}\right) - \left\{\dfrac{1}{100}\right\}$

$\therefore$ Maximum possible range of

$f(x) = \left(0, \dfrac{1}{10}\right) - \left\{\dfrac{1}{100}\right\} = \left(0, \dfrac{1}{100}\right) \cup \left(\dfrac{1}{100}, \dfrac{1}{10}\right)$

i.e., $(0, 10^{-2}) \cup (10^{-2}, 10^{-1})$

37. **(b)** $f(x).f(y) = f(x) + f(y) + f(xy) - 2\ \forall\ x,y \in \mathbb{R}$ ....(i)

Put $x = 1, y = 1$

$\Rightarrow$ $[f(1)]^2 = f(1) + f(1) + f(1) - 2$

$\Rightarrow$ $[f(1)]^2 = 3f(1) - 2$

$\Rightarrow$ $(f(1) - 1)\,(f(1) - 2) = 0$

$\Rightarrow$ $f(1) = 1$ or $f(1) = 2$

Now, put $x = x, y = 1$ in (i), we get $f(x).f(1) = f(x) + f(1) + f(x) - 2\ \forall\, x \in \mathbb{R}$

If $f(1) = 1$, then $f(x) = f(x) + 1 + f(x) - 2\ \forall\, x \in \mathbb{R}$

$\Rightarrow$ $f(x) = 1\ \forall\, x \in \mathbb{R}$

$\Rightarrow$ $f(x)$ is a constant function, which contradicts the given hypothesis that $f(x)$ is not a constant function.

Thus, $f(1) = 2$

38. **(b)** $f\left(\dfrac{2\tan x}{1 + \tan^2 x}\right) = \dfrac{(\cos 2x + 1)(\sec^2 x + 2\tan x)}{2}$

$\Rightarrow$ $f(\sin 2x) = \dfrac{(\cos 2x + 1)(\sec^2 x + 2\tan x)}{2}$

$= \dfrac{2\cos^2 x}{2}\left[\dfrac{1 + \sin^2 x}{\cos^2 x}\right] = (1 + \sin 2x)$

Put $\sin 2x = \dfrac{1}{3}$

$\Rightarrow$ $f\left(\dfrac{1}{3}\right) = 1 + \dfrac{1}{3} = \dfrac{4}{3}$

39. **(c)** $f(x) = \begin{cases} 0 \text{ if } \text{ is rational} \\ x \text{ if } x \text{ is irrational} \end{cases}$ and $g(x) = \begin{cases} 0 \text{ if } x \text{ is irrational} \\ x \text{ if } x \text{ is rational} \end{cases}$

$\therefore$ $(f - g)\,x = \begin{cases} -x \text{ if } x \text{ is rational} \\ x \text{ if } x \text{ is irrational} \end{cases} = h(x)$ (say)

Let $x_1, x_2 \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{Q}$, then $h(x_1) = h(x_2)$

$\Rightarrow -x_1 = -x_2 \qquad \Rightarrow x_1 = x_2$

Similarly $x_1, x_2 \in \overline{\mathbb{Q}}$ such that $h(x_1) = h(x_2)$

$\Rightarrow x_1 = x_2$

Further if $x_1 \in \mathbb{Q}, x_2 \in \overline{\mathbb{Q}}$ such that $h(x_1) = h(x_2)$

$\Rightarrow -x_1 = x_2$

which is false as in this case either both x_1, x_2 are rational or both irrational.

$\Rightarrow (f-g)(x)$ is injective (one–one)

Also for each $x \in \mathbb{R}$.

If $x \in \mathbb{Q}$, then $(f-g)(-x) = x$ and If $x \in \overline{\mathbb{Q}}$, then $(f-g)(x) = x$

$\Rightarrow (f-g)(x)$ is subjective (onto).

40. (d) $T = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix}; a, b, c, d \in \mathbb{R}\right\}$; $f: T \to \mathbb{R}$ such that $f(A) = |A|\ \forall A \in T$

Clearly f can't be injective as for every real number k, there exist at least two matrices having same determinant k.

e.g., $A = \begin{bmatrix} 7 & 4 \\ 6 & 8 \end{bmatrix}; B = \begin{bmatrix} 5 & 2 \\ 4 & 8 \end{bmatrix}$; then $f(A) = |A| = 32$ and $f(B) = |B| = 32$

For every real number k, there is a matrix having determinant k, e.g., $A = \begin{bmatrix} k & 1 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow |A| = k$

$\Rightarrow f$ is subjective.

41. (b) $n(A) = 4$; $n(B) = 6$

Number of one–one functions $= {}^6P_4 = \dfrac{6!}{2!} = 360$ (from A to B)

Number of onto functions from B to A.

= Number of ways of putting 6 balls into 4 boxes, so that no box remains empty.

$= (4)^6 - [{}^4C_1 (3)^6 - {}^4C_2 (2)^6 + {}^4C_3 (1)^6] = 1560$

42. (b) $A = \{1, 2, 3, 4, 5, 6\}$

$f: A \to A$ is an injective function such that $f(\lambda) \neq \lambda$. Then number of such functions is equal to the number of ways of putting 6 letters into 6 envelopes, so that no letter goes to its correct addressed envelope.

$= 6!\left[1 - \dfrac{1}{1!} + \dfrac{1}{2!} - \dfrac{1}{3!} + \dfrac{1}{41} - \dfrac{1}{51} + \dfrac{1}{6!}\right] = 360 - 120 + 30 - 6 + 1 = 265$

43. (b) $f: \mathbb{R} \to \mathbb{R}$ is given by $f(x) = (x+1)^2 - 1, x \geq -1$

$y = (x+1)^2 - 1$

$\Rightarrow (x+1)^2 = y + 1$

$\Rightarrow x + 1 = \pm\sqrt{+}$

But $x \geq -1 \qquad \Rightarrow x + 1 \geq 0$

$\Rightarrow x + 1 = \sqrt{y+1} \Rightarrow x = \sqrt{y+1} - 1$

$\Rightarrow f^{-1}(y) = \sqrt{y+1} - 1$

$\Rightarrow f^{-1}(x) = \sqrt{x+1} - 1$

$\therefore f^{-1}(x) = f(x) \qquad \Rightarrow \sqrt{x+1} - 1 = (x+1)^2 - 1$

$\Rightarrow \sqrt{x+1} = (x+1)^2 \qquad \Rightarrow (x+1)[(x+1)^3 - 1] = 0$

$\Rightarrow x = -1$ or $x = 0 \qquad \Rightarrow S = \{0, -1\}$

44. (d) $h(x) = |kx + 5|$

Domain of $f(x) = [-5, 7]$ and domain $f(h(x)) = [-6, 1]$

$\Rightarrow -5 \leq h(x) \leq 7\ \forall x \in [-6, 1]$

$\Rightarrow -5 \leq |kx + 5| \leq 7\ \forall x \in [-6, 1]$

$\Rightarrow 0 \leq |kx + 5| \leq 7\ \forall x \in [-6, 1]$

$\Rightarrow -7 \leq (kx + 5) \leq 7\ \forall x \in [-6, 1]$

Case (i): If $k > 0$, then $(kx + 5)$ is increasing.

$\Rightarrow k(-6) + 5 = -7$ and $k(1) + 5 = 7$

$\Rightarrow k = 2$

Case (ii): If $k < 0$, then $(kx + 5)$ is decreasing.

$\Rightarrow k(-6) + 5 = 7$ and $k(1) + 5 = -7$

$\Rightarrow k = -\dfrac{1}{3}$ and $k = -12$, which can't hold simultaneously.

$\therefore k = 2$

45. (c) $f(x) = \sin^2 x + \left[\dfrac{1 - \cos\left(2x + \dfrac{2\pi}{3}\right)}{2}\right] + \cos x \cos\left(x + \dfrac{\pi}{3}\right)$

$= \dfrac{1}{2}\left[1 - \cos 2x + 1 - \cos\left(2x + \dfrac{2\pi}{3}\right) + 2\cos x \cos\left(x + \dfrac{\pi}{3}\right)\right]$

$= \dfrac{1}{2}\left[2 - \cos 2x - \cos\left(2x + \dfrac{2\pi}{3}\right) + \cos\left(2x + \dfrac{\pi}{3}\right) + \cos\left(\dfrac{\pi}{3}\right)\right]$

$= \dfrac{1}{2}\left[2 - \cos 2x - \cos\left(2x - \dfrac{\pi}{3}\right) + \cos\left(2x + \dfrac{\pi}{3}\right) + \dfrac{1}{2}\right]$

$= \dfrac{1}{2}\left[2 - \cos 2x + 2\cos 2x \cos\dfrac{\pi}{3} + \dfrac{1}{2}\right] = \dfrac{1}{2}\left[\dfrac{5}{2}\right] = \dfrac{5}{4}$

$\therefore$ 4 gof $(x) = 4g\left(\dfrac{5}{4}\right) = 4.(1298) = 5192$

46. (a) $f(x) = 1 + x^2$; $f(f(x)) = 1 + x^2 - 2x^3 + x^4$

$\Rightarrow 1 + (f(x))^2 = 1 + x^2 - 2x^3 + x^4$

$\Rightarrow (f(x))^2 = x^4 - 2x^3 + x^2$

$\Rightarrow f(x) = \pm\sqrt{x^4 - 2x^3 + x^2}$

$\Rightarrow f(18) = \pm\sqrt{(18)^4 - 2(18)^3 + (18)^2}$

$= \pm 18\sqrt{(18)^2 - 2(18) + 1} = \pm 18\sqrt{(18-1)^2} = \pm 18\,(17)$

$= \pm 06$

47. (c) $f(x) = \cos^{-1}\sqrt{1 - x^2}$ and $f(x) = -\sin^{-1} x$; $f(x) = f(x)$

$\Rightarrow \cos^{-1}\sqrt{1-x^2} = -\sin^{-1} x$

$\Rightarrow \sin^{-1} x = -\cos^{-1}\sqrt{1-x^2}$

$\Rightarrow x \in [-1, 0]$

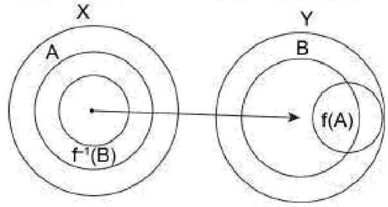
48. (b) $f(x) = x^2 + x + \sin x - \cos x + \log(1 + |x|)$, $f(-x) = -f(x)$

$\Rightarrow f(x) = -f(-x)$

$= -x^2 + x + \sin x + \cos x - \log(1 + |x|)$

49. (c) $f(x) = [x] + \displaystyle\sum_{r=1}^{1000} \dfrac{\{x + r\}}{1000} = [x] + \sum_{r=1}^{1000} \dfrac{\{x\}}{1000} = [x] + 1000\,\dfrac{\{x\}}{1000} = [x] + \{x\} = x$

50. **(b)** $f(f^{-1}(B)) = \{f(x): x \in f^{-1}(B)\} = \{f(x): f(x) \in B\}$



$\Rightarrow$ $f(f^{-1}(B)) \subset B$

Now if $x \in B \subset Y$ $\Rightarrow$ $x \in Y$

It may happen that $f^{-1}(x)$ does not Exist in x as function is not given to be subjective.

$\therefore$ $f(f^{-1}(B)) \neq B$

Also, $f^{-1}(f(A)) = \{x \in X: f(x) \in f(A)\}$ but $f(A) = \{f(x): x \in A\}$

From above, we can't conclude $f^{-1}(f(A) \subset A$.

If the function is non-injective, then it may happen that $x \notin$ A and $f(x) \in f(A)$.

$\Rightarrow$ $f^{-1}(f(A) \not\subset A$ $\Rightarrow$ $f^{-1}(f(A)) \neq A$

51. **(b)** $f: \mathbb{R} \to \mathbb{R}; f(x) = x^2 + 1$

Here it is not given that $f^{-1}(x)$ is a function, so, let us treat $f^{-1}(x)$ as a relation given by $f^{-1}(x) = \pm\sqrt{x-1}$

$\therefore$ $f^{-1}(17) = \pm 4$ and $f^{-1}(3) = \pm\sqrt{2}$

$\Rightarrow$ $\{f^{-1}(17)\} \cup \{f^{-1}(3)\} = \{\pm 4, \pm\sqrt{2}\}$

52. **(c)** $f(x) = \cos nx.\sin\left(\dfrac{5x}{n}\right)$

Period of $\cos nx = \dfrac{2\pi}{|n|}$

Period of $\sin \dfrac{5x}{n} = \dfrac{2\pi}{\left|\dfrac{5}{n}\right|} = \dfrac{2|n|\pi}{5}$

$\therefore$ Period of $f(x)$ = L.C.M $\left(\dfrac{2\pi}{|n|}, \dfrac{2|n|\pi}{5}\right) = 2\pi$ (given).

$\Rightarrow$ L.C.M $\left(\dfrac{1}{|n|}, \dfrac{|n|}{5}\right) = 1 = \dfrac{\text{L.C.M}(1,|n|)}{\text{H.C.F}(|n|,5)} = 1$

$\Rightarrow$ $\dfrac{|n|}{\text{H.C.F}(|n|,5)} = 1$

$\Rightarrow$ H.C.F $(|n|, 5) = |n|$

If g.c.d $(|n|, 5) = 1$ $\Rightarrow$ $|n| = 1$

$\Rightarrow$ $n = 1$

If g.c.d $(|n|, 5) \neq 1$ $\Rightarrow$ $|n| = 5m; m \in \mathbb{N}$

$\Rightarrow$ g.c.d $(5m, 5) = 5$

$\Rightarrow$ $|n| = 5$ $\Rightarrow$ $n = \pm 5$

$\therefore$ $n \in \{\pm 1, \pm 5\}$

53. **(a)** $f(x) = 1 - x^3 - x^4 - 2x^5 = f(x) + h(x)$, where $f(x)$ = even function and $h(x)$ is an odd function

$\Rightarrow$ $h(x) = \dfrac{f(x) - f(x)}{2}$ $\Rightarrow$ $h(-5) = \dfrac{f(-5) - f(5)}{2}$

$= \left[\dfrac{1 + 125 - 625 + 6250 - 1 + 125 + 625 + 6250}{2}\right]$

$= 125 + 6250 = 6375$

54. **(d)** $f: \mathbb{R} \to \mathbb{R}; f(x) = \left(\dfrac{ax+5}{x^2+2}\right)$

$$f'(x) = \frac{(x^2+2)(a) - (ax+5)(2x)}{(x^2+2)^2}$$

$$= \frac{ax^2 + 2a - 2ax^2 - 10x}{(x^2+2)^2} = \frac{2a - 10x - ax^2}{(x^2+2)^2}$$

For a continuous function $f(x)$ to be invertible, $f(x) > 0$ or $f'(x) < 0$; where the or used is exclusive.

$\Rightarrow$ $ax^2 + 10x - 2a < 0$ or exclusive $> 0\ \forall\ x \in \mathbb{R}$

$\therefore$ Disc. $= 100 + 8a^2 > 0$

$\therefore$ For no real value of a, $f(x)$ is invertible. Thus, $a \in \{\}$

55. **(d)** $f(x) = \dfrac{1}{2}$ [cos (sin $4x$) + cos (cos $4x$)]

Period of cos (sin x) + cos (cos x) = $g(x)$, (say) is $\dfrac{\pi}{2}$ as

$$g\left(\frac{\pi}{2} + x\right) = \cos\left(\sin\left(\frac{\pi}{2} + x\right)\right) + \cos\left(\cos\left(\frac{\pi}{2} + x\right)\right)$$

$= \cos(\cos x) + \cos(-\sin x)$

$= \cos(\cos x) + \cos(\sin x) = g(x)$

$\therefore$ $f(x) = \dfrac{1}{2}$ [cos (sin $4x$) + cos (cos $4x$)] has period $= \dfrac{\frac{\pi}{2}}{4} = \dfrac{\pi}{8}$

56. **(a)** $f(x) = \cos\sqrt{k}\ x$; $k = [m]$; Period of $f(x) = \dfrac{2\pi}{\sqrt{k}} = \pi$

$\Rightarrow$ $\sqrt{k} = 2$ $\Rightarrow$ $k = 4$

$\Rightarrow$ $[m] = 4$ $\Rightarrow$ $m \in [4, 5)$

57. **(d)** $F(x) = f(x).\ g\left(\dfrac{x}{5}\right) - g(x).f\left(\dfrac{x}{3}\right)$

Period of $f(x).g\left(\dfrac{x}{5}\right)$ = L.C.M (7, 55) = 385

Period $f(x)f\left(\dfrac{x}{3}\right)$ = L.C.M (11, 21) = 231

$\therefore$ Period of $F(x)$ = L.C.M (385, 231) = 1155

58. **(a)** $\sin[x + 4x + 9x + \ldots\ldots + (n+1)^2 x]$

$$= \sin\left[\frac{(n+1)(n+2)(2n+3)}{6}\right]x$$

Its period $= \dfrac{12\pi}{(n+1)(n+2)(2n+3)} = \dfrac{\pi}{4}$ (given)

$\Rightarrow$ $(n+1)(n+2)(2n+3) = 3 \times 4 \times 7$

$\Rightarrow$ $n = 2$

59. **(b)** $f(x) = 2\{x\} + \sin 2\pi\{x\}$

$\because$ $f(x+1) = 2\{x+1\} + \sin 2\pi\{x+1\}$

$= 2\{x\} + \sin 2\pi\{x\} = f(x)$

$\Rightarrow$ $f(x)$ is periodic with period 1.

60. **(c)** $f(x) = 2\cot 3x + 5\sqrt{1 - \cos 6x}$

$f(x) = 2\cot 3x + 5\sqrt{2}\,|\sin 3x|$

Period = L.C.M $\left(\frac{\pi}{3},\frac{\pi}{3}\right)=\frac{\pi}{3}$

Now, $\sec^2 3x + \text{cosec}^2 3x = \frac{4}{\sin^2 6x} = 4\,\text{cosec}^2 6x$, having period $\frac{\pi}{6}$

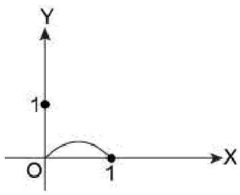
$2\sin 3x + 3\cos 3x$ has period L.C.M $\left(\frac{2\pi}{3},\frac{2\pi}{3}\right)=\frac{2\pi}{3}$

$2\sqrt{1-\cos^2 3x} + \text{cosec}\,6x = 2\,|\sin 3x| + \text{cosec}\,6x$ has period = L.C.M $\left(\frac{\pi}{3},\frac{2\pi}{6}\right)=\frac{\pi}{3}$

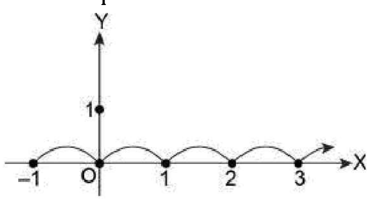
$\therefore$ $f(x)$ can be $2\sqrt{1-\cos^2 3x} + \text{cosec}\,6x$

61. (b) $f(x) = x(1 - x);\ 0 \le x \le 1;$

Graph of $f(x)$ for $x \in [0, 1]$ is as shown below.



If $f(x)$ is extended by $f(x + 1) = f(x)$, then the new function will be represented as shown below.



$\Rightarrow$ The new function will be a periodic function with period 1.

62. (b) $f(x) = \begin{cases} x^2\sin\left(\frac{\pi x}{2}\right); |x|<1 \\ x|x|; \quad |x|\ge 1\end{cases} = \begin{cases} x^2\sin\left(\frac{\pi x}{2}\right); -1<x<1 \\ -x^2 \quad ; \ x\le -1 \\ x^2 \quad ; \ x\ge 1\end{cases}$

$\Rightarrow f(-x) = \begin{cases} x^2\sin\left(\frac{\pi x}{2}\right); 0\le x<1 \\ x^2 \ ; \ x\ge 1\end{cases}$ and $f(-x)$

$= \begin{cases} -x\ \sin\left(—\right); 0\le x\le 1 \\ -x \ ; \ x\ge 1\end{cases}$

$\therefore$ $f(x) = -f(-x)\ \forall\ x \in [1, \infty)$

$\Rightarrow$ $f(x)$ is an odd function

63. (d) $f(x) = x^2 - 1$ and $f(x) = x^3 - 7x^2 + 5x + 6$

$\Rightarrow f'(x) = 2x$ and $g'(x) = 3x^2 - 14x + 5$

Let $P \equiv (h, k)$ and $Q \equiv (u, v)$

Since tangent at P to $f(x)$ is parallel to tangent at Q to $f(x)$

$\Rightarrow 2h = 3u^2 - 14u + 5$

$\Rightarrow 3u^2 - 14u + (5 - 2h) = 0$

Disc. $= (-14)^2 - 4(3)(5 - 2h)$

$= 196 - 60 + 24\,h$

$= 136 + 24\,h > 0$, for $h > -\frac{136}{24}$ or $h > -\frac{17}{3}$

$\therefore$ For $h > -\frac{17}{3}$; u has two real values given by

$u = \frac{14 \pm \sqrt{136+24h}}{6}$ or $u = \frac{7 \pm \sqrt{34+6h}}{3}$

$\therefore$ Corresponding to each point $p(h, k)$ on $f(x)$; $h > \frac{-17}{3}$; there will be two points $Q(u, v)$ on $g(x)$ such that at tangent at P is to the Q.

Thus, there will be the infinitely many pairs (P, Q).

64. (b) $f''(x) > 0\ \forall\ x \in \mathbb{R}$ and $g(x) = f(x^2) - f(x^2 + 3)$

$\Rightarrow g'(x) = f'(x^2).(2x) - f'(x^2 + 3)(2x) = (2x)\,[f'(x^2) - f'(x^2 + 3)]$ ………(1)

$\because$ $f'(x) > 0\ \forall\ x \in \mathbb{R}$

$\Rightarrow$ $f'(x)$ is an increasing function $x \in \mathbb{R}$,

$\Rightarrow f'(x^2) - f'(x^2 + 3) < 0.$ ……(2)

$\therefore$ From (1) and (2), we note that $g'(x) < 0$ for $x > 0$ and $g'(x) > 0$ for $x < 0$

$\Rightarrow$ $g(x)$ is decreasing in $(0, \infty)$.

65. (a) $f(x) = \frac{\left(x^3 - 3x^2 + 3x - 7\right)}{\sqrt{x^{14} - x^{11} + x^{10} - x^7 + x^6 - x^3 + x^2 + 1}}$

$\Rightarrow$ For domain, $f(x) = (x^{14} - x^{11} + x^{10} - x^7 + x^6 - x^3 + x^2 + 1) > 0$

For $x \le 0, f(x) > 0$

For x (0, 1), $x^{10} > x^{11}, x^6 > x^7, x^2 > x^3$

$\Rightarrow f(x) > 0$

For $x \in [1, \infty)$, $x^{14} \ge x^{11}, x^{10} \ge x^7, x^6 \ge x^3$

$\Rightarrow f(x) > 0$ $\quad\therefore\ f(x) > 0\ \forall\ x \in \mathbb{R}$

$\Rightarrow$ Domain $f(x) = (-\infty,\infty)$

66. (b) $f(x) = \log_{(x^2+x+1)}(3x^2 - 4x + 5)$

$\Rightarrow$ For domain, $x^2 + x + 1 > 0$ and $\ne 1$, disc. of $x^2 + x + 1 = -3 < 0$

$\Rightarrow x^2 + x + 1 = 1 > 0\ \forall\ x \in \mathbb{R}$ and $x^2 + x + 1 = 1$ for $x = 0$ or – 1

$\therefore$ For base, $x \in \mathbb{R} - \{0, -1\}$ …….(i)

Also for $3x^2 - 4x + 5 > 0$ which holds $\forall\ x \in \mathbb{R}$ as its disc $= -44 < 0$

$\Rightarrow$ Domain of $f(x) = \mathbb{R} - \{0, -1\}$

67. (c) $f(x) = \sqrt{8 - 3^{x+1} - 3^{1-x}} + \sqrt{\sin^{-1}(2x+1)}$

For domain of $f(x)$, $8 - 3\left(3^x\right) - \frac{3}{\left(3^x\right)} \ge 0$ and $\sin^{-1}(2x + 1) \ge 0$

$\Rightarrow 8(3^x) - 3(3^x)^2 - 3 \ge 0$ and $(2x + 1) \in [0, 1]$

$\Rightarrow 3(t^2) - 8t + 3 \le 0$ and $x \in \left[\frac{-1}{2},0\right]$; where $t = 3^x$

$\Rightarrow\ t \in \left[\frac{4-\sqrt{7}}{3},\frac{4+\sqrt{7}}{3}\right]$ and $x \in \left[\frac{-1}{2},0\right]$

$\Rightarrow\ 3^x \in \left[\frac{4-\sqrt{7}}{3}, \frac{4+\sqrt{7}}{3}\right]$ and $x \in \left[\frac{-1}{2}, 0\right]$

$\Rightarrow\ x \in \left[\log_3\left(\frac{4-\sqrt{7}}{3}\right), \log_3\left(\frac{4+\sqrt{7}}{3}\right)\right]$ and $x \in \left[\frac{-1}{2}, 0\right]$

$\because\ \frac{4+\sqrt{7}}{3} > 1 \Rightarrow \log_3\left(\frac{4+\sqrt{7}}{3}\right) > 0$ and

$\frac{4-\sqrt{7}}{3} = \frac{9}{3\left(4+\sqrt{7}\right)} = \frac{3}{4+\sqrt{7}} < \frac{3}{\sqrt{12}+\sqrt{3}} = \frac{1}{\sqrt{3}}$

$\Rightarrow\ \log_3\left(\frac{4-\sqrt{7}}{3}\right) < \log_2\left(\frac{1}{\sqrt{3}}\right) = \frac{-1}{2}$

$\Rightarrow\ \left[\frac{-1}{2}, 0\right] \subset \left[\log_3\left(\frac{4-\sqrt{7}}{3}\right), \log_3\left(\frac{4+\sqrt{7}}{3}\right)\right]$

$\Rightarrow$ Domain of $f(x) = \left[-\frac{1}{2}, 0\right]$

68. **(c)** $f(x) = \frac{3\cos^2 x + 3\cos x + 4}{\cos^2 x + \cos x + 1} = \frac{3\left[\cos^2 x + \cos x + 1\right] + 1}{\cos^2 x + \cos x + 1}$

$= 3 + \frac{1}{\left(\cos^2 x + \cos x + 1\right)} = 3 + \frac{1}{\left\{\left(\cos x + \frac{1}{2}\right)^2 + \frac{3}{4}\right\}}$

as $\cos x \in [-1, 1]$

$\Rightarrow\ \left(\cos x + \frac{1}{2}\right) \in \left[\frac{-1}{2}, \frac{3}{2}\right]$

$\Rightarrow\ \left(\cos x + \frac{1}{2}\right)^2 \in \left[0, \frac{9}{4}\right]$

$\Rightarrow\ \left\{\left(\cos x + \frac{1}{2}\right)^2 + \frac{3}{4}\right\} \in \left[\frac{3}{4}, 3\right]$

$\Rightarrow\ \frac{1}{\left\{\left(\cos x + \frac{1}{2}\right)^2 + \frac{3}{4}\right\}} \in \left[\frac{1}{3}, \frac{4}{3}\right]$

$\Rightarrow\ 3 + \frac{1}{\left\{\left(\cos x + \frac{1}{2}\right)^2 + \frac{3}{4}\right\}} \in \left[\frac{10}{3}, \frac{13}{3}\right] = [\lambda, \mu]$

$\Rightarrow\ \lambda = \frac{10}{3}, \mu = \frac{13}{3}$

$\therefore\ 6\lambda + 9\mu + 2 = 20 + 39 + 2 = 61$

69. **(b)** $f(x) = x^2 - px + q$ ……..(i)

Given, $q + p = 11$ and $q - p = a$

$\Rightarrow\ q = \frac{11+a}{2}$ and $p = \frac{11-a}{2}$

$\Rightarrow\ f(x) = x^2 - \left(\frac{11-a}{2}\right)x + \left(\frac{11+a}{2}\right)$

$\Rightarrow\ f(a) = a^2 - \left(\frac{11-a}{2}\right)a + \left(\frac{11+a}{2}\right)$

$\Rightarrow\ f(a) = \frac{2a^2 - 11a + a^2 + 11 + a}{2}$

$\Rightarrow\ f(a) = \frac{3a^2 - 10a + 11}{2}$ ……..(2)

If α, β are prime roots of (1), then $\alpha + \beta = p$

$= \frac{11-a}{2}, \alpha\beta = q = \frac{11+a}{2}$

$\because\ p, q \in \mathbb{N}$ (as α, β are prime integers)

$\Rightarrow$ 2/(11 – a) and 2/(11 + a) as $11 - a > 0$

$\Rightarrow\ a \in \{1, 3, 5, 7, 9\}$

Also $\alpha + \beta = p = \frac{11-a}{2}$ and α, β are prime

$\Rightarrow\ \alpha + \beta \geq 4$ $\qquad \Rightarrow\ \frac{11-a}{2} \geq 4$

$\Rightarrow\ 11 - a \geq 8$ $\qquad \Rightarrow\ a \leq 3$

$\Rightarrow\ a = 1$ or 3

For $a = 1$, $\alpha + \beta = p = 5$, $\beta = q = 6$

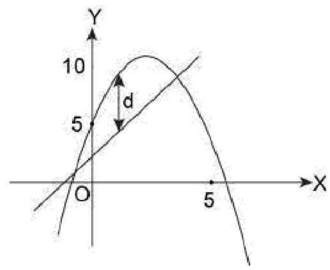
$\Rightarrow\ \alpha = 2, \beta = 3$

For $a = 3$, $\alpha + \beta = p = 4$, $\beta = q = 7$ which is impossible as α, β are prime integers .

$\Rightarrow\ a = 1$

$\therefore$ From (2), we get $f(a) = \frac{3-10+11}{2} = 2$

70. **(c)** The vertical distance (d) between the curves will be maximum when both $-x^2 + 5x + 7 > 0$ and $2x + 3 > 0$ and $2x + 3 > 0$ and $-x^2 + 5x + 7 > 2x + 3$



Further (d) is given by $|(-x^2 + 5x + 7) - (2x + 3)|$

$= -x^2 + 5x + 7 - 2x - 3$

$d(x) = -x^2 + 3x + 4$

$\Rightarrow$ Maximum $d(x) = \frac{-[(3) \ \ -4(-1)(4)]}{4(\ \ 1)} = \frac{-[9+16]}{-4} = \frac{25}{4}$

SECTION–IV (MORE THAN ONE ANSWER CORRECT)

1. **(a), (b), (c), (d)**

(a) $\text{sgn}(e^{-x}) = 1$ as $e^{-x} > 0\ \forall\ x \in \mathbb{R}$

$\Rightarrow$ It is a constant function, and hence, is a periodic function with no fundamental period.

(b) $f(x) = \begin{cases} 1 \text{ if } x \text{ is a rational no.} \\ 0 \text{ if } x \text{ is an irrational no.} \end{cases}$

There exists infinitely many 'h' for which $f(x + h) = f(x)$ $\forall\ x \in \mathbb{R}$.

e.g., if $h = 2$, then $f(x + h) = f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x + h) = f(x) = 0$ if $x \in \mathbb{Q}$

$\Rightarrow$ $f(x + h) = f(x)\ \forall\ x \in \mathbb{R}$, but we can't find smallest positive real 'h' for which $f(x + h) = f(x)\ \forall\ x \in \mathbb{R}$

$\Rightarrow$ $f(x)$ is periodic with no fundamental period.

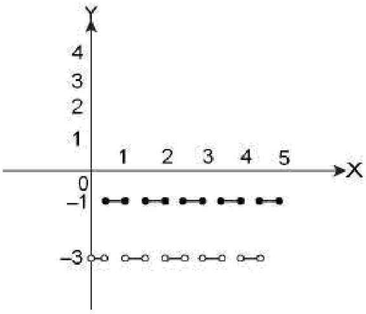
(c) $f(x) = \sqrt{\dfrac{8}{1+\cos} + \dfrac{8}{1-\cos x}} = 4\ |\text{cosec } x|$, which is periodic function with period π.

(d) $\left[x+\dfrac{1}{2}\right]+\left[x-\dfrac{1}{2}\right]+2[-x]$

$= \left[x+\dfrac{1}{2}\right]+\left[\left(x+\dfrac{1}{2}\right)-1\right]+2([-x]) = 2\left[x+\dfrac{1}{2}\right]+2[-x]-1$

$$= \begin{cases} 2x-2x-1=-1 \text{ if } x\in\mathbb{Z} \\ 2n+2+2(-n-1)-1=-1 \text{ if } x=n+\dfrac{1}{2} \\ 2n-2n-2-1=-3; n<x<n+\dfrac{1}{2} \\ 2n+2-2n-2-1=-1; n+\dfrac{1}{2}<x<n+1 \end{cases}$$

The graph of $f(x)$ will be as shown below



Clearly, the period of $f(x)$ is 1

$f(x + 1) = \left[x+1+\dfrac{1}{2}\right]+\left[x+1-\dfrac{1}{2}\right]+2[-x-1]$

$= 1 + \left[x+\dfrac{1}{2}\right]+1+\left[x-\dfrac{1}{2}\right]+2[-x]-2$

$= \left[x+\dfrac{1}{2}\right]+\left[x-\dfrac{1}{2}\right]+2[-x] = f(x)$

2. (a), (b) $f(x) = [x]^2 + [x + 1] - 3$

$\Rightarrow$ $f(x) = [x]^2 + [x] - 2$

$\Rightarrow$ $f(x) = ([x] + 2)([x] - 1)$

$\therefore$ $f(x) = 0$ $\Rightarrow$ $[x] = -2$ or $[x] = 1$

$\Rightarrow$ $f(x) = 0$ for $x \in [-2, -1) \cup [1, 2)$

$\Rightarrow$ $f(x)$ is many–one function, $f(x) = 0$ for infinite number of values of x.

Also range of $f(x)$ consists of only integer values, and hence, $f(x)$ is into.

3. (a), (d) $f(x) = \dfrac{x^2-3x+2}{x^2+x-6} = \dfrac{(x-1)(x-2)}{(x+3)(x-2)} = \dfrac{(x-1)}{(x+3)}$ for x 2

$\therefore$ Domain of $f(x) = \mathbb{R} - \{2, -3\}$

Now, $y = \dfrac{x-1}{x+3}$ $\Rightarrow$ $xy + 3y = x - 1$

$\Rightarrow$ $x(y - 1) = -3y - 1$ $\Rightarrow$ $x = \dfrac{3y+1}{1-y}$

$\therefore$ $x \in \mathbb{R}$ for $y \neq 1$ also $x \neq 2$

$\Rightarrow$ $2 \neq \dfrac{3y+1}{1-y}$

$\Rightarrow$ $2 - 2y \neq 3y + 1$ $\Rightarrow$ $y \neq 1/5$

$\Rightarrow$ Range $= \mathbb{R} - \left\{1, \dfrac{1}{5}\right\}$

4. (c), (d) $f(x) = \sqrt{2-x}+\sqrt{1+x}$

For domain $2 - x \geq 0$ and $1 + x \geq 0$

$\Rightarrow$ $x \leq 2$ and $x \geq -1$

Domain $= [-1, 2]$

Now, $y^2 = 3 + 2\sqrt{(2-x)(1+x)}$

$\Rightarrow$ $y^2 = 3 + 2\sqrt{-x^2+x+2}$

In domain, $[-1, 2]$, $-x^2 + x + 2 \in \left[0, \dfrac{9}{4}\right]$

$\Rightarrow$ $y^2 \in [3, 6]$ and $y > 0$ $\Rightarrow$ $y \in \left[\sqrt{3}, \sqrt{6}\right]$

$\Rightarrow$ Range $= \left[\sqrt{3}, \sqrt{6}\right]$

5. (b), (c) $\log_x y . \log_{xy} y . \log_{x^2 y} y = \dfrac{1}{6}$ ……..(i)

For domain $x > 0, y > 0, x \neq 1, xy \neq 1, x^2 y \neq 1$

Further (i) can be written as $\dfrac{\log y}{\log x}.\dfrac{\log y}{(\log xy)}.\dfrac{\log y}{\log x^2 y} = \dfrac{1}{6}$

$\Rightarrow$ $\dfrac{(\log y)^3}{(\log x)(\log x+\log y).(2\log x+\log y)} = \dfrac{1}{6}$

$\Rightarrow$ $6(\log y)^3 = (\log y)^2 (\log x) + 3(\log x)^2 (\log y) + 2(\log x)^3$

Put $\log y = t$, $\log x = u$

$\Rightarrow$ $6t^3 - ut^2 - 3u^2 t - 2u^3 = 0$

Clearly $t = u$ satisfy it $(t - u)$ is its factor.

$\Rightarrow$ $(t - u)(6t^2 + 5ut + 2u^2) = 0$

Disc. of $6t^2 + 5ut + 2u^2$ *is* $-23u^2 < 0$

$\Rightarrow$ $t = u$ is the only possible root.

$\Rightarrow$ $\log y = \log x$ $\Rightarrow$ $y = x$

$\therefore$ In Explicit form (i) can be written as $y = x$ for domain $x > 0, x \neq 1$

6. (b), (d) $f(x) = (1 + \tan x)\left\{1+\tan\left(\dfrac{\pi}{4}-x\right)\right\} = (1 + \tan x)\left\{1+\dfrac{1-\tan x}{1+\tan x}\right\} = 2$ for $\tan x \neq -1$

Also $g(x)$ is given to be defined for every real x.

$\text{gof}(x) = f(2)$ for $x \neq n\pi + \left(\dfrac{-\pi}{4}\right): n \in \mathbb{Z}$

$\Rightarrow$ gof (x) is constant $\forall\ x \in D_f$

$\Rightarrow$ gof (x) is non-subjective function.

7. (b), (c) $f(x) = \sqrt{x}\sqrt{x-1}$ and $g(x) = \sqrt{x(x-1)}$

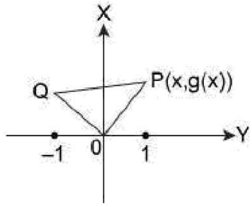
Domain of $f(x) = [1, \infty)$ and Domain of $f(x) = (-\infty, 0]$ $[1, \infty)$

Since Domain of $f(x) \neq$ domain of $f(x)$

$\Rightarrow$ $f(x)$ and $f(x)$ are not equal functions $\forall\ x \in \mathbb{R}$.

Also $f(x) = f(x)\ \forall\ x \in [1, \infty)$, i.e., they are identical in their common domain.

8. (b), (c) Area of Equilateral $\Delta = \frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{4}$



$\Rightarrow a = 1 \qquad \Rightarrow OP = 1$

$\Rightarrow x^2 + (f(x))^2 = 1$

$\Rightarrow x^2 + y^2 = 1 \qquad \Rightarrow y^2 = 1 - x^2$

$\Rightarrow y = \pm\sqrt{1-x^2}$

For y to be defined $x^2 - 1 \le 0$

$\Rightarrow x \in [-1, 1]$

9. (a), (d) $f(x) = \sec^{-1}[1 + \cos^2 x]$

$1 + \cos^2 x \in [1, 2]$

$\Rightarrow [1 + \cos^2 x] \in \{1, 2\}$

$\Rightarrow f(x) = \{\sec^{-1} 1, \sec^{-1} 2\} = \left\{0, \frac{\pi}{3}\right\}$

Domain of $f(x) = \mathbb{R}$ and Range of $f(x) = \left\{0, \frac{\pi}{3}\right\}$

10. (a), (d) $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$\therefore\ 2\tan^{-1}x = \begin{cases} \sin^{-1}\left(\frac{2x}{1+x^2}\right) \\ \text{for } x \in [-1, 1] \end{cases}$ and $2\tan^{-1}x = \begin{cases} \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \\ \text{for } x \in [0, \infty) \end{cases}$

$\Rightarrow \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ for x [0, 1]

$\Rightarrow$ Domain of $f(x) = [0, 1] = D_f$

Hence, range of $f(x) = \left[0, \frac{\pi}{2}\right] = R_f$

11. (b), (c) $\sin\left(\frac{\cos^{-1}x}{y}\right) = 1$

$\Rightarrow \frac{\cos^{-1}x}{y} = 2n\pi + \frac{\pi}{2}; n \in \mathbb{Z}$

Also $y \in Y \subseteq \mathbb{Z} - \{0\}$

$\Rightarrow \frac{\cos^{-1}x}{y} = (4n+1)\frac{\pi}{2} \Rightarrow x = 0, y = 1$, n = 0 or $\cos^{-1}x =$ $(4n + 1)k\pi$; $y = 2k$; $k \in \mathbb{Z} - \{0\}$

$\Rightarrow \cos^{-1}x = 0$ or π

$\Rightarrow (4n + 1)k = 0$ or 1

$\Rightarrow n = 0, k = 0, 1 \qquad \Rightarrow y = 0, 2$

$\therefore\ x \in \{-1, 1\}$ and $y \in \{0, 2\}$

Since $y \in Y_1 \subseteq \mathbb{Z} - \{0\}, y \ne 0, x \ne 1$

$\Rightarrow Y_1 = \{2, 1\}$ and $x \in X_1 = \{-1, 0\}$

12. (a), (c) $\cos\left(\frac{\sin^{-1}x}{y}\right) = 0$; $y\ Y_2 \subseteq \mathbb{Z} - \{0\}$

$\Rightarrow \frac{\sin^{-1}x}{y} = (2n+1)\frac{\pi}{2}; x \in \mathbb{Z}$

$\Rightarrow x = \pm 1, y = \pm 1$ or $\sin^{-1}x = k(2n + 1)\pi$; $y = 2k$; $k \in \mathbb{Z} - \{0\}$

$\Rightarrow (2n + 1)k = 0 \qquad \Rightarrow n = 0, k = 0$

$\Rightarrow x = 0, y = 0$

$\Rightarrow$ No more solution expect for $x = \pm 1, y = \pm 1$

$\Rightarrow X_2 = \{1, -1\}$ and $Y_2 = \{1, -1\}$

13. (a), (b), (d) Clearly for $n \ge 3$, there is no one–one function.

For $n = 1$, there will be two functions namely, $f_1(1) = -1$ and $f_2(1) = 1$

For $n = 2$, there will be two functions namely, $f_1 = \{(1, -1), (2, 1)\}$ and $f_2 = \{(1, 1), (2, -1)\}$

14. (a), (b) $f(1) = -1, f(x) = -1$

Now each one of 2, 3, 4 …..$(n - 1)$ has 2 choice, i.e., –1 and 1

$\Rightarrow$ Total number of possible functions $(2)^{n-2}$ for $n \ge 3$ and It is 1 for $n = 2$ and 1 for $n = 1$

$\Rightarrow (2)^{n-2}$ for $n \ge 2$ and 1 for $n = 1$

15. (a), (b), (c)

For $n = 1$:

$X = \{1\}$ and $Y = \{-1, 1\}$

Since there in no positive integer less than 1 (= n), which is to assign 1

$\therefore$ There will no such function.

For $n \ge 2$:

$X = \{1, 2, 3, \ldots\ldots, n\}$; $Y = \{-1, 1\}$and $f(r) = 1$ for exactly one r $\in \{1, 2, 3, \ldots, (n - 1)\}$

$\Rightarrow f(1) = f(2) = f(3) = \ldots.. = f(r - 1) = -1$

$f(r) = 1$ and $f(r + 1) = f(r + 2) = \ldots = f(n - 1) = -1$

$f(n)$ has two choice, i.e., – 1 and 1

Now, for r we have choice, 1, 2, 3, ….., $(n - 1)$

$\therefore$ Total number of functions = (number of ways of choosing r) $\times$ (number of ways for $f(n)$)

$= (n - 1) \times 2 = 2(n - 1)$

16. (a), (b), (c), (d)

$f(x) = \sqrt{\log_{\frac{1}{2}}\left(\log_5\left([x^2] - 3\right)\right)}$;

for domain $\log_{\_}(\log_5([x^2] - 3)) \ge 0$

$\Rightarrow 0 < \log_5([x^2] - 3) \le 1 \Rightarrow\ 10 < [x^2] - 3 \le 5$

$\Rightarrow 4 < [x^2] \le 8$

$\Rightarrow 5 \le [x^2] \le 8 \qquad \Rightarrow [x^2] \in \{5, 6, 7, 8\}$

$\Rightarrow$ option (a) is correct,

$\Rightarrow x^2 \in [5, 9) \qquad \Rightarrow x \in \left(-3, -\sqrt{5}\right] \cup \left[\sqrt{5}, 3\right)$

option (b) is correct.

$\Rightarrow [x] \in [-3, 2) \qquad \Rightarrow$ <u>option (c) is correct</u>

$\because\ [x^2] \in \{5, 6, 7, 8\}$

$\Rightarrow$ Range of $f(x)$

$= \left\{\sqrt{\log_{\frac{1}{2}}(\log_5(2))}, \sqrt{\log_{\frac{1}{2}}\log_5(3)}, \sqrt{\log_{\frac{1}{2}}(\log_5(4))}, 0\right\}$

option (d) is correct.

17. **(b), (c)** $f(x) + g(x) = e^x$ ....(i)

$f(x) - f(x) = e^{-x}$ ....(ii)

Adding (i) and (ii), we get $f(x) = \dfrac{e^x + e^{-x}}{2}$

$\Rightarrow$ $f(-x) = f(x)$

Subtracting, we get $f(x) = \dfrac{e^x - e^{-x}}{2}$

$\Rightarrow$ $f(-x) = \dfrac{e^{-x} - e^x}{2} = -f(x)$

$\Rightarrow$ $f(x)$ is an even function and $f(x)$ is an odd function.

18. **(a), (b), (c), (d)**

$f(x) = 2x + \cos x$ and $f(x) = \sqrt[3]{x}$, gof $(x) = f(f(x)) = (2x + \cos x)^{1/3}$

Clearly, domain of gof(x) is $\mathbb{R}$, $2x + \cos x \in (-\infty, \infty)$ and $(x)^{1/3}$ is an increasing function.

$\Rightarrow$ gof (x) has range $\mathbb{R}$ and (gof)′(x) = $\dfrac{1}{3}(2x+\cos x)^{-\frac{2}{3}}$

$(2 - \sin x) = \dfrac{(2-\sin x)}{3(2x+\cos)^{\frac{2}{3}}} > 0\ \forall\ x \in \mathbb{R} - \left\{\dfrac{-1}{2}\cos x\right\}$

$\Rightarrow$ gof is an injective function.

Further, $f'(x) = 2 - \sin x > 0\ \forall\ x \in \mathbb{R}$ and $g'(x) = \dfrac{1}{3(x)^{\frac{2}{3}}}$

$> 0\ \forall\ x \in \mathbb{R} - \{0\}$

$\Rightarrow$ $f(x)$ and $f(x)$ are injective.

Range of $f(x) = (-\infty, \infty)$

$\Rightarrow$ $f(x)$ is onto.

Also range of $f(x) = (-\infty, \infty)$

$\Rightarrow$ $g(x)$ is onto

19. **(a), (c), (d)** $f(x) = |x + 1|$, $x \in [-1, \infty)$

$\Rightarrow$ $f(x) = x + 1$; which is injective,

$f(x) = x + \dfrac{1}{2}$; $x \in (0, \infty)$

$f(x) \in (\infty, 2]$ for $x \in (0, 1]$ and $f(x) \in [2, \infty)$ for x $[1, \infty)$

$\Rightarrow$ $g(x)$ is non- injective

$\Rightarrow$ $h(x) = x^2 + 4x - 5$, $x \in (0, \infty)$

$\Rightarrow$ $h'(x) = 2x + 4 > 0$ for $x \in (-2, \infty)$

$\Rightarrow$ $h(x)$ is injective.

$\Rightarrow$ $k(x) = e^{-x}$

$\Rightarrow$ $k'(x) = -e^{-x} < 0\ \forall\ x \in \mathbb{R}$

20. **(a), (c)** $f(x) = \cos[\pi^2]\, x + \cos[-\pi^2]\, x = \cos 9x + \cos 10x$

$\Rightarrow$ $f\left(\dfrac{\pi}{2}\right) = \cos\dfrac{9\pi}{2} + \cos 5 = 0 - 1 = -1$

$\Rightarrow$ $f(\pi) = \cos 9\pi + \cos 10\pi = -1 + 1 = 0$

$\Rightarrow$ $f(-\pi) = \cos(-9\pi) + \cos(-10\pi) = -1 + 1 = 0$

SECTION–V (ASSERTION / REASON)

1. **(c)** Obviously the assertion is correct, but reason is incorrect as when $k < 0$, then $y = k^x$ becomes non-real for x of the form

$\dfrac{(2m+1)}{2n}$, $m, n \in \mathbb{Z}$, $n \neq 0$.

$\Rightarrow$ In this case domain $= \mathbb{R} - \left\{x : x = \dfrac{2m+1}{2n}, m, n \in \mathbb{Z}, n \neq 0\right\}$

and range $= \{k^x;\ x \in D_f\}$

$\therefore$ Domain and range will be changed.

For $k = 1$, $y = k^x$ $\Rightarrow$ $y = 1\ \forall x \in \mathbb{R}$

$\Rightarrow$ Function becomes a constant function, which is not desirable here, as if we want to find its inverse , then it will be impossible as inverse exists for one–one functions.

2. **(d)** $y = k^x$ is strictly increasing for $k > 1$. Its inverse is $y = \log k^x$ and which is strictly increasing as $k > 1$

$\Rightarrow$ Assertion is incorrect, however, reason is correct.

3. **(c)** Fundamental period of $\sin x = 2\pi$,

Fundamental period of $\tan x = \pi$

Fundamental period of $\sin x + \tan x$

$=$ L.C.M $(2, \pi) = 2\pi$

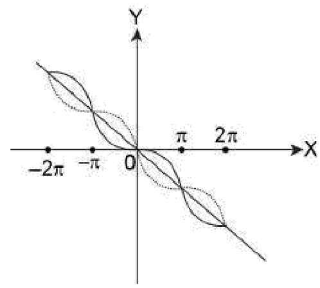
$\Rightarrow$ Assertion is correct.

But L.C.M (T_1, T_2) is period of $f(x) + f(x)$, but not necessary its fundamental period.

e.g., fundamental period of $|\sin x| + |\cos x|$ is $\dfrac{\pi}{2}$ inspite fundamental period of $|\sin x|$ and $|\cos x|$ is π and L.C.M $(\pi, \pi) = \pi$

$\therefore$ Assertion is correct reason incorrect

4. **(a)** Graph of $y = -x + \sin x$ and its inverse are as shown below.



Clearly reason is correct.

By reason, if $f(x)$ and $f^{-1}(x)$ intersect at $y = x$, then $x = -x + \sin x$

$\Rightarrow$ $2x = \sin x$ $\Rightarrow$ $x = 0$ is the only solution.

If $f(x)$ and $f^{-1}(x)$ intersect at $y = -x + \sin x$

$\Rightarrow$ $\sin x = 0$ $\Rightarrow$ $x = n\pi;\ n \in \mathbb{Z}$

$\therefore$ $f(x)$ and $f^{-1}(x)$ intersect at $y = -x$ at each $x = n\pi;\ n \in \mathbb{Z}$

$\Rightarrow$ Assertion is also correct and decided by reason.

5. **(c)** If we consider a function $f(x) = x + \sin x$

$f'(x) = 1 + \cos x$, which is periodic with period 2π, but $f(x)$ is not periodic .

Thus, reason is incorrect, however, assertion is correct.

6. **(b)** $\because$ $f(f(x + T)) = f(f(x))$, if T is period of $f(x)$.

$\Rightarrow$ fog (x) is also periodic with period T

$\Rightarrow$ Reason is correct.

If $f(x) = x + 3\sin x$, then it is non-periodic and $\sin x$ is periodic, and $\sin[(2\pi + x) + 3\sin(2\pi + x)] = \sin[2\pi + x + 3\sin x] = \sin(x + 3\sin x)$

$\Rightarrow$ sin (x + 3 sin x) is periodic with period 2 .

$\Rightarrow$ Assertion is correct, but it is not concluded from reason.

7. (c) If a function is defined on a closed interval, and then it attains its greatest and least values only when it is continuous, e.g., $y = \begin{cases} \tan x; x \in [0,2) \\ 0; x = 0 \end{cases}$ is defined on $\left[0, \frac{\pi}{2}\right]$, but does not attains its greatest value as it discontinuous. Thus reason is in correct. Also if a function is continuous and defined on a closed interval, then its range = co-domain will be a closed interval. Thus assertion is correct.

8. (d) Clearly, reason is correct.
e.g., $y = \tan x$ is increasing but not continuous, however, assertion is incorrect as it true only when $f(x)$ is continuous.

9. (b) Clearly, reason is correct as cos θ is a periodic function, and hence, repeats its maximum value 1 infinitely many times in (k, ∞).
i.e., in continuous domain.
Let $f(x) = xe^{[x]} + 2x^2 - x$ for $x \in (-1, \infty)$
Clearly, $g(0) = 0$

$\Rightarrow$ cos $(f(0))$ = cos 0 = 1 $\Rightarrow$ cos $(xe^{[x]} + 2x^2 - x)$ attains its maximum value 1 in $(-1, \infty)$.
But in reason cos θ is continuous for $\theta \in (k, \infty)$, where as $f(x) = \cos(xe^{[x]} + 2x^2 - x)$ is discontinuous in $x \in (-1, \infty)$

10. (c) $f(x) = \begin{cases} \sin x; x \neq n\pi; n \in \mathbb{Z} \\ 2; \text{otherwise} \end{cases}$ and $f(x) = \begin{cases} x^2 + 1; x \neq 0, 2 \\ 4; x = 0 \\ 5; x = 2 \end{cases}$

$$\Rightarrow f(f(x)) = \begin{cases} (f(x))^2 + 1; f(x) \neq 0, 2 \\ 4 \quad ; f(x) = 0 \\ 5; \ f(x) = 2 \end{cases}$$

$$\Rightarrow f(f(x)) = \begin{cases} \sin^2 x + 1; x \neq n\pi \\ 5 \quad ; x = n\pi \end{cases}$$

$\Rightarrow$ Reason is incorrect.
Now, $\lim\limits_{x \to 0} go f(x) = \lim\limits_{x \to 0} (\sin^2 x + 1) = 1$

$\Rightarrow$ Assertion is correct

11. (a) $y + \cos x = \sin x$

$\Rightarrow$ $y = \sin x - \cos x$

$$\Rightarrow y = \sqrt{2}\left[\sin x.\frac{1}{\sqrt{2}} - \cos x.\frac{1}{\sqrt{2}}\right]$$

$$\Rightarrow y = \sqrt{2}\left[\sin\left(x - \frac{\pi}{4}\right)\right]$$

$\therefore$ Range of $y = \left[-\sqrt{2}, \sqrt{2}\right]$, Also reason is correct and Explains the assertion.

12. (d) $f(x) = 4^x + 2^x + 4^{-x} + 2^{-x} + 3 = \left(2^x + \frac{1}{2^x}\right) + \left(4^x + \frac{1}{4^x}\right) + 3$

$= y + y^2 - 2 + 3 = y^2 + y + 1$

Now $y = 2^x + \frac{1}{2^x} \in [2, \infty)\ \forall\ x \in \mathbb{R}$

$$\Rightarrow y^2 + y + 1 = \left(y + \frac{1}{2}\right)^2 + \frac{3}{4} \geq \left(2 + \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{28}{4} = 7$$

$\Rightarrow$ Range $f(x) = [7, \infty)$.

$\Rightarrow$ Assertion is incorrect, also is correct.

13. (c) Reason is incorrect, even degree polynomial is even only when each term contains even power of x, however, assertion is correct as for even degree polynomial.
$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$
$f(-\infty) = f(\infty) = \pm\infty$ is finite and $f(0) = a_n$

$\Rightarrow$ $f(x)$ is non-injective.

14. (a) If $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$ is an even degree polynomial, then $f(-\infty) = f(\infty) = \pm\infty$ and $f(0)$ is finite

$\Rightarrow$ $f(x)$ is non-injective $\Rightarrow$ $f(x)$ is non-invertible

15. (c) $f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3)\ \forall\ x \in \mathbb{R}$ …(i)

$\Rightarrow$ $f'(x) = 3x^2 + 2xf'(1) + f''(2)$ …(ii)

$\Rightarrow$ $f''(x) = 6x + 2f'(1)$ …(iii)

$\Rightarrow$ $f'''(x) = 6$ $\Rightarrow$ $f'''(3) = 6$ …(iv)

And from (iii), $f'''(2) = 12 + 2f'(1)$ …(v)

Using (iv) and (v) in (i), we have $f(x) = x^3 + x^2 f'(1) + 12x + 2f'(1)\,x + 6$

$\Rightarrow$ $f'(x) = 3x^2 + 2xf'(1) + 12 + 2f'(1)$

$\Rightarrow$ $f'(1) = 3 + 2f'(1) + 12 + 2f'(1)$

$\Rightarrow$ $f'(1) = -5$

$\therefore$ $f(x) = x^3 - 5x^2 + 12x - 10x + 6$

$\Rightarrow$ $f(0) = 6$

$\Rightarrow$ $f(2) - f(1) = [7 - 5(3) + 12(1) + 10(1)]$

$\Rightarrow$ $f(2) - f(1) = -6, f(0) = 6$

$\Rightarrow$ Assertion is correct, reason is incorrect.

16. (a) $A = \{1, 2, 3, \ldots, 2n\}$, then the self invertible functions have pairs of the form (a, b) and (b, a) and not of the for (a, a) {According to question $f(i) \neq i$}, then such ordered pairs can't be selected $\left[{}^{2n}C_2 \times {}^{2n-2}C_2 \times \ldots \times {}^2C_2\right] \times \frac{1}{n!}$
Also the reason is correct and explains the assertion correctly.

17. (b) $f(x)$, $f(x)$ and $h(x)$ are even and odd functions respectively.
$f(f(h(2)) + f(h(f(7))) + h(f(f(2))$
$= f(g)(4)) + f(h(9)) + h(f(3))$
$= f(7) + f(12) + h(0)$
$= 9 - 9 = 0$

$\Rightarrow$ Assertion is correct. Also reason is correct, but does not explain the assertion.

18. (c) $f(x) = \frac{7}{2 + 9a^x}; a > 0 \neq 1, a^x \in (0, \infty)$

$\Rightarrow$ $9a^x \in (0, \infty)$

$\Rightarrow$ $2 + 9a^x \in (2, \infty)$ $\Rightarrow$ $\frac{1}{2 + 9a^x} \in \left(0, \frac{1}{2}\right)$

$\Rightarrow$ $\frac{7}{2 + 9a^x} \in \left(0, \frac{7}{2}\right)$

which contains exactly 3 integer solutions 1, 2 and 3.

$\Rightarrow$ Assertion is correct.

If the end point of range are $\alpha, \beta \in \mathbb{Z}$ and range is $(\alpha\ ,\ \beta)$, then the number of integer solutions will be $(\beta - 1) - \alpha]$.

$\therefore$ Reason is incorrect.

19. **(a)** $f(x) + f(5x + y) + 6xy = f(6x - y) + 2x^2 + 1\ \forall\ x, y \in \mathbb{R}$

Replacing y by $\frac{x}{2}$, we get $f(x) + f\left(\frac{11}{2}x\right) + 3x^2 = f\left(\frac{11}{2}x\right) + 2x^2 + 1$

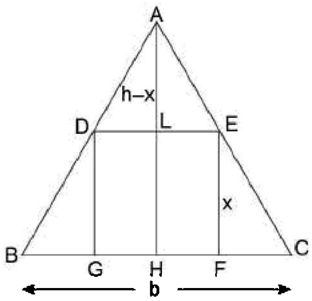
$\Rightarrow f(x) = -x^2 + 1 \qquad \Rightarrow f(5) = -24$

$\Rightarrow$ Assertion as well as reason both are correct and reason explain the assertion correctly.

SECTION–VI (COMPREHENSION TYPE)

A:

1. **(b)** Perimeter of rectangle $DEFG = 2x + 2DE$ ...(i)



In similar Δ's ADL and ABH, $\frac{AL}{AH} = \frac{DL}{BH} \Rightarrow \frac{h-x}{h} = \frac{DL}{\frac{b}{2}}$

$\Rightarrow DL = \frac{b(h-x)}{2h} \qquad DE = 2DL = \frac{b(h-x)}{h}$ ...(ii)

Using (ii) in (i), we get perimeter of rectangle $= 2x + \frac{2b(h-x)}{h} = 2\left\{x + \frac{b(h-x)}{h}\right\}$

2. **(c)** Area of rectangle $DEFG = x.DE = x\left\{\frac{b(h-x)}{h}\right\}$

3. **(c)** Perimeter = Area (numerically)

$\Rightarrow 2\left|x + \frac{b(h-x)}{h}\right| = \left|\frac{xb}{h}(h-x)\right|$

$\Rightarrow 2\left|x + \frac{b(h-x)}{h}\right| = \frac{xb}{h}(h-x) \qquad (\because h > x)$

Given, $b = 2, h = 4$

$\Rightarrow 2\left|x + \frac{2(4-x)}{4}\right| = \frac{2x}{4}(4-x)$

$\Rightarrow \left|x + 2 - \frac{x}{2}\right| = \frac{x}{4}(4-x) \Rightarrow \left|\frac{x}{2} + 2\right| = \frac{x}{4}(4-x)$

$\Rightarrow (x + 4) = \pm\frac{x}{2}(4 - x)$

$\Rightarrow x^2 - 2x + 8 = 0$ or $x^2 - 6x - 8 = 0$

First quadratic equation has no real root, while second one has roots $\frac{6 \pm \sqrt{68}}{2} = 3 \pm \frac{\sqrt{68}}{2}$ among which one is negative and other is greater than $h\ (= 4)$.

$\therefore$ No real value of x is possible

B:

4. **(c)** f: $[0, 2] \to [0, 2]$ is a bijective function, given by $f(x) = ax^2 + bx + c$, which is continuous, and hence, either $f(0) = 2, f(2) = 0$ or

$f(0) = 0, f(2) = 2$

For later case, $c = 0$ which is against the given condition that a, b, c are non-zero.

Thus, $f(2) = 0$

5. **(a)** From above, $f(0) = 2, f(2) = 0$

$\Rightarrow c = 2,\ 4a + 2b + 2 = 0$

$\Rightarrow 2a + b = -1,\ \alpha\beta = \frac{2}{a}, \alpha + \beta = \frac{-b}{a}$

$\Rightarrow \alpha + \beta = \frac{2a+1}{a} = 2 + \frac{1}{a} \Rightarrow \alpha = 2, \beta = \frac{1}{a}$

6. **(c)** $ax^2 + bx + c = 0$

$\Rightarrow a(x - 2)\left(x - \frac{1}{a}\right) = 0$

$\Rightarrow ax^2 - (2a + 1)x + 2 = 0$

$\therefore f(2) = 0$ and $f(0) = 2$

$\Rightarrow f(x)$ is a decreasing function $\forall\ x \in [0, 2]$

$\Rightarrow f'(x) = 2ax - (2a + 1) \le 0\ \forall\ x \in [0, 2]$

$\Rightarrow 2a(x - 1) \le 1\ \forall\ x \in [0, 2]$

Now, $x \in [0, 2] \qquad \Rightarrow -1 \le x - 1 \le 1$

$\Rightarrow -| \le 2a < 0$ or $| \le 2a$

$\Rightarrow a \in \left[\frac{-1}{2}, 0\right)$ or $a \in \left[\frac{1}{2}, \infty\right)$

$\Rightarrow a \ne -\frac{7}{2}$

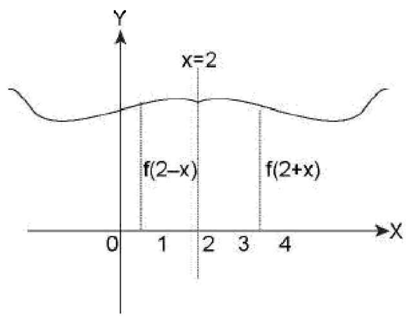
C:

$f(2 - x) = f(2 + x)$ ......(i)

$f(20 - x) = f(x) = \forall\ x \in \mathbb{R}$ ..... (ii)

$f(2 - x) = f(2 + x)$

$f(x)$ is symmetric about the line $x = 2$



$f(2-x) = f(20-(2-x))$ ..[using (ii)]

$= f(18+x)$

$f(2+x) = f(18+x)$ …. [Using (i)]

$\Rightarrow$ $f(x) = f(x+16)$

$\Rightarrow$ $f(x)$ is a function periodic with period 16.

Also $f(2-x) = f(2+x) = f(0) = f(4)$ and the period of $f(x)$ is 16

$\Rightarrow$ $f(0) = f(4) = f(20) = f(36)$

$= \ldots\ldots = f(4+(n-1)(16))$ where $4+(n-1)(16) \le 170$

$\Rightarrow$ $16n - 12 \le 170$

$\Rightarrow$ $16n \le 182$

$\Rightarrow$ $n \le 11\dfrac{3}{8}$

$\therefore$ $f(x) = 5$ for $x \in \{0, 4, 20, \ldots, 176\}$, i.e., 12 value of x.

7. (b)

8. (a) $\because f(x)$ is symmetrical about $x = 2$ and and $f(x)$ is periodic with period 16.

$\Rightarrow$ $f(2) = f(2+16) = f(18)$

$\Rightarrow$ $f(x)$ is also symmetrical about $x = 18$

Also $f(x-2) = f(2-x)$

$\Rightarrow$ $f((20-x)-2) = f(2-(20-x))$

$\Rightarrow$ $f(18-x) = f(x-18)$

9. (c) $f(-2) \ne f(6)$

$\Rightarrow$ $f(-2+8) \ne f(-2)$

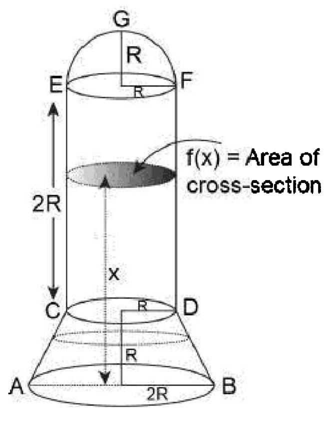
$\Rightarrow$ 8 can't be the periodic with period $f(x)$.

$\Rightarrow$ $f(x)$ can't be periodic with fundamental period of $f(x)$.

D:

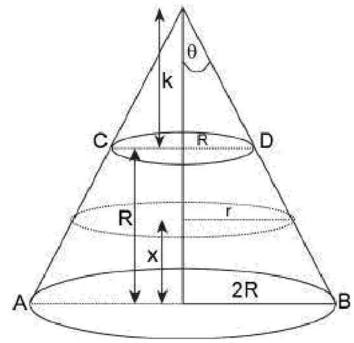
Clearly, $f(x) =$

$$\begin{cases} \text{Area of cross section of frustum for } x \in [0, R] \\ \text{Area of cross-section of cylinder for } x \in [R, 3R] \\ \text{Area of cross-section of hemi-sperial cross-section for } x \in [3R, 4R] \end{cases}$$



$$\Rightarrow \tan\theta = \frac{R}{k} = \frac{r}{k+(R-x)} = \frac{2R}{k+R}$$

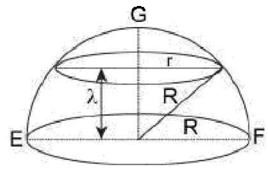
$\Rightarrow$ $k = R$ and $r = (2R-x)$



$\therefore$ Area of cross-section of frustum $= \pi r^2 = \pi(2R-x)^2$

For hemi-sphere r $= \sqrt{R^2-\lambda^2}$; $x = 3R +$

$\Rightarrow$ $r = \sqrt{R^2-(x-3R)^2}$



$\therefore$ Area of cross-section of hemi-sphere $= \pi[R^2-(x-3R)^2]$

Clearly domain of $f(x) = [0, 4R]$ and $f(x)$ is given by $f(x)$

$$= \begin{cases} \pi(2R-x)^2; x \in [0, R] \\ \pi R^2;\ x \in [R, 3R] \\ \pi[R^2-(x-3R)^2]; x \in [3R, 4R] \end{cases}$$

$\therefore$ $f(x)$ is a decreasing and continuous function on $[0, 4R]$

$\Rightarrow$ Range of $f(x) = [f(4R), f(0)] = [0, 4\pi R^2]$

Clearly $f(x)$ is one–one on the intervals $[0, R] \cup [3R, 4R]$

10. (b)

11. (a)

12. (a)

13. (d)

E:

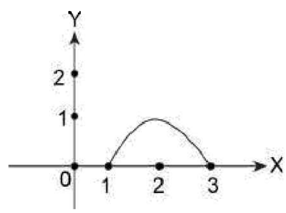
14. (d) Clearly, figure (ii) can be obtained in the following steps.

Step 1: By replacing the function $f(x)$ on x-axis, i.e., $y = f(x)$ changes to $y = -f(x)$.

Step 2: Now lifting the function obtained by 1 unit above the x-axis, i.e., $y = 1 - f(x)$.

Step 3: Shifting the function obtained in step (2) by 1 unit towards left side, i.e., $y = 1 - f(x+1)$.

15. (b) If $f(x)$ is made such that its domain is [1, 3] and range is [0, 1] that is as shown below.



The above graph can be obtained by shifting $f(x)$ by 1 unit towards right side to obtain $y = f(x - 1)$

16. (c) Figure (iii) can be obtained by shifting the graph of $y = f(x)$ towards left side by 2 units to obtained the function $y = f(x + 2)$.

17. (d) **(i)** For domain of $f(-x)$; $-x \in [0, 2]$

$\Rightarrow$ $x \in [-2, 0]$

$\therefore$ Domain $f(-x) = [-2, 0]$ and Range is same, i.e., $[0, 1]$

(ii) For $y = f(x) - 1$; Domain $= [0, 2]$ and Range $= [-1, 0]$

(iii) For $y = f(x) + 2$; Domain $= [0, 2]$ and Range $= [2, 3]$

(iv) For $y = -f(x + 1) + 1$; $(x + 1) \in [0, 2]$

$\Rightarrow$ $x \in [-1, 1]$ and Range $= [0, 1]$

F:

$C = \{(a, b, c)$: a, b, c are three consecutive even whole numbers$\}$

$= \{(2n - 2, 2n, 2n + 2); n \in \mathbb{N}\}$

$*$: $C \times C \to C$ is given by $(a, b, c) * (d, e, f) = (a + d, b + e - 2, c + f - 4)$

Closureness: Let $(a, b, c) \equiv (2n - 2, 2n, 2n + 2)$ and $(d, e, f) = (2m - 2, 2m, 2m + 2)$

$\therefore$ $(a, b, c) * (d, e, f) = (2(m + n) - 4, 2(n + m) - 2, 2(m + n))$

$= [2(m + n - 1) - 2, 2(m + n - 1), 2(m + n - 1) + 2] \in C$

$\Rightarrow$ C is closed w.r.t. *

Commutative: $(a, b, c) * (d, e, f) = (a + d, b + e - 2, c + f - 4)$ and $(d, e, f) * (a, b, c)$

$= (d + a, e + b - 2, f + c - 4)$

Clearly $(a, b, c) * (d, e, f) = (d, e, f) * (a, b, c)$

$\therefore$ (*) is commutative in C

Associativity: Let $(a, b, c), (d, e, f), (g, h, i) \in C$, then $(a, b, c) * ((d, e, f) * (g, h, i))$

$= (a, b, c) * (d + g, e + h - 2, f + i - 4)$

$= (a + d + g, b + e + h - 4, c + f + i - 8)$ and $[(a, b, c) * (d, e, f)] * (g, h, i)$

$= (a + d, b + e - 2, c + f - 4) * (g, h, i)$

$= (a + d + g, b + e + h - 4, c + f + i - 8)$

$\Rightarrow$ * is associative in C.

Existence of identity: Clearly $(0, 2, 4) \in C$ and let $(a, b, c) \in C$, then $(a, b, c) * (0, 2, 4)$

$= (0, 2, 4) * (a, b, c)$

$= (a + 0, b + 2 - 2, c + 4 - 4)$

$= (a, b, c)$

$\Rightarrow$ $(0, 2, 4)$ is an identity element in C w.r.t. *

Existence of inverse: Let $(a, b, c) \in C$, then it is not necessary that for some (d, e, f).

$(a, b, c) * (d, e, f) = (0, 2, 4)$ as if it holds, then $(a + d, b + e - 2, c + f - 4) = (0, 2, 4)$

$\Rightarrow$ $a + d = 0, b + e = 4, c + f = 8$

$\Rightarrow$ $a = d = 0, b = e = 2, c = f = 4$, *i.e.*, $(0, 2, 4)$ is the only element which is the inverse of itself.

18. (a), (d)

19. (a)

20. (b)

21. (c)

G: $A = \{2, 4, 6, 8, 12\}$ and $B = \{3, 7, 11\}$

22. (b) Each element of A has three options, i.e., 3, 7 and 11.

$\therefore$ Total number of functions $= (3)^6$.

23. (c) Pre-image of 3 can be defined ${}^6C_2$ ways and remaining 4 elements can be associated with 7 or 11, i.e., in two ways.

$\therefore$ Total number of possible functions $= {}^6C_2 \times (2)^4 = 15 \times 16 = 240$.

24. (a) Each of the elements of A has two possible images, i.e., 3 and 11.

$\therefore$ No. of possible functions $= (2)^6$

25. (b) x_1, x_2 can be selected in ${}^6C_2$ ways x_3, x_4, x_5 can be selected in ${}^4C_3$ ways.

x_6 can be selected in ${}^1C_1$ way.

$\therefore$ Number of possible functions $= {}^6C_2 \times {}^4C_3 \times {}^1C_1 = 15 \times 4 \times 1 = 60$.

26. (c) $\because f(2) = 7$ and each of 4, 6, 8, 10, 12 has three possibilities of association, i.e., 2, 3, or 11.

$\therefore$ Number of functions $= (3)^5$.

SECTION–VII (COLUMN MATCHING TYPE)

1. (i) → (b); (ii) → (b); (iii) → (b); (iv) → (c)

(i) Fundamental period of $\sin^{-1}(\sin x)$ is 2π

$\therefore$ Fundamental period of $\sin^{-1}(\sin k_x) = \dfrac{2\pi}{|k|} = \dfrac{\pi}{2}$ (given)

$\Rightarrow$ $|k| = 4$ $\qquad \Rightarrow$ $k = \pm 4$

$\therefore$ $(i) \to (b)$

(ii) $f(x) = \tan^{-1} x + \sin^{-1} x + \sec^{-1} x$

Domain of $f(x) = (-\infty, \infty) \cap [-1, 1] \cap \{(-\infty, -1] \cup [1, \infty)\} = \{-1, 1\}$; which contains exactly two points

$\therefore$ (ii) $\to (b)$

(iii) $f(x) = \sin\left(\dfrac{\pi x}{2}\right).\text{cosec}\left(\dfrac{\pi x}{2}\right)$

$$\Rightarrow f(x) = \begin{cases} 1 \text{ for } \dfrac{\pi x}{2} \neq n\pi \\ \text{Not defined at } \dfrac{\pi x}{2} = n\pi \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} 1 \text{ at } x \neq 2n; x \in \mathbb{Z} \\ \text{Not defined at } x = 2n; n \in \mathbb{Z} \end{cases}$$

$\Rightarrow$ $f(x)$ is periodic with period 2

$\therefore$ (iii) $\to$ (b)

(iv) $f(x) = \cos^{-1}[5x]$

For function to be defined $[5x] \in \{-1, 0, 1\}$

$\therefore$ Range of $f(x) = \{\cos^{-1}(-1), \cos^{-1}(0), \cos^{-1}(1)\}$

$= \left\{\pi, \dfrac{\pi}{2}, 0\right\} = \{a, b, c\}$

$\Rightarrow$ $a + b + c = \dfrac{3\pi}{2}$ $\qquad \Rightarrow$ $k = 3$

$\therefore$ (iv) $\to$ (c)

2. (i) → (b); (ii) → (b); (iii) → (d); (iv) → (b)

(i) $x^2 - 3x + \sin^{-1}(\sin 2) > 0$

Let $2 = \pi - \theta$; $\theta \in \left(0, \dfrac{\pi}{2}\right)$

$\Rightarrow x^2 - 3x + \sin^{-1}(\sin(\pi - \theta)) > 0$

$\Rightarrow x^2 - 3x + \sin^{-1}(\sin\theta) > 0$

$\Rightarrow x^2 - 3x + \theta > 0$ as $\theta \in \left(0, \dfrac{\pi}{2}\right)$

$\Rightarrow x^2 - 3x + (\pi - 2) > 0$

$\Rightarrow x \in \mathbb{R} - \left(\dfrac{3-\sqrt{17-4\pi}}{2}, \dfrac{3+\sqrt{17-4\pi}}{2}\right)$

$\Rightarrow$ Positive integer values of x satisfying the given inequality are belonging to set $\{3, 4, 5, 6, \ldots,\}$.

$\Rightarrow$ Smallest positive integer value of $x = 3$

$\therefore$ (i) → (b)

(ii) $2[x] = x + 2\{x\}$

$\Rightarrow 2[x] = [x] + \{x\} + 2\{x\}$

$\Rightarrow [x] = 3\{xy\}$ $\quad\Rightarrow \{x\} = \dfrac{[x]}{3}$

$\Rightarrow [x] \in \{0, 1, 2\}$

$\Rightarrow x \in \left\{0, \dfrac{4}{3}, \dfrac{8}{3}\right\}$ $\quad\Rightarrow$ There are 3 solutions

(ii) → (b)

(iii) $x^2 + y^2 = 1$

Let $x = \cos\theta$, $y = \sin\theta$, then $x + y = (\sin\theta + \cos\theta)$ having its maximum value $= \sqrt{2}$

$\Rightarrow$ Maximum value of $(x + y)^2 = 2$.

$\therefore$ (iii) → (d)

(iv) $f\left(x+\dfrac{1}{2}\right) + f\left(x-\dfrac{1}{2}\right) = f(x)\ \forall \in \mathbb{R}$ ......(i)

Replacing x by $x + \dfrac{1}{2}$, we get $f(x + 1) + f(x) = f\left(x+\dfrac{1}{2}\right)$ .... (2)

Equation (1) + (2) given, $f(x + 1) + f\left(x-\dfrac{1}{2}\right) = 0$

$\Rightarrow f\left(x-\dfrac{1}{2}\right) = -f(x+1) \Rightarrow f(x) = -f\left(x+\dfrac{3}{2}\right)$

$\Rightarrow f(x) = -\left[-f\left(x+\dfrac{3}{2}+\dfrac{3}{2}\right)\right]$

$\Rightarrow f(x) = f(x + 3)$

$\Rightarrow f(x)$ is periodic with period 3.

$\therefore$ (iv) → (b)

3. (i) → (b); (ii) → (a); (iii) → (c); (iv) → (d)

(i) Domain of $f(x)$ is $[-2, 2]$ for domain of $f(|x| + 1)$, $-2 \le |x| + 1 \le 2$

$\Rightarrow -3 \le |x| \le 1$ $\quad\Rightarrow x \in [-1, 1]$

$\therefore$ (i) (b)

(ii) $f(x) = \dfrac{\sin^{-1}x + \cos^{-1}x + \tan^{-1}x}{\pi}$

Domain of $f(x) = [-1, 1]$

$f(x) = \dfrac{\dfrac{\pi}{2} + \tan^{-1}x}{\pi} = \dfrac{1}{2} + \dfrac{1}{\pi}\tan^{-1}x$, which is an increasing function.

$\Rightarrow$ Range of $f(x) = [f(-1), f(1)]$

$= \left[\dfrac{1}{2} + \dfrac{1}{\pi}\tan^{-1}(-1), \dfrac{1}{2} + \dfrac{1}{\pi}\tan^{-1}(1)\right] = \left[\dfrac{1}{4}, \dfrac{3}{4}\right]$

(ii) → (a)

(iii) $f(x) = 3|\sin x| - 4|\cos x|$

Clearly, $f(x)$ is periodic with period π, so, it is sufficient to find the range for $x \in [0, \pi]$

$$\Rightarrow f(x) = \begin{cases} 3\sin x - 4\cos x; 0 \le x \le \dfrac{\pi}{2} \\ 3\sin x + 4\cos x; \dfrac{\pi}{2} < x \le \pi \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} 5\sin(x-\theta); x \in \left[0, \dfrac{\pi}{2}\right] \\ 5\sin(x+\theta); x \in \left[\dfrac{\pi}{2}, \pi\right] \end{cases}; \text{ where}$$

$\theta = \tan^{-1}\left(\dfrac{4}{3}\right)$ and $\theta \in \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

$$\Rightarrow f(x) = \begin{cases} 5\sin(x-\theta); x \in \left[0, \dfrac{\pi}{2}\right] \\ 5\sin(x+\theta); x \in \left[\dfrac{\pi}{2}, \pi\right] \end{cases}$$

Now $\dfrac{-\pi}{2} < -\theta < (x-\theta) < \dfrac{\pi}{2} - \theta < \dfrac{\pi}{4}$ for $x \in \left[0, \dfrac{\pi}{2}\right]$

and $\dfrac{3\pi}{4} < \dfrac{\pi}{2} + \theta < (x+\theta) < \pi + \theta < \dfrac{3\pi}{2}$ for $x \in \left[\dfrac{\pi}{2}, \pi\right]$

$\Rightarrow 5\sin(x - \theta)\uparrow$ for $x \in \left[0, \dfrac{\pi}{2}\right]$ and $5\sin(x + \theta)\downarrow$ for $x \in \left[\dfrac{\pi}{2}, \pi\right]$

$$\Rightarrow f(x) \in \begin{cases} \left[-5\sin\theta, 5\sin\left(\dfrac{\pi}{2}-\theta\right)\right] \text{ for } x \in \left[0, \dfrac{\pi}{2}\right] \\ \left[5\sin(\pi+\theta), 5\sin\left(\dfrac{\pi}{2}+\theta\right)\right] \text{ for } x \in \left[\dfrac{\pi}{2}, \pi\right] \end{cases}$$

$$\Rightarrow f(x) \in \begin{cases} [-4, 3] \text{ for } x \in \left[0, \dfrac{\pi}{2}\right] \\ [-4, -3] \text{ for } x \in \left[\dfrac{\pi}{2}, \pi\right] \end{cases}$$

$\Rightarrow$ Range of $f(x) = [-4, 3]$

$\therefore$ (iii) → (c)

(iv) $f(x) = (\sin^{-1}x).\sin x$

$f(x)$ is defined for $x \in [-1, 1]$

$f'(x) = \left\{(\sin^{-1})\cos x + \dfrac{\sin x}{\sqrt{1-x^2}}\right.$ for $x \in (-1, 1)$

In (–1, 1), cos x, $\dfrac{1}{\sqrt{1-x^2}} > 0$ and in (–1, 0], sin x, $\sin^{-1} x \le 0$ and in [0, 1), sin x, $\sin^{-1} x \ge 0$

$\Rightarrow$ $f'(x) \le 0$ for $x \in (-1, 0]$ and $f'(x) \ge 0$ for $x \in [0, 1)$

$\Rightarrow$ $f(x) \in \left[0, \dfrac{\pi}{2}\sin 1\right]$ for $x\,[-1, 0]$ and $f(x) \in \left[0, \dfrac{\pi}{2}\sin 1\right]$ for $x \in [0, 1)$

$\Rightarrow$ Range of $f(x) = \left[0, \dfrac{\pi}{2}\sin 1\right]$

(iv) $\to$ (d)

4. (i) → (b); (ii) → (d); (iii) → (a); (iv) → (c) , (d)

$f(x) = \sin^{-1} x, f(x) = \cos^{-1} x, h(x) = \tan^{-1} x$

(i) $f\left(\sqrt{x}\right) + g\left(\sqrt{x}\right) = \dfrac{\pi}{2}$

$\Rightarrow$ $\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \dfrac{\pi}{2}$ holds for $\sqrt{x} \in [-1, 1]$ but $\sqrt{x} \ge 0$

$\Rightarrow$ $\sqrt{x} \in [0, 1]$ $\quad\Rightarrow\quad x \in [0, 1]$

$\therefore$ (i) $\to$ (b)

(ii) $f(x) + g\left(\sqrt{1-x^2}\right) = 0$

$\Rightarrow$ $\sin^{-1} x + \cos^{-1}\sqrt{1-x^2} = 0;\ x\,[-1, 1]$

$\Rightarrow$ $\sin^{-1} x = -\cos^{-1}\sqrt{1-x^2}$

$\Rightarrow$ $x \in [-1, 0]$ $\qquad\therefore$ (ii) $\to$ (d)

(iii) $g\left(\dfrac{1-x^2}{1+x^2}\right) = 2h(x) \Rightarrow \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = 2\tan^{-1} x$

$$\Rightarrow x \ge 0, \text{i.e., } x \in [0, \infty) \text{ as } 2\tan^{-1} x = \begin{cases} \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right); x \ge 0 \\ -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right); x < 0 \end{cases}$$

$\therefore$ (iii) $\to$ (a)

(iv) $h(x) + h(1) = h\left(\dfrac{1+x}{1-x}\right)$

$\Rightarrow$ $\tan^{-1} x + \dfrac{\pi}{4} = \tan^{-1}\left(\dfrac{1+x}{1-x}\right)$

$\Rightarrow$ $\tan\left(\dfrac{\pi}{4} + \tan^{-1} x\right) = \dfrac{1+x}{1-x}$

$\Rightarrow$ $\dfrac{1+x}{1-x} = \dfrac{1+x}{1-x}$; which is true $\forall\, x \in \mathbb{R} - \{1\}$

$\Rightarrow$ Then equation holds for $x \in (-\infty, 1)$ as well as for $x \in [-1, 0]$

$\therefore$ (iv) $\to$ (c), (d)

5. (i) → (d); (ii) → (c); (iii) → (b); (iv) → (a), (d)

(i) $sgn(\{x\}) = |1 - x|$ $\quad\Rightarrow\quad |1 - x| = \begin{cases} 0 \text{ for } \{x\} = 0 \\ 1 \text{ for } \{x\} \in (0,1) \end{cases}$

$\Rightarrow$ $|1 - x| = \begin{cases} 0 \text{ for } x \in \mathbb{Z} \\ 1 \text{ for } x \notin \mathbb{Z} \end{cases}$

$\Rightarrow$ $|1 - x| = \begin{cases} 0 \text{ for } x = 1 \\ \text{No solution for } x \ne 1 \end{cases}$

$\Rightarrow$ $x = 1$ is the only solution

$\therefore$ (i) $\to$ (d)

(ii) $f(x) = \sin^{-1}\left(\dfrac{x^2 - 2x}{3}\right) + \sqrt{[x] + [-x]}$

For domain, $\dfrac{x \quad x}{} \in [-1, 1]$

$\Rightarrow$ $x^2 - 2x \in [-3, 3]$

$\Rightarrow$ $x^2 - 2x + 3 \ge 0$ and $x^2 - 2x - 3 \le 0$

$\Rightarrow$ $x \in \mathbb{R} \cap [-1, 3] = [-1, 3]$ .....(i)

Also $[x] + [-x] \ge 0$ but $[x] + [-x] = \begin{cases} -1 \text{ for } x \notin \mathbb{Z} \\ 0 \text{ for } x \in \mathbb{Z} \end{cases}$

$\Rightarrow$ $x \in \mathbb{Z}$ .....(ii)

$\therefore$ From (i) and (ii), domain of $f(x) = \{-1, 0, 2, 3\}$, which contains 5 elements.

$\therefore$ (ii) $\to$ (c)

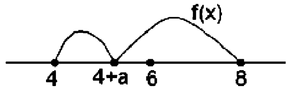
(iii) $f(x + 2) = f(x - 2)\ \forall\, x \in \mathbb{R}$

$\Rightarrow$ $f(x + 4) = f(x)\ \forall\, x \in \mathbb{R}$

$\Rightarrow$ $f(x)$ is periodic with period 4

$\therefore$ $f(8) = 0$ $\quad\Rightarrow\quad f(4) = 0$

But $f(x)$ has exactly 3 roots in the interval [4, 8], implies $f(x)$ also has a root at $x = 4 + a$, for some $a \in (0, 4)$ as shown below.



Since, $f(x)$ is periodic with period 4.

$\Rightarrow$ In [–8, 12], $f(x) = 0$, would have roots given by the set $\{-8, -8 + a, -4, -4 + {}^{a}, 0, a, 4, 4 + a, 8, 8 + a, 12\}$

i.e., $11 = k + 7$ (given)

$\Rightarrow$ $k = 4$

$\therefore$ (iii) $\to$ (b)

(iv) $f(x) = \dfrac{1}{x-1}$ and $f\left(\dfrac{1}{x-1}\right)$

$f(x)$ is not defined at $x = 1$

$\Rightarrow$ $f\left(\dfrac{1}{x-1}\right)$ is not defined at $x = 1$ and where $\dfrac{1}{x-1} = 1$

i.e., at $x = 2$

$\therefore$ $f\left(\dfrac{1}{x-1}\right)$ is not defined at $x = 1$ and $x = 2$

$\therefore$ (iv) $\to$ (a), (d)

6. (i) → (c); (ii) → (a); (iii) → (a), (b); (iv) → (d)

(i) $f(x) = \sqrt{-2\sqrt{2}\left(\cos \pi x\right) - \sqrt{3} - 1}$ is defined for $2\sqrt{2}\cos \pi x + \sqrt{3} + 1 \le 0$

$\Rightarrow$ $\cos \pi x \le -\dfrac{\left(\sqrt{3}+1\right)}{2\sqrt{2}} = -\left[\dfrac{\sqrt{3}}{2}\cdot\dfrac{1}{\sqrt{2}} + \dfrac{1}{2}\cdot\dfrac{1}{\sqrt{2}}\right]$

$= -\left[\cos\left(\dfrac{\pi}{4} - \dfrac{\pi}{6}\right)\right]$

$\Rightarrow \cos \pi x \le -\cos\frac{\pi}{12} = \cos\frac{11\pi}{12}$

$\Rightarrow \pi x \in \left[(2n+1)\pi - \frac{\pi}{12}, (2n+1)\pi + \frac{\pi}{12}\right]$

$\Rightarrow x \in \left[(2n+1) - \frac{1}{2}, (2n+1) + \frac{1}{12}\right]; x \in \mathbb{Z}$

In particular, for n = 0, $x \in \left[\frac{11}{2}, \frac{13}{12}\right]$

$\therefore$ (i) → (c)

(ii) $\lim_{x\to 1^-} \frac{3}{4}[x] + \frac{7}{8}\{x\} = L_1$

$\Rightarrow L_1 = \frac{3}{4}(0) + \frac{7}{8}\left(\lim_{x\to 1^-}(x-0)\right) = \frac{7}{8}(1) = \frac{7}{8} > \frac{3}{4}$ but $< \frac{11}{12}, 1$

$\Rightarrow L_1 \in (0, 1)$

$\therefore$ (ii) → (a)

(iii) $\sqrt{\log_x (0.75)} < 1$ for domain, $x \in (0, 1)$ and $\log_x (0.75) < 1$

$\Rightarrow 0.75 > x$

$\Rightarrow x \in \left(0, \frac{3}{4}\right) \subset (0, 1)$ $\therefore$ (iii) → (a), (b)

(iv) $\frac{|x-1|}{x+2} > 1$

For $x < -2$, $-(x-1) < (x+2)$

$\Rightarrow 2x > -1$

$\Rightarrow x > \frac{-1}{2}$; which is against the case condition.

For $-2 < x < 1$, $-(x-1) > x + 2$

$\Rightarrow 2x < -1$ $\Rightarrow x < \frac{-1}{2}$

$\Rightarrow x \in \left(-2, \frac{-1}{2}\right)$ in this case. For $x \ge 1$; $x - 1 > x + 2$

$\Rightarrow -1 > 2$; which is an absurd

$\Rightarrow x \not> 1$

$\therefore$ If the given inequality holds, then x lies in $\left(-2, \frac{-1}{2}\right)$

$\therefore$ (iv) → (d)

SECTION–VIII (INTEGER TYPE)

1. $\left|\sqrt{x^2+x+3} - x\right| = q$ …...(i)

Case (i): If $x \le 0$, then $\sqrt{x^2+x+3} > x$

$\Rightarrow \left|\sqrt{x^2+x+3} - x\right| = \sqrt{x^2+x+3} - x$

Case (ii): If $x > 0$, then $\sqrt{x^2+x+3} > x$

If $x^2 + x + 3 > x^2$, i.e., if $x + 3 > 0$ which is true.

$\therefore \left|\sqrt{x^2+x+3} - x\right| = \sqrt{x^2+x+3} - x$

Thus equation (1) becomes, $\sqrt{x^2+x+3} - x = q$

$\Rightarrow \sqrt{x^2+x+3} = x + q \Rightarrow x^2 + x + 3 = x^2 + 2qx + q^2$

$\Rightarrow x = \frac{q^3-3}{1-2q} = \frac{q^2-a}{1-bq}\left(b \ne \frac{1}{q} \text{given}\right)$

$\Rightarrow a = 3, b = 2$

Also $y = \sqrt{x^2+x+3} = x + q = |x + q|$

$\left(\because \sqrt{x^2+x+3} > 0\right)$

$= \left|\frac{q^2-3}{1-2q} + q\right| = \left|\frac{q^2-q+3}{1-2q}\right| = \left|\frac{q^2-q+3}{|1-2q|}\right|$

$(\because q^2 - q + 3 > 0 \ \forall q)$

$= \frac{q^2-q+c}{|1-bq|}$ (given) $\Rightarrow c = 3$

$\therefore$ abc = (3)(2)(3) = 2(3)²

$\Rightarrow$ Number of positive integer divisors of (abc) = 2 × 3 = 6

2. $f(x) = {}^{15-x}C_{3x-1} + {}^{20-4x}C_{5x-7}$ …..(i)

$15 - x \ge 3x - 1 \ge 0$; $15 - x \ne 0$ …(ii)

And $20 - 4x \ge 5x - 7 \ge 0$; $20 - 4x \ne 0$ ….(iii)

From (ii), $x \in \left[\frac{1}{3}, 4\right] \cap \left[\frac{7}{5}, 3\right]; x \in \mathbb{Z}$

$\Rightarrow x \in \left[\frac{7}{5}, 3\right]; x \in \mathbb{Z}$ $\Rightarrow x \in \{2, 3\}$

$\Rightarrow f(x) \in \{{}^{13}C_5 + {}^{12}C_3, {}^{12}C_8 + {}^8C_8\} = \{1507, 496\}$

$\Rightarrow$ Maximum value of $f(x) = 1507$

3. $f(x) = x^{99} + x^{79} - x^{69} + x^9 + 1$

When $f(x)$ is divided by $(x^3 - x)$, the remainder will be a quadratic polynomial $(ax^2 + bx + c)$ (say) = $f(x)$.

$\therefore f(x) = (x^3 - x).q(x) + (ax^2 + bx + c)$

$\Rightarrow f(0) = c = 1; f(1) = a + b + 1 = 3$

$\Rightarrow a + b = 2$ and $f(-1) = a - b + 1 = -1$

$\Rightarrow a - b = -2$

$\Rightarrow a = 0, b = 2, c = 1$ $\Rightarrow f(x) = 2x + 1$

$\Rightarrow f(20) = 41$

4. $f\left(x + \frac{1}{x}\right) = x^3 + x^{-3} = x^3 + \frac{1}{x^3}$

$\because \left(x + \frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right)$ for $x + \frac{1}{x} = 7$; 343

$= \left(x^3 + \frac{1}{x^3}\right) + 21$

$\Rightarrow x^3 + \frac{1}{x^3} = 322$ $\therefore f(7) = 322$

5. $\because f(x)$ is an even function

$\Rightarrow f(-x) = f(x)$ $\therefore f(x) = f\left(\frac{x+1}{x+2}\right)$

$\Rightarrow$ Either $x = \frac{x+1}{x+2}$ or $\frac{x+1}{x+2} = -x$

$\Rightarrow (x + 1) = \pm(x^2 + 2x)$

$\Rightarrow x^2 + x - 1 = 0$ or $x = \frac{-3 \pm \sqrt{5}}{2}$

$\Rightarrow$ There are 4 distinct real values.

6. $f(x + y) = f(x).f(y)\ \forall\ x \in \mathbb{N}$ and $f(1) = 2$.

$\Rightarrow f(x + 1) = f(x).f(1) \quad \Rightarrow f(x + 1) = 2f(x)$ given

$\Rightarrow f(x + 1) = 2f(x)\ \forall\ x \in \mathbb{N}$

$\therefore \sum_{n=1}^{10} f(x) = f(1) + f(2) + f(3) + \ldots + f(10) = 2 + 2(2) + 2(2)^2 + 2(2)^3 + \ldots + 2(2)^9 = 2 + (2)^2 + (2)^3 + (2)^4 + \ldots + (2)^{10}$

$= 2\left[\dfrac{(2)^{10}-1}{(2-1)}\right] = 2046$

7. $f(x) = \left(4\cos^4 x - 2\cos 2x - \dfrac{1}{2}\cos 4x - x^9\right)^{\frac{1}{9}}$

$= \left[4\cos^4 x - 2\left(2\cos^2 x - 1\right) - \dfrac{1}{2}\left(2\cos^2 2x - 1\right) - x^9\right]^{\frac{1}{9}}$

$= \left[4\cos^4 x - 4\cos^2 x + 2 + \left(\dfrac{1}{2}\right) - \left(4\cos^4 x + 1 - 4\cos^2 x\right) - x^9\right]^{\frac{1}{9}}$

$= \left[\dfrac{3}{2} - x^9\right]^{\frac{1}{9}}$

$\Rightarrow g(g(x)) = \left[\dfrac{3}{2} - (g(x))^9\right]^{\frac{1}{9}} = \left[\dfrac{3}{2} - \left(\dfrac{3}{2} - x^9\right)^9\right]^{\frac{1}{9}} = x$

$\therefore f(f(2016)) = 2016$

8. $\dfrac{\sin x + \sin 3x + \sin 5x + \sin 7x}{\cos x + \cos 3x + \cos 5x + \cos 7x}$

$= \dfrac{2\sin 4x\cos x + 2\sin 4x\cos 3x}{2\cos 4x\cos x + 2\cos 4x\cos 3x} = \tan 4x$, which has period $\dfrac{\pi}{4} = \dfrac{\pi}{k}$ (given)

$\Rightarrow k = 4$

9. Period of $\sin^{-1}(\sin x)$ is 2π

$\Rightarrow$ Period of $\sin^{-1}(\sin 4\pi x)$ is $\dfrac{2\pi}{4\pi} = \dfrac{1}{2} = k$

$\Rightarrow 4k = 2$

10. $f(x + p) = 1 + \{1 - 3f(x) + 3f^2(x) - f^3(x)\}^{1/3}$

$\Rightarrow f(x + p) = 1 + (1 - f(x)) = 2 - f(x)$

$\Rightarrow f(x + p) = 2 - [2 - f(x - p)]$

$\Rightarrow f(x + p) = f(x - p)$

$\Rightarrow f(x) = f(x + 2p)$

$\Rightarrow$ Period of $f(x) = 2p = \lambda p$ (given)

$\Rightarrow \lambda = 2$

11. $f(x - 1) + f(x + 3) = f(x + 1) + f(x + 5)$

Replacing x by $x + 2$, we get $f(x + 1) + f(x + 5) = f(x + 3) + f(x + 7)$

Adding above two equations, we get $f(x - 1) = f(x + 7)$

$\Rightarrow f(x) = f(x + 8)$

$\Rightarrow$ Period of $f(x) = 8$

12. $f(x) = \dfrac{x^3}{3} + (m - 1)x^2 + (m + 5) + 11$ and $f^{-1}(x)$ exists only for k integer values of m.

$f'(x) = x^2 + 2(m - 1)x + (m + 5)$

Disc. $= 4(m - 1)^2 - 4(m + 5) = 4[m^2 + 1 - 2m - m - 5]$

$= 4[m^2 - 3m - 4] = 4(m - 4)(m + 1) \le 0$ for $m \in [-1, 4]$

$\Rightarrow f'(x) \ge 0\ \forall\ x \in \mathbb{R}$ if $m \in (-1, 4)$

$\Rightarrow f(x)$ is invertible for integer values of $m \in \{-1, 0, 2, 3, 4\}$

$\Rightarrow k = 6$

13. Given $f(x, y) = f(2x + 2y, 2y - 2x)\ \forall\ x, y \in \mathbb{R}, f(x) = f(2^x, 0)$ and $f(x)$ is periodic with period k.

$\Rightarrow f(x) = f(2^x, 0) = f(2.2^x + 2(0), 2(0) - 2.2^x) = f(2^{x+1}, -2^{x+1})$

$= f(2.2^{x+1} - 2.2^{x+1}, -2.2^{x+1} - 2.2^{x+1}) = f(0, -2^{x+3})$

$= f(2.(-2^{x+3}), -2.2^{x+3}) = f(-2^{x+4}, -2^{x+4}) = f(-2^{x+6}, 0)$

$= f(-2^{x+7}, 2^{x+7}) = f(0, 2^{x+9}) = f(2^{x+10}, 20^{x+10}) = f(2^{x+12}, 0)$

$= f(x + 12)$

$\Rightarrow f(x)$ is periodic with period 12.

$\Rightarrow k = 12$

14. Let $y = \dfrac{ax^2 + 6x - 8}{a + 6x - 8x^2}$; We want y to be real for every real x

$\Rightarrow (8y + a)x^2 - 6yx + 6x - 8 - ay = 0$ for $x, y \in \mathbb{R}$ ……(i)

If $8y + a = 0$, i.e., $y = -\dfrac{a}{8}$

$\Rightarrow -6\left(\dfrac{-a}{8}\right)x + 6x - 8 - a\left(\dfrac{-a}{8}\right) = 0$

$\Rightarrow \dfrac{3a}{4}x + 6x - 8 + \dfrac{a^2}{8} = 0$

$\Rightarrow x\left(\dfrac{3a}{4} + 6\right) = 8 - \dfrac{a^2}{8}$

$\Rightarrow x = \dfrac{64 - a^2}{8} \times \dfrac{4}{3a + 24} = \dfrac{64 - a^2}{6a + 12} \in \mathbb{R}$ for $a \ne -2$

$\therefore$ If $a \ne -2$ and $y = -\dfrac{a}{8}$, then $x \in \mathbb{R}$

$\Rightarrow$ For $a = -2$ and $y = \dfrac{1}{4}$; $x \in \mathbb{R}$

$\Rightarrow a \ne -2$ ……(ii)

If $(8y + a) \ne 0$, then for real roots 'x' of (i); Disc. $\ge 0\ \forall\ y \in \mathbb{R}$

$\Rightarrow (6 - 6y)^2 + 4(8y + a)(8 + ay) \ge 0\ \forall\ y \in \mathbb{R}$

$\Rightarrow (9 + 8a)y^2 + (a^2 + 46)y + (9 + 8a) \ge 0\ \forall\ y \in \mathbb{R}$

If $9 + 8a = 0$, then $y \ge 0\ \forall\ y \in \mathbb{R}$, which is false, so, $a \ne -\dfrac{9}{8}$ …(iii)

Now for $9a + 8a \ne 0$

$\Rightarrow (a^2 + 46)^2 - 4(9 + 8a)(9 + 8a) \le 0$

$\Rightarrow (a^2 + 46)^2 - [2(9 + 8a)]^2 \le 0$

$\Rightarrow -2(9 + 8a) \le (a^2 + 46) \le 2(9 + 8a)$

$\Rightarrow a^2 + 16a + 64\ 0$ and $a^2 - 16a + 28 \le 0$

$\Rightarrow (a + 8)^2\ 0$ and $(a - 2)(a - 14) \le 0$

$\Rightarrow a \in [2, 14]$ ….(iv)

$\therefore$ From (ii), (iii) and (iv), we have $a \in [2, 14] = [m, n]$ (given)

$\Rightarrow m = 2, n = 14$

$\Rightarrow \sqrt{mn - 3} = 5$

15. $y = \dfrac{x-4}{16-\lambda^2}$; $x \in [x_1, x_2]$; where $x_1 = 4 - \lambda$; $x = x_2$ such that

$\Delta y = \dfrac{1}{4-\lambda}$

$\Delta y = y_2 - y_1 = f(x_2) - f(x_1)$

$\Rightarrow \dfrac{1}{4-\lambda} = \dfrac{x_2-4}{16-\lambda^2} - \dfrac{x_1-4}{16-\lambda^2}$

$\Rightarrow \dfrac{1}{4-\lambda} = \dfrac{x_2-4}{16-\lambda^2} - \dfrac{(4-\lambda)-4}{16-\lambda^2}$

$\Rightarrow \dfrac{1}{4-\lambda} = \dfrac{x_2-4-4+\lambda+4}{16-\lambda^2}$

$\Rightarrow 4 + \lambda = x_2 + \lambda - 4 \quad \Rightarrow x_2 = 8$

16. $f(x) = \cos^2 x + \cos^2\left(\dfrac{\pi}{3}+x\right) - \cos x.\cos\left(\dfrac{\pi}{3}+x\right)$; $g\left(\dfrac{3}{4}\right) = 2$; go $f(x)$ =?

$f(1) = \cos^2 1 + \cos^2\left(\dfrac{\pi}{3}+1\right) - \cos 1 \cos\left(\dfrac{\pi}{3}+1\right)$

$\Rightarrow f(1) = \cos^2 1 + \cos\left(\dfrac{\pi}{3}+1\right).\left[\cos\left(\dfrac{\pi}{3}+1\right) - \cos 1\right]$

$\Rightarrow f(1) = \cos^2 1 + \cos\left(\dfrac{\pi}{3}+1\right).\left[-2\sin\left(\dfrac{\pi}{6}+1\right)\sin\dfrac{\pi}{6}\right]$

$\Rightarrow f(1) = \cos^2 1 - \cos\left(\dfrac{\pi}{3}+1\right)\sin\left(\dfrac{\pi}{6}+1\right)$

$\Rightarrow f(1) = \cos^2 1 - \dfrac{1}{2}\left[\sin\left(\dfrac{\pi}{2}+2\right) + \sin\left(-\dfrac{\pi}{6}\right)\right]$

$\Rightarrow f(1) = \cos^2 1 - \dfrac{1}{2}\left[\cos 2 - \dfrac{1}{2}\right]$

$= \cos^2 1 - \dfrac{1}{2}\left(2\cos^2 1 - 1\right) + \dfrac{1}{4} = \dfrac{3}{4}$

$\therefore$ $gof(1) = f(f(1)) = f(3/4) = 2$

17. f: {1, 2, 3} → {1, 2, 3, 4, 5} such that $f(i) \le f(j)$ whenever i < j and $i, j \in$ {1, 2, 3}

Case (i): $f(1) = f(2) < f(3)$

Now select any 2 numbers out of 1, 2, 3, 4, 5 and arrange them in ascending order, the smallest will be the value $f(1)$ and $f(2)$ and greatest of 5 values $f(4)$ can take any of 5 values.

$\therefore$ Number of functions = ${}^5C_2 \times 5 = 50$

Case (ii): $f(1) < f(2) = f(3)$

$\Rightarrow$ Number of functions = ${}^5C_2 \times 5 = 50$

Case (iii): $f(1) = f(2) = f(3)$

$\Rightarrow$ Number of functions = ${}^5C_1 \times 5 = 25$

Case (iv): $f(1) < f(2) < f(3)$

$\Rightarrow$ Number of functions = ${}^5C_3 \times 5 = 50$

$\therefore$ Total number of possible functions = 50 + 50 + 25 + 50 = 175

18. Let $p(x) = ax^2 + bx + c$ then, $p(x) \ge 0\ \forall\ x \in \mathbb{R}$

$\Rightarrow a > 0,\ b^2 - 4ac \le 0$ ...(i)

Also $p(1) = 0$ and $p(2) = 2$ ...(ii)

$\because\ p(1) = 0$

$\Rightarrow$ 1 is a root

$\Rightarrow$ $p(x)$ must have repeated root 1

$\Rightarrow \dfrac{p(x)}{a} = x^2 + \dfrac{b}{a}x + \dfrac{c}{a} = (x-1)^2$

$\Rightarrow \dfrac{b}{a} = -2, \dfrac{c}{a} = 1$

$\Rightarrow b = -2a,\ c = a$

$\therefore\ p(x) = ax^2 - 2ax + a = a(x-1)^2$

$\because\ p(2) = 2$

$\Rightarrow a = 2$

$\Rightarrow p(x) = 2(x-1)^2$

$\Rightarrow p(0) + p(3) = 2(1) + 2(2)^2 = 10$

19. $x^4 - 4x^3 + 6x^2 - 4x = 2008$

$\Rightarrow (x-1)^4 = 2008$

$\Rightarrow (x-1)^2 = \pm\sqrt{2008}$

$\therefore$ Imaginary roots are given by $(x-1)^2 = -\sqrt{2008} = \sqrt{2008}i^2$

$\Rightarrow x - 1 = \pm (2008)^{1/4}\, i$

$\Rightarrow x = 1 \pm i(2008)^{1/4}$

$\Rightarrow x_1x_2 = p = 1 + (2008)^{1/2}$

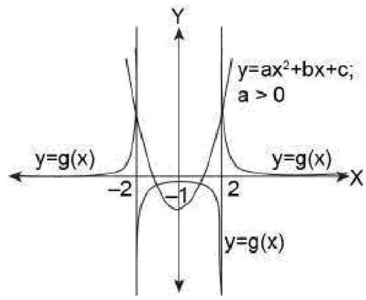
$\Rightarrow p \in (1 + 44.8, 1 + 44.9)$

$\Rightarrow p \in (45.8, 45.9)$

$\Rightarrow [p] = 45$

20. $f(x) = ax^2 + bx + c$, $a > 0$ and $f(x) = \dfrac{1}{x^2-4}$

Graph of $y = f(x)$ and $y = ax^2 + bx + c$ drawn on same frame of reference the maximum number of point of intersection is as shown below.



Clearly maximum number of points of intersection is 4.

CHAPTER 3

Graph Theory

INTRODUCTION

The word Graph originated from the Greek word 'GRAPHO' which means to write. One may define it as 'Things written or drawn in a specified way, e.g., photograph, painting etc. You may also call it as a diagram showing the relation between variable quantities'. In discrete mathematics a graph is a representation of a set of 'points or nodes' (objects) where some pairs of points are connected by links (straight or curved). The interconnected point objects are called as vertices and the connecting links are known as edges.

It is better to say that drawing 'Graph' is a visual or descriptive art, which is vividly descriptive and easily understandable. It provides maximum information in a minimum space about the function (thing which it represents) and being a visual representation always leaves an impression lasting longer than that of any other manner of representation like mathematical, analytical etc. As the photograph of a person conveys his identity better than any amount of description of his physical measurement, color etc. Similarly the graph of a function provides us better understanding of the nature and behavior of a function. Graphs not only provide information about functions but also help in solving many problems related to number of solution of some equations and area enclosed by curves.

So, we are going to analyze the graphical approach of understanding the function and we will try to graph a function with the help of properties of function like domain, range, limits, continuity, differentiability derivative, mono-tonicity, maxima/minima of functions. Thus, to sketch the graph of some unknown function, the complete knowledge of differential calculus is required along with the understanding of some very basic functions and their curves. And with the above tools available to us we will try to portray the graph in the best possible way with the added application of our individual logical skills.

GRAPHS OF FUNCTION

Graph of a function means the curve obtained by plotting all the ordered pairs (x, y) which satisfy the equation of the given function, $y = f(x)$. Let us classify the basic curves function wise.

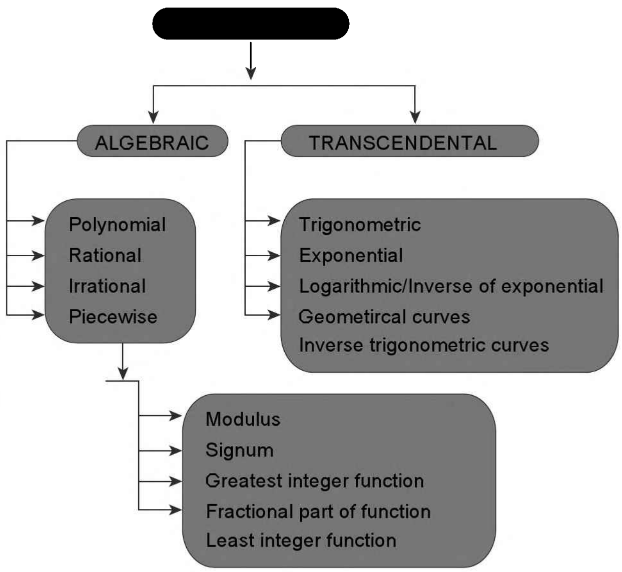


FIGURE 3.1

ALGEBRAIC FUNCTIONS

Polynomial Function

A function of the form: $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$; where $n \in \mathbb{N}$ and $a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$ is known as polynomial function in x. The natural number n is called as degree of polynomial. Depending on the value of n these are classified into different categories, e.g., linear ($n = 1$), quadratic ($n = 2$), cubic ($n = 3$), biquadratic ($n = 4$).

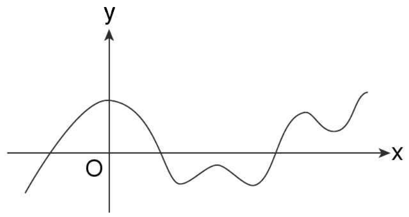


FIGURE 3.2

(a) **Straight line (linear polynomial):** Every first degree equation in x, y represents a straight line. So, the general equation of a line is $ax + by + c = 0$. To draw a straight line, find the points where it meets with the coordinate axes by putting $y = 0$ and $x = 0$ respectively in its equation. By joining these two points, we get the sketch of the line

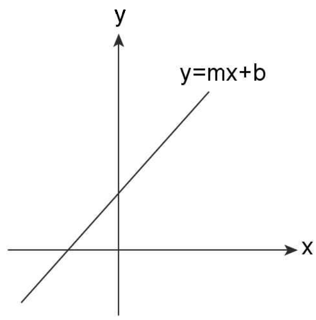


FIGURE 3.3

If a polynomial contains only one term, then it is called as monomial. Some of basic monomial functions are described below:

(b) **Identity function or graph of $f(x) = x$:** A function f defined by $f(x) = x$ for all $x \in \mathbb{R}$, is called identity function.

Here, $y = x$ clearly represents a straight line passing through the origin and inclined at an angle of 45° with x-axis. The domain and range of identity functions are both equal to $\mathbb{R}$.

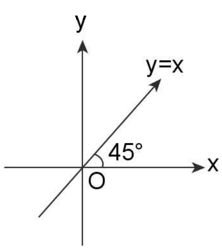


FIGURE 3.4

(c) **Graph of $f(x) = x^2$:** A function given by $f(x) = x^2$ is a pure quadratic monomial called as 'square function'.

The domain of square function is $\mathbb{R}$ and its range is $\mathbb{R}^+ \cup \{0\}$ or $[0, \infty)$. Since $y = x^2$ is an even function, so, its graph is clearly a parabola symmetrical about y-axis with vertex at origin (0, 0) as shown below.

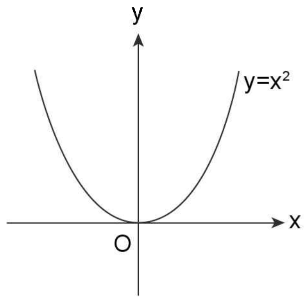


FIGURE 3.5

(d) **Graph of $f(x) = x^3$:** A function given by $f(x) = x^3$ is called a cubic monomial function with the domain and range both equal to $\mathbb{R}$. The following table represents few inputs and corresponding outputs of the above function.

x	.....	–3	–2	–1	0	1	2	3	.....
y	.....	–27	–8	–1	0	1	8	27	.....

It is clearly an odd continuous function, thus, its graph must be symmetric about origin as shown below.

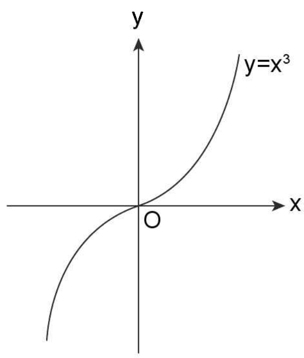


FIGURE 3.6

(e) **Power functions:**

(i) Graph of $f(x) = x^{2n}$; $n \in \mathbb{N}$: If $n \in \mathbb{N}$, then function f given by $f(x) = x^{2n}$ is an even function.

So, its graph is always symmetrical about y-axis. To sketch these curves for various possible n and to know the relative position it is required to analyze and understand the following facts.

The value of x^{2n} decreases by increasing the exponent (power) in the domain $x \in (-1, 1)$, whereas it increases by increasing the power for all $x \in (-\infty, -1] \cup (1, \infty)$.

i.e., $x^2 > x^4 > x^6 > x^8 > \ldots \ \forall\, x \in (-1, 1)$ and
$x^2 < x^4 < x^6 < x^8 < \ldots 1\ \forall\, x \in (-\infty, -1] \cup (1, \infty)$.
The relative graphs of $y = x^2$, $y = x^4$, $y = x^6$,...etc. are drawn as shown in the following diagram:

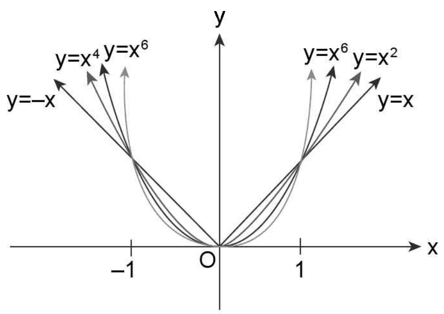


FIGURE 3.7

Rectify the GRAPH

(ii) Graph of $\boldsymbol{f(x) = x^{2n-1}}$; $n \in \mathbb{N}$: If $n \in \mathbb{N}$, then function f given by $f(x) = x^{2n-1}$ is an odd function.
So, its graph is always symmetrical about origin or in opposite quadrants. To study the relative positioning of these graphs for different values of $n \in \mathbb{N}$, we need to compare the values of x, x^3, x^5,..... etc.

For instance $\forall\, x \in (1, \infty)$; $x < x^3 < x^5 < \ldots$whereas for $x \in (0, 1)$ $x > x^3 > x^5 > \ldots$

But when $x \in (-1, 0)$, then $x < x^3 < x^5 < \ldots$

And for $x \in (-\infty, -1)$ we find that $x > x^3 > x^5 > \ldots$

Then put a point mirror at the origin and you will find that for every point (x_1, x_1^3) on the curve, there is an image point $(-x_1, -x_1^3)$. i.e., the curve $y = x^3$, i.e., symmetrical about origin. In the light of the above facts we get the graphs of $f(x) = x$, $f(x) = x^3$, $f(x) = x^5$,.. etc. as shown below.

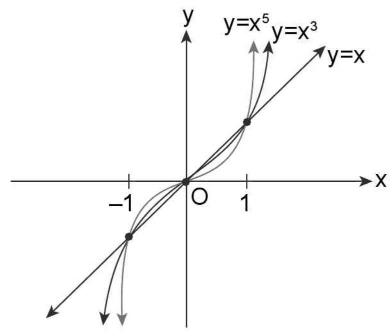


FIGURE 3.8

ILLUSTRATION 1: Find the area of the region enclosed by $f(x) = x^5$ and the semi-circular arc of the circle $x^2 + y^2 = 2$, lying below the straight line $y = x$.

SOLUTION: The semi-circular arc described in the problem is drawn as shown in Figure 3.9. The graph of $f(x) = x^5$ passes through $A(-1, -1)$, $O(0, 0)$, $B(1, 1)$ as drawn in Figure 3.10.

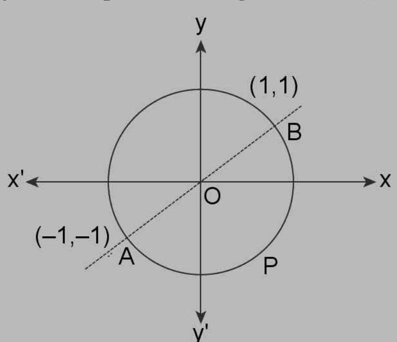


FIGURE 3.9

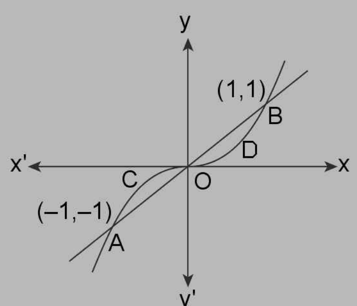


FIGURE 3.10

Clearly, the area $\widehat{ACOA}$ is equal to area $\widehat{ODBO}$ (say Δ) because $f(x) = x$ and $f(x) = x^3$ both the curves are symmetric about origin.

Thus, the required area is shaded in Figure 3.11 must be equal to

Area of semicircle – $\Delta + \Delta$ = Area of semicircle $= \pi(\sqrt{2})^2 = 2\pi$ square unit.

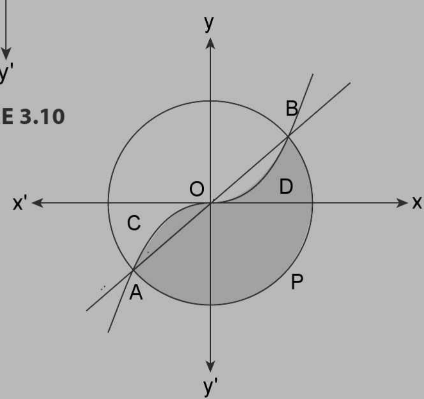


FIGURE 3.11

RATIONAL FUNCTIONS

The function, written as the quotient of two polynomial functions is known as a rational function. Given two polynomial functions $P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_n x^n$ and $Q(x) = b_0 + b_1x + b_2x^0 + \ldots + b_m x^m$, the function $f(x) = \dfrac{P(x)}{Q(x)}$ is a rational function of x. Domain = $\mathbb{R} - \{x : Q(x) \neq 0\}$. i.e., domain = $\mathbb{R}$, except for those points for which denominator = 0. Graphs of the some basic rational functions are discussed as below:

(a) **Graph of $f(x) = 1/x$:** *A* function defined by $f(x) = 1/x$ is called the reciprocal function or rectangular hyperbola, with coordinate axes as asymptotes. The domain and range of $f(x) = 1/x$ is $\mathbb{R} - \{0\}$.
Indeed $f(x)$ is an odd function, and hence, its graph is symmetrical in opposite quadrants. The reciprocal function is a decreasing function. Also we observe $\lim\limits_{x\to 0^+} f(x) = +\infty$ and $\lim\limits_{x\to 0^-} f(x) = -\infty$. As $x \to \pm\infty$ $\Rightarrow f(x) \to 0$. Therefore, co-ordinate axes are asymptotes of function. Thus, $f(x) = 1/x$ could be shown in the graph shown below.

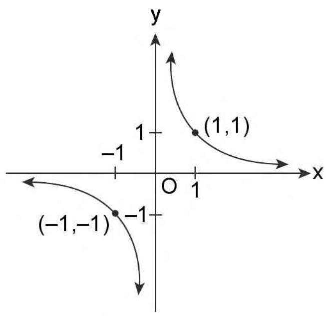


FIGURE 3.12

(b) **Graph of $f(x) = 1/x^2$:** Here $f(x) = 1/x^2$ is an even function, so, its graph is symmetrical about y-axis. Domain of $f(x)$ is $\mathbb{R} - \{0\}$ and range is $(0, \infty)$. Clearly $f(x)$ is a decreasing function in $(0, \infty)$, whereas $f(x)$ is an increasing function in $(-\infty, 0)$. Also $y \to \infty$ as $x \to 0^{\pm}$ and $y \to 0$ as $x \to \pm\infty$, therefore, co-ordinate axes are asymptotes of function.
Thus, graph of $f(x) = 1/x^2$ is as shown in the Figure 3.13.

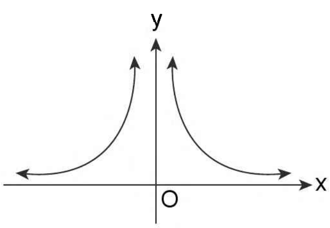


FIGURE 3.13

(c) **Graph of $f(x) = \dfrac{1}{x^{2n-1}}$; $n \in \mathbb{N}$:** Here, $f(x) = \dfrac{1}{x^{2n-1}}$ is an odd function, so, its graph is symmetrical in opposite quadrants. Also, $y \to \infty$ when $x \to 0^+$ and $y \to -\infty$ as $x \to 0^-$. Therefore, co-ordinate axes are asymptotes of function.
Thus, the graph for $f(x) = \dfrac{1}{x^3}$; $f(x) = \dfrac{1}{x^5}$, ..., etc., will be similar to the graph of $f(x) = \dfrac{1}{x}$ which has asymptotes as coordinate axes, shown as in Figure 3.14.

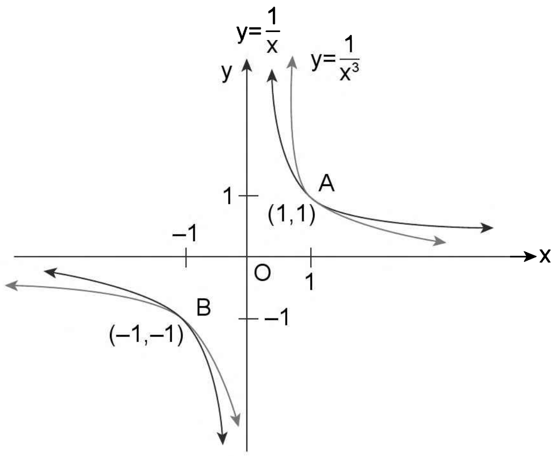


FIGURE 3.14

(d) **Graph of $f(x) = \dfrac{1}{x^{2n}}$; $n \in \mathbb{N}$:** We observe that the function $f(x) = \dfrac{1}{x^{2n}}$ is an even function, so, its graph is symmetrical about y-axis. Also, $y \to \infty$ as $x \to 0^{\pm}$ and $y \to 0$ as $x \to \pm\infty$.
The values of y decreases as the values of x increases for $x > 0$. Thus, the graph of $f(x) = \dfrac{1}{x^4}$; $f(x) = \dfrac{1}{x^6}$, ..., etc. will be similar as the graph of $f(x) = \dfrac{1}{x^2}$, which has asymptotes as coordinate axis, as shown in Figure 3.15.

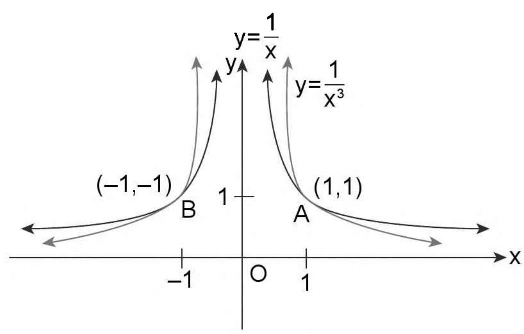


FIGURE 3.15

ILLUSTRATION 2: Find the number of solution of equation $x^3.\{ x \} = 1$, lying in the interval [–2, 5]. Here {.} denotes fractional part function.

SOLUTION: Given equation $x^3.\{ x \} = 1 \Rightarrow \{ x \} = 1/x^3$

Therefore, by drawing both the curves $y = \{ x \}$ and $y = 1/x^3$ as shown below in Figure 3.16

we can count the number of points of intersections as four. Consequently the given equation has only four real solutions lying in the interval [– 2, 5].

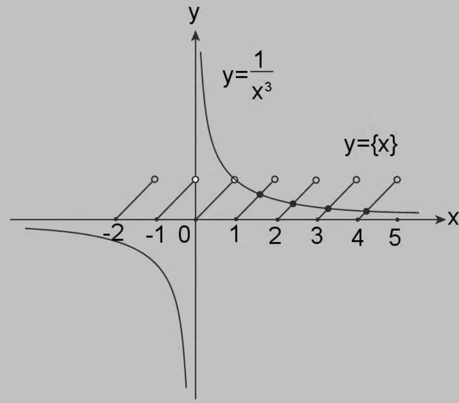


FIGURE 3.16

IRRATIONAL FUNCTION

The algebraic functions having at least one term containing non-integer rational power of x are termed as irrational functions of x.

Graph of Some Basic Irrational Function

(a) **Graph of $f(x) = x^{1/2n}$; $n \in \mathbb{N}$:** Since, $f(x) = x^{1/2n}$ is defined for all $x \in [0, \infty)$ and generates positive values only, therefore, both the domain and range of $f(x)$ must be $[0, \infty)$.

Also being inverse function of the function $f(x) = x^{2n}$, the graph of $f(x) = x^{1/2n}$ is the mirror image of the graph of $f(x) = x^{2n}$ about the line $y = x$; $\forall\ x \in [0, \infty)$.

Thus, the graphs of $f(x) = x^{1/2}$, $f(x) = x^{1/4}$,.... are as shown in the Figure 3.17.

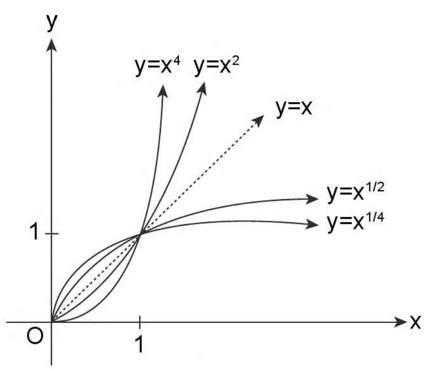


FIGURE 3.17

For instance; the graph of $f(x) = \sqrt{x}$, i.e., $x^{1/2}$ is the portion of the parabola $y^2 = x$, which lies above x-axis. its domain and range both are given as $[0, \infty)$. It is shown below.

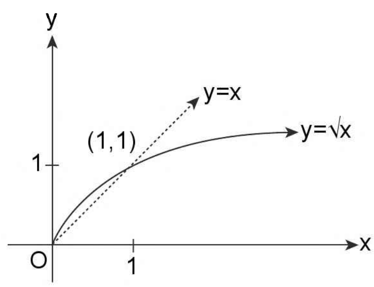


FIGURE 3.18

(b) **Graph of $f(x) = x^{1/2n-1}$**; when $n \in \mathbb{N}$

The function $f(x) = x^{1/2n-1}$ is defined for all $x \in \mathbb{R}$ as the odd root of every real number is defined as real number. Also we can see that $f(x)$ returns positive values for positive input and negative values corresponding to negative inputs. Thus, the domain as well and range of $f(x)$ is $\mathbb{R}$.

Also $f(x) = x^{1/2n-1}$ being the inverse function of $f(x) = x^{2n-1}$, the graph of $f(x) = x^{1/2n-1}$ is the mirror image of the graph of $f(x) = x^{2n-1}$ about the line $y = x\ \forall\ x \in \mathbb{R}$. Consequently the graph of, $f(x) = x^{1/3}$, $f(x) = x^{1/5}$,..., shall be as shown in the following Figure 3.19.

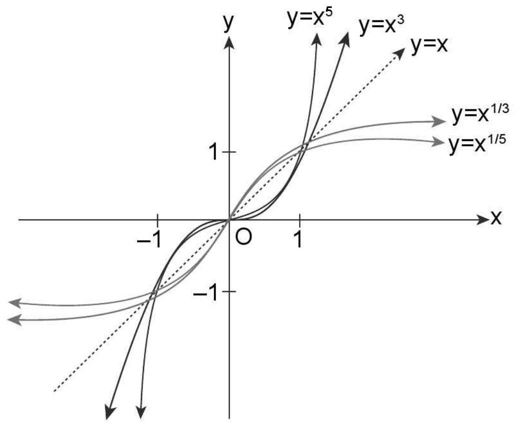


FIGURE 3.19

e.g., Graph of $f(x) = x^{1/3}$. As discussed above, is inverse of $g(x) = x^3$. Therefore, the graph of $f(x) = x^{1/3}$ is image of $g(x)$ about $y = x$. where domain and and range of $f(x)$ is $\mathbb{R}$. Thus, the graph of $f(x) = x^{1/3}$ is as shown in the Figure 3.20.

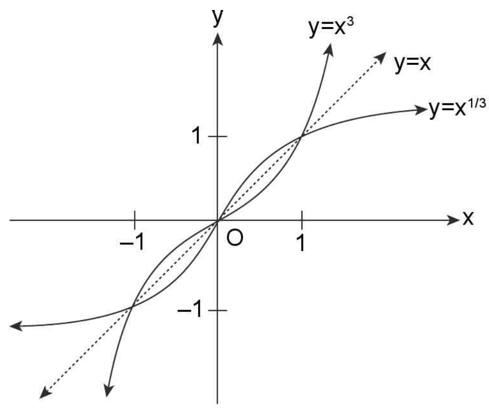


FIGURE 3.20

ILLUSTRATION 3: A function $f(x)$ is defined as $f(x) = \begin{cases} x^{1/3}; & x \geq 0 \\ -x^{1/3}; & x < 0 \end{cases}$, whereas $g(x)$ is given by $g(x) = \begin{cases} -\dfrac{1}{x}; & x < 0 \\ x(x-1); & x \geq 0 \end{cases}$. Find the number of points of intersections of $f(x)$ and $g(x)$.

SOLUTION: Clearly for $x \in [0, \infty)$; $f(x)$ is reflection of x^3 in $y = x$ and then taking its even extension . Complete graph of $f(x)$ and $g(x)$ are shown below in Figure 3.21 and 3.22 respectively.

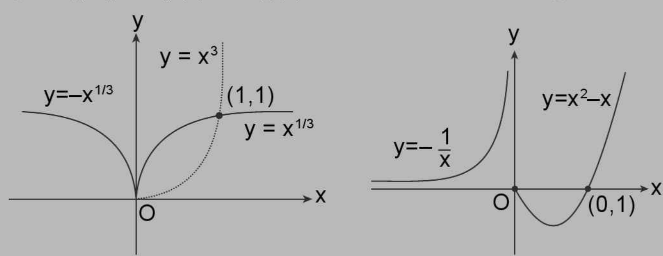


FIGURE 3.21

Clearly, sketching both functions together as drawn below, we observe that they have three points of intersections.

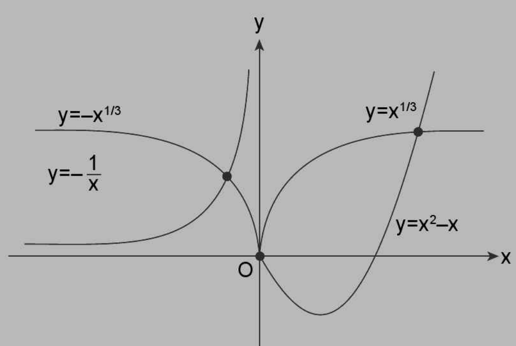


FIGURE 3.22

■ PIECE-WISE DEFINED FUNCTIONS

(a) **Modulus function or absolute value function:**
The function $f: \mathbb{R} \to \mathbb{R}$ is said to be modulus function or absolute value function if

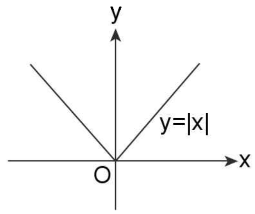


FIGURE 3.23

$f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{if } x = 0 \\ -x, & \text{if } x < 0 \end{cases}$; $|x|$ is pronounced as modulus of x or in brief mod x.

Note: $|x|$ is also defined as $\sqrt{x^2}$ and max $\{x, -x\}$.

(b) **Signum function:** The function $f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

is called a signum function and is denoted by *sgn* (x). Graph of signum function, i.e., $y = sgn(x)$ is shown. Hence, *sgn* ($f(x)$) is:

$$f(x) = \begin{cases} \dfrac{|f(x)|}{f(x)} = 1 \;; & f(x) > 0 \\ -\dfrac{f(x)}{f(x)} = -1; & f(x) < 0 \\ 0 \;; & f(x) = 0 \end{cases}$$

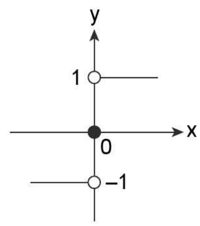


FIGURE 3.24

ILLUSTRATION 4: Draw the graph of *sgn* (sin x)

SOLUTION: $sgn\ (\sin x) = \begin{cases} 1, & \text{when } \sin x > 0 \\ -1, & \text{when } \sin x < 0 \\ 0, & \text{when } \sin x = 0 \end{cases}$

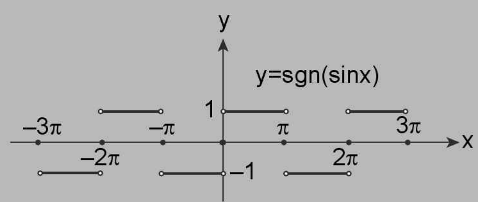


FIGURE 3.25

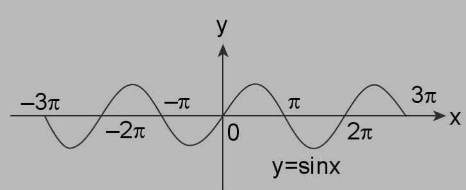


FIGURE 3.26

ILLUSTRATION 5: Find the number of real solutions of equation $f(x) = g(x)$ where f and g are functions defined as $f(x) = \begin{cases} |x+1|; & x \leq 0 \\ \sqrt{1-x^2}; & x > 0 \end{cases}$; $g(x) = \begin{cases} -x^{1/5}; & x < 0 \\ x^2; & x \geq 0 \end{cases}$

SOLUTION: The function $f(x)$ can be defined as $f(x)=\begin{cases} -(1+x); & x<-1 \\ x+1; & -1\le x\le 0 \\ \sqrt{1-x^2}; & 0<x\le 1 \end{cases}$

Thus, its graph is given below.

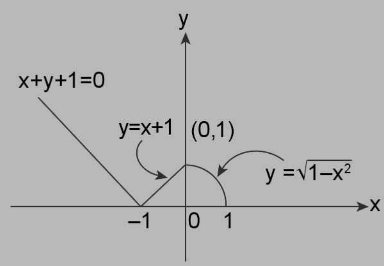


FIGURE 3.27

The graph of $g(x)$ is as shown below

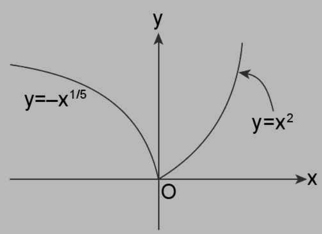


FIGURE 3.28

Sketching them together or superimposing $g(x)$ on $f(x)$, we get $f(x) = g(x)$ has three solutions

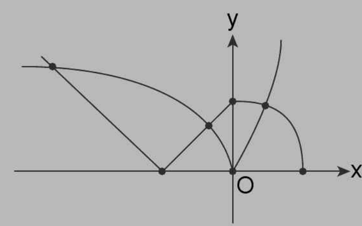


FIGURE 3.29

(c) **Greatest integer function:** For any real number x, we denote $[x]$, the greatest integer less than or equal to x, it is also known as integral part of the number x.

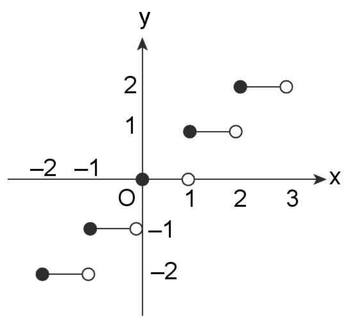


FIGURE 3.30

i.e., $f(x) = [x] = -1$ for $-1 \le x < -2$
$[x] = 0$ for $0 \le x < 1$
$[x] = 1$ for $1 \le x < 2$
$[x] = 2$ for $2 \le x < 3$

The domain of the function is: $(-\infty, +\infty)$ and the range is : set of *all integers*.

(d) **Least integer function:** It is also called the ceiling of x and it is represented by $\lceil x \rceil$. It is the least integer greater than or equal to the number. Therefore, $y = \lceil x \rceil = I + 1$ if $I < x \le I + 1$

e.g., $\lceil 1.5 \rceil = 2, \lceil 2.9 \rceil = 3, \lceil -2.3 \rceil = -2$, $\lceil -0.6 \rceil = 0, \lceil 0.25 \rceil = 1$, thus, $[x]$ converts $x = (I + f)$ into I while $\lceil x \rceil$ converts it into $I + 1$. But when x is an integer $[x] = x = \lceil x \rceil$. Thus,

$\lceil x \rceil = -2$ for $-3 < x \le -2$
$\lceil x \rceil = -1$ for $-2 < x \le -1$
$\lceil x \rceil = 0$ for $-1 < x \le 0$
$\Rightarrow \lceil x \rceil = 1$ for $0 < x \le 1$
$\lceil x \rceil = 2$ for $1 < x \le 2$
$\lceil x \rceil = 3$ for $2 < x \le 3$

and can be expressed graphically as shown in Figure 3.31.

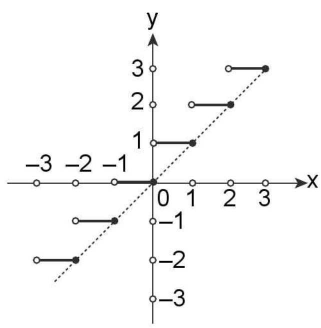


FIGURE 3.31

(e) **Fractional—part function:** Since we know that $x \ge [x]$. the difference between the number x and it's integral value $[x]$ is called the fractional part of x and is symbolically denoted as $\{x\}$.

Thus, $\{x\} = x - [x]$

e.g., $\{2.5\} = 0.5$, $\{-1.9\} = -1.9 - [-1.9] = -1.9 + 2 = 0.1 = 1 - (0.9) = 1 - \{1.9\}$

Since $f(x) = \{x\}$,

$\{x\} = x$ for $0 \le x < 1$

$\{x\} = x - 1$ for $1 \le x < 2$
$\Rightarrow \{x\} = x - 2$ for $2 \le x < 3$
$\{x\} = x + 1$ for $-1 \le x < 0$
$\{x\} = x + 2$ for $-2 \le x < -1$

and can be expressed graphically as shown in Figure 3.32:

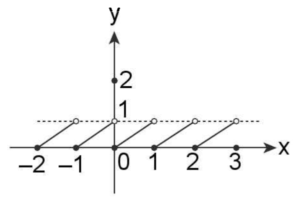


FIGURE 3.32

ILLUSTRATION 6: Obtain the number of real solution of the equation $\left[\frac{x}{2}\right] = \log x$, where [.] denotes the integer function of x.

SOLUTION: Consider $f(x) = \left[\frac{x}{2}\right] = \begin{cases} -1; -2 \le x < 0 \\ 0;\ 0 \le x < 2 \\ 1;\ 2 \le x < 4 \end{cases}$

and $g(x) = \log x$

Superimposing both the curves together, clearly the equation has only one solution, Figure 3.32. i.e., $x = 1$.

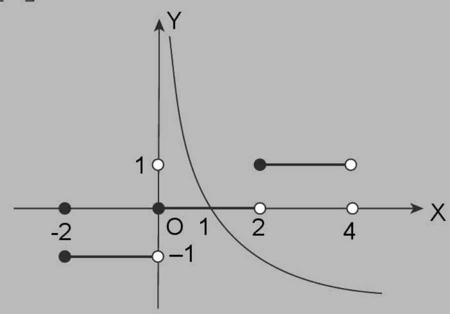


FIGURE 3.33

■ TRANSCENDENTAL FUNCTIONS

Trigonometric Curves

sin$x$, cos x, tan x, cot x, sec x, cosec x are trigonometric curves. Their graphs are as follows:

(a) $f(x) = \sin x$: Let $f(x) = \sin x$, increases strictly from – 1 to 1 as x increases from $-\pi/2$ to $\pi/2$, decreases for x changing from $\pi/2$ to $3\pi/2$ and so on, so, the graph of sin x and cosec x will be as shown in the Figure 3.34.

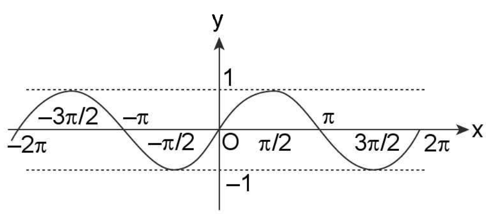


FIGURE 3.34

Properties:

1. Domain of sin x is $\mathbb{R}$ and range is $[-1, 1]$.
2. It is an odd function.
3. sin x is periodic function with period 2π.
4. Principal domain is $[-\pi/2, \pi/2]$.
5. It is a continuous function and increases in first and fourth quadrants while decreases in second and third quadrants.

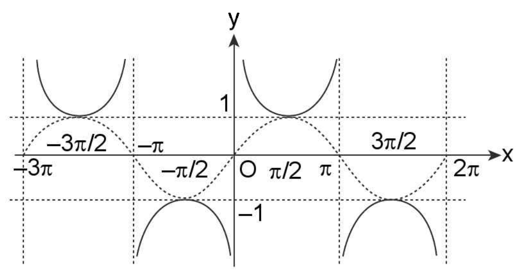


FIGURE 3.35

(b) $f(x)$ = cosec x: cosec x is reciprocal of sin x.

Properties:

1. The domain is $\mathbb{R} - \{n\pi \mid n \in \mathbb{Z}\}$.
2. Range of cosec$x$ is $\mathbb{R} - (-1, 1)$.
3. Principal domain is: $[-\pi/2, \pi/2] - \{0\}$
4. The cosec x is periodic with period 2π; $n \in \mathbb{Z}$
5. It is odd function discontinuous at $x = n\pi$
6. It decreases in first and fourth quadrants while increases in second and third quadrants.

(c) **$f(x)$ = cos x:** As discussed, cos$x$ decreases strictly from 1 to – 1 as x increases from 0 to π, increases strictly from – 1 to 1 as x increases from π to 2π and so on.

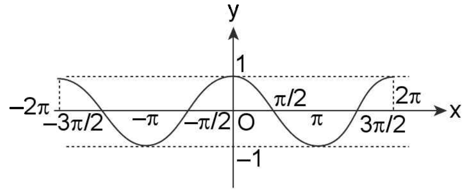


FIGURE 3.36

Properties:

1. The domain of cos x is ℝ and the range is [–1, 1].
2. Principal domain is [0, π].
3. It is an even function, i.e., symmetric about y-axis.
4. cos x is periodic with period 2π.
5. It is continuous function decreases in Ist and IInd quadrant increases in IIIrd and IVth quadrant.

(d) $f(x)$ = sec x: sec x is reciprocal of cos$x$.

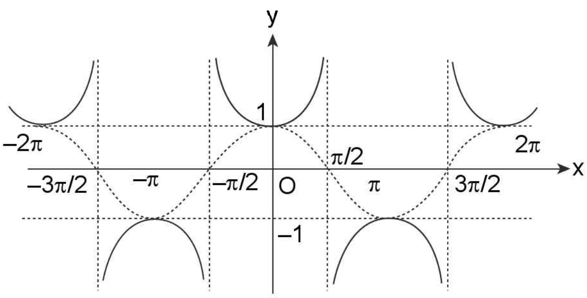


FIGURE 3.37

Properties:

1. The domain of sec x is $\mathbb{R} - \left\{(2n+1)\frac{\pi}{2}\middle|\, n \in Z\right\}$ and range is ℝ – (– 1, 1).
2. The sec x is periodic with period 2π.
3. Principal domain is [0, π] – {π/2}
4. It is discontinuous at x = (2n +1) π/2; n ∈ ℤ.

(e) y = tan x:

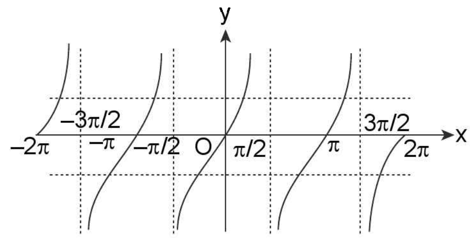


FIGURE 3.38

1. The domain of tan x is $\mathbb{R} - \{(2n+1)\pi/2\}$ and range ℝ or (– ∞, ∞). Principal domain is (– π/2, π/2).
2. It is periodic with period π.
3. It is discontinuous at x = $\mathbb{R} - \{(2n+1)\pi/2\}$ and it is strictly increasing function in its domain.

(f) y = cot x: cot x is reciprocal of tan x.

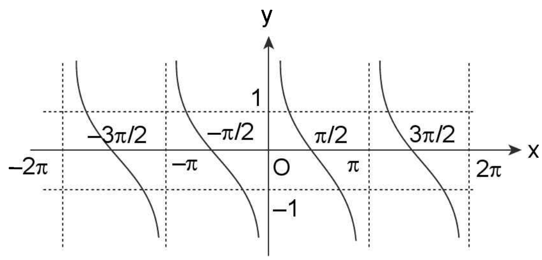


FIGURE 3.39

1. The domain of $f(x)$ = cot x = ℝ ~ {nπ}; Range = ℝ.
2. It is periodic with period π and x = nπ, n ∈ ℤ as asymptotes.
3. Principal domain is (0, π).
4. It is discontinuous at x = nπ; n ∈ ℤ.
5. It is strictly decreasing function in its domain

ILLUSTRATION 7: Determine the number of solutions of the equation tan x = log x lying in the interval [0, 3π].

SOLUTION: Sketching both the graph on same set of axes, we get clearly four real solutions of the given equation.

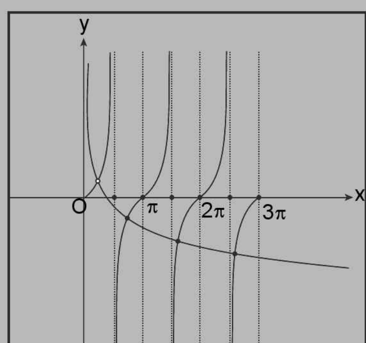


FIGURE 3.40

ILLUSTRATION 8: Find the number of real values of x satisfying the equation $\sin x - x^3 = 0$.

SOLUTION: $f(x) = x^3$ is an increasing function and $f(x) = x^3 \in [-1, 1]$ when $x \in [-1, 1]$.

Thus, solution exist only in the interval $[-1, 1]$ and represented by number of point of intersection of the graph $y = x^3$ and $y = \sin x$.

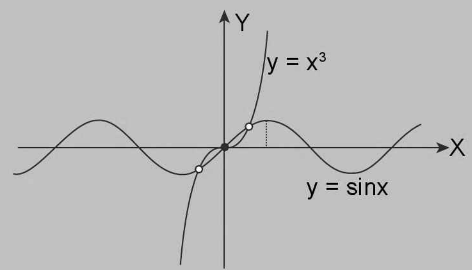


FIGURE 3.41

Clearly, from the graph we can see the three real solutions.

Exponential Function

Here, $f(x) = a^x$, $a > 0$, $a \neq 1$, and $x \in \mathbb{R}$, where domain = $\mathbb{R}$, range = $(0, \infty)$.

Case I. **$a > 1$:** Here, $f(x) = y = a^x$ increases with the increase in x,

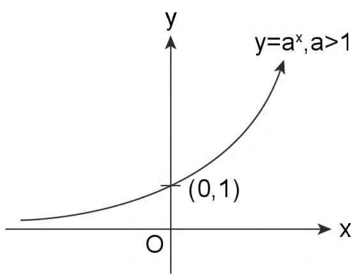


FIGURE 3.42

That is, $f(x)$ is increasing function on $\mathbb{R}$.

For example, $y = 2^x$, $y = 3^x$, $y = 4^x$,... . We have; $2^x < 3^x < 4^x <$.... for $x > 1$ and $2^x > 3^x > 4^x >$.... for $0 < x < 1$.

and they can be shown as shown in Figure 3.43..

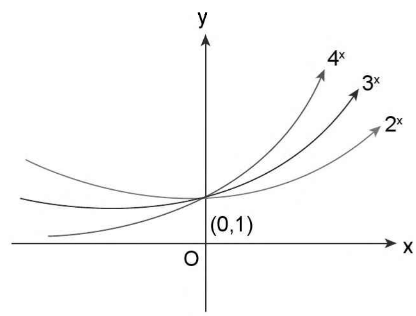


FIGURE 3.43

Case II. **$0 < a < 1$:** Here, $f(x) = a^x$ decrease with the increase in x, i.e., $f(x)$ is decreasing function on $\mathbb{R}$.

'In general, exponential function increases or decreases as $(a > 1)$ or $(0 < a < 1)$ respectively'.

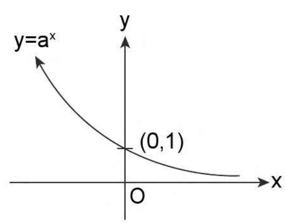


FIGURE 3.44

ILLUSTRATION 9: Find the number of ordered pairs (x, y); $x, y \in \mathbb{R}$ satisfying the equation $y = x^{1/3}$; and $y^2 = 4^{-x}$ simultaneously.

SOLUTION: Graph of $x^{1/3}$ is shown below in Figure 3.45.

The other curve given is $y^2 = 4^{-x}$

$\Rightarrow y = \pm\sqrt{4^{-x}} = \pm 2^{-x}$

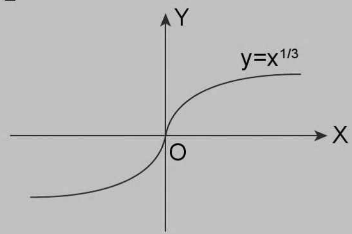


FIGURE 3.45

is a pair of exponential functions $y = 2^{-x}$ and $y = -2^{-x}$ which are mirror image of each other in x-axis.

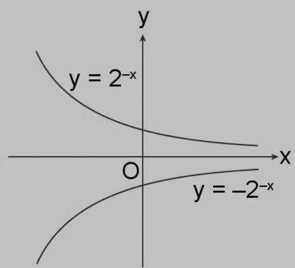


FIGURE 3.46

Drawing them on same axes exactly one ordered pair satisfies both.

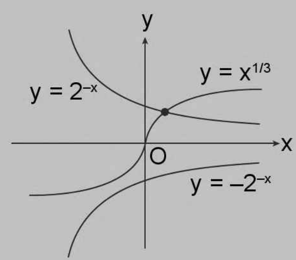


FIGURE 3.47

LOGARITHMIC FUNCTION

If 'a' is a positive real number, then the function that associates every positive real number to $\log_a x$. That is, $y = \log_a x$ is called the logarithmic function and it is defined as the inverse of exponential function to the same base.

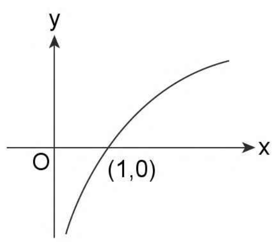
e.g., $y = \log_{10}(x^2 + x + 1)$

The domain of the logarithmic function is $(0, \infty)$

The range of the logarithmic function is : $(-\infty, +\infty)$

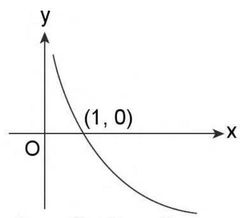
Case I. **$a > 1$:** Here, $f(x) = y = \log_a x$ increases with increase in x, i.e., $f(x)$ is increasing function on $\mathbb{R}$.

e.g., $y = \log_2 x$, $y = \log_3 x$, $y = \log_4 x$,... we have; $\log_2 x > \log_3 x > \log_4 x >$.... for $x > 1$ and $\log_2 x < \log_3 x < \log_4 x <$.... for $0 < x < 1$ and they can be represented as shown in Figure 3.48.



y=log$_a$x(for a>1)

FIGURE 3.48



y=log$_a$x (for 0<a<1)

FIGURE 3.49

Case II. **$0 < a < 1$:** Here, $f(x) = \log_a x$ decreases with increase in x, i.e., $f(x)$ is a decreasing function on $\mathbb{R}$.

"In general, logarithmic function increases or decreases as $(a > 1)$ or $(0 < a < 1)$ respectively" as shown in Figure 3.50.

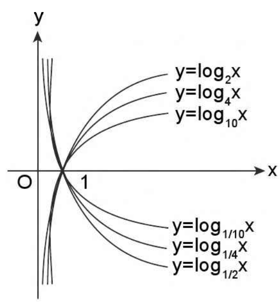


FIGURE 3.50

ILLUSTRATION 10: Find the number of real solutions of the equation $x^3 - 4x = \log_2 x$.

SOLUTION: Sketching the graph of cubic polynomial $f(x) = x^3 - 4x$ as it has 3 real roots $x = 0, -2, 2$ and $g(x) = \log_2 x$ on the same reference frame, we get

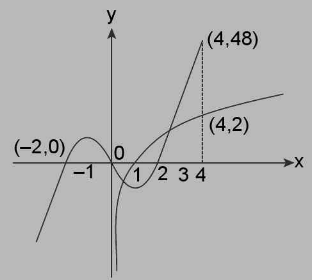


FIGURE 3.51

$\because\ f(4) = x^3 - 16 = 48$ and $g(4) = 2 \quad \Rightarrow f(4) > g(4)$

Thus, from the graph, we can see that curves intersect twice. Thus, given equation has exactly two real roots.

■ INVERSE CIRCULAR FUNCTIONS

(a) **$\sin^{-1} x$:** Its domain is $[-1, 1]$. Its range is $[-\pi/2, \pi/2]$. It is an increasing function.
It is an odd function, i.e., $\sin^{-1}(-x) = -\sin^{-1}x\ \forall\ x \in [-1, 1]$

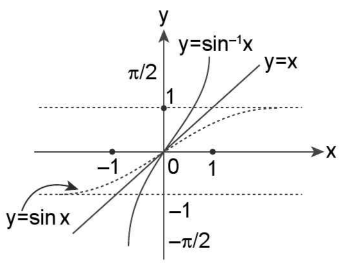


FIGURE 3.52

(b) **$\cos^{-1} x$:** Its domain is $[-1, 1]$. Its range is $[0, \pi]$. It is a decreasing function. It neither an even function nor odd function $\cos^{-1}(-x) = \pi - \cos^{-1}x$.

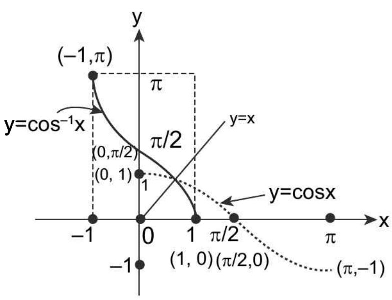


FIGURE 3.53

(c) **$\tan^{-1} x$:** Its domain is $\mathbb{R}$. Its range is $(-\pi/2, \pi/2)$. It is an increasing function.
It is an odd function, $\tan^{-1}(-x) = -\tan^{-1}x\ \forall\ x \in \mathbb{R}$.

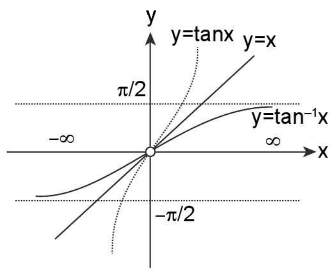


FIGURE 3.54

(d) **$\cot^{-1} x$:** Its domain is $\mathbb{R}$. Its range is $(0, \pi)$. It is a decreasing function. It is neither an even function nor odd function $\cot^{-1}(-x) = \pi - \cot^{-1}x$.

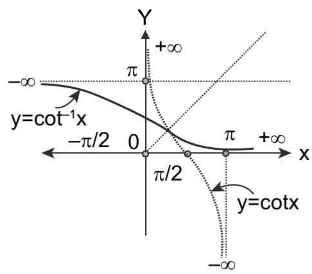


FIGURE 3.55

(e) **$\sec^{-1} x$:** Its domain is $\mathbb{R} - (-1, 1)$. Its range is $[0, \pi] - \{\pi/2\}$. It is an increasing function. It neither an even function nor odd function $\sec^{-1}(-x) = \pi - \cot^{-1}x$.

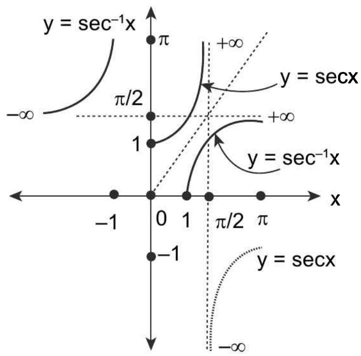


FIGURE 3.56

(f) cosec^{-1}*x*: Its domain is $\mathbb{R}$ – (–1, 1). Its range is [– π/2, π/2] ~ {0}. It is an decreasing function. It is an odd function cosec^{-1}(–*x*) = – cosec^{-1}*x*.

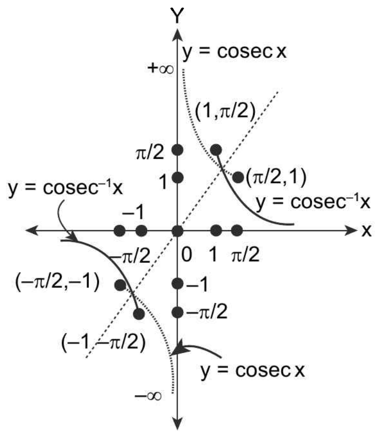


FIGURE 3.57

ILLUSTRATION 11: Given f and g as two functions defined as $f(x)=\begin{cases}|x| & \text{if } x<0\\ \sin^{-1}x & \text{if } x\geq 0\end{cases}$ and $g(x)=\cot^{-1}x$.

Find the number of real roots of equation $f(x) = g(x)$.

SOLUTION: Clearly $f(x)=\begin{cases}-x; & x<0\\ \sin^{-1}x; & x\geq 0\end{cases}$ and $g(x) = \cot^{-1}x$

Sketching both curves on same *x-y* plane and counting the point of intersection, we get two solutions.

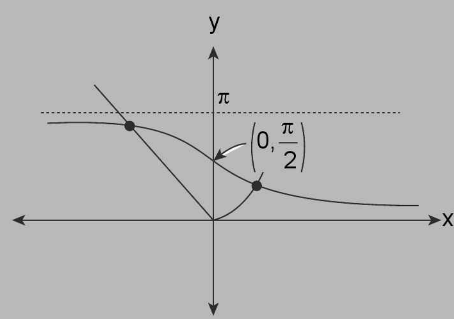


FIGURE 3.58

ILLUSTRATION 12: Find the set of common solutions satisfying the inequality $x^2 - 8|x| + 7 > 0$ and $\frac{\pi^2}{4}+\frac{\pi}{2}\tan^{-1}x-6(\tan^{-1}x)^2\geq 0$

SOLUTION: Given $|x|^2 - 8|x| + 7 > 0$

$\Rightarrow$ $(|x| - 1)(|x| - 7) > 0$ $\qquad \Rightarrow$ $|x| < 1$ or $|x| > 7$

$\Rightarrow$ $x \in (-1, 1) \cup (-\infty, -7) \cup (7, \infty)$ and

$$\frac{\pi^2}{4}+\frac{\pi}{2}\left(\tan^{-1}x\right)-6\left(\tan^{-1}x\right)^2\geq 0$$

$\Rightarrow$ $24(\tan^{-1}x)^2 - 2\pi\tan^{-1}x - \pi^2 \leq 0$

$\Rightarrow$ $(6\tan^{-1}x + \pi)(4\tan^{-1}x - \pi) \leq 0$

$\Rightarrow$ $-\frac{\pi}{6}\leq\tan^{-1}x\leq\frac{\pi}{4}$ $\Rightarrow$ $-\frac{1}{\sqrt{3}}\leq x\leq 1$

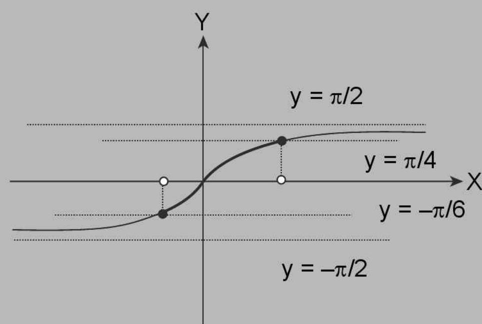


FIGURE 3.59

ILLUSTRATION 13: Solve the inequality $\tan^{-1}x\leq\cot^{-1}x<\frac{3\pi}{4}$.

SOLUTION: **Method 1:** $\because$ $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$

Thus, the inequality $\tan^{-1} x \le \cot^{-1} x < \frac{3\pi}{4}$

$\Rightarrow \tan^{-1} x \le \frac{\pi}{2} - \tan^{-1} x < \frac{3\pi}{4} \Rightarrow \tan^{-1} x \le \frac{\pi}{4}$ & $\tan^{-1} x > \frac{-\pi}{4} \Rightarrow -\frac{\pi}{4} < \tan^{-1} x \le \frac{\pi}{4}$

$\because$ tan x is an increasing function in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore \tan\left(\frac{-\pi}{4}\right) < x \le \tan\frac{\pi}{4} \Rightarrow -1 < x \le 1$. Consequently the solution set is $(-1, 1]$

Method II: Graphically

Drawing the graph $y = \frac{3\pi}{4}, y = \tan^{-1} x$ and $y = \cot^{-1}x$ on the same x-y plane as shown in the Figure 3.60.

Clearly, the values of x satisfying the given inequality are $x \in (-1, 1]$.

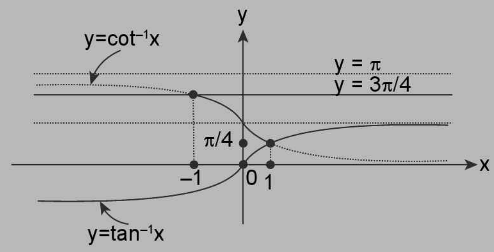


FIGURE 3.60

(g) $y = \sin(\sin^{-1} x) = \cos(\cos^{-1} x) = x$

Its domain is $[-1, 1]$. Its range is $[-1, 1]$. It is a non-periodic, monotonically increasing, odd function

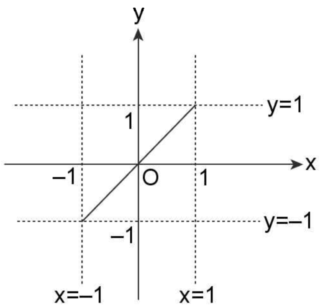


FIGURE 3.61

(h) $y = \text{cosec}(\text{cosec}^{-1} x) = \sec(\sec^{-1} x) = x$: Its domain is $\mathbb{R} - (-1, 1)$. Its range is $\mathbb{R} - (-1, 1)$.

It is a non-periodic, monotonically increasing, odd function.

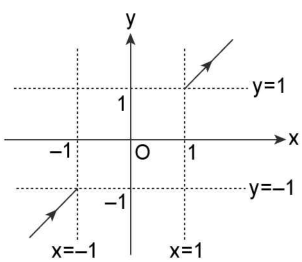


FIGURE 3.62

(i) $y = \tan(\tan^{-1} x) = \cot(\cot^{-1}x) = x$

Its domain is R. Its range is $\mathbb{R}$. It is a non-periodic, monotonically increasing, odd function.

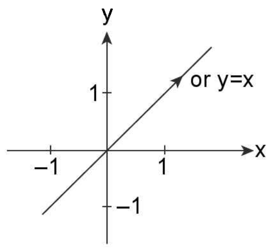


FIGURE 3.63

(j) $y = \sin^{-1}(\sin x)$: Its domain is $\mathbb{R}$ and its range is $[-\pi/2, \pi/2]$.

It is a periodic function with period 2π. It is an odd function.

$$= \begin{cases} x; & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ -\pi - x; & -\frac{3\pi}{2} \le x \le -\frac{\pi}{2} \\ \pi - x; & \text{when } \frac{\pi}{2} \le x \le \frac{3\pi}{2} \end{cases}$$

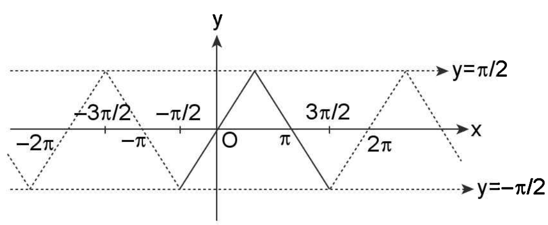


FIGURE 3.64

(k) $y = \cos^{-1}(\cos x)$: Its domain is $\mathbb{R}$. Its range is $[0, \pi]$. It is a periodic function with period 2π. It is an even function.

$$= \begin{cases} x; & 0 \le x \le \pi \\ 2\pi - x; & 0 \le 2\pi - x \le \pi \text{ or } \pi \le x \le 2\pi \end{cases}$$

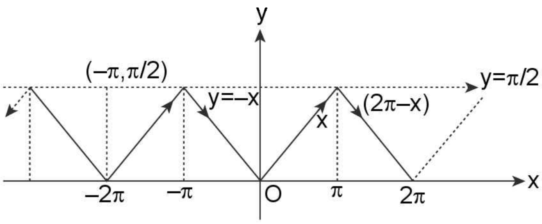


FIGURE 3.65

(l) $y = \tan^{-1}(\tan x)$: Its domain is $\mathbb{R} - \{(2n+1)\pi/2\}$. Its range is $(-\pi/2, \pi/2)$

It is a periodic function with period π.

It is an odd function and monotonically increasing function and $\left\{x; -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}\right\}$

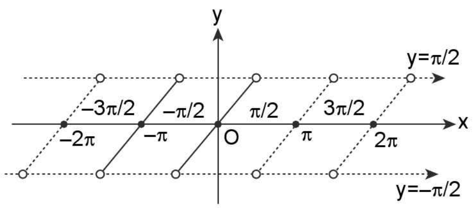


FIGURE 3.66

(m) $y = \sec^{-1}(\sec x)$: Its domain is $\mathbb{R} - \{(2n+1)\pi/2\}$

Its range is $[0, \pi] - \{\pi/2\}$. It is a periodic function with period 2π. It is an even function.

$$= \begin{cases} x; & x \in \left[0, \dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2}, \pi\right] \\ 2\pi - x; & 2\pi - x \in \left[0, \dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2}, \pi\right] \text{or} [\pi, 2\pi] - \left\{\dfrac{3\pi}{2}\right\} \end{cases}$$

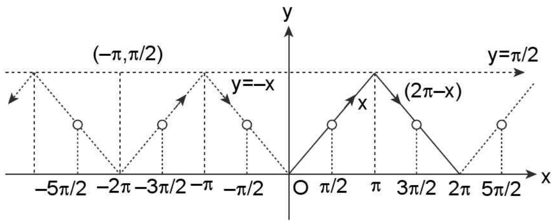


FIGURE 3.67

(n) $y = \text{cosec}^{-1}(\text{cosec}\, x)$: Its domain is $\mathbb{R} - \{n\pi : n \in \mathbb{Z}\}$ Its range is $[-\pi/2, \pi/2] - \{0\}$. It is a periodic function with period 2π. It is an odd function.

$$= \begin{cases} x & -\dfrac{\pi}{2} \le x < 0 \ \cup \ 0 < x \le \dfrac{\pi}{2} \\ \pi - x & -\dfrac{\pi}{2} \le \pi - x < 0 \ \cup \ 0 \le \pi - x \le \dfrac{\pi}{2} \end{cases}$$

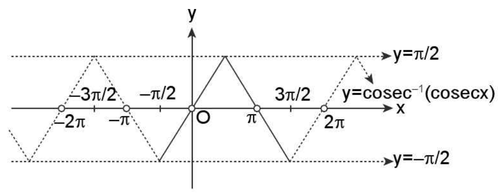


FIGURE 3.68

(o) $y = \cot^{-1}(\cot x)$. Its domain is $\mathbb{R} - \{n\pi\}$. Its range is $(0, \pi)$. It is a periodic function with period π. It is neither even nor odd function.

$$= \begin{cases} x; \ 0 < x < \pi \\ x + \pi; -\pi < x < 0 \text{ and so on.} \\ x - \pi; \pi < x < 2\pi \end{cases}$$

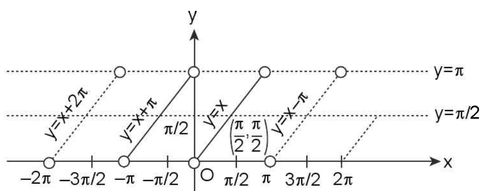


FIGURE 3.69

ILLUSTRATION 14: Find the number of real values of x satisfying the equation $\sin^{-1}(\sin x) = \cos x$ or $\sin x = \sin(\cos x)$ belonging to the interval $[-2\pi, 3\pi]$.

SOLUTION: Sketching the graph of both the functions $y = \sin^{-1}(\sin x)$ and $y = \cos x$ on same reference frame, we get 5 points of intersection in the interval $[-2\pi, 3\pi]$ as obvious from the Figure 3.70, Thus, the given equation has 5 real solutions $\in [-2\pi, 3\pi]$

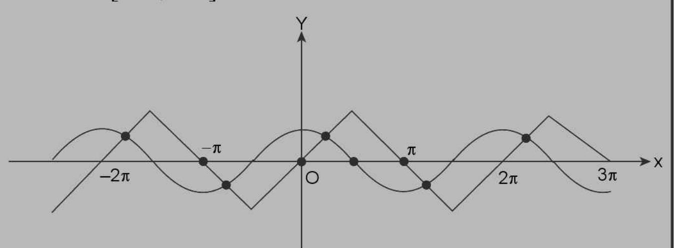


FIGURE 3.70

STANDARD CONIC SECTIONS

Circle

The general equation of a circle is $x^2+y^2+2gx+2fy+c=0$

To draw a sketch of a circle, we write the equation in the standard form $(x+g)^2+(y+f)^2 = \left(\sqrt{g^2+f^2-c}\right)^2$, whose centre is $(-g, -f)$; and radius $r=\sqrt{g^2+f^2-c}$.

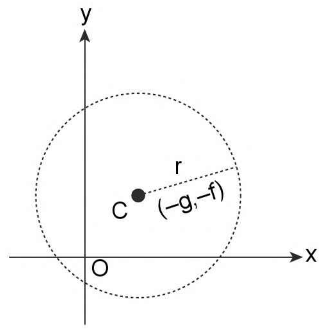


FIGURE 3.71

NOTE

(i) If centre is origin and radius is a, then the equation of circle is $x^2+y^2=a^2$.
It is not function in x but it is a relation in x and symmetric in all four quadrants.

(ii) *If the end points of diameter are* (x_1, y_1) *and* (x_2, y_2)*, then the equation of circle is* $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$;.

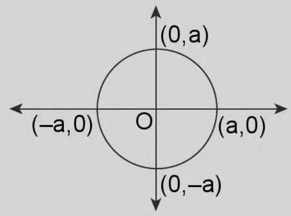


FIGURE 3.72

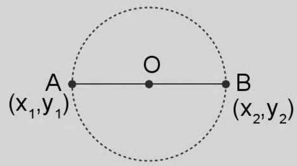


FIGURE 3.73

REPRESENTATION AS A FUNCTION

Clearly, the above equation of circle are not functions rather are many–many relations. After converting them in to functions following cases arise that may be worth noticing.

(i) $y=\sqrt{a^2-x^2}$

Domain: $[-a, a]$; Range: $[0, a]$. Even function in x, Symmetrical about y-axis.

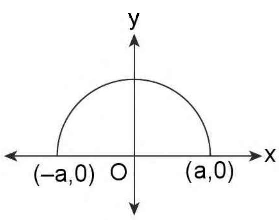


FIGURE 3.74

(ii) $y=-\sqrt{a^2-x^2}$

Domain: $[-a, a]$, Range: $[-a, 0]$
Even function in x, Symmetrical about y-axis.

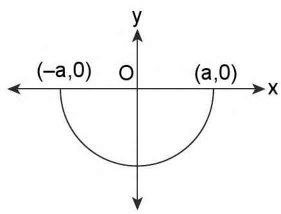


FIGURE 3.75

(iii) $x=\sqrt{a^2-y^2}$

Domain: $[0, a]$, Range: $[-a, a]$

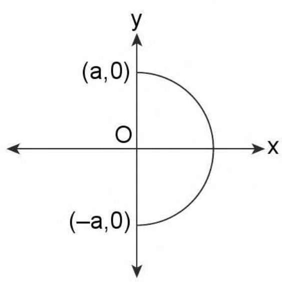


FIGURE 3.76

It is not function of x but a relation, symmetric about x-axis.

(iv) $x = -\sqrt{a^2 - y^2}$

Domain: $[-a, 0]$; Range: $[-a, a]$. It is a relation, symmetric about x-axis.

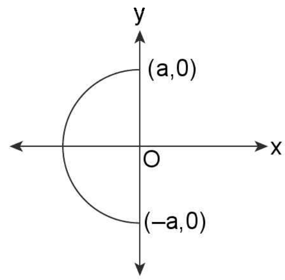


FIGURE 3.77

ILLUSTRATION 15: Find the area of region lying above the curve $y = \begin{cases} -x\sqrt{3}; & x < 0 \\ x\sqrt{3}; & x > 0 \end{cases}$ and inside the circle $x^2 + y^2 - 4y = 0$.

SOLUTION: $\because\ x^2 + y^2 - 4y = 0 \ \Rightarrow \quad (x - 0)^2 + (y - 2)^2 = 4$

Clearly $x^2 + y^2 - 4y = 0$ represents circle with centre (0, 2) and radius 2.

Sketching both the curve as shown in Figure 3.78, we need to find the area of shaded region $= 2(\text{area of } \Delta OAB) + \text{area of circular sector } \Delta\widehat{BDC}$

$$= \frac{1}{2}(2)(2)\sin\left(\frac{2\pi}{3}\right) + \frac{1}{2}(2)^2\frac{2\pi}{3} = \left(\sqrt{3} + \frac{4\pi}{3}\right)$$

Square units.

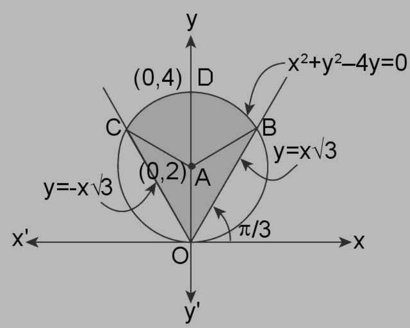


FIGURE 3.78

Parabola

It is the locus of points moving on a plane such that its distance from a fixed point is equal to its distance from a fixed straight line, point and straight line being on plane of moving point. Taking the fixed straight line $x = -a$, $a > 0$ and fixed point $(a, 0)$, we get the equation of parabola $y^2 = 4ax$.

Nature of curve:

(a) It passes through (0, 0)

(b) It is symmetrical about axis of x. (called as axis of parabola)

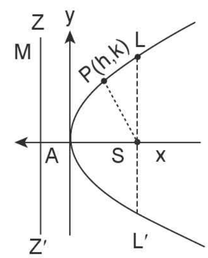
(c) No part of the curve lies on the negative side of axis of x.

(d) Curve turns at (0, 0) which is called the vertex of the curve.

(e) The curve extends to infinity. It is not a close curve.

(i) $\boldsymbol{y^2 = 4ax}$

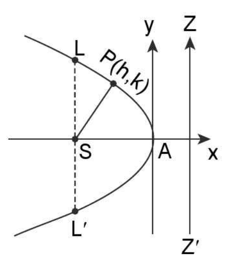
Shape



(Right handed parabola)

FIGURE 3.79

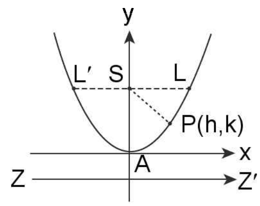
(ii) $\boldsymbol{y^2 = -4ax}$



(Left handed parabola)

FIGURE 3.80

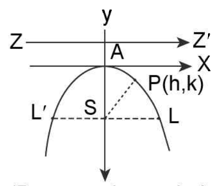
(iii) $x^2 = 4ay$



(Upwards parabola)

FIGURE 3.81

(iv) $x^2 = -4ay$



(Downwards parabola)

FIGURE 3.82

(v) $(y-k)^2 = 4a(x-h)$; $a > 0, h, k > 0$

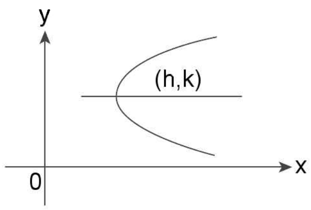


FIGURE 3.83

2. $y^2 = 4a\,(x - h)$; $a > 0$, $h > 0$

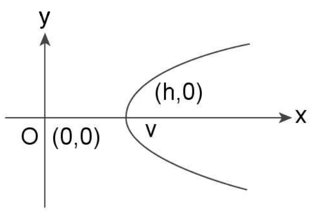


FIGURE 3.84

3. $y^2 = 4a\,(x + h)$; a and h are positive

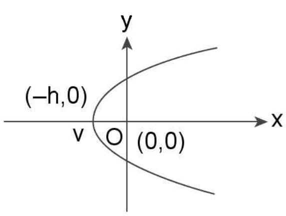


FIGURE 3.85

4. $y^2 = -4a(x-h)$; $a, h > 0$

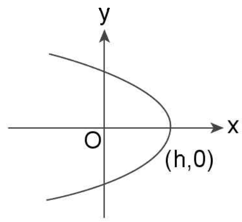


FIGURE 3.86

5. $y^2 = -4a(x+h)$; $a, h > 0$; where $a, h > 0$

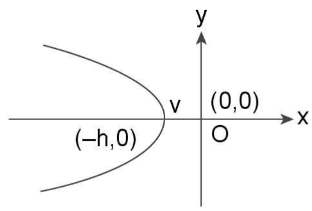


FIGURE 3.87

6. $x^2 = 4a(y+k)$; $a > 0, k > 0$

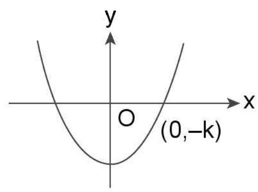


FIGURE 3.88

7. $x^2 = 4a(y-k)$; $a, k > 0$

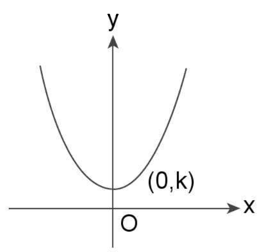


FIGURE 3.89

8. $x^2 = -4a\,(y+k)$; $a, k > 0$

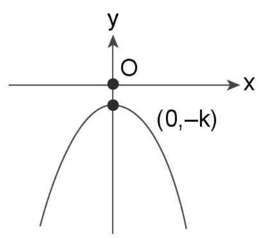


FIGURE 3.90

9. $x^2 = -4a(y-k)$; $a, k > 0$

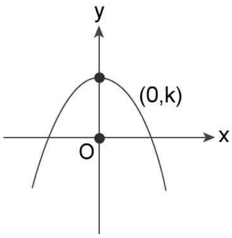


FIGURE 3.91

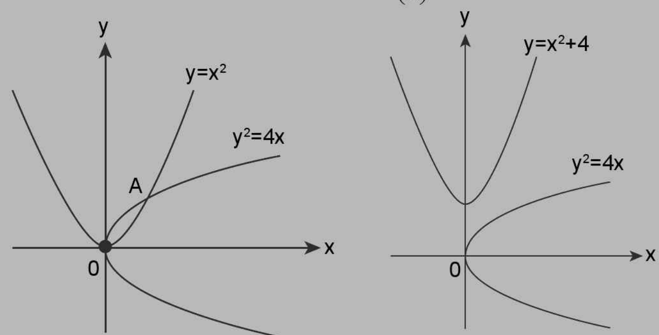
We observe that the function $y = x^2$ is symmetrical about y-axis. As a rule, if $f(x)$ contains even powers of x only, then the curve is symmetric about y-axis.

We observe that the function $y^2 = x$ is symmetrical about x-axis. As a rule, if a function contains even powers of y only, then the function is symmetrical about x-axis.

ILLUSTRATION 16: Find the number of points at which the two curves given by $y^2 = 4x$ and $y = x^2 + k$ meet for the following values of 'k'.

(a) $k = 0$ (b) $k = 4$

SOLUTION: (a) (b)



Two points of intersection 0 and A

No point of intersection

FIGURE 3.92

NOTE

In fact, in reference to the above illustration, we can also find out the value of 'k' for which the two curves touch each other at exactly one point, but that would require use of calculus and is covered in our book of differential calculus.

Ellipse

(i) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1; (a^2 > b^2)$

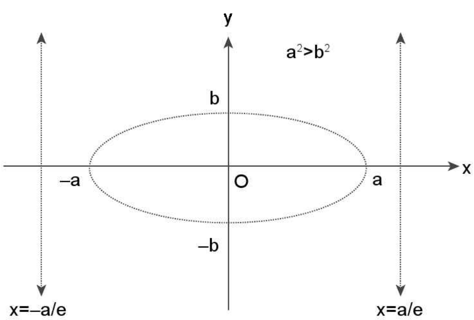


FIGURE 3.93

(ii) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1; (a^2 < b^2)$

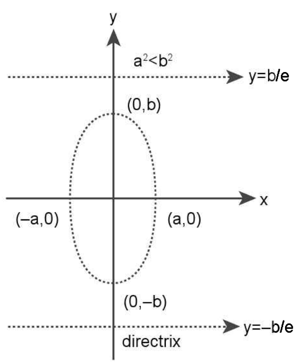


FIGURE 3.94

(iii) $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1; (a^2>b^2)$

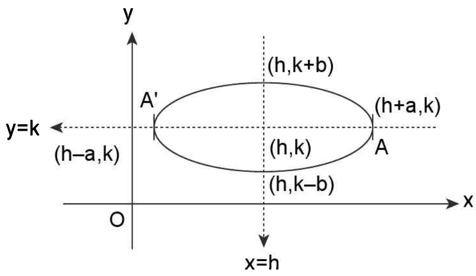


FIGURE 3.95

ILLUSTRATION 17: Compute the area enclosed by curves $3y=2\sqrt{9-x^2}$ and $y=\sqrt{9-x^2}$

SOLUTION: Indeed $y=\sqrt{9-x^2}$ is semicircle of radius 3 unit centered at origin lying above x-axis. Whereas $y=\dfrac{2}{3}\sqrt{9-x^2}$ is semi ellipse above x-axis with centre as origin and major axis as x-axis of length 6 and minor axis as y-axis of length 4. Therefore, the required area is given by shaded region in the Figure 3.96.

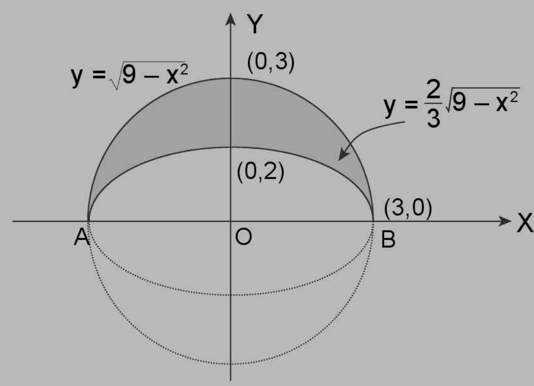


FIGURE 3.96

The area of shaded region = (area of semi-circle) – (area of semi ellipse)

$=\dfrac{\pi a^2}{2}-\dfrac{\pi ab}{2}=\dfrac{\pi a}{2}(a-b)=\dfrac{3\pi}{2}(3-2)$

$=\dfrac{3\pi}{2}$ Square unit.

ILLUSTRATION 18: Compute the area enclosed by the boundaries formed by the curves $f(x)$ defined as $f(x)=\begin{cases}\cos^{-1}\cos x; & 0\le x\le\pi\\ \sqrt{8\pi x-3\pi^2-4x^2}; & \pi<x\le\dfrac{3\pi}{2}\end{cases}$ and x-axis.

SOLUTION: Given function can be defined as $f(x)=\begin{cases}\cos^{-1}(\cos x); & 0\le x\le\pi\\ \sqrt{8\pi x-3\pi^2-4x^2}; & \pi<x\le\dfrac{3\pi}{2}\end{cases}$

$\Rightarrow\quad f(x)=\begin{cases}x; & 0\le x\le\pi\\ \sqrt{8\pi x-3\pi^2-4x^2}; & \pi<x\le\dfrac{3\pi}{2}\end{cases}$. For $x\in\left(\pi,\dfrac{3\pi}{2}\right]$; $f(x)$ is quarter ellipse whose complete equation is given by $\dfrac{(x-\pi)^2}{(\pi/2)^2}+\dfrac{y^2}{\pi^2}=1$ with major axis $x=\pi$ of length 2π and minor axis $y=0$ of length π and centre $(\pi, 0)$. Therefore, the area of shaded region is as shown in the Figure 3.97.

$$= \text{Area of } \Delta OAB + \text{Area of quarter ellipse } (BCAB)$$

$$= \frac{1}{2}(\pi)(\pi) + \frac{1}{4}\pi(\pi)\left(\frac{\pi}{2}\right) = \frac{\pi^2}{2} + \frac{\pi^3}{8} = \frac{\pi^2(4+\pi)}{8}$$

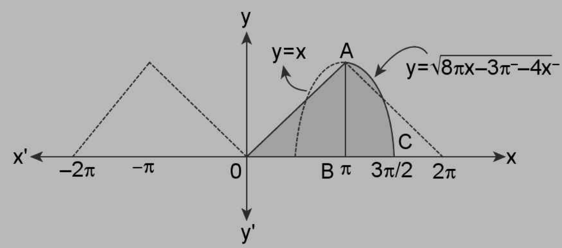


FIGURE 3.97

Hyperbola

(i) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

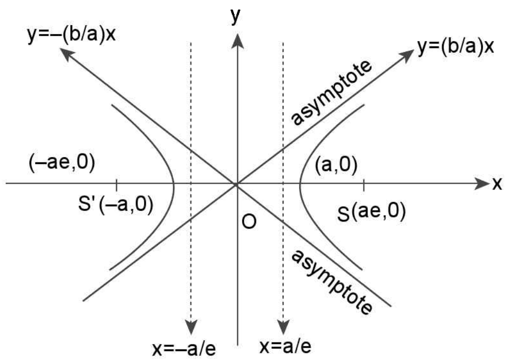


FIGURE 3.98

(ii) $-\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

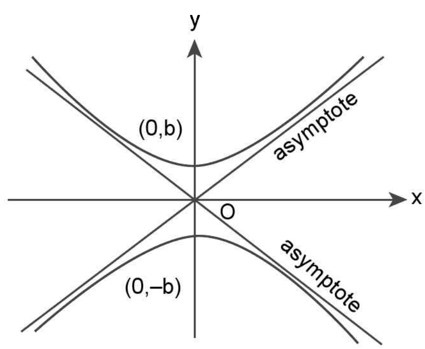


FIGURE 3.99

(iii) $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$

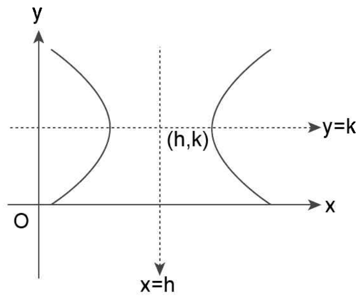


FIGURE 3.100

(iv) $x^2 - y^2 = a^2$
(Rectangular hyperbola)

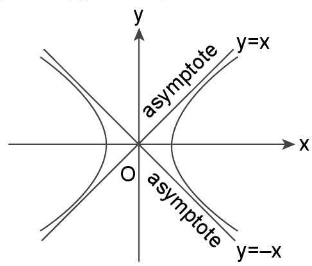


FIGURE 3.101

(v) $xy = c^2$

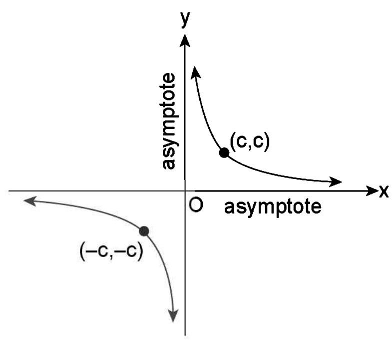


FIGURE 3.102

ILLUSTRATION 19: Find the number of points where the curves $x^2 + y^2 = 9$ and $x^2y^2 = 1$ intersect.

SOLUTION: Clearly, $x^2 + y^2 = 9$ is a circle with centre (0, 0) and radius 3 units, whereas equation $x^2y^2 = 1$

$\Rightarrow\ y = \pm\dfrac{1}{x}$ or $xy = \pm 1$ is a pair of conjugate rectangular hyperbola with co-ordinate axes as asymptotes graphing them together on same x-y plane as shown below.

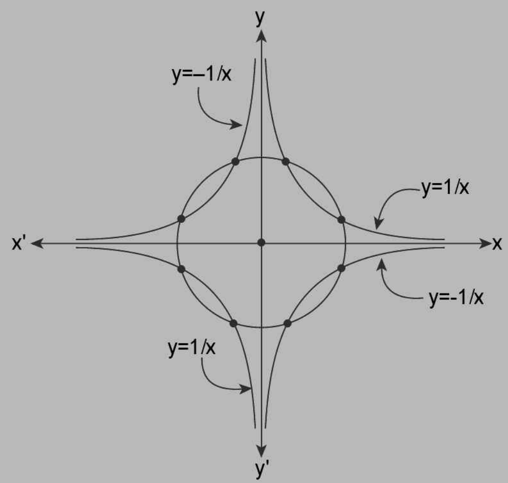


FIGURE 3.103

Clearly both curves have 8 points of intersections.

Aliter: $x^2 + \dfrac{1}{x^2} = 9$

$\Rightarrow\ x^4 - 9x^2 + 1 = 0$

$x^2 = \dfrac{9 \pm \sqrt{81-4}}{2}$ $\Rightarrow\ x^2 = \dfrac{9 \pm \sqrt{77}}{2}$

both values are positive. Thus, x has four real roots. For each x, we get two corresponding points, thus, eight points of intersection.

TEXTUAL EXERCISE–1: (SUBJECTIVE)

1. Sketch the following curves and show their relative nature on same graph sheet:

(a) $y = x$ (b) $y = x^2$

(c) $y = |x|$ (d) $y = \sqrt{x}$

(e) $y = x^{1/3}$

2. Sketch the curve of the following functions:

(a) $y = -\sqrt{9-x^2}\ , x \in [-3, 0)$

(b) $y = -\sqrt{9-x^2}\ , x \in [0, 3]$

(c) $y = \sqrt{9-x^2}\ , x \in [-3, 0)$

(d) $y = \sqrt{9-x^2}\ , x \in x \in [0, 3]$

3. Sketch the curve of the following functions:

(a) $y = -4\sqrt{x-3} + 5, \quad x \geq 3$

(b) $y = 4\sqrt{x-3} + 5, \quad x \geq 3$

(c) $y = -4\sqrt{3-x} + 5, \quad x \leq 3$

(d) $y = 4\sqrt{3-x} + 5, \quad x \leq 3$

4. Sketch the curve of the following functions:

(a) $\sqrt{-\dfrac{x^2}{16} + 1} = \dfrac{y}{3}, \quad x \in [-4, 0)$

(b) $-\sqrt{-\dfrac{x^2}{16} + 1} = \dfrac{y}{3}, \quad x \in [-4, 0)$

(c) $\sqrt{-\frac{x^2}{16}+1}=\frac{y}{3}, \quad x \in [0,4]$

(d) $-\sqrt{-\frac{x^2}{16}-1}=\frac{y}{3}, \quad x \in [0,4]$

5. Sketch the curve of the following functions:

(a) $\sqrt{\frac{x^2}{16}-1}=\frac{y}{3}, \quad x \geq 4$

(b) $-\sqrt{\frac{x^2}{16}-1}=\frac{y}{3}, \quad x \geq 4$

(c) $\sqrt{\frac{x^2}{16}-1}=\frac{y}{3}, \quad x \leq -4$

(d) $-\sqrt{\frac{x^2}{16}-1}=\frac{y}{3}, \quad x \leq -4$

6. Draw the graph of the following functions:

(i) $y = |x-3| + |x+1| + 2x$

(ii) $y = x\,|x| - 2x - 3$

7. Sketch the graph of the following functions using transformation:

(a) $y = \cos x + |\cos x|$

(b) $y = 1 + \log_x \sqrt{x}$

(c) $y = \log_{10} x + |\log_{10} x|$

8. Draw the graph of the following functions:

(a) $y = x^2 - 2|x+1| - 1$

(b) $y = |2x^2 - 3x + |x-1||$

(c) $y = -|x^2 - 2x|$

9. Draw the graph of the following functions:

(a) $y = 4^{(|x|+x)/2x}$

(b) $y = \dfrac{\cos\left(|x| - \frac{3\pi}{2}\right)}{2\sin x}$

(c) $y = (2-x)\,|x+3|$

10. Draw the graph of following functions:

(a) $f(x) = \log|x| - \log x^2$

(b) $f(x) = \frac{1}{2}\left(\frac{|\sin x|}{\cos x} + \frac{|\cos x|}{\sin x}\right)$

11. Construct the graph for:

(i) $f(x) = \begin{cases} x-1; & x<0 \\ \frac{1}{4}; & x=0 \\ x^2; & x>0 \end{cases}$

(ii) $f(x) = \begin{cases} 2x+3; & -3 \leq x < -2 \\ x+1; & -2 \leq x < 0 \\ x+2 & 0 \leq x \leq 1 \end{cases}$

12. Construct the graph of the function:

(i) $f(x) = |x-1| + |x+1|$

(ii) $f(x) = \begin{cases} 3^x; & -1 \leq x \leq 1 \\ 4-x; & 1 \leq x < 4 \end{cases}$

(iii) $f(x) = [x] + |x-1|;\ -1 \leq x \leq 3$ (where [.] denotes greatest integer function).

(iv) $f(x) = \begin{cases} x^4; & x^2 < 1 \\ x; & x^2 \geq 1 \end{cases}$

13. Draw the following curves given by implicit relations:

(a) $16x^2 + 24xy + 9y^2 - 5x - 10y + 1 = 0$

(b) $4x^2 - 4xy + y^2 - 12x + 6y + 9 = 0$

(c) $x^2 - xy - 6y^2 = 6$

(d) $xy = a(x+y)$

(e) $x^2 + xy + y^2 + x + y = 1$

14. Draw the following graphs and find out what will be the graph of

(i) $y = \sin^4 x$,

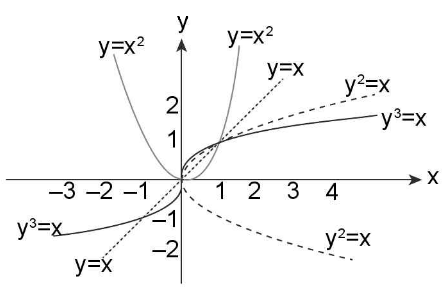
(ii) $y = \sin^5 x$

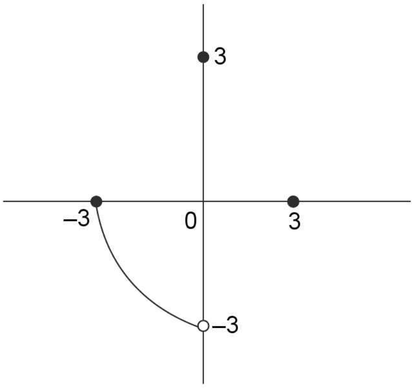
(ii) $y = \sin^{2n} x$ *as* $n \to \infty\ n \in I$

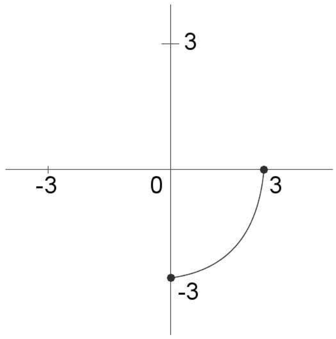
(iii) $y = \sin^{(2n+1)} x$ *as* $n \to \infty\ n \in I$

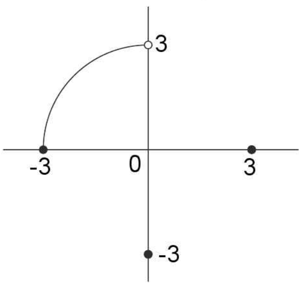
Answer Keys

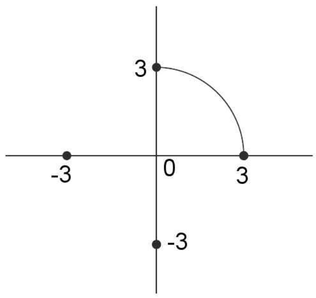
1.

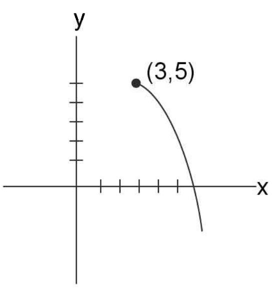


2. (a)


(b)


(c)


(d)


3. (a)


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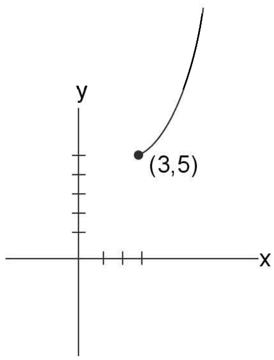

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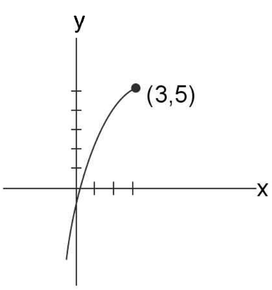

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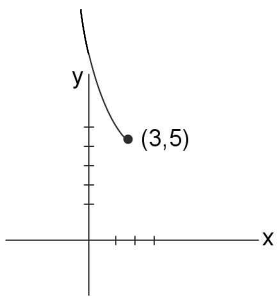

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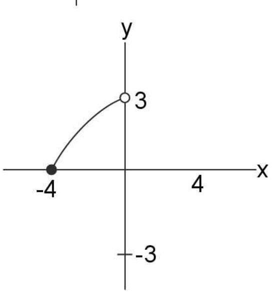

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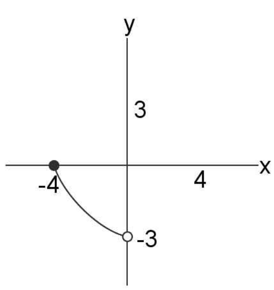
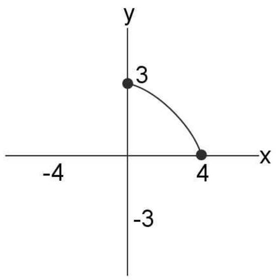
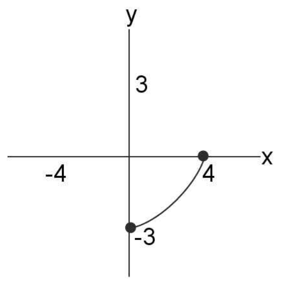
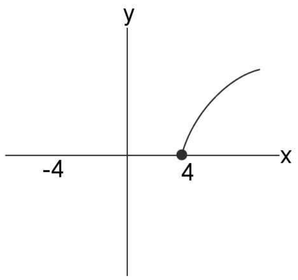
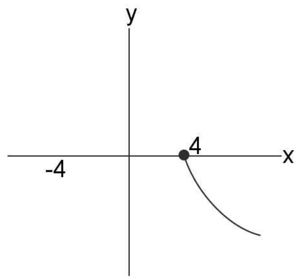
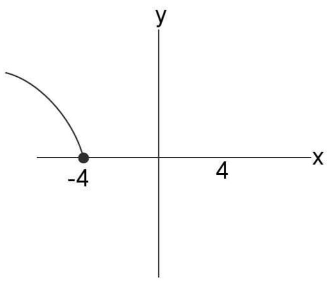


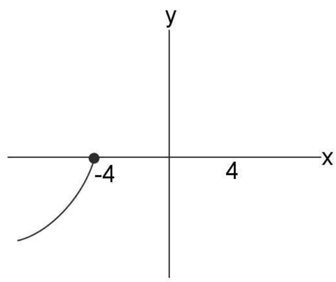
(c) 

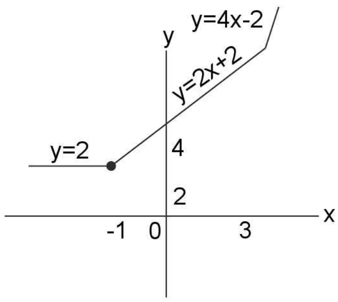
(d) 

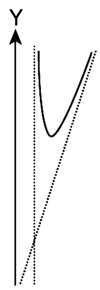
5. (a) 

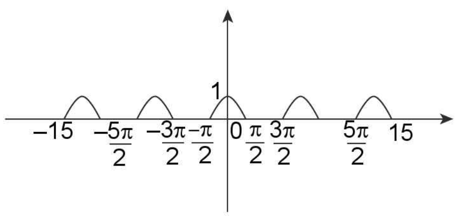
(b) 

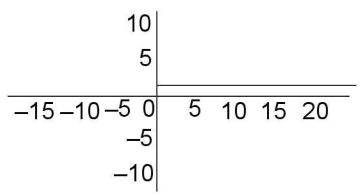
(c) 

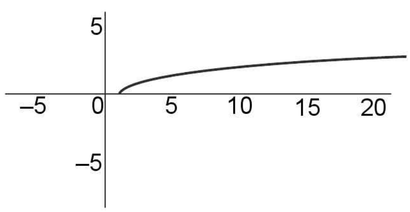
(d) 

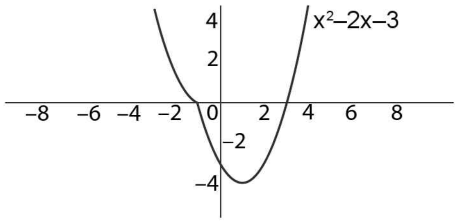
6. (i) 

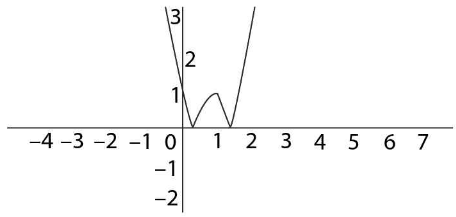
(ii) 

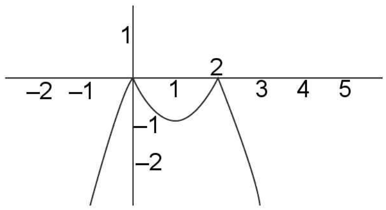
7. (a) 

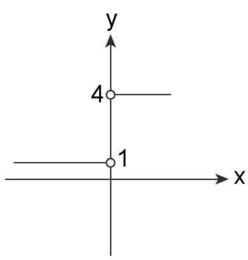
(b) 

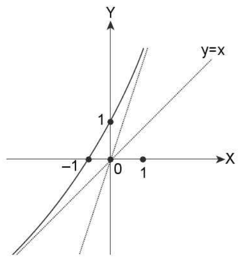
(c) 

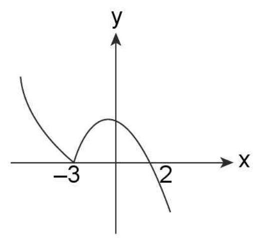
8. (a) 

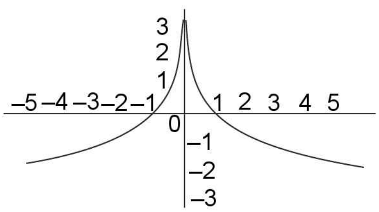
(b) 

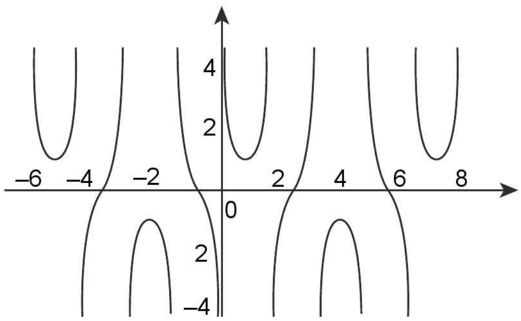
(c) 

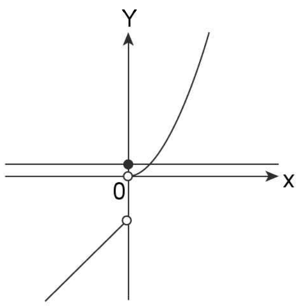
9. (a) 

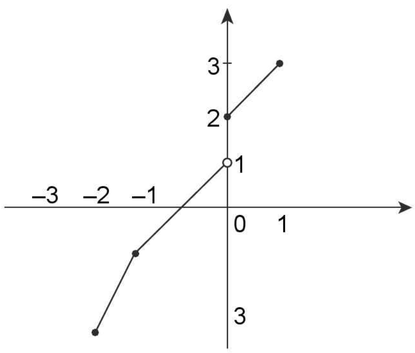
(b) 

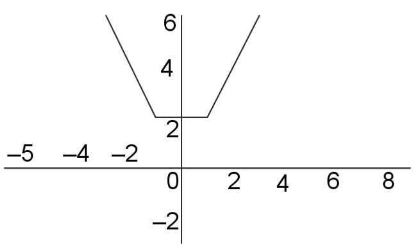
(c) 

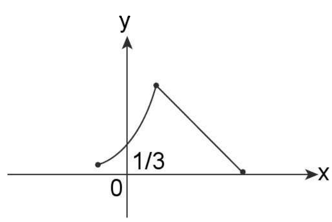
10. (a) 

(b) 

11. (a) 

(b) 

12. (i) 

(ii) 

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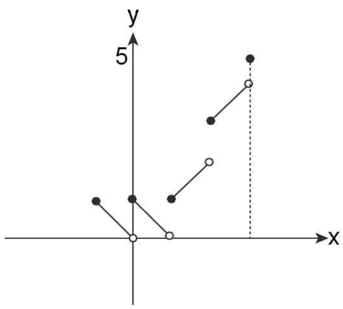

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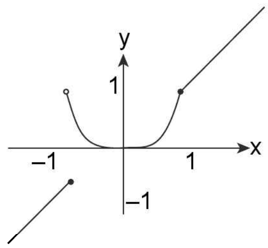

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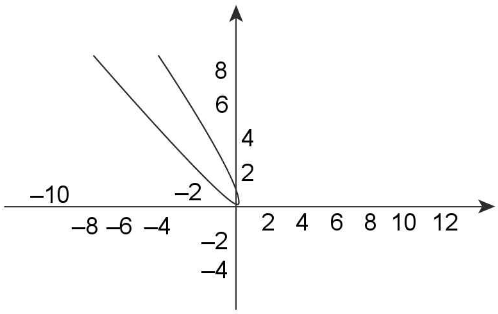

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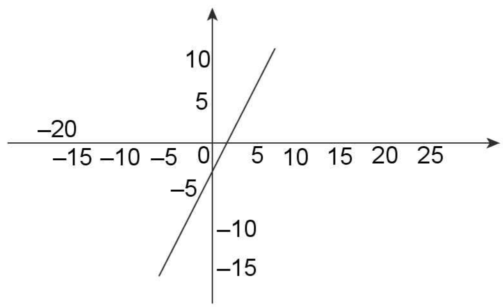

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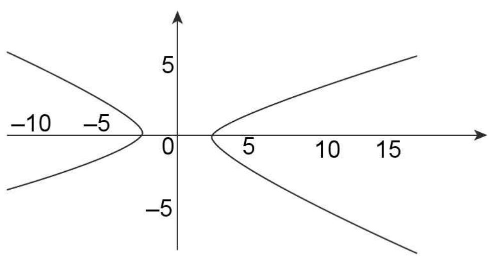

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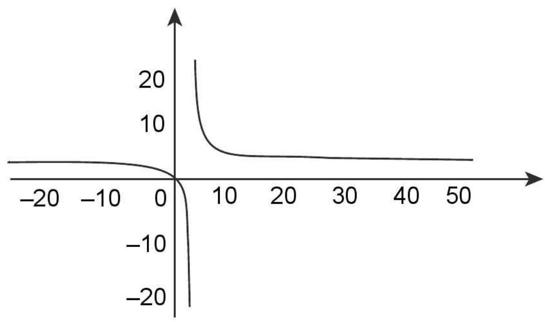

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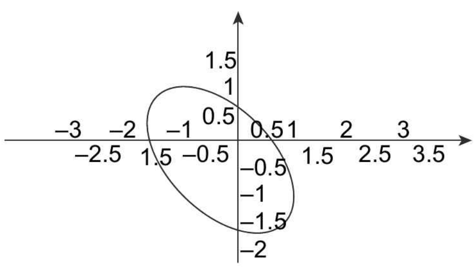

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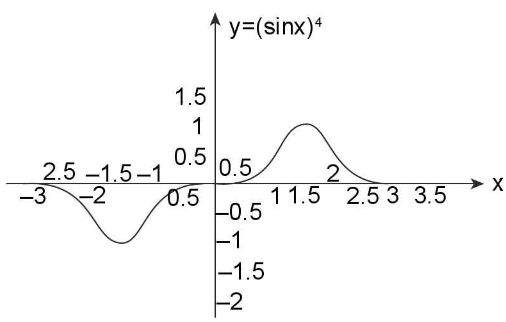

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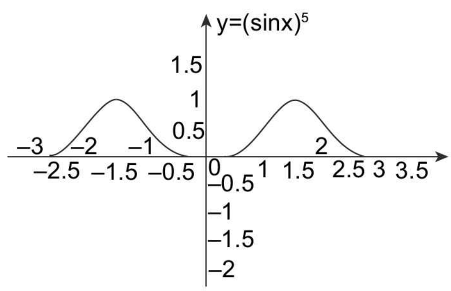
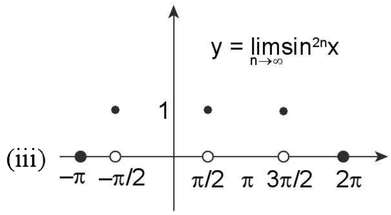
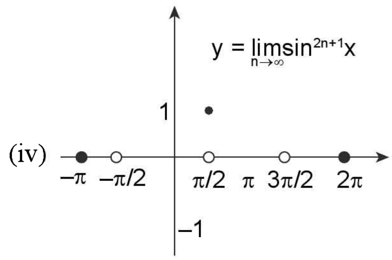






■ TRANSFORMATION OF GRAPHS

In this chapter, we shall study the change in the nature of graph of the function going through different standard transformations. These transformations are very helpful in solving questions geometrically. For this purpose we shall take the help of graph of a function as shown below.

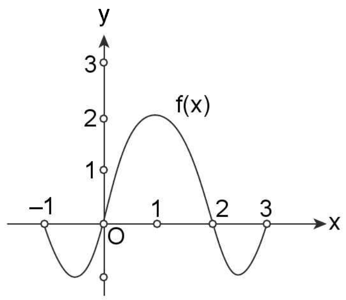


FIGURE 3.104

1. **Graph of $y = f(x) + k$:** Graph of $y = f(x) + k$ can be obtained by translating graph of $f(x)$ by $|k|$ unit along y-axis in the direction same as sign of k, i.e., upward when $k > 0$ and downward when $k < 0$.

   This is because each out put of the function is added by k. Therefore, each point of graph shifts vertically by k unit.

For example, graph of $y = f(x) + 1$ is as shown in the Figure 3.105 and graph of $y = f(x) - 1$ is shown in Figure 3.106.

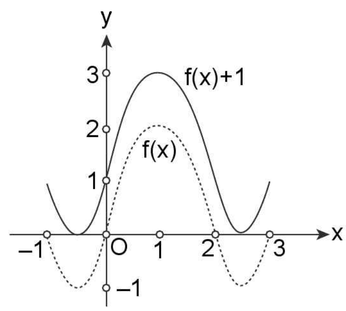


FIGURE 3.105

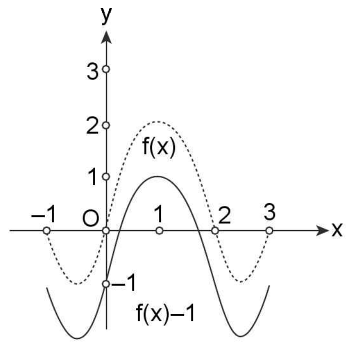


FIGURE 3.106

ILLUSTRATION 20: Draw the graph of $y = e^x$, and hence, draw the graph of $y = e^x + 1$ and $y = e^x - 1$

SOLUTION: We know, $y = e^x$ could be plotted as shown below:

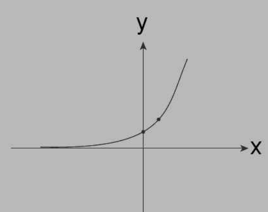


FIGURE 3.107

$y = e^x + 1$ is plotted by shifting the graph of $y = e^x$ by 1 unit in upward direction.

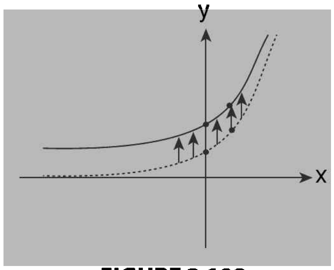


FIGURE 3.108

Also $y = e^x - 1$ can be plotted by shifting the graph of $y = e^x$ downwards by 1 unit.

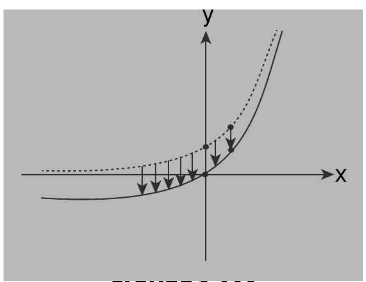


FIGURE 3.109

ILLUSTRATION 21: Find the number of solutions of equation $1 + \cos x = \log_2 x$

SOLUTION: The number of solutions of the above equation can be found by the number of points of intersection of the graphs $y = 1 + \cos x$ and $y = \log_2 x$

The graph of $y = \cos x$ is shown in Figure 3.110.

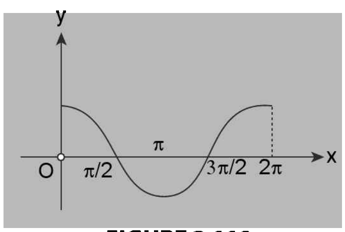


FIGURE 3.110

The graph of $y = 1 + \cos x$ can be obtained by shifting the curve of $y = \cos x$ by 1 unit in the upward direction.

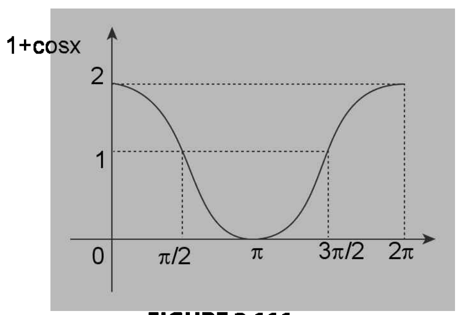


FIGURE 3.111

The graph of $y = \log_2 x$ is shown in Figure 3.112.

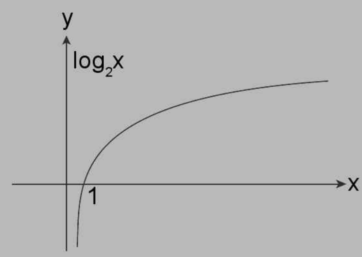


FIGURE 3.112

Now; maximum of $1 + \cos x = 2$

Now $\log_2 x \le 2 \Rightarrow x \le 4$

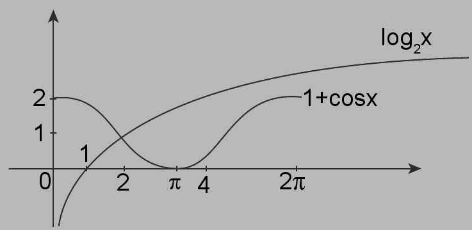


FIGURE 3.113

Clearly, only one solution is possible.

2. **Graph of $y = f(x + k)$:** Graph of $y = f(x + k)$ can be obtained by translating graph of $y = f(x)$ by $|k|$ units in the direction opposite to the sign of k along x-axis. That is, a addition and subtraction to independent variable leads to horizontal shift because each output $f(x)$ of the original function is obtained by the transformed function $f(x + k)$ at the input $x - k$.

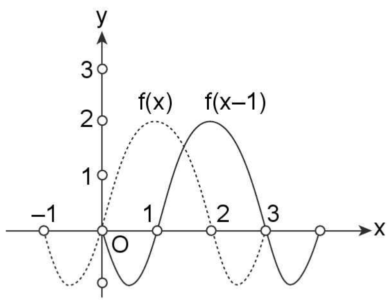


FIGURE 3.114

For example, graph of $f(x - 1)$ is as shown in the Figure 3.114 and that of $f(x + 1)$ is shown in Figure 3.115. Because if the origin is shifted to (1, 0). And in new system of axes, $f(x + k)$ behaves as $f(x)$.

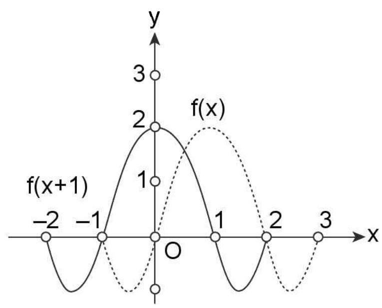


FIGURE 3.115

ILLUSTRATION 22: Draw the graph of $y = x^2$, and hence, draw the graph of $y = x^2 + 2x + 1$ and also the graph of $y = x^2 - 2x + 1$, and therefore, predict the co-ordinates of vertex of the parabola $y = x^2 + 2x + 1$ and $y = x^2 - 2x + 1$.

SOLUTION: We know; $y = x^2$ could be plotted as

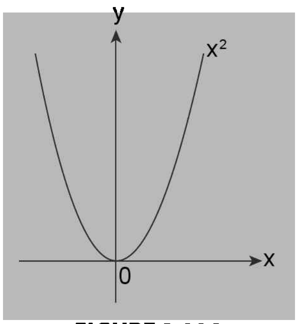


FIGURE 3.116

Now $y = x^2 + 2x + 1 = (x+1)^2$ would be plotted by shifting $y = x^2$ by 1 unit in the left direction as shown in the Figure 3.117. Therefore, the vertex of the parabola would be (–1, 0).

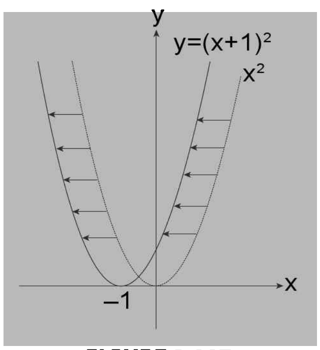


FIGURE 3.117

Now $y = x^2 - 2x + 1 = (x - 1)^2$ would be plotted by shifting $y = x^2$ by 1 unit in the right direction, and hence, the vertex of new parabola lies at (1, 0).

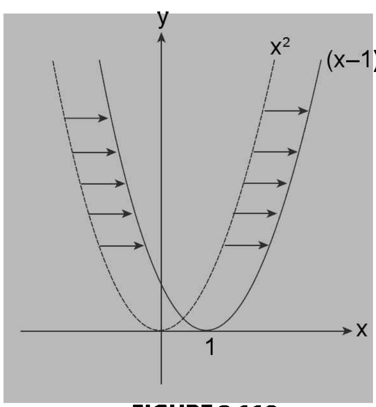


FIGURE 3.118

3. **Graph of $y = k(f(x))$:** Graph of $y = k(f(x))$ can be obtained by vertically stretching or contracting the graph of $f(x)$ depending on the value of k. It is because each output of the obtained function becomes k times that of the original function. Hence, due to this transformation no stretching/compression is produce along x-axis. There are few important facts that are worth noticing in this case.

(i) no change in the domain of the function.

(ii) the behaviour of the function at $y = 0$ (i.e., x intersect/roots) remains same as the original function.

(iii) the range of the function may be change.

The detail of the transformation can be understood by considering the following two cases as described below.

Case I: When $0 < |k| < 1$, compressing the graph of $f(x)$, i.e., $f(x)$ is transformed to k times $f(x)$ along y-axis towards x-axis.

e.g., graph of $y = \frac{1}{2} f(x)$ is shown in the Figure 3.119 and graph of $y = -\frac{1}{2} f(x)$ is shown in Figure 3.120.

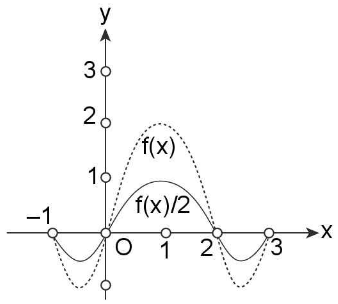


FIGURE 3.119

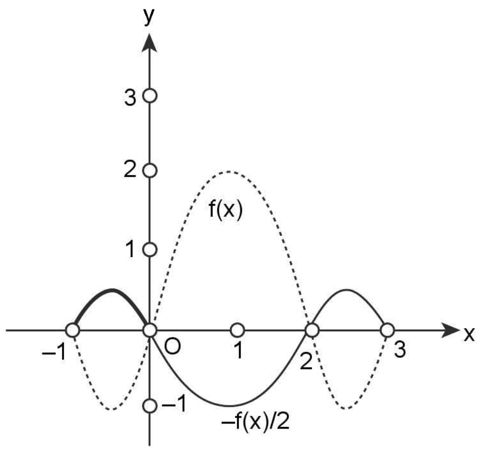


FIGURE 3.120

Case II: When $|k| > 1$, stretching the graph of $f(x)$, along y-axis away from x-axis.

That is, $f(x)$ is transformed to k times $f(x)$.

For example, graph of $y = 2 f(x)$ is as shown in the Figure 3.121 and graph of $y = -2 f(x)$ is as shown in Figure 3.122.

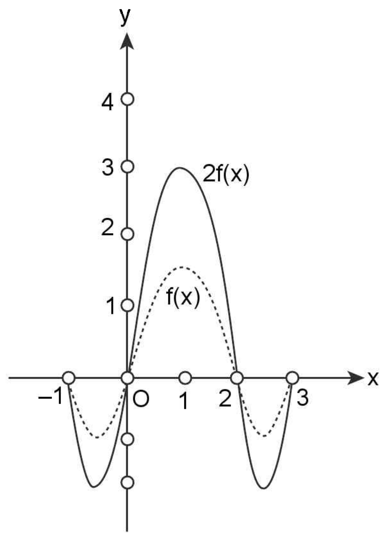


FIGURE 3.121

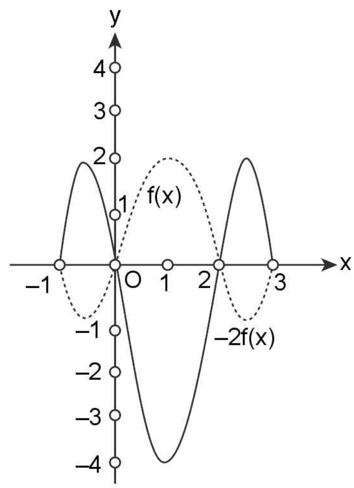


FIGURE 3.122

ILLUSTRATION 23: Find the number of solutions of $(x + 3)^2 = 2\ln x$.

SOLUTION: The number of solutions of the above equation can be found by the number of point of intersection of the graphs $y = (x + 3)^2$ and $y = 2\ln x$.

Clearly, $x > 0$ (Domain)

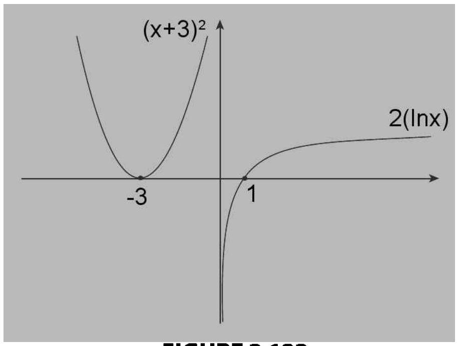


FIGURE 3.123

Clearly, there is no point of intersection, and hence, no solution.

ILLUSTRATION 24: Draw the graph of $\{x\}$, and hence, draw the graphs of 3 $\{x\}$ and also $\frac{1}{2}\{x\}$, where $\{x\}$ denotes the fractional part of x.

SOLUTION:

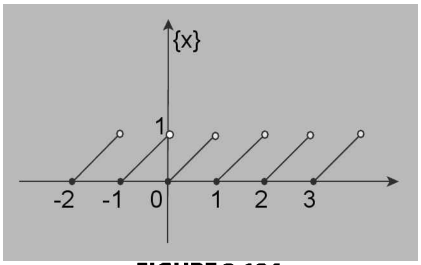


FIGURE 3.124

The graph of $3\{x\}$ can be drawn by stretching 3 times, the graph of $\{x\}$ along y-axis, away from x-axis.

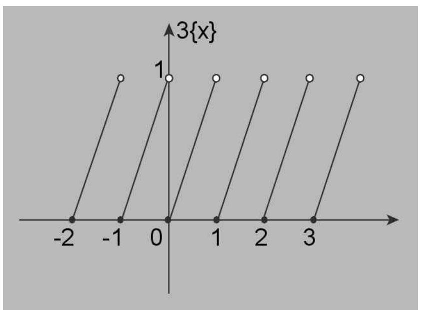


FIGURE 3.125

The graph of $\frac{1}{2}\{x\}$ can be drawn by compressing to 1/2 times the graph of $\{x\}$ along y-axis, towards x-axis.

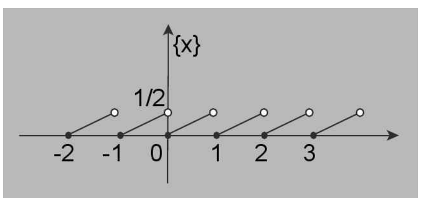


FIGURE 3.126

4. Graph of $y = f(kx)$: Graph of $y = f(kx)$ can be obtained by compressing or stretching the graph of $y = f(x)$ along x-axis towards y-axis or away from y-axis depending on the value of k as described below.

Case I: When $|k| > 1$, compressing the graph of $f(x)$ horizontally towards y-axis. For example, graph of $f(2x)$ is shown in Figure 3.127 and graph of $f(-2x)$ is shown in Figure 3.128.

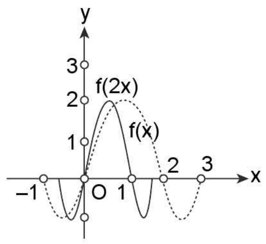


FIGURE 3.127

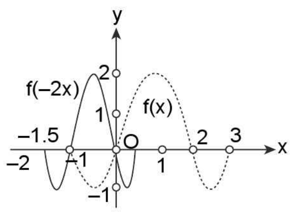


FIGURE 3.128

Case II: When $0 < |k| < 1$, stretching the graph of $y = f(x)$ along x-axis away from y-axis.
e.g., graph of $f(x/2)$ is shown in Figure 3.129 and graph of $f(-x/2)$ is shown in Figure 3.130.

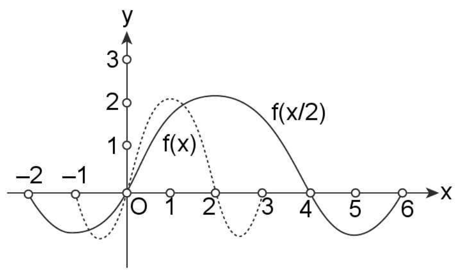


FIGURE 3.129

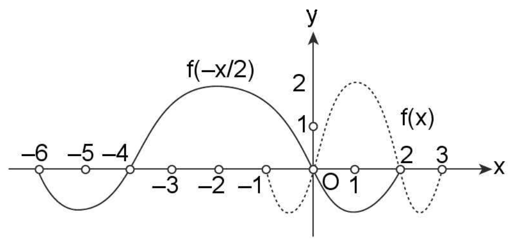


FIGURE 3.130

The above effect of horizontal stretching and compression towards y-axis is due to the fact that the transformed function takes up the same values for the input x/k, as the original function takes up for input x. that is why the domain of function is either stretched or compressed as may be the case, whereas the range of the function remains unchanged in this transformation. This transformation also explains why the period of the periodic function $f(kx)$ becomes $1/|k|$ times the period of $f(x)$.

ILLUSTRATION 25: A function $f:\left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \to \mathbb{R}$ such that $f(x) = \tan x$. Now draw $f(2x)$, $f(-2x)$ and $f\left(\frac{-x}{2}\right)$

SOLUTION: The graph of $\tan x$ for $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ is as shown below.

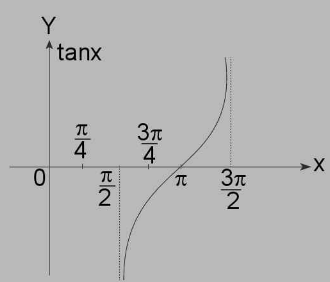


FIGURE 3.131

Since domain of function f is $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

$\therefore \quad f(2x) \Rightarrow 2x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

$\Rightarrow \quad x \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

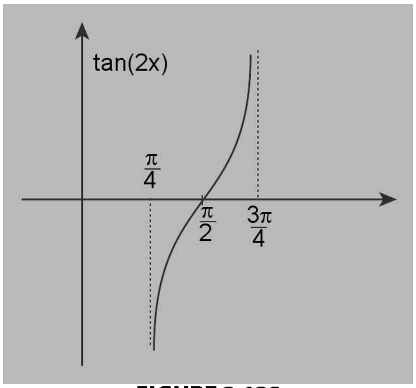


FIGURE 3.132

Since domain of function f is $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

$\therefore \quad f(-2x) \Rightarrow -2x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

$\Rightarrow \quad x \in \left(\frac{-3\pi}{4}, \frac{-\pi}{4}\right)$

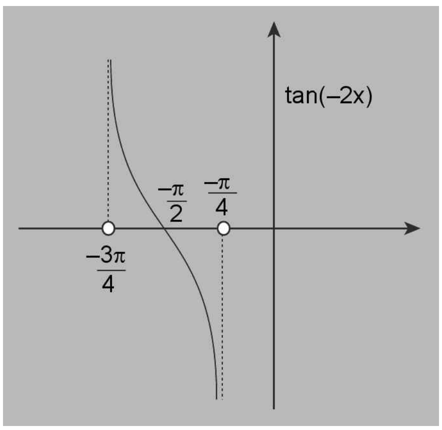


FIGURE 3.133

Since domain of function f is $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

$$f\left(\frac{x}{2}\right) \Rightarrow \frac{x}{2} \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \Rightarrow x \in (\pi, 3\pi)$$

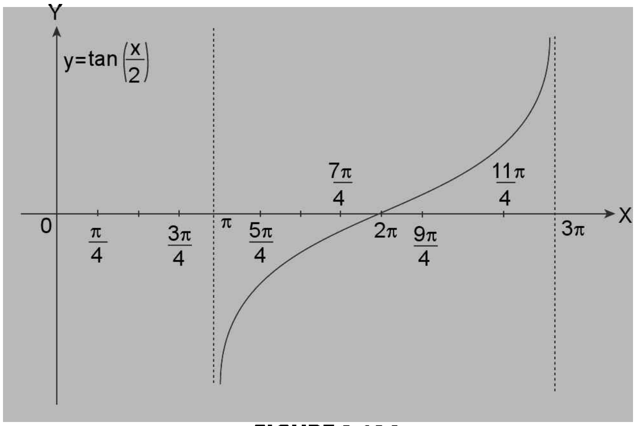


FIGURE 3.134

For $f\left(\frac{-x}{2}\right); -\frac{x}{2} \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \quad \Rightarrow \quad x \in (-3\pi, -\pi)$

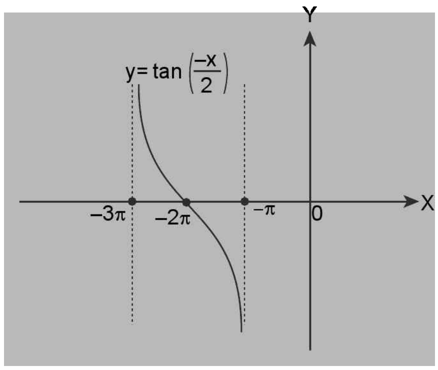


FIGURE 3.135

ILLUSTRATION 26: Draw the graph of $\sin^{-1}(\sin x)$, and hence, draw the graphs of $-2(\sin^{-1}(\sin x))$ and $\frac{-1}{2}(\sin^{-1}(\sin x))$.

SOLUTION: The graph of $\sin^{-1}(\sin x)$ is given by

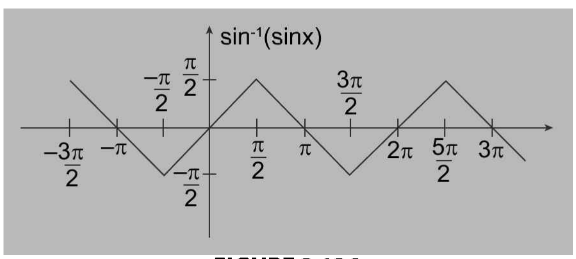


FIGURE 3.136

Therefore, the graph of $-2(\sin^{-1}(\sin x))$ can be obtained by taking mirror image of the graph of $\sin^{-1}(\sin x)$ about x-axis, and there by, stretching the resulting graph 2 times along y-axis, away from the x-axis.

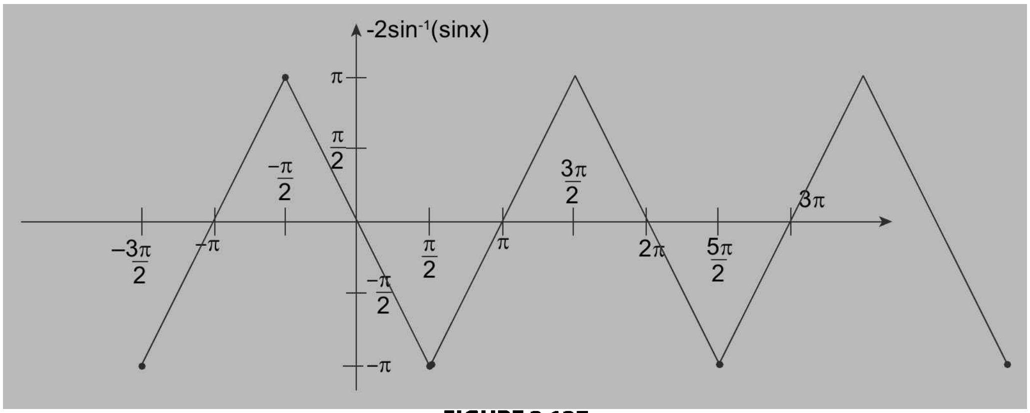


FIGURE 3.137

Therefore, the graph of $\frac{-1}{2}(\sin^{-1}(\sin x))$ can be obtained by taking mirror image of the graph of $\sin^{-1}(\sin x)$ about x-axis and thereby compressing the resulting graph 2 times along y-axis, towards the x-axis.

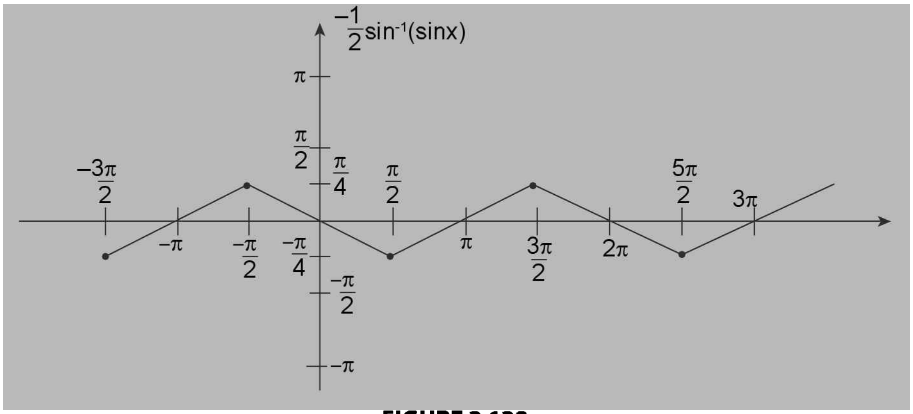


FIGURE 3.138

ILLUSTRATION 27: Find the number of solutions of $\sin^{-1}(-x-1) = \cos^{-1}x$

SOLUTION: Graph of $\sin^{-1}x$

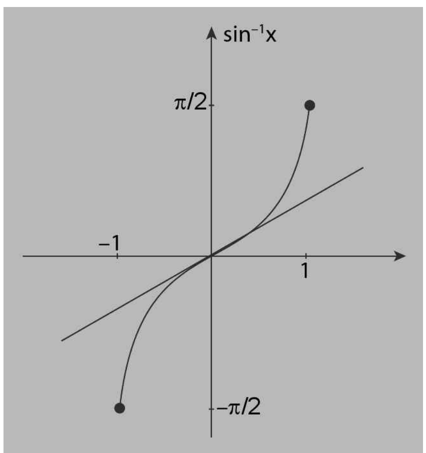


FIGURE 3.139

Graph of $\sin^{-1}(-x)$

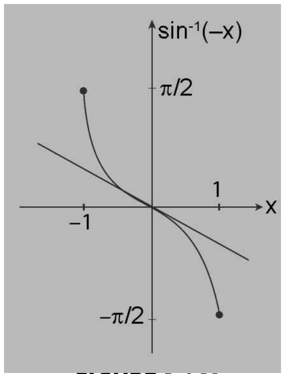


FIGURE 3.140

Graph of $\sin^{-1}(-x-1)$

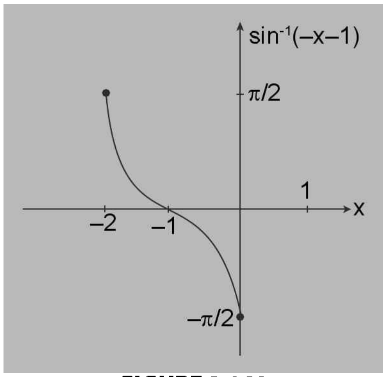


FIGURE 3.141

Graph of $\cos^{-1} x$

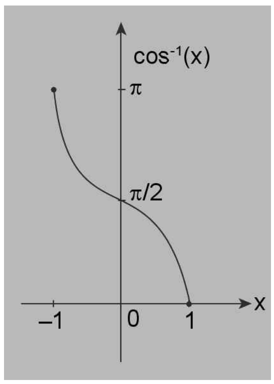


FIGURE 3.142

Clearly, the two graphs do not have a point of intersection, and hence, no solution.

5. **Graph of $y = |f(x)|$:** Graph of $y = |f(x)|$ can be obtained by reflecting the portion of the graph of $f(x)$ lying below x-axis on x-axis as a mirror and keeping the portion of graph above x-axis as it is. Since $y = f(x)$ when $f(x) \geq 0$ and $y = -f(x)$ when $f(x) < 0$. Hence, the graph of $y = |f(x)|$ is as shown in Figure 3.143 and 3.144.

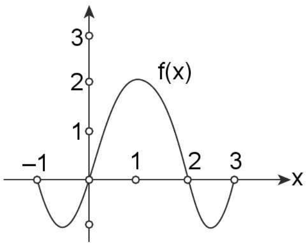


FIGURE 3.143

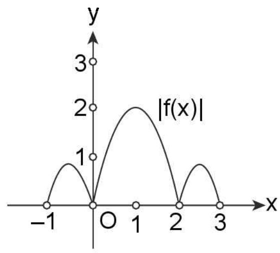


FIGURE 3.144

ILLUSTRATION 28: Draw the graphs of the following functions.

(i) $y = |\sin x|$ (ii) $y = |\ln x|$ (iii) $y = |[x]|$

SOLUTION: (i) Graph of $y = |\sin x|$. Graph of $y = |f(x)|$ is obtained by reflecting the portion of graph of $f(x)$ below x-axis, taking x-axis as mirror. So, graph of $y = |\sin x|$ would be as shown below.

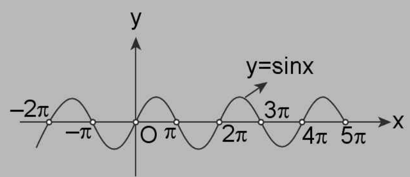


FIGURE 3.145

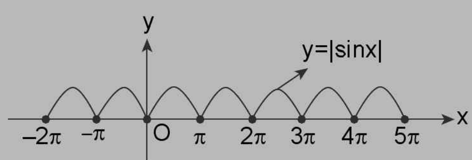


FIGURE 3.146

(ii) Graph of $y = |\ln x|$: Graph of $y = |f(x)|$ is obtained by reflecting the portion of graph below x-axis taking x-axis as mirror.

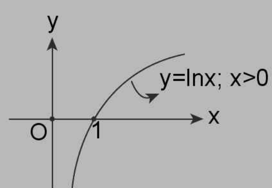


FIGURE 3.147

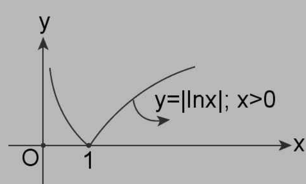


FIGURE 3.148

(iii) Graph of $y = |[x]|$

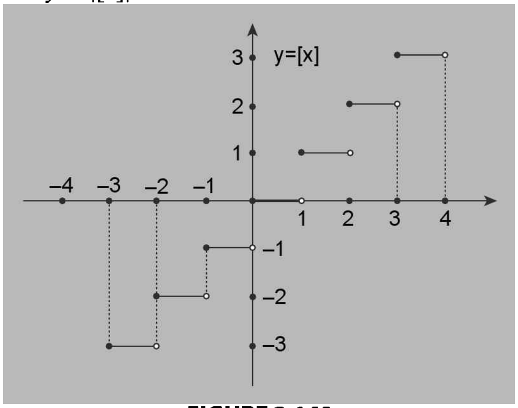


FIGURE 3.149

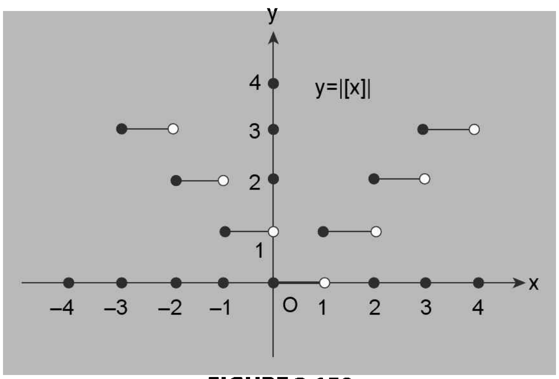


FIGURE 3.150

ILLUSTRATION 29: Draw the graph of $f(x) = ||||x| - 1| - 2| - 3|$, and hence, find value of 'k' (where k is a constant) so that the number of solutions of $f(x) = k$ is maximum.

SOLUTION: Graph of $y = x$

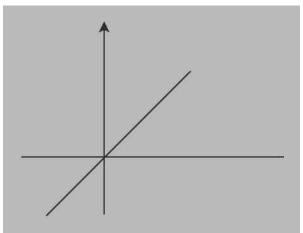


FIGURE 3.151

Graph of $y = |x|$

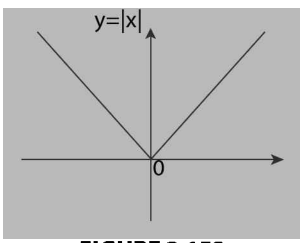


FIGURE 3.152

Graph of $y = |x| - 1$

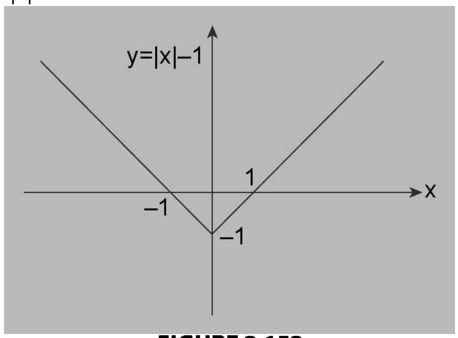


FIGURE 3.153

Graph of $y = ||x| - 1|$

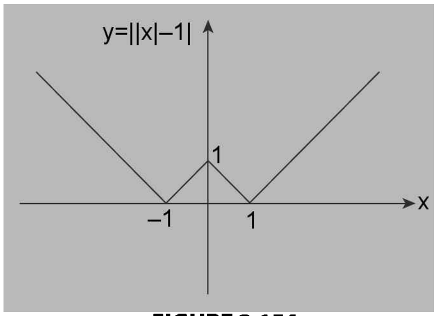


FIGURE 3.154

Graph of $y = ||x| - 1| - 2$

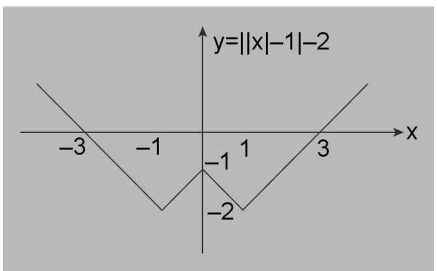


FIGURE 3.155

Graph of $y = ||x| - 1| - 2|$

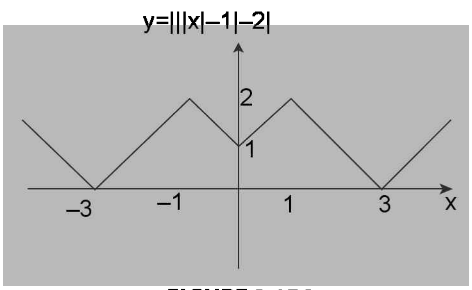


FIGURE 3.156

Graph of $y = |||x| - 1| - 2| - 3$

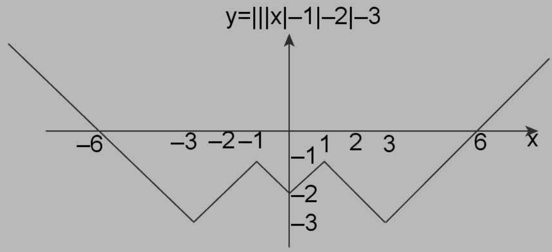


FIGURE 3.157

Graph of $y = ||||x| - 1| - 2| - 3|$

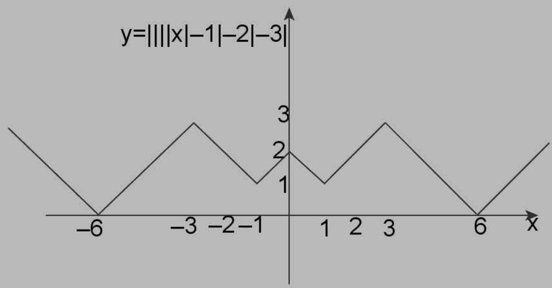


FIGURE 3.158

Now, clearly $f(x) = k$ will have maximum (8) solutions for $k \in (1,2)$.

6. **Graph of $y = f(|x|)$:** Graph of $y = f(|x|)$ can be obtained by keeping the portion of graph of $f(x)$ on right side of y-axis and replacing the portion of the graph of $y = f(x)$ on left side of y-axis by the reflection of right graph on y-axis. Graph of $y = f(|x|)$ is as shown in Figure 3.59 and 3.160.

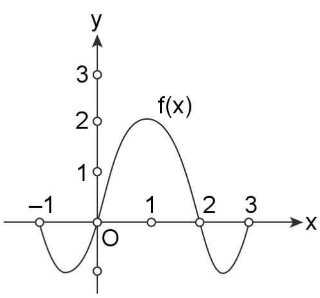


FIGURE 3.159

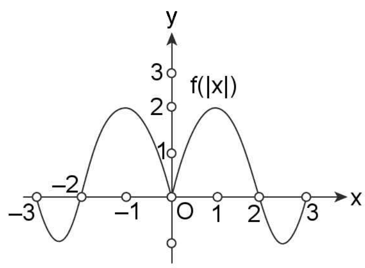


FIGURE 3.160

ILLUSTRATION 30: Draw the graph of the following functions:

(i) $y = \ln |x|$ (ii) $y = \sin |x|$

SOLUTION: **Graph of y = ln $|x|$:** Graph of $y = f(|x|)$ is obtained by reflecting the right side graph of $y = f(x)$ on y-axis.

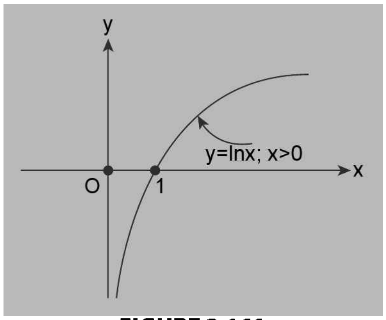


FIGURE 3.161

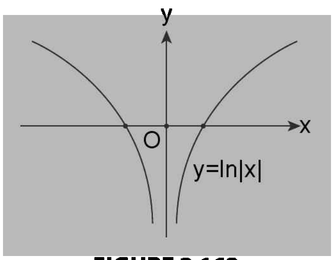


FIGURE 3.162

Graph of sin |x|:

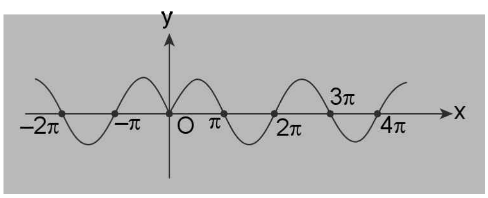


FIGURE 3.163

ILLUSTRATION 31: Draw the graph of the function $f(x) = \tan |x|$ and show that it is non-periodic in its domain.

SOLUTION: The graph of tan x is as shown below.

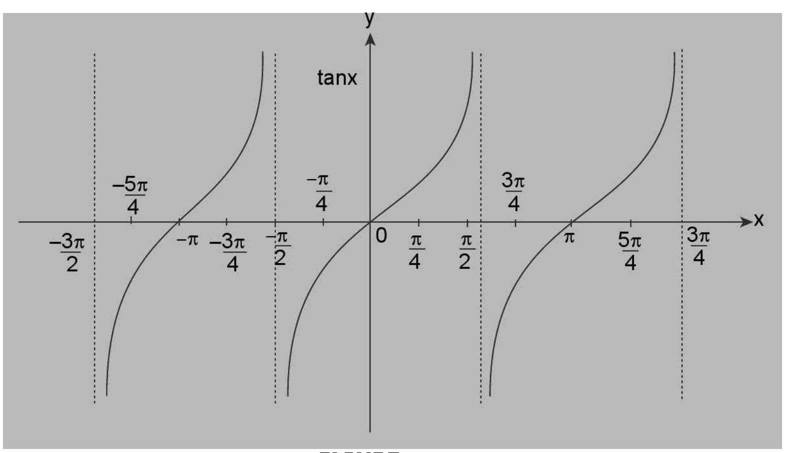


FIGURE 3.164

The graph of tan x for $x > 0$ is as shown in Figure 3.165.

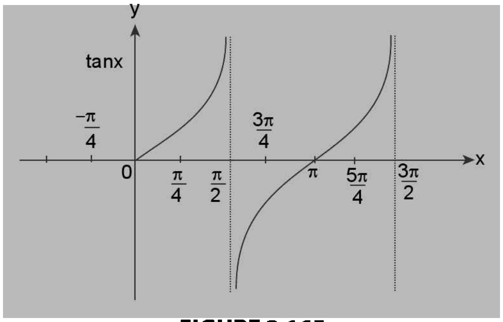


FIGURE 3.165

The graph of tan $|x|$ can be obtained by taking the mirror image of the graph of tan x for $x > 0$ about the y-axis, and hence, the resulting graph of tan $|x|$ is as shown in Figure 3.166.

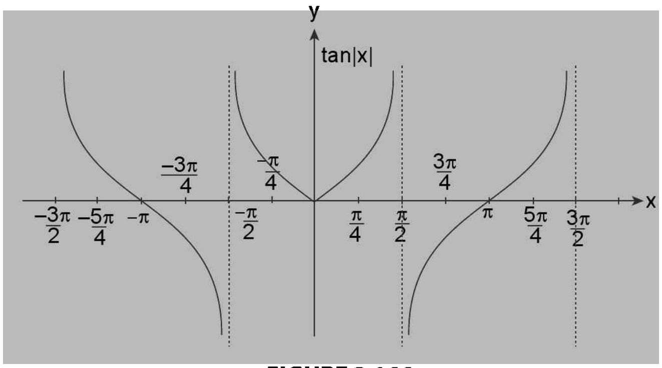


FIGURE 3.166

As is seen in the graph; tan $|x|$ is not a periodic function.

ILLUSTRATION 32: Find the number of solutions of $x = 5\sin(\pi x)$. Also find the number of solutions of $|x| = 5 \sin \pi x$. Hence, or otherwise, prove that the number of solutions of $x = 5\sin(\pi x)$ and $|x| = 5 \sin\pi x$ are not equal.

SOLUTION: The graph of $y = 5\sin(\pi x)\ \forall\ x \in [-5, 5]$ is as shown in Figure 3.167.

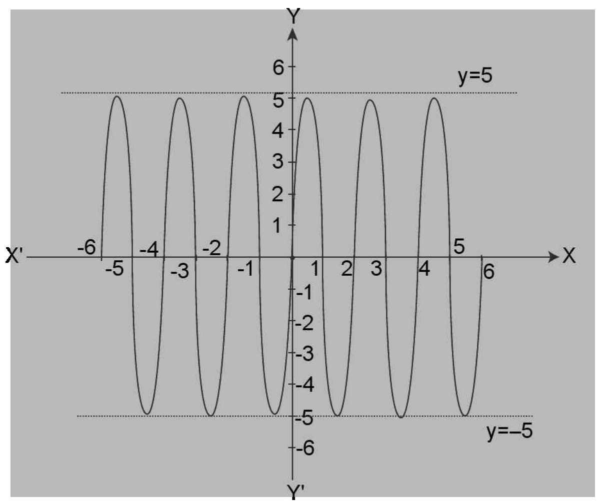


FIGURE 3.167

Therefore, the number of solutions of $x = 5\sin(\pi x)\ \forall\ x \in [-5, 5]$ can be observed as the point of intersection of the graph of $y = x$ and $y = 5\ \sin(\pi x)\ \forall\ x \in [-5, 5]$ drawn in same reference frame as shown in Figure 3.168.

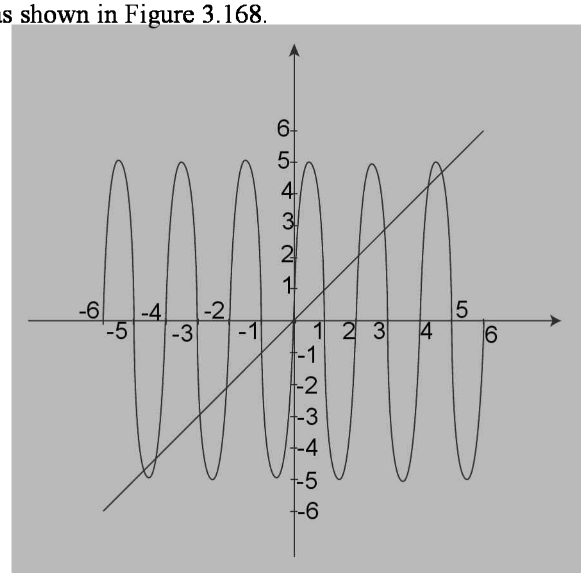


FIGURE 3.168

It is evident from the graph that the equation $x = 5\ \sin(\pi x)\ \forall\ x \in [-5, 5]$ has 11 solutions.

Similarly, the number of solutions of $|x| = 5\sin(\pi x)\ \forall\ x \in [-5, 5]$ can be observed as the point of intersection of the graph of $y = |x|$ and $y = 5\ \sin(\pi x)$ simultaneously.

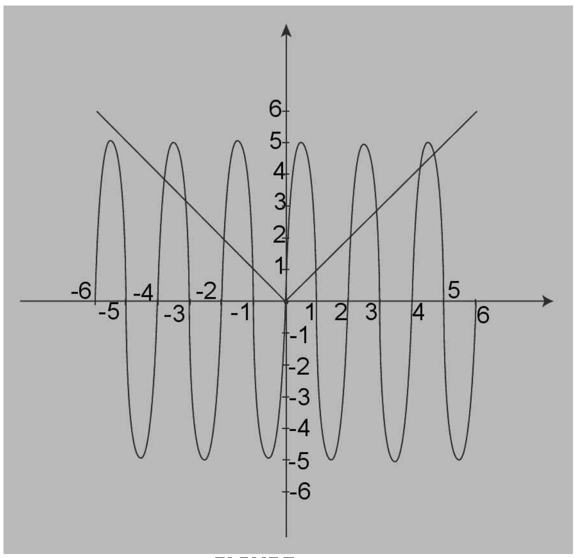


FIGURE 3.169

It is evident from the graph $|x| = 5\sin(\pi x)\ \forall\ x \in [-5, 5]$ has 10 solutions.

Thus, proving the fact that the number of solutions of $x = 5\sin(\pi x)$ and $|x| = 5\ \sin \pi x$ are not equal.

7. Graph of $y = |f|x||$ can be obtained in two steps:

Step 1: Using graph of $y = f(x)$, draw the graph of $f|x|$.

Step 2: Using graph of $y = f|x|$, draw the graph of $y = |f|x||$.

ILLUSTRATION 33: Draw the graph of $y = |\ln|x||$

SOLUTION: **Graph of $y = |\ln|x||$:** Is obtained by reflecting the portion of graph of $y = \ln|x|$ below x-axis on x-axis as shown in Figure 3.170.

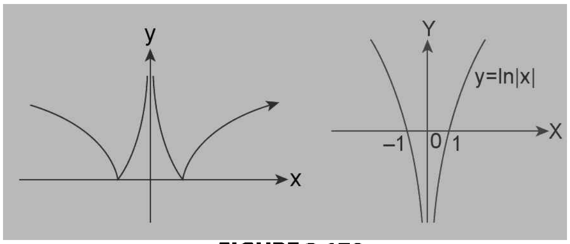


FIGURE 3.170

ILLUSTRATION 34: Sketch the graph of $y = |x^2 - 5|x| - 6|$, and hence, find the number of solutions of $|x^2 - 5|x| - 6| = 8$.

SOLUTION: Let $f(x) = x^2 - 5x - 6$

Graph of $f(x)$

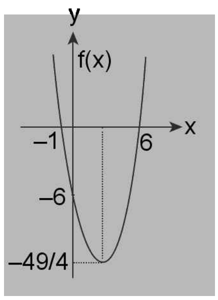


FIGURE 3.171

Graph of $f|x|$

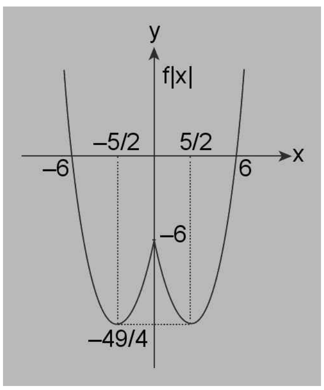


FIGURE 3.172

Graph of $|f|x||$

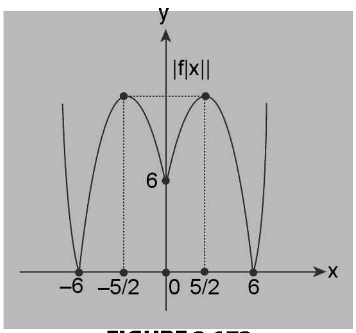


FIGURE 3.173

Drawing $y = 8$ and $y = |f|x||$ on the same reference frame, we get

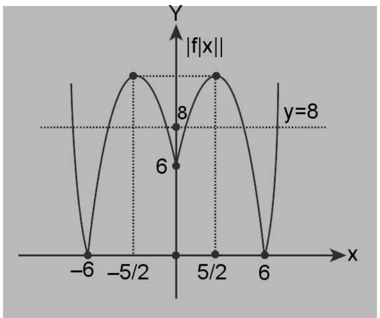


FIGURE 3.174

Clearly from the graph, we can say that $|x^2 - 5|x| - 6| = 8$ has 6 solutions.

ILLUSTRATION 35: If $f(x) = \begin{cases} 1; x \geq 1 \\ 2x - 1; 0 < x < 1 \\ -1; x \leq 0 \end{cases}$ and $g(x) = \log_b |ax|$ and $h(x) = g|x|$

Then find the number of solutions of $|f|(x)|| = g(x)$ when (a, b) is given by

(i) $(1, e)$ (ii) $(1, 1/e)$ (iii) $\left(10, \frac{1}{10}\right)$

Also, find the points of intersection, if possible.

Also find the number of solutions of $|f|x|| = |h(x)|$ when (a, b) is given by

(iv) $(1/10, 1/10)$ (v) $(10, 1/10)$

SOLUTION: The graph of $f(x)$ is as shown in Figure 3.175.

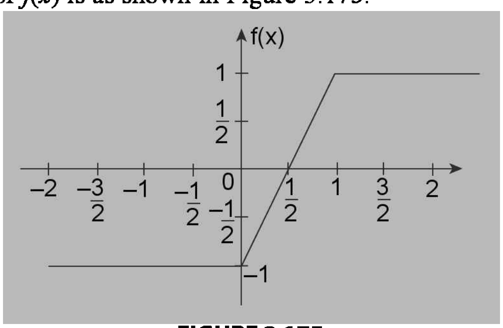


FIGURE 3.175

Now, the graph of $f|x|$ can be obtained by erasing the graph of $f(x)$ on the left hand side of the y-axis and taking the mirror image of the graph of the right hand side of the y-axis about y-axis.

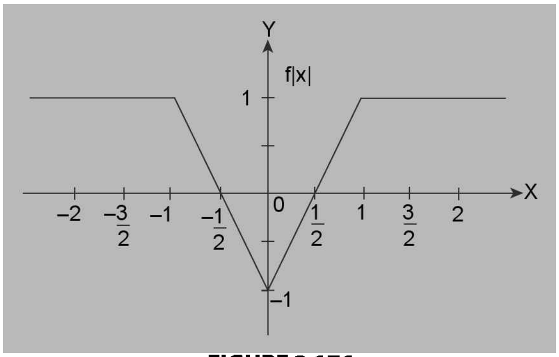


FIGURE 3.176

Now, the graph of $|f|x||$ can be obtained by taking the modulus of the graph of $f|x|$, i.e., reflecting the portion below x-axis on x-axis.

Hence, the graph of $|f|x||$ is as shown in Figure 3.177.

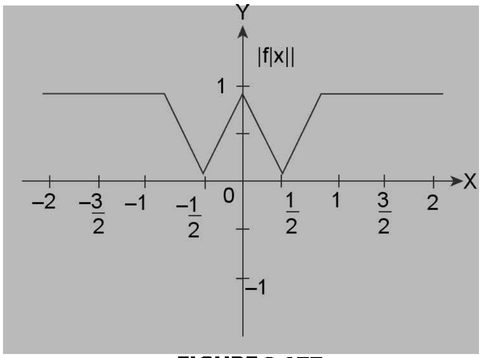


FIGURE 3.177

(i) The number of solutions of $|f|x|| = g(x)$, where $g(x) = \log_b |ax|$ and (a, b) is given by $(1, e)$. Hence, the function $g(x) = \log_e|x|$

The number of solutions of $|f|x|| = g(x)$ can be seen by drawing both the graphs on the same reference plane.

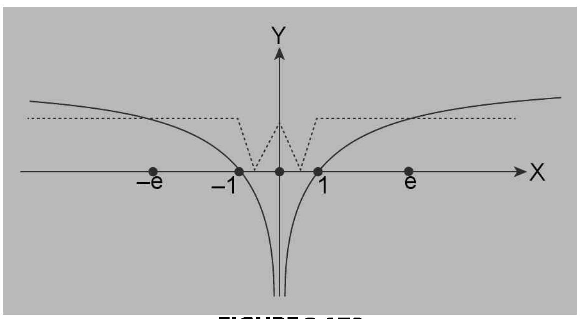


FIGURE 3.178

Hence, points of the intersection are $x = \pm e$

(ii) $(a, b) = (1, 1/e) \Rightarrow g(x) = \log_{1/e}|x|$ and to find the solution of $|f|x|| = \log_{1/e}|x|$

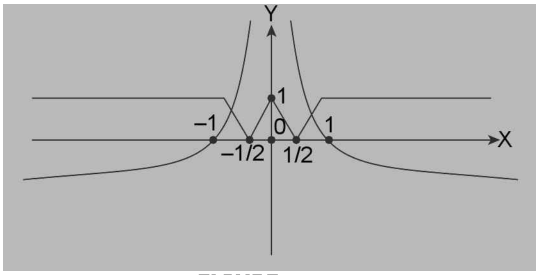


FIGURE 3.179

Here, we have only two points of intersection lying between (–1, –1/2) and (1/2, 1)

(iii) $(a, b) = (10, 1/10) \Rightarrow g(x) = \log_{(1/10)}|10\,x|$

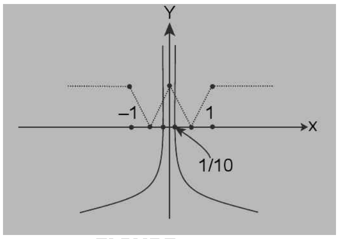


FIGURE 3.180

$\Rightarrow g(x) = \log_{1/10}|10x|$

As is evident from the graph, only two solutions, one each in (–1/10, 0) and (0, 1/10)

(iv) $(a,b)=\left(\frac{1}{10},\frac{1}{10}\right) \Rightarrow g(x)=\log_{1/10}\left|\frac{1}{10}x\right|$

$\Rightarrow$ Graph of $g(x)$ or $g|x|$ or $h|x|$ is as shown in Figure 3.181.

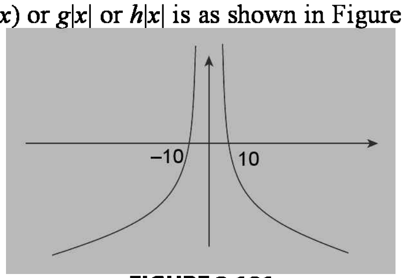


FIGURE 3.181

Graph of $|h(x)|$ and $|f|x||$

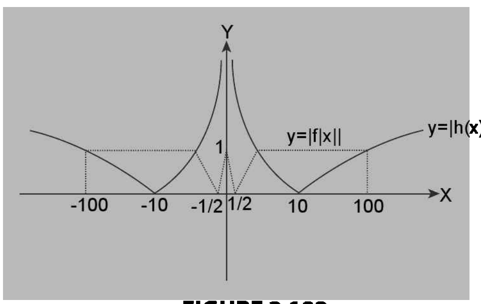


FIGURE 3.182

$$\therefore \quad |h(x)| = |f|x|| \quad \Rightarrow \quad \left|\log_{1/10}\left|\frac{1}{10}x\right|\right| = 1$$

$$\Rightarrow \quad \log_{1/10}\left|\frac{1}{10}x\right| = \pm 1$$

$$\Rightarrow \quad \left|\frac{1}{10}x\right| = \left(\frac{1}{10}\right)^{\pm 1} = 10 \text{ or } \frac{1}{10} \qquad \Rightarrow \quad |x| = 100 \text{ or } 1 \qquad \Rightarrow \quad x = \pm 100 \text{ or } \pm 1$$

Hence, 4 solutions $x = \pm 100$ and $x = \pm 1$

(v) $(a, b) = (10, 1/10)$

$\Rightarrow \quad g(x) = \log_{1/10}|10\,x| \qquad\qquad \therefore \quad |h(x)| = |g|x|| = |g(x)|$

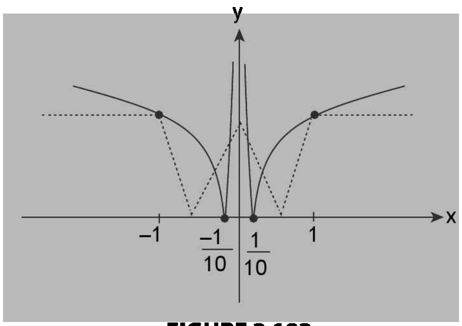


FIGURE 3.183

Hence, 4 solutions $x = \pm 1$, one each in $(-1/10, 0)$ and $(0, 1/10)$

ILLUSTRATION 36: Sketch the graph of $y = |e^{-|x|} - 1/4|$

SOLUTION: We know, the graph of $y = e^{-x}$

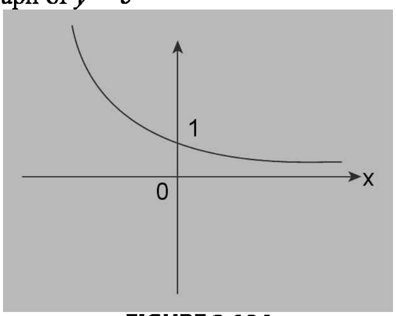


FIGURE 3.184

Now graph of $y = e^{-|x|}$

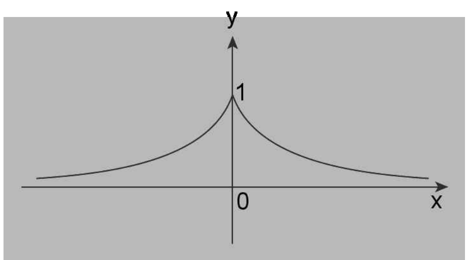


FIGURE 3.185

Graph of $y = e^{-|x|} - 1/4$

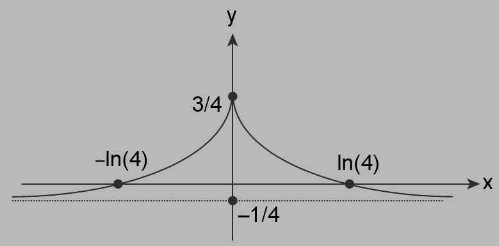


FIGURE 3.186

Hence, graph of $y = |e^{-|x|} - 1/4|$

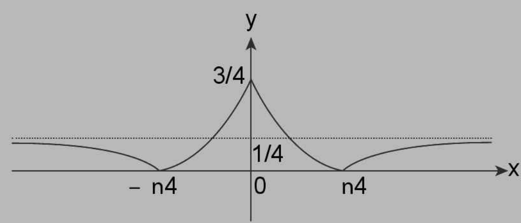


FIGURE 3.187

8. Graph of $|y| = f(x)$: Graph of $|y| = f(x)$ is obtained by discarding the portion of graph of $f(x)$ below x-axis ($\because |y| \geq 0$) and reflecting the portion of graph of $f(x)$ above x-axis on x-axis.
If $f(x) < 0$, graph of $|y| = f(x)$ would not exist.
And if $f(x) \geq 0$, $|y| = f(x)$ would give $y = \pm f(x)$.
Hence, graph of $|y| = f(x)$ would exist only in the regions where $f(x)$ is non-negative and will be reflected about x-axis only in those regions. Regions where $f(x) < 0$ will be neglected. The graph of $|y| = f(x)$ is shown in Figure 3.188

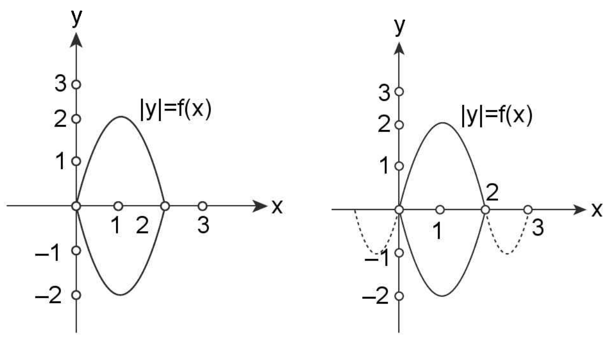


FIGURE 3.188

ILLUSTRATION 37: Draw the graph of the following functions: (i) $|y| = \ln|x|$ (ii) $|y| = \tan x$

SOLUTION: Graph of $|y| = f(x)$ is obtained by reflecting the portion of graph above x-axis taking x-axis is as mirror and neglecting the portion of graph below x-axis. (i) graph of $|y| = \ln|x|$

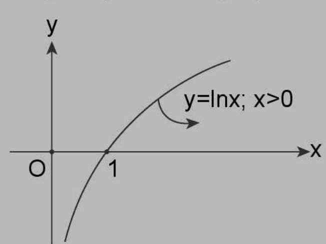


FIGURE 3.189

Graph of $|y| = \ln|x|$: It can be obtained by reflecting the graph of $|y| = \ln x$; $x > 0$ on y-axis.

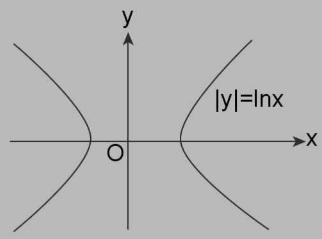


FIGURE 3.190

(ii) Graph of $|y| = \tan x$

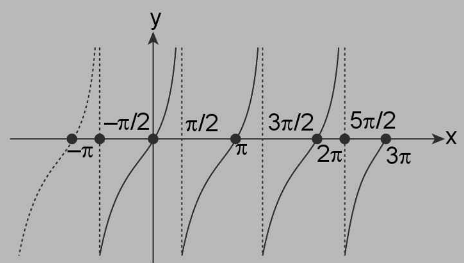


FIGURE 3.191

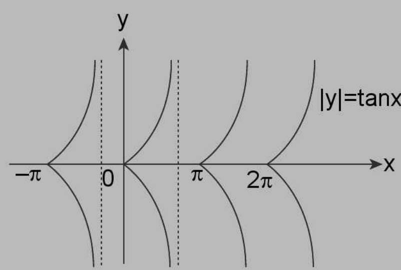


FIGURE 3.192

9. **Graph of $|y| = |f(x)|$:** Graph of $|y| = |f(x)|$ can be obtained by the following two steps:

(i) Using graph of $f(x)$, plot the graph of $|f(x)|$.

(ii) Now, take the mirror image of the graph of $y = |f(x)|$ on x-axis, and hence, we obtain the graph of $|y| = |f(x)|$.

ILLUSTRATION 38: Draw the graph of the following functions:

SOLUITON: Graph of $|y| = |\ln|x||$ is obtained by reflecting the graph of $y = |\ln|x||$ on x-axis as shown in Figure 3.193.

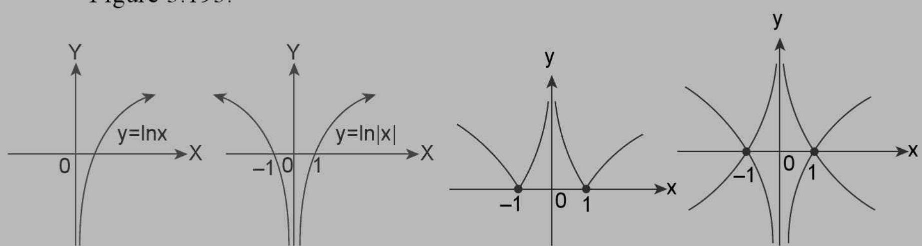


FIGURE 3.193

TEXTUAL EXERCISE–2: (SUBJECTIVE)

1. If a function is defined as $f(x) = \begin{cases} x & ; \quad 0 \le x \le 1 \\ 2 - x & ; \quad 1 < x \le 2 \\ (x-2)(x-4); & \quad 2 \le x \le 4 \end{cases}$ then sketch the graph of $y = f(x)$, and hence, sketch the curves of the following functions:

(a) $y = f(-x)$ (b) $y = -f(x)$

(c) $y = -f(-x)$ (d) $y = f(x + 2)$

(e) $y = f(x) + 3$ (f) $y = f(x+2) + 3$
(g) $y = 2f(x)$ (h) $y = f(2x)$
(i) $y = 2f(3x)$ (j) $y = |f(x)|$
(k) $y = f(|x|)$ (l) $y = |f(|x|)|$
(m) $y = f(x) + |f(x)|$ (n) $y = f(x) - |f(x)|$
(o) $y = \dfrac{|f(x)|}{f(x)}$ (p) $y = sgn(f(x))$
(q) $y = f^2(x)$ (r) $y^2 = f(x)$
(s) $y = (f(x))^{1/2}$

2. Sketch the graph of the following functions using transformation:
(a) $y = \log_e(-x)$ (b) $y = 2\sin x$
(c) $y = \sin 2x$ (d) $y = \sin(|x|)$
(e) $|y| = \log x$ (f) $y = \log |x|$
(g) $y = |\log x|$

3. Sketch the graph of the following functions:
(a) $y = x^2 + 4$ (b) $x^2 = 4y - 6$
(c) $x = 1 + \sqrt{4y - y^2}$ (d) $y = 3 + \sqrt{4x - x^2 + 5}$
(e) $y = \sqrt{x^2 - 16}$ (f) $x = 1 - \sqrt{y^2 - 4x + 8}$
(g) $y = \sin x + \cos x$ (h) $y = \dfrac{2}{3}\sqrt{9 - x^2}$

4. Sketch the graph of the following functions:
(a) $f(x) = 4\sin(3x)$
(b) $f(x) = \ln(4x - 5)$
(c) $f(x) = 3\log_2(-2x) - 5$

5 Sketch the graph of each of the following functions starting with the graph of $y = \sqrt{x}$ using suitable graphical transformation.
(a) $y = 2\sqrt{x} - 3$
(b) $y = 2\sqrt{x+5}$
(c) $f(x) = 2 + \sqrt{x-3}$
(d) $f(x) = -\sqrt{3-2x}$
(e) $y = \begin{cases} 3^x - 1, & if\ x < 0 \\ \sqrt{2x - x^2}, & if\ x \geq 0 \end{cases}$

6. Sketch the graph of each of the following functions starting with the graph of $y = \sqrt{1-x^2}$, using suitable graphical transformation.
(a) $f(x) = \sqrt{1 - (x/2)^2}$
(b) $f(x) = 1 + 3\sqrt{1 - (x-2)^2/9}$
(c) $f(x) = -3 + \sqrt{4 - (x-1)^2}$

7. Sketch the graph of the following functions:
(i) $y = (x-1)^{2/3}$
(iii) $y = \sqrt[3]{x-8}$
(iii) $y = \sqrt[3]{2x+8} - 5$

8. Sketch the graph of the following functions:
(i) $y = \log_3(4 - x)$
(ii) $y = \log_{1/2}(4 - 2x)$

9. Sketch the graph of the following functions:
(i) $y = \begin{cases} 3^x - 1, & if\ x < 0 \\ \sqrt{2x - x^2}, & if\ x \geq 0 \end{cases}$
(ii) $y = \begin{cases} 2 - \sqrt{1-x^2}, & if\ x \leq 1 \\ 2 + \log_{1/2} x, & if\ x > 1 \end{cases}$

10. The graphs of the curves
(a) $y = x(4 - x)$
(b) $y = 3x - 5$
(c) $y = |x + 1| - 2$
Through the following successive transformations, Find the resulting equations of each curve after the transformation T_3 have been exerted.
(T_1) A vertical translation of 2 units up.
(T_2) A reflection about the y-axis.
(T_3) A horizontal translation of 3 unit to the left.

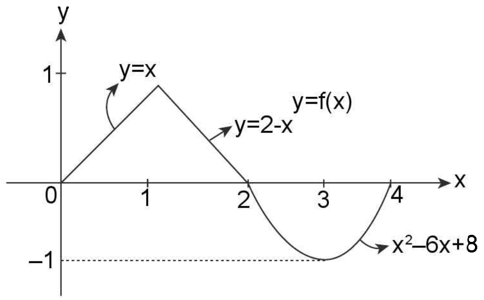
11. Draw the following graphs in succession
(a) $y = |1 - |1 - x||$
(b) $y = 1 - |1 - |1 - x||$
(c) $y = 1 - |1 - |1 - |1 - x|||$

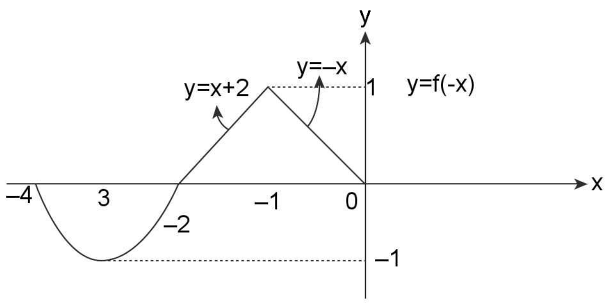
12. Draw the graph of the functions:
(a) $|y| = |\log_{1/3}(x/3)|$ (b) $|y| = |3 - |x - 2||$

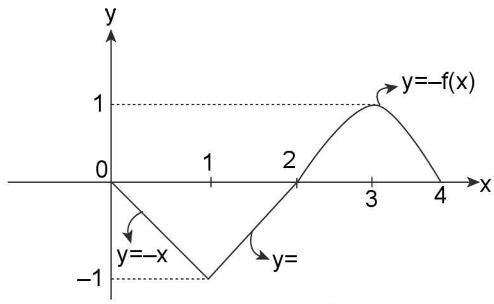
13. Construct the following curves:
(i) $\dfrac{y}{x+2} = -1$
(ii) $|y| + x = -4$
(iii) $|x| + |y| = 5$
(iv) $|y - 4| = |x - 1|$

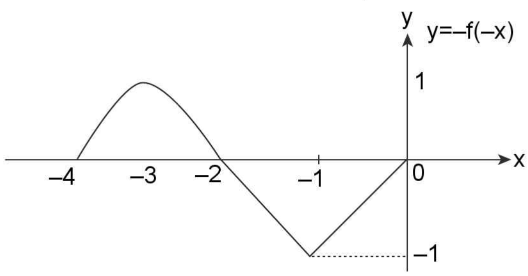
14. Sketch the graph of the following functions:
(a) $y = \cos(x - 1)$ (b) $y = \sin^2 x - 2\sin x$
(c) $y = 2^{\sin x}$ (d) $y = |2 - |x - 1||$
(e) $y = x^2 - 2|x|$

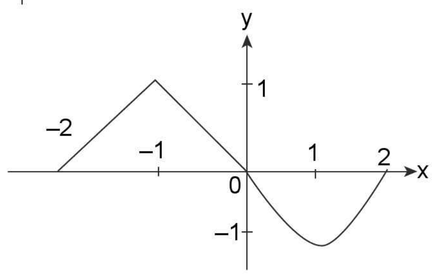
Asnwer Keys

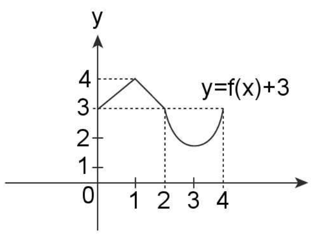
1. 

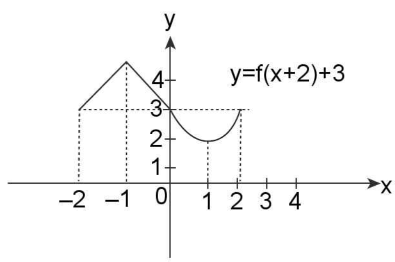
(a) 

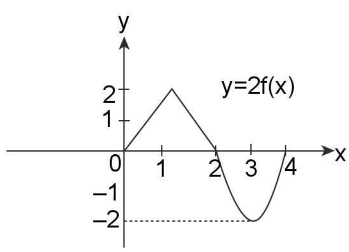
(b) 

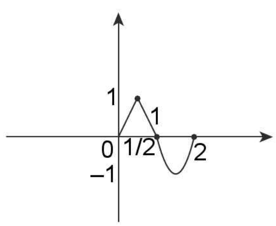
(c) 

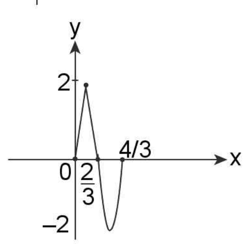
(d) 

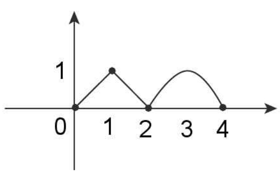
(e) 

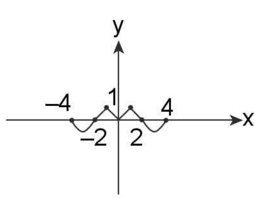
(f) 

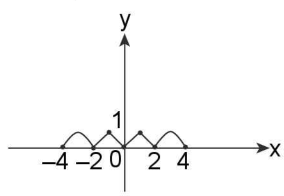
(g) 

(h) 

(i) 

(j) 

(k) 

(l) 

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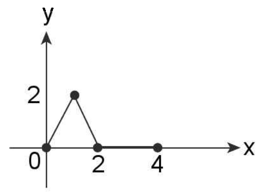

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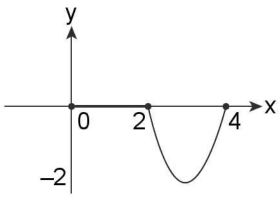

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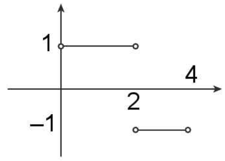

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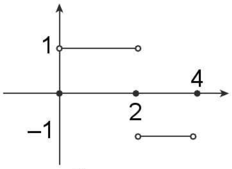

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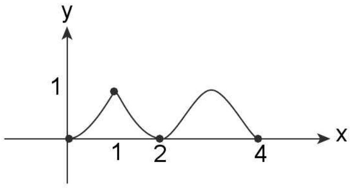

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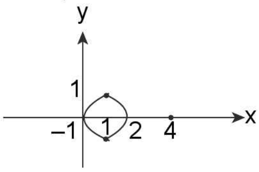
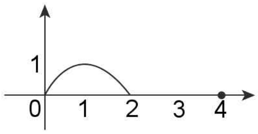
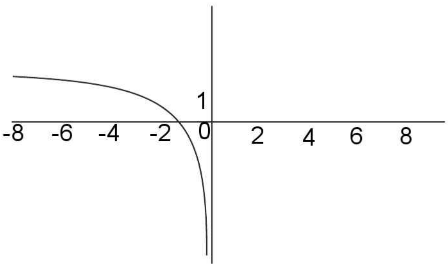


(s)


2. (a)


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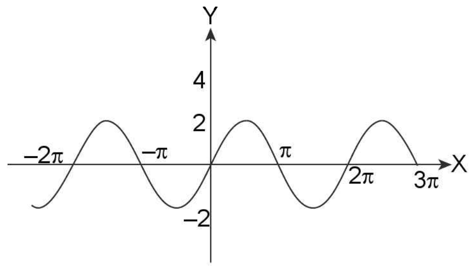

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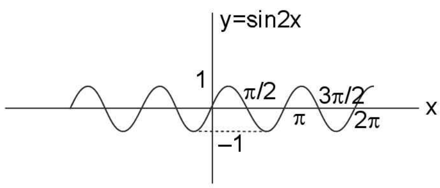
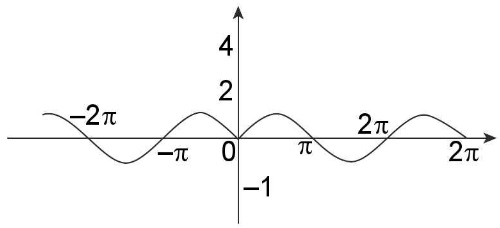


(d)


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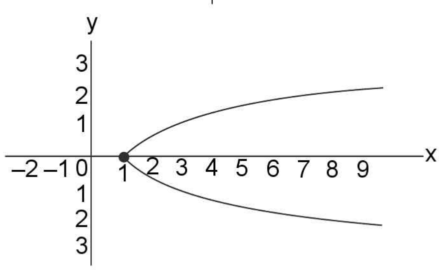

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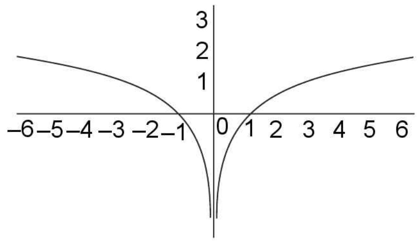

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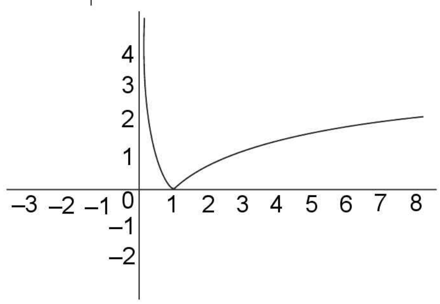

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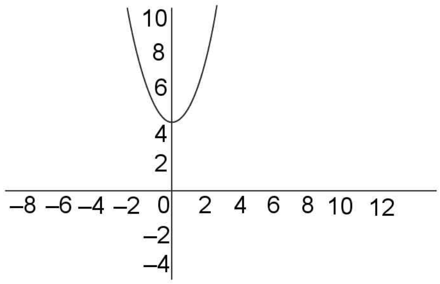

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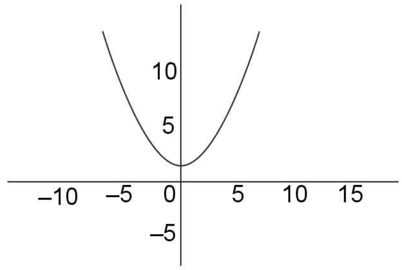

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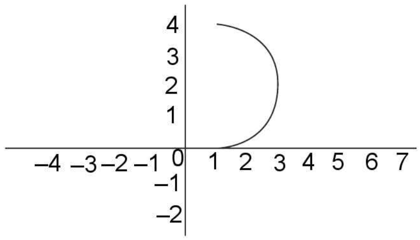

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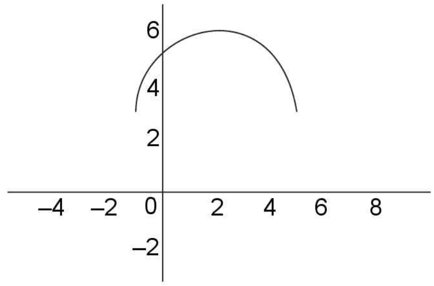

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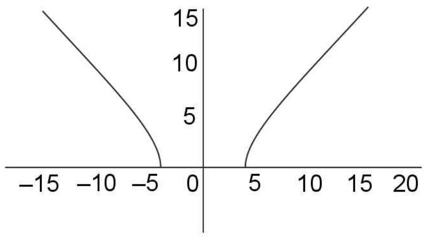

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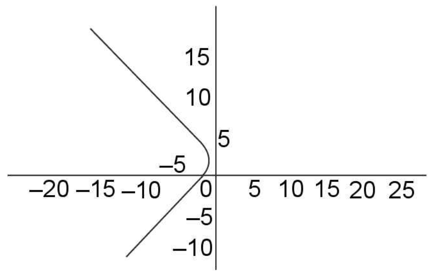

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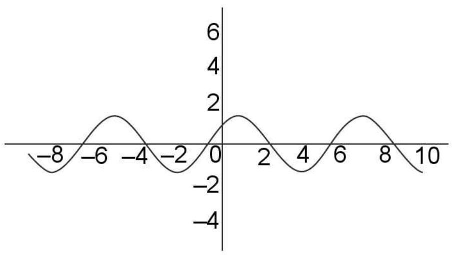

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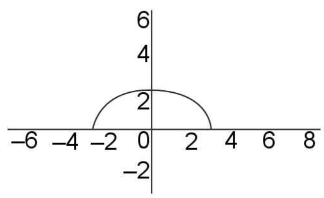

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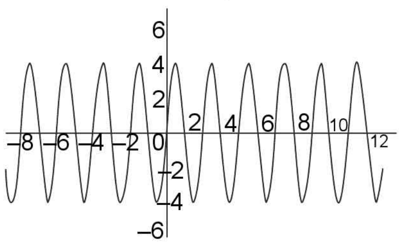

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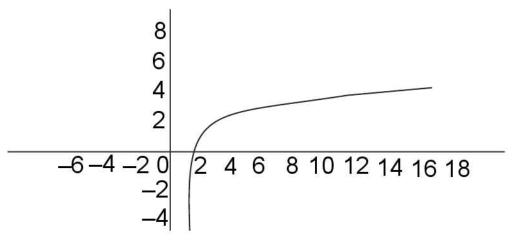

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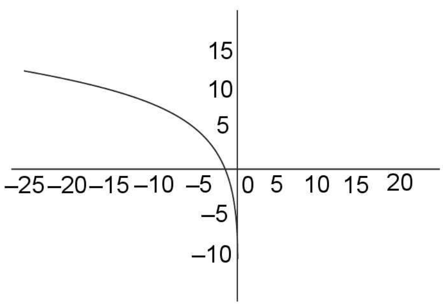

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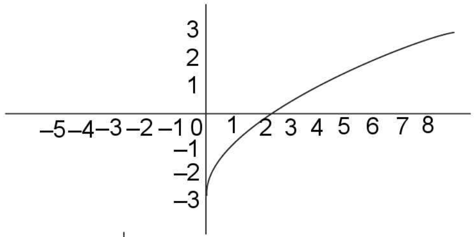

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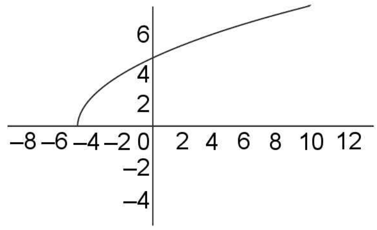

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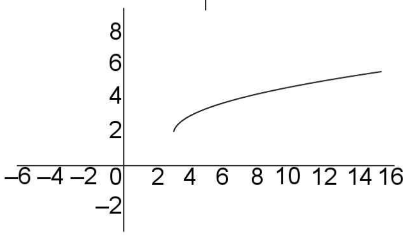

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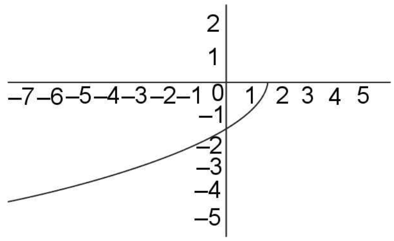

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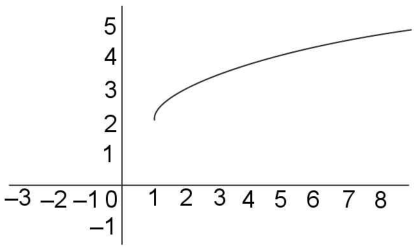

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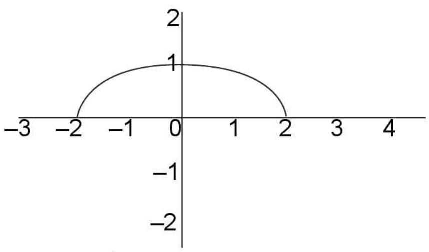

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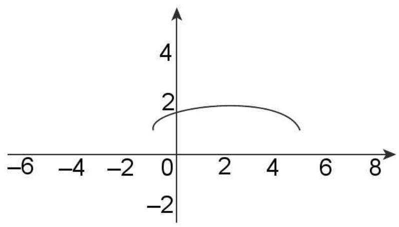

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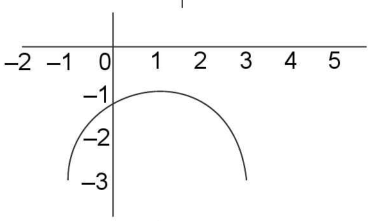
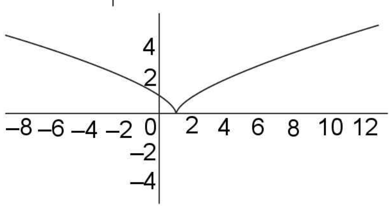


7. (a)



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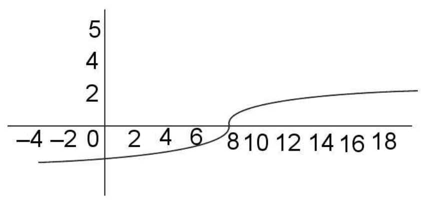
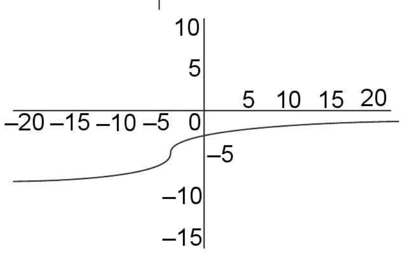
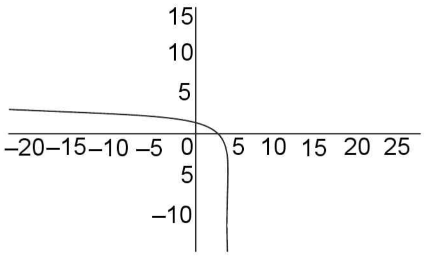
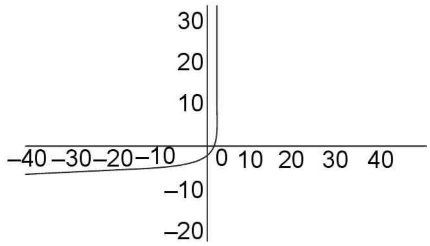
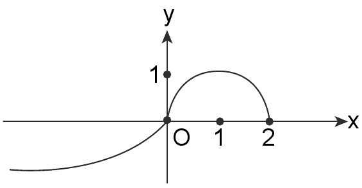


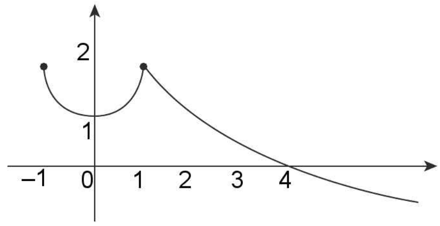
(c)

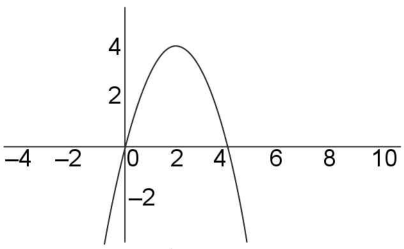


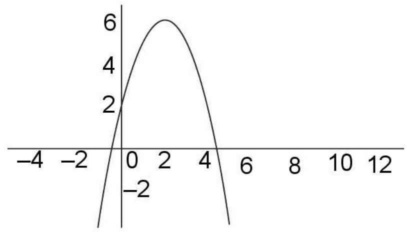
8. (a) 

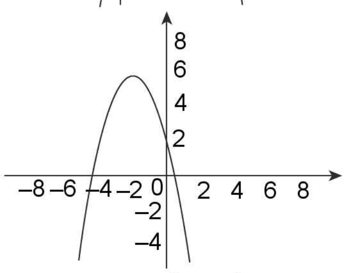
(b) 

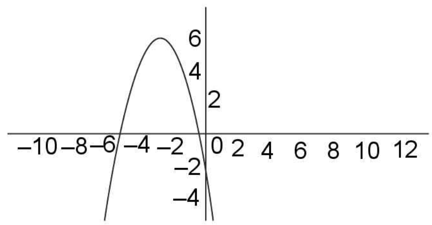
9. (i) 

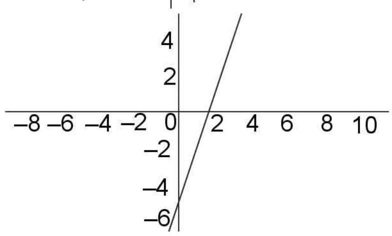
(ii) 

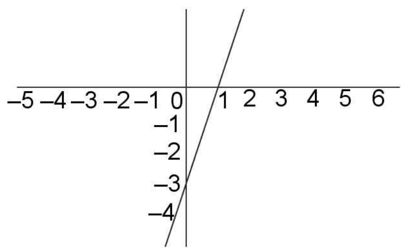
10. (a) (i) 

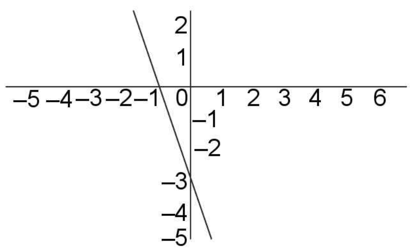
(ii) 

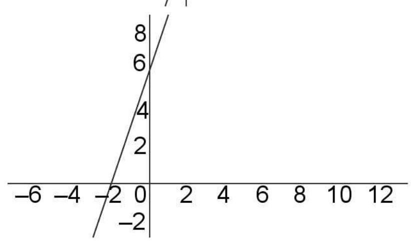
(iii) 

(iv) 

(b) (i) 

(ii) 

(iii) 

(iv) 

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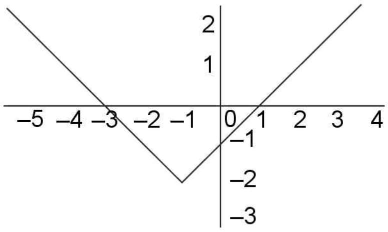

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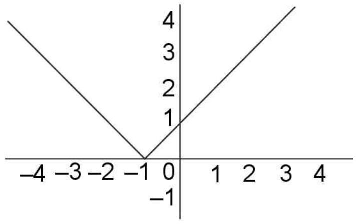

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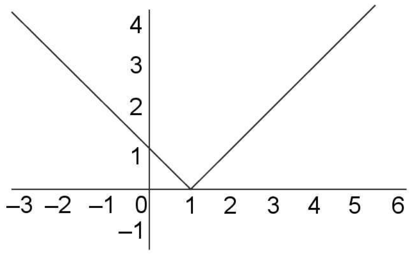

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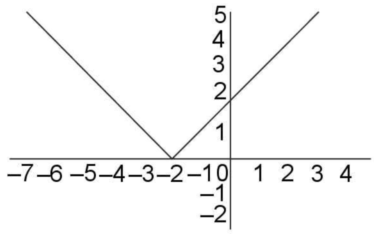
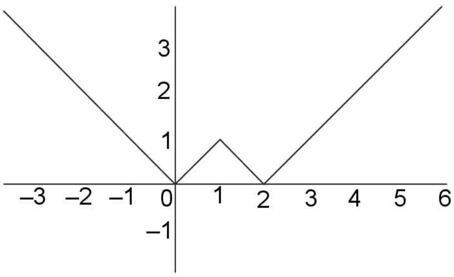


11. (a)



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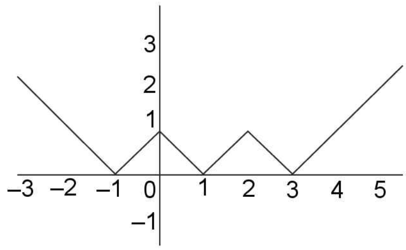

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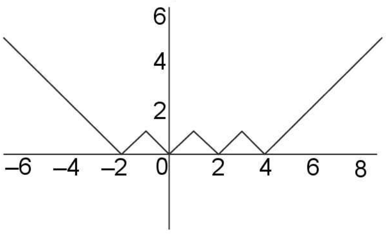
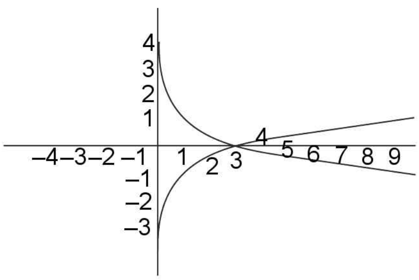
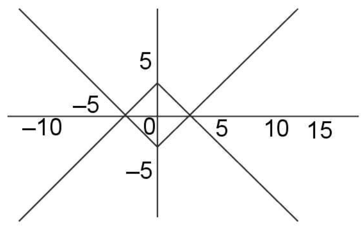


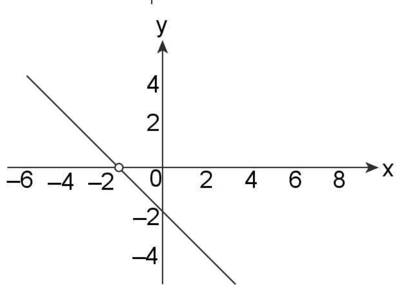
12. (a)



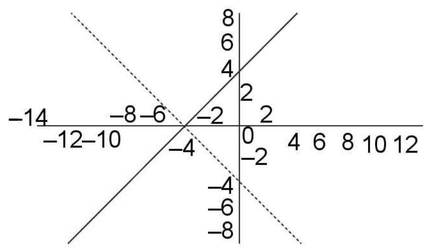
(b)

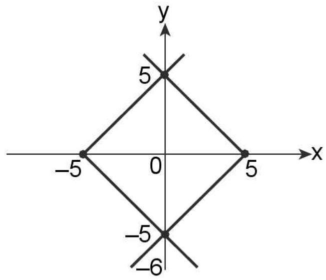


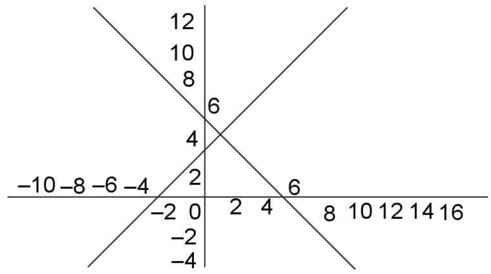
13. (i)

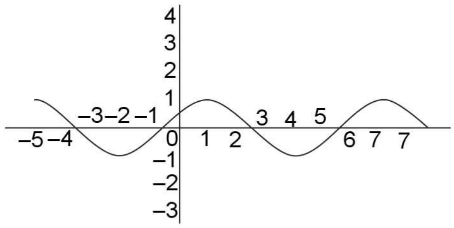


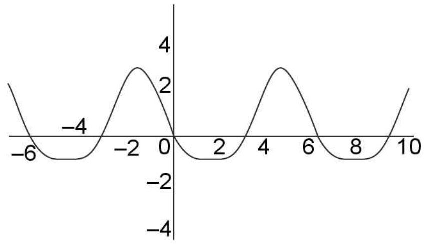
(ii)

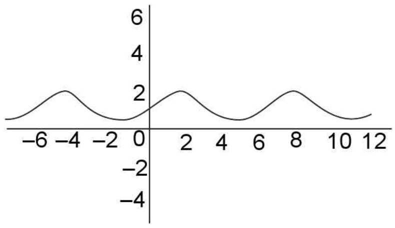


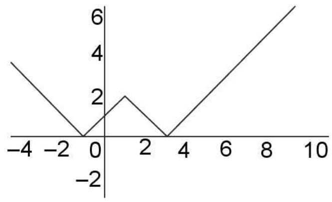
(iii) 

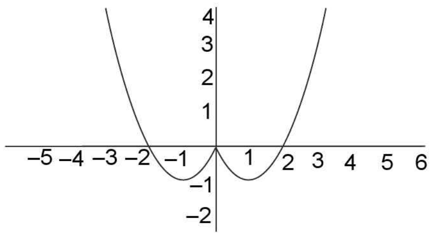
(iv) 

14. (a) 

(b) 

(c) 

(d) 

(e) 

Transformation of Graph Continued

10. Graph of $y = [f(x)]$ where [] denotes the GINT function: It is clear that if $n \le f(x) < n + 1$, $n \in \mathbb{Z}$, then $[f(x)] = n$. Thus, we would first locate all points on the graph having integer ordinates and then draw lines parallel to the x-axis in the direction of increasing ordinate from each such point upto the ordinate of next such points. And this process is to be done for all the portion on the curve where $f(x)$ lies between two successive integers. (see in Figures 3.194 and 3.195).

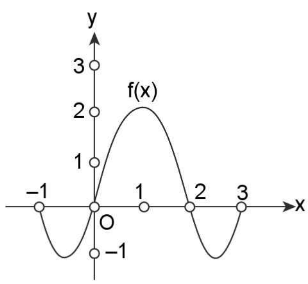


FIGURE 3.194

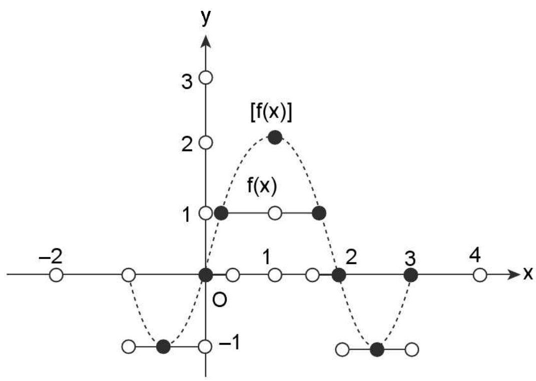


FIGURE 3.195

Clearly, it is evident that if at any such point the ordinate (output) of the curve decreases on both sides, then no projection is drawn from that point in either direction. Thus, that point remains an isolated point on the transformed curve.

ILLUSTRATION 39: Draw the graph of the following functions:

SOLUTION: **Graph of $y = [|x|]$:** $y = \begin{cases} [x] & \textit{for } x \geq 0 \\ [-x] & \textit{for } x < 0 \end{cases}$

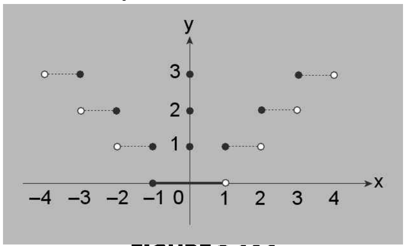


FIGURE 3.196

ILLUSTRATION 40: Sketch the curve $y = [x^2 - 2]$ (where [.] denotes GINT function) for $|x| \leq 2$

SOLUTION: The graph of $y = x^2 - 2$ could be plotted as shown in Figure 3.197.

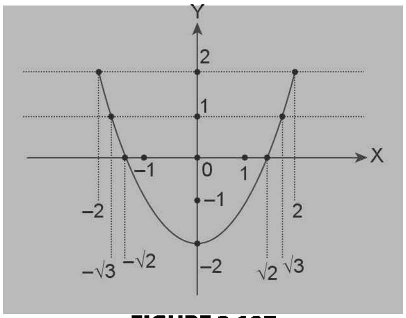


FIGURE 3.197

Hence, the graph of $y = [x^2 - 2]$ is as shown in Figure 3.198.

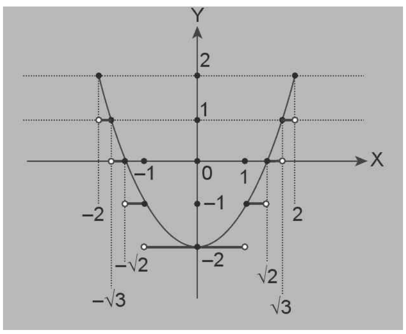


FIGURE 3.198

ILLUSTRATION 41: Find the solutions set of the equation $[e^x] = [\sin x]$ where [.] represents the greatest integer function.

SOLUTION: The graph of $y = \sin x$ is as shown in Figure 3.199.

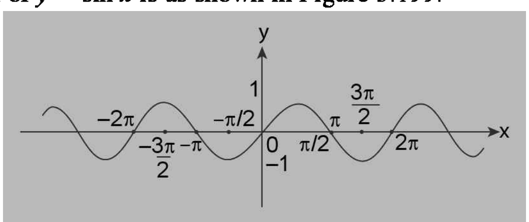


FIGURE 3.199

Hence, the graph of $y = [\sin x]$ is as shown in Figure 3.200.

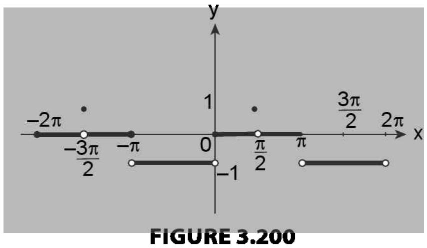


FIGURE 3.200

Now, graph of $y = e^x$ is as shown in Figure 3.201.

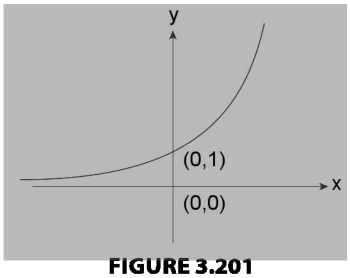


FIGURE 3.201

Hence, the graph of $y = [e^x]$ is shown in Figure 3.202.

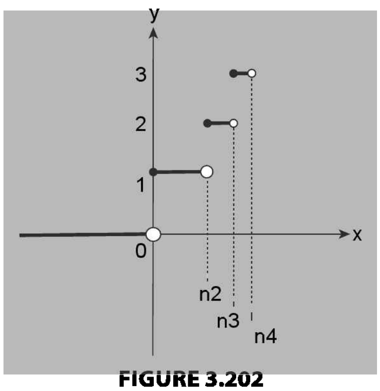


FIGURE 3.202

Now, plotting the two curves together, we infer that the solutions set of $[e^x] = [\sin x]$ can be pictorially seen as the points of intersection of $y = [e^x]$ and $y = [\sin x]$.

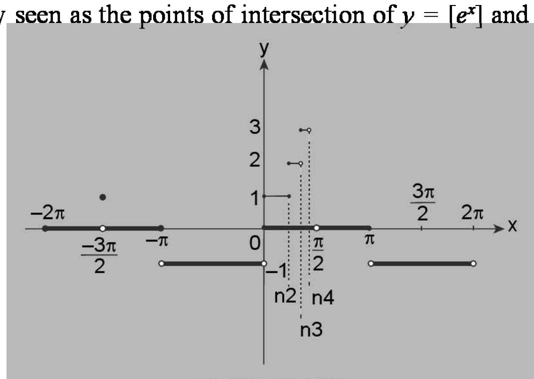


FIGURE 3.203

Clearly; the graphs have no point of intersection for $x \geq 0$.
And for $x < 0$; the two graphs; intersect for $[-2n\pi, -2n\pi + \pi] - \{-2n\pi + \pi/2\}$; where $n \in \mathbb{N}$

11. Graph of $y = f([x])$: To obtain the graph of $y = f([x])$, it is necessary to observe the following facts.

When $x \in [0, 1)$ the value of $[x] = 0$.

$\Rightarrow f([x]) = f(0)$ for all $x \in [0, 1)$. Thus, the graph becomes $y = f(0)\ \forall\ x \in [0, 1)$.

Similarly $f([x]) = f(1)$ for all $x \in [1, 2)$. Hence, the graph becomes $y = f(1)\ \forall\ x \in [1, 2)$.

In general, $f([x]) = f(n)\ \forall\ x \in [n, n + 1), n \in \mathbb{Z}$. It implies that first of all we should locate all such points on the graph of $y = f(x)$ having integer x-coordinates say $(n, f(n))$ and then draw a line segments of unit length parallel to the x-axis projecting rightwards given by $y = f(n)$ through the points $(n, f(n))$ excluding right end point, for all n lying in the domain of the function. Illustrated given in Figure 3.204.

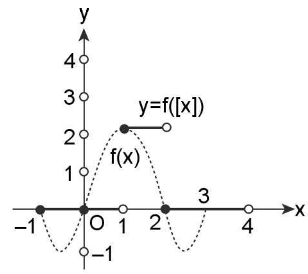


FIGURE 3.204

ILLUSTRATION 42: Find the solution set of $e^{[x]} = [e^x]$, where $[x]$ denotes the greatest integer function.

SOLUTION: Graph of e^x is as shown in Figure 3.205.

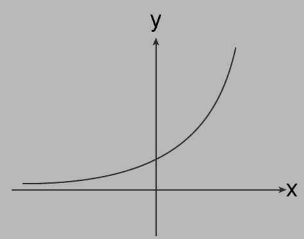


FIGURE 3.205

Hence, the graph of $[e^x]$ is as shown in Figure 3.206.

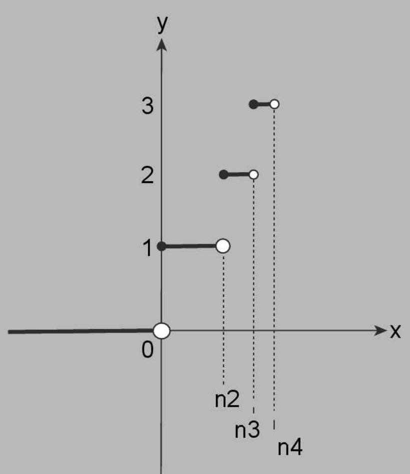


FIGURE 3.206

And the graph of $e^{[x]}$ will be given by as shown in Figure 3.207.

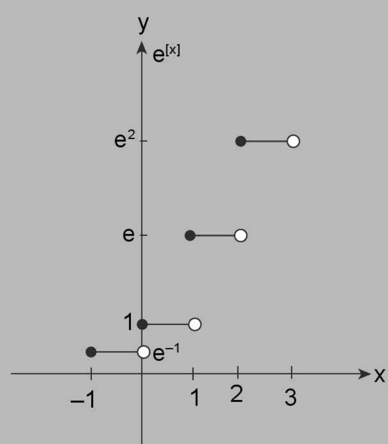


FIGURE 3.207

Now, plotting the two graphs together, we can see that

For $x < 0$; $e^{[x]} > 0$ and $[e^x] = 0$, and hence, no solution is possible.

For $x \in [0, \ln 2)$; $e^{[x]} = e^0 = 1$ and $[e^x] = 1 \quad \Rightarrow e^{[x]} = [e^x] = 1\ \forall\ x \in [0, \ln 2)$

Hence $x \in [0, \ln 2)$ is a solution set.

For $x \in [\ln 2, 1)$; $e^{[x]} = 1$, where as $[e^x] = 2$ and $x \geq 1$; $[e^x] \in \mathbb{Z}$ and $e^{[x]} \notin \mathbb{Z}$ and is irrational.

$\therefore$ No more point of intersection.

Hence, the solution is $x \in [0, \ln 2)$

ILLUSTRATION 43: Sketch the curve $y = \sin [x]$; where [] denotes the GINT function and also, suggest whether $\sin [x]$ is a periodic function or not?

SOLUTION: The curve for $y = \sin [x]$; could be plotted as shown in Figure 3.208.

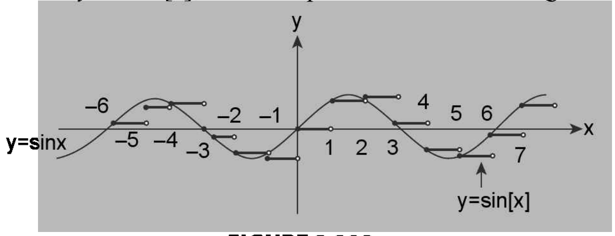


FIGURE 3.208

We know, that $\sin x$ is periodic with a period of π, but $[x]$ only takes integral values, hence, $[x]$ can never be a multiple of π, hence, $\sin [x]$ is not a periodic function.

ILLUSTRATION 44: Sketch the curve for $y = \cos[x]$ and $y = [x]^2$, and hence, find the number of solutions to the equation $\cos[x] = [x]^2$ where [] denotes the GINT function.

SOLUTION: The curve $y = \cos[x]$ could be plotted as,

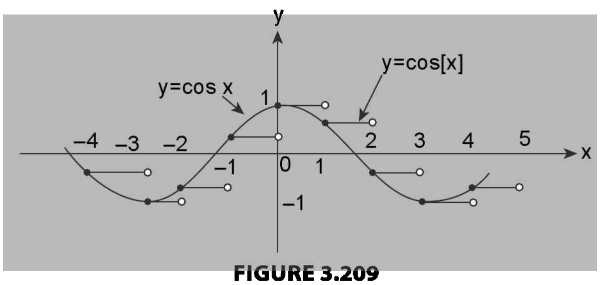


FIGURE 3.209

And the curve $y = [x]^2$ for $x \in [-2, 2]$ is given by

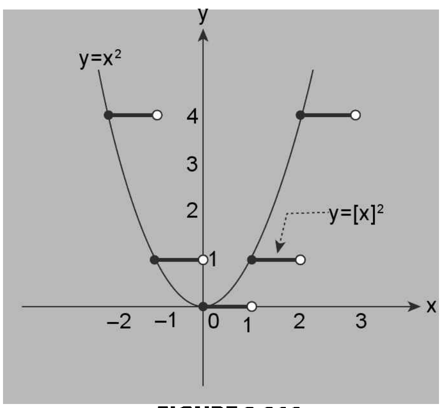


FIGURE 3.210

Plotting the two graphs together, we can see that the two graphs never intersect each other, and hence, the equation $\cos[x] = [x]^2$ does not have any solution.

ILLUSTRATION 45: Sketch the curve $y = [\sin[x]]$; where [] denotes the GINT function when $x \in [0, 2\pi]$.

SOLUTION: The graph of the function $y = \sin[x]$; where [] denotes the GINT function when $x \in [0, 2\pi]$ is as shown in Figure 3.211.

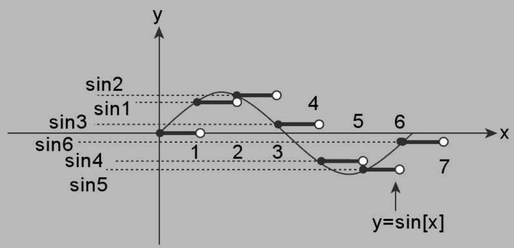


FIGURE 3.211

Now, since sin 1, sin 2, sin 3 are all less than 1 and greater than 0, therefore, $y = [\sin[x]] = 0$ for $x \in [0, 4)$.

Also, since sin 4, sin 5, sin 6 are all greater than –1 and less than 0, therefore, $y = [\sin[x]] = -1$ for $x \in [4, 2\pi]$.

Hence, the graph of $y = [\sin[x]]$ for $x \in [0, 2\pi]$ is given in Figure 3.212.

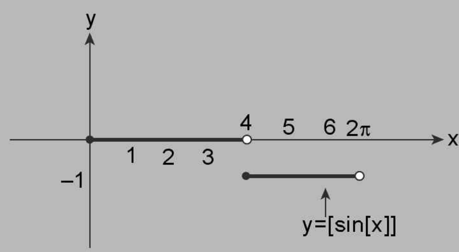


FIGURE 3.212

12. Graph of $[y] = f(x)$, where [ ] denotes the GINT function: It is clear that $[y] = f(x)$, would make sense only when $f(x)$ is an integer, and if $f(x)$ is not an integer graph of $[y] = f(x)$ would not exist. As such to obtain the graph of $[y] = f(x)$ first of all we would locate those points on graph of $f(x)$ for which ordinate is integer.

This can be done by considering horizontal lines through every integer on y-axis within the range of function.

Now from the point of intersection of horizontal lines and the curve we draw one unit vertical line segments, excluding the top end point.

Suppose $f(x_0) = 2$, then for $[y] = f(x)$ at point $(x_0, 2)$, we get $[y] = 2$ consequently $y \in [2, 3)$,

That is, $\{x_0, y): y \in [2, 3)\}$

i.e., we draw a line segment $x = x_0$ for $y \in [2, 3)$ and so on. Therefore, the graph of $y = f(x)$ and $[y] = f(x)$ are as shown in Figures 3.213 and 3.214, respectively.

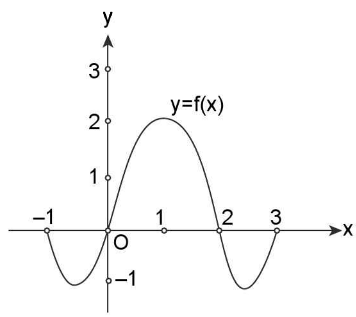


FIGURE 3.213

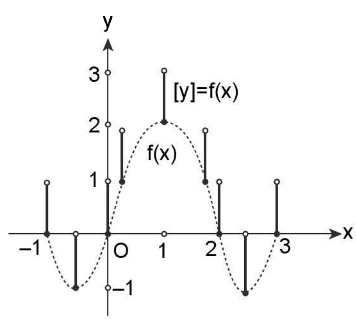


FIGURE 3.214

ILLUSTRATION 46: Draw the graph of $[y] = \sin^{-1}x$.

SOLUTION: The graph of $y = \sin^{-1}x$ is as shown in Figure 3.215.

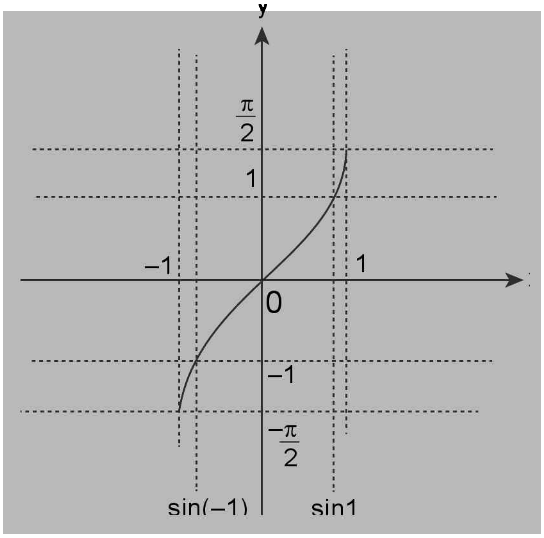


FIGURE 3.215

Hence, the graph of $[y] = \sin^{-1}x$ is as shown in Figure 3.216.

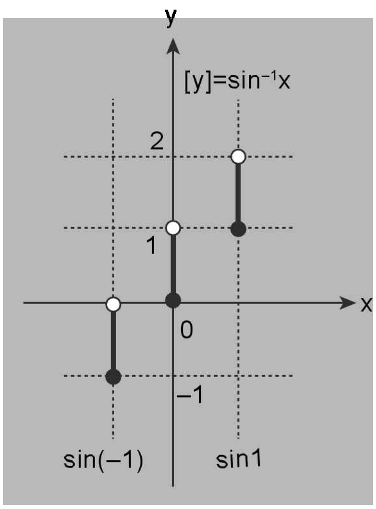


FIGURE 3.216

13. Graph of $[y] = [f(x)]$, where [] denotes the GINT function: The graph of $[y] = [f(x)]$ can be plotted by the following two simple steps.

Step I. Draw the graph of $y = [f(x)]$ as discussed earlier.

Step II. Apply [] on y.

ILLUSTRATION 47: Draw the curve $[y] = [\sin x]$ where [] represents the GINT function.

SOLUTION: The graph of $y = \sin x$ is as shown in Figure 3.217.

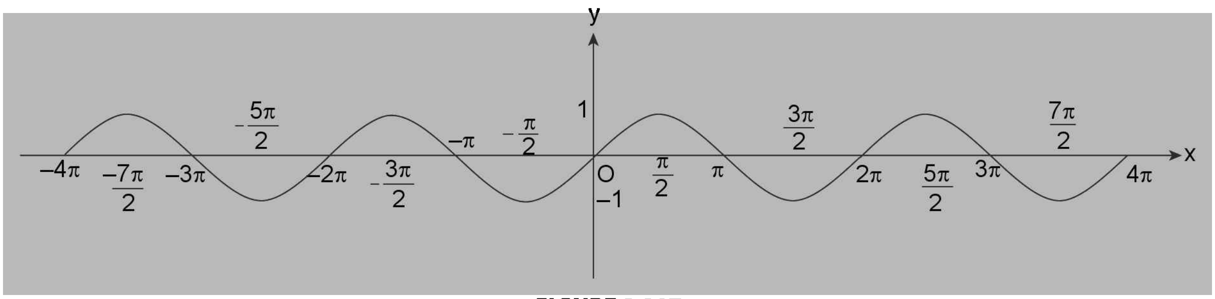


FIGURE 3.217

The graph of $y = [\sin x]$ is as shown in Figure 3.218.

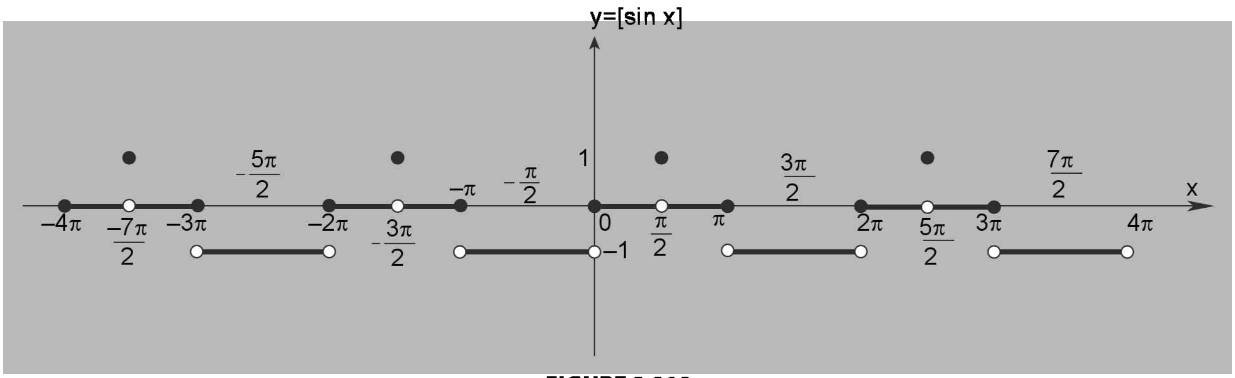


FIGURE 3.218

The graph of $[y] = [\sin x]$ is as shown in Figure 3.219.

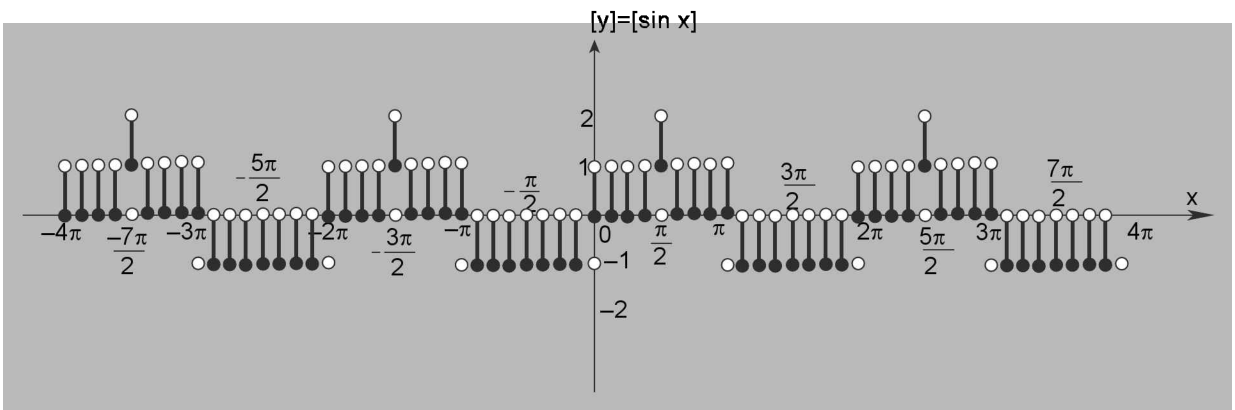


FIGURE 3.219

ILLUSTRATION 48: Draw the curve for $[y] = [2^x]$

SOLUTION: The graph of $y = 2^x$ is as shown in Figure 3.220.

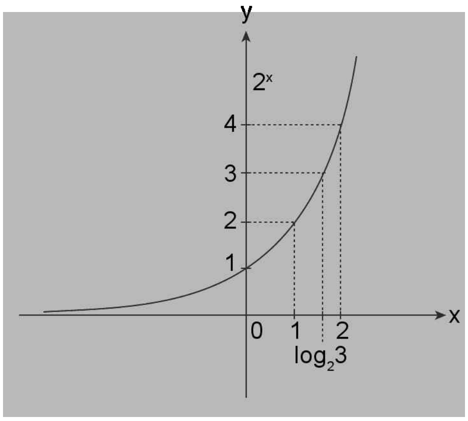


FIGURE 3.220

Therefore, the graph of $y = [2^x]$ must be as shown in Figure 3.221.

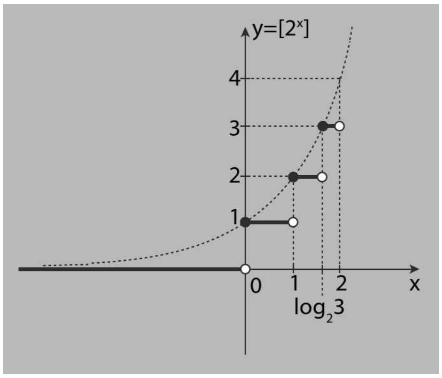


FIGURE 3.221

When $x < 0$	$\Rightarrow [y] = 0$	$\Rightarrow y \in [0,1)$
When $x \in [0,1)$	$\Rightarrow [y] = 1$	$\Rightarrow y \in [1,2)$
When $x \in [1, \log_2 3)$	$\Rightarrow [y] = 2$	$\Rightarrow y \in [2,3)$
$x \in [\log_2 3, 2)$	$\Rightarrow [y] = 3$	$\Rightarrow y \in [3,4)$

consequently the graph of $[y] = [2^x]$ should be as shown in Figure 3.222.

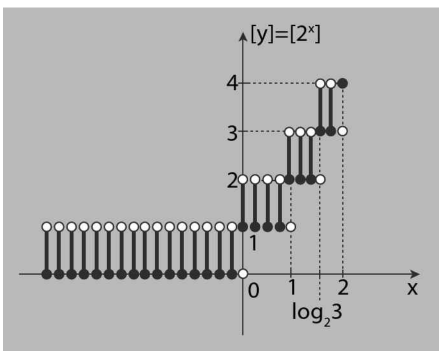


FIGURE 3.222

14. Graph of $y = f(\{x\})$; where $\{x\}$ represents the fractional part of x: We are well aware of the fact that $\{x\}$ is a periodic function with a fundamental period of 1. Therefore, the graph of $f(x - [x])$ or $f(\{x\})$ can be obtained from the graph of $f(x)$ by using the following mentioned method, Retain only the part of the graph of $f(x)$ for the values of x lying between interval $[0, 1)$. Now, since $\{x\}$ is periodic, hence, the graph of $f\{x\}$ can be obtained by repeating the retained portion of graph in the interval $[0, 1)$ (taking periodicity 1). The resultant obtained function is a graph for $y = f(\{x\})$. **Graph of $f(x)$** as in Figure 3.223.

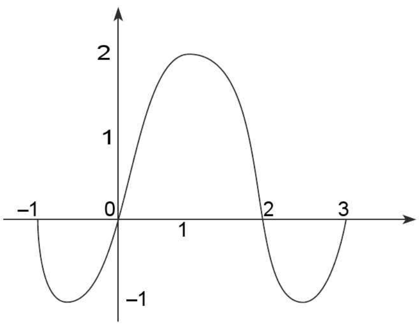


FIGURE 3.223

Graph of $f(x)$ from [0,1) as in Figure 3.224.

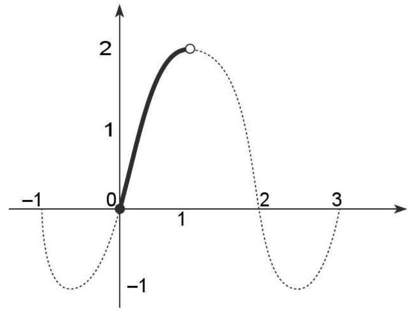


FIGURE 3.224

Graph of $f\{x\}$ can be obtained by repeating the part of graph in the interval $[0, 1)$ (taking periodicity 1). Hence, the graph of $f\{x\}$ is as shown in Figure 3.225.

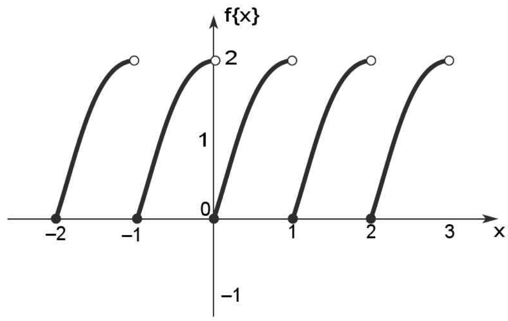


FIGURE 3.225

ILLUSTRATION 49: Sketch the curve $y = \ln \{x\}$

SOLUTION: The graph of $\ln x$ is given as:

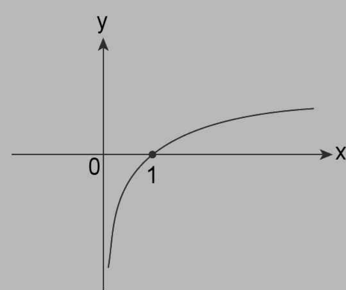


FIGURE 3.226

The graph of $\ln\{x\}$ can be obtained by repeating the part of graph in the interval $[0,1)$ (taking periodicity 1).

Hence, the graph of $f\{x\}$ is as shown below.

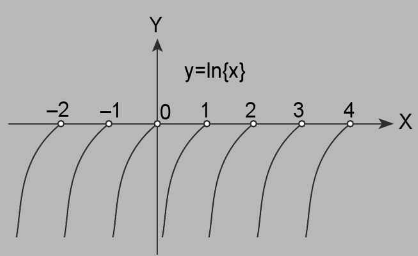


FIGURE 3.227

15. Graph of $y = \{f(x)\}$: Plot the horizontal lines $y = 0, 1$ and for all the integral values of y. Now cut the points of graphs for which $f(x) \in [n, n + 1)$; $n \in W - \{0\}$, put these portions on x-axis transforming them vertically. We need to do the above mentioned step because $\{f(x)\}$ must lie in $[0, 1)$.

NOTE

Graph of {y} = {f(x)}
We know that, the range of graph of y= {f(x)} is [0, 1), and hence, there is no difference between the graphs of {y} ={f(x)} and the graph of y = {f(x)}.

ILLUSTRATION 50: Draw the graph of $y = f(x)$ where $f(x) = x^2$.
Also draw the graph of $y = \{f(x)\} = \{x^2\}$.

SOLUTION: The graph of $y = x^2$ for $x \in [-2, 2]$ is given as in Figure 3.228.

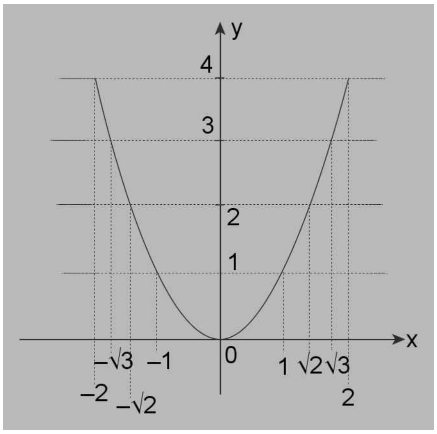


FIGURE 3.228

The graph of $y = \{x^2\}$ for $x \in [-2, 2]$ is given as in Figure 3.229.

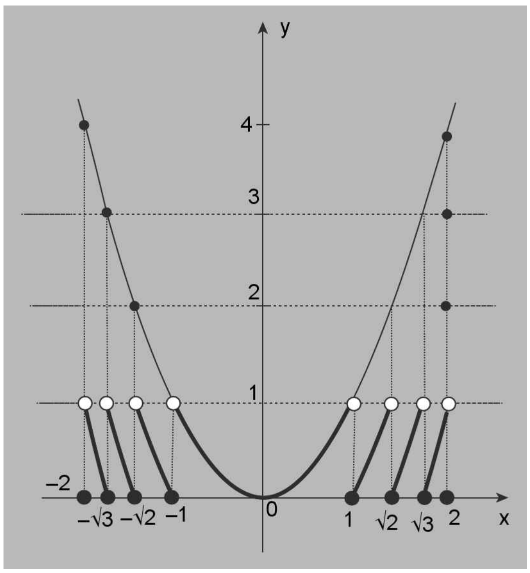


FIGURE 3.229

ILLUSTRATION 51: Draw the graph of $y = f(x)$ where $f(x) = \dfrac{2}{\pi}\sin^{-1}\sin\left(\dfrac{\pi x}{2}\right)$.

Also draw the graph of $y = \{f(x)\}$.

SOLUTION:

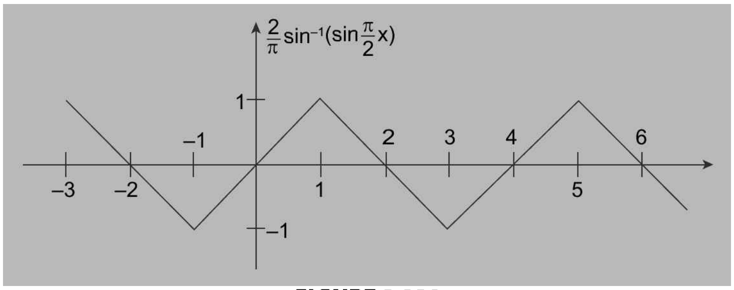


FIGURE 3.230

The graph of $y = \left\{\dfrac{2}{\pi}\sin^{-1}\sin\left(\dfrac{\pi x}{2}\right)\right\}$.

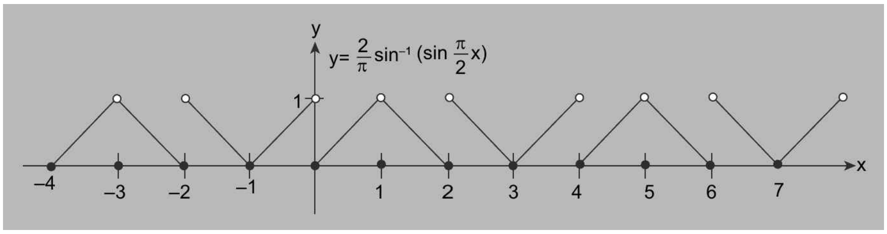


FIGURE 3.231

ILLUSTRATION 52: Find the number of solutions of $\left\{\sin\dfrac{\pi x}{2}\right\} = \dfrac{2^x}{2^{[x]}}$.

Where $\{x\}$ and [] represents the fractional part and the greatest integer function respectively

SOLUTION: The graph of $y = \sin x$ is as shown in Figure 3.232.

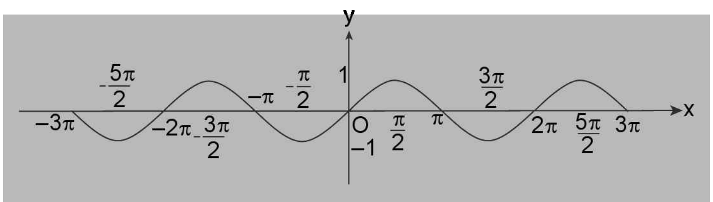


FIGURE 3.232

The graph of $y = \sin\dfrac{\pi x}{2}$ is as shown in Figure 3.233.

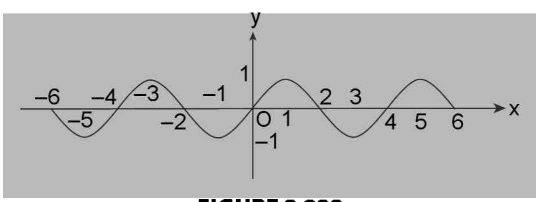


FIGURE 3.233

The graph of $y = \left\{\sin\frac{\pi x}{2}\right\}$ is as shown in Figure 3.234.

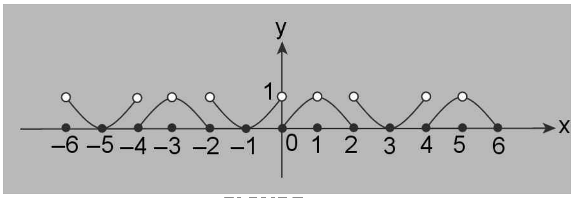


FIGURE 3.234

The graph of $y = 2^x$ is as shown in Figure 3.235.

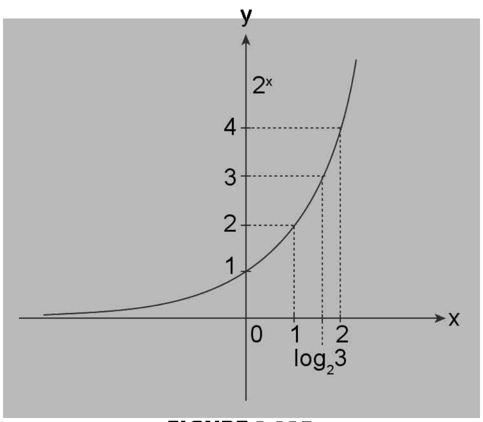


FIGURE 3.235

Now, $y = \frac{2^x}{2^{[x]}} = 2^{\{x\}}$ for all real values of x, which is a periodic function with a fundamental period of 1.

Its graph can be plotted using the graph of $y = 2^{\{x\}}$ for $x \in [0, 1)$.

Hence, the graph of $y = 2^{\{x\}}$ is as shown in Figure 3.236.

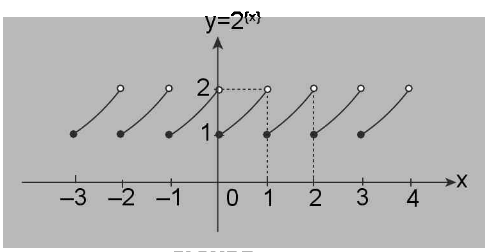


FIGURE 3.236

As is evident from the graphs $y = \frac{2^x}{2^{[x]}}$ is always greater than or equal to 1 whereas $y = \left\{\sin\frac{\pi}{2}x\right\}$ is always less than 1. Hence, the two graphs never intersect each other and thereby, number of solutions is zero.

16. **Graph of {y} = f(x) where {x} represents the fractional part:** To obtain the graph of $\{y\} = f(x)$, retain only the portion of the graph for which $y \in [0, 1)$ and ignore all the parts of the curve for which y obtains other values of $f(x)$. Graph of $y = f(x)$ as shown in Figure 3.237.

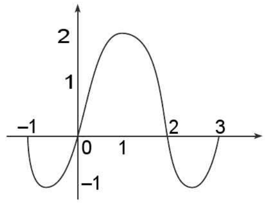


FIGURE 3.237

Highlighting the graph of $y = f(x)$ for which $y \in [0, 1)$.

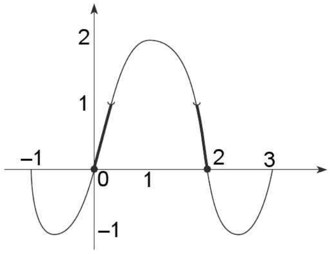


FIGURE 3.238

Retaining only that part of the graph and deleting the rest of the graph, we obtain the graph of $\{y\} = f(x)$ as shown in Figure 3.239.

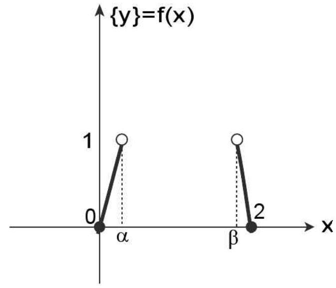


FIGURE 3.239

Graph of y = {f{x}}

To sketch the graph of $y = \{f\{x\}\}$, we follow the given algorithm.

Step 1: Draw the graph of $y = f\{x\}$

Step 2: Then apply the { } on $f\{x\}$, and hence, we obtain the graph of $\{f\{x\}\}$

ILLUSTRATION 53: Sketch the graph of $y = \{2 \cos \{x\}\}$

SOLUTION: The graph of $y = 2 \cos x$ is shown as in Figure 3.240.

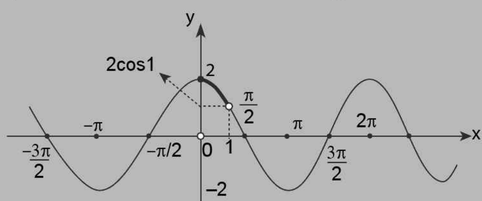


FIGURE 3.240

Thus, the graph of $y = 2 \cos \{x\}$ shall be obtained as shown in Figure 3.241.

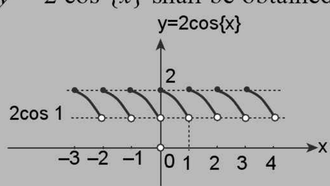


FIGURE 3.241

Now since $\cos 1 > \cos \pi/3$ [Since $1 < \pi/3$ and $\cos x$ is a decreasing function in $[0, \pi/2)$].

$\Rightarrow$ $\cos 1 > 1/2 \Rightarrow 2 \cos 1 > 1$

Alone when $x = 0$; $\cos 0 = 1 \Rightarrow 2 \cos 0 = 2$

Therefore, the graph of $y = \{2\cos\{x\}\}$ can be plotted by applying the { } on $y = 2 \cos\{x\}$ and it is shown as in Figure 3.242.

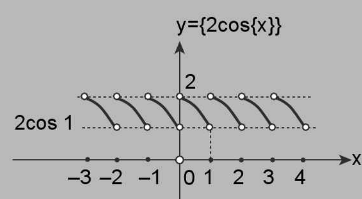


FIGURE 3.242

TEXTUAL EXERCISE–3: (SUBJECTIVE)

1. Draw the graphs of the following functions involving greatest integer functions:
   (i) $y = [x/2] - (x/2)$
   (ii) $y = [2x]$
   (iii) $y = 2[|x|]$
   (iv) $y = \{2x\}$
   (v) $y \{x\}^2$
   (vi) $y = [1/[|x|]]$
   (vii) $y = [e^{-x}] \ln x$

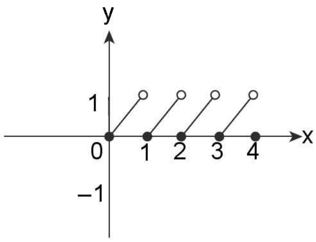
2. If a function is defined as $y(x) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 < x \le 2 \\ (x-2)(x-4) & 2 \le x \le 4 \end{cases}$ then sketch the following functions:
   (a) $y = [f(x)]$
   (b) $y = f([x])$
   (c) $[y] = f(x)$
   (d) $y = f[\{x\}$
   (e) $y = \{f(x)\}$
   (f) $y = \{f\{x\}\}$
   (g) $\{y\} = f(x)$

3. Draw the graph of the following functions
   (i) $y = 2\{x\}^2 - 5\{x\} + 3$
   (ii) $y = [\sin x - \cos x]$
   (iii) $y = [\sin x + \cos x]$
   (iv) $y = \cot\left(\frac{\pi}{2} - \pi\{x\}\right)$

4. Draw the graphs of following functions involving greatest integer functions:
   (a) $y = \sin(x - [x])$
   (b) $y = sgn\ (x - [x])$
   (c) $y = [x] + [-x]$
   (d) $y = [|\sin x|]$
   (e) $y = [\sin |x|]$
   (f) $y = |[\sin x]|$
   (g) $y = \sin[x] - \cos[x]$

Answer Keys

1. (i)



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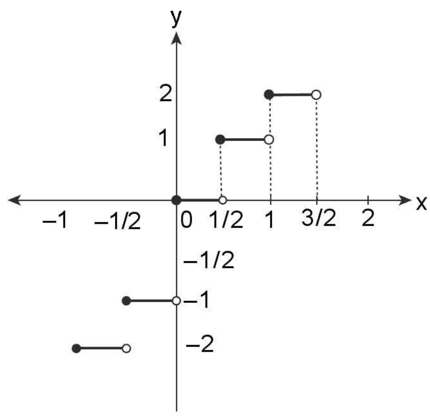

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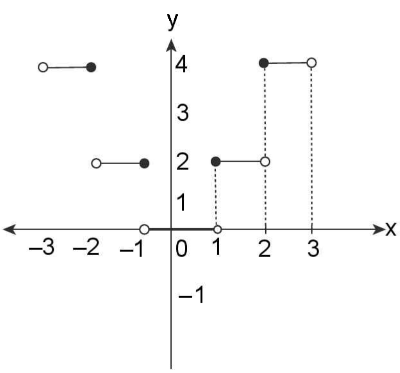

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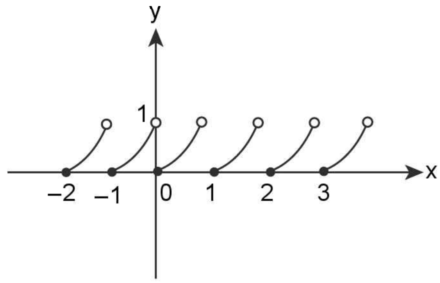
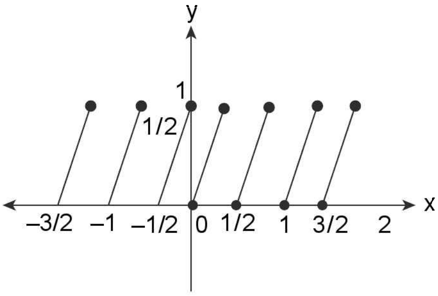
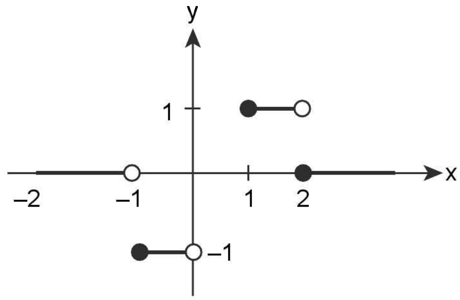
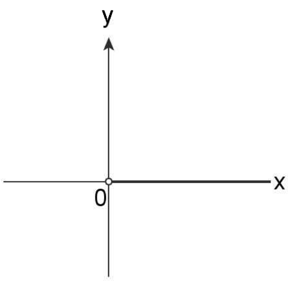
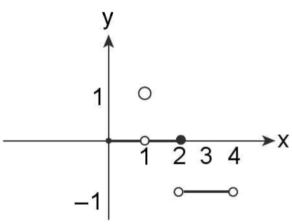
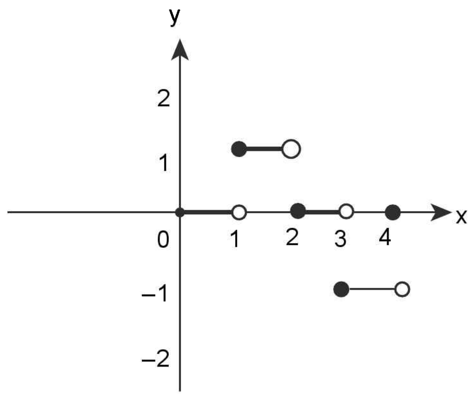


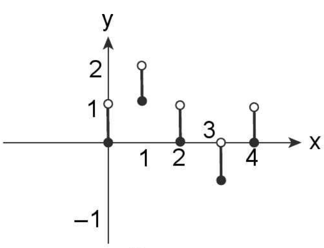
(v) 

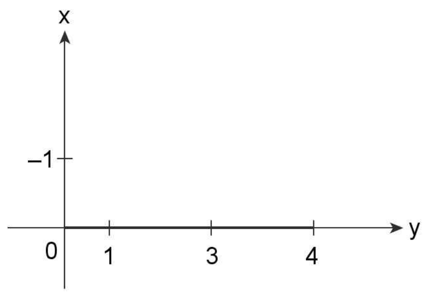
(vi) 

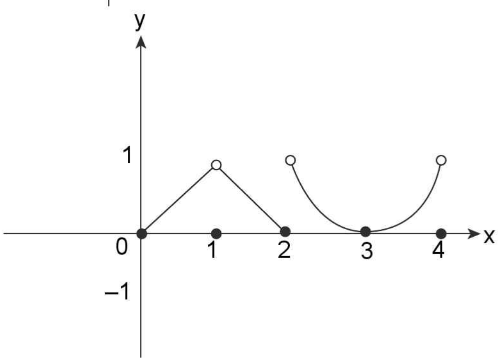
(vii) 

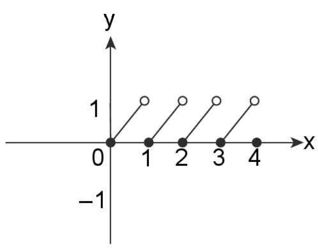
2. (a) 

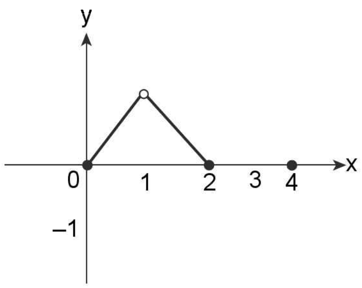
(b) 

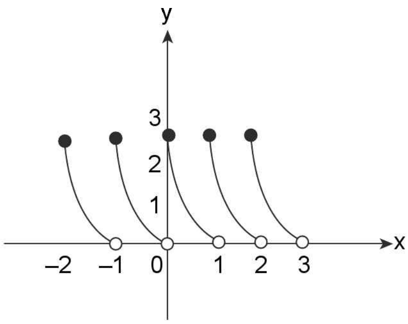
(c) 

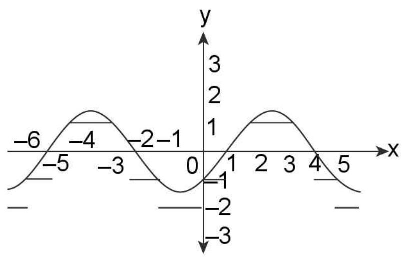
(d) 

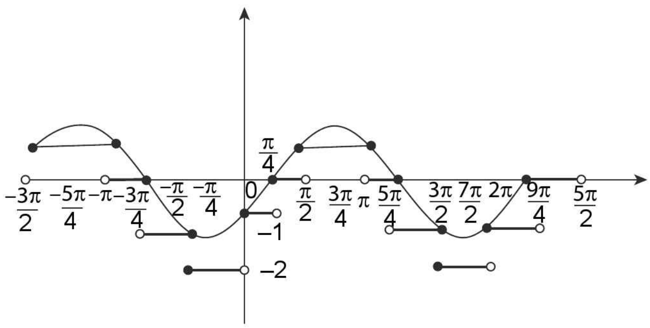
(e) 

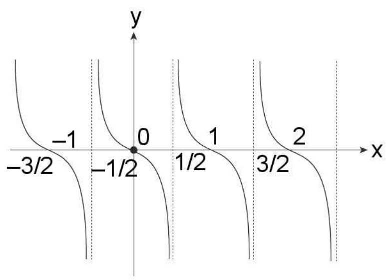
(f) 

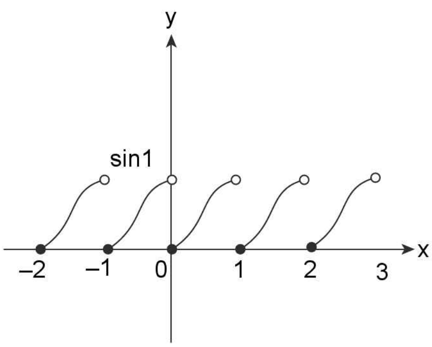
(g) 

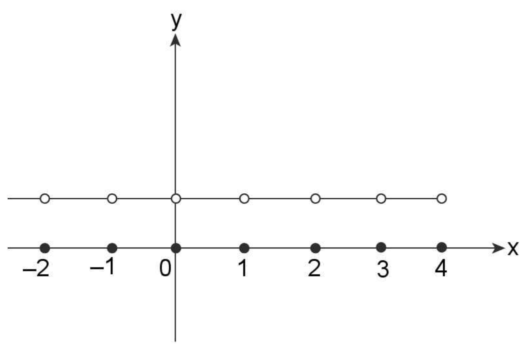
3. (i) 

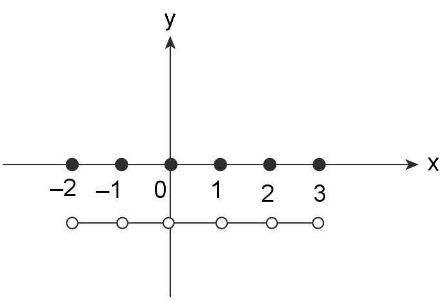
(ii) 

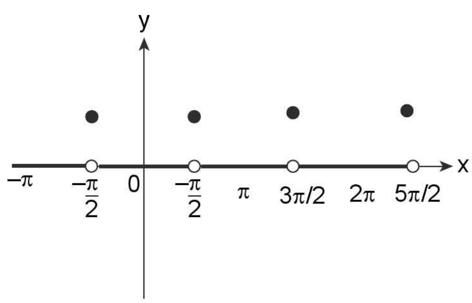
(iii) 

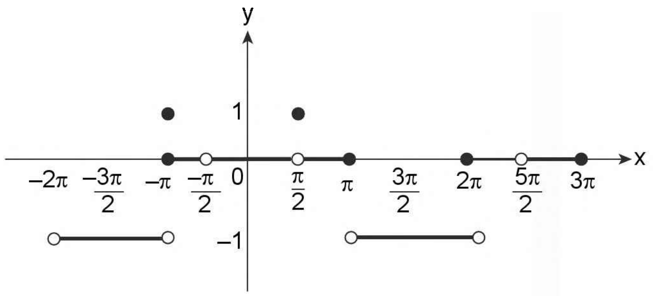
(iv) 

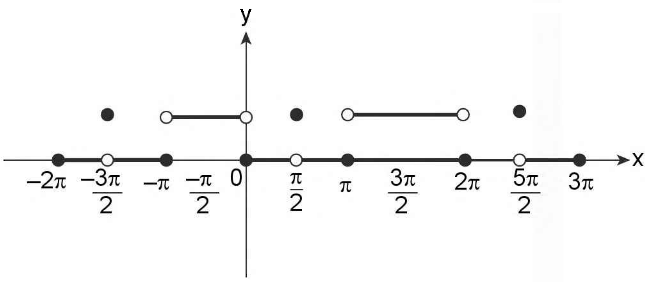
4. (a) 

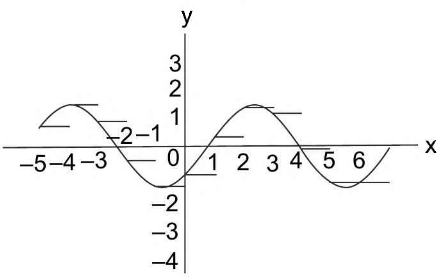
(b) 

(c) 

(d) 

(e) 

(f) 

(g) 

INEQUALITY

Linear Equation and Inequality

Method of tracing the region represented by inequality

Each curve $f(x, y) = 0$, categories the points of the entire x-y plane into following five set of points.

(i) $R_1 = \{(x, y) : f(x, y) = 0,\ x, y \in \mathbb{R}\}$, i.e., the points lying on the curve $f(x, y) = 0$

(ii) $R_2 = \{(x, y) : f(x, y) > 0,\ x, y \in \mathbb{R}\}$, i.e., the points lying outside of curve $f(x, y) = 0$.

(iii) $R_3 = \{(x, y) : f(x, y) < 0,\ x, y \in \mathbb{R}\}$, i.e., the points lying inside of $f(x, y) = 0$.

(iv) $R_4 = \{(x, y) : f(x, y) \geq 0,\ x, y \in \mathbb{R}\}$, i.e., the points lying on the curve $f(x, y) = 0$ or outside of curve $f(x, y) = 0$.

(v) $R_5 = \{(x, y) : f(x, y) \leq 0,\ x, y \in \mathbb{R}\}$, i.e., the points lying on the curve $f(x, y) = 0$ or inside of curve $f(x, y) = 0$.

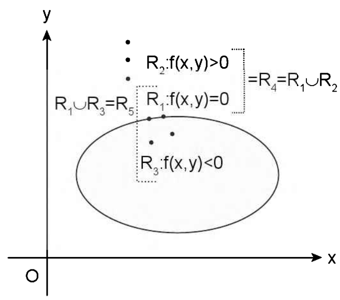


FIGURE 3.243

NOTE

We must observe the difference between the regions R_2 and R_4. In region R_2, the curve f(x,y) = 0 is shown by dotted lines so as to represent the fact that the points lying on f(x, y) = 0 do not satisfy the curve f(x, y) > 0 whereas, in region R_4, the curve f(x, y) = 0 is shown by bold lines so as to represent the fact that the points lying on f(x, y) = 0 satisfies the curve f(x, y) ≥ 0. Also the graphs R_3 and R_5 differ in the same way.

ALGORITHM

To identify the region represented by a given inequality $f(x, y) \geq 0$.

Step I: Consider the equality and draw the curve using the symmetry and other concepts of curve sketching and transformation of graphs.

Step II: Consider any points (α, β) not lying on the curve preferably (0, 0) {or point on coordinate axis} and determine the sign of $f(\alpha, \beta)$.

Step III: If $f(\alpha, \beta) > 0$, then $f(x, y) \geq 0$ represents the region containing (α, β). If $f(\alpha, \beta) < 0$, then the region which does not contain point (α, β) will be represented by inequality $f(x, y) > 0$.

Similarly, we can identify the region represented by a given inequality $f(x, y) \leq 0$. For example, The inequality $(x-a)^2 + (y-b)^2 < r^2$ represents the interior or a circle.

The inequality $(x-a)^2 + (y-b)^2 > r^2$ represents the exterior of a circle. (i.e., region lying outside the circle).

We take the independent variables along x-axis and dependent variables along the y-axis in the cartesian coordinate system.

ILLUSTRATION 54: Mark the region represented by $3x + 4y < 12$.

SOLUTION: Converting the inequality into equation, we get $3x + 4y = 12$. This line meets the coordinate axes at (4, 0) and (0, 3), respectively.

Join these points to obtain straight line represented by $3x + 4y = 12$

This straight line divides the plane in two parts besides from itself. One part contains the origin and the other does not contain the origin. Clearly (0, 0) satisfy the inequality $3x + 4y < 12$. So, the region represented by $3x + 4y < 12$ is region containing the origin containing the origin, i.e., the half plane below the line $3x + 4y = 12$.

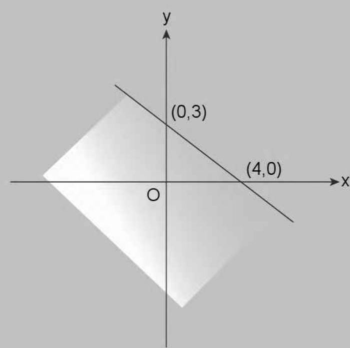


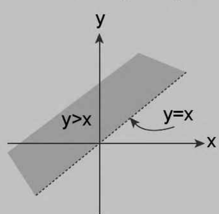
FIGURE 3.244

ILLUSTRATION 55: Sketch the region represented by the following inequalities:

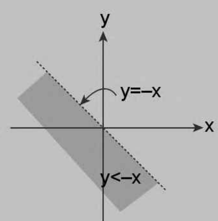
(a) $y > x$ (b) $y < -x$ (c) $y \geq |x - 1|$ (d) $x - 3y + 15 \leq 0$, $y + 5 \leq 5x$ and $4y + 28 \geq 7x$

(e) $(x^2 - x) < (y - xy)$

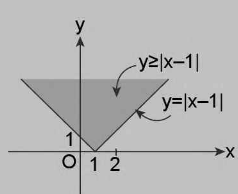
SOLUTION: (a)



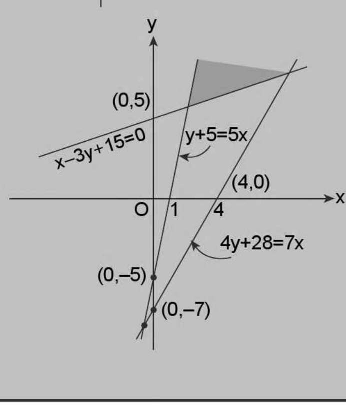
(b)



(c)



(d)



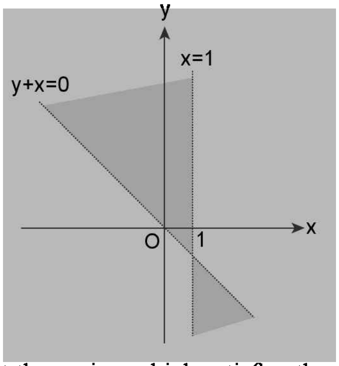
(e) 

ILLUSTRATION 56: Plot the region which satisfies the equation $[x^2 + y^2] - 2\ \text{sgn}\ (x^2 + y^2) = 0$.

SOLUTION: As is obvious from the expression $[x^2 + y^2] - 2\ \text{sgn}\ (x^2 + y^2)$, we understand that the required graph will be symmetric about x-axis as well as y-axis.

Also $\text{sgn}\ (x^2 + y^2) = \begin{cases} 0;\ x = y = 0 \\ 1;\text{otherwise} \end{cases}$

Case I: when $x = y = 0$, then the above equation is satisfied, and hence, (0,0) is a point on the required graph.

Case II: When x, y are not simultaneously zero $[x^2 + y^2] - 2\ \text{sgn}(x^2 + y^2) = 0$.

$\Rightarrow\ [x^2 + y^2] = 2 \quad \Rightarrow\ (x^2 + y^2) \in [2, 3)$

$\Rightarrow\ \left(\sqrt{2}\right)^2 \le x^2 + y^2 < \left(\sqrt{3}\right)^2$; which will represent a circular ring.

Hence, the overall graph will be as shown in Figure 3.245.

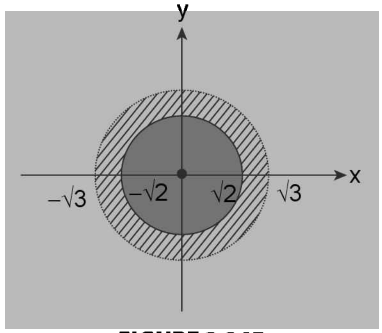


FIGURE 3.245

ILLUSTRATION 57: Plot the region which satisfies the equation $[x^2] + [y^2] = 1$

SOLUTION: Obviously $[x^2]$ or $[y^2]$ can never be negative, and hence, $[x^2]$ or $[y^2]$ can never take value greater than 1 for the given equation to be satisfied.

Also observe that $[x^2] + [y^2] = 1$ will be symmetric about x-axis as well as about y-axis,

Now when $[x^2] = 0$; $[y^2] = 1$

$\Rightarrow$ When $x^2 \in [0,1)$; $y^2 \in [1, 2)$. In the first quadrant.

$\Rightarrow$ $x \in [0,1)$; $y \in [1, \sqrt{2})$ and vice-versa, i.e., $x \in [1, \sqrt{2})$ and $y \in [0, 1)$

$\therefore$ The graph in the I[st] quadrant will be as shown below.

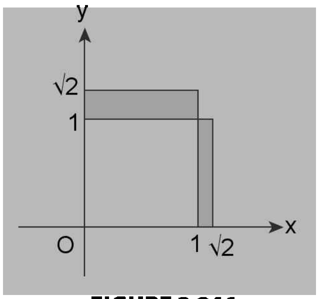


FIGURE 3.246

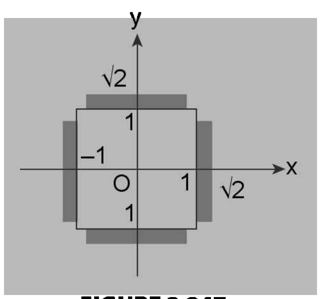


FIGURE 3.247

ILLUSTRATION 58: Plot the region which satisfies the equation $(|x| - |y|)(x - y) \le 4$

SOLUTION: ***Case I:*** $x, y \ge 0 \quad \therefore \ (x - y)^2 \le 4 \; -2 \le x - y \le 2$

and the graphs can be plotted as shown in Figure 3.248.

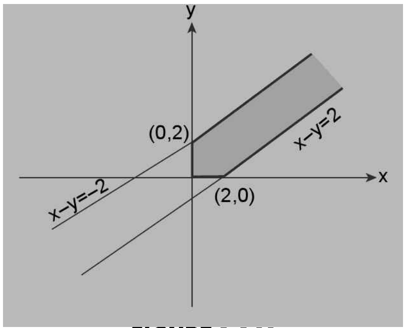


FIGURE 3.248

Case II: $x, y \le 0 \quad \therefore \ ((-x) - (-y))\ (x - y) \le 4$

$\Rightarrow \ (y - x)\ (x - y) \le 4$ $\qquad \Rightarrow (x - y)^2 \ge -4$ which is true $\forall\ x, y \le 0$

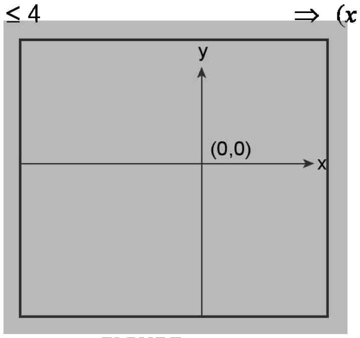


FIGURE 3.249

Case III: If $x \le 0$ and $y \ge 0$; then we get $((-x) - (y))\ (x - y) \le 4 \Rightarrow y^2 - x^2 \le 4$

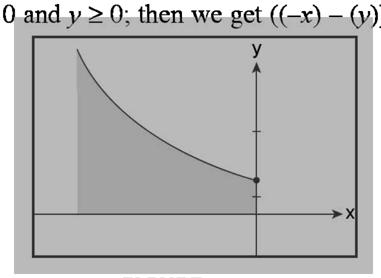


FIGURE 3.250

Case Iv: If $x \ge 0$ and $y \le 0$; then we get $(x - (-y))\ (x - y) \le 4 \Rightarrow x^2 - y^2 \le 4$

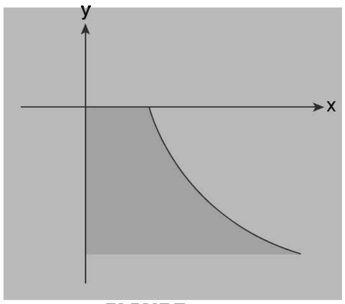


FIGURE 3.251

Hence, the graph of $(|x| - |y|)(x - y) \le 4$ will be given as in Figure 3.252.

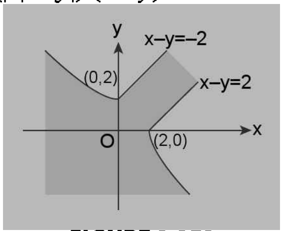


FIGURE 3.252

ILLUSTRATION 59: Without using calculus, find the area of the region covered by $|x + y| + |x - y| \le 4$ and $x^2 + y^2 \ge 4y$.

SOLUTION: First of all, let us try to plot the region satisfying $|x + y| + |x - y| \le 4$

Now; we observe that if $f(x, y) = |x + y| + |x - y|$

Then $f(-x, -y) = |-x - y| + |-x + y| = |x + y| + |x - y| = f(x, y)$ ....(1)

Also $f(y, x) = |y + x| + |y - x| = f(x, y)$ ....(2)

And $f(-x, y) = |-x + y| + |-x - y| = f(x, y)$ ...(3)

And $f(x, -y) = |x - y| + |x + y| = f(x, y)$ ...(4)

Equations (1), (2), (3) and (4) respectively indicates the symmetry of the graph of $f(x, y) = |x + y| + |x - y|$ about origin, about the line $y = x$, about y-axis and about x-axis, respectively.

Hence, we draw the required graph in the first quadrant only and that too below, the line $y = x$;

In this region ; we have $x, y \ge 0$ and $y \le x$

$\Rightarrow\ |x + y| + |x - y| = 2x \le 4 \Rightarrow x \le 2$

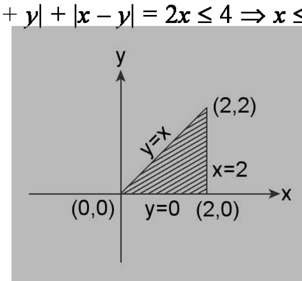


FIGURE 3.253

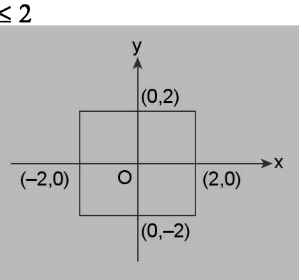


FIGURE 3.254

Hence, using symmetry, the overall graph will be a square of length 4 units with centre at origin as shown in Figure 3.254.

Now; $x^2 + y^2 \ge 4y \ \Rightarrow\ x^2 + (y - 2)^2 \ge 4$, which will represent the outside area of the circle of radius 2 and its boundary having centre at (0,2).

$\therefore$ The common area bounded by the two inequalities will be shown by the shaded region in the Figure 3.255.

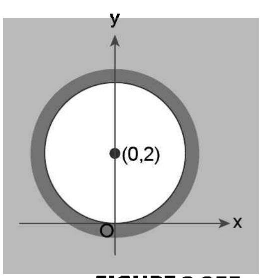


FIGURE 3.255

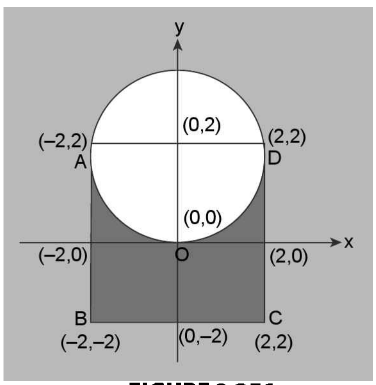


FIGURE 3.256

$\therefore$ Required area (area of region $ABCD\,O$) = (area of square $ABCD$) – (area of semi-circle AOD) = $(16 - 2\pi)$ square units.

ILLUSTRATION 60: Sketch the region bounded by $9 \le x^2 + y^2 \le 3\,(|x| + |y|)$

SOLUTION: The graph of $x^2 + y^2 \ge 9$ will be as shown below.

We observe the graph is a circle with centre at origin and hence, symmetric about both the axes.

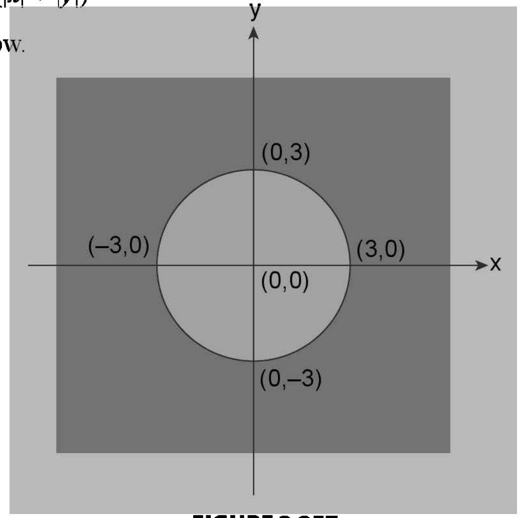


FIGURE 3.257

Now consider $x^2 + y^2 \le (|x| + |y|)$

Here, also we observe that $f(x, y) = f(-x, -y) = f(x, -y) = f(-x, y) = f(y, x)$

Hence, the graph is symmetric about both the axes first of all we plot the curve $x^2 + y^2 \le 3\,(|x| + |y|)$ only in first quadrant.

$\Rightarrow$ $(x^2 - 3x + 9/4) + (y^2 - 3y + 9/4) \le 9/2$

$\Rightarrow$ $(x - 3/2)^2 + (y - 3/2)^2 \le (3/\sqrt{2})^2$ which is a circle with centre at (3/2, 3/2) and radius $\dfrac{3}{\sqrt{2}}$

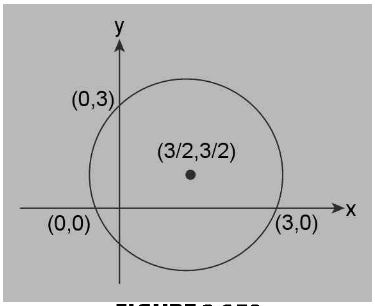


FIGURE 3.258

Therefore, the curve $x^2 + y^2 \le 3\,(|x| + |y|)$ for $x, y \in \mathbb{R}$ will be as shown in Figure 3.259.

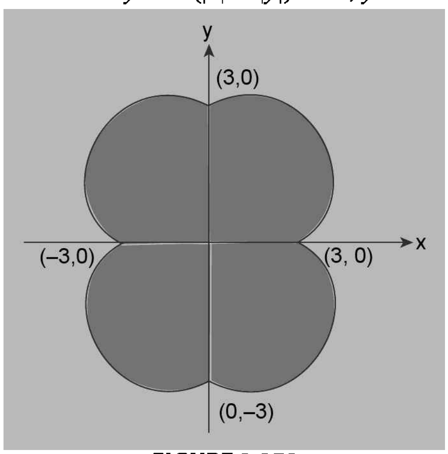


FIGURE 3.259

Now plotting the two curves together on the same reference plane, we shade the region satisfying $9 \le x^2 + y^2 \le 3\,(|x| + |y|)$.

Now; to find the area, we find the shaded area in the first quadrant only

Area of curve ($ADBCA$) = (area of semi-circle $ABCA$) – (area of arc $ABDA$)

$= \dfrac{\pi\left(\dfrac{3}{\sqrt{2}}\right)^2}{2}$ – (area of quarter circle $OADBO$ – area of ΔOAB)

$= \dfrac{9\pi}{4} - \left(\dfrac{\pi}{4}\times(3)^2 - \dfrac{1}{2}\times 9\right) = \dfrac{9}{2}$. Hence, total shaded area

$= 4\times\dfrac{9}{2} = 18$ square units.

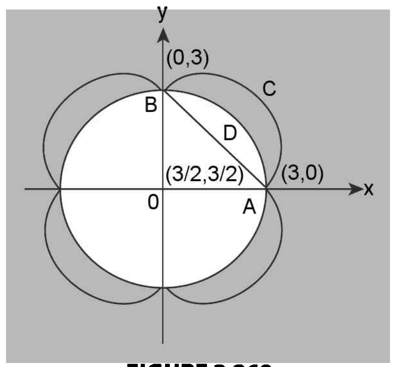


FIGURE 3.260

ILLUSTRATION 61: $||x| - |y|| + |x - y| \le 4$

SOLUTION: We observe that $f(x, y) = f(y, x)$

Hence, the graph is symmetric about $y = x$

Also, $f(-x, -y) = f(x, y)$

And hence the graph is symmetric about origin. Hence, we need to plot the curve in the first quadrant (below $y = x$) and also in the second quadrant and using that, we can plot the complete graph.

Case I: For $x, y \ge 0$ and $y \le x$

$|x - y| + |x - y| \le 4 \;\Rightarrow\; x - y \le 2$

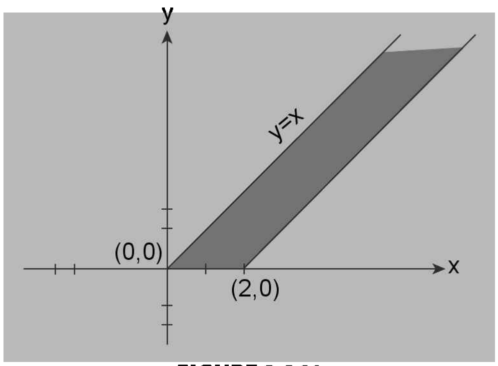


FIGURE 3.261

Case II: If $x \le 0$ and $y \ge 0$ (second quadrant) and $y + x \ge 0$, i.e., $y \ge -x$

Then $|x| - |y|| + |x - y| \le 4$

$\Rightarrow\; |-x - y| + |x - y| \le 4 \qquad \Rightarrow\; |x + y| + |y - x| \le 4$

$\Rightarrow\; x + y + y - x \le 4 \Rightarrow 2y \le 4 \qquad \Rightarrow\; y \le 2$

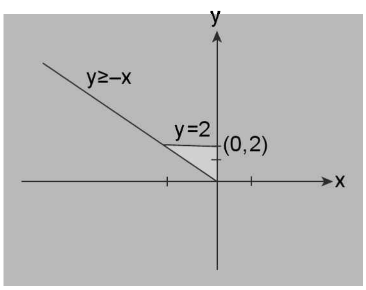


FIGURE 3.262

Case III: If $x \le 0$ and $y \ge 0$ and $y + x \le 0$, i.e., $y \le -x$ then $||x| - |y|| + |x - y| \le 4$

$\Rightarrow\ |-x - y| + |y - x| \le 4 \qquad \Rightarrow\ |x + y| + |y - x| \le 4$

$\Rightarrow\ -x - y + y - x \le 4 \qquad \Rightarrow\ -2x \le 4 \Rightarrow x \ge -2$

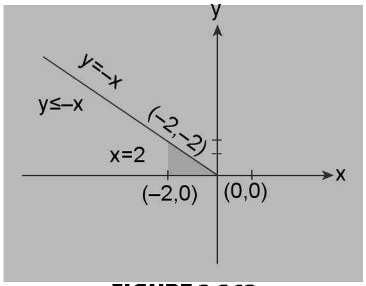


FIGURE 3.263

And using the above three figures, we can draw the required region

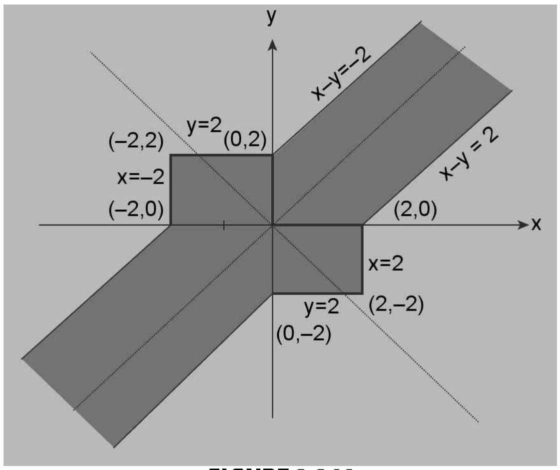


FIGURE 3.264

ILLUSTRATION 62: $3^{|x|}\,|y| + 3^{|x|-1} \le 1;\ 3|x| \le 1;$

SOLUTION: Now $3^{|x|}\,|y| + \dfrac{3^{|x|}}{3} \le 1 \ \Rightarrow\ 3^{|x|}\left(|y| + \dfrac{1}{3}\right) \le 1 \ \Rightarrow\ |y| + \dfrac{1}{3} \le 3^{-|x|}$

As we observe, $f(x, y) = f(-x, -y) = f(x, -y) = f(-x, y)$

Hence, the graph is symmetric about both the axis.

For $x, y \ge 0$; we have $y + \dfrac{1}{3} \le 3^{-x} \ \Rightarrow\ y \le 3^{-x} - (1/3)$

Graph of $y = 3^{-x}$ and $y = 3^{-x} - (1/3)$ are shown in Figure 3.265.

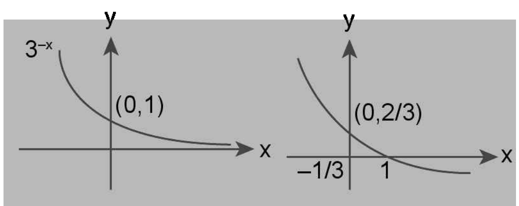


FIGURE 3.265

For $x \in [0, 1/3]$; we represent $y \le 3^{-x} - 1/3$ by the shaded region.

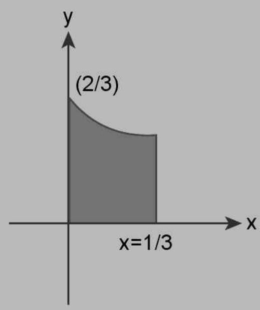


FIGURE 3.266

Now since the graphs is symmetric about both the axes; hence, the required graph will as shown in Figure 3.267.

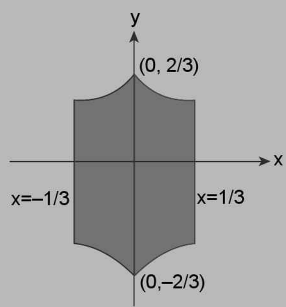


FIGURE 3.267

TEXTUAL EXERCISE–4: (SUBJECTIVE)

1. Sketch the region represented by following linear inequations:

(a) $y > x$ (b) $y < -x$

(c) $y \geq |x - 1|$ (d) $x > |y - 1|$

(e) $|x - y| \leq 2$ (f) $|x + y| \geq 3$

(g) $|x| - |y| \geq 1$ (h) $|x| + |y| \leq 2$

(i) $|x - 2| + |y + 1| \geq 3$

2. Sketch the curve of $y = [x + y]^2$. Hence, find the area of the region represented by $[x + y]^2 = 4$ and lying in the first quadrant.

3. Sketch the following graphs:

(i) $y \leq [x]–[x – (1/2)]$ (ii) $[x] + [y] = 1$

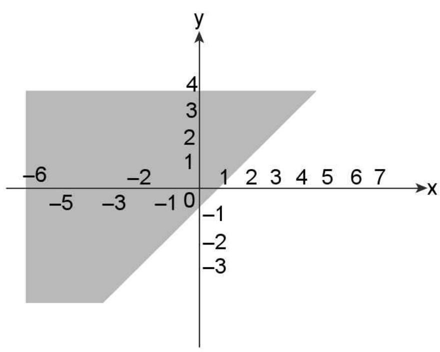
(iii) $[x] – [y] = 1$ (iv) $[x][y] = 1$

(v) $[y] = [x]$ (vi) $[x + 1] = [y + 2]$

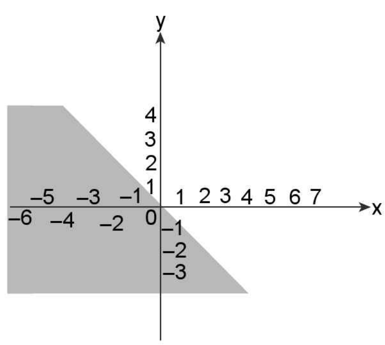
4. Sketch on $x – y$ plane: $[|x^2 – 2|]^2 + [|y + 1|]^2 = 0$

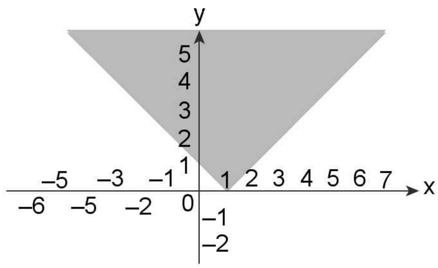
Answer Keys

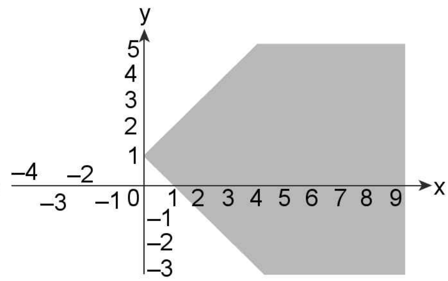
1. (a)

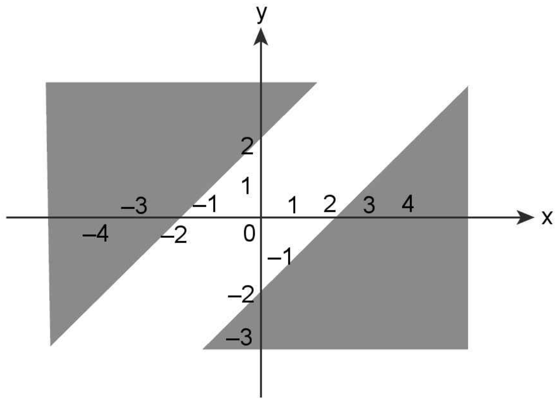


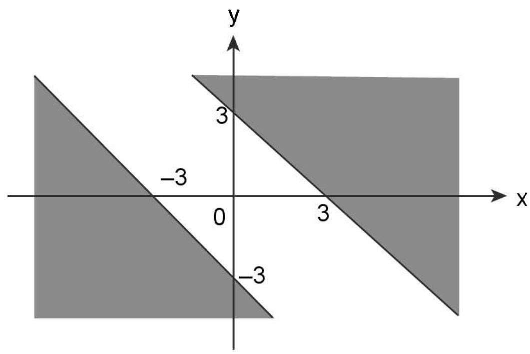
(b)

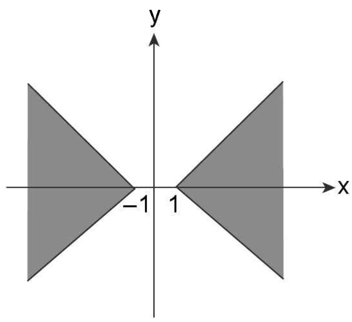


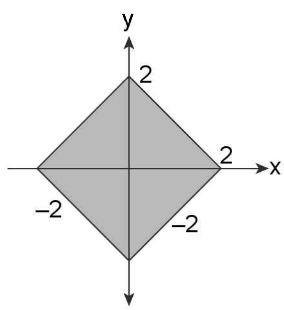
(c) 

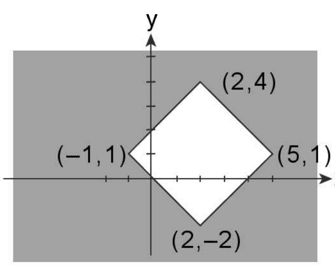
(d) 

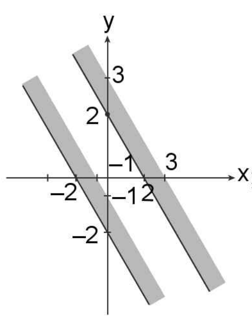
(e) 

(f) 

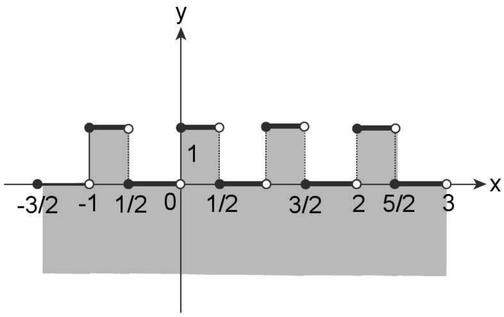
(g) 

(h) 

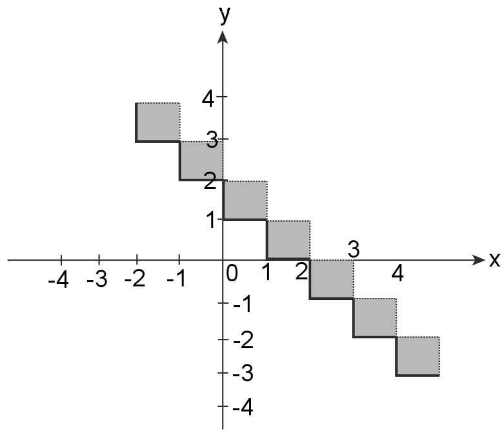
(i) 

2. ; area = 5/2

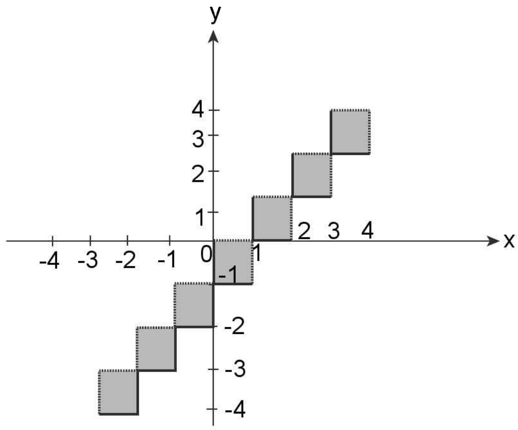
3. (i)



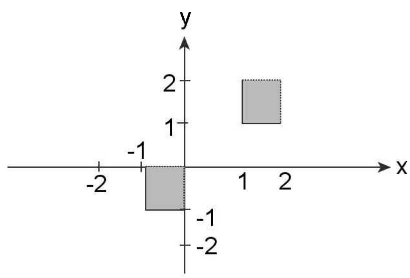
(ii)



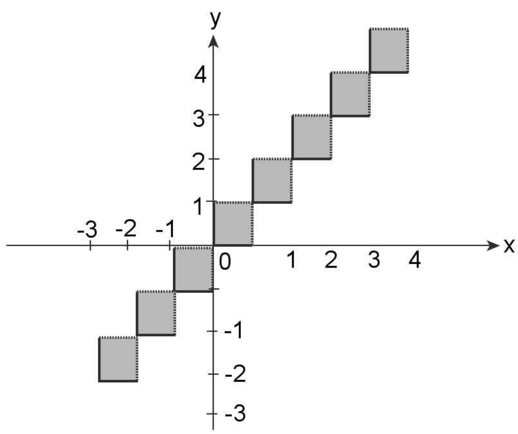
(iii)



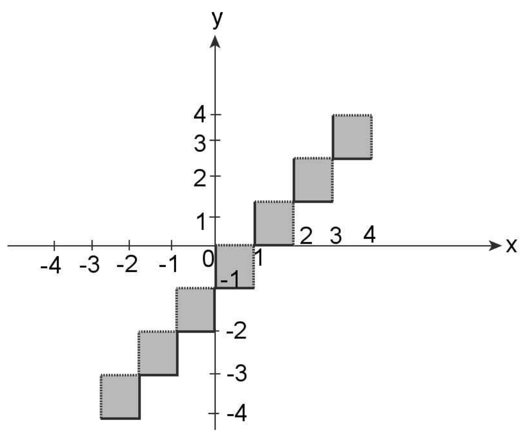
(iv)



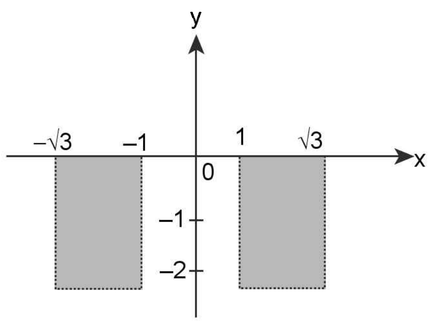
(v)



(vi)



4.



■ GRAPHS OF RECIPROCAL FUNCTION

In order to draw the graph of reciprocal of a function $f(x)$, it is necessary to keep in mind the following facts relating the output $f(x)$ and $\dfrac{1}{f(x)}$.

1. $\dfrac{1}{f(x)}$ is not defined when $f(x) = 0$. As it tends to $\pm\infty$ whenever $f(x)$ approaches to 0 respectively from positive/negative side.
   Therefore, the lines $x = \alpha$, where $f(\alpha) = 0$ (i.e., α is root of $f(x)$) are tangents to the curve $\dfrac{1}{f(x)}$ at $\pm\infty$ known as vertical asymptote of the function $\dfrac{1}{f(x)}$.
2. If $\lim\limits_{x\to\pm\infty} f(x) = k$, then $y = 1/k$ will be the horizontal asymptote of the function $1/f(x)$.
3. $f(x)$ and $\dfrac{1}{f(x)}$ meet each other whenever $f(x) = \pm 1$.
4. The output of the function $f(x)$ and $\dfrac{1}{f(x)}$ have same sign, i.e., if $f(x)$ is positive/negative in an interval I; then $1/f(x)$ is also positive/negative in that interval.
5. As the function $f(x)$ and $1/f(x)$ have opposite monotonicity, therefore, as and when $f(x)$ increases/decreases, the reciprocal function $1/f(x)$ decreases/increases, respectively. Therefore, if $f(x)$ is small/big in magnitude at a point x_1, then $\dfrac{1}{f(x)}$ is big/small in magnitude at that point.
   Therefore, to obtain an approximate sketch of the function $1/f(x)$, we should follow the algorithm given below:

Step 1: Draw the graph of $f(x)$ and locate its roots (say α, β etc.) as well as the points where it takes the value ± 1 (say γ, δ etc.).

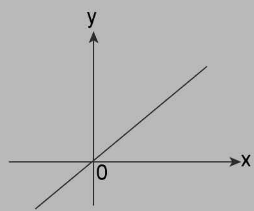
Step 2: Draw the vertical asymptotes $x = \alpha$, $x = \beta$ etc.

Step 3: From the points on $f(x)$ (say $(\gamma, 1)$), draw the curve $1/f(x)$, taking care of its monotonicity, so that it approaches to the neighbouring asymptotes.

Step 4: Find the $\lim\limits_{x\to\pm\infty} \dfrac{1}{f(x)}$, (say it is '$k$'). Then $y = 1/k$ will be the horizontal asymptote of the function $1/f(x)$.

ILLUSTRATION 63: Sketch the graph of $y = 1/x$

SOLUTION: Graph of $y = x$ Graph of $y = 1/x$



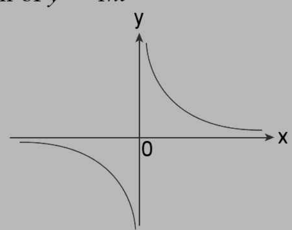


FIGURE 3.268

ILLUSTRATION 64: Sketch the graph of $y = \dfrac{1}{(x+1)(x)(x-2)}$.

SOLUTION: The graph of $y = (x + 1)(x)(x - 2)$ is as shown is in Figure 3.269.

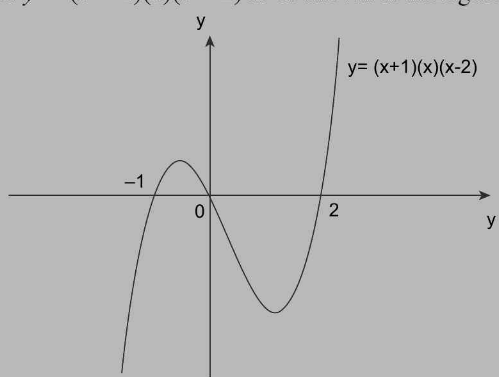


FIGURE 3.269

Hence, the graph of $y = \dfrac{1}{(x+1)(x)(x-2)}$ is as given in Figure 3.270.

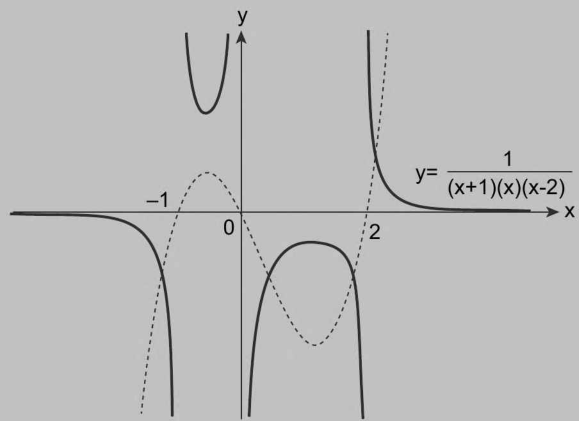


FIGURE 3.270

ADDITION OF GRAPHS

As we already know that the addition of two functions, $f(x)$ and $g(x)$, i.e., $y = f(x) + g(x)$ is defined for those values of x which lie in the domain of both the functions.

For the construction of the graph $y = f(x) + g(x)$, we follow the algorithm given below.

Step 1: We draw the graph of $f(x)$ and $g(x)$ on the same axes.

Step 2: At any x_0 lying in the common domain, we draw vertical arrows on the two functions, hence, getting the values of $f(x_0)$ and $g(x_0)$.

Step 3: We now draw a vertical arrow at the same x_0 with a height equal to $f(x_0) + g(x_0)$.

Step 4: To get the value of $y = f(x) + g(x)$ at several other values of x, we repeat the steps (2) and (3) for different values of x lying in the common domain, and thereby, joining these points by a smooth curve, we get the desired curve.

ILLUSTRATION 65: Construct the graph of $y = [x] + \{x\}$ using addition of graphs.

SOLUTION: Let us consider $y_1 = [x]$ and $y_2 = \{x\}$

Let us consider $y_1 = [x]$ and $y_2 = \{x\}$

Graph of $y_1 = [x]$

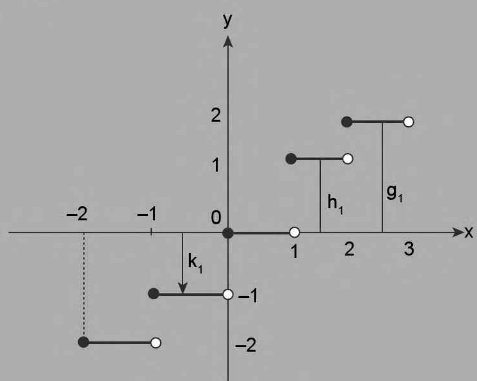


FIGURE 3.271

Graph of $y_2 = \{x\}$

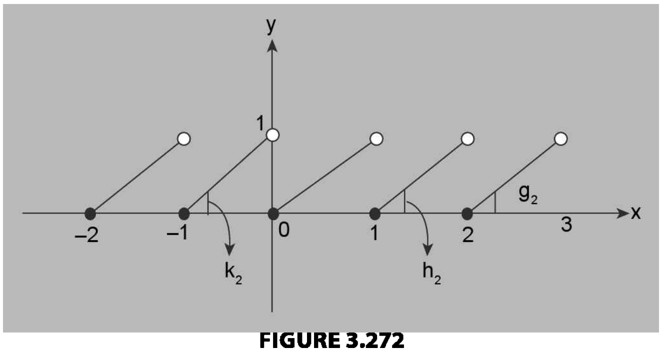


FIGURE 3.272

Adding the heights h_1 and h_2; g_1 and g_2; k_1 and k_2 obtained on the two different graphs, for some point x in the common domain, we get

Graph of $y = [x] + \{x\}$

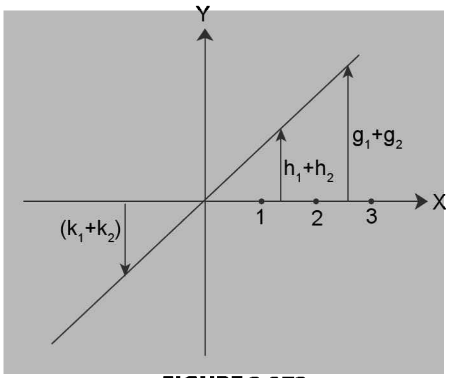


FIGURE 3.273

ILLUSTRATION 66: Using addition of graphs, draw the graphs of $y = x + \sin x$.

SOLUTION: The graph of $y = x$ is as shown in Figure 3.274.

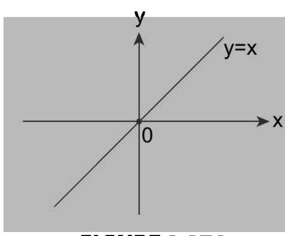


FIGURE 3.274

The graph of $y = \sin x$ is as shown in Figure 3.275.

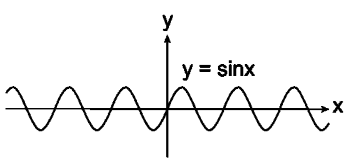


FIGURE 3.275

Hence, the graph of $y = x + \sin x$ is as shown in Figure 3.276.

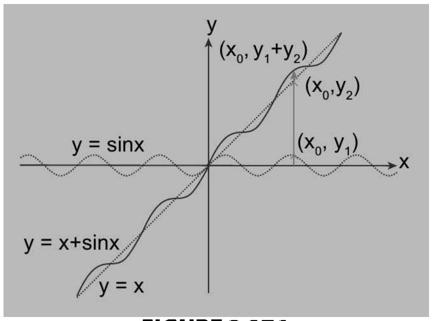


FIGURE 3.276

ILLUSTRATION 67: Draw the graphs of $y = e^x$ and $y = e^{-x}$, and hence, draw the graph of $y = e^x + e^{-x}$ and also the graph of $y = e^x - e^{-x}$.

SOLUTION: The graph of $y = e^x$ is as shown in Figure 3.277.

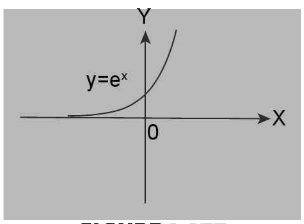


FIGURE 3.277

The graph of $y = e^{-x}$ is as shown in Figure 3.278.

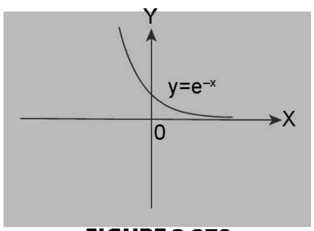


FIGURE 3.278

The graph of $y = -e^{-x}$ is as shown in Figure 3.279.

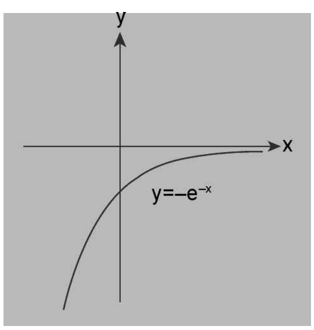


FIGURE 3.279

The graph of $y = e^x + e^{-x}$ is as shown in Figure 3.280.

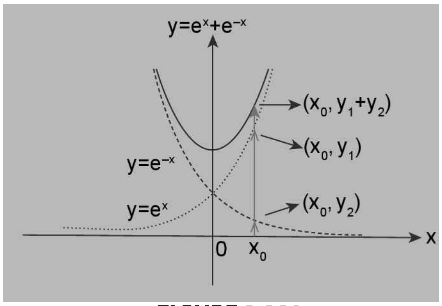


FIGURE 3.280

The graph of $y = e^x - e^{-x}$ is as shown in Figure 3.281.

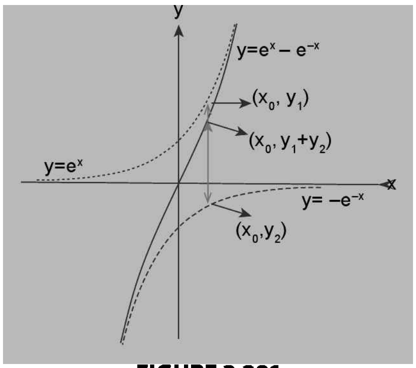


FIGURE 3.281

ILLUSTRATION 68: Draw the graph of $f(x) = x^2$ and $g(x) = 1/x$, and hence, $h(x) = f(x) + g(x)$.

SOLUTION: The graph of $y = x^2$ is as shown in Figure 3.282.

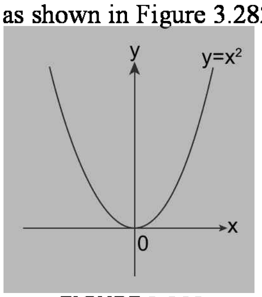


FIGURE 3.282

The graph of $y = 1/x$ is as shown in Figure 3.283.

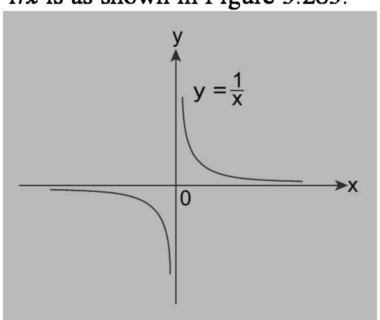


FIGURE 3.283

The graph of $y = x^2 + 1/x$ is as shown in Figure 3.284.

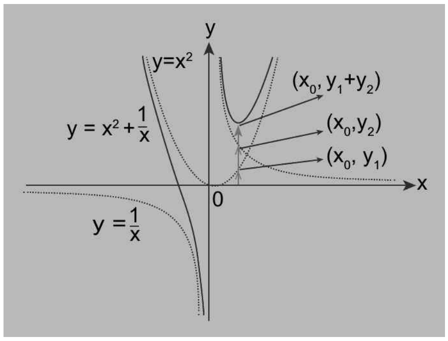


FIGURE 3.284

ILLUSTRATION 69: Draw the graph of $f(x) = \sin[x]$ and $g(x) = [\sin[x]]$, and hence, $h(x) = f(x) + g(x)$ for $x \in [0, 2\pi]$.

SOLUTION: We already know, the graph of $f(x) = \sin[x]$ is as shown in Figure 3.285.

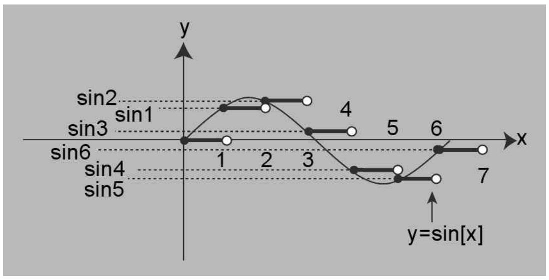


FIGURE 3.285

Also, the graph of $g(x) = [\sin[x]]$ is as shown in Figure 3.286.

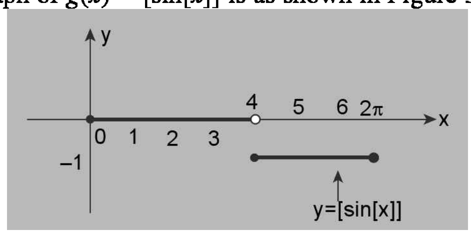


FIGURE 3.286

Now, the function $h(x) = f(x) + g(x) = \sin[x] + [\sin[x]]$ and its graph is as shown in Figure 3.287.

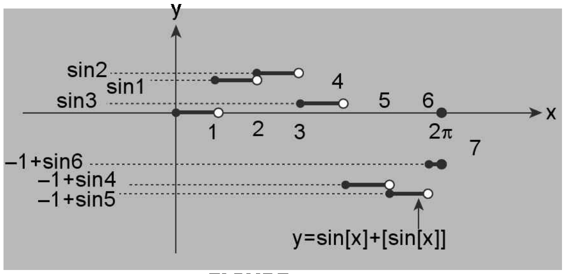


FIGURE 3.287

MULTIPLICATION OF GRAPHS

As we already know that the multiplication of two functions, $f(x)$ and $g(x)$, i.e., $y = f(x).\ g(x)$ is defined for those values of x which lie in the domain of both the functions. For the construction of the graph $y = f(x).\ g(x)$, we follow the algorithm given below.

Step 1: We draw the graph of $f(x)$ and $g(x)$ on the same axes.

Step 2: At any x_0 lying in the common domain, we draw vertical arrows on the two functions, hence, getting the values of $f(x_0)$ and $g(x_0)$.

Step 3: We now draw a vertical arrow at the same x_0 with a height equal to $f(x_0).\ g(x_0)$.

Step 4: To get the value of $y = f(x).\ g(x)$ at several other values of x, we repeat the steps (2) and (3) for different values of x lying in the common domain and thereby, joining these points by a smooth curve, we get the desired curve.

NOTE

As such it is not very easy to plot these graphs.

SPECIAL CASE OF ENVELOPED GRAPHS

When one of the functions is a trigonometric function like $\sin x$, $\cos x$, $|\sin x|$, etc.

That is, we wish to draw functions like
$y = f(x)\sin x$, $y = f(x)\cos x$, $y = f(x)\,|\sin x|$, etc.

ILLUSTRATION 70: Construct the graph of the functions $y = x \sin x$

SOLUTION: We know that $-1 \le \sin x \le 1$ or we can say that $|\sin x| \le 1$.

Hence, $|x \sin x| \le |x|$.

It is obvious that $y = 0$ for the values $x = k\pi$, $k = 0, \pm1, \pm2, \ldots$ for which $\sin x = 0$. and so the graph of the function y crosses the positive x-axis at the point $x = k\pi$, $k \in 0, 2\ \forall\ k \in \mathbb{Z}$

Also, for $x > 0$, $|x \sin x| \le |x| \Rightarrow -x \le x \sin x \le x$.

Hence, for positive values of the independent variable 'x', the amplitude of the graph of the function y does not extend above the straight line $y = x$ or below the straight line $y = -x$, In this case, the points of the graph of the function y that correspond to the values of $x > 0$ for which $\sin x = \pm1$, i.e., the graph $y = x.\ \sin x$ touches the graph $y = \pm x$ at the points where

$y = \sin x = \pm1$, i.e., at $x = (4n+1)\dfrac{\pi}{2}, n \in W$.

Also, for $x < 0$, $|x \sin x| \le |x| \Rightarrow x \le x \sin x \le -x$.

Hence, for negative values of the independent variable 'x', the amplitude of the graph of the function y does not extend above the straight line $y = -x$ or below the straight line $y = x$. In this case, the points of the graph of the function y that correspond to the values of $x < 0$ for which $\sin x = \pm1$ i.e., the graph $y = x.\ \sin x$ touches the graph $y = \pm x$ at the points where

$y = \sin x = \pm1$ i.e., at $x = (4n+1)\dfrac{\pi}{2}, n \in \mathbb{Z} - W$.

The graph of $y = x$ is given as in Figure 3.288.

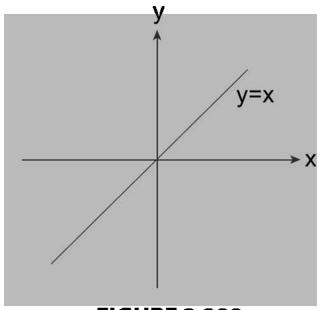


FIGURE 3.288

Now, creating an envelope whose boundaries are given by $y = \pm x$, we get the envelope as

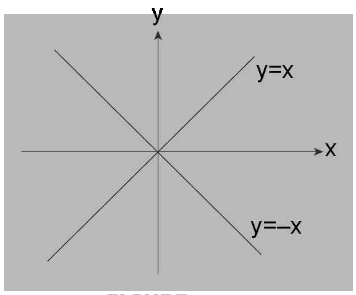


FIGURE 3.289

Now, as discussed earlier, the graph of $y = x \sin x$ will be contained in/on the graph $y = \pm x$. Hence, the graph of $y = x \sin x$ will be as shown in Figure 3.290.

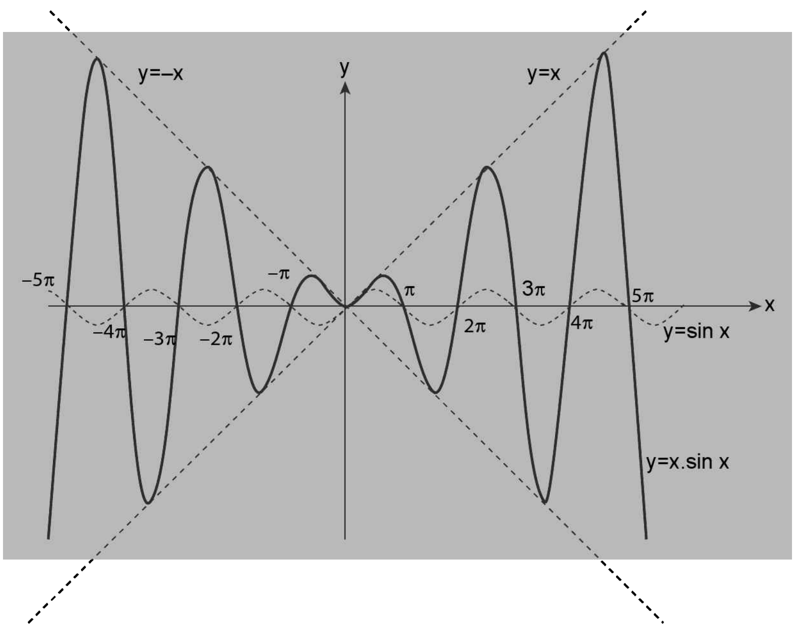


FIGURE 3.290

ILLUSTRATION 71: Construct the graph of the functions $y = (1.1)^x \sin x$.

SOLUTION: The graph of $y = (1.1)^x$ is as shown in Figure 3.291

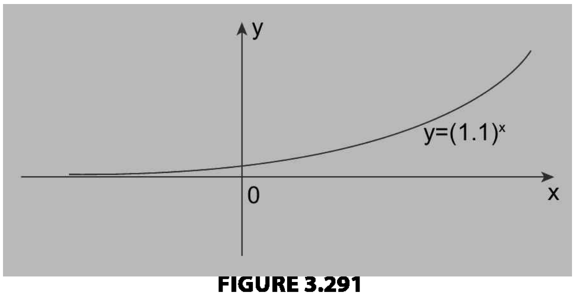


FIGURE 3.291

Hence, the envelope will be formed by $y = \pm\,(1.1)^x$, and its graph will be shown in Figure 3.292.

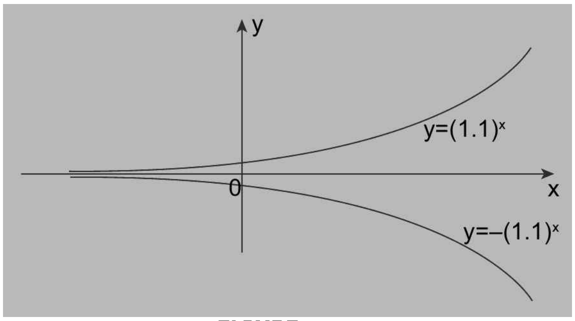


FIGURE 3.292

Now, as discussed earlier, the graph of $y = (1.1)^x \sin x$ will be contained in/on the graph $y = \pm\,(1.1)^x$. Hence, the graph of $y = (1.1)^x \sin x$ is as shown in Figure 3.293.

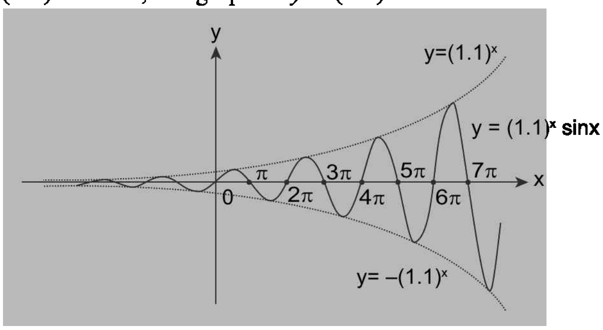


FIGURE 3.293

ILLUSTRATION 72: Construct the graph of the functions $y = \left(\dfrac{x}{5}\right)^2 \cos x$.

SOLUTION: The graph of $y = \left(\dfrac{x}{5}\right)^2$ is as shown in Figure 3.294.

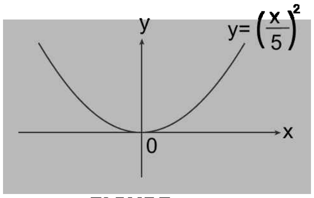


FIGURE 3.294

Hence, the envelope will be formed by $y = \pm\left(\frac{x}{5}\right)^2$, and its graph will be as shown in Figure 3.295.

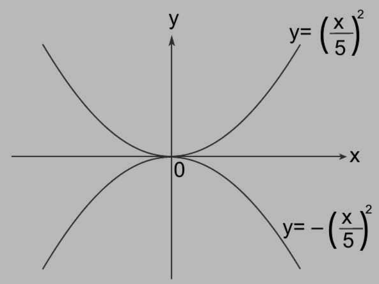


FIGURE 3.295

Now, as discussed earlier, the graph of $y = \left(\frac{x}{5}\right)^2 \cos x$ will be contained in/on the graph $y = \pm\left(\frac{x}{5}\right)^2$. Hence, the graph of $y = \left(\frac{x}{5}\right)^2 \cos x$ will be as shown in Figure 3.296.

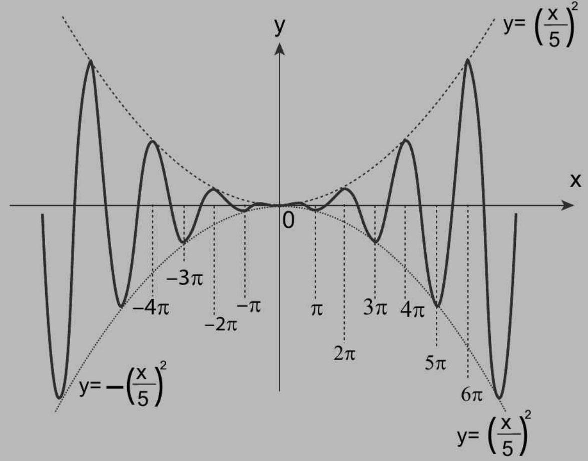


FIGURE 3.296

■ MAXIMUM AND MINIMUM FUNCTIONS

Maximum/minimum function of a given group of functions is that function which takes up the largest value/least value in the common domain. In order to select maxima/minimum function out of a group of given functions we can use the following definition:

$$\text{Max}\,\{f(x), g(x)\} = \frac{f(x)+g(x)+|f(x)-g(x)|}{2} = f(x)$$

$\forall\, x$ where $f(x) \geq g(x)$ and $g(x)\ \forall\, x$ where $f(x) \leq g(x)$

$$\text{Min.}\ \{f(x), g(x)\} = \frac{f(x)+g(x)-|f(x)-g(x)|}{2} = g(x)$$

$\forall\, x$ where $f(x) \geq g(x)$ and $f(x)\ \forall\, x$ where $f(x) \leq g(x)$

Above definition can also be extended to group of more than two functions.

ILLUSTRATION 73: Let $h(x)$ = max. $\{x, x^2\}$. Then write the equivalent definition of $f(x)$.

SOLUTION: Given $f(x) = x$ and $g(x) = x^2$ and sketching their graphs as shown in Figure 3.297.

In order to find the maximum value of two functions

$f(x)$ and $g(x)$, we get $h(x) = \begin{cases} x^2; \, x \le 0 \text{ or } x \ge 1 \\ x \,; \; 0 \le x \le 1 \end{cases}$

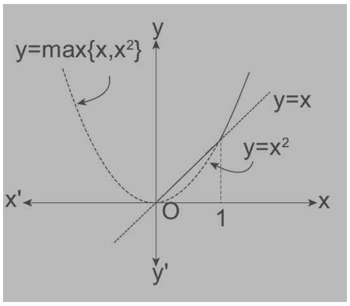


FIGURE 3.297

ILLUSTRATION 74: Sketch the graph of the following functions:

(a) y = max.(sin x, cos x); $\forall x \in \left[-\pi, \frac{3\pi}{2}\right]$ (b) y = max.{tan x, cot x}

SOLUTION: (a) To sketch the graph of y = max {sin x, cos x} $\forall x \in \left[-\pi, \frac{3\pi}{2}\right]$ draw the graph of y = sin x and y = cos x in described domain and obtaining their points of intersections, i.e., $x = \frac{-3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}$ and considering the upper boundary for maximum function, we get

$$y = \begin{cases} \sin x; & x \in \left[-\pi, -\frac{3\pi}{4}\right) \\ \cos x; & x \in \left[-\frac{3\pi}{4}, \frac{\pi}{4}\right) \\ \sin x; & x \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right) \\ \cos x; & x \in \left[\frac{5\pi}{4}, \frac{3\pi}{2}\right] \end{cases}$$

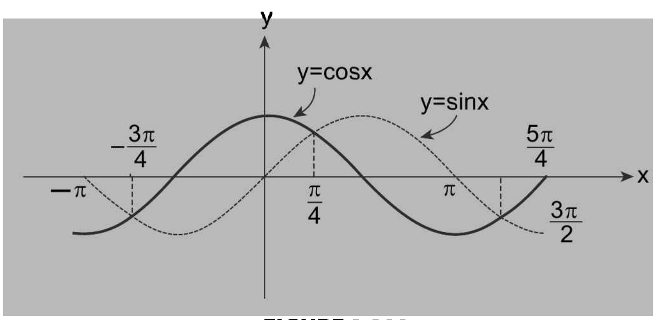


FIGURE 3.298

(b) $y = \max\{\tan x, \cot x\}$. Of course the given function is periodic with period π as both $\tan x$ and $\cot x$ are periodic with period π.

Thus, sketching $y = \tan x$ and $y = \cot x$ both, we get

$$y = \begin{cases} \cot x; & x \in \left(0 < x < \frac{\pi}{4}\right) \\ \tan x; & x \in \left[\frac{\pi}{4} < x < \frac{\pi}{2}\right) \\ \cot x; & x \in \left(\frac{\pi}{2} < x < \frac{3\pi}{4}\right) \\ \tan x; & x \in \left[\frac{3\pi}{4} < x < \pi\right) \end{cases}$$ and then it repeats periodically with period π.

It is worth noticing that at $x = \frac{n\pi}{2}; n \in \mathbb{Z}$, the given function is not defined.

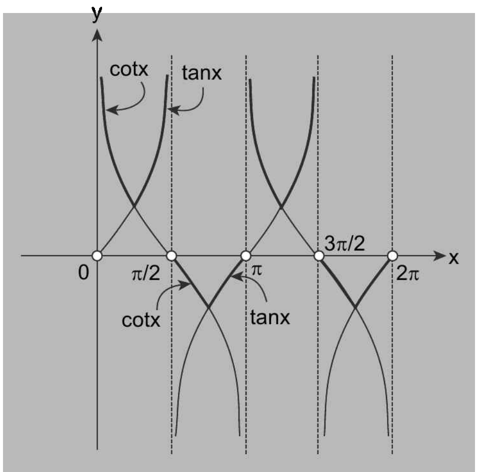


FIGURE 3.299

ILLUSTRATION 75: Plot the curve $y = \max\{|x| + |y|, 3|x| - |y|\}$

SOLUTION: ***Case I:*** If $|x| + |y| \geq 3|x| - |y|$, then $2|y| \geq 2|x|$

$\Rightarrow$ $|y| \geq |x|$ and the corresponding region could be plotted as shown in Figure 3.300.

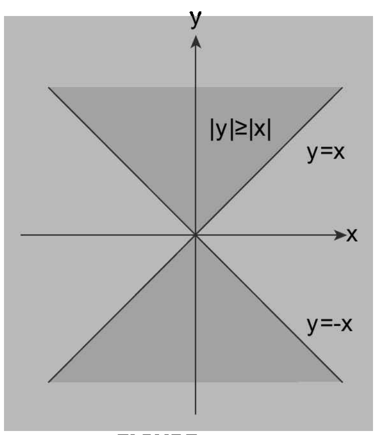


FIGURE 3.300

And in that region, we have $y = |x| + |y|$

Case I: $y \geq 0$

$\Rightarrow\ y = |x| + y \Rightarrow |x| = 0 \Rightarrow x = 0$

Case II: $y < 0 \ \Rightarrow\ |y| = -y \ \Rightarrow\ y = |x| - y$

$\Rightarrow\ y = \dfrac{|x|}{2}$ not possible in the given region.

$\therefore$ The curve $y = |x| + |y|$ is non-negative infinite segment of y-axis.

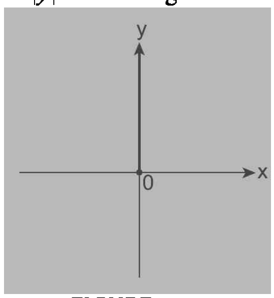


FIGURE 3.301

Case II: When $|x| + |y| \leq 3|x| - |y| \ \Rightarrow\ 2|y| \leq 2|x| \ \Rightarrow\ |y| \leq |x|$

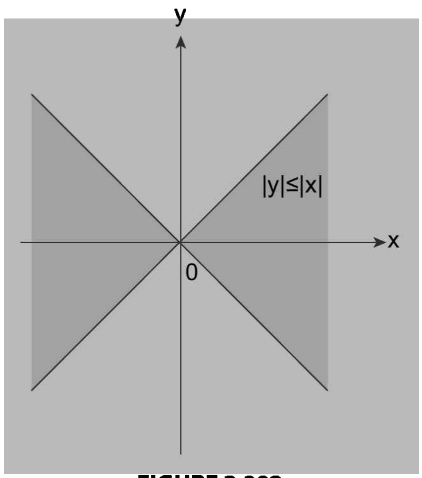


FIGURE 3.302

And in this region, we must equate $y = 3|x| - |y|$.

Case II: ***(a)*** when $y \geq 0 \ \Rightarrow\ y = 3|x| - y \ \Rightarrow\ 2y = 3|x| \ \Rightarrow\ y = \dfrac{3|x|}{2}$

Which is impossible for $|x| > 0$ as $|y| \leq |x|$

Thus, the only point satisfying $y = 3|x| - |y|$ and $y \geq 0$ is (0, 0)

Case II (b) When $y < 0 \ \Rightarrow\ y = 3|x| - (-y) \ \Rightarrow\ 3|x| = 0 \ \Rightarrow\ x = 0$

$\Rightarrow\ |x| = 0$, but $|y| \leq |x| \ \Rightarrow\ |y| \leq 0$, impossible as $y < 0 \ \Rightarrow\ |y| > 0$

$\therefore$ Thus, the given shaded region could have no point satisfying the inequalities

Hence,the graph of $y = \max\{|x| + |y|, 3|x| - |y|\}$ is shown in Figure 3.300, i.e., non-negative infinite segment of y-axis.

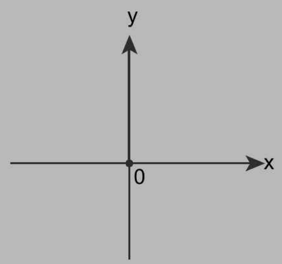


FIGURE 3.303

ILLUSTRATION 76: Draw the curve y = min. $\{e^x, e^{-x+1}, 3/2\}$

SOLUTION:

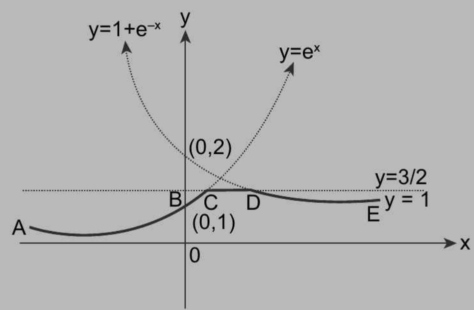


FIGURE 3.304

The dark curve ($ABCDE$) represents the function y = min. $\{e^x, 1 + e^{-x}, 3/2\}$

TEXTUAL EXERCISE–5: (SUBJECTIVE)

1. Sketch the graph of the following functions:
   (i) $y=\dfrac{x-5}{x+3}$ (ii) $y=\dfrac{3x-1}{x+2}$

2. Construct the graph of the following functions:
   (i) $y=\dfrac{2}{|x-1|-2}$ (ii) $y=\dfrac{|x-2|}{|x|-2}$
   (iii) $y = 2-\dfrac{4}{|x-1|}$

3. Draw the graph of following functions:
   (i) y = max $\{2 - x, 2 + x, 5\}$
   (ii) y = max $\{|x|, |x - 4|, 2 - |x - 2|\}$
   (iii) $y=min\left\{e^{2x},\dfrac{1}{2},\left(\dfrac{1}{7}+e^{-x}\right)\right\}$

4. Construct the graph of the following functions:
   (i) $y=\dfrac{x^4+4}{x^2}$
   (ii) $y=\sec x+\dfrac{1}{x^2},-\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$
   (iii) $y=\tan x+\dfrac{1}{x},-\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$

5. Sketch the graph of the following function:
   (a) $y = |x| \cos^2 x$
   (b) $y=\dfrac{3\sin \pi x}{2x}$
   (c) $y=\sqrt{-x}\,\sin\left(\dfrac{1}{x}\right)$

6. Sketch the following curves and show their relative nature on same graph sheet:
   (i) $y=\dfrac{1}{x}$ and $y=\dfrac{1}{x^2}$
   (ii) $y=\dfrac{1}{(x-1)(x-3)}$ and $y=\dfrac{1}{x(x+2)}$
   (iii) $y=\dfrac{1}{x^2+1}$ and $y=\dfrac{x-1}{x+1}$

7. Using general concept of curve sketching draw the following curves:

(a) $y = \dfrac{x^2 - 1}{x^2 + 1}$ (b) $y = \dfrac{2x}{x^2 + 1}$

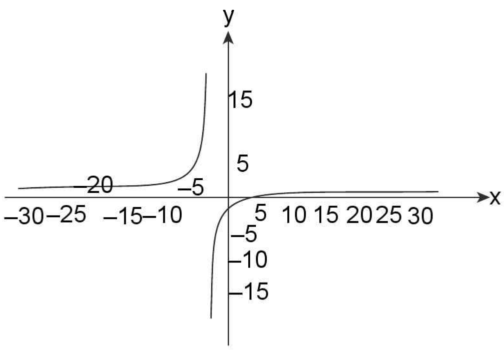
(c) $y = \dfrac{x^2 - 1}{x^2 + 2}$

8. Sketch the following:

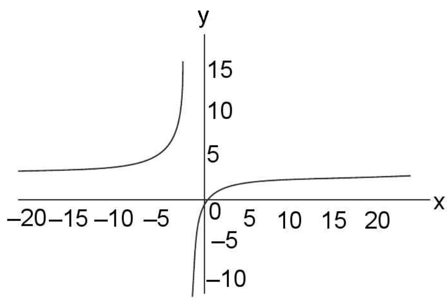
(i) $y = \max(x, 1 - x)$
(ii) $y = \max(|x|, |x - 1|)$
(iii) $y = \max(x, x^3)$
(iv) $y = \max(\sin x, \cos x)$
(v) $y = \min(\sin x, \cos x)$
(vi) $y = \max(x, x^2)$
(vii) $y = \min(x, x^2)$
(xiii) $y = \max(|x - 1|, |x|, |x + 1|)$
(xiv) $y = \min(|x - 1|, |x|, |x + 1|)$
(xv) $y = \min(|\sin x|, |\cos x|, 1/2)$
(xvi) $y = \max(\sin x, \cos x, 1/2)$
(xvii) $y = \max(x - [x], -x - [-x])$
(xviii) $f(x) = \min(x - [x], -x - [-x])$
(xix) $y = \min(|x| + |y|, |x| - |y|)$
(xx) $y = \max(|x| + |y|, |x| - |y|)$
(xxi) $y = \max(|x|, |y|)$
(xxii) $\min(|x|, |y|) = 1$

Answer Kyes

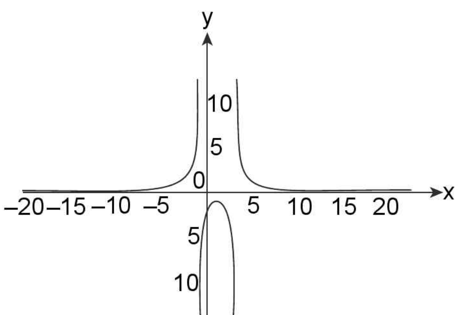
1. (i)



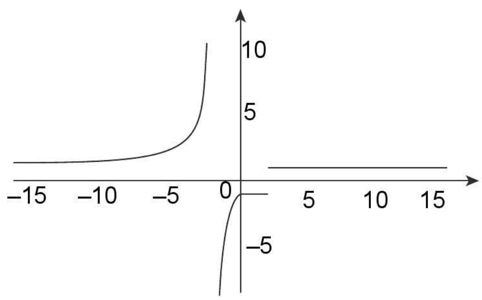
(ii)



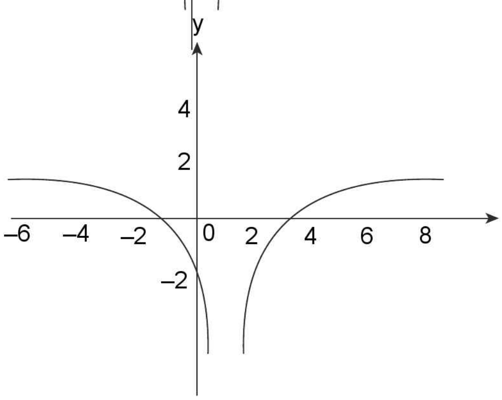
2. (i)



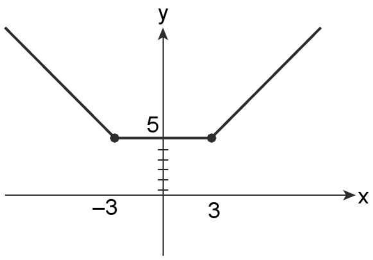
(ii)



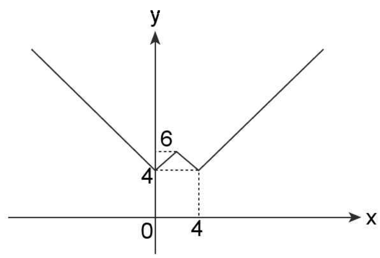
(iii)



3. (i)



(ii)



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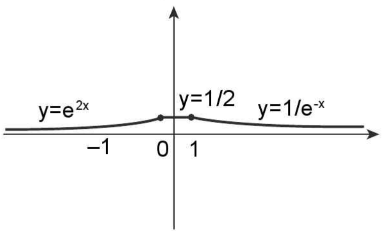
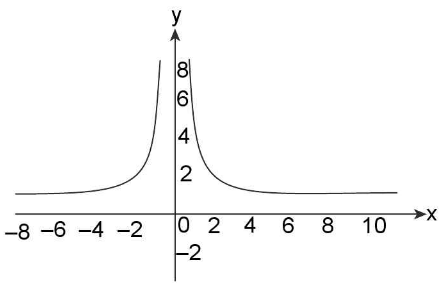


4. (i)



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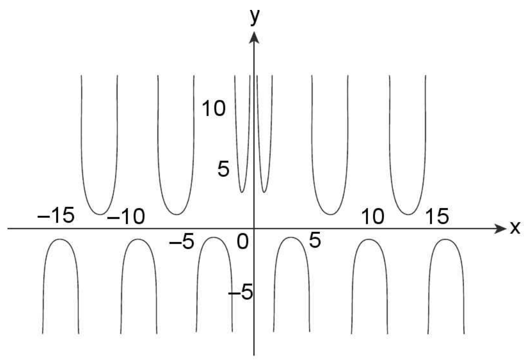
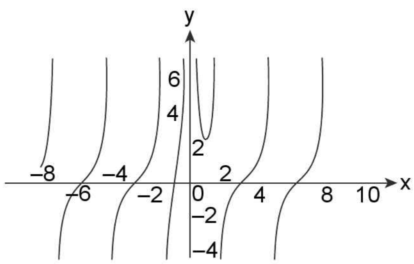
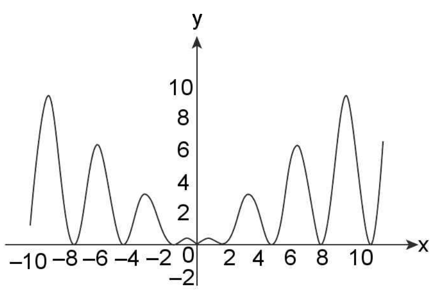


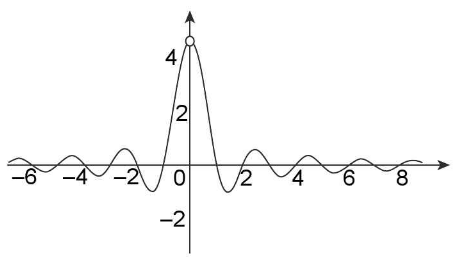
(iii)



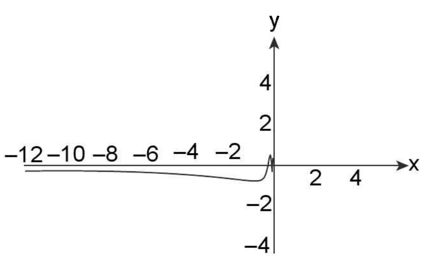
5. (a)



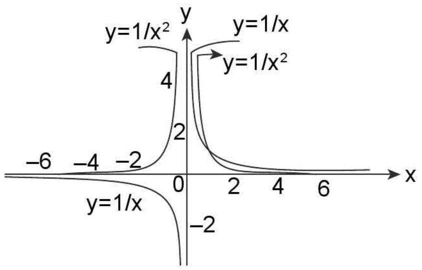
(b)



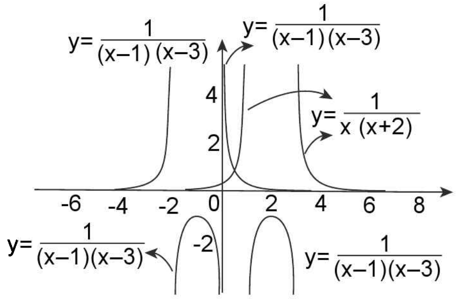
(c)



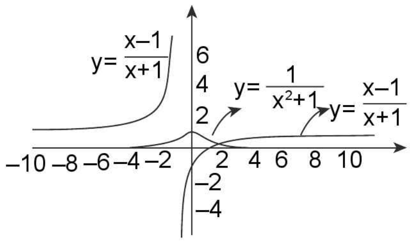
6. (i)



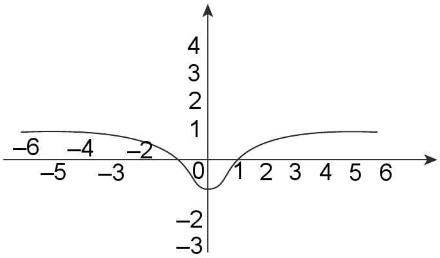
(ii)



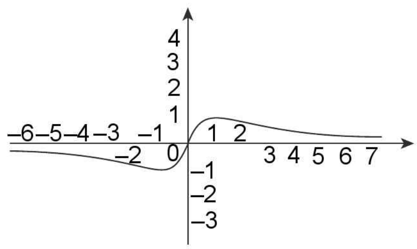
(iii)



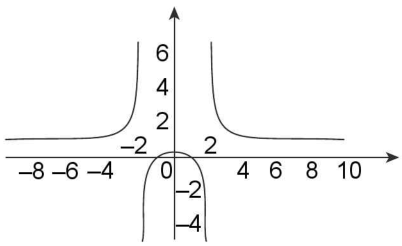
7. (a)



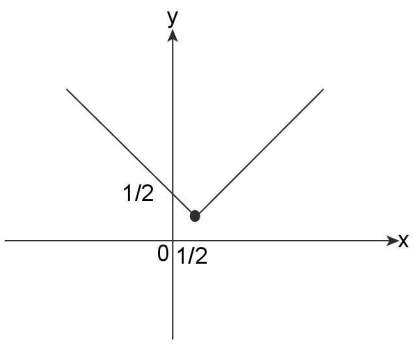
(b)



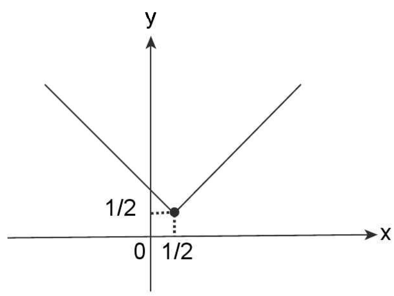
(c)



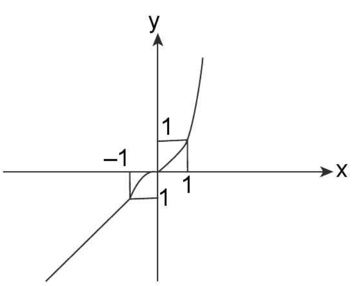
8. (i)



(ii)



(iii)



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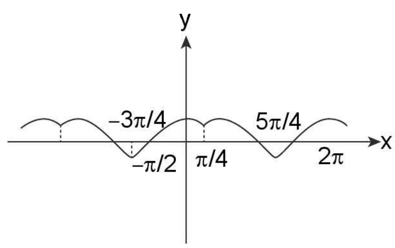

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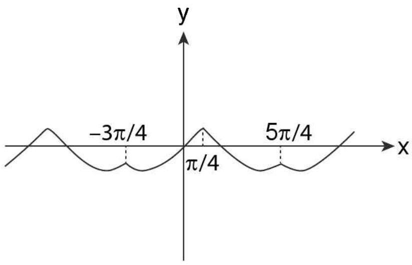

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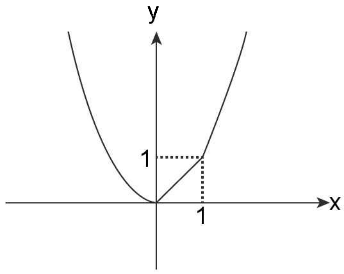

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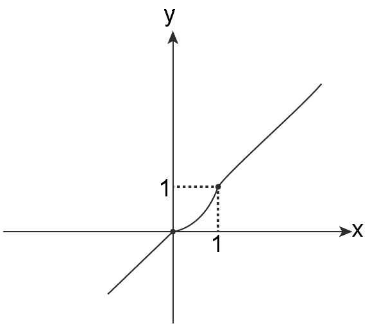

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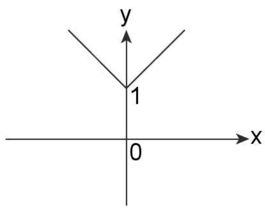

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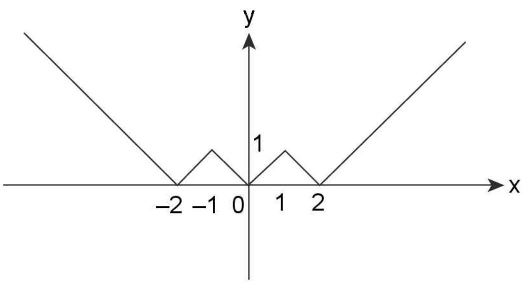

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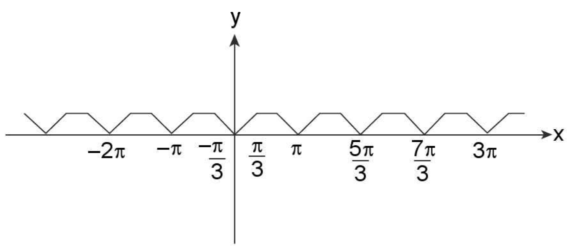

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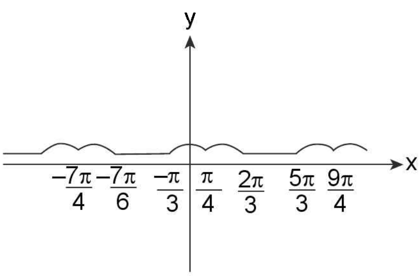

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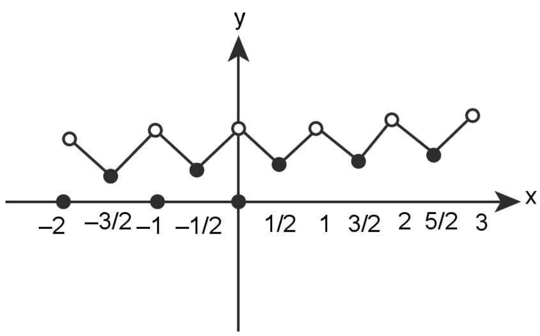

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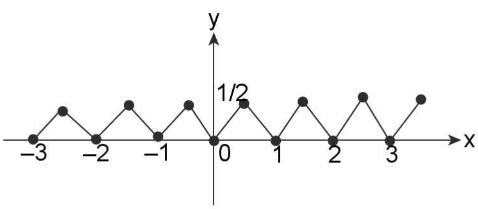

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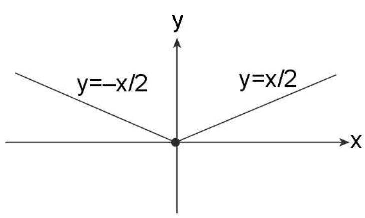

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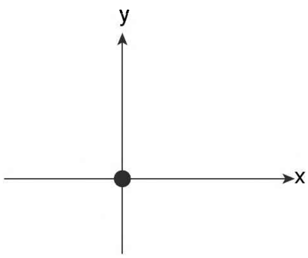

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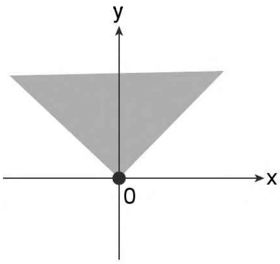

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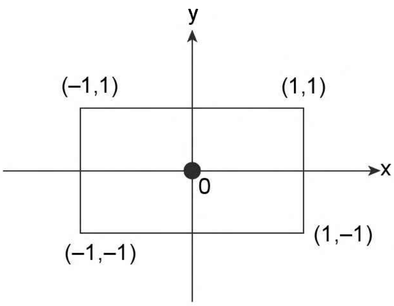


SKETCHING OF NON-STANDARD CURVES

The analysis and sketching of functions by elementary methods and transformation of standard curves has been already considered in chapter functions. Now we will carry out more profound and comprehensive study of various properties of function using the knowledge of differential calculus and explain the shape of its graph with the help of monotonicity convexity, concavity and extrema of the function.

To plot unknown curves the following investigations will be of immense help:

1. Find the domain of the function and if possible range of the function.
2. By putting $y = 0$ in the equation of the given curve, find points where the curve crosses the x-axis. Similarly by putting $x = 0$ in the equation of the given curve we can find points where the curve crosses the y-axis. i.e., the roots of the function are abscissa of point of intersection with x-axis and the value of function at $x = 0$ is the point of intersection with y-axis.
3. Test the periodicity of the function, if the function is periodic. i.e., if $f(x + T) = f(x)$ for real and finite positive constant T, then graph of $f(x)$ is repeated throughout the domain after interval of length T.

SYMMETRY/MONOTONICITY/CURVATURE OF FUNCTION

(a) $f(-x, y) = f(x, y)$, i.e., even w.r.t x, so, symmetric about y-axis. If all the powers of x in equation of the given curve are even, then it is symmetric about y-axis, e.g., $x^2 = 4ay$ is symmetric about y-axis.

(b) $f(x, -y) = f(x, y)$, i.e., even w.r.t y, so, symmetric about x-axis. If all the powers of y in equation of the given curve are even, then it is symmetric about x-axis. That is, the shape of the curve above x-axis, is exactly identical to its shape below x-axis e.g., $y^2 = 4ax$ is symmetric about x-axis.

(c) $f(-x, -y) = f(x, y)$, i.e., symmetry about origin. If by putting $-x$ for x and $-y$ for y the equation of curve remains same, then it is symmetric in opposite quadrants. e.g., $xy = c^2$, $x^2 + y^2 = a^2$.

(d) $f(x, y) = f(y, x)$ i.e., symmetry about line $y = x$. If the equation of a given curve remains unaltered by interchanging x and y then it is symmetric about the line $y = x$ which passes through the origin and makes an angle of 45° with positive direction of x-axis.

(e) $f(a - x) = f(a + x) \Rightarrow$ symmetry about line $x = a$.

(f) $f(x) = f(2a - x) \Rightarrow$ symmetry about line $x = a$.

5. Test the function for continuity, find out the discontinuities and their character.
6. Test the monotonicity of the function, i.e., find the points at which $\frac{dy}{dx} = 0$. At these points the tangent to the curve is parallel to x-axis. Find the interval in which $\frac{dy}{dx} > 0$. In this interval, the function is monotonically increasing and the interval in which $\frac{dy}{dx} < 0$. In this interval, the function is monotonically decreasing. Put $\frac{dy}{dx^2} = 0$ and check the sign of $\frac{d^2y}{dx^2}$ at the

points so obtained to find the points of maxima and minima.

If f is a continuous function over the interval (a, b) then local maxima or minima, if they exists, must occur at values of x, called critical values, such that $f'(x) = 0$ or $f'(x)$ does not exists (i.e., not defined).

7. Find the second derivative of the function and test the curvature of function, i.e., point at which the function is concave up $\frac{d^2y}{dx^2} > 0$ and concave down $\frac{d^2y}{dx^2} < 0$.

8. Find the points of inflexion on its graph of the function, compute the values of the function and of its derivative at these points. Find the intervals of convexity/concavity of the graph of the function.

9. Find the limit of function at the end point of the domain (if domain is open interval) and find the asymptotes of the function.

10. If the point (0, 0) satisfies the equation of the curve, then it passes through the origin and in such case to find the equations of the tangents at the origin, equate the lowest degree term to zero. e.g., $y^2 = 4ax$ passes through the origin. The lowest degree term in the equation is $4ax$. Equating $4ax$ to zero, we get $x = 0$. So, $x = 0$, i.e., y-axis is tangent at the origin to $y^2 = 4ax$.

11. Find the value of y in terms of x from the equation of the curve and find the value of x for which y is imaginary. Similarly, find the value of x in terms of y and determine the values of y for which x is imaginary. The curve does not exists for these values of x and y.

12. Graph the function using the results of this investigation. If it is necessary to specify certain regions of the curve, calculate the coordinates of several additional points (in particular, x-intercepts and y-intercepts). With the above information the function can be sketched conveniently.

ILLUSTRATION 77: Graph the function: $y = x^3 - 5x^2 + 4x$

SOLUTION: **Step 1: Factories the function, if possible.**

$y \neq x^3 - 5x^2 + 4x = x\,(x^2 - 5x + 4) = x\,(x^2 - x + 4 - 4x) = x\,(x - 1)\,(x - 4)$

Step 2: $y = 0 \Rightarrow x\,(x - 1)\,(x - 4) = 0 \;\Rightarrow\; x = 0$ or $x = 1$ or $x = 4$

$\Rightarrow$ The function intersects the x-axis at three points, i.e., $x = 0$, $x = 1$ and $x = 4$.

Step 3: Find points of local maxima/minima $y' = 3x^2 - 10x + 4$.

$y' = 0$ for points of local maxima/minima

$\Rightarrow\; 3x^2 - 10x + 4 = 0$

$$\Rightarrow\; x = \frac{10 \pm \sqrt{100 - 48}}{6} = \frac{10 \pm 7.2}{6}$$

$\Rightarrow\; 2.86$ or 0.47

$$\Rightarrow\; \frac{d^2y}{dx^2} = 6x - 10$$

$\Rightarrow\; f''(2.86) = +\text{ve}$

$\Rightarrow\; x = 2.86$ is point of local minima and local minimum value is $f(2.86) = -5.86$

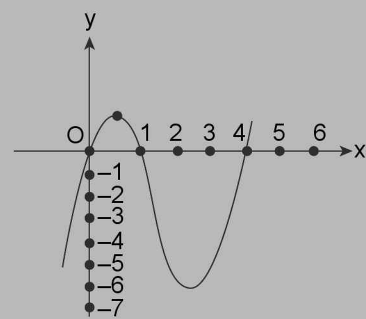


FIGURE 3.305

$\Rightarrow\; f''(0.47) = -\text{ve}$

$\Rightarrow\; x = 0.47$ is a point of local maxima and local maximum value is $f(0.47) = 0.88$

Step 4: $\lim\limits_{x \to -\infty} y = \lim\limits_{x \to \pm\infty} x^3 \left(1 - \frac{1}{x}\right)\left(\frac{1-4}{x}\right) = \pm\infty$. Thus, from the above information the graph of the function can be drawn as follows:

ILLUSTRATION 78: Sketch $y = (x - 1)\,(x - 2)$

SOLUTION: Here $y = (x - 1)\,(x - 2)$

(i) Put $y = 0 \Rightarrow x = 1, 2$ (ii) $y = x^2 - 3x + 2$

$\Rightarrow \dfrac{dy}{dx} = 2x - 3$ and $\dfrac{d^2y}{dx^2} = 2$ $\quad\therefore$ minimum at $x = 3/2$ (as $\dfrac{d^2y}{dx^2} > 0$)

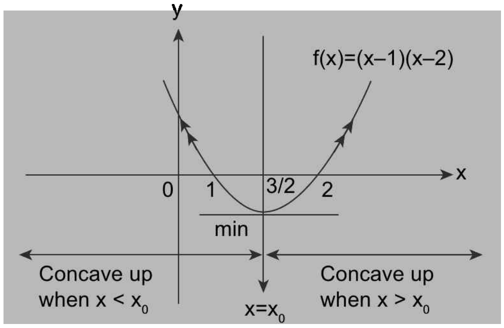


FIGURE 3.306

(iii) Increase when $x > 3/2$ and decreases when $x < 3/2$

(iv) Concave upwards for $x > 3/2$ or $x < 3/2$

ILLUSTRATION 79: Sketch the graph for the function: $y = |x + 3|\,(x + 1)$.

SOLUTION: Here, $y = |x + 3|\,(x + 1) = \begin{cases} (x+3)(x+1); & x \geq -3 \\ -(x+3)(x+1); & x < -3 \end{cases}$

$\Rightarrow x = -1, -3$ when $y = 0$ ...(i)

Also $\dfrac{dy}{dx} = \begin{cases} 2x+4; & x \geq -3 \\ -2x-4; & x < -3 \end{cases}$ and $\dfrac{d^2y}{dx^2} = \begin{cases} 2; & x \geq -3 \\ -2; & x < -3 \end{cases}$

Increasing when $x < -3$ or $x > -2$ and Decreasing when $-3 < x < -2$

Local maximum at $x = -3$ and local minimum at $x = -2$

Concave down when $x < -3$ and concave up when $x > -3$

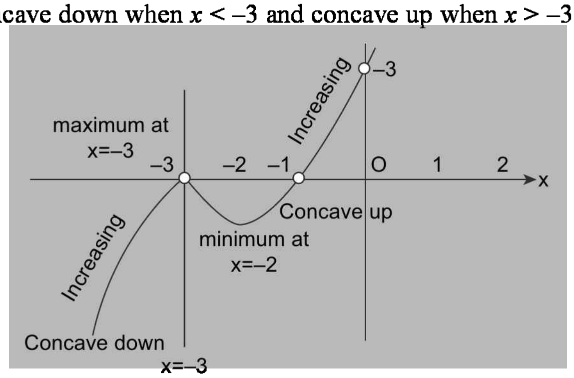


FIGURE 3.307

ILLUSTRATION 80: Sketch the graph for $f(x) = \dfrac{x+1}{x^2+3}$

SOLUTION: Here $y = \dfrac{x+1}{x^2+3}$

$\Rightarrow x = -1$, when $y = 0$

$y = 1/3$, when $x = 0$

$$\frac{dy}{dx} = \frac{-x^2 - 2x + 3}{(x^2 + 3)^2} = \frac{-(x+3)(x-1)}{(x^2+3)^2}$$

$\Rightarrow$ Increasing when, $-3 < x < 1$

Decreasing when, $x < -3$ or $x > 1$

Also $\dfrac{d^2y}{dx^2} = \dfrac{2(x^3 + 3x^2 - 9x - 3)}{(x^2+3)^3}$

Minimum at $x = -3$ as; $\dfrac{d^2y}{dx^2} = \dfrac{1}{36} > 0$ and Maximum at $x = 1$ as; $\dfrac{d^2y}{dx^2} = -\dfrac{1}{4} < 0$

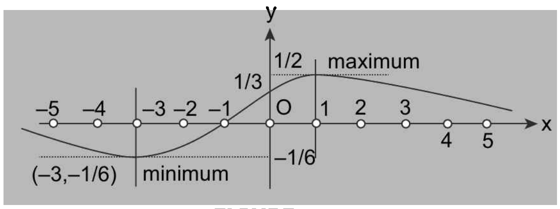


FIGURE 3.308

ILLUSTRATION 81: Plot the curves of the following functions and also indicate their extrema.

(a) $f(x) = \dfrac{3}{4}x^4 - x^3 - 9x^2 + 7$ (b) $f(x) = x(x+1)^3(x-3)^2$

SOLUTION: (a) The given function is a polynomial function, and hence, it is defined and differentiable over the entire number scale. Therefore, the critical points can be obtained by equating the derivative $f'(x) = 3x^3 - 3x^2 - 18x = 3x(x+2)(x-3)$ to zero. We find the critical points to be $x_1 = -2$, $x_2 = 0$, $x_3 = 3$ (they should always be arranged in an increasing order). Let us now investigate the sign of the derivative in the neighbourhood of each of these points. Since there are no critical points to the left of the point $x = -2$, the derivative at all the points $x < -2$ has one and the same sign; it is negative. Analogously, in the interval $(-2, 0)$ the derivative is positive, in the interval $(0, 3)$ it is negative, at $x > 3$ it is positive. Hence, at the points $x_1 = -2$ and $x_3 = 3$. We have minima $f(-2) = -9$ and $f(3) = -40\frac{1}{4}$, and at the point $x_2 = 0$, maximum $= f(0) = 7$. Hence, the graph will be as given in Figure 3.309.

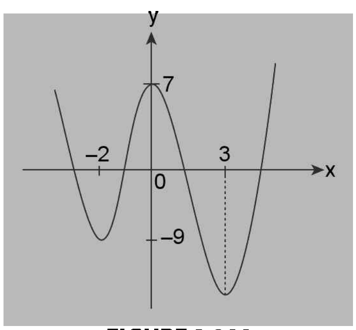


FIGURE 3.309

(b) The critical points are the roots of the derivative $f'(x)$, since the function is defined and differentiable throughout the number scale.

$$f'(x) = (x+1)^3(x-3)^2 + 3x(x+1)^2(x-3)^2 + 2x(x+1)^3(x-3)$$
$$= 3(x+1)^2(x-3)(2x^2-3x-1)$$

Equating this expression to zero, we find the critical points.

$$x_1 = -1,\ x_2 = \frac{\left(3-\sqrt{17}\right)}{4},\ x_3 = \frac{\left(3+\sqrt{17}\right)}{4},\ x_4 = 3$$

Let us tabulate the signs of the derivative in the intervals between the critical points.

Intervals	$x < x_1$	$x_1 < x < x_2$	$x_2 < x < x_3$	$x_3 < x < x_4$	$x_4 < x$
Sign of $f'(x)$	–	–	+	–	+

As is seen from the table, there is no extremum at the point $x_1 = -1$, there is a minimum at the point $x = \frac{3-\sqrt{17}}{4}$, a maximum at the point $x = \frac{3+\sqrt{17}}{4}$ and a minimum at the point $x = 3$.

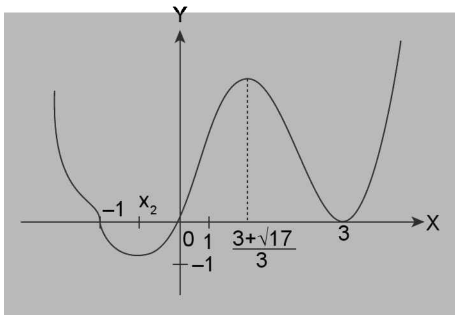


FIGURE 3.310

ILLUSTRATION 82: Investigate and graph the following functions $y = \sqrt[3]{x} - \sqrt[3]{x+1}$.

SOLUTION: The function is defined and continuous over the entire number scale and is negative everywhere, since $\sqrt[3]{x} < \sqrt[3]{x+1}$.

The graph has neither vertical, nor inclined asymptotes, since the order of magnitude of y is less than unity as $x \to \infty$. Determine the horizontal asymptote:

$$\lim_{x\to\pm\infty} y = \lim_{x\to\pm\infty}\left(\sqrt[3]{x} - \sqrt[3]{x+1}\right) = \lim_{x\to\pm\infty}\frac{-1}{\sqrt[3]{x^2} + \sqrt[3]{x(x+1)} + \sqrt[3]{(x+1)^2}} = 0$$

Hence, the straight line $y = 0$ is the horizontal asymptote of the graph.

The first derivative $y' = \frac{1}{3\sqrt[3]{x^2}} - \frac{1}{3\sqrt[3]{(x+1)^2}} = \frac{\sqrt[3]{(x+1)^2} - \sqrt[3]{x^2}}{3\sqrt[3]{x^2(x+1)^2}}$ becomes zero at the point $x_1 = -\frac{1}{2}$ and infinite at the points $x_2 = -1$, $x_3 = 0$.

The second derivative $y'' = \frac{1}{3}\left(-\frac{2}{3}\right)\frac{1}{\sqrt[3]{x^5}} - \frac{1}{3}\left(-\frac{2}{3}\right)\frac{1}{\sqrt[3]{(x+1)^5}} = \frac{-2\left[\sqrt[3]{(x+1)^5} - \sqrt[3]{x^5}\right]}{9\sqrt[3]{[x(x+1)]^5}}$ does not vanish and is infinite at the same points $x_2 = -1$, $x_3 = 0$. Compile a table

x	-1	$\left(-1,-\frac{1}{2}\right)$	$-\frac{1}{2}$	$\left(-\frac{1}{2},0\right)$	0	$(0,\infty)$	1
y'	$-\infty$	$-$	0	$+$	∞	$+$	
y''	∞	$+$	$+\frac{16}{9\sqrt[3]{2}}$	$+$	∞	$-$	
	vertical tangent		point of minima		vertical tangent		
y							-0.26

With the aid of this table, and of the asymptote $y = 0$, construct the graph of the function as shown below.

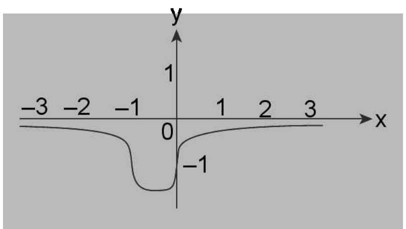


FIGURE 3.311

ILLUSTRATION 83: Investigate and graph the following functions $y = x^2e^{1/x}$:

SOLUTION: The function is defined positive and continuous on each of the intervals $(-\infty, 0)$ and $(0, \infty)$. The point $x = 0$ is a discontinuity.

Since $\lim\limits_{x\to 0^+} y = \lim\limits_{x\to 0^+} x^2 e^{\frac{1}{x}} = \lim\limits_{t\to+\infty} \frac{e^t}{t^2} = \infty$ where $\left(t = \frac{1}{x}\right)$,

The straight line $x = 0$ is a vertical asymptote. But $\lim\limits_{x\to 0^-} y = \lim\limits_{x\to 0^-} x^2 e^{\frac{1}{x}} = 0$.

There are no inclined asymptotes since the function $y = x^2e^{1/x}$ has the second order of smallness with respect to x as $x \to \pm\infty$.

Let us find the extrema of the function, for which purpose we evaluate the derivative

$y' = 2xe^{1/x} - e^{1/x} = 2e^{1/x}(x - 1/2)$; whence we find the only critical point $x = 1/2$

Since for $x \neq 0$

$$y''(x) = 2e^{1/x} - \frac{2}{x}e^{1/x} + \frac{1}{x^2}e^{1/x} = \frac{1}{x^2}e^{1/x}\left(2x^2 - 2x + 1\right) > 0$$

On each of the intervals of the domain of definition, the graph of the function is concave upwards, and at the point $x = 1/2$ the function has a minimum.

$= y\left(\frac{1}{2}\right) = \frac{1}{4}e^2 \approx 1.84; f(-1) = e^{-1} = 0.37, f(1) = e \simeq 2.72.$

From the information obtained we can sketch the graph as Figure 3.312.

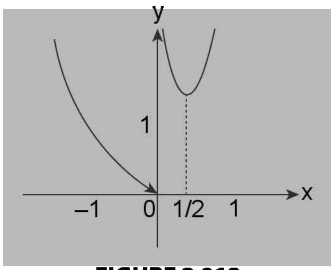


FIGURE 3.312

ILLUSTRATION 84: Investigate and graph the following functions $y = arc\ \sin\dfrac{1-x^2}{1+x^2}$.

SOLUTION: The function is defined and continuous throughout the number scale, since at any x; $\left|\dfrac{1-x^2}{1+x^2}\right| \le 1$.

Since the function is even, we may confine ourselves to the investigation of the function at $x \ge 0$.

As the function is continuous, the graph has no vertical asymptotes, but it has a horizontal asymptote.

$\lim\limits_{\to+\infty} = \text{arc}\sin(-1) = -\!-$

The first derivative $y' = \dfrac{1}{\sqrt{1-\dfrac{(1-x^2)^2}{(1+x^2)^2}}} \times \dfrac{-2x(1+x^2)-2x(1-x^2)}{(1+x^2)^2} = -\dfrac{1}{2|x|} \times \dfrac{4x}{(1+x^2)^2}$ is negative for $x > 0$, therefore, the function decreases. The derivative is non-existent at the point $x = 0$. By virtue of the symmetry of the graph about the y-axis, there will be a maximum at the point $y(0) = \dfrac{\pi}{2}$.

Notice that at the point $x = 0$ the right derivative is equal to –1, and the left one is +1

The second derivative = $y''(x) = \dfrac{4x}{(1+x^2)^2} > 0$ for all $x > 0$ and $\dfrac{-4x}{(1+x^2)^2} > 0\ \forall\, x < 0$.

Hence, in the interval (– ∞, 0) and (0, ∞) the graph of the function is concave upwards. Also note that the curve intersects with the x-axis at the points $x = \pm 1$.

Taking into consideration the results of the investigation, construct the graph of the function as is shown in Figure 3.313.

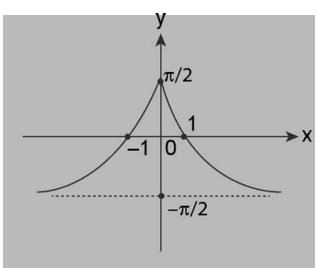


FIGURE 3.313

ILLUSTRATION 85: Draw the graph of $ex \ln (ex)$, where $x > 0$

SOLUTION: **Step 1:** $y = ex \ln ex = 0$

$\Rightarrow x = \frac{1}{e}$ is the roots of function $ex (\ln ex)$

Step 2: $y' = e\left\{x\frac{1}{ex}.e + \ln ex\right\} = 0$

$\Rightarrow e(1 + \ln ex) = 0 \Rightarrow \ln ex = -1 \Rightarrow ex = e^{-1} \Rightarrow x = \frac{1}{e^2}$

Next, $y'' = \frac{e}{ex}.e = \frac{e}{x} \Rightarrow y'' > 0 \ \forall \ x > 0$

$\Rightarrow f(x)$ has minima at — and $y_{\min} = e.\frac{1}{e^2}\ln e\frac{1}{e^2} = \frac{1}{e}(-1) = -\frac{1}{e}$

Further $y' > 0 \ \forall \ x > \frac{1}{e^2} \Rightarrow f(x)$ increases and $y' < 0 \ \forall \ 0 < x < \frac{1}{e^2} \Rightarrow f(x)$ decreases.

Step 3: (a) $\lim\limits_{x\to\infty} ex \ln ex = \infty$

(b) $\lim\limits_{x\to 0} ex \ln ex = \lim\limits_{x\to 0} e\left[\frac{1+\ln x}{\frac{1}{x}}\right]\left(\frac{\infty}{\infty}\text{form}\right) = 0 = \lim\limits_{x\to\infty}\frac{\left(\frac{h}{x}\right)}{\left(-\frac{1}{x^2}\right)} = 0$

With these information we can draw the graph of $ex \ln (ex)$ as follows:

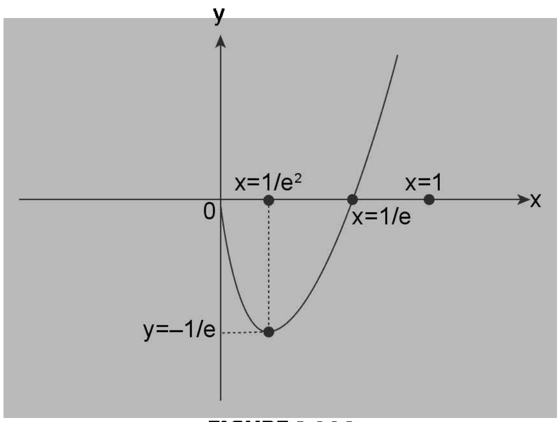


FIGURE 3.314

ILLUSTRATION 86: Draw the graph of $f(x) = \frac{\ln(ex)}{ex}$, where $x > 0$

SOLUTION: **Step 1:** Domain of the $f(x)$ is $(0, \infty)$

$f(x) = 0 = \frac{\ln ex}{ex}$

$\Rightarrow$ Root of equation is $1/e$

Step 2: $f'(x) = \frac{(1 - \ln ex)}{ex^2} = 0$

$\Rightarrow x = 1$ is the critical point.

Clearly, $f'(x) > 0$ when $x < 1$ and $f'(x) < 0$ when $x > 1$

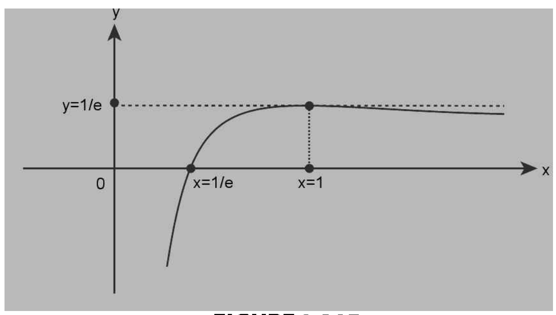


FIGURE 3.315

So, the graph of $f(x)$ will be as shown figure:

Hence, $x = 1$ is point of maxima and the maximum value of $f(x) = \frac{1}{e}$

Step 3: $\lim_{x\to\infty} \frac{\ln ex}{ex} = \lim_{x\to\infty} \frac{\frac{e}{ex}}{e} = 0$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\ln ex}{ex} = -\infty$

TEXTUAL EXERCISE–6: (SUBJECTIVE)

1. Draw the graph of the following functions:

(a) $y = \frac{2}{x^2+1}$ (b) $y = \frac{2}{x^2-4}$

(c) $y = \frac{2x}{x^2-1}$ (d) $y = \frac{x^2+2}{x^2-4}$

2. Sketch the curve

(a) $y = (1 - x^{2/3})^{3/2}$

(b) $y = (x - 1)\, x^{2/3}$

(c) $x^{2/3} + y^{2/3} = a^{2/3}$

3. Sketch the curves

(i) $y = \frac{e^x + e^{-x}}{2}$ (ii) $y = \frac{e^x - e^{-x}}{2}$

(iii) $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ (iv) $y = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

4. Sketch the curves

(a) $y = 2 \sin x + \cos 2x$

(b) $y = e^{-x^2}$ (Gaussian curve)

(c) $y^2 = x^3$ (Semi-cubical curve)

(d) $y = x^4 - x^6$

(e) $y = \frac{\sin 2x}{2} + \cos x$.

5. Sketch the curves

(a) $y^2 = (x - 1)(x - 2)(x - 3)$.

(b) $y = 1 + x^2 - \frac{1}{2} x^4$

(c) $y = (x + 1)(x - 2)^2$

(d) $y = \frac{a^2 x}{a^2 + x^2}$

(e) $y = ex \log ex$

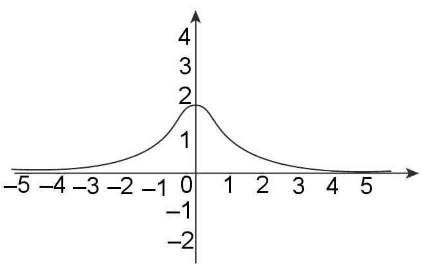
(f) $y = \frac{\log ex}{ex}$

(g) $y = x^x$

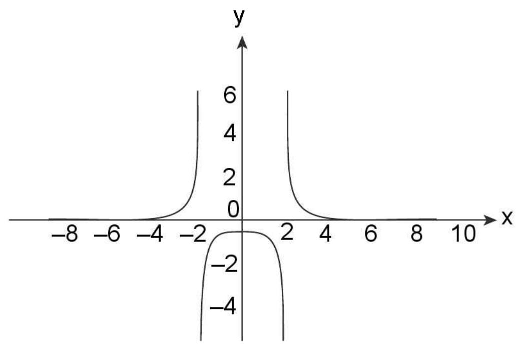
(h) $y^2 x^2 = (x - 1)(x - 2)(x - 3)$

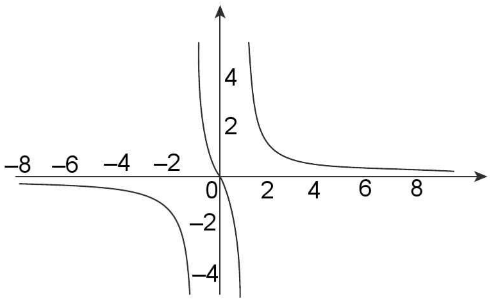
Answer Keys

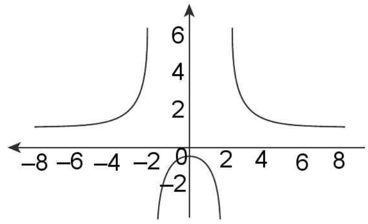
1. (a)

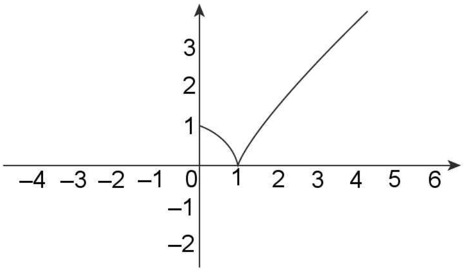


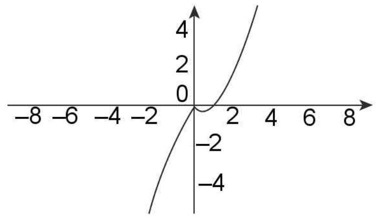
(b)

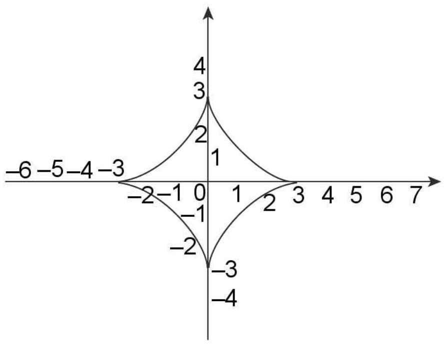


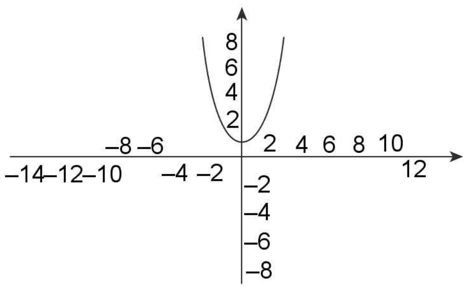
(c) 

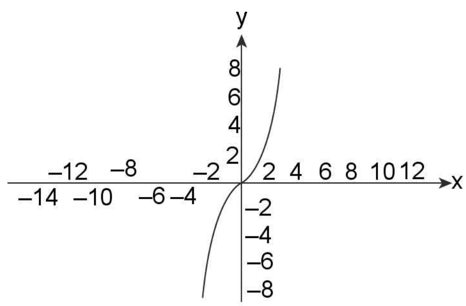
(d) 

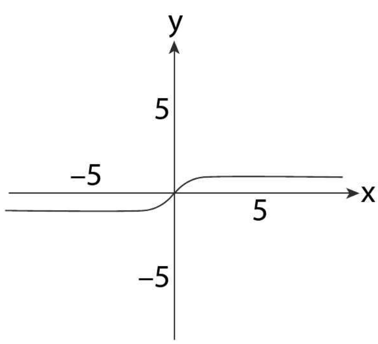
2. (a) 

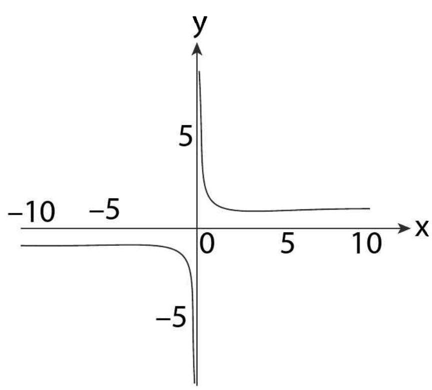
(b) 

(c) 

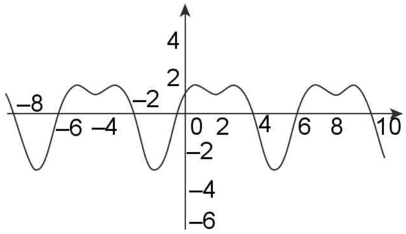
3. (a) 

(b) 

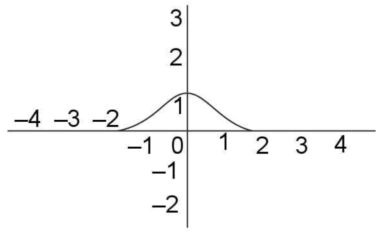
(c) 

(d) 

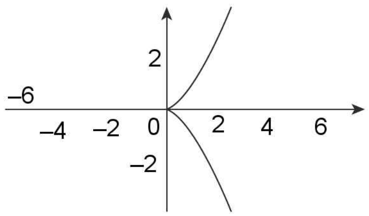
4. (a)



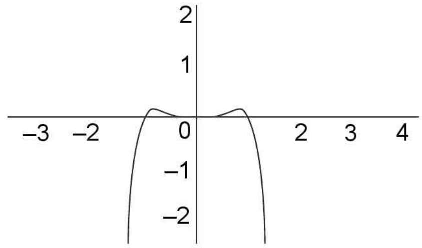
(b)



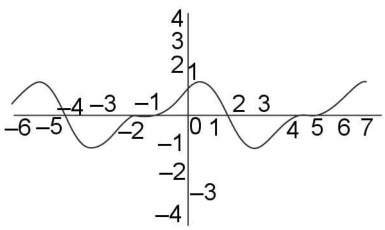
(c)



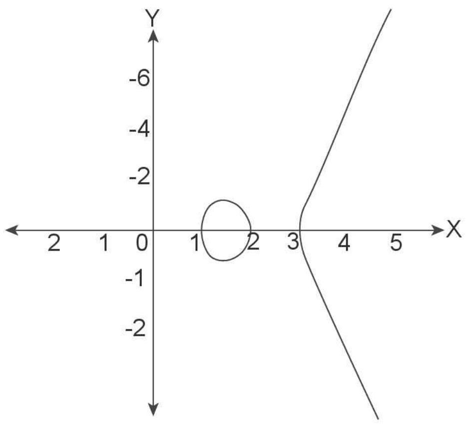
(d)



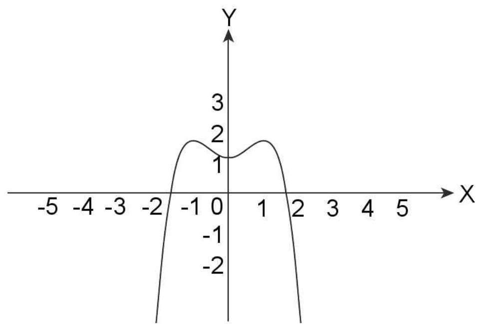
(e)



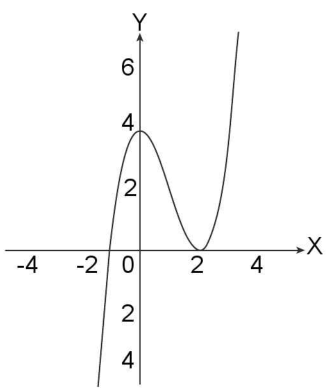
5. (a)



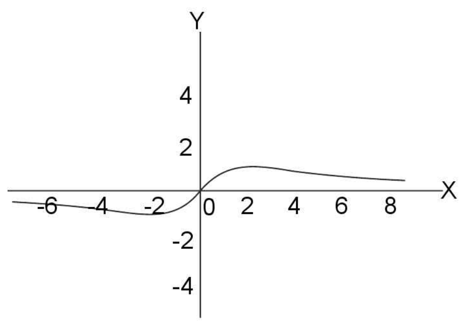
(b)

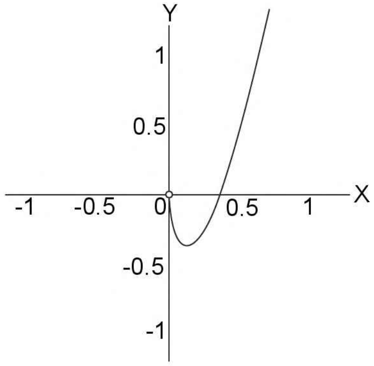


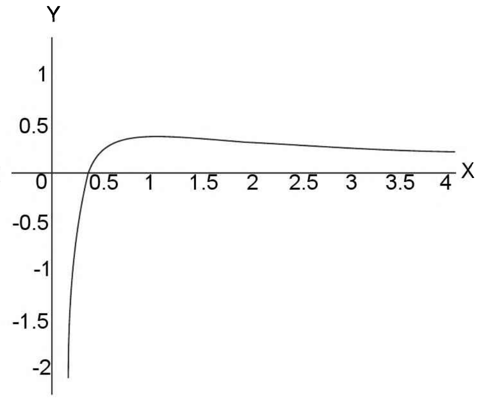
(c)

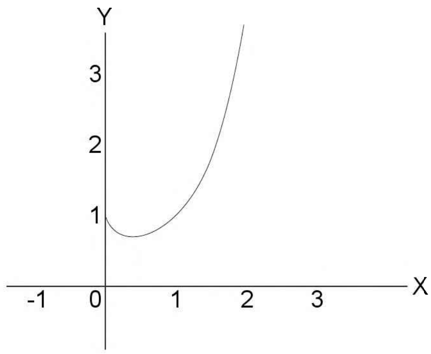


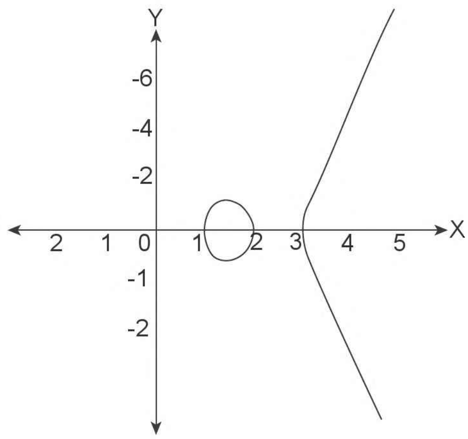
(d)



(e)


(f)


(g)


(h)


ASYMPTOTES

So far in this topic, we are done with all basic curves and various geometric transformations over them. As of now we have also learned tracing of simple curves having expression as an explicit function of x. But there are some known curves $f(x, y) = 0$, with implicit nature as well, e.g., $f(x, y) = b^2x^2 + a^2y^2 - a^2b^2 = 0$; $g(x, y) = b^2x^2 - a^2y^2 - a^2b^2 = 0$ or $h(x, y) = x^4y^2 + 2x^2y^2 + y^2 - 1 = 0$. Indeed these curves have two branches for instance $f(x,y)\colon\ y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$,

$g(x,y)\colon\ y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$ and $h(x,y)\colon y = \pm\frac{1}{1+x^2}$.

More generally speaking an equation $F(x, y) = 0$ can be solved for y in terms of x so as to generate $y = f_k(x)$; where $k = 1, 2, \ldots, n$. These n functions obtained are termed as n branches of curve $F(x, y) = 0$.

Consider any branch say $y = f(x)$, if both domain (D_f) and range (R_f) are finite sets, then it is called finite branch function, for instance both branches of ellipse $f(x, y) = 0$ and if atleast one of D_f or R_f comes out to be infinite set, then it is known as infinite branch function, e.g., both the branches $g(x,y)\colon\ y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$ and $h(x,y)\colon y = \pm\frac{1}{1+x^2}$. It is so because as a point $P(x, y)$ on an infinite branch curve moves away from origin either x or y or both tend to infinity.

Definition

A straight line at a finite distance from origin is said to be an asymptote (Rectilinear asymptote) of an infinite branch function $y = f(x)$ of some curve, if the perpendicular distance of any variable point $P(x, y)$ on the curve from the line tends to zero when x or y or both tend to infinity (i.e., it tends to meet the given curve at infinity) as shown in Figures 3.316 and 3.317.

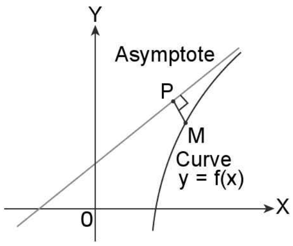


FIGURE 3.316

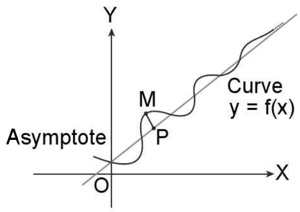


FIGURE 3.317

Mathematically

Let $y = f(x)$ be a curve and let (x, y) be a point on it.

Tangent at (x, y) is given by $Y - y = \dfrac{dy}{dx}(X - x)$

$$\Rightarrow \quad Y = \frac{dy}{dx}.X + \left(y - x\frac{dy}{dx}\right) \qquad \ldots(i)$$

Now , if asymptotes exists, then $x \to \infty$

$$\Rightarrow \quad \frac{dy}{dx} \text{ and } \left(y - x\frac{dy}{dx}\right) \to \text{finite limit say } m \text{ and } c.$$

Say, $\dfrac{dy}{dx} \to m$ and $y - x\dfrac{dy}{dx} \to c$

$\therefore$ Equation (i) reduces to $y = mx + c$ is an asymptote of equation.

In Figure (3.318) asymptote is parallel to x-axis and is called a horizontal asymptote

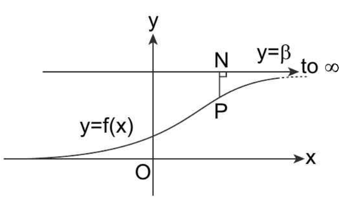


FIGURE 3.318

In Figure (3.319) asymptote is parallel to y-axis and is called a vertical asymptote.

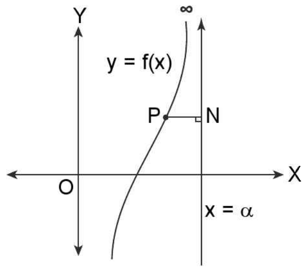


FIGURE 3.319

If it is neither parallel to x-axis nor to y-axis, then it is called as oblique asymptote.

For instance the straight line $y = 0$ is a horizontal asymptote to the curve $y = \dfrac{1}{1 + x^2}$

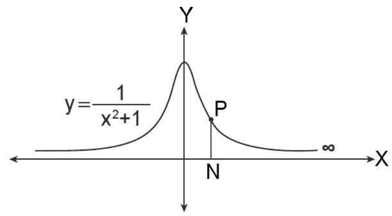


FIGURE 3.320

The lines $y = \pm\dfrac{b}{a}x$ are pair of asymptotes of hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

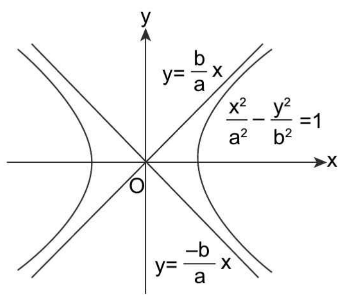


FIGURE 3.321

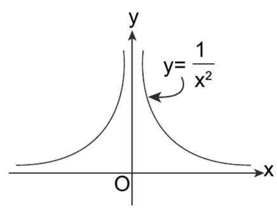


FIGURE 3.322

The line $x = 0$ is vertical asymptote and $y = 0$ is horizontal asymptote of $y = \dfrac{1}{x^2}$ $y = 0$; π are asymptotes of $y = \cot^{-1}x$.

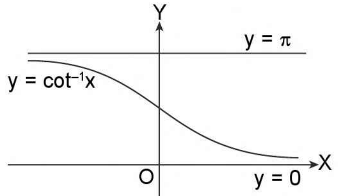


FIGURE 3.323

HORIZONTAL ASYMPTOTE (ASYMPTOTES PARALLEL TO X-AXIS)

Consider $y = \beta$ be a line parallel to the x-axis, and P be any point on the curve $f(x, y) = 0$, also PN be the perpendicular from P on $y = \beta$. Then $PN = |y - \beta|$. If $PN \to 0$ as P tends to infinity along the curve, then $y = \beta$ is an asymptote. This means that $y = \beta$ is an asymptote provided that, as P tends to infinity along the curve, $y \to \beta$ and $x \to \infty$ (for at least one co-ordinate of P must tend to infinity in order that P may tends to infinity). This means that in order to find the asymptotes parallel to the x-axis, we have to look for those real numbers 'β' such that $\lim\limits_{x\to\infty} f(x, \beta) = 0$.

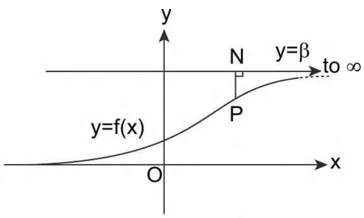


FIGURE 3.324

Let $y = f(x)$ be a given curve, then horizontal asymptote is/are given by $y = \underset{x\to\infty}{Lt} f(x)$ and $y = \underset{x\to-\infty}{Lt} f(x)$.

For example: Consider the function $f(x) = \dfrac{3x-5}{\sqrt{2x^2+3}}$

Then $\underset{x\to\infty}{\text{Lt}} f(x) = \underset{x\to\infty}{\text{Lt}} \dfrac{3x-5}{\sqrt{2x^2+3}} = \underset{x\to\infty}{\text{Lt}} \dfrac{\frac{3x}{x} - \frac{5}{x}}{\sqrt{2+\frac{3}{x^2}}}$

(dividing each numerator and denominator by x)

$$= \underset{x\to\infty}{\text{Lt}} \frac{3-\frac{5}{x}}{\sqrt{2+\frac{3}{x^2}}} = \frac{3}{\sqrt{2}} \text{ and } \underset{x\to-\infty}{\text{Lt}} f(x) = \underset{x\to-\infty}{\text{Lt}} \frac{3x-5}{\sqrt{2x^2+3}}$$

$$= \underset{x\to-\infty}{\text{Lt}} \frac{3x-5}{\sqrt{x^2}\sqrt{2+3/x^2}}.$$

$$= \underset{x\to-\infty}{\text{Lt}} \frac{3x-5}{|x|\sqrt{2+3/x^2}} \qquad \left(\begin{array}{l}\because x<0 \\ \Rightarrow \sqrt{x^2} = |x| = -x\end{array}\right)$$

$$= \underset{x\to-\infty}{\text{Lt}} \frac{3-5/x}{-\sqrt{2+3/x^2}} = -\frac{3}{\sqrt{2}}$$

The two horizontal asymptotes are $y = \dfrac{3}{\sqrt{2}}$ and $y = -\dfrac{3}{\sqrt{2}}$. It is as shown in Figure 3.325.

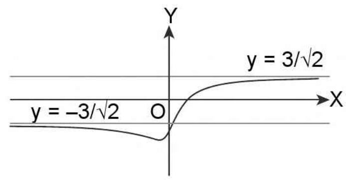


FIGURE 3.325

PROCEDURE TO FIND HORIZONTAL ASYMPTOTE TO ALGEBRAIC CURVE

Let $F(x, y) = 0$ be a rational algebraic curve of degree n. Arranging $F(x, y)$ in descending powers of x, the equation $F(x, y) = 0$ can be written as

$$x^p g_0(y) + x^{p-1} g_1(y) + x^{p-2} g_2(y) + \ldots = 0 \qquad \ldots(1)$$

where $p \le n$, and $g_0(y), g_1(y), g_2(y), \ldots$ are polynomials in y.

Dividing (1) throughout by x^p, i.e., highest power term of x, the equation can be re-written as

$$g_0(y) + (1/x)g_1(y) + (1/x^2)g_2(y) + \ldots = 0 \qquad \ldots(2)$$

Taking limits as $x \to \infty$, and replacing $\lim y = \beta$, yields $g_0(\beta) = 0$ so that β is a root of $g_0(\beta) = 0$

If $\beta_1, \beta_2, \beta_3, \ldots$ are the roots of $g_0(\beta) = 0$, then $y = \beta_1$, $y = \beta_2$, $y = \beta_3$, ... are the asymptotes parallel to the x-axis. Therefore, the joint equation of the asymptotes parallel to the x-axis is

$(y - \beta_1)(y - \beta_2)(y - \beta_3) \ldots = 0$

i.e., $g_0(y) = 0$. Consequently since $g_0(y)$is the coefficient of the highest power of x in $F(x, y)$, we have the following rule for finding asymptotes parallel to the x-axis.

Algorithm

1. If the coefficient of term containing highest power of x is constant, then there will be no asymptote parallel to x-axis.
2. For finding the asymptotes parallel to x-axis put the coefficient of term containing highest power of x equal to zero.

ILLUSTRATION 87: Find the horizontal asymptotes to curves

(i) $x^2y^2 + xy - 4x^2 = 0$ (ii) $x^3y^2 - x^2y^3 - 3yx^3 + 2x^3 = 0$

SOLUTION: (i) Given equation is $x^2y^2 + xy - 4x^2 = 0$ or $x^2(y^2 - 4) + xy = 0$

Term containing the highest power of x is $x^2(y^2 - 4)$ and coefficient of x^2 is $y^2 - 4$

Putting $y^2 - 4 = 0$, we get, $y + 2 = 0$ and $y - 2 = 0$

$\therefore$ $y = -2$ and $y = 2$ are two horizontal asymptotes to given curve.

(ii) Given equation is $x^3y^2 - x^2y^3 - 3yx^3 + 2x^3 = 0$ or $x^3(y^2 - 3y + 2) - x^2y^3 = 0$

Equating the coefficient of term containing the highest power of x, i.e., of x^3 equal to zero, we get $y^2 - 3y + 2 = 0$ or $(y - 2)(y - 1) = 0$

$\Rightarrow$ $y = 2$ and $y = 1$ are two horizontal asymptotes to the given curve.

VERTICAL ASYMPTOTES (ASYMPTOTES PARALLEL TO Y-AXIS)

Consider $x = \alpha$ be a line parallel to the y-axis, and P be any point on the curve $F(x, y) = 0$, also PN be the perpendicular from P on $x = \alpha$. Then $PN = |x - \alpha|$. If $PN \to 0$ as P tends to infinity along the curve, then $x = \alpha$ is an asymptote. This means that $x = \alpha$ is an asymptote provided that as P tends to infinity along the curve, $x \to \alpha$, and $y \to \infty$ (for at least one co-ordinate of P must tend to infinity in order that P may tends to infinity). This means that in order to find the asymptotes parallel to the y-axis, we have to look for those real numbers α which are such that $\lim\limits_{y\to\infty} F(\alpha, y) = 0$.

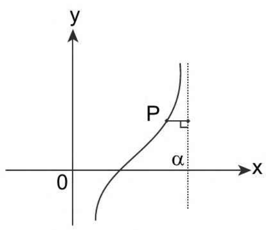


FIGURE 3.326

To find the vertical asymptotes of the infinite branch function $y = f(x)$ of the curve $F(x, y) = 0$, express x in terms of y, i.e., $x = g(y)$ and vertical asymptotes can be obtained by applying limit $y \to \infty$ and $y \to -\infty$ on $g(y)$, i.e., $x = \operatorname*{Lt}\limits_{y\to-\infty} g(y)$ and $x = \operatorname*{Lt}\limits_{y\to\infty} g(y)$ give us vertical asymptotes. Any curve $y = f(x)$ can have any number (or infinite number) of vertical asymptotes, as a function can be many one, and hence, $y \to \pm\infty$ can occur at infinitely many inputs x.

e.g., $y = \tan x$ has infinitely many vertical asymptotes at $x = (2n+1)\frac{\pi}{2}$; $n \in \mathbb{Z}$ as shown Figure 3.327.

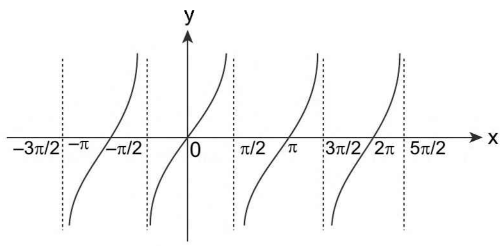


FIGURE 3.327

ILLUSTRATION 88: Find the horizontal and vertical asymptotes to curve.

(i) $y = \dfrac{1}{x-5}$ (ii) $y = \dfrac{3x-4}{x+2}$ (iii) $y = \dfrac{1}{x(x-2)}$

SOLUTION: (i) $y = \dfrac{1}{(x-5)}$; clearly $y = \operatorname*{Lt}\limits_{x\to\pm\infty}\left(\dfrac{1}{x-5}\right) = 0$ is the only horizontal asymptote, and $y > 0$ for $x > 5$ and $y < 0$ for $x < 5$.

Now, $- = - \Rightarrow x = 5 + \dfrac{1}{y}$

$\therefore$ Vertical asymptotes are given by $x = \underset{y\to\infty}{\text{Lt}}\left(5+\frac{1}{y}\right)$ and $x = \underset{y\to-\infty}{\text{Lt}}\left(5+\frac{1}{y}\right)$.

i.e., $x = 5$ and $x > 5$ when $y \to \infty$; $x < 5$ when $y \to -\infty$. Graphically as shown in Figure 3.328.

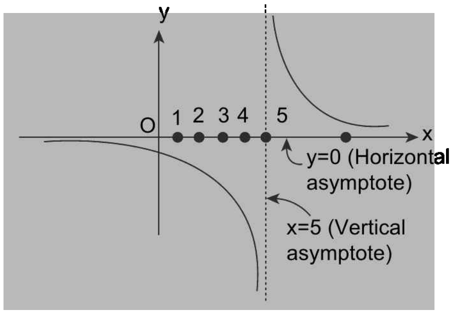


FIGURE 3.328

(ii) $y = \dfrac{3x-4}{x+2}$, horizontal asymptotes are given by $y = \underset{x\to\pm\infty}{\text{Lt}}\left(\dfrac{3x-4}{x+2}\right) = \underset{x\to\pm\infty}{\text{Lt}}\left(\dfrac{3-4/x}{1+2/x}\right) = 3$

$\Rightarrow$ $y = 3$ is the only horizontal asymptote and $y > 0 \Rightarrow (3x-4)(x+2) > 0$

$\Rightarrow$ $x \in (-\infty,-2) \cup \left(\dfrac{4}{3},\infty\right)$ and $y < 0 \Rightarrow x \in \left(-2,\dfrac{4}{3}\right)$

Now, $y \to \infty$ for $x + 2 \to 0 \Rightarrow x \to -2$

$\therefore$ $x = -2$ is the only vertical asymptote, when $x \to -2^-$; $y \to +\infty$ and when $x \to -2^+$; $y \to -\infty$

Graphically shown in Figure 3.329.

(iii) $y = \dfrac{1}{x(x-2)}$. For horizontal asymptotes, $y = \underset{x\to\pm\infty}{\text{Lt}} \dfrac{1}{x(x-2)} = 0$

$\therefore$ $y = 0$ is the only horizontal asymptote and $y < 0$ for $x \in (0, 2)$.

And $y > 0$ for $x \in (-\infty, 0) \cup (2, \infty)$.

$\therefore$ $y \to 0^+$ for $x \to \pm\infty$. Also $y \to -\infty$ as $x \to 0^+$ and $y \to \infty$ as $x \to 0^-$

and $y \to \infty$ as $x \to 2^+$ and $y \to -\infty$ as $x \to 2^-$

$\therefore$ $x = 0$ and $x = 2$ are to vertical asymptotes. Graphically shown in Figure 3.330.

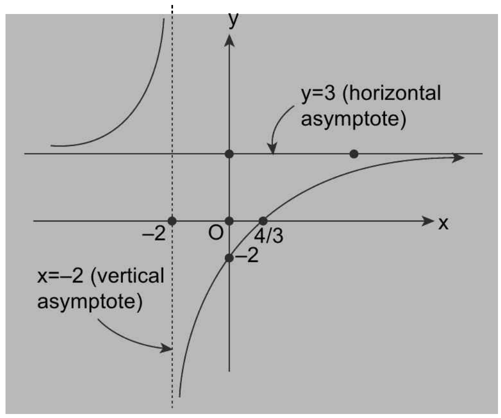


FIGURE 3.329

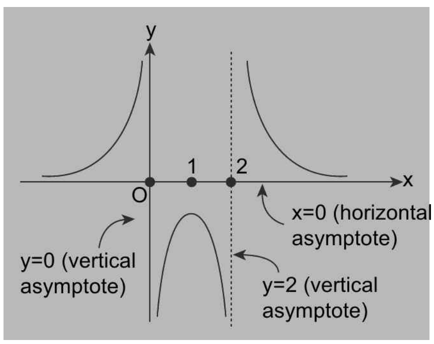


FIGURE 3.330

PROCEDURE TO FIND VERTICAL ASYMPTOTE

Let $f(x, y) = 0$ be a rational algebraic curve of degree n. Arranging $F(x, y)$ in descending powers of y, the equation $F(x, y) = 0$ can be written as

$$y^p f_0(x) + y^{p-1} f_1(x) + y^{p-2} f_2(x) + \ldots = 0 \qquad \ldots(1)$$

where $p \le n$ and $f_0(x), f_1(x), f_2(x), \ldots$ are polynomials in x.

Dividing (1) throughout by y^p, i.e., highest power term of y, the equation can be re-written as

$$f_0(x) + (1/y) f_1(x) + (1/y\ ) f_2(x) + \ldots. = 0 \qquad \ldots(2)$$

Taking limits as $y \to \infty$, and replacing lim $x = \alpha$, yields $f_0(\alpha) = 0$ so that α is a root of $f_0(x) = 0$.

If $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots.$ are the root of $f_0(x) = 0$, then $x = \alpha_1, x = \alpha_2, x = \alpha_3 \ldots\ldots$ are the asymptotes parallel to the x-axis, Therefore, the joint equation of the asymptotes parallel to the x-axis is

$$(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)\ldots\ldots. = 0$$

i.e., $f_0(x) = 0$. Consequently since $f_0(x)$ is the co-efficient of the highest power of y in $F(y, x)$, we have the following rule for finding asymptotes parallel to the y-axis.

(i) If the coefficient of term containing highest power of y is non-zero, then there will be no asymptote parallel to y-axis.

(ii) Equating the coefficient of term containing highest power of y equal to zero gives us vertical asymptotes.

ILLUSTRATION 89: Find the vertical asymptotes to the curves

(i) $x^2y^2 - 5xy + 6y = 0$ (ii) $3y^2 - 5xy + 2y = 0$ (iii) $x^3y^2 + 3xy - y^2x = 0$

SOLUTION: (i) Given curve is $x^2y^2 - 5xy + 6y = 0$. Coefficient of highest power of y, i.e., of y^2 is x^2

$\therefore$ Vertical asymptotes are given by $x^2 = 0 \Rightarrow x = 0$ (y-axis)

(ii) Given curve is $3y^2 - 5xy + 2y = 0$. Coefficient of highest power of y (i.e., y^2) is 3, which is constant. Thus, there is no asymptote parallel to y-axis.

(iii) Given curve is $x^3y^2 + 3xy - y^2x = 0$ or $y^2(x^3 - x) + 3xy = 0$

Coefficient of highest power of y (i.e., y^2) is $x^3 - x$

$\therefore$ Vertical asymptotes are given by $x^3 - x = 0$

$\Rightarrow x(x^2 - 1) = 0 \qquad \Rightarrow x = 0, x = -1, x = 1$

$\therefore$ There are three vertical asymptotes, namely $x = 0$, $x = -1$ and $x = 1$.

OBLIQUE ASYMPTOTES

Let $y = f(x)$ be a given infinite branch function of some curve $F(x, y) = 0$ and $P(x, y)$ be any variable point on it, then equation of tangent to curve at point $P(x, y)$ will be

$$Y - y = \frac{dy}{dx}(X - x) \quad \text{or} \quad Y = y + \frac{dy}{dx}(X - x)$$

$$\text{or} \quad Y = \frac{dy}{dx}X + \left(y - x\frac{dy}{dx}\right) \qquad \ldots..(1)$$

$\because$ There exists an oblique asymptote for the given curve, as $x, y \to \infty$, slope of tangent (m) should be finite and also y-intercept should be finite

$$\Rightarrow \quad m = \underset{\substack{x\to\infty \\ y\to\infty}}{\text{Lt}} \left(\frac{dy}{dx}\right) \quad \text{and} \quad c = \underset{\substack{x\to\infty \\ y\to\infty}}{\text{Lt}} \left(y - x\frac{dy}{dx}\right)$$

$\therefore$ the straight line $y = mx + c$ obtained is indeed the required oblique asymptote.

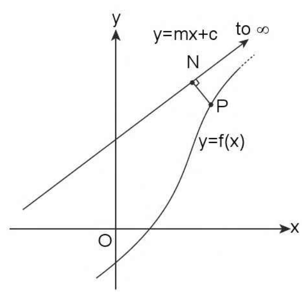


FIGURE 3.331

Procedure to Find Oblique Asymptotes

Step 1: Obtain $\frac{dy}{dx}$ and find $m = \underset{\substack{x\to\infty \\ y\to\infty}}{\text{Lt}} \left(\frac{dy}{dx}\right)$

Step 2: Also find $c = \underset{\substack{x\to\infty \\ y\to\infty}}{\text{Lt}} \left(y - x\frac{dy}{dx}\right)$

Step 3: Consequently $y = mx + c$ gives us oblique asymptote.

ILLUSTRATION 90: Find the asymptotes to the given curves:

(i) $y = 5x - \frac{2}{x}$ (ii) $2y = 3\sqrt{x^2+4}$

SOLUTION : (i) Equation of curve is $y = 5x - \frac{2}{x}$ or $xy = 5x^2 - 2$

For horizontal asymptote:

Since the coefficient of x^2, i.e., 5 is a non-zero constant, so, there is no asymptote parallel to x-axis.

For vertical asymptote:

Put coefficient of y, i.e., x equals to zero, i.e., $x = 0$ (y-axis) is the only vertical asymptote.

For oblique Asymptote:

$$y = 5x - \frac{2}{x} \qquad \Rightarrow \frac{dy}{dx} = 5 + \frac{2}{x^2}$$

$$\therefore \quad m = \underset{\substack{x\to\infty \\ y\to\infty}}{\text{Lt}} \left(\frac{dy}{dx}\right) = \underset{\substack{x\to\infty \\ y\to\infty}}{\text{Lt}} \left(5 + \frac{2}{x^2}\right) = 5 \text{ and } c = \underset{\substack{x\to\infty \\ y\to\infty}}{\text{Lt}} \left(y - x\frac{dy}{dx}\right) = \underset{\substack{x\to\infty \\ y\to\infty}}{\text{Lt}} \left(y - x\left(5 + \frac{2}{x^2}\right)\right)$$

$$\Rightarrow c = \underset{\substack{x\to\infty \\ y\to\infty}}{\text{Lt}} \left(y - 5x - \frac{2}{x}\right) = \underset{x\to\infty}{\text{Lt}} - \left(\frac{2}{x} + \frac{2}{x}\right) = \underset{x\to\infty}{\text{Lt}} \left(-\frac{4}{x}\right) = 0$$

$\therefore$ $y = 5x + 0$ or $y = 5x$

$\therefore$ The straight lines $x = 0$, $y = 5x$ are the asymptotes to the curve.

(ii) Given curve $2y = 3\sqrt{x^2+4}$ or $4y^2 = 9(x^2 + 4)$ $\Rightarrow$ $4y^2 = 9x^2 + 36$

$\Rightarrow$ There is no asymptote parallel to coordinate axis as coefficient of x^2 and y^2 are non-zero constant.

$$\text{Now, } y = \frac{3}{2}\sqrt{x^2+4} \Rightarrow \frac{dy}{dx} = \frac{3}{2}\left(\frac{1}{2}\right).\frac{1}{\sqrt{x^2+4}}(2x) \Rightarrow \frac{dy}{dx} = \frac{3x}{2\sqrt{x^2+4}}$$

$$\therefore \quad As\, x \to \infty \;\; m = \underset{x\to\infty}{\text{Lt}} \frac{3x}{2\sqrt{x^2+4}} = \underset{x\to\infty}{\text{Lt}} \frac{3}{2}\frac{x}{|x|\sqrt{1+4/x^2}} = \underset{x\to\infty}{\text{Lt}} \frac{3}{2}\frac{1}{\sqrt{1+4/x^2}} = \frac{3}{2}$$

$$\text{and } As\, x \to \infty \;\; m = \underset{x\to-\infty}{\text{Lt}} \frac{3}{2}\frac{x}{|x|\sqrt{1+4/x^2}} = \underset{x\to-\infty}{\text{Lt}} \frac{3}{2}\frac{x}{(-x)\sqrt{1+4/x^2}} = -\frac{3}{2} \text{ and}$$

$$c = \underset{\substack{y\to\infty \\ x\to\infty}}{\text{Lt}} \left(y - x\frac{dy}{dx}\right) = \underset{\substack{x\to\infty \\ y\to\infty}}{\text{Lt}} \frac{3}{2}\left(\sqrt{x^2+4} - x^2.\frac{1}{\sqrt{x^2+4}}\right) = \underset{x\to\infty}{\text{Lt}} \frac{6}{\sqrt{x^2+4}} = 0$$

$\therefore$ $y = -\frac{3}{2}x + 0$ and $y = \frac{3}{2}x + 0$, i.e., $y = \pm\frac{3}{2}x$ are the only asymptotes to the given curve.

■ ANOTHER METHOD TO FIND OBLIQUE ASYMPTOTE FOR SECOND DEGREE CURVE

Let $F(x, y) = 0$ be the given curve, and the straight line $y = mx + c$ be the oblique asymptote(s). Put $y = mx + c$ in $F(x, y) = 0$ to obtain a polynomial in x. Substituting the highest power of x equal to zero gives us the possible slopes and second highest power of x equal to zero gives us the possible values of y-intercept 'c'.

ILLUSTRATION 91: Find the asymptotes of the curve $y = \dfrac{3x^2 - 2x + 4}{x}$.

SOLUTION: Given equation is $xy = 3x^2 - 2x + 4$ or $3x^2 - xy - 2x + 4 = 0$

Horizontal asymptote: As coefficient of $x^2 = 3$ (non-zero constant) there is no asymptote parallel to x-axis.

Vertical asymptote: Coefficient of y is $x = 0$ (y-axis) is only vertical asymptote.

Oblique asymptote: Let $y = mx + c$ be the oblique asymptote.

$\therefore$ $3x^2 - x(mx + c) - 2x + 4 = 0$

$\Rightarrow$ $x^2(3 - m) - (2 + c)x + 4 = 0$

$\therefore$ slope of asymptote is given by $3 - m = 0$ and y-intercept is given by $2 + c = 0$

$\Rightarrow$ $m = 3,\ c = -2$

$\therefore$ $y = 3x - 2$ is the only oblique asymptote to the given curve.

REMARKS

1. *A curve of n-degree in x and y can't have more than n-asymptotes.*
2. *For a given value of x, the values of y for different branches of the curve f(x, y) = 0 will be different. Therefore, we may get several different values of m and correspondingly several different values of* $\lim_{x\to\infty}(y - mx)$. *This shows that a curve may have more than one asymptote.*
3. *The above method determines not only oblique asymptotes but also horizontal asymptotes. It does not, however, determine the vertical asymptote.*

■ METHOD TO FIND OBLIQUE ASYMPTOTES FOR ALGEBRAIC CURVES OF ANY DEGREE

Let the equation of a rational algebraic curve of degree 'n' in x and y be of the form

$(a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_ny^n) + (b_1x^{n-1} + b_2x^{n-2}y + b_3x^{n-3}y^2 + \ldots + b_ny^{n-1}) + (c_2x^{n-2} + c_3x^{n-3}y + \ldots + c_ny^{n-2}) + \ldots + = 0$

i.e., $H_n + H_{n-1} + H_{n-2} + \ldots + H_2 + H_1 + H_0 = 0$ ......(1)

Where H_r is a homogeneous polynomial of degree r in x, y

Therefore, we can write $H_r = x^r\,\phi_r\,(y/x)$, where ϕ_r is a polynomial of degree at the most r in y/x.

Consequently equation (1) can be written in the form

$$x^n\phi_n\left(\frac{y}{x}\right) + x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right) + x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right) + \ldots + x\phi_1\left(\frac{y}{x}\right) + \phi_0\left(\frac{y}{x}\right) = 0 \qquad \ldots (2)$$

Dividing (2) throughout by x^n, we have

$$\phi_n\left(\frac{y}{x}\right) + \left(\frac{1}{x}\right)\phi_{n-1}\left(\frac{y}{x}\right) + \left(\frac{1}{x^2}\right)\phi_{n-2}\left(\frac{y}{x}\right) + \ldots + \left(\frac{1}{x^n}\right)\phi_0\left(\frac{y}{x}\right) = 0 \qquad \ldots (3)$$

On taking limits of both sides of (3) as $x \to \infty$, $\dfrac{y}{x} \to m$, we have equation

$(a_0 + a_1m + a_2m^2 + \ldots + a_nm^n) +$
$1/x(b_1 + b_2m + b_3m^2 + \ldots + b_nm^{n-1}) +$
$1/x^2(c_2 + c_3m + \ldots + c_nm^{n-2}) + \ldots + = 0$

i.e., $\phi_n(m) = 0$ ... (4)

Here, $\phi_n(m)$ denotes the polynomial of degree n in 'm' by which we determines the (possible) slopes of the asymptotes .

If we determine the polynomial of degree $n-1$ in m by $\phi_{n-1}(m)$, the polynomial of degree $n - 2$ in 'm' by $\phi_{n-2}(m)$ and so on. Following algorithm helps to determine asymptotes of an algebraic curve.

Step I: Putting $\phi_n(m) = 0$, gives us at most n real roots m_1, m_2, m_3, ... m_n, i.e., slope of asymptotes.

Step II: Now to find y-intercept 'c' corresponding to a given slope $m_i (= m)$ we use

$c\phi_n'(m) + \phi_{n-1}(m) = 0$, if m is a non-repeated root, $\frac{c^2}{2!}\phi_n''(m) + \frac{c}{1!}\phi_{n-1}'(m) + \phi_{n-2}(m) = 0$, if m is a root twice repeated $\frac{c^3}{3!}\phi_n'''(m) + \frac{c^2}{2!}\phi''_{n-1}(m) + \frac{c}{1!}\phi'_{n-2}(m) + \phi_{n-3}(m) = 0$, if m is a root thrice repeated and so on.

Step III: Asymptotes are given by $y = mx + c$

ILLUSTRATION 92: Find all the asymptotes to the curve $(x^3y - x^2y^2) + (4x^2y + 2xy^2) - 8xy + 12x + 10 = 0$

SOLUTION: Given curve is $(x^3y - x^2y^2) + (4x^2y + 2xy^2) - 8xy + 12x + 10 = 0$

$$\Rightarrow x^4\left(\frac{y}{x} - \left(\frac{y}{x}\right)^2\right) + 2x^3\left(2\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2\right) - 8x^2\left(\frac{y}{x}\right) + (12x+10)\left(\frac{y}{x}\right)^0 = 0$$

Asymptotes || to x-axis are given by $y = 0$ (equating the coefficient of, highest power of x, i.e., x^3 equal to 0).

Asymptotes || to y-axis are given by $2x - x^2 = 0$ (equating the coefficient of $y^2 = 0$)

$\Rightarrow$ $x(2 - x) = 0$

$\Rightarrow$ $x = 0$ and $x = 2$ are two asymptotes parallel to y-axis

$\phi_4(m) = m - m^2$

and $\phi_3(m) = 4m + 2m^2$, $\phi_2(m) = -8m$, $\phi_1(m) = 12$, $\phi_0(m) = 10$

Now slopes of oblique asymptotes are given by $\phi_4(m) = 0$

i.e., $m - m^2 = 0 \Rightarrow$ m $(1 - m) = 0$

$\Rightarrow$ $m = 0$ and $m = 1$

But $m = 0$ corresponds to asymptote parallel to x-axis, i.e., $y = 0$

$\therefore$ $m = 1$ corresponds to oblique asymptote not parallel to x-axis, corresponding y-intercept 'c' is given by $c\phi_4'(m) + \phi_3(m) = 0$

$\Rightarrow$ $c(1 - 2m) + (4m + 2m^2) = 0$

$\Rightarrow$ $c(1 - 2) + (4 + 2) = 0$ $\qquad\Rightarrow$ $c = 6$

$\therefore$ $y = x + 6$ is the oblique asymptote

Thus, $y = 0$, $x = 0$, $x = 2$, $y = x + 6$ are asymptotes to given curve.

ASYMPTOTE BY EXPANSION

If the equation of the curve is of the form $y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + ...$, then $y = mx + c$ will be an asymptote of the given curve.

ILLUSTRATION 93: Find the asymptote for $y = 2x + 5 + 3/x$.

SOLUTION: Here, $y = 2x + 5 + 3/x$ is of the form, $y = mx + c + \frac{A}{x} + \frac{B}{x^2} + ...$

$\Rightarrow$ $y = 2x + 5$ is an asymptote of the curve $y = 2x + 5 + 3/x$.

ILLUSTRATION 94: Find the asymptotes to the curve $y^3 = 8x^3 + 24x^2 + 16x$.

SOLUTION : Given the curve $y^3 = 8x^3 + 24x^2 + 16x$.

$$\Rightarrow\ y^3 = 8x^3\left(1+\frac{3}{x}+\frac{2}{x^2}\right) \quad \Rightarrow\ y = 2x\left(1+\frac{3}{x}+\frac{2}{x^2}\right)^{1/3}$$

$$\Rightarrow\ y = 2x\left[1+\frac{1}{3}\left(\frac{3}{x}+\frac{2}{x^2}\right)+\frac{\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\left(\frac{3}{x}+\frac{2}{x^2}\right)^2}{2!}+....\right]$$

$$\Rightarrow\ y = 2x\left[1+\frac{1}{3}\left(\frac{3}{x}+\frac{2}{x^2}\right)-\frac{1}{9}\left(\frac{3}{x}+\frac{2}{x^2}\right)^2+....\right]$$

$$\Rightarrow\ y = 2x\left(1+\frac{1}{x}+\frac{5}{3x^2}+.....\right)$$

$\therefore\ y = 2(x+1)$ is the oblique asymptote to the given curve .

THE POSITION OF THE CURVE WITH RESPECT TO AN ASYMPTOTE

Let the equation of the curve be of the form:

$$y = mx + c + \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+....., \text{ then,}$$

(a) The curve lies above the asymptote if

(i) $A \neq 0$ and A and x have same signs.

(ii) $A \neq 0, B > 0$

(iii) $A = 0, B = 0, C \neq 0$ and C and x have same signs.

(b) The curve lies below the asymptote if

(i) $A \neq 0$ and A and x have opposite signs.

(ii) $A = 0, B < 0$

(iii) $A = 0, B = 0, C \neq 0$ and C and x have opposite signs.

ILLUSTRATION 95: For the curve $y^4 - x^4 + 3x^3 = 0$

(i) show that the curve lie above asymptote $y = x - 1/2$ for $x < 0$

(ii) show that the curve lie below asymptote $y = x - 1/2$ for $x > 0$

SOLUTION: Given equation of curve is $y^4 - x^4 + 2x^3 = 0$

$$\Rightarrow\ y^4 = x^4 - 2x^3 \Rightarrow\ y^4 = x^4\left(1-\frac{2}{x}\right) \quad \Rightarrow\ y = x\left[1-\frac{2}{x}\right]^{1/4}$$

$$\Rightarrow\ y = x\left(1+\frac{1}{4}\left(\frac{-2}{x}\right)+\frac{1}{4}\frac{\left(\frac{1}{4}-1\right)\left(\frac{-2}{x}\right)^2}{2!}+\frac{\left(\frac{1}{4}\right)\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)\left(-\frac{2}{x}\right)^3}{3!}+....\right)$$

$$\Rightarrow\ y = x\left(1-\frac{1}{2x}-\frac{3}{8x^2}-\frac{7}{16x^3}-...\right) \qquad \Rightarrow\ y = \left(x-\frac{1}{2}-\frac{3}{8x}-\frac{7}{16x^2}-...\right)$$

Comparing with $y = mx + c + \frac{A}{x}+\frac{B}{x^2}+...$, we get

the oblique asymptote given by $y = \left(x-\frac{1}{2}\right)$, $A = -3/8$, $B = -7/16$

Since both A and B are negative.

$\therefore$ The curve lies above the asymptote for $x < 0$ because x and A are of same sign.

And the curve lies below the asymptote for $x > 0$ since x and A are opposite sign.

$\therefore$ The curve lies above asymptote $y = \left(x - \frac{1}{2}\right)$ for $x < 0$

and the curve lies below asymptote $y = \left(x - \frac{1}{2}\right)$ for $x > 0$.

SINGULAR POINTS (MULTIPLE POINTS)

Introduction

We know that there are many relations which are multiple valued (one–many) that is why their equations are of the form $F(x, y) = 0$ (and not of the form $y = f(x)$. An equation of the form $F(x, y) = 0$ may sometimes be not solvable so as to yield $y = f(x)$, for it may give rise to several values of y corresponding to a single value of x, and these different values of y give rise to different branches of the curve.

Definition

A point on a curve is said to be a multiple point of order r, if r branches of the curve pass through this point. If P is the multiple point of order r, then there will be r tangents at P, one of each of the r branches. These r tangents may be real, imaginary, distinct, coincident.

DOUBLE POINTS

A point on a curve is said to be a double point of the curve, if two branches of the curve pass through this point. Double points have two tangents, they may be real, imaginary, distinct or coincident.

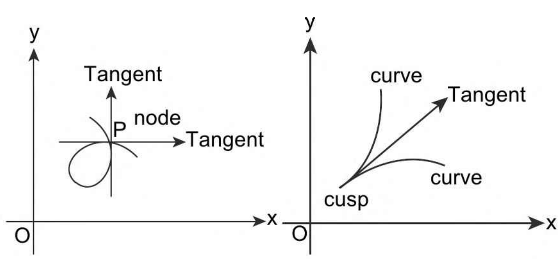


FIGURE 3.332

Types of Double Points

(a) Node: If the two branches of a curve through the double point and the tangents to them at the point are real and distinct, then the double point is called a node. For instance, $y^2 = (x - 2)(x - 5)^2$. The curve has two branches given by analytic equation $y = \pm|x - 5|\sqrt{x - 2}$. The point (2, 0) and (5, 0) is common to both. The graph of this curve is as shown in figure 3.333 indeed the two branches have different tangents at the point (5, 0), therefore, this point shall be called as Node.

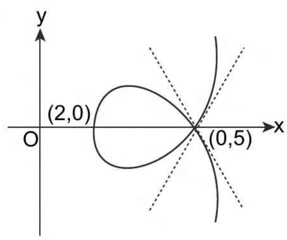


FIGURE 3.333

(b) Cusp: If the two branches of the curve pass through the double point and the tangent to them at the point are real and coincident, then the double point is called cusp.

The graph of a continuous function $y = f(x)$ has a cusp at a point $x = c$ if the concavity is same on the both side of c and either

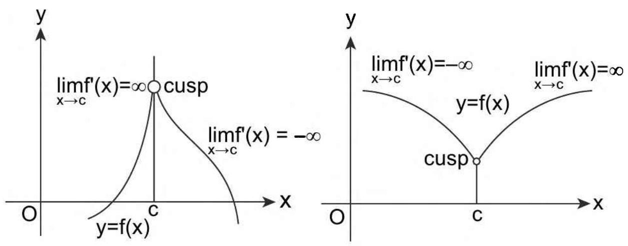


FIGURE 3.334

1. $\lim\limits_{x \to c^-} f'(x) = \infty$ and $\lim\limits_{x \to c^+} f'(x) = -\infty$
2. $\lim\limits_{x \to c^-} f'(x) = -\infty$ and $\lim\limits_{x \to c^+} f'(x) = \infty$

A cusp can either be local maximum (1) or a local minima as in (2).

For instance, the curves described by equation $y^2(2a - x) = x^3$, also has two functional branches given by functional formulae $y = \pm|x|\sqrt{\dfrac{x}{2a-x}}$; having origin as common point to both the branches. The graph of this curve is as shown in the figure given below. Clearly, both the branches of the curve have a common tangent there, namely $y = 0$. Thus, the origin is a cusp here.

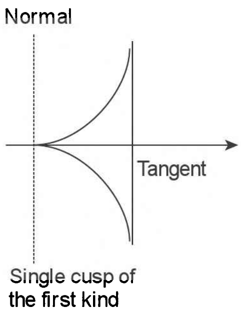


FIGURE 3.335

(c) Isolated point: If there are no real point on the curve in the neighborhood of a point P, then it is called an isolated or a conjugate point. e.g., $y^2 = x(x + a)^2$, $a > 0$ has two functional branches given by functional formulae $y = \pm|x+a|.\sqrt{x}$; The graph of this curve is as shown in figure. Here the points $(-a, 0)$ and $(0,0)$ are common to both the branches. Tangent at $(-a, 0)$ is imaginary with joint equation as $y^2 + (x + a)^2 = 0$) since there is no other point on the curve in a any neighborhood of $(-a, 0)$.Whereas the tangents to both the branches at $(0, 0)$ are vertical (the slope of tangent at this point does not exits finitely).

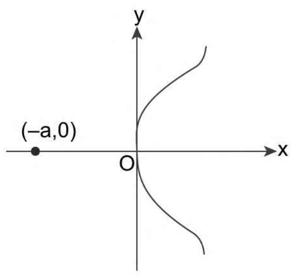


FIGURE 3.336

NECESSARY CONDITIONS FOR THE EXISTENCE OF DOUBLE POINTS

Let (x, y) be a point on the given curve $f(x, y) = 0$. The necessary and sufficient conditions for (x, y) to be double points are $f = 0, \dfrac{\partial f}{\partial x} = 0, \dfrac{\partial f}{\partial y} = 0$ at (x, y).

Now , if $\dfrac{\partial^2 f}{\partial x \partial y}$, i.e., $f^2{}_{xy}$, $\dfrac{\partial^2 f}{\partial x^2}$, i.e., f_{xx} and $\dfrac{\partial^2 f}{\partial y^2}$, i.e., f_{yy} are not all zero then,

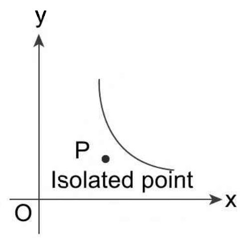


FIGURE 3.337

(i) Double point will be a node if

$$\left(\frac{\partial^2 f}{\partial x\, \partial y}\right)^2 = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) > 0$$

or $f_{xy}^2 - f_{xx} f_{yy} > 0$

(ii) The double point will be an isolated point if $f_{xy}^2 - f_{xx} f_{yy} < 0$

(iii) The double point will be a cusp if $f_{xy}^2 - f_{xx} f_{yy} = 0$
Here , $f_{xx} = f_{xy} = f_{yy} =$ if at (x, y) then it will be a multiple point of order greater than 2.

TYPES OF CUSPS

As we have understood that if two branches of a curve pass through a point and the tangents are coincident it is known as cusp. Therefore, normal to the branches at a cusp would also be coincident. Cusp can be classified as below.

(a) Single cusp of first species: If the branches of the curve lie on the same side of the common normal, then the cusp is called a single cusp. But if the branches of the curve lie on the both sides of common tangents, then the cusp is called as single cusp of first kind.

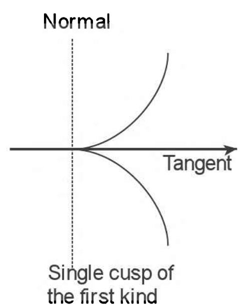


FIGURE 3.338

(b) Single cusp of second species: If the branches of the curve lie on the same side of the common normal, but the branches of the curve also lie on the same sides of common tangents, then the cusp is known as single cusp of second kind.

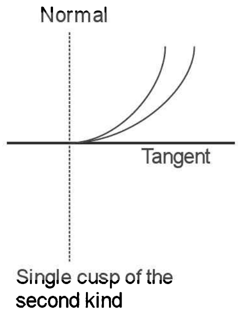


FIGURE 3.339

(c) Double cusp of first species: A curve is said to have a double cusp of the first kind at a point if both the branches of the curve extend to both the sides of the normal at the point and lie on opposite sides of the tangent at that point as shown in Figure 3.340.

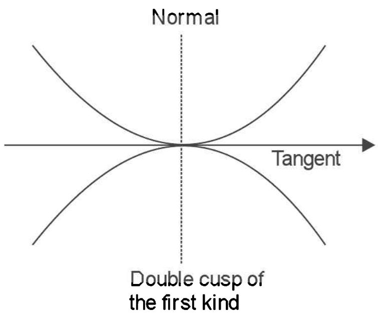


FIGURE 3.340

(b) Double cusp of second species: A curve is said to have a double cusp of the second kind at a point if both the branches of the curve extend to both sides of the normal but are on the same side of the tangent as shown in Figure 3.341.

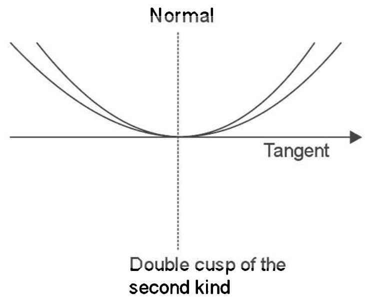


FIGURE 3.341

(c) Point of Oscu-inflexion: A curve is said to have a point of oscu-inflexion if there is a cusp of the first kind on one side of the normal and a cusp of the second kind on the other side of the normal. In such a case the point is also a point of inflexion, as the name suggests. In figure we have shown a point of oscu-inflexion in which the curve has a cusp of the first kind on the right of the normal and a cusp of the second species on the left of the normal. We can also have a point of oscu-inflexion in which there is a cusp of the first species on the left of the normal and a cusp of the second species on the right of the normal see Figures 3.342 and 3.343.

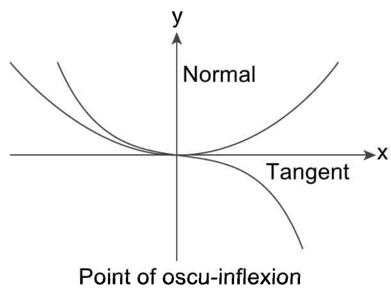


FIGURE 3.342

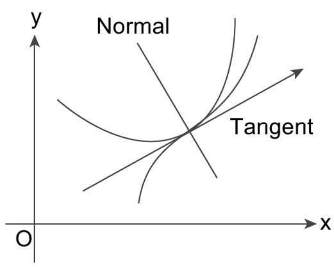


FIGURE 3.343

REMARKS

From the above discussion it is obvious that the origin is a multiple point if and only if f(x, y) does not contain any constant term and terms of degree one.

1. *The equations of the tangents at the origin to the curve $x^4 + y^4 = a^2xy$ are $x = 0$, $y = 0$, so that the origin is a node.*

2. *The equation of the tangents at the origin to the curve $x^4 + y^4 + 2x^3 = x^2 + 2xy + y^2$ is $x^2 + 2xy + y^2 = 0$, i.e., $(x + y)^2 = 0$. Since the tangents are coincident, the origin is a cusp.*
3. *The equation of the tangents at the origin to the curve $(x^2 + y^2)(x - a) + b^2x^3 = 0$. Since the tangents are imaginary ($y = \pm ix$), therefore, the origin is a conjugate point on the curve.*
4. *The equation of the tangents at the origin to the curve $3x^4 + y^4 + 3y(x^2 - y^2) = 0$ is $y(x^2 - y^2) = 0$. Since there are three tangents at the origin, namely $y = 0$, $x \pm y = 0$, therefore, the origin is a triple point on the curve.*

ILLUSTRATION 96: Show that the origin is a single cusp of the first species on the curve $(y - 2x^2)^2 = x^3$.

SOLUTION: It is obvious that (0, 0) is a point on the curve. The equation of the tangents at (0, 0) are given by $y^2 = 0$, showing that (0, 0) is a cusp.

Since the left hand side can never be negative, it follows that x can never be negative, i.e., the curve lies entirely to the right of the normal at the cusp (i.e., $x = 0$). This implies that the origin is a single cusp.

Writing the equation of the curve as $y = 2x^2 \pm x^{3/2}$.

We find that for every positive value of x there are two values of y. Since $x^{3/2} > x^2$ for small positive values of x, therefore, for every small positive value of x, the two values of y are of opposite sign. Hence, the cusp is of first species.

ILLUSTRATION 97: Find the position and nature of the cusp on the curve $y^2 + 3x^4 = 4x^2y + x^5$.

SOLUTION: Writing the equation of the curve as $f(x, y) = y^2 + 3x^4 - 4x^2y - x^5 = 0$

We have $f_x = 12x^3 - 8xy - 5x^4$.

$f_y = 2y - 4x^2$

Solving the equations $f(x, y) = 0$, $f_x = 0$, $f_y = 0$ together, we find that (0, 0) is the only multiple point.

Equating to zero the lowest degree terms, we find that $y = 0$ is a cuspidal tangent.

Solving $f(x, y) = 0$, as a quadratic in y, we have $y = 2x^2 \pm x^2\sqrt{(1 + x)}$. ... (1)

From (1), we find that for each small positive value of x, there are two values of y which are both positive (because $2x^2 > x^2\sqrt{(1 + x)}$)

Again, for every numerically small but negative value of x, there are two values of y which are both positive.

Thus, in the neighborhood of the origin, the curve lies on both sides of the y-axis, but entirely above the x-axis (which is the cuspidal tangent).

Hence, the origin is a double cusp of the second species.

ILLUSTRATION 98: Compute the area of this figure contained between the curve, $xy^2 = 8 - 4x$ and its asymptote.

SOLUTION: $x = \dfrac{8}{4 + y^2}$

$$\Rightarrow A = 2\int_0^\infty \frac{8}{4 + y^2}dy = \frac{2\,.\,8}{2}\,.\,\tan^{-1}\frac{y}{2}\bigg]_0^\infty = 4\pi$$

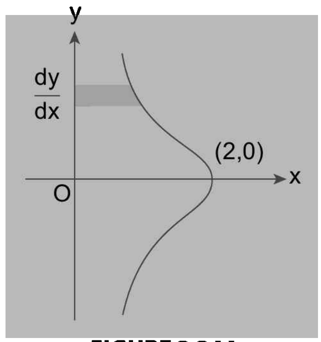


FIGURE 3.344

ILLUSTRATION 99: Sketch the graph of the following functions:

(a) $y = \dfrac{2}{x^2 + 1}$ (b) $y = x + \dfrac{1}{x}$, $x \in \mathbb{R} - \{0\}$

(c) $y = 2x^2 - \log|x|$, $x \neq 0$

SOLUTION: (a) Given function is $y = \dfrac{2}{x^2+1}$

Clearly the domain of the function is $(-\infty. \infty)$ and the range is $(0, \infty)$.

It has no roots and at $x = 0$, $y = 2$ which means intersects the y-axis at (0, 2). Since it is even function w.r.t x, therefore, the graph will be symmetric about y-axis. As we know that $f(x) = x^2 + 1$ is a parabola opening upwards with vertex (0, 2) and it increases for $x > 0$ and decreases for all $x < 0$.

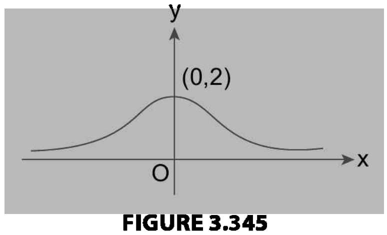


FIGURE 3.345

So, the given function $y = \dfrac{2}{x^2+1}$ will increase for all $x < 0$ and decrease for $x > 0$. Also $\lim\limits_{x\to\infty} f(x) = 0$, so, x-axis is the horizontal asymptote.

(b) Clearly the function does not exist at $x = 0$, and therefore, discontinuous at $x = 0$.

$$y' = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2} \quad \Rightarrow \quad y' = 0 \quad \Rightarrow \quad x = -1, 1$$

Also is undefined at $x = 0$. But y at $x = 0$ is also undefined, therefore, $x = 0$ is not a critical point. Therefore, only critical points are the stationary points $x = -1, 1$.

We now use the second derivative test to decide the nature of extrema if they exist.

$$y'' = \frac{2}{x^3} \quad \Rightarrow \quad y'' \text{ at } x = -1 \text{ is } -2 < 0 \text{ and } y'' \text{ at } x = 1 \text{ is } 2 > 0$$

Clearly y'' is positive for $x > 0$ (so, function is concave up) and negative for $x < 0$ (so function is concave down).

Hence, $x = -1$ is a point of local maximum and $x = 1$ is a point of local minimum . Now there are two consecutive extrema , maximum at $x = -1$ and minimum at $x = 1$. Also, y at $x = -1$ is -2 and at $x = 1$ is 2.

The value of y at a point of local maximum is less than the value of y at a point of local minimum.

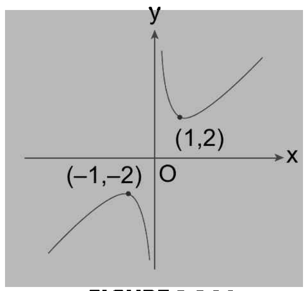


FIGURE 3.346

If minimum value > maximum value at two consecutive extrema, there must be a discontinuity between them.

(c) Clearly y is not defined at $x = 0$ as log 0 is undefined

$$\frac{dy}{dx} = 4x - \frac{1}{x} = \frac{4x^2-1}{x} = \frac{4(x-1/2)(x+1/2)}{x}. \qquad \left[\because \frac{d}{dx}\log|x| = \frac{1}{x}\right]$$

Since, y is not defined at $x = 0$, $x = 0$ is not a critical point though $\dfrac{dy}{dx}$ is undefined at $x = 0$. The only critical points are $x = -1/2$ and 1/2 where $\dfrac{dy}{dx} = 0$.

Now $\dfrac{d^2y}{dx^2} = 4 + \dfrac{1}{x^2} > 0$ at $x = -1/2$ and 1/2

$\therefore$ y has local minima at both points $x = -1/2$ and $x = 1/2$ shows an illusion that two minima occur consecutively but note that there is a discontinuity at $x = 0$ between them.

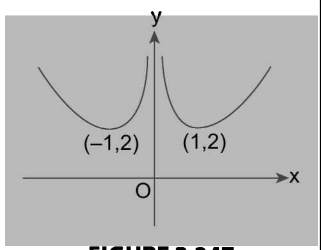


FIGURE 3.347

ILLUSTRATION 100: Find the asymptotes of the following curves $y = \dfrac{3x}{x-1} + 3x$.

SOLUTION: The curve has a vertical asymptote $x = 1$

since $\lim\limits_{x \to 1^-} y = \lim\limits_{x \to 1^-} \left(\dfrac{3x}{x-1} + 3x \right) = -\infty$

$$\lim_{x \to 1^+} y = \lim_{x \to 1^+} \left(\frac{3x}{x-1} + 3x \right) = \infty$$

Find the inclined asymptotes:

$$m = \lim_{x \to \pm\infty} \frac{y}{x} = \lim_{x \to \pm\infty} \left(\frac{3}{x-1} + 3 \right) = 3$$

$$c = \lim_{x \to \pm\infty} (y - mx) = \lim_{x \to \pm\infty} \left(\frac{3x}{x-1} + 3x - 3x \right) = 3$$

Thus, the straight line $y = 3x + 3$ is an inclined asymptote as shown in the figure.

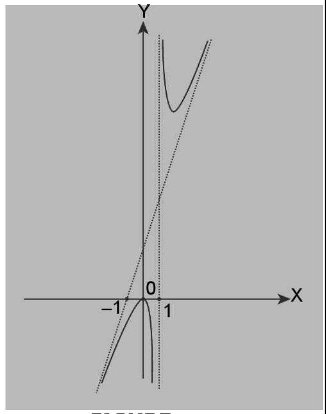


FIGURE 3.348

ILLUSTRATION 101: Find the asymptotes of the following curves $y = \sqrt{1+x^2} + 2x$.

SOLUTION: The curve has no vertical asymptotes, since the function is continuous everywhere. Let us look for inclined asymptotes. The limits will be different as $x \to +\infty$ and $x \to +\infty$, therefore, we have to consider two cases separately.

Find the right asymptote (as $x \to +\infty$):

$$k_1 = \lim_{x \to +\infty} \frac{\sqrt{1+x^2} + 2x}{x} = \lim_{x \to +\infty} \frac{\sqrt{\frac{1}{x^2} + 1} + 2}{1} = 3$$

$$b_1 = \lim_{x \to +\infty} \left(\sqrt{1+x^2} + 2x - 3x \right) = \lim_{x \to +\infty} \left| \sqrt{1+x^2} - x \right|$$

$$= \lim_{x \to +\infty} \frac{1 + x^2 - x^2}{\sqrt{1+x^2} + x} = 0.$$

Thus, as $x \to +\infty$ the curve has an asymptote $y = 3x$.

Find the left asymptote (as $x \to +\infty$):

$$k_2 = \lim_{x \to -\infty} \frac{\sqrt{1+x^2} + 2x}{x} = \lim_{x \to -\infty} \frac{|x| \sqrt{\frac{1}{x^2} + 1} + 2x}{x} = 1$$

$$b_2 = \lim_{x \to -\infty} \left| \sqrt{1+x^2} + 2x - x \right| = \lim_{x \to -\infty} \frac{1}{\sqrt{1+x^2} - x} = 0$$

Since both summands $\left(\sqrt{1+x^2} \text{ and } (-x) \right)$ in the denominator are positive at $x < 0$, and so, the curve has an asymptote $y = x$, as $x \to -\infty$.

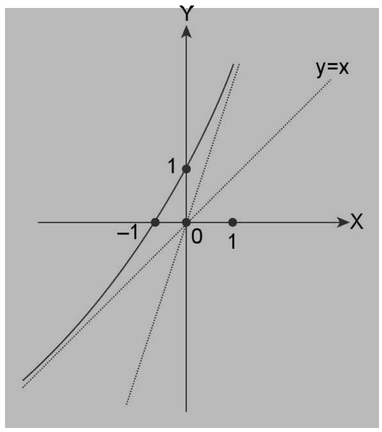


FIGURE 3.349

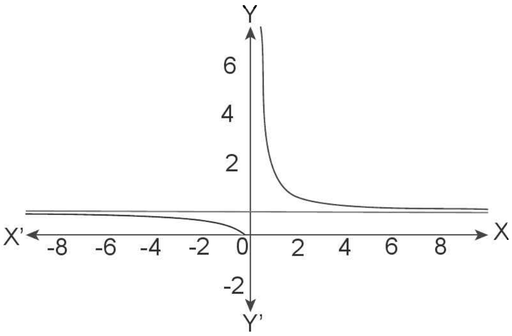
TEXTUAL EXERCISE–7: (SUBJECTIVE)

1. Show the curve $y = e^{1/x}$ has a vertical and horizontal asymptote.
2. Find the asymptote of the curve $x^2y + xy^2 = a^3$.
3. Show the asymptote of the curve $xy(x^2 - y^2) + x^2 + y^2 - 1 = 0$ cut at 8 points.
4. Find the asymptote of the curve $y^3 = x^2(x - a)$.
5. For the curve $y^5 = x^5 + 2x^4$; show
   (i) The curve lies above the asymptote $y = x + 2/5$ if $x < 0$.
   (ii) The curve lies below the asymptote $y = x + 2/5$ if $x > 0$.
6. Show the curve $y = e^{1/x}$ has a vertical and horizontal asymptote.
7. Find the asymptote to the curve $y = x + 1/x$ and then sketch.
8. Sketch the curve
   (a) $y^2 (a + x) = x^2 (a - x)$
   (b) $y^2x^2 = x^2 - a^2$
   (c) $y^2(x^2 - 1) = (2x - 1)$
   (d) $x^3 + y^3 = 3ax^2$ $(a > 0)$.
   (e) $y^2 (x^2 - 1) = x^2 - 4$
   (f) $(y^2 - 1)(x^2 - 1) = x$
   (g) $(y^2 - 1)(x^2 - 1) = y^2 + x^2$
   (h) $(y^2 - 1)(x^2 + 1) = y^2 + x^2$
9. Sketch the curve
   (a) $y = (1 - x^2)^{-1}$
   (b) $y = \dfrac{x}{(1-x^2)^2}$
   (c) $y = 2x - 1 + \dfrac{1}{(1+x)}$
   (d) $y^2 = x^2 \left(\dfrac{a+x}{b-x}\right)$
   (e) $y = \dfrac{8a^3}{x^2 + 4a^2}$
   (f) $y = \dfrac{\cos x}{\cos 2x}$
   (g) $y^2 = 8x^2 - x^4$
   (h) $y^2 = \dfrac{x-1}{x+1}$
10. Sketch the graph of the following functions:
   (a) $y = x^{2/3}$
   (b) $y = \dfrac{2x}{x^2+1}$
   (c) $y = \dfrac{x^2 - 1}{x^2+1}$
   (d) $y = \dfrac{x^2+x-1}{x^2+x+1}$
   (h) $y = x^{1/x}$
11. Investigate and graph the following functions:
   (a) $y = x^6 - 3x^4 + 3x^2 - 5$
   (b) $y = \dfrac{2x^3}{x^2-4}$
   (c) $y = \dfrac{1-x^3}{x^2}$
   (d) $y = x + \ln\,(x^2 - 1)$
   (e) $y = x^2e^{1/x}$
   (f) $y = 1 + x^2 - \dfrac{x^4}{2}$
   (g) $y = \dfrac{1}{x} + 4x^2$
   (h) $y = \dfrac{x^3}{x^2-1}$
   (i) $y = \sqrt[3]{x^2} - \sqrt[3]{x^2 - 4}$
   (j) $y = x^2 \ln(x+2);\quad y = x^3e^{-4x}$

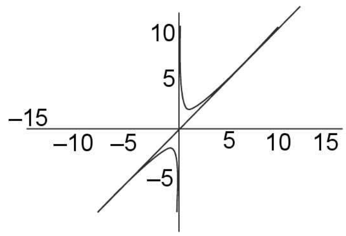
Answer Keys

1. $y = e^{1/x}$ has two asymptotes $x = 0$ and $y = 1$.
2. The asymptotes are $x = 0$, $y = 0$, $x + y = 0$
4. $y = x - a/3$ is the only asymptote of the given curve.

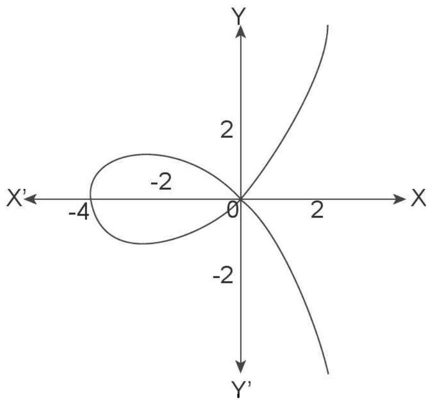
6.


Y
6
4
2
X' -8 -6 -4 -2 0 2 4 6 8 X
-2
Y'

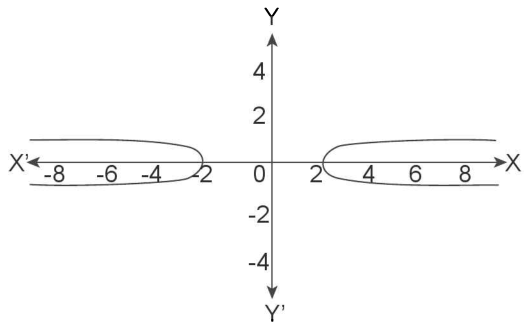
7.


10
5
−15
−10 −5 5 10 15
−5

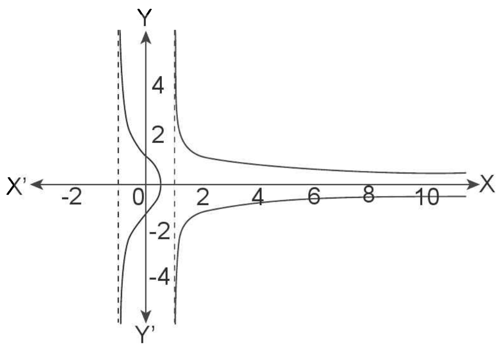
8. (a)


Y
2
-2
X' -4 0 2 X
-2
Y'

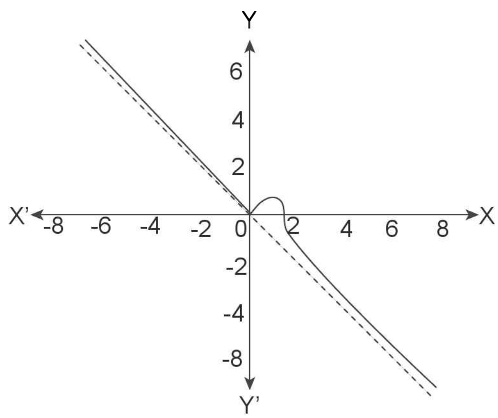
(c)


Y
4
2
X' -8 -6 -4 -2 0 2 4 6 8 X
-2
-4
Y'

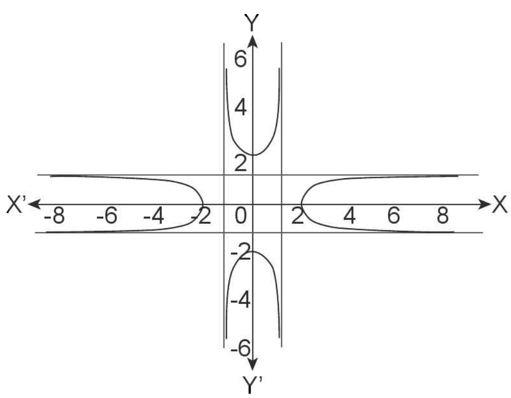
(d)


Y
4
2
X' -2 0 2 4 6 8 10 X
-2
-4
Y'

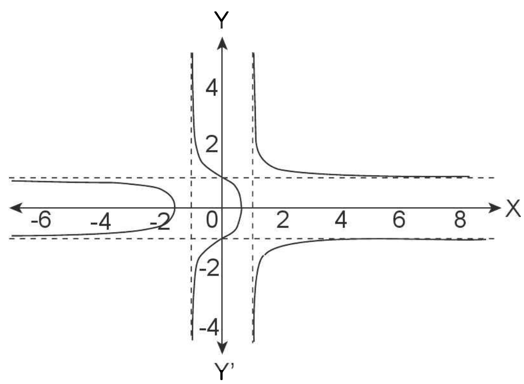
(e)

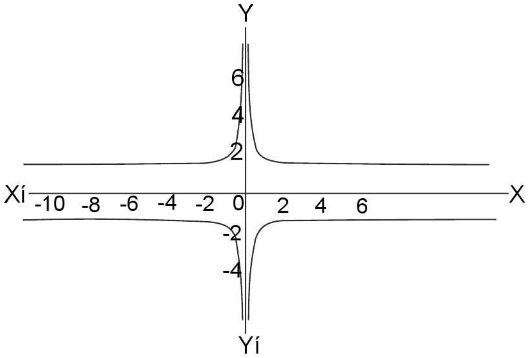

Y
6
4
2
X' -8 -6 -4 -2 0 2 4 6 8 X
-2
-4
-8
Y'

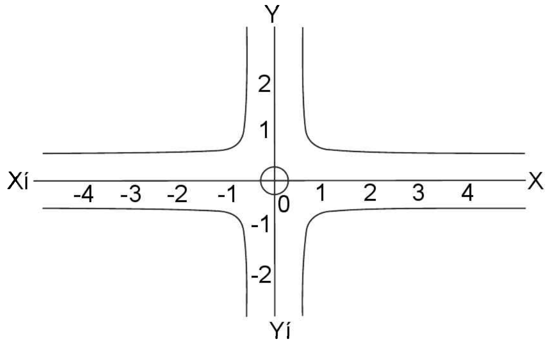
(g)

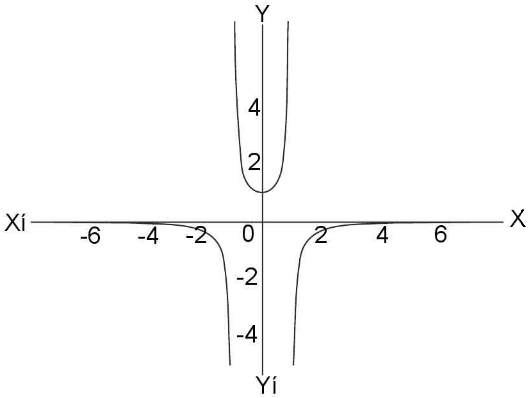

Y
6
4
2
X' -8 -6 -4 -2 0 2 4 6 8 X
-2
-4
-6
Y'

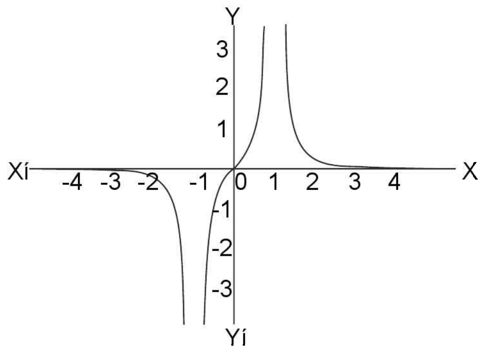
(h)

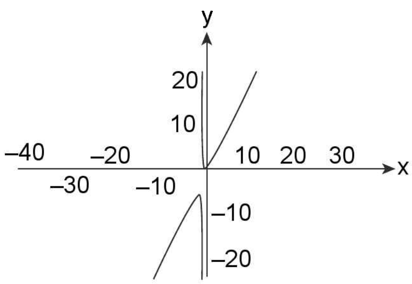

Y
4
2
-6 -4 -2 0 2 4 6 8 X
-2
-4
Y'

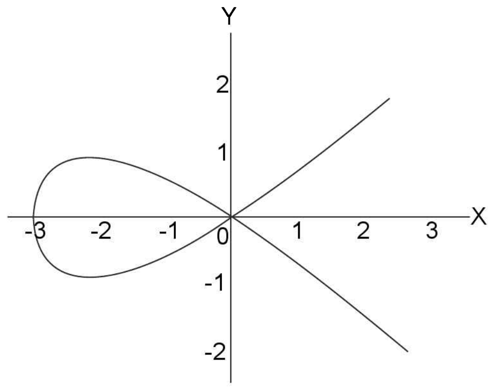
(i)

Y
6
4
2
Xí -10 -8 -6 -4 -2 0 2 4 6 X
-2
-4
Yí

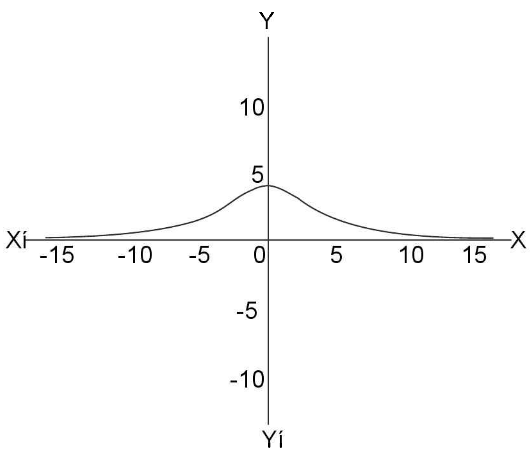
(j)

Y
2
1
Xí -4 -3 -2 -1 0 1 2 3 4 X
-1
-2
Yí

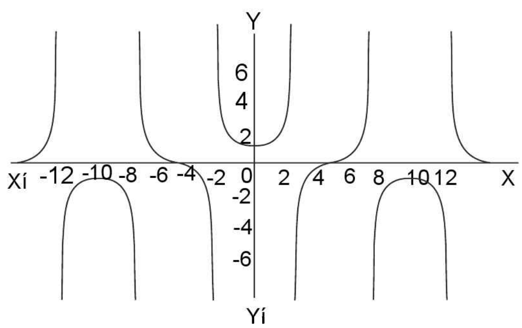
9. (c)

Y
4
2
Xí -6 -4 -2 0 2 4 6 X
-2
-4
Yí

(d)

Y
3
2
1
Xí -4 -3 -2 -1 0 1 2 3 4 X
-1
-2
-3
Yí

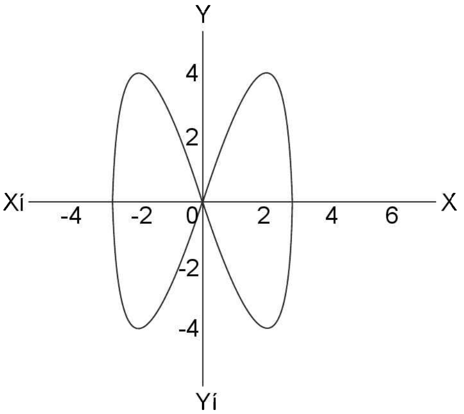
(e)

y
20
10
–40 –20 10 20 30 x
–30 –10
–10
–20

(g)

Y
2
1
-3 -2 -1 0 1 2 3 X
-1
-2

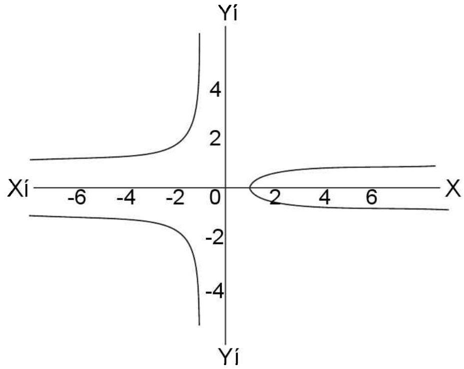
(h)

Y
10
5
Xí -15 -10 -5 0 5 10 15 X
-5
-10
Yí

(i)

Y
6
4
2
Xí -12 -10 -8 -6 -4 -2 0 2 4 6 8 10 12 X
-2
-4
-6
Yí

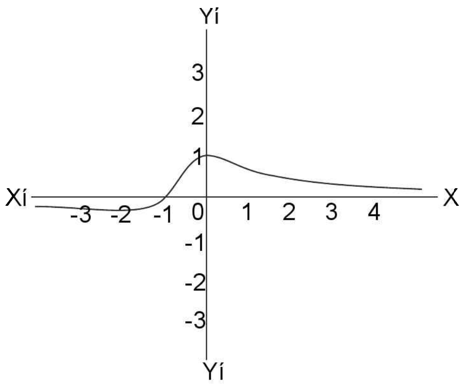
(j)


Y
4
2
Xí
-4
-2
0
2
4
6
X
-2
-4
Yí

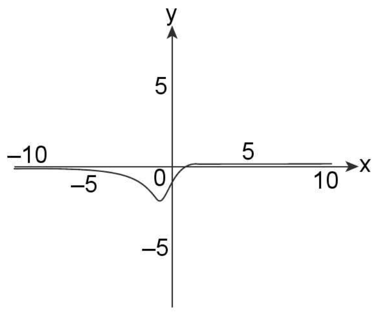
(k)


Yí
4
2
Xí
-6
-4
-2
0
2
4
6
X
-2
-4
Yí

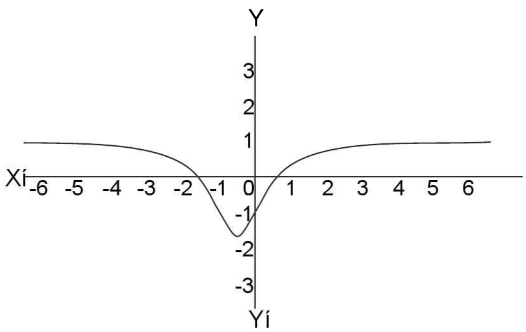
10. (a)


Yí
3
2
1
Xí
-3
-2
-1
0
1
2
3
4
X
-1
-2
-3
Yí

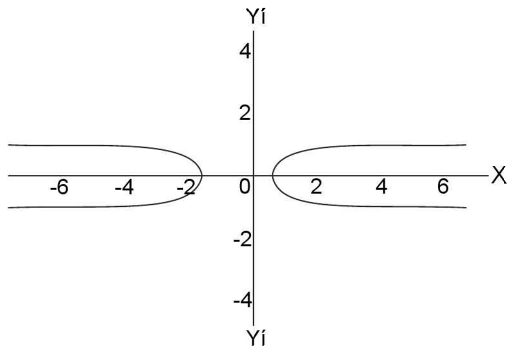
(b)


y
5
–10
5
–5
0
10
x
–5

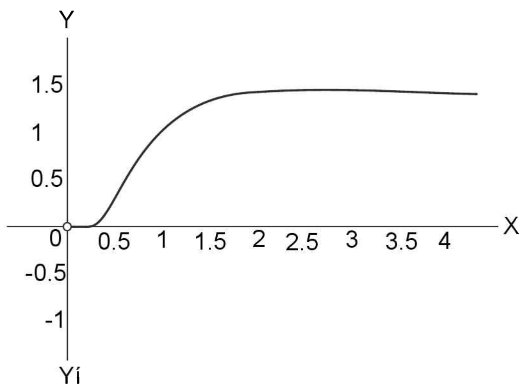
(c)


Y
3
2
1
Xí
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
-1
-2
-3
Yí

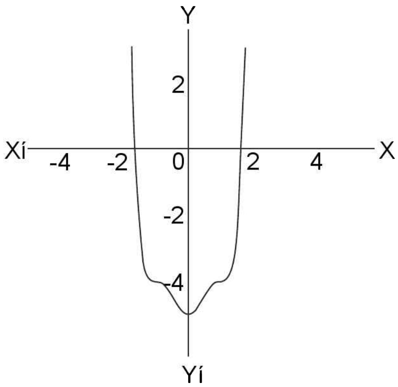
(d)


Yí
4
2
-6
-4
-2
0
2
4
6
X
-2
-4
Yí

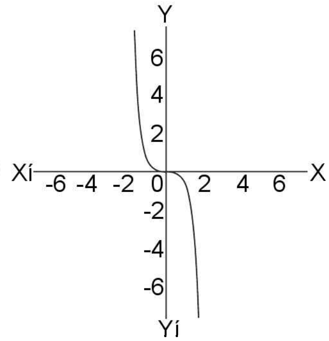
(h)


Y
1.5
1
0.5
0
0.5
1
1.5
2
2.5
3
3.5
4
X
-0.5
-1
Yí

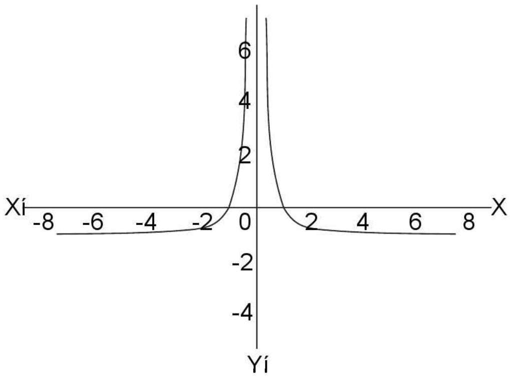
11. (a)


Y
2
Xí
-4
-2
0
2
4
X
-2
-4
Yí

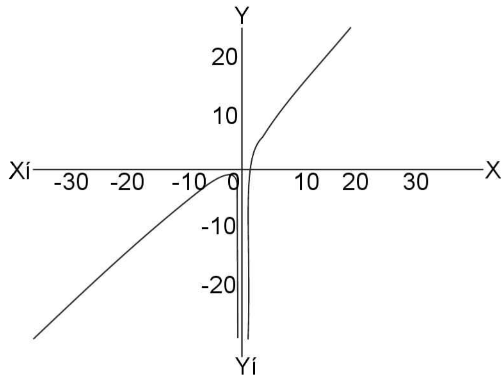
(b)


Y
6
4
2
Xí
-6 -4 -2 0
2 4 6
X
-2
-4
-6
Yí

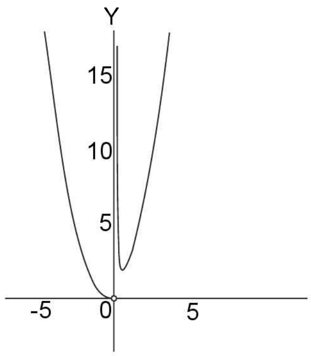
(c)


6
4
2
Xí
-8 -6 -4 -2 0 2 4 6 8
X
-2
-4
Yí

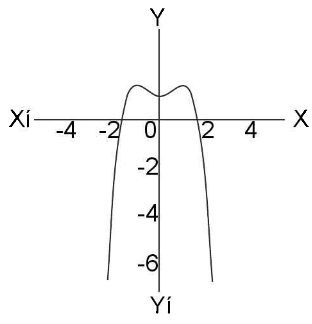
(d)


Y
20
10
Xí
-30 -20 -10 0
10 20 30
X
-10
-20
Yí

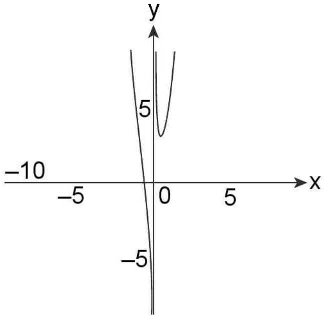
(e)


Y
15
10
5
-5
0
5

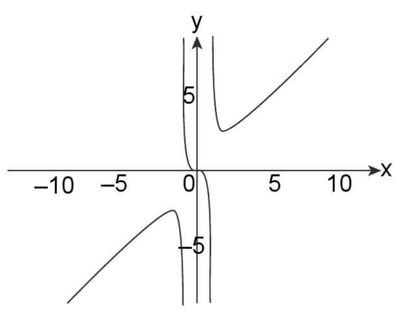
(f)


Y
Xí
-4 -2 0 2 4
X
-2
-4
-6
Yí

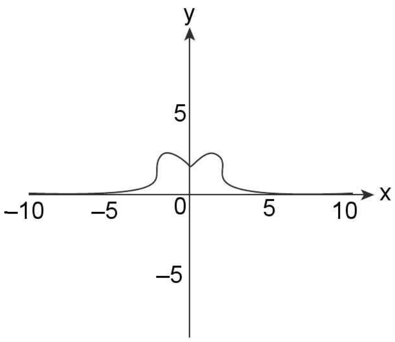
(g)


y
5
–10
–5
0
5
x
–5

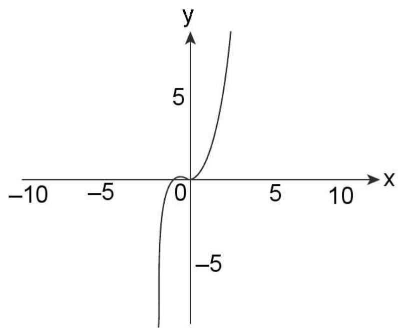
(h)


y
5
–10 –5 0 5 10
x
–5

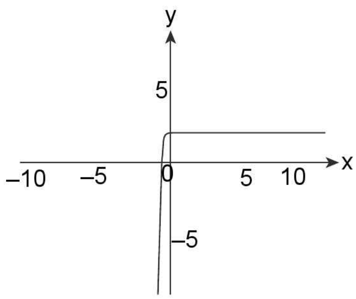
(i)



(j)



(k)



MULTIPLE–CHOICE QUESTIONS

SECTION–I

OBJECTIVE-TYPE SOLVED EXAMPLES

1. Find the area enclosed by max $(|x + 2y + 4|, |y - 2x|) = 3$

(a) 42 sq units (b) 7.2 sq units
(c) 33 sq units (d) None of these

Solution: (b)

To find the area enclosed by max$\{|x + 2y + 4|, |y - 2x|\} = 3$, first of all let us observe where

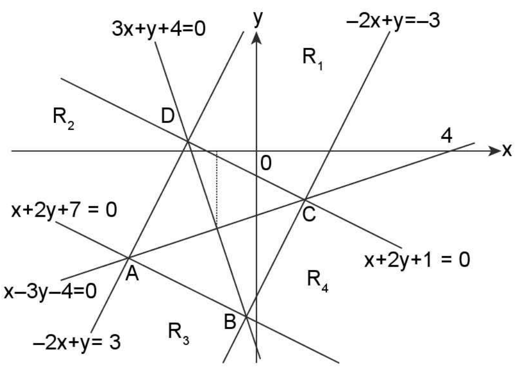
$|x + 2y + 4| = |y - 2x|$

$\Rightarrow x + 2y + 4 = \pm (y - 2x)$

$\Rightarrow 3x + 2y + 4 = 0$ ....(i)

$\Rightarrow x - 3y - 4 = 0$ ....(ii)

Lines (i) and (ii) are perpendicular to each other with their point of intersection at $\left(\frac{-4}{5}, \frac{-8}{5}\right)$ as shown below.



In region R_1; (0, 0) lies and $|x + 2y + 4| = 4$ at (0, 0); $|y - 2x| = 0$ at (0, 0)

$\Rightarrow |x + 2y + 4| > |y - 2x|$

$\Rightarrow |x + 2y + 4| = 3$

$\Rightarrow x + 2y + 4 = \pm 3$

$\Rightarrow x + 2y + 7 = 0$ and $x + 2y + 1 = 0$

In region R_2: (– 2, 0) lies, $|x + 2y + 4| = 2$ at (– 2, 0) and $|y - 2x| = 4$ at (–2, 0)

$\Rightarrow |y - 2x| = 3$ $\Rightarrow y - 2x = \pm 3$

In region R_3: (0, 5) lies and $|x + 2y + 4| = |-6| = 6$; $|y - 2x| = 5$

$\Rightarrow |x + 2y + 4| > |y - 2x|$ $\Rightarrow |x + 2y + 4| = 3$

$\Rightarrow x + 2y + 4 = \pm 3$

$\Rightarrow x + 2y + 1 = 0$ and $x + 2y + 7 = 0$

In region R_4: (0, – 3) lies and $|x + 2y + 4| = 2$; $|y - 2x| = 3$

$\Rightarrow |y - 2x| > |x + 2y + 4|$

$\Rightarrow |y - 2x| = 3$ $\Rightarrow (y - 2x) = \pm 3$

From above we can conclude that

AB: $x + 2y + 7 = 0$ in region R_3

BC: $-2x + y = -3 = 0$ in region R_4

CD: $x + 2y + 1 = 0$ in region R_1

DA: $-2x + y = 3$ in region R_2 and $A \left(\frac{-13}{5}, \frac{-11}{5}\right)$; $B \equiv \left(\frac{-1}{5}, \frac{-17}{5}\right)$; $C = (1, -1)$; $D\left(\frac{-7}{5}, \frac{1}{5}\right)$

Clearly, $AB \perp BC$; $CD \perp DA$; $BC \perp CD$

$\Rightarrow ABCD$ is a rectangle.

Now $BC = \sqrt{\left(\frac{-1}{5} - 1\right)^2 + \left(\frac{-17}{5} + 1\right)^2} = \frac{6}{\sqrt{5}}$;

$AB = \sqrt{\left(\frac{-13}{5} + \frac{1}{5}\right)^2 + \left(\frac{-11}{5} + \frac{17}{5}\right)^2} = \frac{6}{\sqrt{5}}$

$\Rightarrow ABCD$ is a rectangle having its area $= \left(\frac{6}{\sqrt{5}}\right)^2 = \frac{36}{5}$ sq. units = 7.2 units.

2. Find two values of θ in the interval $(0, \pi)$ satisfying the equation $(1 - \cot\theta)(1 + \cot\theta)\operatorname{cosec}^2\theta + 2^{\cot^2\theta} = 0$. Also, find the total number of roots of the equation

(a) $\pi/6, 5\pi/6, 4$ (b) $\pi/3, 2\pi/3, 4$
(c) $\pi/6, 5\pi/6, 6$ (d) None of these

Solution: (a) Here $(1 - \cot^2\theta)(1 + \cot^2\theta) + 2^{\cos^2\theta} = 0$

Putting $\cot^2\theta = x$; we get, $(1 - x)(1 + x) + 2^x = 0$

$\Rightarrow 1 - x^2 + 2^x = 0$ $\Rightarrow 2^x = x^2 - 1$

By observation, we can get $x = 3$ as one of the roots of the above equation.

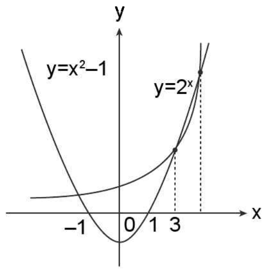
($\because 2^3 = 8$ and $3^2 - 1 = 8$)

Now we have been asked two values of θ in the interval.

$\Rightarrow \cot^2\theta = 3 \Rightarrow \cot\theta = \pm\sqrt{3}$

$\Rightarrow \theta = \pi/6$ or $5\pi/6$

Now, sketching the curves $y = 2^x$ and $y = x^2 - 1$ on the same axis, we get



Here, we see that one value of x is negative whereas the other two are positive.

Now $\cot^2\theta = x$ will not be satisfied for any values of θ for negative value of x.

Whereas $\cot^2\theta = x$ will have two value of θ lying between $(0,\pi)$ for every positive value of x.

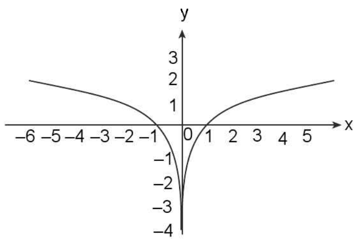
Hence, total 4 solutions.

3. Find the number of points of intersection of the graphs of $(x-1)^2 + y^2 - 9 = 0$ and $|y| = |\ln(|x| - 1)|$

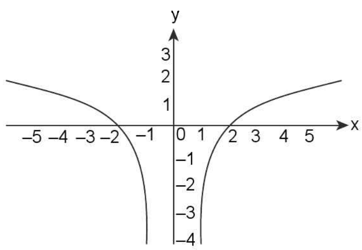
(a) 4 (b) 6

(c) 7 (d) None of these

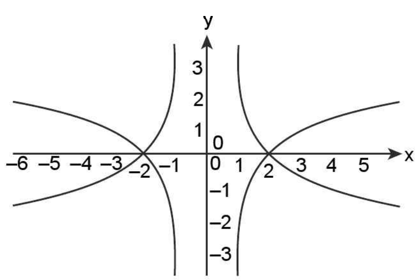
Solution: (c) The graph of $y = \ln|x|$ is as shown below.



The graph of $y = \ln(|x| - 1)$ is as shown below.

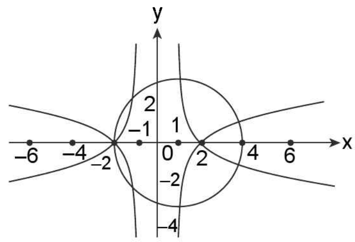


The graph of $|y| = |\ln(|x| - 1)|$ is as shown below.



Now $(x-1)^2 + y^2 = 9$ will be a circle with centre at (1,0) and radius $= 3$.

Plotting the two curves on the same axis, we get



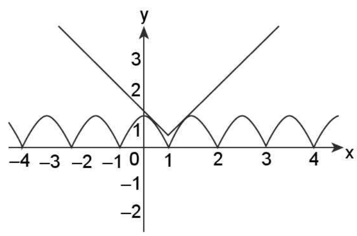
Hence, with the help of the diagram, we can see that the number of points of intersection of the two curves is 7.

4. Find the all possible value of 'a' for which $|\cos 2x| = |x - \pi/4| + a$ does not have any real solution.

(a) $\left(0, \dfrac{3\sqrt{3}-\pi}{6}\right)$ (b) $\left(0, \dfrac{3\sqrt{3}+\pi}{6}\right)$

(c) $\left(\dfrac{3\sqrt{3}-\pi}{6}, \infty\right)$ (d) $\left(\dfrac{3\sqrt{3}+\pi}{6}, \infty\right)$

Solution: (c) We would need to consider the limiting case when the graphs of $y = |\cos 2x|$ and $y = |x - \pi/4| + a$ touch each other.



If A is the point of contact, then at point A, the slopes of the two curves are the same.

Hence, $\dfrac{d}{dx}(|\cos 2x|) = \dfrac{d}{dx}\left(\left|x - \dfrac{\pi}{4}\right| + a\right)$

Considering the case when $x > \pi/4$ and $x < \pi/2$, then $|\cos 2x| = -\cos 2x$ and $|x - \pi/4| = x - \pi/4$

$\Rightarrow \dfrac{d}{dx}(-(\cos 2x)) = \dfrac{d}{dx}\left(\left(x - \dfrac{\pi}{4}\right) + a\right)$

$\Rightarrow$ $2\sin 2x = 1 \Rightarrow \sin 2x = 1/2$

$\Rightarrow$ $\sin 2x = \sin \pi/6 \Rightarrow 2x = n\pi + (-1)^n \pi/6$

Now $x \in (\pi/4, \pi/2)$

$\Rightarrow$ $2x \in \left(\dfrac{\pi}{2}, \pi\right)$ $\Rightarrow$ $2x = \pi - \pi/6 = 5\pi/6$

$\Rightarrow$ $x = 5\pi/12$

Now since the point A also lies on $y = x - \pi/4 + a$, hence, we have

$$\Rightarrow \left|\cos\frac{5\pi}{6}\right| = x - \frac{\pi}{4} + a$$

$$\Rightarrow \frac{\sqrt{3}}{2} = \frac{5\pi}{12} - \frac{\pi}{4} + a \quad \Rightarrow \frac{\sqrt{3}}{2} = \frac{\pi}{6} + a$$

$$\Rightarrow a = \frac{\sqrt{3}}{2} - \frac{\pi}{6} \quad \Rightarrow a = \frac{3\sqrt{3}-\pi}{6}$$

If $a > \frac{3\sqrt{3}-\pi}{6}$, then the given equation will not have any solutions.

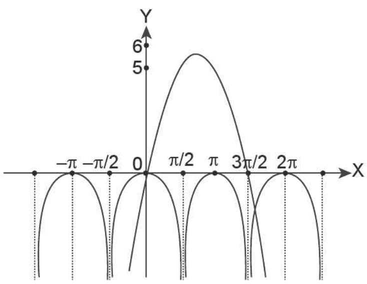
Thus, the required set of value of 'a' is given by $\left(\frac{3\sqrt{3}-\pi}{6}, \infty\right)$.

5. Find the number of points of intersection of the curves given by $y = \ln|\cos x|$ and $y = -x(x - 3\pi/2)$ in $(-3, 6)$

(a) 2 (b) 3

(c) 4 (d) None of these

Solution: (c) The number of points of intersection can be found out by drawing the two curves simultaneously.



As is evident from the graph the two curve intersect at 4 points.

6. Which of the following statement*(s)* is/are true for the function $f(x) = (x-1)^2(x-3)+1$ defined on the interval $[0, 3]$.

(a) Range of $f(x)$ is $[-2, 0]$

(b) The absolute maximum is achieved at two different points.

(c) The absolute minimum is not achieved.

(d) The values of 'a' for which $f(x) = a$ has 3 distinct roots is given by $\left(-\frac{5}{27}, 1\right)$.

Solution: (b, d) $f(x) = (x-1)^2(x-3)+1$

$f(x) = 2(x-1)(x-3) + (x-1)^2$

$\therefore$ $f(x) = 0 \Rightarrow x = 1$ or $7/3$

$f'(x) = (3x-7) + 3(x-1) = 6x - 10 = 2(3x-5)$

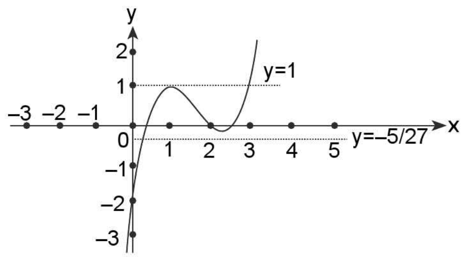
$f'(x)|_{x=1} = 2(-2) = -4$

$f'(x)|_{x=7/3} = 2(2) = 4$

$\therefore$ $x = 1$ is a point maxima and $x = 7/3$ is a point of minima and $f(x)|_{x=1} = 1, f(x)|_{x=7/3} = 7/3 = \frac{-5}{27}$

Checking at end point of the interval, we have $f(x)|_{x=0} = -2$ and $f(x)|_{x=3} = 1$

Hence, the graph of $y = f(x)$ will be given as shown below.



From the diagram we can see that range of $f(x)$ is $[-2, 1]$

Hence, option (a) is incorrect.

The absolute maximum is achieved at two different point ($x = 1$, and 3) so, option (b) is correct.

The absolute minimum is achieved at $x = 0$, so, option (c) is incorrect.

And $f(x) = a$ has 3 distinct root for $a \in \left(-\frac{5}{27}, 1\right)$.

Hence, option (d) is correct.

7. Given $f(x) = \begin{cases} \frac{5x^2-35x+50}{2x^2-3x-35}; & x \neq 5 \\ \frac{15}{17}; & x = 5 \end{cases}$ be a function, then which of the following statement(s) is/are correct ?

(a) $f(x)$ is continuous every where except for $x = 5$

(b) $f(x)$ is an increasing function on $\left(-\infty, -\frac{7}{2}\right) \cup \left(-\frac{7}{2}, \infty\right)$

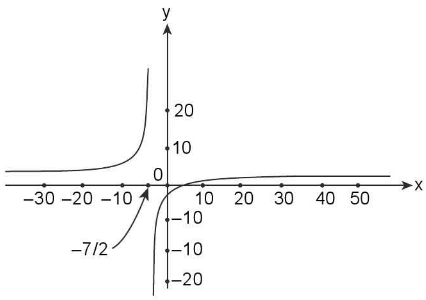
(c) $f(x)$ has exactly two vertical asymptotes

(d) $f(x)$ has only one horizontal asymptote

Solution: (d): $f(x) = \begin{cases} \frac{5(x-2)(x-5)}{(2x+7)(x-5)}; & x \neq 5 \\ \frac{15}{17} & ; x = 5 \end{cases}$

$$= \begin{cases} \frac{5(x-2)}{(2x+7)}; x \neq 5 \\ \frac{15}{17} \quad ; x = 5 \end{cases}$$

Obviously $f(x)$ is not defined for $x = -7/2$. Hence $x = -7/2$ will be a vertical asymptote. Also $x = -7/2$ will be a point of discontinuity. Hence, option (a) is incorrect. The graph of $f(x)$ is as shown below



As is evident from the graph $f(x)$ is an increasing function at every point in $\left(-\infty, -\dfrac{7}{2}\right)$ and at every point $\left(-\dfrac{7}{2}, \infty\right)$, but for $x_1 \in \left(-\infty, -\dfrac{7}{2}\right)$ & $x_2 \in \left(-\dfrac{7}{2}, \infty\right)$

$\Rightarrow f(x_1) > f(x_2)$ for $x_1 < x_2$

$\Rightarrow f(x)$ is an increasing function on

$$\left(-\infty, -\frac{7}{2}\right) \cup \left(-\frac{7}{2}, \infty\right)$$

Hence option (b) is incorrect.

Also $x = -7/2$ is the only vertical asymptote, hence, option (c) is incorrect

Let $y = \dfrac{5(x-2)}{2x+7}$ $\quad\Rightarrow\quad 2xy + 7y = 5x - 10$

$\Rightarrow\ 2xy - 5x = -7y - 10 \Rightarrow\ x(5 - 2y) = 7y + 10$

$\Rightarrow\ x = \dfrac{7y+10}{5-2y}$ $\quad\Rightarrow\quad f^{-1}(x) = \dfrac{7x+10}{5-2x}$

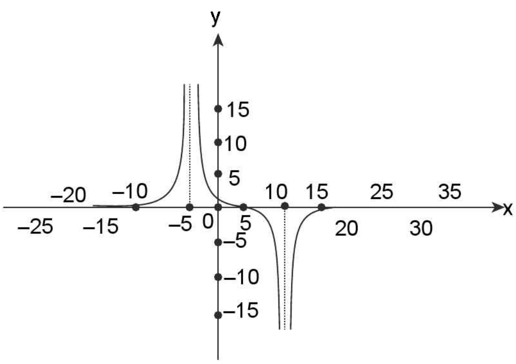
So, $f^{-1}(x)$ is not defined on $x = 5/2$

$\Rightarrow f(x)$ does not achieve 5/2

Hence, $x = 5/2$ is the only horizontal asymptote.

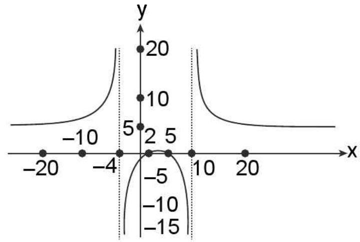
Hence, option (d) is correct.

8. If the graph of $y = f(x)$ is shown as below

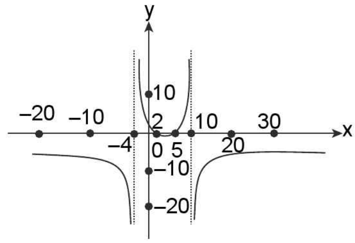


Given $f(x) = \dfrac{p(x)}{q(x)}$; where $p(x)$ and $q(x)$ are quadratic expressions. Also known that $f(x)$ is a rational function with $f(2) = f(5) = 0$; then which of the following represents the approximate graph of $y = f(x)$.

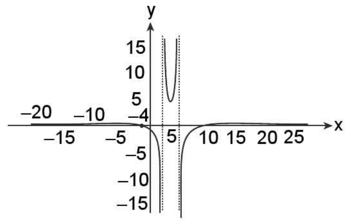
(a)



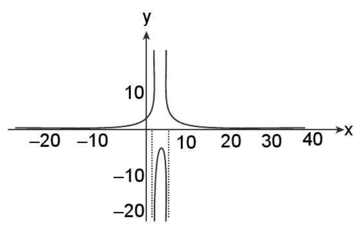
(b)



(c)



(d)



Solution: **(a)** As is evident from the graph of $y = f'(x)$ The graph of $f'(x)$ has vertical asymptote at $x = -4$ and $x = 10$

Hence, the graph of $f(x)$ is not defined at $x = -4$ and at $x = 10$.

Hence, $q(x) = a(x + 4)(x - 10)$; $a \in \mathbb{R} \sim \{0\}$

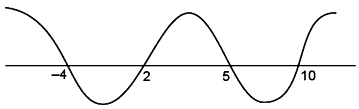
Also given $f(2) = f(5) = 0$

$\Rightarrow p(x) = b(x - 2)(x - 5)\ \forall\ b \in \mathbb{R} \sim \{0\}$

Hence, $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{b}{a}\dfrac{(x-2)(x-5)}{(x+4)(x-10)}$

Now, using the graph of $y = f'(x)$, we need to find the sign of $\dfrac{b}{a}$

If b/a is positive, then Wavy curve of $f(x)$ will be shown as below.



Also $x \to \pm\infty \Rightarrow f(x) \to 1$

Clearly $f(x)$ will have a point of local maxima in the interval $x = (2,5)$

Hence, for a point $c \in (2, 5)$; we have $f(c) = 0$ and $f'(c^-) > 0$.

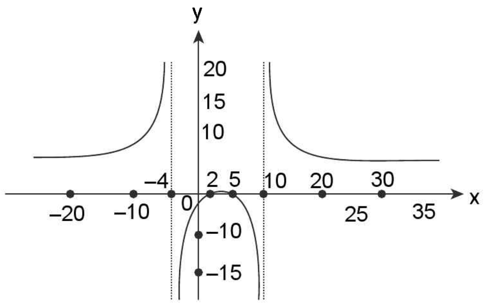
and $f'(c^+) < 0$ and this can be verified from the graph.

Hence, b/a is positive.

Therefore, we need not consider the case when 'b/a' is negative.

Now approximate graph of $y = f(x)$ can be obtained by taking $f(x) = \dfrac{(x-2)(x-5)}{(x+4)(x-10)}$.

And the graph will be given as shown below.



Which is same as in option (a).

9. If $f(x)$ is a function defined as $f(x) = \begin{cases} x+6 & \text{if } x < -2 \\ x^2 & \text{if } -2 \le x < 2 \\ (x-4)^2 & \text{if } x \ge 2 \end{cases}$. Then which of the following statement(s) is/are correct:

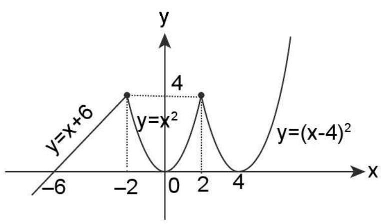
(a) $f(x)$ is continuous at $x \in \mathbb{R}$.

(b) The function is non-differentiable at exactly 3 points.

(c) $f'(x)$ changes its sign exactly 4 times.

(d) $f(x)$ has exactly two local minima and two local maxima.

Solution: (a, c, d) The graph of $y = f(x)$ is as shown below



As is evident from the graph $f(x)$ is continuous at each $x \in \mathbb{R}$, so, option (a) is correct.

The function $f(x)$ is non-differentiable at $x = -2, 2$.

Hence, option (b) is incorrect.

$f'(x)$ changes its sign at $x = -2, x = 0, x = 2$, and $x = 4$

Hence, option (c) is correct.

$f(x)$ has local minima at $x = 0$ and $x = 4$

Also $f(x)$ has local maxima at $x = -2$ and $x = 2$

Therefore, option (d) is correct.

10. Find the area bounded by the curves $\sqrt{|x-y|} \ge \sqrt{x+y}$ and $x^2 + y^2 = 25$

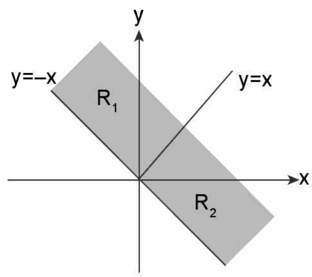
(a) $\dfrac{25\pi}{8}$ (b) $\dfrac{25\pi}{16}$

(c) $\dfrac{25\pi}{4}$ (d) $\dfrac{25\pi}{2}$

Solution: (c) From the inequality we have $x + y \ge 0$

$\Rightarrow y \ge -x$

Case I. $y \ge x$



Considering region R_1; we have $|x - y| = y - x$

Now the inequality $\sqrt{|x-y|} \ge \sqrt{x+y}$

$\Rightarrow \sqrt{y-x} \ge \sqrt{x+y}$

$\Rightarrow y - x \ge x + y \qquad \Rightarrow -2x \ge 0 \Rightarrow x \leq 0$

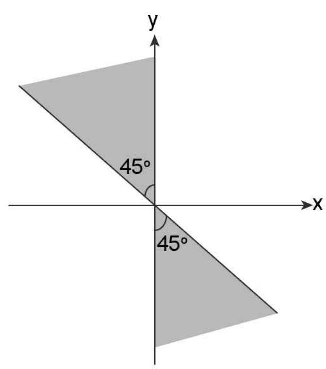
Case II: $x > y$

Considering Region R_2; we have $|x - y| = x - y$

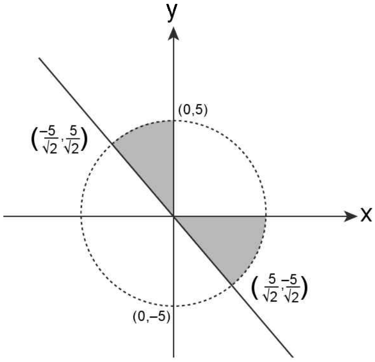
$\therefore$ we have $\sqrt{x-y} \ge \sqrt{x+y}$

$\Rightarrow x - y \ge x + y \qquad \Rightarrow y \le 0$

$\therefore$ The region that satisfy $\sqrt{|x-y|} \ge \sqrt{x+y}$ is shown as below



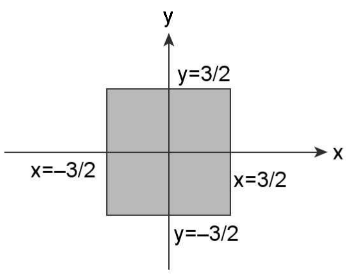
The area bounded by the two curves is as shown below.



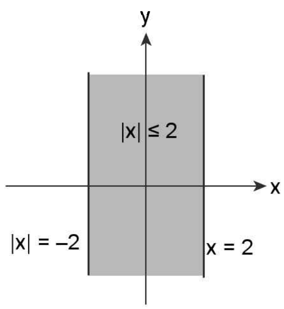
$$\text{Shaded area} = \frac{\pi \times 5^2}{4} = \frac{25\pi}{4}$$

11. Find the region on the x-y plane which satisfies exactly 3 out of the 4 following inequalities:
(i) $|x + y| + |x - y| \le 3$ (ii) $|x| \le 2$
(iii) $y \ge \sqrt{x^2 - 2x + 1}$ (iv) $|y| \le 2$
(a) 8 (b) 9
(c) 7 (d) None of these

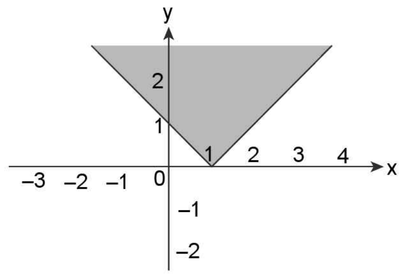
Solution: (**b**) As already known the inequality $|x + y| + |x - y| \le 3$ is plotted as shown below.



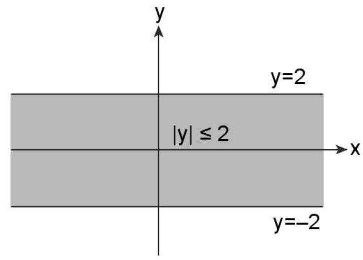
and the graph of $|x| \le 2$ is plotted as shown below.



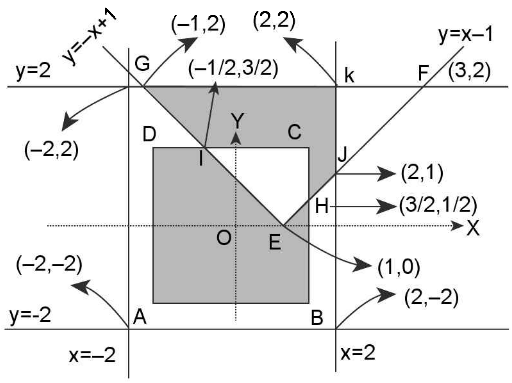
Also $y \ge \sqrt{x^2 - 2x + 1} \Rightarrow y \ge |x - 1|$ and its graph is shown as below.



Similarly, the graph of $|y| \le 2$ is as shown below.



Hence, the region satisfying exactly 3 out of the 4 above mentioned inequalities is shown in figure below.



Now, we need to find the area of the shaded region in the figure shown above.

Area = area ($ABHEID$) + area ($HJKGICH$)
= [area ($ABCD$) – area ($EHCI$)] + [area (EFG) – area ($EHCI$) – area (JFK)]

Now area ($ABCD$) = 9 square units

$$\text{Area }(EHCI) = \frac{1}{2}\begin{vmatrix} 1 & 0 \\ 3/2 & 1/2 \\ 3/2 & 3/2 \\ -1/2 & 3/2 \\ 1 & 0 \end{vmatrix} = \frac{7}{4} \text{ square units}$$

$$\text{Area }(EFG) = \frac{1}{2}\begin{vmatrix} 1 & 0 \\ 3 & 2 \\ -1 & 2 \\ 1 & 0 \end{vmatrix} = \frac{1}{2} \times 8 = 4$$

Area (JFK) = 1/2 square units

$\therefore$ Required Area $= \left(9-\frac{7}{4}\right)+\left(4-\frac{7}{4}-\frac{1}{2}\right)=9$ square units

12. The curve $y = f(x)$ is symmetric about the lines $(10^4 - 1)x + 2(10^4)y + (10^4 + 1) = 0$ and $2(10^4)x + (1 - 10^4)y + [1 - 3(10^4)) = 0$. If (5, 6) lies on the curve, then which of the following points always lies on $f(x)$:

(a) (11, –11) (b) (–3, –8)

(c) (6, 5) (d) $(10^5, 3 \times 10^5)$

Solution: (**b**) Solving both the straight lines together, by observing coefficients carefully, we get (1, –1) is the point of intersection and both the straight lines are perpendicular. If $y = f(x)$ is symmetric about two perpendicular lines then it is also symmetric about their point of intersection. Applying the fact $y = f(x)$ is symmetric about the point (1, –1), and therefore, image of (5, 6) in (1, –1) say (α,β) also lies on $y = f(x)$. Thus, $\alpha + 5 = 2$ and $\beta + 6 = -2$

$\Rightarrow$ $\alpha = -3$ and $\beta = -8$. Consequently (–3, –8) always lies on the curve $y = f(x)$.

13. If the equation $|x^2 - 5x + 6| - \lambda x + 7\lambda = 0$ has exactly 3 solutions, then λ is equal to

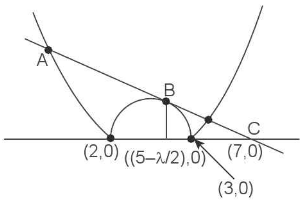
(a) $-7+\sqrt{23}$ (b) $-9+4\sqrt{5}$

(c) $-7-\sqrt{23}$ (d) $-9-4\sqrt{5}$

Solution: (**b**) $-x^2 + 5x - 6 = \lambda x - 7\lambda$ must have equal roots

$\Rightarrow \lambda = -9 \pm 4\sqrt{5};\ x = \frac{5-\lambda}{2} \in (2,3)$

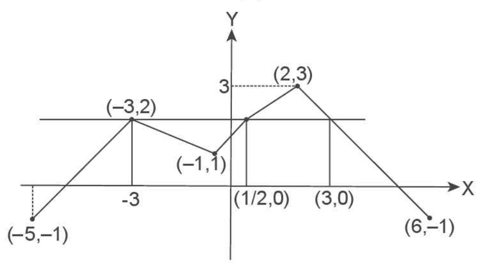
$\Rightarrow \lambda \in (-1, 1) \qquad \Rightarrow \lambda = -9+4\sqrt{5}$



14. The graph of $y = f(x)$ is shown, then the number of solutions of $f(f(x)) = 2$ will be

(a) 1 (b) 2

(c) 3 (d) 4



Solution: (**c**) Let $f(x) = t$

$\Rightarrow f(t) = 2$, so, $t = -3,\ 1/2,\ 3$

Now, $f(x) = -3 \rightarrow$ no solution

$f(x) = 1/2 \rightarrow$ 2 solutions

$f(x) = 3 \rightarrow$ 1 solution.

15. The number of real roots of the polynomial equation $x^8 + x^6 + 4x^2 - 2x + 4 = 0$ is

(a) 0 (b) 2

(c) 4 (d) None of these

Solution: (**a**) Given equation is $x^8 + x^6 + 4x^2 - 2x + 4 = 0$

$\Rightarrow x^8 + x^6 + 4x^2 + 4 = 2x$

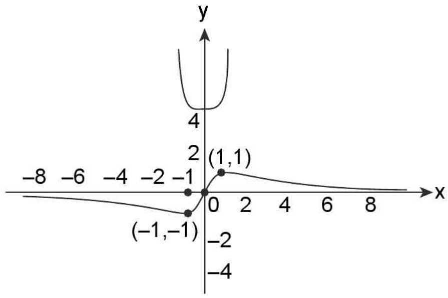
$\Rightarrow x^6(x^2 + 1) + 4(x^2 + 1) = 2x$

$\Rightarrow (x^2 + 1)(x^6 + 4) = 2x$

$$\Rightarrow x^6 + 4 = \frac{2x}{x^2+1}$$

Now range of L.H.S. $[4, \infty)$ whereas range of R.H.S. $[-1, 1]$.

Clearly, no real solutions



16. Number of real values of x satisfying the equation $f(|x|) = x - [x + 1]$; where $f(x)$ is defined as

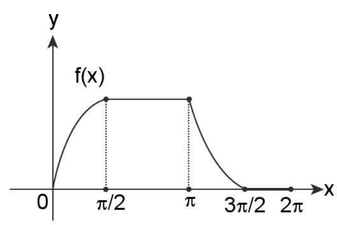
$$f(x)=\begin{cases} \max(\sin t); & 0 \le t \le x; \quad \forall x \in [0,\pi] \\ \min(1+\sin t); & \pi < t \le x; \quad \forall x \in [\pi,2\pi] \end{cases}$$ is

equal to ([] is greatest integer function)

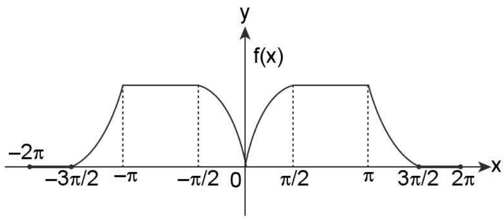
(a) 3 (b) Infinitely many

(c) 2 (d) None of these

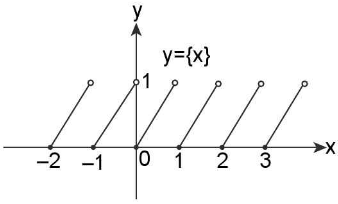
Solution: (**d**) The graph of $f(x)$ is as shown below



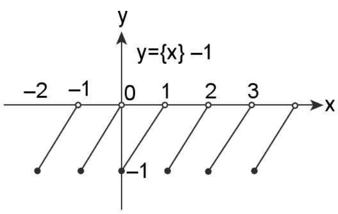
Hence, the graph of $f|x|$ will be as shown below.



Now let $g(x) = x - [x + 1] = x - [x] - 1 = \{x\} - 1$
Graph of $\{x\}$ is as shown below.



Graph of $g(x) = \{x\} - 1$ is shown as below.

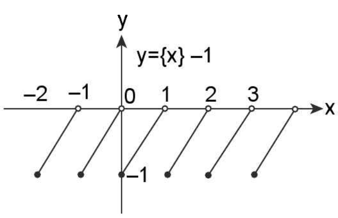


Now as is evident from the graph $y = f(|x|)$ is always greater than or equal to 0, whereas $g(x)$ is always less than 0.
Hence, the two graphs never intersect each other, and hence, no real value of x satisfies $f(|x|) = x - [x + 1]$

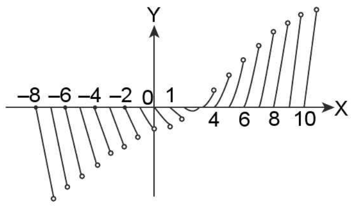
17. If $\{x\}$ denotes the fractional part of x, then find the number of real solutions of the equation $(x - 3)\{x\} = \{x\} - 1 \ \forall\ x > -4$

(a) 16 solutions (b) ∞ solutions
(c) 5 solutions (d) None of these

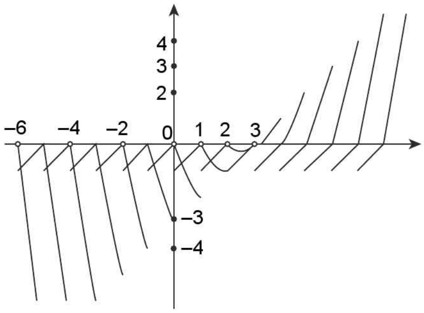
Solution: (c) The graph of $y = \{x\} - 1$ is as given below.



The graph of $y = (x - 3)\{x\}$ is as given below.



Sketching the two curves in the same frame of reference, we get



As is evident from the graph, we have 6 solutions.

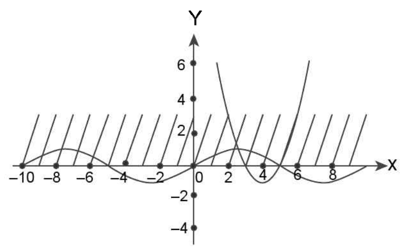
18. A function $f : \mathbb{R} \to \mathbb{R}$ is defined such that

$$f(x) = \begin{cases} 2\sin^2\left(\dfrac{2}{\pi}x\right); & 0 < x \le 1 \\ 3\{x\}; & 1 < x \le \dfrac{4}{3} \\ x^2 - 8x + 15; & x \ge \dfrac{4}{3} \end{cases}$$

and $2f(0) = 1$, the number of real solution of equation $2f(|x|) = 1$ is

(a) 9 (b) 8
(c) 11 (d) None of these

Solution: (c) Graph of $y = f|x|$ is as shown in the figure. We can see that $f(0) = \dfrac{1}{2}$, and therefore, line $y = \dfrac{1}{2}$ cuts the graph at 11 distinct, points therefore, the equation $2f(|x|) = 1$ has 11 distinct solutions.



19. The number of ordered pairs (x, y) satisfying the following two curves simultaneously $|y| = 3\sin\left(\dfrac{\pi x}{2}\right)$ and $|y| = \log_{10}|x^3|$ is equal to

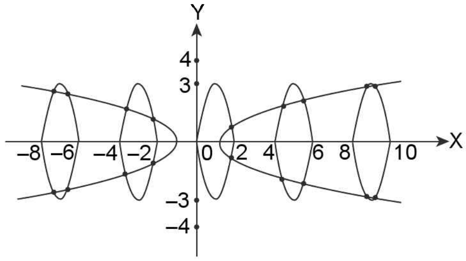
(a) 18 (b) 17
(c) 16 (d) None of these

Solution: (a) $\because |y| = \log_{10}|10^3| = 3$

$\Rightarrow y = \pm 3$ and $|y| = 3\sin\left(\dfrac{\pi x}{2}\right)$

$\Rightarrow y \in [-3, 3]$

$\Rightarrow$ No point of intersection after $x = 10$ and before $x = -10$.



SECTION–II

SUBJECTIVE-TYPE SOLVED EXAMPLES

1. Investigate and graph the following functions $y = \dfrac{1}{2}\sin 2x + \cos x$:

Solution: The function is defined and continuous throughout the number scale and has a period 2π. Therefore, in investigating we may confine ourselves to the interval $[0, 2\pi)$. The graph of the function has no asymptote by virtue of continuity and periodicity. Find the first derivative $y' = \cos 2x - \sin x$.

Now, $y' = 0$

$$\Rightarrow x_1 = \frac{\pi}{6},\ x_2 = \frac{5\pi}{6},\ x_3 = \frac{3\pi}{2}$$

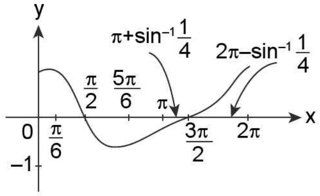
Evaluate the second derivative; $y'' = -2\sin 2x - \cos x$.

On the interval $[0, 2\pi]$, it has four roots

$$x_1 = \frac{\pi}{2},\ x_2 = \pi + \text{arc}\ \sin\left(\frac{1}{4}\right),\ x_3 = \frac{3\pi}{2},$$

$$x_4 = 2\pi - \text{arc}\ \sin\left(\frac{1}{4}\right)$$

The results of the above investigations enable us to construct the graph of the function as shown below.

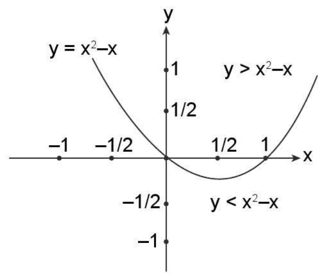


2. Plot the curve $y = \max\{x^2 - y, .x\}$

Solution: when $x^2 - y \geq x$

i.e., for $y \leq x^2 - x$

Now $y = x^2 - x$

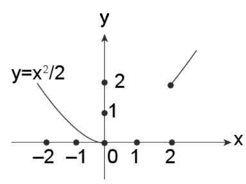


Now for the region representing $y \leq x^2 - x$, max $\{x^2 - y, x\} = x^2 - y \quad \Rightarrow y = x^2 - y \Rightarrow y = \dfrac{x^2}{2}$

$\Rightarrow \dfrac{x^2}{2} \leq x^2 - x \quad \Rightarrow x \in (-\infty, 0] \cup [2, \infty)$

$\therefore$ In region $y \leq (x^2 - x)$

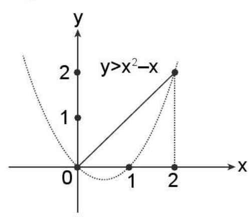
$\Rightarrow y = \dfrac{x^2}{2}$ will be as shown below.



$\therefore\ y > x^2 - x \quad \Rightarrow x > x^2 - x$

$\Rightarrow x^2 - 2x < 0 \quad \Rightarrow x \in (0, 2)$

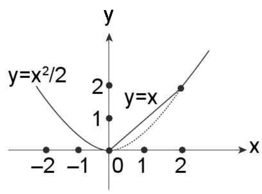
The corresponding graph in region $y > x^2 - x$, will be shown below by thick curve shown below.



And for the region satisfying $y > x^2 - x$

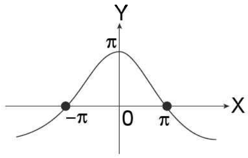
⇒ max $\{x^2 - y, x\} = x$, then $y = x$

Hence, the graph of $y = \max\{x^2 - y, x\}$ will be as shown by thick curve given below.

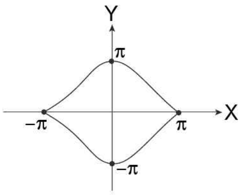


3. Find the number of point of intersection of the graphs $|y| = |x|^2 - 2|x| - 3|$ and $|y| = \pi \cos x/2$

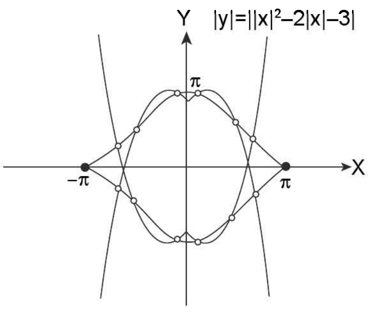
Solution: $y = \pi\cos\left(\frac{x}{2}\right)$



$|y| = \pi \cos x/2$



Now sketching the two curves together, we get



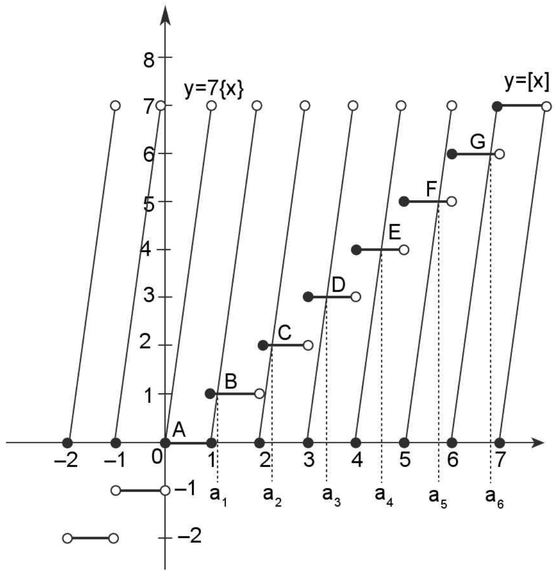
Clearly, the two curves intersect at 12 points

4. Find the number of solutions of $\frac{[x]}{\{x\}} = 7$; where $[x]$ denotes the greatest integer function and $\{x\}$ denotes the fractional part of x

Solution: Clearly, x can't be an integer

$(\because \{x\} = 0 \ \forall \ x \in \mathbb{Z})$

⇒ $[x] = 7\{x\}$



Now, clearly there are only 6 point of intersection, and hence, only 6 values of x which satisfy the above equation, as A(0, 0) corresponds to $x = 0 \in \mathbb{Z}$.

Now, these solutions can be found out diagram magically

$a_1 = 1 + 1/7$, $a_4 = 4 + 4/7$

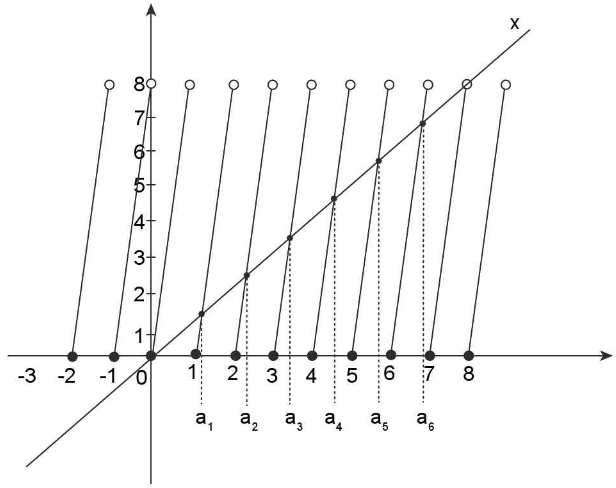
$a_2 = 2 + 2/7$, $a_5 = 5 + 5/7$

$a_3 = 3 + 3/7$, $a_6 = 6 + 6/7$

Aliter, $[x] = x - \{x\}$, then $[x] = 7\{x\}$

⇒ $x - \{x\} = 7\{x\}$

⇒ $x = 8\{x\}$



As shown in the diagram above, there are 6 number of solutions.

Since, $0 < \{x\} < 1$

Now, $[x] = 7\{x\} \in (0, 7)$

⇒ $0 < [x] < 7$

If $[x] = 1$, then $\{x\} = 1/7$

$\Rightarrow x = [x] + \{x\} = 1+1/7 = 8/7$

Similarly when $[x] = 2$; $\{x\} = 2/7 \Rightarrow x = 2 + 2/7$

$[x] = 3$; $\{x\} = 3/7 \Rightarrow x = 3 + 3/7$

$[x] = 4$; $\{x\} = 4/7 \Rightarrow x = 4 + 4/7$

$[x] = 5$; $\{x\} = 5/7 \Rightarrow x = 5 + 5/7$

$[x] = 6$; $\{I\} = 6/7 \Rightarrow x = 6 + 6/7$

5. Sketch the function $y = \dfrac{2}{1+x^2}$

Solution: ***Step 1:*** Domain of function is $(-\infty, \infty)$ $f(0) = 2$ and function is symmetrical about y-axis.

Step 2: $\dfrac{dy}{dx} = \dfrac{-4x}{(1+x^2)^2} = 0$; $\dfrac{d^2y}{dx^2} = \dfrac{4(3x^2-1)}{(1+x^2)^3}$

$\Rightarrow f'(x) = 0$ at $x = 0$ and $f''(x) = 0$ at $x = \pm\dfrac{1}{\sqrt{3}}$

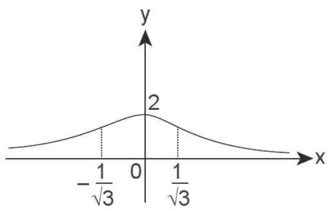
$\Rightarrow f'(x)$ is concave downwards for $x \in \left(-\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$

and $\dfrac{dy}{dx} > 0$ *where* $x < 0$ (y increasing),

$\dfrac{dy}{dx} < 0$ when $x > 0$ (y decreasing) and

Point $x = 0$ is point of maxima.

Step 3: $\lim\limits_{x\to\pm\infty} y = 0$, so, graph of the function is



6. Trace the curve $(x^2 - a^2)(y^2 - b^2) = a^2 b^2$

Solution: 1. The curve is symmetrical about both the axes.

2. Origin is a point on the curve, the tangents at the origin being $b^2x^2 + a^2y^2 = 0$. Since the tangents at the origin are imaginary, therefore, the origin is a conjugate point. The curve does not meet the axes at any other point.

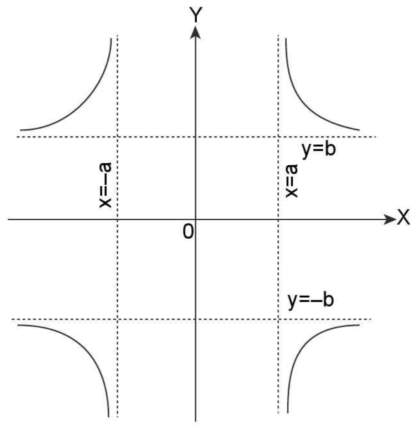
3. Re-writing the equation in the form $y = \pm\sqrt{\dfrac{b^2x^2}{x^2-a^2}}$ we find that $|x| > a$. Similarly it can be seen that $|y| > b$. Therefore, no portion of the graph (except the conjugate point (0, 0)) lies in the region bounded by the lines $x = \pm a$, $y = \pm b$.

4. The lines $x = \pm a$, $y = \pm b$ are the asymptotes of the curve. The portion of the curve lying in the first quadrant is above the asymptote $y = b$, and on the right of the asymptote $x = a$.

5. Since $2x(y^2 - b^2) + 2y(dy/dx)(x^2 - a^2) = 0$, therefore, if $x > 0$, $y > 0$, then $dy/dx < 0$. Therefore, in the first quadrant, y decreases as x increases. Also $y \to +\infty$ as $x \to a$, and $y \to b$ as $x \to +\infty$.

6. The portion or the graph lying in the first quadrant can now be drawn.

By using symmetries, the graph can be completed. The graph is as shown in the figure given below.



7. Find the interval in which all the roots of the equation $x^2 - 40x + 76 + 68\{x\} = 0$ lie in the interval ($\{x\}$ represents fractional part of x).

Solution: Let $f(x) = x^2 - 40x + 76$ and $g(x) = -68\{x\}$

Now $-68 < g(x) \leq 0$. Now $f(x) = 0$

$\Rightarrow x^2 - 40x + 76 = 0 \Rightarrow x = 2, 38$; and $f(x) = -68$

$\Rightarrow x^2 - 40x + 76 = -68$

$\Rightarrow x^2 - 40x + 144 = 0 \Rightarrow x = 4, 36$.

Thus, $f(x) \in [-68, 0]$ for $x \in [2, 4] \cup [36, 38)$

$\therefore$ $f(x) = g(x)$ will have all its roots in the interval $[2, 38)$ and the roots of $f(x) = g(x)$ can not lie beyond the roots of $f(x) = 0$ and $f(x) = 68$. Thus, the required interval is $[2, 4] \cup [36, 38)$.

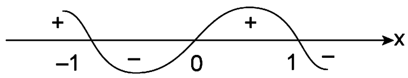
8. Using the first derivative, find the extrema of the following functions $f(x) = 3\sqrt[3]{x^2} - x^2$

Solution: The function is defined and continuous throughout the number scale.

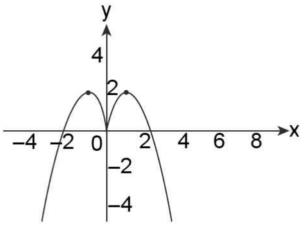
Let us find the derivative: $f'(x) = 2\left(\dfrac{1}{\sqrt[3]{x}} - x\right)$

From the equation $f'(x) = 0$, we find the roots of the derivative $x = \pm 1$.

Furthermore, the derivative goes to infinity at the point $x = 0$. Thus, the critical points are $x_1 = -1, x_2 = 0$, $x_3 = 1$. The results of investigating the sign neighborhood of these points are given in figure.



The investigation shows that the function has two maxima: $f(-1) = 2, f(1) = 2$ and a minimum $f(0) = 0$.



9. Using the second derivative, find out the character of the extrema of the following functions $y = 2\sin x + \cos 2x$

Solution: Since the function is a periodic one, we may confine ourselves to the interval $[0, 2\pi]$. Find the first and second derivatives: $y' = 2\cos x - 2\sin 2x = 2\cos x\,(1 - 2\sin x)$; $y'' = -2\sin x - 4\cos 2x$.

From the equation $2\cos x(1 - 2\sin x) = 0$, determine the critical points on the interval $[0, 2\pi]$; $x_1 = \pi/6$, $x_2 = \pi/2$, $x_3 = 5\pi/6$, $x_4 = 3\pi/2$

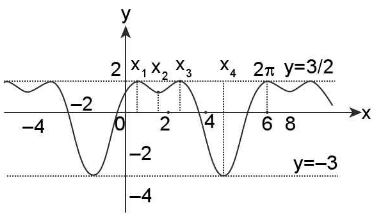
Now, find the sign of the second derivative at each critical point:

$y''(\pi/6) = -3 < 0$; hence, we have a maximum $y(\pi/6) = 3/2$ at the point $x_1 = \pi/6$

$y''(\pi/2) = 2 > 0$; hence, we have a minimum $y(\pi/2) = 1$ at the point $x_2 = \pi/2$

$y''(5\pi/6) = -3 < 0$; hence, we have a maximum $y(5\pi/6) = 3/2$ at the point $x_3 = 5\pi/6$

$y''(3\pi/2) = 6 > 0$; hence, we have a minimum $y(3\pi/2) = -3$ at the point $x_4 = 3\pi/2$ as shown in the figure given below.



10. Find the asymptotes of the following curves $y = xe^{\frac{1}{x}}$:

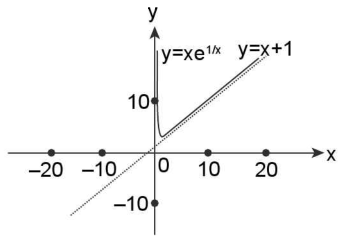
Solution: The curve has a vertical asymptote $x = 0$, since $\lim\limits_{x\to 0^+} y = \lim\limits_{x\to 0^+} xe^{\frac{1}{x}} = \lim\limits_{t=\frac{1}{x}\to+\infty} \dfrac{e^t}{t} = +\infty$

Find the inclined asymptotes:

$$m = \lim_{x\to\pm\infty} \frac{y}{x} = \lim_{x\to\pm\infty} e^{\frac{1}{x}} = e^0 = 1 \text{ and}$$

$$c = \lim_{x\to\pm\infty}\left(xe^{\frac{1}{x}} - x\right) = \lim_{x\to\pm\infty} \frac{e^{\frac{1}{x}} - 1}{1/x} = \lim_{1/x=z\to 0} \frac{e^z - 1}{z} = 1$$

Thus, the straight line $y = x + 1$ will be an inclined asymptote of the curve as shown in the figure



11. Find the asymptotes of the following curves $y = \sqrt{1 + x^2}\,\sin\dfrac{1}{x}$:

Solution: The curve has no vertical asymptotes; since it is continuous at $x \neq 0$ and in the neighbourhood of the point $x = 0$ the function is bounded.

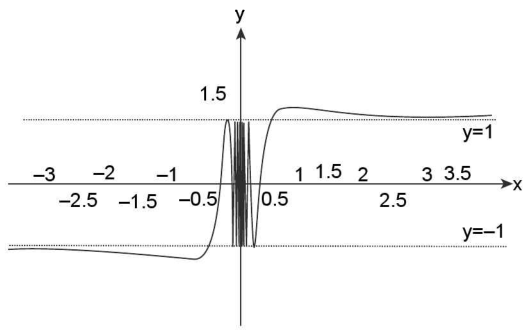
Let us find the inclined asymptotes. We have

$$m = \lim_{x\to+\infty} \frac{y}{x} = \lim_{x\to+\infty} \frac{|x|\sqrt{1 + \frac{1}{x^2}}\,\sin\frac{1}{x}}{x} = \pm 1.0 = 0$$

Then $c = \lim\limits_{x\to+\infty}(y - mx) = \lim\limits_{x\to+\infty} |x|\sqrt{1 + \dfrac{1}{x^2}}\,\sin\dfrac{1}{x}$

$$= \begin{cases} 1; & \text{as } x \to +\infty \\ -1; & \text{as } x \to -\infty \end{cases}$$

Thus, the curve has two horizontal asymptotes: $y = +1$ and $y = -1$ (as shown in the figure). The same result can be obtained proceeding from symmetry about the origin and keeping in mind that the function y is odd.



12. Trace the curve $y^2(x - a) = x^2(x + a)$, $a > 0$.

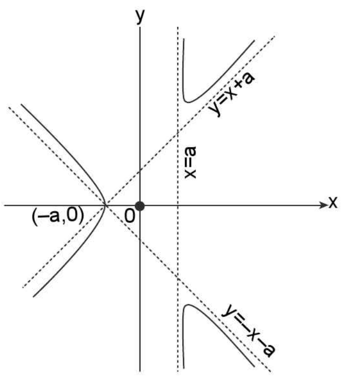
Solution: 1. The curve is symmetrical with respect to the x-axis.

2. The curve meets the x-axis at the points given by $x = 0$, and $x = -a$, i.e., at $(0, 0)$ and $(-a, 0)$. It does not meet the y-axis except at the origin. The tangents at $(0, 0)$ are given by $x^2 + y^2 = 0$. Since the tangents at $(0, 0)$ are imaginary, therefore, $(0, 0)$ is a conjugate point. The tangent at $(-a, 0)$ can be easily seen to be the line $x + a = 0$.
3. If $-a < x < a$ and $x \neq 0$, then $y^2 < 0$, so that no part of the graph except the origin lies in the region bounded by the ordinates $x = \pm a$.
4. The asymptotes can be easily seen to be the lines $x = a$, $y = \pm (x + a)$. The two branches of the curve are $y = \pm x\,[(x + a)/(x - a)]^{1/2}$.

   For the branch $y = x[(x + a)/(x - a)]^{1/2}$, we have $y = x + a + a^2/2x + \ldots$

   So that if $x > 0$, then the curve is above the line $y = x + a$, and if $x < 0$, then the curve is below $y = x + a$. Similarly, it can be seen that if $x > 0$, then the curve is below $y = -x - a$, and if $x < 0$, then the curve is above $y = -x - a$. Also, if $x > 0$, then the curve is to the right of $x = a$

   The graph of the curve can now be easily seen to be as is shown in the figure given below.

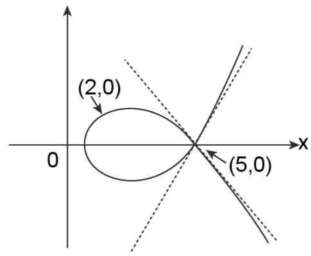


13. Sketch the graph of the curve $y^2 = (x - 2)(x - 5)^2$.

Solution: 1. The curve is symmetrical with respect to the x-axis

2. If $x < 2$, then $y^2 < 0$, so, no portion of the curve lies to the left to the ordinate $x = 2$.
3. The graph meets the x-axis at the points $x = 2$ and $x = 5$. It does not meet the y-axis. The tangents at $(2, 0)$ is the line $x = 2$. The tangents at $(5, 0)$ are the lines $y = \pm\sqrt{3}(x - 5)$. This can be seen by shifting the origin to $(5,0)$ by the transformation $x = X + 5$, $y = Y$. The equation of the curve transforms to $Y^2 = X^2(X + 3)$. Equating to zero the lowest degree terms, the tangents are given by $Y^2 = 3X^2$, i.e., $y^2 = 3(x - 5)^2$]. Since there are two distinct tangents at $(5, 0)$, therefore, it is a node.
4. There are no asymptotes.
5. As $x \to +\infty$, $y^2 \to +\infty$

   The graph can now be easily seen to be as in the figure shown below:



14. Find the asymptotes of the following curves $y = \dfrac{3x}{2}\ln\left(e - \dfrac{1}{3x}\right)$:

Solution: The function is defined and continuous at $e - \dfrac{1}{3x} > 0$. i.e., at $x < 0$ and $x > \dfrac{1}{3e}$.

Since the function is continuous at every point of the domain of definition, vertical asymptotes can exist only on finite boundaries of the domain of definition.

As $x \to 0^-$, we have $\lim\limits_{x \to 0^-} y = \lim\limits_{x \to 0-} \dfrac{3x}{2}\ln\left(e - \dfrac{1}{3x}\right)$

$$= -\frac{1}{2}\lim_{z \to \infty}\frac{\ln(e + z)}{z} = 0\ ;\ \text{Putting}\ \left(z = -\frac{1}{3x}\right)$$

i.e., the straight line $x = 0$ is not a vertical asymptote.

As $x \to \left(\dfrac{1}{3e}\right)^+$, we have

$$\lim_{x \to \left(\frac{1}{3e}\right)^+} y = \frac{3}{2}\lim_{x \to \left(\frac{1}{3e}\right)^+} x\ln\left(e - \frac{1}{3x}\right) = -\infty$$

i.e., the line $x = 1/(3e)$ is a vertical asymptote.

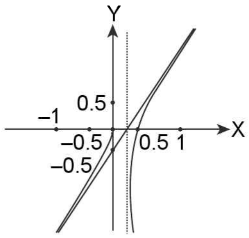
Now let us find the inclined asymptotes

$$m = \lim_{x \to \pm\infty}\frac{y}{x} = \frac{3}{2}\lim_{x \to \pm\infty}\ln\left(e - \frac{1}{3x}\right) = \frac{3}{2}$$

$$c = \lim_{x \to \pm\infty}[y - mx] = \frac{3}{2}\lim_{x \to \pm\infty} x\left[\ln\left(e - \frac{1}{3x}\right) - 1\right]$$

$$= \frac{-1}{2e}\lim_{x \to \pm\infty}\frac{\ln\left(1 - \dfrac{1}{3xe}\right)}{\left(1 - \dfrac{1}{3xe}\right)} = -\frac{1}{2e}$$

Hence, the straight line $y=\frac{3x}{2}-\frac{1}{2e}$ is an inclined asymptote as shown in the figure given below:



15. Investigate and graph the following functions $y = x^6 - 3x^4 + 3x^2 - 5$:

Solution: The function is defined and continuous throughout the number scale, therefore, the curve has no vertical asymptote. The function is even, since $f(-x) = f(x)$. Consequently, its graph is symmetric about the y-axis, and therefore, it is sufficient to investigate the function only on the interval $[0, \infty)$.

There are no inclined asymptotes, since as $x \to \infty$ the quantity y turns out to be an infinitely large quantity of the sixth order with respect to x.

Investigate the first derivatives $y' = 6x^5 - 12x^3 + 6x = 6x(x^4 - 2x^2 + 1) = 6x(x^2 - 1)^2$; the critical points are:

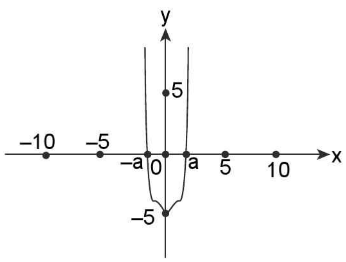
$$x_1 = -1, \qquad x_2 = 0, \qquad x_3 = 1$$

Since in the interval $[0, \infty)$ the derivative $y' \geq 0$, the function increases.

Investigate the second derivative $y'' = 30x^4 - 36x^2 + 6 = 6(5x^4 - 6x^2 + 1)$

The positive roots of the second derivative $x_1 = \frac{1}{\sqrt{5}}$, $x_2 = 1$.

Using the results of the investigation and taking into consideration the symmetry principle, we construct the graph of the function. As is seen from the graph, the function has roots $x = \pm a$, where $a \approx 1.6$



16. Investigate and graph the following functions $y = x + \ln(x^2 - 1)$

Solution: The function is defined and continuous at all values of x for which $x^2 - 1 > 0$ or $|x| > 1$, i.e., on two intervals $(-\infty, -1)$ and $(1, +\infty)$

We seek the vertical asymptotes:

$$\lim_{x\to -1^-} y = \lim_{x\to -1^-} \left[x+\ln(x^2-1)\right] = -\infty$$

$$\lim_{x\to 1^+} y = \lim_{x\to 1^+} \left[x+\ln(x^2-1)\right] = -\infty$$

Thus, the curve has two vertical asymptotes: $x = -1$ and $x = +1$.

Find inclined asymptotes:

$$m = \lim_{x\to\pm\infty} \frac{y}{x} = \lim_{x\to\pm\infty} \frac{x+\ell n(x^2-1)}{x}$$

$$= \lim_{x\to\pm\infty} \left[1+\frac{\ell n(x^2-1)}{x}\right] = 1 \text{ and}$$

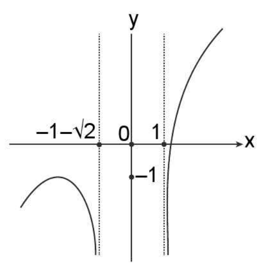
$$c = \lim_{x\to\pm\infty} [y-x] = \lim_{x\to\pm\infty} \ln\left(x^2-1\right) = \infty$$

Hence, the curve has neither inclined nor horizontal asymptotes.

Since the derivative $y' = 1+\frac{2x}{x^2-1}$ exists and is finite at all points of the domain of definition of the function, only the zeros of the derivative. $x_1 = -1 - \sqrt{2}$; $x_2 = -1 + \sqrt{2}$ can be critical points. At the point $x_2 = -1 + \sqrt{2}$ the function is not defined; hence, there is only one critical point $x_1 = -1 - \sqrt{2}$ belonging to the interval $(-\infty, -1)$. In the interval $(1, \infty)$ derivative $y' > 0$, and hence, the function increases. The second derivative $y'' = -\frac{2(x^2+1)}{(x^2-1)^2} < 0$, hence, the curve is convex upwards everywhere, and at the point $x_1 = -1 - \sqrt{2} \approx -2.41$ the function has a maximum $y(-1 - \sqrt{2}) \approx -1 - \sqrt{2} + \ln(2 + 2\sqrt{2}) \approx -0.84$.

To plot the graph in the interval $(1, \infty)$, where there are no characteristics points, we choose the following additional points $x = 2$; $y = 2 + \ln 3 \approx 3.10$ and $x = 1.2$; $y = 1.2 + \ln 0.44 \approx 0.38$.

The graph of the function is shown below.



17. Find the asymptotes of the curve $y = \dfrac{x^2+2x-1}{x}$ and hence, sketch.

Solution: Here, the curve $y = \dfrac{x^2+2x-1}{x}$ could be written as $x^2 + 2x - yx - 1 = 0$

(i) No asymptote parallel to x-axis

(ii) Asymptote parallel to y-axis $\Rightarrow x = 0$

(iii) Oblique asymptote

Let $y = mx + c$ be oblique asymptote

$\therefore \quad x^2 + 2x - x(mx + c) - 1 = 0$

$x^2 - mx^2 + 2x - cx - 1 = 0$

$\Rightarrow \quad x^2(1-m) + x(2-c) - 1 = 0$

For oblique asymptote equate the highest power and second highest power of x to zero.

i.e., Coefficient of $x^2 = 0 \Rightarrow m = 1$

Coefficient of $x = 0$

$\Rightarrow c = 2$

$y = x + 2$ is oblique asymptote to $y = x - 1/x + 2$

(iv) Neither symmetric about axis nor about origin

(v) Domain is $\mathbb{R} - \{0\}$

(vi) Range is $\mathbb{R}$

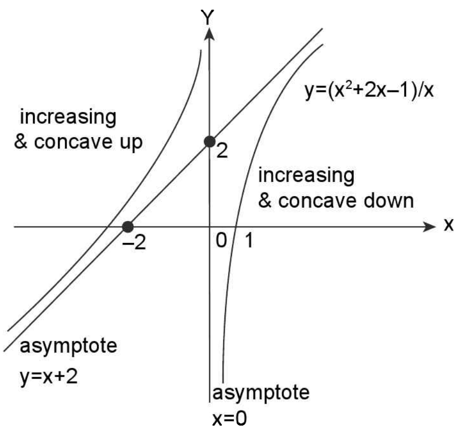
(vii) $\dfrac{dy}{dx} = 1 + \dfrac{1}{x^2} \quad \Rightarrow \quad \dfrac{dy}{dx} > 0$ for all $x \in \mathbb{R} - \{0\}$

(viii) $\dfrac{d^2y}{dx^2} = -\dfrac{2}{x^3} > 0$ for $x < 0$ and < 0 for $x > 0$

$\Rightarrow$ when $x > 0$, curve is concave down.

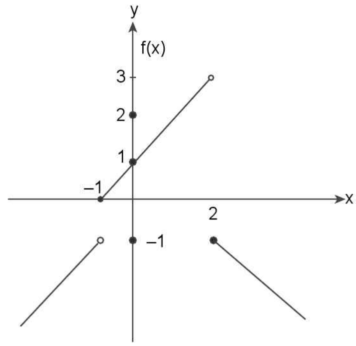
when $x < 0$, curve is concave up.

From the above information, we can plot the curve $y = x - 1/x + 2$ as shown below:



18. Let $f(x) = \begin{cases} x & ; x < -1 \\ x+1 & ; -1 \le x < 2 \\ 1-x & ; x \ge 2 \end{cases}$ and $g(x) = \begin{cases} |x| & ; x \le 2 \\ 3x & ; x > 2 \end{cases}$, then find $fog(x)$ and $gof(x)$

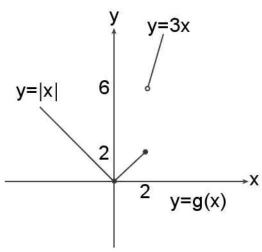
Solution: The graph of $f(x)$ is as shown below.



Clearly; $f(x) \le 2\ \forall\ x \in R \sim (1,2)$ and $f(x) > 2\ \forall\ x \in (1, 2)$

The graph of $g(x)$ is as shown below.

Clearly $g(x) < 0$ is not satisfied for any value of (x)



Now $0 \le g(x) < 2 \Rightarrow x \in [0, 2)$ and $g(x) \ge 2$

$\Rightarrow x \in \mathbb{R} \sim (-2, 2)$

To find fog (x), i.e., $f(g(x))$

$$= \begin{cases} g(x); & g(x) < -1 \\ g(x)+1; & -1 \le g(x) < 2 \text{ or } 0 \le g(x) < 2 \\ 1-g(x); & g(x) \ge 2 \end{cases}$$

$$= \begin{cases} g(x)+1\ ; & x \in (-2,2) \\ 1-g(x)\ ; & x \in \mathbb{R} \sim (-2,2) \text{ or } (-\infty,-2] \cup [2,\infty) \end{cases}$$

Also for $x \in (-2,2)$; $g(x) = |x|$

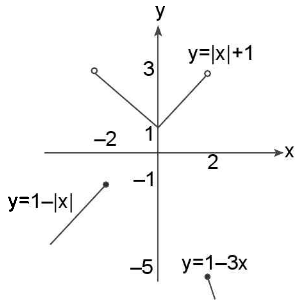
for $x \in (-\infty,-2]$; $g(x) = |x|$

and for $x = 2$; $g(x) = |x|$

and for $x \in (2,\infty)$; $g(x) = 3x$

$$\Rightarrow \quad f(g(x)) = \begin{cases} 1-|x| & ; x \in (-\infty,-2] \\ |x|+1 & ; x \in (-2,2) \\ 1-|x| & ; x = 2 \\ 1-3x & ; x \in (2,\infty) \end{cases}$$

$\therefore$ The graph of $f(g(x))$ will be as shown below



To find $gof(x)$

$$g(f(x)) = \begin{cases} f|(x)|; & f(x) \le 2 \\ 3f(x); & f(x) > 2 \end{cases} = \begin{cases} |f(x)|; x \in \mathbb{R} \sim (1,2) \\ 3f(x); x \in (1,2) \end{cases}$$

Also for $x < -1$; $f(x) = x$

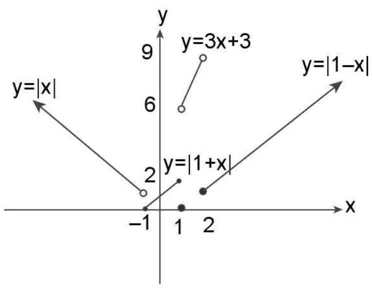
for $x \in [-1, 1]$; $f(x) = x + 1$

for $x \in (1, 2)$; $f(x) = x + 1$

and for $x \ge 2$; $f(x) = 1 - x$

$$\therefore \begin{cases} |x| & ; x < -1 \\ |x+1| & ; x \in [-1,1] \\ 3(x+1) & ; x \in (1,2) \\ |1-x| & ; x \ge 2 \end{cases}$$

$\therefore$ The graph of $g(f(x))$ will be as shown below



19. Graph the function $y = x^3 - 5x^2 + 4x$

Solution: ***Step 1:*** Factorize the function, if possible, $y = x^3 - 5x^2 + 4x$

$= x(x^2 - 5x + 4)$

$= x(x^2 - x + 4 - 4x)$

$= x(x - 1)(x - 4)$

Step 2: $y = 0 \Rightarrow x(x - 1)(x - 4) = 0$

$\Rightarrow$ $x = 0$ or $x = 1$ or $x = 4$

$\Rightarrow$ The function intersects the x-axis at the three point

i.e., $x = 0$ or $x = 1$ or $x = 4$.

Step 3: Find points of local maximum/minima

$y' = 3x^2 - 10x + 4$

$y' = 0$ for points of local maximum/minima.

$\Rightarrow$ $3x^2 - 10x + 4 = 0$

$$\Rightarrow x = \frac{10 \pm \sqrt{100-48}}{6} = \frac{10 \pm 7.2}{6} = 2.86 \text{ or } 0.47$$

$$\Rightarrow \frac{d^2y}{dx^2} = 6x - 10 \qquad \Rightarrow f''(2.86) = +\text{ve}$$

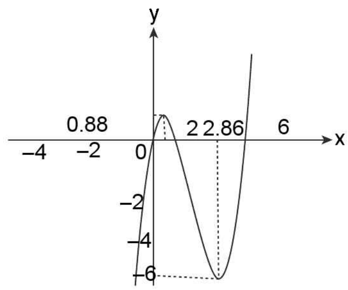
$\Rightarrow$ $x = 2.86$ is a point of local minima and local minimum value is $f(2.86) = -6.06$ and $f''(0.47) = -\text{ve}$

$\Rightarrow$ $x = 0.47$ is a point of local maxima and local maximum value is $f(0.47) = 0.88$

Step 4: $\lim\limits_{x\to\infty} y = \lim\limits_{x\to\infty} x^3\left(1-\frac{1}{x}\right)\left(1-\frac{4}{x}\right) = \infty$

and $\lim\limits_{x\to-\infty} y = \lim\limits_{x\to-\infty} (x)^3\left(1-\frac{1}{x}\right)\left(1-\frac{4}{x}\right) = -\infty$

Thus, from the above information, the graph of the function can be drawn as follow:



20. Graph $y = (x - 1)(x - 3)^2$

Solution: ***Step 1:*** $y = 0$

$\Rightarrow$ $x = 1, x = 3$ are roots of function where graphs cuts x-axis, $f(0) = (-1)(-3)^2 = -9$

$\Rightarrow$ graph cuts y-axis at pt $(0,-9)$

Step 2: $\frac{dy}{dx} = 2(x-1)(x-3) + (x-3)^2 = 0$

$\Rightarrow$ $(x - 3)(2x - 2 + x - 3) = 0$

$\Rightarrow$ $(x - 3)(3x - 5) = 0$

$\Rightarrow$ Critical points are $x = 3, \frac{5}{3}$

$$\Rightarrow \frac{d^2y}{dx^2} = 3(x-3) + (3x-5) = 6x - 14$$

Now, $\left(\frac{d^2y}{dx^2}\right)_{x=3} = +ve$; hence, local minima at $x = 3$

and $y_{\min} = 0$

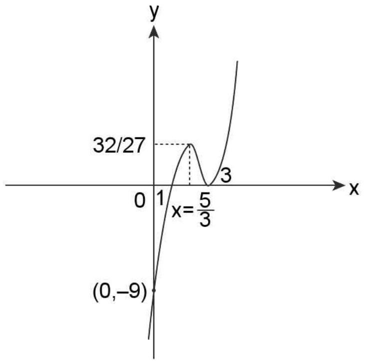
Also, $\left(\frac{d^2y}{dx^2}\right)_{x=\frac{5}{3}} = -ve$, hence, local maxima at $x = \frac{5}{3}$

and $\left(y_{\max}\right) = \frac{2}{3} \times \frac{16}{9} = \left(\frac{32}{27}\right)$

Step 3: $\lim_{x \to \infty}(x-1)(x-3)^2 = +\infty$

and $\lim_{x \to -\infty}(x-1)(x-3)^2 = -\infty$

From the above informations, we can easily draw the graph as shown below.



21. Trace the curve $y^2 = (x-1)(x-2)(x-4)$

Solution: **1.** The curve is symmetrical about the x-axis

2. The curve meets the x-axis at the points (1, 0), (2, 0) and (4, 0). The curve does not meet the y-axis as $y^2(0) < 0$.

3. When $x < 1$, $y^2 < 0$, no part of the graph lies to the left of the line $x = 1$. In particular, no part of the graph lies in the second and the third quadrants.
Also, If $2 < x < 4$, $y^2 < 0$, so, no part of the graph lies in the region bounded by the lines $x = 2$ and $x = 4$.

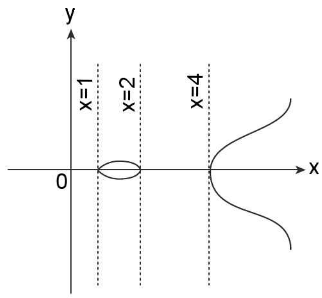
4. The equations of the tangents at the points (1, 0), (2, 0) and (4, 0) are $x = 1$, $x = 2$ and $x = 4$, respectively.

5. There are no multiple points.

6. There are no asymptotes.

7. Since y vanishes at $x = 1$ and $x = 2$, therefore, y^2 must have a maximum for some values of x lying between 1 and 2.
In fact, $\frac{d(y^2)}{dx} = 3x^2 - 14x + 14$, so that
$\frac{d\left(y^2\right)}{dx} = 0$ when $x = \frac{1}{3}\left(7 \pm \sqrt{7}\right)$
Let $x_1 = \frac{1}{3}\left(7-\sqrt{7}\right), x_2 = \frac{1}{3}\left(7+\sqrt{7}\right)$
Then $1 < x_1 < 2$, and $2 < x_2 > 4$
Also y^2 is a maximum when $x = x_1$
The portion of graph between $x = 1$ and $x = 2$ is, therefore, an oval.

8. When x takes values greater than 4, y^2 keeps on increasing. For large values of x, $y^2 \approx x^3$, and $\lim_{x \to \infty}$ $y \to \pm\infty$. The graph of the curve can now be easily seen to be as is shown in the figure given below.



22. Trace the curve $y = x^2(x-3a)$, $a > 0$

Solution: **1.** The curve does not possess any symmetry

2. If $x < 3a$, then $y < 0$ In particular, no portion of the graph lies in the second quadrant

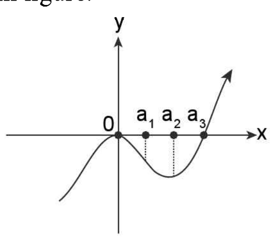
3. The curve meets the x-axis at points given by $x = 0, 3a$, i.e., at (0, 0), $(3a, 0)$.It meets the y-axis only at the origin

4. The tangents at the origin is the line $y = 0$. Since $y < 0$ for small values of x, therefore, it follows that the shape of the graph in the neighborhood of the origin is as shown in figure

5. $dy/dx = 3x(x-2a)$ and $\frac{d^2y}{dx^2} = 6(x-a)$
Since $dy/dx > 0$ when $x < 0$, therefore, y is increasing in $(-\infty, 0]$ and concave upwards for $x > a$
Since $dy/dx < 0$ in $[0, 2a]$, therefore, y is decreasing in $[0, 2a]$ and concave downwards for $x < a$
Again, since $dy/dx > 0$ if $x > 2a$, therefore, y is increasing in $(2a, \infty)$.
Also, y has a maximum at (0, 0) and a minimum at $(2a, -4a^3)$ and $x = a$ is a point of inflexion.

6. $y \to -\infty$ as $x \to -\infty$ and $y \to +\infty$ as $x \to +\infty$

7. There are no asymptotes
The graph of the curve can now be seen to be as shown in figure.



23. Sketch the graph of $y^2 = x^2\,(4 - x^2)$

Solution: ***Step 1:*** (a) Domain of given relation is $(-\infty,\infty)$

(b) Roots of equation $x^2\,(4 - x^2) = 0$ are $x = 0$ and $x = \pm 2$

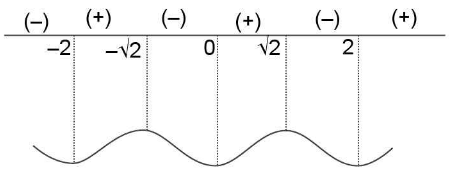
(c) We can observe that since $f(x)$ contains only even power terms of x and y, hence, graph will be symmetrical about x as well as y-axis.

$\therefore\ f(-x) = f(x)$

Step 2: $2y.\dfrac{dy}{dx} = x^2(-2x) + (4 - x^2)2x$

$= 2x\,(-x^2 + 4 - x^2) = 2x\,(4 - 2x^2)$

$\dfrac{dy}{dx} = \dfrac{2x(2-x^2)}{\sqrt{x^2(4-x^2)}}$ for $y > 0$ and $\dfrac{-2x(2-x^2)}{\sqrt{x^2(4-x^2)}}$ for $y < 0$



So, critical points are $x = \pm\sqrt{2}$, $x = 0$, $x = \pm 2$

Curve has local minima at $x = -2, 0, 2$ for $y > 0$

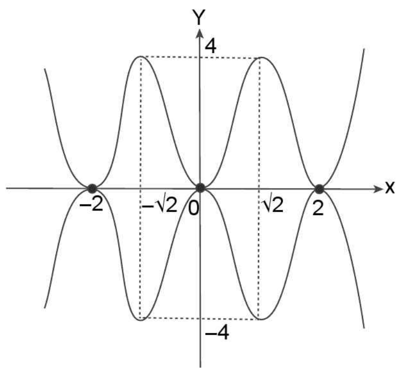
$(y)_{x=-2} = 0;\ (y)_{x=0} = 0;\ (y)_{x=2} = 0$

Curve has maxima at $x = \pm\sqrt{2}$ for $y > 0$

and $(y)_{x=-\sqrt{2}} = 4$, $(y)_{x=+\sqrt{2}} = 4$

Step 3: $\lim\limits_{x\to\infty} f(x) = \infty$

So, the graph of $f(x)$ will be as shown below:



24. Draw the graph of $y^2 = x^3 - x$

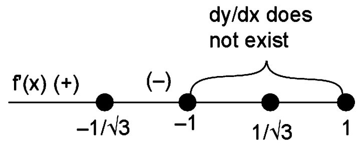
Solution: ***Step 1:*** $y^2 = x^3 - x = x\,(x^2 - 1)$

$y^2 = x\,(x-1)\,(x+1)$

$\Rightarrow$ Roots of this equation are $x = 0$, $x = 1$,$x = -1$

Step 2: $2y\dfrac{dy}{dx} = 3x^2 - 1$; $y = \pm\sqrt{x^3 - x}$

$\Rightarrow\ \dfrac{dy}{dx} = \dfrac{\pm(3x^2-1)}{2\sqrt{x^3-x}}$. Critical points are $x = \pm\dfrac{1}{\sqrt{3}}, 1$



Domain of relation is $[-1, 0] \cup [1, \infty)$

$$\frac{dy}{dx} = \pm\frac{3}{2}\frac{\left(x^2 - \frac{1}{3}\right)}{\sqrt{x(x-1)(x+1)}}$$

Point $x = -\dfrac{1}{\sqrt{3}}$ is of maxima and $y = \sqrt{x^3 - x}$

increasing in $x \in \left[-1, -\dfrac{1}{\sqrt{3}}\right] \cup [1,\infty)$ and decreasing

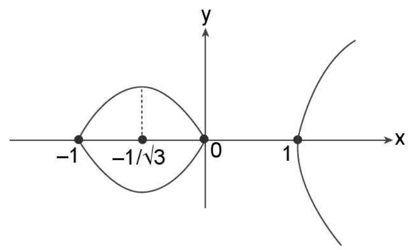
in $x \in \left(-\dfrac{1}{\sqrt{3}}, 0\right)$

$(y)_{x=-\frac{1}{\sqrt{3}}} = -\dfrac{1}{3\sqrt{3}} + \dfrac{1}{\sqrt{3}} = \dfrac{2}{3\sqrt{3}}$ for $y = \sqrt{x\ \ - x}$;

$(y)_{x=0} = (y)_{x=1} = (y)_{x=-1} = 0$ for $y = \pm\sqrt{x^3 - x}$

Step 3: $\lim\limits_{x\to+\infty} \sqrt{x^3 - x} = \infty$

So, graph of $y^2 = x^3 - x$ will be as shown below.



25. Plot the points on x-y plane which satisfy the following equations.

(i) $y = x|y|$ (ii) $y - |y| + x + |x| = 0$

(iii) $x - |x| = y - |y|$ (iv) $|2x + 3y| + |2x - 3y| = 10$

Solution: (i) $y = x|y|$

Case I: $y = 0$

$\Rightarrow x \in \mathbb{R}$ $\Rightarrow$ Entire x-axis

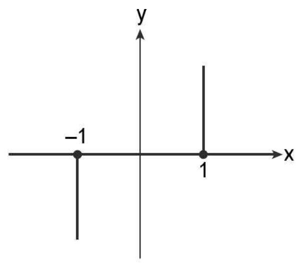
Case II: $y > 0$

$\Rightarrow x\,y = y \Rightarrow x = 1$

Case III: $y < 0$

$x(-y) = y \Rightarrow x = -1$

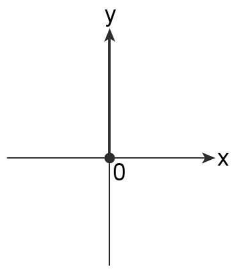
Thus, the graph of $y = x|y|$ is as shown below:



(ii) $y - |y| + x + |x| = 0$

Case I: $y \geq 0$; $x \geq 0$

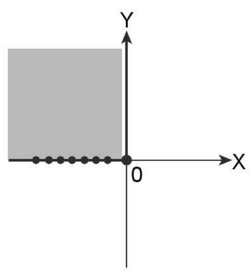
$\Rightarrow$ $|y| = y$ and $|x| = -x \Rightarrow$ $2x = 0 \Rightarrow x = 0$



Case II: $y \geq 0$; $x < 0$

$\Rightarrow$ $|y| = y$ and $|x| = -x$

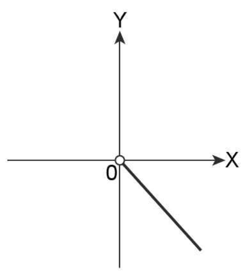
$\Rightarrow$ $0 = 0$



Case III: $y < 0$; $x \geq 0$

$|y| = -y$ and $|x| = x$

$\Rightarrow$ $2y + 2x = 0 \Rightarrow y = -x$

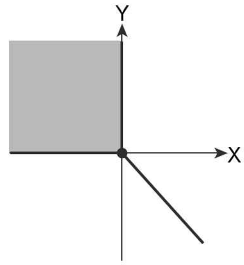


Case IV: $y < 0$; $x < 0$

$\Rightarrow$ $|y| = -y$ and $|x| = -x$

$\Rightarrow$ $2y = 0 \Rightarrow y = 0$ $\Rightarrow$ No reason

Hence, the overall graph is as shown figure:

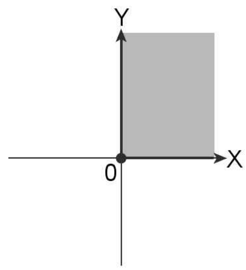


(iii) $x - |x| = y - |y|$

Case I: $x \geq 0$ and $y \geq 0$

$\Rightarrow$ $0 = 0$

$\Rightarrow$ Entire Ist quadrant including positive x and y-axis.



Case II: $x \geq 0$ and $y < 0$

$\Rightarrow$ $0 = 2y$

$\Rightarrow y = 0$

$\Rightarrow$ No region

Case III: $x < 0$ and $y \geq 0$

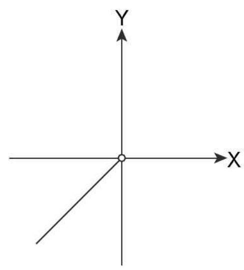
$\Rightarrow$ $2x = 0$

$\Rightarrow x = 0$

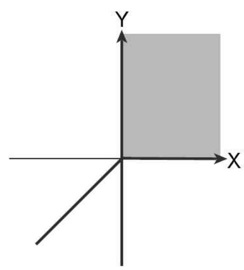
$\Rightarrow$ No region

Case IV: $x < 0$ and $y < 0$

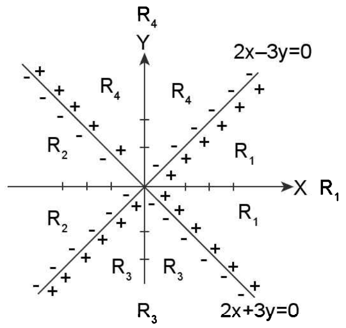
$\Rightarrow$ $2x = 2y \Rightarrow x = y$



Hence, the overall graph will be as shown below



(iv) $|2x + 3y| + |2x - 3y| = 10$



Let R_1 be the region defined by $2x + 3y \geq 0$ and $2x - 3y \geq 0$

Similarly $R_2 : 2x + 3y < 0$ and $2x - 3y < 0$

$R_3 : 2x + 3y < 0$ and $2x - 3y \geq 0$

$R_4 : 2x + 3y \geq 0$ and $2x - 3y < 0$

Case I: Considering region R_1; we have

$2x + 3y + 2x - 3y = 10$

$\Rightarrow 4x = 10 \qquad \Rightarrow x = 5/2$

Case II: Considering region R_2; we have

$-2x - 3y - 2x + 3y = 10$

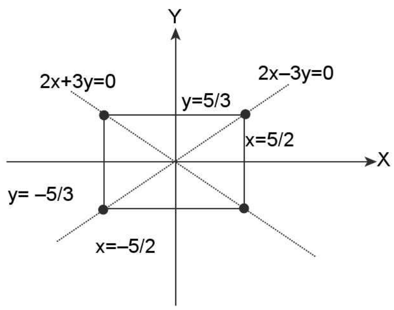
$\Rightarrow -4x = 10 \qquad \Rightarrow x = -5/2$

Case III: Considering region R_3; we have

$-2x - 3y + 2x - 3y = 10$

$\Rightarrow -6y = 10 \qquad \Rightarrow y = -5/3$

Case III: Considering region R_4; we get $2x + 3y - 2x + 3y = 10 \qquad \Rightarrow y = 5/3$



Assertions and reason

26. A: The number of real roots of the equation $5^{1+\cos\pi x} + 4^{1-|x|} = 2$ is 2.

R: If $k > 1$ and f, g be real valued functions such that $f(x) + g(x) \geq 0$, then $k^{f(x)} + k^{g(x)} = 2$ will have $x = \alpha$ as solution if $f(\alpha) = g(\alpha)$

(a) A is true, R is a correct explanation for A

(b) A is true, R is true but R is not a correct explanation for A

(c) A is true, R is false

(d) A is false, R is true

Solution: (a) $\dfrac{k^{f(x)} + k^{g(x)}}{2} \geq \left(k^{f(x)+g(x)}\right) \geq 1$

$\Rightarrow k^{f(x)} + k^{g(x)} \geq 2$. Equally holds when $f(x) = g(x) = 0$.

By above reason, $1 + \cos\pi x = 1 - |x| = 0$

$\Rightarrow \cos\pi x = -1$, $|x| = 1 \Rightarrow x = \pm 1$ are two solutions

Comprehension Passage

A: Let the lines represented by the equation $x^2y^2 - x^2 - y^2 + 1 = 0$ form a square $ABCD$. A point 'P' is taken on the same plane of square such that atleast two sides of all the triangles PAB, PBC, PCD and PDA are equal.

27. The number of possible positions of the point 'P' is.

(a) 7 (b) 5

(c) 17 (d) 9

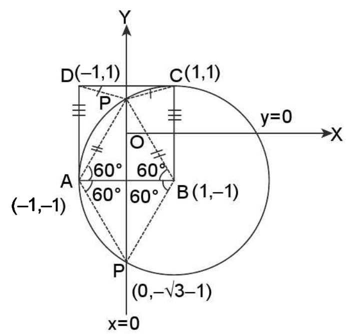
Solution: (d) $x^2y^2 - x^2 - y^2 + 1 = 0$

$\Rightarrow x^2(y^2 - 1) - 1(y^2 - 1) = 0$

$\Rightarrow (y^2 - 1)(x^2 - 1) = 0 \Rightarrow y = \pm 1, x = \pm 1$

$\Rightarrow$ $ABCD$ square is obtained bounded by $x = \pm 1$ and $y = \pm 1$.

Clearly 8 possible points will be the intersection of the circle (with centre at vertex and radius as side of the square) and the line which is perpendicular bisector of the sides. Centre of square will be one such point. Thus, total 9 such points will be there.



28. For all the possible position of point 'P'. How many of the given triangles have all the three sides equals

(a) 0 (b) 4

(c) 8 (d) 5

Solution: (c) Clearly for all the different position of 'P' except origin there will be one equilateral triangle.

29. One of the possible position of the 'P' such that atleast one of the given triangle is equilateral is given by

(a) $(0, 0)$ (b) $(-\sqrt{3} + 1, 0)$

(c) $(0, \sqrt{3} - 2)$ (d) $(\sqrt{3} + 1, \sqrt{3} - 1)$

Solution: (b) Different positions of P except origin will be $(0, -\sqrt{3} - 1)$, $(0, \sqrt{3} - 1)$, $(\sqrt{3} - 1, 0)$, $(-\sqrt{3} - 1, 0)$, $(0, \sqrt{3} + 1)$, $(0, -\sqrt{3} + 1)$, $(\sqrt{3} + 1, 0)$ and $(-\sqrt{3} + 1, 0)$

TUTORIAL EXERCISE

SECTION–III

ONLY ONE CORRECT ANSWER

1. The number of real solutions of the equation $e^x + x = 0$ is:
   (a) 2 (b) 1
   (c) 3 (d) None of these

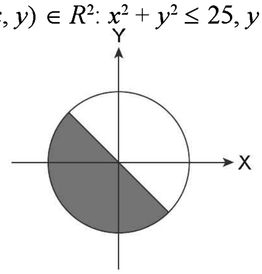
2. The number of real solutions of the equation $\log_a x = |x|$, $0 < a < 1$, is
   (a) 1 (b) 2
   (c) 3 (d) None of these

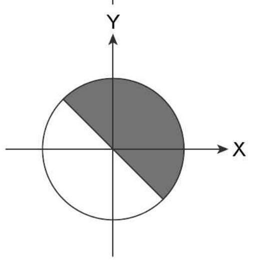
3. If high voltage (440 V) current is applied on the field which is given by the graph $y + |y| - x - |x| = 0$. On which of the following curve Mr. Calculus can move so that he remains safe?
   (a) x^2 (b) $sgn\,(-e^x)$
   (c) $\log_{1/3} x$ (d) $|m + |x||\ (m > 3)$

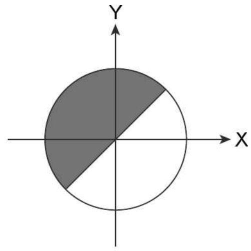
4. The graph of the function $f(x) = -\left(\dfrac{|x|^3 + |x|^2 + |x|}{1 + x^2}\right)$ will lie on
   (a) 1st and 2nd quadrant (b) 2nd and 3rd quadrant
   (c) 3rd and 4th quadrant (d) None of these

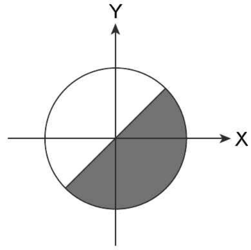
5. If $|f(x) + 6 - x^2| = |f(x)| + |4 - x^2| + 2$, then $f(x)$ is necessarily non-negative in
   (a) $[-2, 2]$ (b) $(-\infty, -2) \cup (2, \infty)$
   (c) $[-\sqrt{6}, \sqrt{6}]$ (d) None of these

6. Which one of the following graphs best represents the $\{(x, y) \in R^2: x^2 + y^2 \le 25, y \ge -x\}$?

(a) 

(b) 

(c) 

(d) 

7. The graph of the relation $x^4 + y^3 = 1$ is symmetric with respect to
   (a) the x-axis (b) the y-axis
   (c) the origin (d) the line $y = x$

8. The number of solutions of the equation $[x]^2 + [x + 3] = 4$ is: where $[x]$ denotes greatest integer function
   (a) Two (b) Four
   (c) Infinite (d) None of these

9. The number of solutions of the equation; $\sin x = e^{-x}$:
   (a) one solution (b) two solutions
   (c) three solutions (d) infinite solution

10. Let $f(x) = \max\{\tan x, \cot x\}$. Then number of roots of the equation $f(x) = \sqrt{3}$ in $(0, 2\pi)$ is
    (a) 2 (b) 4
    (c) 0 (d) None of these

11. The number of solutions of the equation, $\sin x = \dfrac{x}{10}$ are
    (a) 3 (b) 6
    (c) 7 (d) 10

12. Number of solutions of $5^{|x|} = |3 - |x||$ is
    (a) 0 (b) 2
    (c) 4 (d) infinite

13. How many roots does the following equation posses, $x^2 - 2x - \log_2 |1 - x| = 3$?
(a) 2 (b) 3
(c) 4 (d) 5

14. The equation $e^{2x} + e^x - 2e = 0$ has
(a) no real root in the interval [0, 1]
(b) at least one real root in [0, 1]
(c) two real roots in [0, 1]
(d) none of these

15. The set of real value(s) of p for which the equation, $|2x + 3| + |2x - 3| = px + 6$ has more than two solutions is:
(a) $[0, 4\}$ (b) $(-4, 4)$
(c) $R - \{4, -4, 0\}$ (d) $\{0\}$

16. Number of solutions of $\sin x/2 + 2\pi x = x^2 + \pi^2$ is
(a) 0 (b) 1
(c) 2 (d) None of these

17. The number of solutions of the equation $2\cos(e^x) = 5^x + 5^{-x}$ are
(a) no solutions (b) one solution
(c) two solution (d) infinitely many solutions

18. The graph of the curve $y = \dfrac{1}{x}$ and $y = px$ intersects at
(a) one point for any value of p
(b) two points for any values of p
(c) two points for $p \geq 0$
(d) None of these

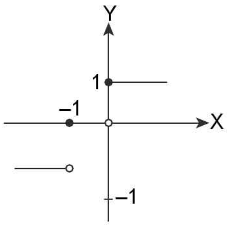
19. The general solution of $|\sin x| = \cos x$; $n \in \mathbb{Z}$ is given by
(a) $n\pi + \dfrac{\pi}{4}$ (b) $2n\pi \pm \dfrac{\pi}{4}$
(c) $n\pi \pm \dfrac{\pi}{4}$ (d) $n\pi - \dfrac{\pi}{4}$

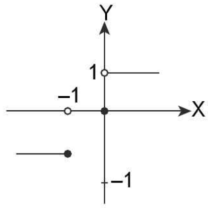
20. The equation $||x - 2| + a| = 4$ can have four distinct real solutions for x if a belongs to the interval
(a) $(-\infty, -4)$ (b) $(-\infty, 0)$
(c) $[4, \infty)$ (d) None of these

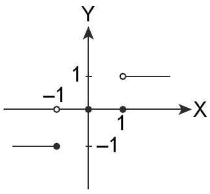
21. Let $f_1(x) = \begin{cases} x & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } x > 1 \\ 0 & \text{for otherwise} \end{cases}$ and $f_2(x) = f_1(-x)$ for all x $f_3(x) = -f_2(x)$ for all x $f_4(x) = f_3(-x)$ for all x
Which of the following is necessarily true?
(a) $f_4(x) = f_1(x)$; for all x
(b) $f_1(x) = -f_3(-x)$; for all x
(c) $f_2(-x) = f_4(x)$; for all x
(d) $f_1(x) + f_3(x) = 0$; for all x

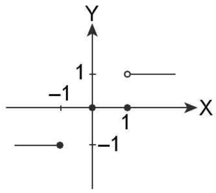
22. The number of solutions of the equation $x^2 - 2[x] = 0$ ([.] denotes greatest integer function) is
(a) one (b) two
(c) four (d) infinity

23. Which of the following represents the graph $f(x) = sgn([x + 1])$ where [ ] denotes the GINT function.

(a)


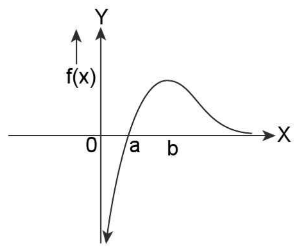
(b)


(c)


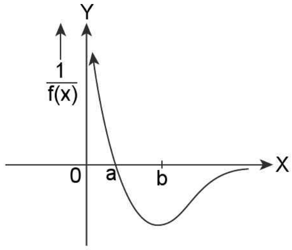
(d)


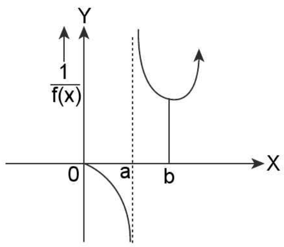
24. Suppose f is defined from $R \to [-1,1]$ as $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$; where R is the set of real number. Then the statement which does not hold is
(a) f is many–one and onto
(b) f increase for $x < 0$ and decrease for $x < 0$
(c) minimum value is attained
(d) maximum value is not attained even though f is bounded

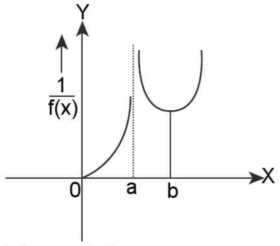
25. The graph of function $y = f(x)$ is as shown in the figure given below.



Then the graph of the function $\dfrac{1}{f(x)}$ will be best represented by

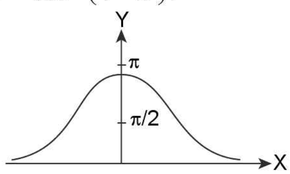
(a)


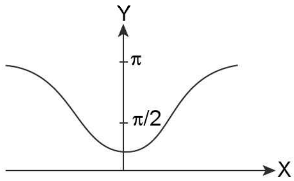
(b)


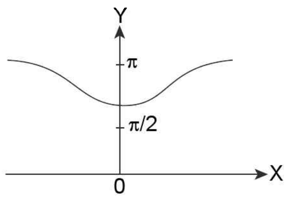
(c)


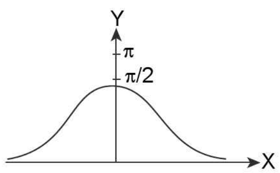
(d) None of these

26. Which of the following best represents graph of $f(x) = \cot^{-1}(4 - x^2)$?

(a)


(b)


(c)


(d)


27. Let $[x]$ = greatest integer $\leq x$ and $\{x\} = x - [x]$. If $f(x) = \{x\}\,[x]$, then $f(x) = 1$ for:
(a) no value of x
(b) infinite number of values of x
(c) one value of x
(d) None of these

28. The number of roots of the equation $x \sin x = 1$, $x \in [-2\pi, 0) \cup (0, 2\pi]$ is
(a) 2 (b) 3
(c) 4 (d) 0

29. The number of points (x, y), where the curves $|y| = \ln|x|$ and $(x - 1)^2 + y^2 - 4 = 0$ cut each other is
(a) 2 (b) 3
(c) 1 (d) 6

30. The number of solutions of equation $2^{\cos x} = |\sin x|$, when $x \in [0, 2\pi]$ is
(a) 4 (b) 3
(c) 0 (d) 2

31. The number of solutions of the equation $\sin \pi x = |\ln|x||$ is
(a) infinite (b) 8
(c) 6 (d) 0

32. The number of roots of the equation $1 + \log_2(1 - x) = 2^{-x}$ is
(a) 0 (b) 1
(c) 2 (d) infinitely many

33. Number of roots of $|\sin|x|| = x + |x|$ in $[-4\pi, 2\pi]$ is
(a) 2 (b) 3
(c) 4 (d) 5

34. The equation $3^{x-1} + 5^{x-1} = 12$ has
(a) No solution (b) one solution
(c) two solution (d) None of these

35. The number of solution of the equation $\cos[x] = 2^{4x-1}$, $x \in [0, 2\pi]$
(a) 1 (b) 2
(c) 3 (d) 4

36. The number of real roots of the equation $x^2 + x + 3 + 2\sin x = 0$, $x \in [-\pi, \pi]$
(a) 2 (b) 4
(c) 6 (d) None of these

37. The total number of real roots of the equation $|x - x^2 - 1| = |2x - 3 - x^2|$ is
(a) 0 (b) 1
(c) 2 (d) infinitely many

38. The equation $\dfrac{x^2}{1-|x-2|} = 1$ has
(a) one real solution (b) two real solution
(c) three real solutions (d) no real solution

39. The graph of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{|x|^3 + |x|}{1 - x^2}$ may lie in
(a) 1st quadrant (b) 2nd quadrant
(c) 3rd quadrant (d) Each of 4th quadrant

40. The number of solutions of the equation $\tan 4x = \cos x$; for $0 < x < \pi$ is

(a) 1 (b) 2
(c) 5 (d) 8

41. The number of real solutions of the equation $\log_{0.5}|x| = 2|x|$ is

(a) 1 (b) 2
(c) 0 (d) None of these

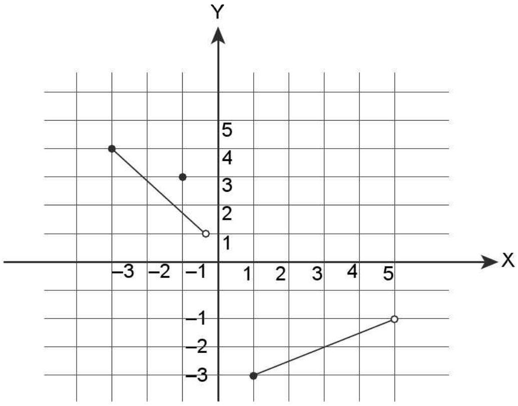
42. Let $f(x) = \max\{1 + \sin x, 1, 1 - \cos x\}, x \in [0, 2\pi]$ and $g(x) = \max\{1, |x - 1|\}; x \in \mathbb{R}$, then

(a) $g(f(0)) = 1$ (b) $f(f(1)) = 1$
(c) $f(g(1)) = 1$ (d) $f(g(0)) = \sin 1$

43. If f: $\mathbb{R} \to \mathbb{R}$, g: $\mathbb{R} \to \mathbb{R}$ be two given functions, then $2\min\{f(x) - g(x), 0\}$ equals

(a) $f(x) + g(x) - |g(x) - f(x)|$
(b) $f(x) + g(x) + |g(x) - f(x)|$
(c) $f(x) - g(x) + |g(x) - f(x)|$
(d) $f(x) - g(x) - |g(x) - f(x)|$

44. If the graph of $y = f(x)$ is as shown in the figure below, then answer the questions that follow:



(i) Find $f(-2)$

(a) 5/2 (b) 7/2
(c) 3 (d) None of these

(ii) The domain of the function $f(x)$

(a) $[-3, -1] \cup [1, 5)$ (b) $[-3, -1) \cup [1, 5)$
(c) $[-3, -1] \cup [1, 5]$ (d) None of these

(iii) The range of the function $f(x)$

(a) $[-3, -1] \cup [1, 4]$ (b) $[-3, -1] \cup [1, -4)$
(c) $[-3, -1) \cup (1, 4]$ (d) None of these

45. Find the area bounded by the following inequalities:

(i) $x - 2y \le 4$, $2x + y > 5$, $y \ge x$, $x < 6$, $y < 8$

(a) $\frac{313}{12}$ (b) $\frac{337}{12}$
(c) ∞ (d) None of these

(ii) $4x + y \le 5$; $x - 4y > 9$; $y < -x + 1$; $xy \ge 0$; $|x| \le 3$, $|y| \le 3$

(a) $\frac{7}{8}$ (b) $\frac{9}{8}$
(c) 1 (d) None of these

46. The number of solutions of the equation $[x]^2 + [x + 3] = 4$ is (where $[x]$ denotes greatest integer function)

(a) two (b) four
(c) infinite (d) None of these

47. The number of solutions of the equation $x^2 - 2 - [x] = 0$ ([.] denotes greatest integer function) is

(a) one (b) two
(c) three (d) infinity

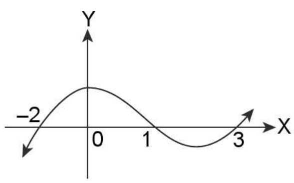
48. The number of real solutions of equation $\max(x^2 - 4x) = 1$ is

(a) 1 (b) 2
(c) 3 (d) infinitely many solution

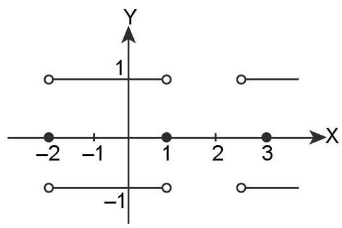
SECTION–IV

MORE THAN ONE ANSWER CORRECT

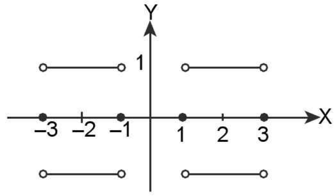
1. The graph of the function $y = f(x)$ is as shown in the figure. Then which one of the following graphs is correct?



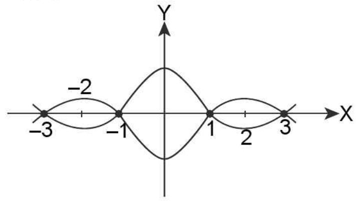
(a) $|y| = sgn(f(x))$



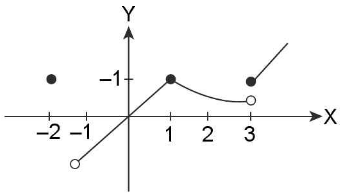
(b) $|y| = sgn\,(-f(|x|))$



(c) $|y| = |f|x||$



(d) $y = x\,sgn\,(f(x))$



2. The equation $2|x| - 1 = 3|\sin \pi x|$ has

(a) 4 solutions

(b) no solution lies in $[2, \infty)$

(c) 6 solutions

(d) two solution lies in $(0, 1/2)$

3. Let $f(x) = x^2 + 3x + 5$ and $g(x) = |x - k|$, then which of the following statements is/are correct?

(a) $f(x) = g(x)$ has exactly one root for exactly two distinct values of 'k'.

(b) $f(x) = g(x)$ has no real root for $k \in (-3, 1)$.

(c) $f(x) = g(x)$ has two real roots for $k > 5$.

(d) None of these

4. Let $f(x) = ||x - 1| -2|$ and $g(x) = x + |x - 1| + \dfrac{|x-2|}{x-2}$; then which of the following statements is/are correct

(a) $f(x) = g(x)$ has exactly 3 real roots

(b) $f(x) = g(x)$ has exactly 2 real roots

(c) $y = g(x)$ has one point of discontinuity

(d) $y = f(x)$ has 3 points of non-differentiability

5. Let $(x^2 + y^2)y - ax^2 = 0$ and $(x^2 + y^2) = a^2(x^2 - y^2)$; $a \neq 1$ be two curves; then which of the following statements is/are correct?

(a) The two curves intersect at atmost two distinct points.

(b) The two curves intersect at 3 distinct points for exactly one particular value of 'a'.

(c) The two curves intersect at 3 distinct points for $a \in \mathbb{R} \sim [-1, 1]$.

(d) The curves $(x^2 + y^2)y - ax^2 = 0$ has exactly 1 cusp.

6. Consider two curves

C_1: $100(x^2 + y^2) = a^2x^2y^2$ and C_2: $y = (a^3 - x^3)^{1/3}$

Which of the following statements is/are correct?

(a) C_1 and C_2 intersect in 3 distinct points for exactly 2 values of 'a'.

(b) C_1 and C_2 intersect in 4 distinct points for infinite values of 'a'.

(c) C_1 and C_2 intersect in 2 distinct points for infinite values of 'a'.

(d) Exactly 2 of the above 3 statements are correct.

7. Which of the following statement(s) is/are correct?

(a) $\sin^{-1}\left(\dfrac{1}{x}\right) = \cos^{-1}\left(\dfrac{1}{x}\right)$ has 1 real root

(b) $\sin^{-1}\left(\dfrac{1}{x}\right) = \tan^{-1}\left(\dfrac{1}{x}\right)$ has 1 real root

(c) $\sin^{-1}\left(\dfrac{1}{x}\right) = \text{cosec}^{-1}\left(\dfrac{1}{x}\right)$ has 2 real roots

(d) $\sin^{-1}\left(\dfrac{1}{x}\right) = \cot^{-1}\left(\dfrac{1}{x}\right)$ has 1 real root

SECTION–V

ASSERTION AND REASON TYPE

1. A: $f(x) = ax^3 + 2x^2 - bx + c$ is symmetric about the line $x = 1$, then $a + 2b = 8$

R: An odd degree polynomial can never be symmetric about any vertical line every quadratic polynomial is symmetric about its axis.

2. A: $y = f(x)$ is a +ve valued function $x \in \mathbb{R}$; $\int_a^b f(x)x = \int_{a-k}^{b-k} f(x+k)dx$

R: The graph of $f(x + k)$ is obtained by shifting the graph of $f(x)$ by $|k|$ unit horizontally in direction opposite to sign of k.

3. A: $f(x) = x^4 - 4x^3 + 4x^2 + k - 1$ has no real root $k \in (1, \infty)$.

R: graph of $y = g(x) + c$ can be obtained by vertical scaling of graph of $y = g(x)$ by $|c|$ unit upwards if $c > 0$ and downward if $c < 0$.

4. Let f: $\mathbb{R} \to \mathbb{R}$ defined by $f(x) = \cos \pi[x]$, where $[x]$ denotes the greatest integer function less than or equal to x.

A: $f(x)$ is a periodic function.

R: $[x]$ is a periodic function.

5. Let g: $\mathbb{R} \to \mathbb{R}$ defined by $g(x) = \{e^x\}$, where $\{x\}$ denotes fractional part function.

A: $g(x)$ is periodic function.

R: $\{x\}$ is periodic function.

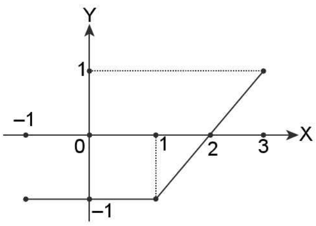
6. **A:** The function $f(x) = \tan\left(\frac{3\pi}{2}[x]\right)$ where $[x]$ is the greatest integer function, is a periodic.

R: $g(x) = [x]$ is a periodic.

SECTION–VI

COMPREHENSION PASSAGE

A: The diagram given below represents graph of a function $f(x)$. Observe its properties carefully and based on it answer the following questions:



1. $|f(x)|$ is an increasing function in the interval

(a) (2, 3) (b) (1, 3)
(c) (0, 2) (d) None of these

2. $-|f(x)|$ is non-differentiable at x = (in the interval (–1, 3))

(a) 3 and 2 (b) 1 and 2
(c) 0 and 2 (d) None of these

3. $f(-|x|)$ is

(a) increasing function
(b) a decreasing function
(c) a constant function
(d) positive $\forall\, x \in$ Domain

4. $g(x) = \frac{1}{2}(|f(x)| + f(x)))$ is a constant function in the interval

(a) [–1, 3] (b) [0, 3]
(c) [–1, 2] (d) None of these

5. $h(x) = \frac{1}{2}(|f(x)| - f(x))$ is a constant function in the interval

(a) [–1, 2] (b) [–1, 1] ∪ [2, 3)
(c) [1, 3) (d) None of these

6. $g(x) = f\left(\frac{|x|}{x}\right)$ is

(a) an increasing function
(b) a decreasing function
(c) a constant function
(d) None of these

B: Consider a rational function $f(x)$ by $f(x) = \frac{(x^2-1)(x-3)}{x^2(x+2)}$.

7. Which of the following is a horizontal asymptote of $y = f(x)$?

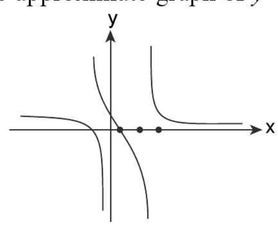
(a) $y = 2$ (b) $y = 1$
(c) $y = -1$ (d) None of these

8. Which of the following is/are vertical asymptote(s) on $y = f(x)$?

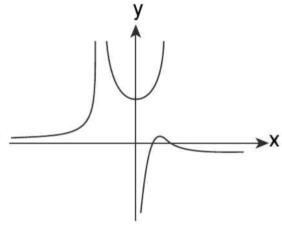
(a) $x = 0$ (b) $x = 2$
(c) $x = -2$ (d) All of these

9. The approximate graph of $y = f(x)$ is given by

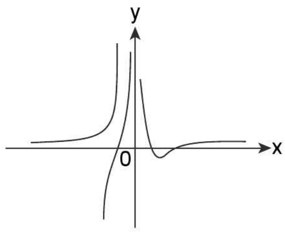
(a)



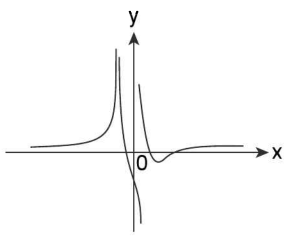
(b)



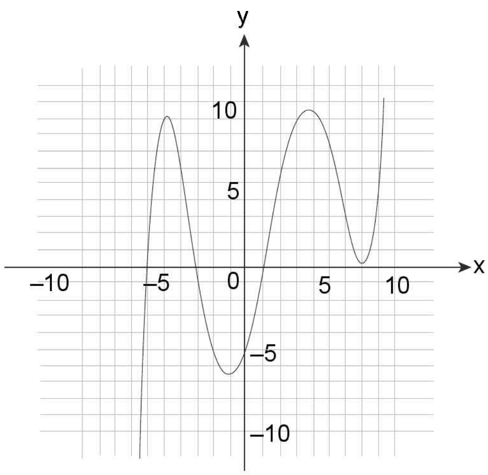
(c)



(d)



C: Consider the function as shown in the figure given below. The shown figure is a polynomial of degree 5. The points marked with black dots have integral coordinates.



10. What is the value of $f(-5)\, f(4)$

(a) 0 (b) 36

(c) 81 (d) None of these

11. Determine the function $f(x)$

(a) $f(x) = \dfrac{(x-1)(x+3)(x+6)(x-7)^2}{192}$

(b) $f(x) = \dfrac{(x-1)(x+3)(x+6)(x-7)^2}{192}$

(c) $f(x) = (x - 1)\,(x - 3)\,(x + 6)\,(x - 7)^2$

(d) $f(x) = \dfrac{(x-1)(x-3)(x+6)(x-7)^2}{48}$

12. Determine the values of $f(f(-5))$

(a) 12 (b) 15

(c) 30 (d) None of these

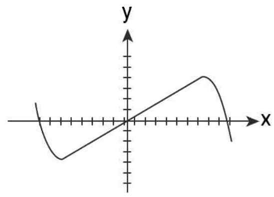
D: Let us consider a function given by

$$f(x) = \begin{cases} (x-3)^3 + 3; & \text{if } x \in (3,5] \\ x; & \text{if } x \in [-3,3] \\ -(x+3)^2 - 3; & \text{if } x \in [-5,-3) \end{cases}$$

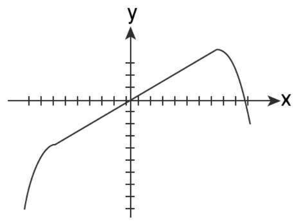
Then on the basis of the information given above answer the questions that follow.

13. Which of the following gives an approximate graph of $f(x)$?

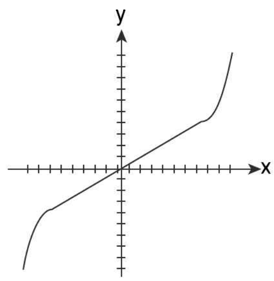
(a)



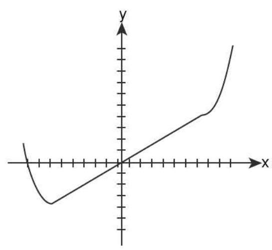
(b)



(c)



(d)

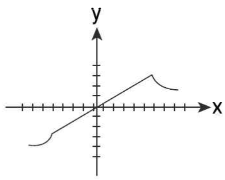
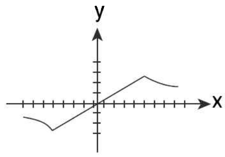


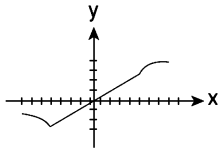
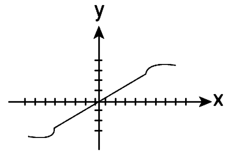
14. Let $g(x)$ be a function such that $g(x) = \underbrace{f.f.f....f}_{n \text{ compositions}}(x)$; where $n \in \mathbb{N}$, then $g(x)$ can never achieve which of the following values?

(a) 0 (b) 4

(c) 8 (d) None of these

15. Which of the following gives as approximate graph of $f^{-1}(x)$?

(a)  (b) 

(c)  (d) 

E: For $x, y \in \mathbb{R}$ the functions are defined here under; $f_1(x, y) = |x| + |y|$, $f_2(x, y) = \min(x + y, x - y)$, $f_3(x, y) = [x] + [y]$ (where [.] denotes the greatest integer function).

16. Which of the following is always true?
(a) $f_1(x, y) < f_2(x, y)$; $x, y \in \mathbb{R}$
(b) $f_2(x, y) < f_3(x, y)$; $x, y \in \mathbb{Z}^+$ (set of +ve integers)
(c) $f_1(x, y) = -f_3(x, y)$; $x, y \in \mathbb{Z}^-$ (set of –ve integers)
(d) None of these

17. In $h_1(x) = f_1(\sin\{x\}, \sin \pi x)\ \forall x \in \mathbb{R}$ (where {.} represents fractional part function). $h_2(x) = f_2(x, \sin x)\ \forall\ 0 < x < 1$, $h_3(x) = f_3(x, -x)\ \forall x \in \mathbb{R}$, then
(a) h_1: (x) is non-periodic function
(b) h_2: $(0, 1) \to (0, 1)$ then h_2 is bijective
(c) h_3 : (x) is periodic function
(d) None of these

18. Which of the following can never be true?
(a) $f_1(x, y) > f_3(x, y)$; $x, y \in \mathbb{R}$
(b) $f_2(x, y) = x + y$; $x, y < 0$
(c) $f_3(x, y) = f_3(-x, -y)$; $x, y \in \mathbb{R}$
(d) $f_1(x, y) = f_3(x, y)$; $x, y \in \mathbb{R} - \mathbb{Z}$

F: Let H_1: $x^2 - 4y^2 + 4$ and $H_2 : x^2 - 9y^2 - \dfrac{1-a}{1+a}$ then answer the following questions

19. The exhaustive set of values of a for which there exist at least one ordered pair (x, y) which satisfy the inequality $H_1 \le 0$ and $H_2 \le 0$ satisfy simultaneously is
(a) $a \in (-\infty, \infty)$
(b) $a \in [-5/4, -1) \cup (-1, 1)$
(c) $a \in (-1, 1)$
(d) $a \in (-\infty, -1) \cup (-1, \infty)$

20. The exhaustive set of values of a for which $H_1 \le 0$ and $H_2 \ge 0$ satisfy simultaneously is
(a) $a \in \mathbb{R} \sim \{-1\}$ (b) $a \in \left(\dfrac{5}{4}, -1\right)$
(c) $a \in \left(-\dfrac{5}{4}, -1\right)$ (d) None of these

21. Number of integral values of a for which $H_1 \ge 0$ and $H_2 \ge 0$ does not possess a common solution is
(a) 0 (b) 1
(c) infinitely many (d) 2

G: $f(x)$ is a continuous function satisfying $x^4 - 4x^2 \le f(x) \le 2x^2 - x^3\ \forall x \in [0, 2]$ such that the area bounded by $y = f(x)$, $y = x^4 - 4x^2$ and the line $x = t$, $(0 \le t \le 2)$ is n times the area bounded by $y = f(x)$, $y = 2x^2 - x^3$ and the line $x = t$, $(0 \le t \le 2)$, then

22. At what point $f(x)$ attains a local minima, when $n = 1$?
(a) 2 (b) $\dfrac{3+\sqrt{73}}{8}$
(c) $\dfrac{-3-\sqrt{73}}{8}$ (d) 2

23. For $n = 2$, $f(x)$ attains a point of inflexion at $x =$
(a) $\sqrt{2}$ (b) 3/2
(c) 1 (d) None of these

24. The value of $\displaystyle\int_{-1}^{1} f(x)dx$ is
(a) $\dfrac{2}{15(n+1)}(23-10n)$ (b) $\dfrac{2}{15(n+1)}(10n-17)$
(c) $\dfrac{2}{15(n+1)}(23+10n)$ (d) $\dfrac{2}{15(n+1)}(10n+17)$

SECTION–VII

COLUMN MATCHING

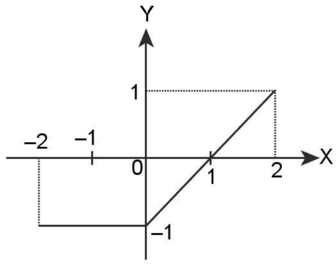
1. Match the column
[.] and {.} represent the greatest integer and fractional part functions respectively

Column-I
(i) Number of solutions of $[x] = \cos^{-1}x$
(ii) Number of solution of $\sin^{-1}x = sgn(x)$
(iii) Number of solutions of $\{x\} = e^{x^2}$
(iv) Number of solutions of $1 - x = \{x\}$

Column-II

(a) 3
(b) 2
(c) 1
(d) 0

2. The graph of the function $y = f(x)$ is as follows.

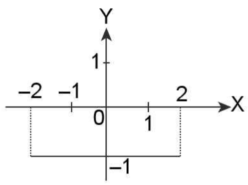


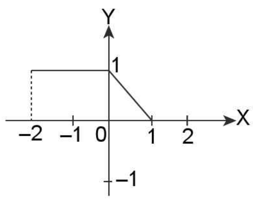
Match the *function* mentioned in Column-I with the respective *graph* given in Column-II.

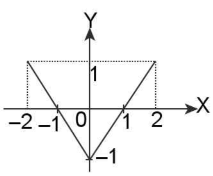
Column-I

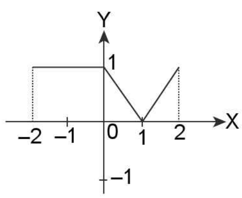
(i) $y = |f(x)|$
(ii) $y = f(|x|)$
(iii) $y = f(-|x|)$
(iv) $y = \frac{1}{2}(|f(x)| - f(x))$

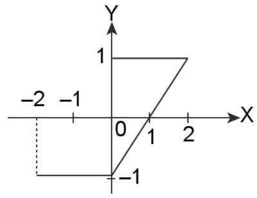
Column-II

(a)


(b)


(c)


(d)


(e)


3. Find the number of points of intersection of the two graphs given by

Column-I

(i) $y = \frac{x^3}{3} + 3x^2 + 5x$ and $x = 2y - y^2$
(ii) $|y| = x^3 - 3x + 2$ and $|y| = \cos(\sin x)$
(iii) $y = |\ln |x^2 - x||$ and $y = \left|\left|\frac{1}{x}\right| - 2\right|$
(iv) $y = \cos e^x$ and $y = \tan e^x$ $\forall x \in \left(-\infty, \frac{\pi}{2}\right)$
(v) $y = \sin e^x$ and $y = \sec e^x$ $\forall$ $x \in (-\infty. 2)$

Column-II

(a) 8
(b) 6
(c) 2
(d) 0
(e) 6

4. If $f(x) = x^4 - 5x^2 + 7$ be a function and $|y| = x^4 - 5x^2 + 7$; then find the values of 'k' for which the number of point of intersection; of $|y| = x^4 - 5x^2 + 7$ and $y = k$ is given by 'n'. Then match the entries in column I (value of n) to the entire in column II (all possible values of 'k' corresponding to a particular 'n').

Column-I

(i) $n = 0$
(ii) $n = 1$
(iii) $n = 2$
(iv) $n = 3$
(v) $n = 4$

Column-II

(a) $(-\infty, -7) \cup (7, \infty) \cup \{\pm 3/4\}$
(b) $\{\pm 7\}$
(c) ϕ
(d) $\left(-\frac{3}{4}, \frac{3}{4}\right)$
(e) $\left(-7, -\frac{3}{4}\right) \cup \left(\frac{3}{4}, 7\right)$

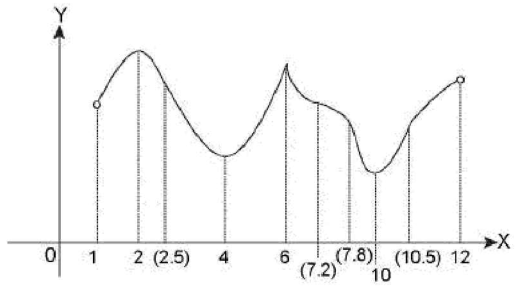
SECTION–VIII

INTEGER TYPE

1. If $y = f(x)$ is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$ except for 4 points of non-differentiability and $f'(0^+) > 0$, then find the number of points of non-differentiability of the graph $y = f(|x|)$; $x \in \mathbb{R}$.

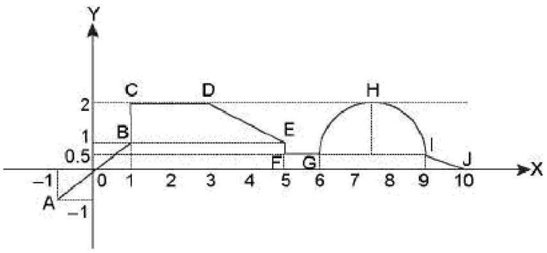
2. Let $f(x)$: $[2, 10] \to \mathbb{R}$ be a continuous function having points of non-differentiability at $x = 3, 4, 5, 6, 7, 8, 9$, then find the number of points of non-differentiability of $y = f(x - 2) = g(x)$ (say) in $[2, 10)$, assuming that $f(x)$ is non-oscillating.

3. In the graph shown below,



$x = x_1, x_2$ are points of local maxima, $x = y_1, y_2$ are points local minima, $x = z_1, z_2, z_3, z_4$ are points of inflexion and $x = u_1$ is points of non-differentiability, then evaluate $\sum x_i + \sum y_i - \sum z_i + u_1$.

4. Let $y = f(x)$ has the graphs as shown below.



Find the area enclosed by the graph of $|y| = |f(|x|)|$ with x-axis is $\left(a + \frac{b\pi}{c}\right)$ square units, where $g.c.d.(b, c) = 1$, then find $a + c - 4b$.

5. Let $f(x) = \begin{cases} |x-1| \text{ for } 0 \le x < 2 \\ e^{x-2} \text{ for } 2 \le x < 2 + ln6 \\ x \text{ for } x \ge 2 + ln6 \end{cases}$ and $g(x) = \begin{cases} \left|\sin \frac{x\pi}{4}\right| \text{ for } 0 \le x < 2 \\ 2x - 3 \text{ for } 2 \le x < 4 \\ \log_2 x \text{ for } x \ge 4 \end{cases}$; then find the number of points where $f(x) = g(x)$.

6. $f(x) = \begin{cases} -x - \frac{\pi}{2}; & x \le \frac{-\pi}{2} \\ -\cos x, & \frac{-\pi}{2} < x \le 0 \\ x - 1, & 0 < x \le 1 \\ \ln x, & x > 1 \end{cases}$ and $g(x) = \begin{cases} 1 - x; & x < 1 \\ (1-x)(2-x); & 1 \le x \le 2 \\ 3 - x; & x > 2 \end{cases}$; then the number of solutions of equation of $f(x) = g(x) + \ln 2$ is k_1, k_2 = number of points of non-differentiability of $y = f(x)$, k_3 = number of points of non-differentiability of $y = g(x)$, then find $\left(\frac{k_1 + k_2}{k_3}\right)$.

7. Let $f(x) = |x^2 - 9| - |x - a|$. Find the number of integers in the range of a so that $f(x) = 0$ has 4 distinct real root.

8. If f: $\mathbb{R} \to \mathbb{R}$ be an injective mapping and p, q, r are non-zero distinct real quantities satisfying $f\left(\frac{p}{r}\right) = f\left(\frac{p-q}{q-r}\right)$ and $f\left(\frac{q}{r}\right) = f\left(\frac{r}{p}\right)$.
If the graph of $g(x) = px^2 + qx + r$ passes through M $(1, 6)$, then find the value of q.

9. Let f: $\mathbb{R} \to \mathbb{R}$ defined by $f(x) = \min.(|x|, 1 - |x|)$, then find the number of solution of equation $3f(x) - 1 = 0$.

10. If the function $f(x) = ax + b$ has its own inverse and passing through the point $(2, 3)$, then evaluate $(2a + b)$.

11. Let $f(x) = \text{sgn}(arc \cot x) + \tan\left(\frac{\pi}{2}[x]\right)$, where $[x]$ is the greatest integer function less than or equal to x. Then find the number of solutions of equation $2f(x) = 1 + 2\{x\}$; where $\{x\}$ denotes fractional parts of x in $[-4, 5]$.

Answer Keys

SECTION–III

1. (b) **2.** (a) **3.** (d) **4.** (c) **5.** (a) **6.** (b) **7.** (b) **8.** (d) **9.** (d) **10.** (b)
11. (c) **12.** (b) **13.** (c) **14.** (b) **15.** (d) **16.** (c) **17.** (a) **18.** (d) **19.** (b) **20.** (a)
21. (b) **22.** (c) **23.** (a) **24.** (b) **25.** (b) **26.** (b) **27.** (b) **28.** (c) **29.** (b) **30.** (a)
31. (c) **32.** (b) **33.** (d) **34.** (b) **35.** (a) **36.** (d) **37.** (b) **38.** (d) **39.** (d) **40.** (c)
41. (b) **42.** (a) **43.** (d) **44.** (i) (a) (ii) (a) (iii) (c)
45. (i) (b) (ii) (b) **46.** (d) **47.** (c) **48.** (d)

SECTION–IV

1. (a,b,c) **2.** (b,c) **3.** (a,b,c) **4.** (b,c,d) **5.** (c,d) **6.** (a,b,c) **7.** (a,c,d)

SECTION–V

1. (a) **2.** (a) **3.** (a) **4.** (d) **5.** (d) **6.** (d)

SECTION–VI

1. (a) **2.** (b) **3.** (c) **4.** (c) **5.** (b) **6.** (c) **7.** (b) **8.** (a,c) **9.** (c) **10.** (c)
11. (b) **12.** (c) **13.** (c) **14.** (d) **15.** (d) **16.** (b,c) **17.** (c) **18.** (a) **19.** (d) **20.** (a)
21. (d) **22.** (c) **23.** (c) **24.** (b)

SECTION–VII

1. (i) → (d), (ii) → (a), (iii) → (d), (iv) → (b)
2. (i) → (d) (ii) → (c) (iii) → (a) (iv) → (b)
3. (i) → (b) (ii) → (b) (iii) → (b) (iv) → (c) (v) → (d)
4. (i) → (d) (ii) → (c) (iii) → (a) (iv) → (b) (v) – (3)

SECTION–VIII

1. 9 **2.** 5 **3.** 0 **4.** 5 **5.** 3 **6.** 4 **7.** 17 **8.** 8 **9.** 4 **10.** 3
11. 5

HINTS AND SOLUTIONS

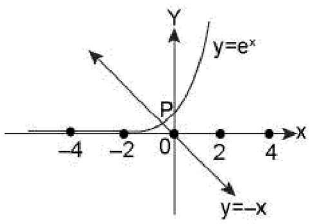
TUTORIAL EXERCISE: (SECTION III)

1. (b) $e^x + x = 0$

$\Rightarrow$ $e^x = -x$

$\Rightarrow$ The solution will be the abscissa of points of intersection of $y = e^x$ and $y = -x$.

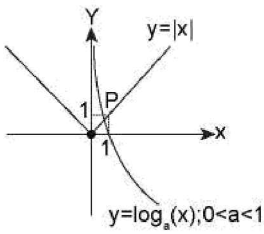
The two graphs drawn on same frame of reference will be as shown below.



Clearly, only one point of intersection 'P', and hence, only one solution.

2. (a) $\text{Log}_a x = |x|$, $0 < a < 1$

Graph of $y = \log_a (x)$ and $y = |x|$ for $0 < a < 1$ drawn on same frame of reference will be as shown below.



Clearly, there is only one point of intersection 'P' and, hence, only one solution.

3. (d) $y + |y| - x - |x|$

$\Rightarrow$ $y + |y| = x + |x|$ ……….(i)

Case (i): $y \geq 0, x \geq 0$ $\Rightarrow$ $y = x$

Case (ii): $y \geq 0, x \leq 0$ $\Rightarrow$ $y = 0$

$\Rightarrow$ + *ve* x-axis

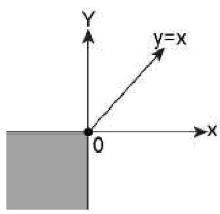
Case (iii): $y \leq 0, x \leq 0$ $\Rightarrow$ $0 = 0$

$\Rightarrow$ whole III[rd] quadrant

Case (iv): $y \leq 0, x \geq 0$ $\Rightarrow$ $x = 0$

$\Rightarrow$ –ve y-axis

The complete graph represented by (1) will be as shown below.



The graphs of $y = x^2$ and $y = \log_{1/3} x$ interests the line $y = x$ which has current.

Also $y = \text{sgn}(-e^x) = -1\ \forall\ x \in \mathbb{R}$, as $e^x > 0\ \forall\ x \in \mathbb{R}$

$\Rightarrow$ $-e^x < 0\ \forall\ x \in \mathbb{R}$

$\Rightarrow$ $\text{sgn}(-e^x) = -1\ \forall\ x \in \mathbb{R}$

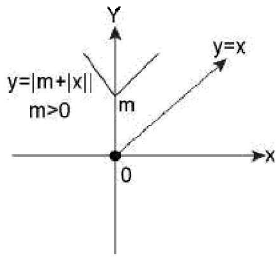
$\Rightarrow$ $\text{sgn}(-e^x) = -1$ is $y = -1$, which intersect the third quadrant having current.

Now $y = |m + |x||$ $(m > 3)$

$\Rightarrow$ $y = (m + |x|)$ as $m > 3$, $|x| \geq 0$

$\Rightarrow$ $m + |x| > 0$

Its graphs are as shown below



Thus, the curve $y = |m + |x|$ is disjoint from the current carrying curve, and hence, safer for Mr. calculus .

4. (c) $f(x) = -\left(\dfrac{|x|^3 + |x|^2 + |x|}{1 + x^2}\right)$

Clearly, $f(x) \leq 0\ \forall\ x \in D_f$ and $D_f = \mathbb{R}$

$\Rightarrow$ $f(x) \leq 0\ \forall\ x \in \mathbb{R}$

$\Rightarrow$ Graph of $f(x)$ would lie in III[rd] and IV[th] quadrants.

5. (a) $|f(x) + 6 - x^2| = |f(x)| + |4 - x^2| + 2$

By triangle inequality, $|f(x) + 6x - x^2| \leq |f(x) + |6 - x^2| = |f(x)| + |(4 - x^2) + 2| \leq |f(x) + |4 - x^2| + 2$

Given, $|f(x) + 6 - x^2| = |f(x) + |4 - x^2| + 2$

$\Rightarrow$ $2(4 - x^2) \geq 0$ and $f(x).(6 - x^2) \geq 0$

$\left(\begin{array}{l} \because |a+b| \leq |a| + |b| \\ \text{and equality holds } a.b \geq 0 \end{array}\right)$

$\Rightarrow$ $4 - x^2 \geq 0$ and $f(x).[4 - x^2 + 2] \geq 0$

$\Rightarrow$ $4 - x^2 \geq 0$ and $f(x) \geq 0$

$\Rightarrow$ $f(x) \geq 0$ for $x \in [-2, 2]$

6. (b) $\{(x, y) \in R^2 : x^2 + y^2 \leq 25, y \geq -x\}$ represents the point on and in interior of circle $x^2 + y^2 = 25$, which are above the line $y = -x$.

7. (b) $x^4 + y^3 + 1$; contains even power of x

$\Rightarrow$ $f(x, y) = f(-x, y)$

$\Rightarrow$ Curve is symmetric about y-axis

8. (d) $[x]^2 + [x + 3] = 4$ …(i)

$\Rightarrow$ $[x]^2 + [x] + 3 = 4$

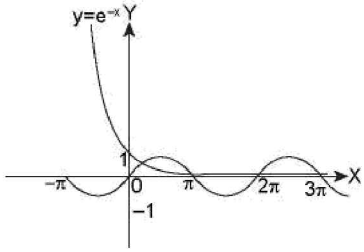
$\Rightarrow$ $[x]^2 + [x] - 1 = 0$

$\Rightarrow$ $[x] = ..$

$\Rightarrow$ $[x] = \dfrac{-1 \pm \sqrt{5}}{2}$, which is impossible as L.H.S is an integer, whereas R.H.S is non-integer.

$\Rightarrow$ No solution.

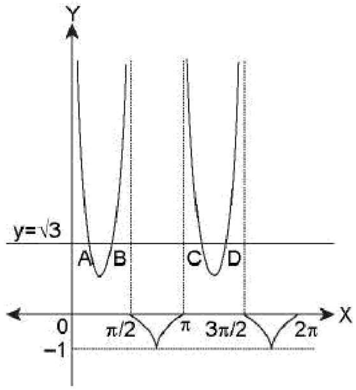
9. (d) Graphs of $y = \sin x$ and $y = e^{-x}$ drawn on same frame of reference are as shown below.



Clearly $y = e^{-x} \in (0, 1]$ for $x \geq 0$ and $\in (1, \infty)$ for $x < 0$

$\Rightarrow$ $y = e^{-x}$ and $y = \sin x$ would intersect each other infinitely many times in the intervals $(0, \infty)$ and do not intersects in $(-\infty, 0)$. Thus, there are infinitely many solutions.

10. (b) $f(x) = \max. \{\tan x, \cot x\}$

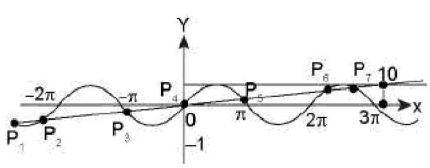


Clearly, $f(x) = \max. \{\tan x, \cot x\}$ and $y = \sqrt{3}$ intersect each other at four points A, B, C and D.

Hence, there will be four solutions in $(0, 2\pi)$.

11. (c) $\sin x = . \dfrac{x}{10}$ ...(i)

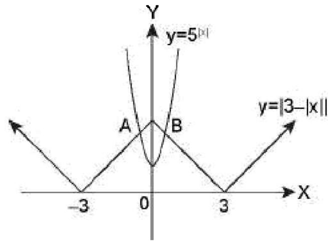
$\because$ $\sin x \in [-1, 1]$; for (1) to hold $\dfrac{x}{10} \in [-1, 1]$, i.e., $x \in [-10, 10]$



Clearly, there will be 7 points of intersection 3 on each side of y-axis along with origin.

12. (b) $5^{|x|} = |3 - |x||$

$y = 5^{|x|}$ and $y = |3 - |x||$ drawn on same reference plane are as shown below.



The graphs of functions $y = 5^{|x|}$ and $y = |3 - |x||$ would intersect each other at A and B.

In (3,), $y = 5^{|x|} = 5^x$ $\Rightarrow$ $y' = 5^x \ln 5 > 0$

$\Rightarrow$ $y = 5^x$ is an increasing function and $y'(3) = (5)^3 \ln 5 = (125)(\ln 5) > 125$.

Also, in $(3, \infty)$, $y = |3 - |x|| = |3 - x| = x - 3$

$\Rightarrow$ $y' = 1$

$\Rightarrow$ The graph of $y = 5^{|x|}$ would always remain above the graph of $y = |3 - |x||$ for $x > 3$, as the slope of tangent to $y = 5^{|x|}$ goes increasing, whereas that of $y = |3 - |x||$ remains constant for $x > 3$.

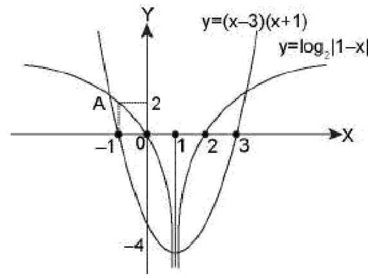
Similar is the case for $x < -3$. Thus, there will be only 2 solutions .

13. (c) $x^2 - 2x - \log_2 |1 - x| = 3$

$\Rightarrow$ $x^2 - 2x - 3 = \log_2 |1 - x|$

$\Rightarrow$ $(x - 3)(x + 1) = \log_2 |1 - x|$

The graphs of $f(x) = (x - 3)(x + 1)$ and $g(x) = \log_2 |1 - x|$ drawn on same frame of reference are given below.



Clearly, there will be 4 points of intersection, and hence, 4 solutions.

14. (b) Given equation is $e^{2x} + e^x - 2e = 0$ $e^x = \dfrac{-1 \pm \sqrt{1+8e}}{2(1)}$ but $e^x > 0$

$\Rightarrow$ $e^x = \dfrac{\sqrt{1+8e} - 1}{2}$

$\Rightarrow$ $x = ln\left(\dfrac{\sqrt{1+8e} - 1}{2}\right)$ is the only solution.

Now, $1 < e < 3$

$\Rightarrow$ $9 < 1 + 8e < 25$ $\Rightarrow$ $3 < \sqrt{1+8e} < 5$

$\Rightarrow$ $1 < \dfrac{\sqrt{1+8e} - 1}{2} < 2$

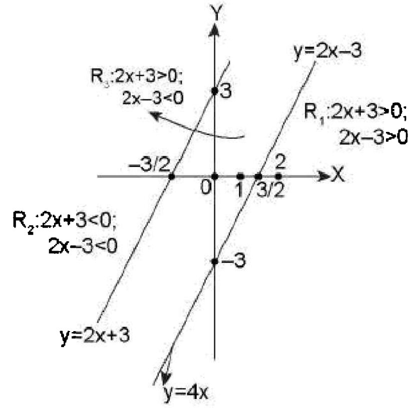
$\Rightarrow$ $0 < \ln\left(\dfrac{\sqrt{1+8e} - 1}{2}\right) < \ln 2 < 1$

$(\because \ln 2 < \ln e = 1)$

$\Rightarrow$ $x \in (0, 1)$. Thus, only one solution in (0, 1) or [0, 1]
The statement that equation has at least one root in [0, 1] is also true.

15. (d) $|2x + 3| + |2x - 3| = px + 6$. Let $f(x) = |2x + 3| + |2x - 3|$ and $g(x) = px + 6$

Graph of $f(x) = |2x + 3| + |2x - 3|$ will be as shown below.



In Region R_1: Thus, $2x + 3 \geq 0$ and $2x - 3 \geq 0$ for $x \in \left[\frac{3}{2}, \infty\right)$

$\therefore$ In intervals $\left[\frac{3}{2}, \infty\right)$; $y = |2x + 3| + |2x - 3| = 2x + 3 + 2x - 3$

$\Rightarrow$ $y = 4x$

In Region R_2: $2x + 3 \leq 0$; $2x - 3 \leq 0$ for $x \in \left(-\infty, \frac{-3}{2}\right]$

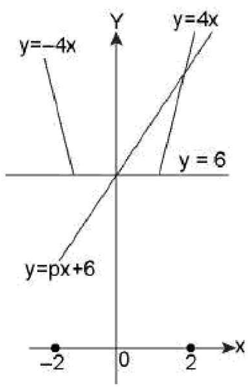
$\therefore$ In interval $\left(-\infty, \frac{-3}{2}\right)$; $y = -2x - 3 - 2x + 3$

$\Rightarrow$ $y = -4x$

In Region R_3: $2x + 3 \geq 0$; $2x - 3 \leq 0$ for $x \in \left[\frac{-3}{2}, \frac{3}{2}\right]$

$\therefore$ In interval $\left[\frac{-3}{2}, \frac{3}{2}\right]$, $y = 2x + 3 - 2x + 3$

$\Rightarrow$ $y = 6$



Clearly $y = px + 6$ and $y = |2x + 3| + |2x - 3|$ would intersect each other at more than two points (infinitely many points).

If $y = px + 6$ coincide with $y = 6$

$\Rightarrow$ $p = 0$, i.e., $p \in \{0\}$

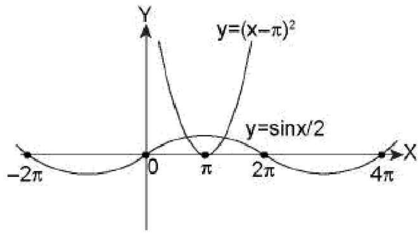
16. (c) $\sin\left(\frac{x}{2}\right) + 2x = x^2 + \pi^2$

$\Rightarrow$ $\sin\left(\frac{x}{2}\right) = x^2 - 2\pi x + \pi^2$

$\Rightarrow$ $\sin\left(\frac{x}{2}\right) = (x - \pi)^2$

Let $f(x) = \sin\left(\frac{x}{2}\right)$ and $g(x) = (x - \pi)^2$

The graphs of $f(x)$ and $g(x)$ drawn on same reference plane are as shown below.



Clearly, there will be two solutions only.

17. (a) $2\cos(e^x) = 5^x + 5^{-x}$ ...(i)

$\because$ $5^x, 5^{-x} > 0\ \forall\ x \in \mathbb{R}$,

By A.M. ≥ G.M., $5^x + 5^{-x}$ $2\sqrt{5^x.5^{-x}}$, i.e., $5^x + 5^{-x} \geq 2$, whereas $2\cos(e^x) \leq 2$

$\therefore$ Equation (1) is satisfied when both sides are equal to 2

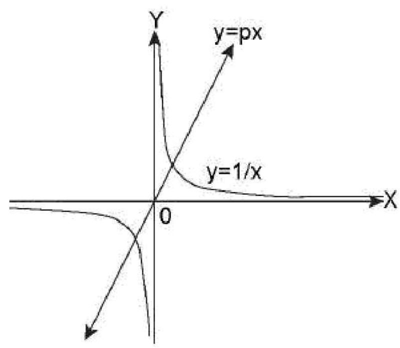
$\Rightarrow$ $\cos(e^x) = 1$ $\Rightarrow$ $e^x = 2n\pi$; $n \in \mathbb{N}$

$\Rightarrow$ $x = \ln(2n\pi)$; $n \in \mathbb{N}$, but $5^x + 5^{-x} = 2$ for $x = 0$

$\therefore$ (1) does not hold for any $x \in \mathbb{R}$

$\Rightarrow$ No solution.

18. (d) The graphs of $y = \frac{1}{x}$ and $y = px$ drawn on same reference frame are shown below.

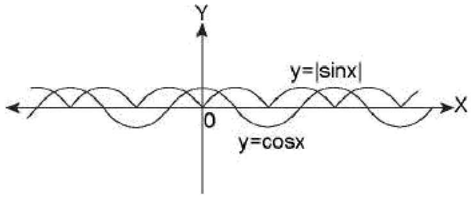


For very value of $p > 0$, $y = px$ and $y = \frac{1}{x}$ intersect each other exactly at two points. As for $p = 0$, x-axis is asymptote and for $p < 0$, $y = px$ completely lie in I, IInd and IIIrd quadrant in which no portion of $y = \frac{1}{x}$ lies.

19. (b) $|\sin x| = \cos x$ ...(1)

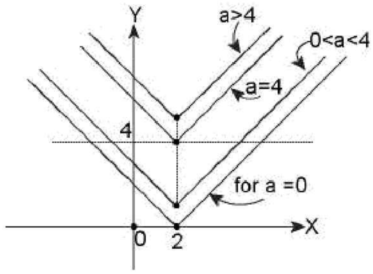
Let $f(x) = |\sin x|$ and $g(x) = \cos x$

The graphs of $y = |\sin x|$ and $y = \cos x$ drawn on same reference plane are as shown below.



Clearly, the two graphs intersect at $x = 2n\pi \pm \dfrac{\pi}{4}$; for each $n \in \mathbb{Z}$.

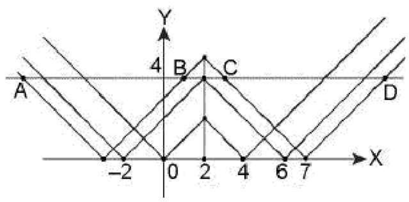
20. (a) The graph of $y = ||x - 2| + a|$ for $a \geq 0$ will of the form as shown as below.



Clearly, for $a \geq 0$, we get at most two solutions of equation $||x - 2| + a| = 4$

$\Rightarrow$ For more than two solutions $a < 0$.

The graphs of $y = ||x - 2| + a|$ for $a < 0$ are as shown below.



Clearly, for 4 different solutions of $||x - 2| + a| = 4$; $a \in (-\infty, -4)$.

21. (b) $f_1(x) = \begin{cases} x\,;\, 0 \le x \le 1 \\ 1\,;\, x > 1 \\ 0\,;\text{otherwise} \end{cases}$; $f_2(x) = f_1(-x)\ \forall\, x, f_3(x) = -f_2(x)$

$\forall\, x; f_4(x) = f_3(-x)\ \forall\, x$

$$f_2(x) = f_1(-x) = \begin{cases} -x;\ 0 \le -x \le 1 \\ 1;\ -x > 1 \\ 0;\ \text{otherwise} \end{cases} = \begin{cases} -x; -1 \le x \le 0 \\ 1;\ x < -1 \\ 0\ ;\text{otherwise.} \end{cases}$$

$$f_3(x) = -f_2(x) = \begin{cases} x;\ -1 \le x \le 0 \\ -1\,;\, x < -1 \\ 0\ ;\text{otherwise} \end{cases}$$

$$f_4(x) = f_3(-x) = \begin{cases} -x; -1 \le -x \le 0 \\ 1\ ;\ -x < -1 \\ 0\ ;\text{otherwise} \end{cases} = \begin{cases} -x\ ;\ 0 \le x \le 1 \\ -1\ ;\ x > 1 = -f_1(x) \\ 0\ ;\text{otherwise} \end{cases}$$

Thus, $f_1(x) = -f_4(x) \Rightarrow$ (a) is incorrect

$f_1(x) = -f_3(-x)$

$\Rightarrow$ (b) is correct and (d) is incorrect.

Also $f_4(x) = f_3(-x) = -f_2(-x)$

$\Rightarrow$ (c) is incorrect.

22. (c) $x^2 - 22[x] = 0 \Rightarrow x^2 - 22(x - \{x\}) = 0$

$\Rightarrow x^2 - 22x + 22\{x\} = 0 \Rightarrow x^2 - 22x = -22\{x\}$

$\Rightarrow \dfrac{22x - x^2}{22} = \{x\} \in [0, 1)$

$\Rightarrow \dfrac{22x - x^2}{22} \in [0, 1)$

$\Rightarrow (22x - x^2) \in [0, 22) \Rightarrow 0 \le 22x - x^2 < 22$

$\Rightarrow x^2 - 22x \le 0$ and $x^2 - 22x + 22 > 0$

$\Rightarrow x \in [0, 22]$ and

$x \in \left(-\infty, \dfrac{22 - \sqrt{(22)^2 - 88}}{2}\right) \cup \left(\dfrac{22 + \sqrt{(22)^2 - 88}}{2}, \infty\right)$

$\Rightarrow x \in [0, 22]$ and $x \in \left(-\infty, \dfrac{22 - 19}{2}\right) \cup \left(\dfrac{22 + 19}{2}, \infty\right)$

$\Rightarrow x \in [0, 1.5) \cup (20.5, 22]$ (consider super set for more accuracy)

For $x \in$ [0, 1); $[x] = 0$

$\therefore\ x^2 = 22[x] = 0 \Rightarrow x = 0$

For $x \in$ [1, 1.5); $[x] = 1$

$\therefore\ x^2 = 22[x] = 22 \Rightarrow x = \sqrt{22}$

$\Rightarrow [x] \neq 1$ $\therefore$ No solution

For $x \in$ (20. 5, 21); $[x] = 20$

$\therefore\ x^2 = 22[x] = 440 \Rightarrow x = \sqrt{440} \simeq 20.9$

$\Rightarrow [x] = 21$

For $x \in$ [21, 22); $[x] = 21$

$\therefore\ x^2 = 22[x] = 462 \Rightarrow x = \sqrt{462} \simeq 21.4$

$\Rightarrow [x] = 21$

For x = 22; $[x] = 22$

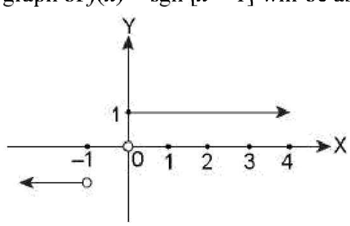
$\therefore\ x^2 = 22[x] = (22)^2 \Rightarrow x = 22; [x] = 22$

Thus, $x = 0, x = \sqrt{440}, x = \sqrt{462}, x = 22$ are four solutions .

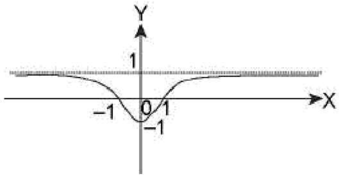
23. (a) $f(x) = \text{sgn}\,[x + 1] = \begin{cases} 1\ \text{for}\ [x+1] > 0 \\ 0\ \text{for}\ [x+1] = 0 \\ -1\ \text{for}\ [x+1] < 0 \end{cases}$

$$= \begin{cases} 1\ \ \text{for}\ x+1 \ge 1 \\ 0\ \ \text{for}\ (x+1) \in [0,1) \\ -1\ \text{for}\ (x+1) \in (-\infty, 0) \end{cases} = \begin{cases} 1\ \ \text{for}\ x \ge 0 \\ 0\ \ \text{for}\ x \in [-1, 0) \\ -1\ \text{for}\ x \in (-\infty, -1) \end{cases}$$

$\therefore$ The graph of $f(x) = \text{sgn}\,[x + 1]$ will be as shown below.



24. **(b)** $f: \mathbb{R} \to [-1, 1]$ defined as $f(x) = \dfrac{x^2-1}{x^2+1}$



Clearly $f(\pm 1) = 0$ $\Rightarrow$ ± 1 are roots of $f(x) = 0$

$\Rightarrow$ $f(x) = 1 - \dfrac{2}{x^2+1} \Rightarrow f'(x) = -2\left(\dfrac{-1}{(x^2+1)^2}\right)(2x)$

$= \dfrac{4x}{(x^2+1)^2} > 0$ for $x > 0$ and < 0 for $x < 0$;

$f''(x) = \dfrac{-4(3x^2-1)}{(x^2+1)^3} < 0$ for $x < \dfrac{-1}{\sqrt{3}}$ or $x > \dfrac{1}{\sqrt{3}}$ and > 0

for $x \in \left(\dfrac{-1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$

Also as $x \pm \infty$, $f(x)$ 1

$\Rightarrow$ $y = 1$ is a horizontal asymptotes.

$\Rightarrow$ Thus, the graph of $f(x)$ will be as shown above. Clearly $f(x)$ is many–one and onto from $\mathbb{R}$ to $[-1, 1]$

$f(x)$ is bounded, but if does not attain its maximum value.

However, $f(x)$ decrease for $x < 0$ and increase for $x > 0$

25. **(b)** From, the graph we can observe , that

(i) $\lim\limits_{x \to 0^+} f(x) = -\infty$

$\Rightarrow$ $\lim\limits_{x \to 0^+} \left(\dfrac{1}{f}(x)\right) = 0^-$

(ii) $f(a) = 0$ and $f(x) < 0$ for $x \in (0, \text{a})$; $f(x) > 0$ for $x \in (a, \infty)$

$\Rightarrow$ $\dfrac{1}{f(x)} < 0$ for $x \in (0, \text{a})$; $\dfrac{1}{f(x)} > 0$ for $x \in (a, \infty)$;

$\lim\limits_{x \to a^-} \dfrac{1}{f(x)} = -\infty$ and $\lim\limits_{x \to a^+} \dfrac{1}{f(x)} = \infty$

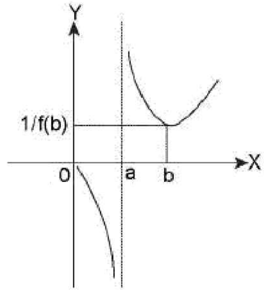
(iii) $f(x) \to 0^+$ *as* $x \to \infty$

$\Rightarrow$ $\dfrac{1}{f(x)} \to +\infty \; x \to \infty$

(iv) $\dfrac{d}{dx}\left(\dfrac{1}{f(x)}\right) = \dfrac{-1}{[f(x)]^2} . f'(x)$

$\Rightarrow$ $\dfrac{1}{f(x)}$ decrease for $x \in (0, a)$ and (a, b) increase for $x \in (\text{b}, \infty)$

Collecting information from above we conclude that the rough graph of $1/f(x)$ would be as represents below.



26. **(b)** $f(x) = \cot^{-1}(4 - x^2) \in (0, \pi)$

We observe the following:

(i) $f(x)$ is symmetric about y-axis

(ii) $\cot^{-1}(4 - x^2) \in \left(0, \dfrac{\pi}{2}\right)$ for $4 - x^2 > 0$, i.e., $x \in (-2, 2)$

(iii) $\cot^{-1}(4 - x^2) \in \left(\dfrac{\pi}{2}, \pi\right)$ for $4 - x^2 < 0$, i.e., $x \in (-\infty, -2) \cup (2, \infty)$

(iv) $\cot^{-1}(4 - x^2) = \dfrac{\pi}{2}$ for $4 - x^2 = 0$ i.e., $x = \pm 2$

(v) $f'(x) = \dfrac{-1}{1+(4-x^2)^2}.(-2x) = \dfrac{2x}{1+(4-x^2)^2} > 0$ for $x > 0$

and < 0 for $x > 0$

$\Rightarrow$ $x = 0$ is a critical point and is a point of local minima, and the local minimum value of $f(x)$ is

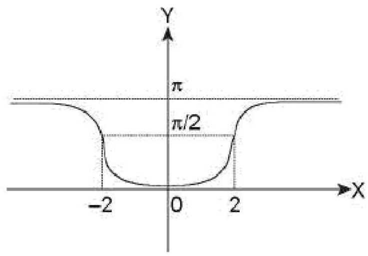
$\text{Cot}^{-1}(4) < \cot^{-1}(0) = \dfrac{\pi}{2}$

(vi) $\cot^{-1}(4 - x^2) \to \pi$ for $4 - x^2 \to -\infty$

$\Rightarrow$ $x^2 \to \infty$ $\Rightarrow$ $x \to \pm\infty$

$\Rightarrow$ $y = \pi$ is a horizontal asymptote.

Thus, the graph of $f(x)$ will be as shown below.



27. **(b)** $f(x) = \{x\}.[x]$

$\therefore$ $f(x) = 1$ $\Rightarrow$ $\{x\}[x] = 1$

$\Rightarrow$ $[x] > 0$ $\Rightarrow$ $[x] \geq 1$

$\Rightarrow$ $x \in [1, \infty)$ and $x \notin \mathbb{Z}$ as otherwise $\{x\} = 0$

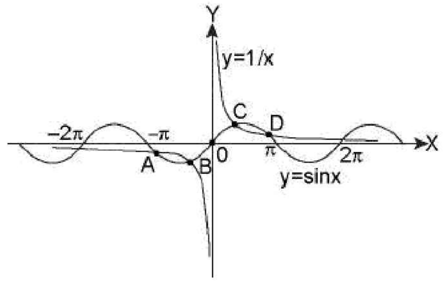
Now $\{x\} = \dfrac{1}{[x]}$ and $[x] > 1$; $x \notin \mathbb{Z}$

$\Rightarrow$ $x = 2 + \dfrac{1}{2}; 3 + \dfrac{1}{3}, 4 + \dfrac{1}{4}, 5 + \dfrac{1}{5}, ...$

$\Rightarrow$ There will be infinitely many solutions.

28. **(c)** Given equation is $x \sin x = 1$ or $\sin x = \dfrac{1}{x}$

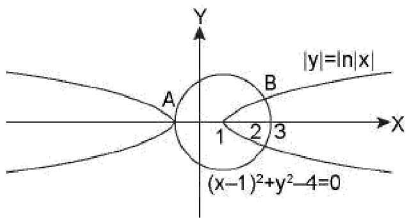
Draw the graph of $f(x) = \sin x$ and $g(x) = \dfrac{1}{x}$ on same frame of reference as shown below.



Clearly, there will be infinitely many points of intersection of $f(x) = \sin x$ and $g(x) = 1/x$.

In fact all points of intersection belong to $(-\infty, -1) \cup (1, \infty)$ and in $[-2\pi, 0) \cup (0, 2\pi]$, there lie exactly 4 points of intersection, and hence, 4 roots.

29. (b) The graphs of $|y| = \ln |x|$ and $(x - 1)^2 + y^2 - 4 = 0$ are as shown below.

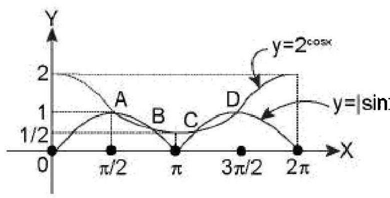


Clearly there will be 3 points of intersection, and hence, 3 solutions, i.e., 3 points (x , y)

30. (a) Given equation is $2^{\cos x} = |\sin x|$

$y = |\sin x|$ and $y = 2^{\cos x}$ are periodic functions with period π and 2 respectively.

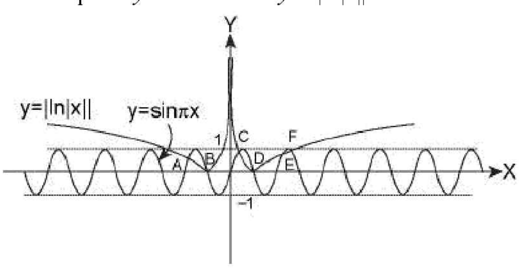
$\therefore$ $2^{\cos x} = |\sin x|$ will also be periodic with period 2π . Thus, it is sufficient to drawn the graph in $[0, 2\pi]$ as shown below.



Clearly in $[0, 2\pi]$ there are 4 points of intersection, and hence, 4 solutions.

31. (c) $\sin \pi x = |\ln|x||$

Graph of $y = \sin \pi x$ and $y = |\ln|x||$ are as shown below.



Clearly there will be 6 points of intersection.

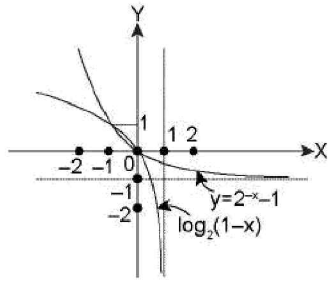
32. (b) Given equation is $1 + \log_2 (1 - x) = 2^{-x}$

$\Rightarrow$ $\log_2 (1 - x) = 2^{-x} - 1$ ...(1)

Let $f(x) = \log_2 (1 - x)$; $(1 - x) > 0$, i.e., $x \in (-\infty, 1)$

and $g(x) = 2^{-x} - 1$ ...(2)

The graphs of $y = f(x)$ and $y = g(x)$ drawn on same frame of reference are as shown below.

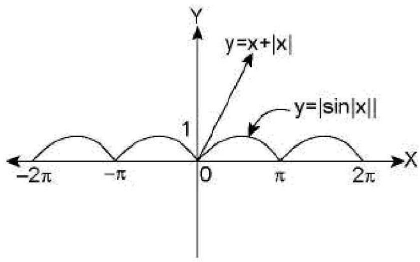


Clearly, the two functions intersect each other at two points.

Thus, there will be 2 solutions.

33. (d) Let $f(x) = |\sin |x||$ and $g(x) = x + 1$; $x \in [-4\pi, 2\pi]$

The graphs of $f(x)$ and $g(x)$ drawn on same frame of reference are as shown below.



Hence, $f(x) = |\sin |x|| = \begin{cases} |\sin x|; x \geq 0 \\ |-\sin x|; x < 0 \end{cases}$ and $g(x)$

$= \begin{cases} 2x \text{ for } x \geq 0 \\ 0 \text{ for } x < 0 \end{cases}$

Clearly $f(x) = g(x)$ at $x = -4\pi, -3, -2\pi, -\pi, 0$

$\therefore$ There will be total 5 solutions.

34. (b) $\because$ $3^{x-1} + 5^{x-1}$ is a continuous and increasing function with range $(0, \infty)$.

Thus, $3^{x-1} + 5^{x-1}$ takes every real numbers > 0 exactly once. Hence, the equation $(3)^{x-1} + (5)^{x-1} = 12$ has Exactly one solution.

35. (a) $\cos [x] = 2^{4x-1}$; $x \in [0, 2\pi]$

$\cos [x] \leq 1$ and $2^{4x-1} > 1 = 2^0$ for $x > \frac{1}{4}$

$\therefore$ $\cos [x] = 2^{4x-1}$ can hold in $\left[0, \frac{1}{4}\right]$ but in $\left[0, \frac{1}{4}\right]$; $[x] = 0$

$\Rightarrow$ $\cos 0 = 2^{4x-1}$ for $x \in \left[0, \frac{1}{4}\right]$

$\Rightarrow$ $1 = 2^{4x-1}$ for $x \in \left[0, \frac{1}{4}\right]$

$\Rightarrow$ $x = \frac{1}{4}$ is the only solution.

36. (d) Let $f(x) = x^2 + x + 3$ and $g(x) = -2\sin x;\ x \in [-\pi, \pi]$

$\because\ f(x) \in \left[\dfrac{11}{4}, \infty\right) \quad \Rightarrow f(x) > 2$, whereas $g(x) \in [-2, 2]$

$\Rightarrow\ f(x) \neq g(x)$ for any $x \in \mathbb{R}$

$\Rightarrow$ There is no real root of given equation.

37. (b) Given equation is $|x - x^2 - 1| = |2x - 3 - x^2|$ ……(i)

$\Rightarrow\ (-x^2 + x - 1) \in \left(-\infty, -\dfrac{3}{4}\right]$

$\Rightarrow\ |x - x^2 - 1| \in \left[\dfrac{3}{4}, \infty\right)$

Also $(-x^2 + 2x - 3) \in (-\infty, -2]$

$\Rightarrow\ |2x - 3 - x^2| \in [2, \infty)$

Also $-x^2 + x - 1 < 0,\ -x^2 - 2x - 3 < 0$

$\therefore$ (1) becomes, $x^2 - x + 1 = x^2 - 2x + 3$

$\Rightarrow\ x = 2$ is the only solution.

38.(d) $\dfrac{x^2}{1-|x-2|} = 1 \quad \Rightarrow \quad x^2 = 1 - |x - 2|$

$$\Rightarrow x^2 = \begin{cases} 3-x \text{ for } x \geq 2 & x^2+x-3=0 \text{ for } x \geq 2 \\ x-1 \text{ for } x<2 & \text{and } x^2-x+1=0 \text{ for } x<2 \end{cases}$$

But $x^2 + x - 3$ for $x \geq 2$ and $x^2 - x + 1 \geq \dfrac{3}{4}$ for $x \in \mathbb{R}$,

Hence, for $x < 2$

$\therefore$ (1) is not satisfied for any real value x.

39. (d) $f(x) = \dfrac{|x|^3 + |x|}{1-x^2}$; $D_f = \mathbb{R} - \{-1, 1\}$

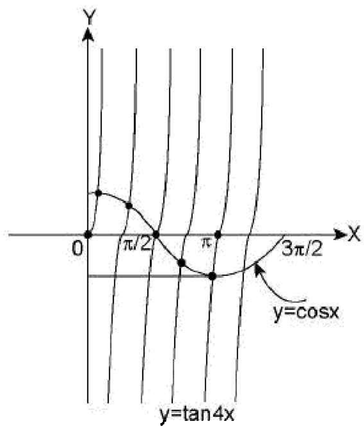
$$\Rightarrow f(x) = \frac{|x|\left(|x|^2+1\right)}{1-x^2} = \frac{|x|\left(1+x^2\right)}{\left(1-x^2\right)}$$

$$\Rightarrow f(x) = \begin{cases} +\text{ve for } x \in (-1,1)-\{0\} \\ 0 \text{ at } x=0 \\ -\text{ve for } x \in (-\infty,-1) \cup (1,\infty) \end{cases}$$

$\Rightarrow\ f(x)$ lie in each of 4 quadrants.

40. (c) $\tan 4x = \cos x;\ 0 < x < \pi$.

The graph of $y = \tan 4x$ and $y = \cos x$ for $x \in (0, \pi)$ drawn on same frame of reference is as shown below.

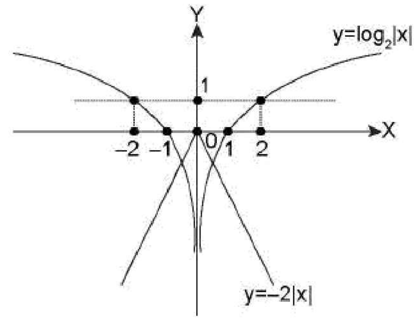


Clearly, there are 5 points of intersection of $y = \tan 4x$ and $y = \cos x$ in $(0, \pi)$.

41. (b) $\log_{0.5}|x| = 2|x| \quad \Rightarrow \log_2{}^{-1}|x| = 2|x|$

$\Rightarrow\ -\log_2|x| = 2|x| \quad \Rightarrow \log_2|x| = -2|x|$

Graphs of $y = \log_2|x|$ and $y = -2|x|$ drawn on same frame are as shown below.



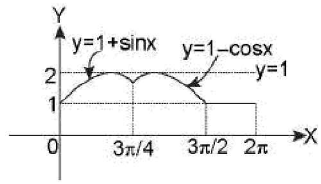
Clearly, there are two points of intersection, and hence, 2 solutions.

42. (a) $f(x) = \max.\{1 + \sin x, 1, 1 - \cos x\};\ x \in [0, 2\pi]$

$g(x) = \max.\ \{1, |x - 1|\};\ x \in \mathbb{R}$

Clearly, $g(x) = \begin{cases} |x-1| \text{ for } x \in (-\infty, 0] \cup [2, \infty) \\ 1 \text{ for } x \in (0,2) \end{cases}$

For $f(x)$, let us draw its graph as shown below.



$$\therefore\ f(x) = \begin{cases} 1+\sin x;\ 0 \leq x \leq \dfrac{3\pi}{4} \\ 1-\cos x;\ \dfrac{3\pi}{4} < x \leq \dfrac{3\pi}{2} \\ 1;\ \dfrac{3\pi}{2} < x \leq 2\pi \end{cases}$$

$\therefore\ g(f(0)) = g(1) = 1 \quad \Rightarrow$ (*a*) is correct

$f(f(1)) = f(1 + \sin 1)$

$\left(\because \dfrac{3\pi}{4} > 2, 1 + \sin 1 < 2 \Rightarrow 1 + \sin 1 < \dfrac{3\pi}{4}\right)$

$= 1 + \sin(1 + \sin 1)$

Also $f(g(1)) = f(1) = 1 + \sin 1$

$\Rightarrow$ (c) is incorrect

Further $f(g(0)) = f(1) = 1 + \sin 1$

$\Rightarrow$ (d) is incorrect .

43. (d) $f: \mathbb{R} \to \mathbb{R};\ g: \mathbb{R} \to \mathbb{R}$

We know that min. $\{f_1(x), f_2(x)\}$

$$= \frac{\left(f_1(x) + f_2(x)\right) - \left|f_1(x) - f_2(x)\right|}{2}$$

$\therefore$ min $\{f(x) - g(x), 0\}$

$$= \frac{\left(f(x) - g(x) + 0\right) - \left|f(x) - g(x) - 0\right|}{2}$$

$$= \frac{\left(f(x) - g(x)\right) - \left|f(x) - g(x)\right|}{2}$$

44. (i) (a) Clearly, $f(-2) = \frac{5}{2}$ as on line joining (–3, 4) and (–1, 1), i.e., $y - 1 = \frac{-3}{2}(x + 1)$

(ii) (a) The function is defined on $[-3, -1] \cup [1, 5)$

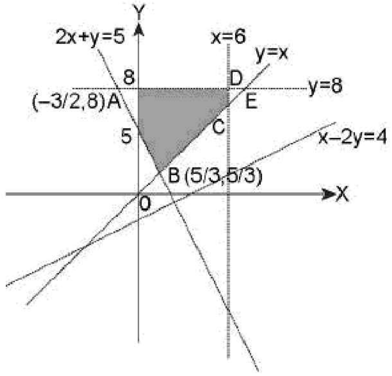
$\Rightarrow$ It is the domain of function.

(iii) (c) The function take all values belonging to interval $[-3, -1)$ or $(1, 4]$

$\therefore$ Range of function is $[-3, -1) (1, 4]$

45. (i) (b) Given inequalities are $x - 2y \le 4$, $2x + y > 5$, $y \ge x$, $x < 6$, $y < 8$

The common region is as shown below.

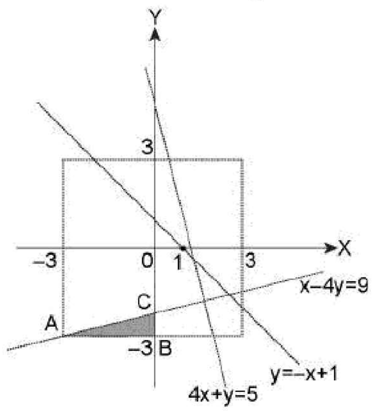


The required area = (Area of ΔABE) – (Area of ΔCDE)

$= \frac{1}{12}(361) - \frac{1}{2}(2)(2) = \frac{361-24}{12} = \frac{337}{12}$ square units.

(ii) (b) Given inequalities are $4x + y \le 5$; $x - 4y > 9$; $y < -x + 1$; $xy \ge 0$; $|x| \le 3$; $|y| \le 3$.

The common bounded region is as shown below.



$\therefore$ The Required area = area of ΔABC

$= \frac{1}{2}(AB)\times(BC) = \frac{1}{2}\times 3\times\left|\left(3-\frac{9}{4}\right)\right|$

$= \frac{1}{2}\times(3)\times\left(\frac{3}{4}\right) = \frac{8}{9}$ sq. units

46. (d) Given Equation is $[x]^2 + [x + 3] = 4$

$\Rightarrow$ $[x]^2 + [x] - 1 = 0$; Let $[x] = y$

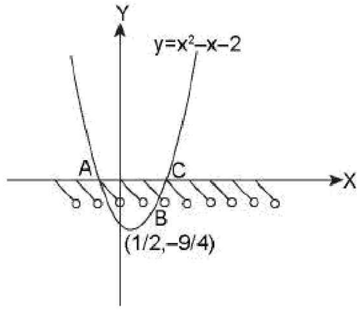
$\Rightarrow$ $y^2 + y - 1 = 0$ $\Rightarrow$ $y = \frac{-1\pm\sqrt{(1)^2+4}}{2} = \frac{1-\pm\sqrt{5}}{2}$

But $y \in \mathbb{Z}$ $\Rightarrow$ There exists no solution.

47. (c) $x^2 - 2 - [x] = 0 \Rightarrow x^2 - 2 - (x - \{^x\}) = 0$

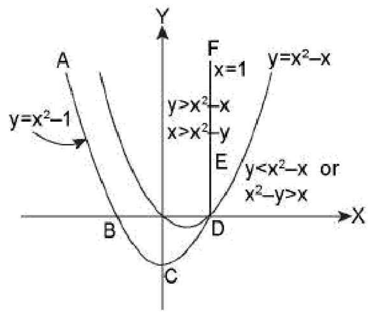
$\Rightarrow$ $x^2 - x - 2 = -\{x\}$ $\Rightarrow$ $(x - 2)(x + 1) = -\{x\}$

The graphs of function $y = (x - 2)(x + 1)$ and $y = -\{x\}$ drawn on same reference plane are as shown below.



Clearly, the two graphs $y = x^2 - x - 2$ and $y = -\{x\}$ intersect each other at 3 points, hence, there will be 3 solutions.

48. (d) max. $\{x^2 - y, x\} = 1$



$x^2 - y = x$

$\Rightarrow$ $y = x^2 - x$

Now $x^2 - y \ge x$

$\Rightarrow$ $y \le x^2 - x$

In region, $x^2 - y > x$, max $\{x^2 - y, x\} = 1$

$\Rightarrow$ $x^2 - y = 1$

$\Rightarrow$ $y = x^2 - 1$

In region, $y \ge x^2 - x$ or $x \ge x^2 - y$, max $\{x^2 - y, x\} = x = 1$

Thus, all points lying on parabola $y = x^2 - 1$ on $x \in (-\infty, 1)$ and on straight line $x = 1$ for $y \ge 0$ satisfy the given equation.

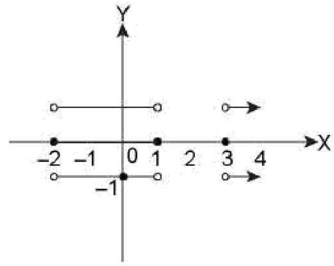
Thus, there will be infinitely many real solutions (x, y).

SECTION–IV: MORE THAN ONE ANSWERS CORRECT

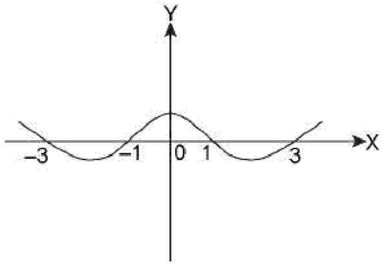
1. (a), (b), (c)

(a) $\operatorname{sgn} f(x) = \begin{cases} 1 \text{ for } x \in (-2, 1) \cup (3, \infty) \\ -1 \text{ for } x \in (1, 3) \cup (-\infty, -2) \\ 0 \text{ for } x = -2, 1, 3 \end{cases}$

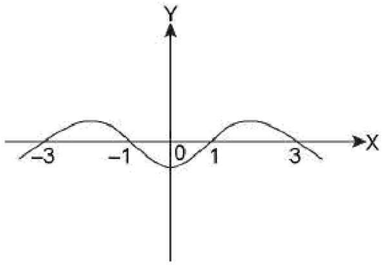
$\Rightarrow$ $|y| = \operatorname{sgn} f(x)$ will be as shown below.



(b) $y = f(|x|)$ is as shown below.

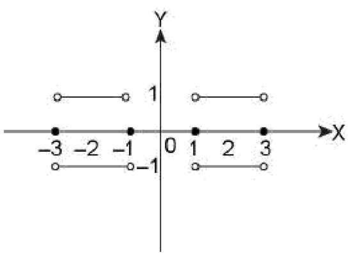


$\Rightarrow$ $y = (-f(|x|))$ will be as shown below.

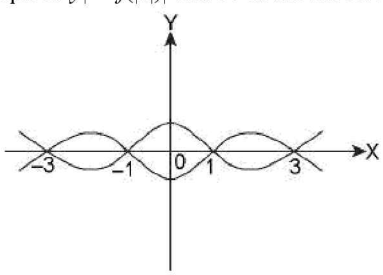


$$\Rightarrow \text{sgn}(-f(|x|)) = \begin{cases} -1 \text{ for } x \in (-\infty,-3) \cup (-1,1) \cup (3,\infty) \\ 0 \text{ for } x = -3,-1,3 \\ 1 \text{ for } x \in (-3,-1) \cup (1,3) \end{cases}$$

Thus, the graph of $|y| = \text{sgn}(-f(|x|))$ will be as shown below.

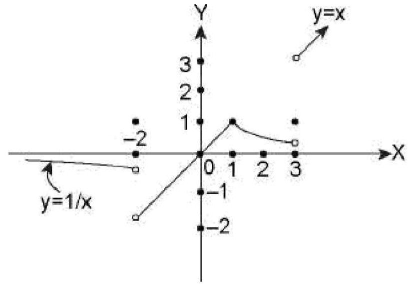


(c) Graph of $|y| = |f(|x|)|$ will be as shown below.



(d) $$y = x^{\text{sgn} f(x)} = \begin{cases} x \text{ for } x \in (-2,1) \cup (3,\infty) \\ x^{-1} \text{ for } x \in (-\infty,-2) \cup (1,3) \\ 1 \text{ for } x = -2,1,3 \end{cases}$$

$\therefore$ The graph of $y = x^{\text{sgn} f(x)}$ will be as shown below.

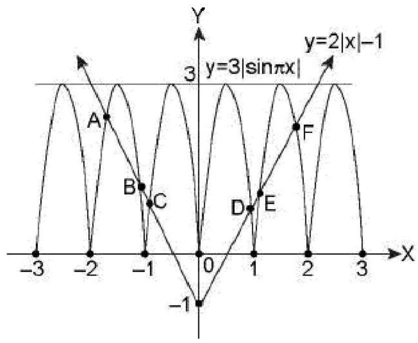


Clearly, option (d) is incorrect.

2. **(b), (c)** Given equation is $2|x| - 1 = 3|\sin \pi x|$

Period of $y = 3|\sin \pi x|$ is $\dfrac{\pi}{\pi} = 1$

Draw graphs of $y = 2|x| - 1$ and $y = 3|\sin x|$ for $x \in [0, 1]$ as shown below.

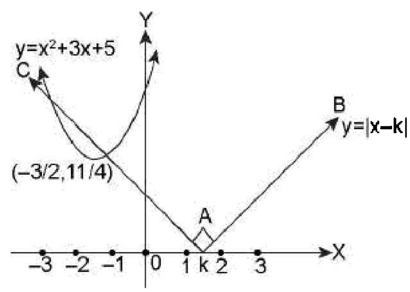


Clearly, there are 6 points of intersection, and hence, 6 solutions in (–2, 2) and no solution in [2, ∞).

Also no solution in $\left(0, \dfrac{1}{2}\right)$.

3. **(a), (b), (c)** $f(x) = x^2 + 3x + 5$; $g(x) = |x - k|$

Graph of $y = f(x)$ and $y = g(x)$ drawn on same reference plane are as shown below.



$f'(x) = 2x + 3 = \pm 1$

$\Rightarrow$ $2x = -3 \pm 1$ $\Rightarrow$ $x = \dfrac{-3 \pm 1}{2} = -1, -2$

Equation of tangent to parabola $y = x^2 + 3x + 5$ with slope 1 and -1 will be at $(-1, 3)$ and $(-2, 3)$, respectively and are given by $(y - 3) = 1\ (x + 1)$ and $(y - 3) = -1(x + 2)$, i.e., $y = x + 4$ and $y = -x + 1$

They will intersect x-axis at $(-4, 0)$ and $(1, 0)$ respectively.

$\therefore$ Branch AB will touch $y = f(x)$ for $k = -4$ and AC will touch $y = f(x)$ for $k = 1$

$\Rightarrow$ $f(x) = g(x)$ has exactly one root for exactly two distinct values of k given by -4 and 1.

Clearly $f(x) \neq g(x)$ for $k \in (-4, 1)$, and hence, $f(x) = g(x)$ has no root in $(-3, 1)$.

AB would intersect $y = f(x)$ for $k \in (-\infty, -4)$ and AC would intersect $y = f(x)$ for $k \in (1, \infty)$.

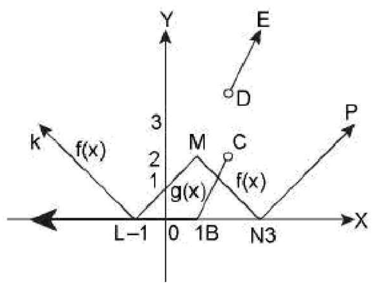
$\therefore$ $f(x) = g(x)$ has two roots for $k \in (-\infty, -4) \cup (1, \infty)$

4. (b), (c), (d) $f(x) = ||x - 1| - 2|$ and $g(x) = x + |x - 1| + \dfrac{|x-2|}{(x-2)}$;

$$g(x) = \begin{cases} 2x \text{ for } x > 2 \\ 2x - 2 \text{ for } 1 < x < 2 \text{ and } f(x) \\ 0 \text{ for } x < 1 \end{cases}$$

$$= \begin{cases} -x-1 \text{ for } x \leq -1 \\ x+1 \text{ for } -1 < x \leq 1 \\ -x+3 \text{ for } 1 < x \leq 3 \\ x-3 \text{ for } x > 3 \end{cases}$$

The graphs of $y = f(x)$ and $y = g(x)$ drawn on same reference frame are as shown below.



Clearly, $f(x) = g(x)$ holds at L, Q has exactly 2 solutions. Also clearly, $y = g(x)$ has a point of discontinuity at $x = 2$.

Further we observe that $y = f(x)$ has 3 points of non-differentiability, i.e., L, M and N.

5. (c), (d) Given curves are $(x^2 + y^2)y - ax^2 = 0$ ……(i) and $(x^2 + y^2) = a^2(x^2 - y^2)$ …..(2), $a \neq 1$

For curve (1):

(i) $x^2(y - a) = -y^3$ $\quad\Rightarrow\quad x = \pm\sqrt{\dfrac{y^3}{a-y}}$

$\Rightarrow$ $x \in \mathbb{R}$ for $\dfrac{\quad}{a \quad y} \geq 0 \Rightarrow y^3(a - y) \geq 0; y \neq a$

$\Rightarrow$ $y^3\ (y - a) \leq 0; y \neq a$ $\quad\Rightarrow\quad y(y - a) \leq 0; y \neq a$

$\Rightarrow$ $y \in [0, a)$ for $a > 0$ and $y \in (a, 0]$ for $a < 0$

(ii) Also, the curve is symmetric about y-axis.

(iii) Curve intersect x-axis, where $y = 0$

$\Rightarrow$ $x = 0$, i.e., at origin

(iv) Horizontal asymptote is given by $y - a = 0$

(v) There is no vertical asymptote.

(vi) $x^2\left(\dfrac{dy}{dx}\right) + (y - a)\ (2x) = -3y^2\ \dfrac{dy}{dx}$

$\Rightarrow$ $\dfrac{dy}{dx}\ [x^2 + 3y^2] = 2x(a - y)$

$\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{2x(a-y)}{x^2+3y^2} = \dfrac{(2x)}{(x^2+3y^2)}\left(\dfrac{y^3}{x^2}\right) = \dfrac{2y^3}{x(x^2+3y^2)}$

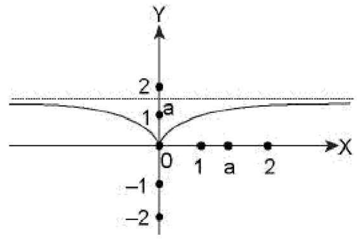
$\therefore$ $\dfrac{dy}{dx} = 0$ $\quad\Rightarrow\quad x = 0$ or $y = a$; but $y \neq a$

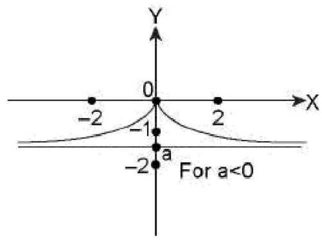
$\Rightarrow$ $x = 0$ $\quad\Rightarrow\quad y = 0$

$\Rightarrow$ $\dfrac{dy}{dx}$ is not defined at origin.

There is no tangent parallel to x-axis. Also, clearly $\dfrac{dy}{dx} > 0$ for $x.y > 0$ and $\dfrac{dy}{dx} < 0$ for $x.y < 0$

Thus, the graph of curve (1) will be of the type as shown below.





For curve (2):

(i) $x^2(1 - a^2) = -y^2(1 + a^2)$

$x^2 = \dfrac{y^2(1+a^2)}{(a^2-1)} \Rightarrow y = \pm\sqrt{\left(\dfrac{a^2-1}{a^2+1}\right)}x$

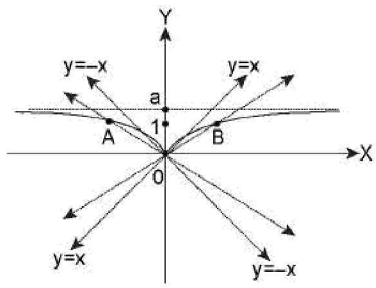
Which represent a pair of straight line through origin having their slopes of magnitude less than 1.

Also $y = \pm\sqrt{\dfrac{x^2(a^2-1)}{(a^2+1)}}$

$\Rightarrow$ $a^2 - 1 \geq 0$

$\Rightarrow$ $a \in (-\infty, -1) \cup (1, \infty)$ as given $a \neq 1$

$\therefore$ The graphs of $y = f(x)$ and $y = g(x)$ drawn on same frame of reference will be as shown below (for $a > 1$)



Clearly, $y = f(x)$ and $y = g(x)$ intersect each other at exactly 3 points for a $\in (-\infty, -1) \cup (1,\infty)$.
Also $(x^2 + y^2).y - ax^2 = 0$ has exactly one cusp.

6. (a), (b), (c)

C_1: $100\,(x^2 + y^2) = a^2\, x^2y^2$ ...(1)

C_2: $y = (a^3 - x^3)^{1/3}$ ...(2)

For C_1 :

(i) $x^2\,(100 - a^2\, y^2) + 100\, y^2 = 0$ or $y^2(100 - a^2\, x^2) + 100\, x^2 = 0$

$\Rightarrow$ Horizontal asymptotes are given by $y = \pm\,\dfrac{10}{a}$ and vertical asymptotes are $x = \pm\,\dfrac{10}{a}$

(ii) C_1 passes through origin.

(iii) C_1 is symmetric about x-axis and y-axis.

(iv) C_1: $\dfrac{1}{y^2}+\dfrac{1}{x^2}=\dfrac{a^2}{100} \quad \Rightarrow \quad \dfrac{1}{y^2}=\dfrac{a^2}{100}-\dfrac{1}{x^2}$

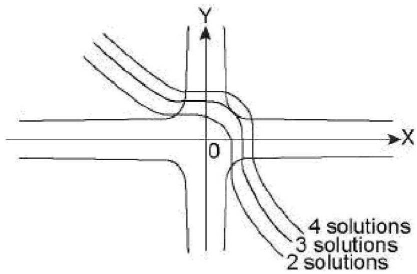
$\Rightarrow\ y^2 = \dfrac{100x^2}{\left(a^2x^2-100\right)}$

$\Rightarrow\ y = \pm\dfrac{10x}{\sqrt{a^2x^2-100}} \Rightarrow x < \dfrac{-10}{a}$ or $> \dfrac{10}{a}$ at x

$= \dfrac{15}{a}, y = \dfrac{\pm\dfrac{150}{a}}{\sqrt{225-100}}$

$\Rightarrow\ y = \pm\dfrac{150}{a\left(5\sqrt{5}\right)} = \pm\dfrac{30}{\sqrt{5}a} \qquad y = \pm\,\dfrac{6\sqrt{5}}{a}$

(v) C_1 is symmetric about $y = x$



For C_2: $y^3 = a^3 - x^3$

$\Rightarrow$ Intersect x-axis, where $x = a$ and intersect y-axis, where $y = a$;

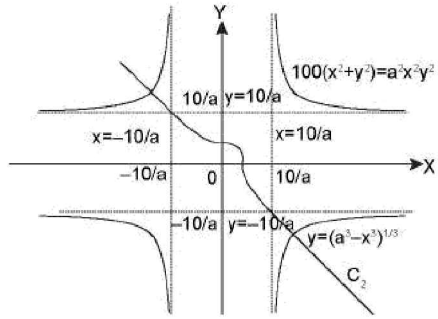
Also C_2 is symmetric about $y = x$ and also $3y^2\,\dfrac{dy}{dx} = -3x^2$

$\Rightarrow\ \dfrac{dy}{dx} = \dfrac{-x^2}{y^2} < 0$

$\Rightarrow\ \dfrac{dy}{dx} = 0$ at $x = (0, a)$

$\Rightarrow\ \dfrac{dy}{dx} \to \infty$ at $(a, 0)$

Graphs of $y = f(x)$ and $y = g(x)$ drawn on same reference frame are as shown below.



At $x = y$ for C_1: $200\, y^2 = a^2\, y^4$

$\Rightarrow\ \dfrac{200}{a^2} = y^2 \Rightarrow y = \dfrac{\sqrt{200}}{a}$

For C_3: $y^3 = a^3 - y^3 \ \Rightarrow\ y^3 = \dfrac{a^3}{2}$

$\Rightarrow\ y = \dfrac{a}{3\sqrt{2}}$

Clearly, there will be 4 points of intersection for ${}^yC_2 > {}^yC_1$, i.e., $\dfrac{a}{3\sqrt{2}} > \dfrac{\sqrt{200}}{a}$

$\Rightarrow\ a^2 > 3\sqrt{2}\,\sqrt{200} = k$ (say)

$\Rightarrow\ a \in (-\infty, -\sqrt{k}) \cup (\sqrt{k}, \infty)$

i.e., there will be infinitely many values of ‘a’

Also for 3 points of intersection, ${}^yC_2 = {}^yC_1$

$\Rightarrow\ \dfrac{a}{3\sqrt{2}} = \dfrac{\sqrt{200}}{a} \Rightarrow a \pm \sqrt{3\sqrt{2}.\sqrt{200}}$, i.e., exactly two real value of ‘a’.

Also there will be two points of intersection for ${}^yC_2 < {}^yC_1$

$\Rightarrow\ \dfrac{a}{3\sqrt{2}} < \dfrac{\sqrt{200}}{a} \Rightarrow a^2 < 3\sqrt{2}.\sqrt{200} = k$ (say)

$\Rightarrow\ a \in \left(-\sqrt{k}, \sqrt{k}\right)$

$\Rightarrow$ There will be 2 points of intersection for infinitely many values ‘a’.

7. (a), (c), (d)

(a) $\sin^{-1}\dfrac{1}{x} = \cos^{-1}\dfrac{1}{x} \Rightarrow \dfrac{1}{x} \in (0,1]$

$\Rightarrow\ \sin^{-1}\dfrac{1}{x} + \sin^{-1}\dfrac{1}{x} = \dfrac{\pi}{2} \ \Rightarrow\ 2\sin^{-1}\dfrac{1}{x} = \dfrac{\pi}{2} \Rightarrow \sin^{-1}\dfrac{1}{x} = \dfrac{\pi}{4}$

$\Rightarrow\ \dfrac{1}{x} = \dfrac{1}{\sqrt{2}} \Rightarrow x = \dfrac{1}{\sqrt{2}} \ \Rightarrow$ only one solution

(b) Let $\sin^{-1}\dfrac{1}{x} = \tan^{-1}\dfrac{1}{x} = \theta$; where $\theta \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$

$\Rightarrow\ \sin\theta = \dfrac{1}{x} = \tan\theta \neq 0$

$\Rightarrow\ \cos\theta = 1 \Rightarrow \theta = \dfrac{\pi}{2} \notin \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$

$\therefore$ No solution

(c) $\sin^{-1}\dfrac{1}{x} = \text{cosec}^{-1}\dfrac{1}{x} \Rightarrow \dfrac{1}{x} = -1 \text{ or } 1$

$\Rightarrow x = -1 \text{ or } 1 \qquad \Rightarrow$ 2 real solutions.

(d) Let $\sin^{-1}\left(\dfrac{1}{x}\right) = \cot^{-1}\left(\dfrac{1}{x}\right) = \theta$

$\Rightarrow \theta \in \left(0, \dfrac{\pi}{2}\right)$

$\Rightarrow \sin\theta = \cot\theta = \dfrac{1}{x} > 0; x\,[1, \infty)$

$\sin^2\theta = \cos\theta \qquad \Rightarrow \cos^2\theta + \cos\theta - 1 = 0$

$\Rightarrow \cos\theta = \dfrac{-1 \pm \sqrt{1+4}}{2} \qquad \Rightarrow \cos\theta = \dfrac{-1-\sqrt{5}}{2}, \dfrac{\sqrt{5}-1}{2}$ but

$\dfrac{-1-\sqrt{5}}{2} < -1 \Rightarrow \cos\theta = \dfrac{\sqrt{5}-1}{2}$

$\Rightarrow \sin\theta = \sqrt{1 - \left(\dfrac{\sqrt{5}-1}{2}\right)^2}$

$\Rightarrow \sqrt{\dfrac{4-\left(6-2\sqrt{5}\right)}{4}} = \sqrt{\dfrac{2\sqrt{5}-2}{4}} = \dfrac{\sqrt{\sqrt{5}-1}}{\sqrt{2}}$

$\Rightarrow \cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{\sqrt{\sqrt{5}-1}}{\sqrt{2}} = \dfrac{1}{x}$

$\Rightarrow x = \dfrac{\sqrt{2}}{\sqrt{5}-1} > 1 \text{ as } 2 > \sqrt{5}-1 \text{ as } 3 > \sqrt{5}$

$\therefore \sin^{-1}\left(\dfrac{1}{x}\right) = \cot^{-1}\left(\dfrac{1}{x}\right)$ has only one solution $x = \dfrac{\sqrt{2}}{\sqrt{\sqrt{5}-1}}$

SECTION–V: ASSERTION AND REASON TYPE

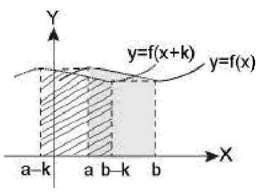
1. (a) Reason is clearly true.

$\therefore$ a = 0 and $y = 2x^2 - bx + c$ should have $x = 1$ as its axis.

$\Rightarrow \dfrac{b}{4} = 1 \qquad \Rightarrow b = 4$

$\Rightarrow a + 2b = 8$

2. (a) Clearly Assertion is true, and therefore, area enclosed by $y = f(x)$ and x-axis between ordinate x = a and $x = b$ is same as that of enclosed by $y = f(x + k)$ and x-axis between ordinates $x = a - k$ and $x = b - k$.

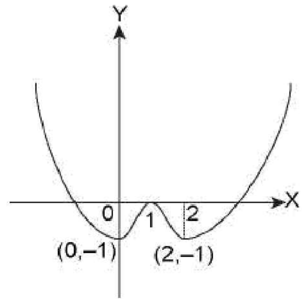


3. (a) Let $f(x) = g(x) + k$

$g(x) = x^4 - 4x^3 + 4x^2 - 1$

$g'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x-1)(x-2)$

Clearly $x = 1$ is local maxima and $x = 0, 2$ are local minima and the graph is shown in the figure.

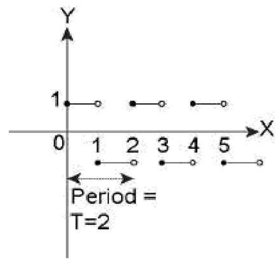


$f(1) = 1$ and $f(0) = f(2) = -1$

$g(x)$ if shifted upwards by k where $k > 1$ lies above x-axis.

$\therefore f(x) = g(x) + k$ has no real roof if $k > 1$

4. (d) We have $f(x) = \cos \pi[x] = \begin{cases} 1, x \in [2n, 2n+1) \\ -1, x \in [2n+1, 2n+2) \end{cases}$ where $n \in \mathbb{Z}$



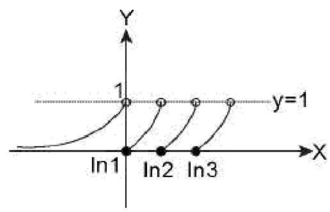
Graph of $f(x) = \cos \pi[x]$

Clearly, $f(x)$ is periodic with period = 2.

$\Rightarrow$ Assertion is false but reason is true.

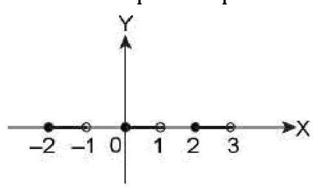
5. (d) $y = \{e^x\} = \begin{cases} e^x & ; & -\infty < x < 0 \\ e^x - 1 & ; & 0 \le x < \ln 2 \\ e^x - 2 & ; & \ln 2 \le x < \ln 3 \end{cases}$ and so on

Clearly $f(x)$ is a periodic on R.



$\Rightarrow$ Assertion is false but reason is true.

6. (d) A is false as it has a period equal to 2 and R is true



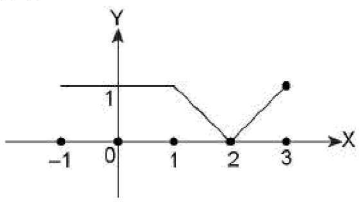
$$f(x) = \begin{cases} 0; & -2 \le x < -1 \\ \text{N.D}; & -1 \le x < 0 \\ 0 & 0 \le x < 1 \\ \text{N.D}; & 1 \le x < 2 \\ 0 & 2 \le x < 3 \end{cases}$$

⇒ period 2

SECTION–VI: COMPREHENSION PASSAGE

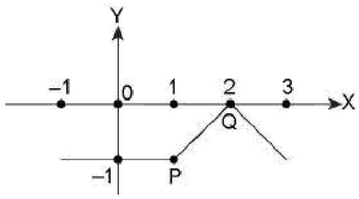
Passage A:

1. (a) $y = |f(x)|$



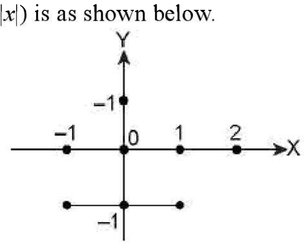
Clearly, $|f(x)|$ increases in (2, 3)

2. (b) $y = -|f(x)|$ has following graph in the interval $(-1, 3)$, $f(x)$ has sharp turns at P and Q, having abscissa $x = 1$ and $x = 2$ respectively.



Thus, $x = 1$ and $x = 2$ are the points of non-differentiability.

3. (c) $y = f(-|x|)$ is as shown below.



Clearly $y = f(-|x|) = -1 \ \forall\ x \in [-1, 1]$

⇒ It is a constant function.

4. (c) $g(x) = \dfrac{1}{2}(|f(x)| + f(x))$

$$= \begin{cases} \frac{1}{2}(1+(-1)); -1 \le x \le 1 \\ \frac{1}{2}((2-x)+(x-2)); 1 \le x \le 2 \\ \frac{1}{2}(f(x)+f(x)) = (x-2); 2 \le x \le 3 \end{cases} \quad \text{or } g(x)$$

$$= \begin{cases} 0\,; -1 \le x \le 2 \\ (x-2); 2 \le x \le 3 \end{cases}$$

⇒ $g(x)$ is a constant function in $[-1, 2]$

5. (b) $h(x) = \dfrac{1}{2}(|f(x)| - f(x)) = \begin{cases} 1; -1 \le x \le 1 \\ 2-x; 1 \le x \le 2 \\ 0\,; 2 \le x \le 3 \end{cases}$

⇒ $h(x)$ is a constant function in $[-1, 1] \cup [2, 3]$

6. (c) $g(x) = f\left(\dfrac{|x|}{x}\right) = \begin{cases} f(1) \text{ for } x > 0 \\ f(-1) \text{ for } x < 0 \\ \text{Not defined for } x = 0 \end{cases} \quad g(x)$

$$= \begin{cases} -1 \text{ for } x > 0 \text{ and } x < 0 \\ \text{Not defined at } x = 0 \end{cases}$$

⇒ $g(x)$ is a constant function on domain $(-, \infty) - \{0\}$

Passage B:

$$f(x) = \frac{(x^2-1)(x-3)}{x^2(x+2)}$$

7. (b) $y = \dfrac{(x^2-1)(x-3)}{x^2(x+2)}$

Horizontal asymptotes are given by $y = c$; where $C = \lim\limits_{x\to\infty} \dfrac{(x^2-1)(x-3)}{x^2(x+2)}, \lim\limits_{x\to-\infty} \dfrac{(x^2-1)(x-3)}{x^2(x+2)}$

$$\therefore\ C = \lim_{x\to\infty} \frac{\left(1-\frac{1}{x^2}\right)\left(1-\frac{3}{x}\right)}{\left(1+\frac{2}{x}\right)} = 1$$

∴ $y = 1$ is a horizontal asymptote

8. (a), (c) Vertical asymptote(s) are given by $x = c$; where $c = \lim\limits_{x\to\pm\infty} x$

Now $y \pm \infty$ ⇒ $x^2(x + 2) \to 0$

⇒ $x = 0$ or $x = -2$

⇒ $x = 0$ and $x = -2$ are two vertical asymptotes to $y = f(x)$

9. (c) **(i)** $f(x) = 0$ has roots, $x = \neq 1$ and $x = 3$

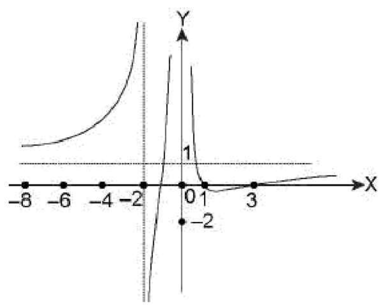
(ii) Horizontal asymptote is $y = 1$

(iii) Vertical asymptotes are $x = 0$, $x = -2$

(iv) $f(x) = \dfrac{(x^2-1)(x-3)}{x^2(x+2)}$

⇒ $f(x) > 0$ for $x \in (-\infty, -2) \cup (-1, 1) \cup (3, \infty)$ and $f(x) < 0$ for $x \in (-2, -1) \cup (1, 3)$.

From above, we conclude that the graph of $y = f(x)$ will be as shown below.



Passage C:

10. **(c)** From the graph, $f(-5) = 9$ and $f(4) = 9$

$\Rightarrow f(-5).f(4) = 81$

11. **(b)** Clearly $x = -6, -3, 1, 7, 7$ are roots of $f(x) = 0$.

$\therefore$ Let $f(x) = a(x + 6)(x + 3)(x - 1)(x - 7)^2$

Now $f(-5) = 9 \quad \Rightarrow \quad a(1)(-2)(-6)(-12)^2 = 9$

$$\Rightarrow a = \frac{9}{144 \times 12} = \frac{1}{192}$$

$$\Rightarrow f(x) = \frac{(x-1)(x+3)(x+6)(x-7)^2}{192}$$

12. **(c)** $f(f(-5)) = f(9) = \dfrac{(8)(12)(15)(14)}{192} = 30$

Passage D:

$$f(x) = \begin{cases} (x-3)^3 + 3; \text{if } x \in (3,5] \\ x \,; \text{if } x \in [-3,3] \\ -(x+3)^2 - 3; \text{if } x \in [-5,3) \end{cases}$$

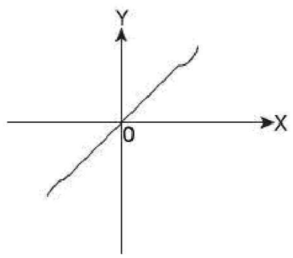
13. **(c)** $f'(x) = \begin{cases} 3(x-3)^2 \text{ if } x \in (3,5) \\ 1 \text{ if } x \in (-3,3) \\ -2(x+3) \text{ if } x \in (-5,3) \end{cases}$

$\Rightarrow f(x)$ is increasing function on $\mathbb{R}$

Also $f''(x) = \begin{cases} 6\,(x-3) \text{ if } x \in (3,5) \\ 0 \;\text{ if }\; x \in (-3,3) \\ -2 \text{ if } x \in (-5,-3) \end{cases}$

$\Rightarrow f(x)$ is concave upwards on (3, 5) concave downwards on (–5, –3) linear on (–3, 3).

Thus, the graph of $f(x)$ would be as shown below.



14. **(d)** **Case (i):** $x \in (3, 5]$

$\Rightarrow f(x) = (x - 3)^3 + 3 \quad \Rightarrow \quad f(x) \in (3, 11]$

$\Rightarrow f(f(x)) = (f(x) - 3)^3 + 3 = (x - 3)^9 + 3$

$\Rightarrow f(f(x)) = [f(f(x)) - 3]^3 + 3 = (x - 3)^{27} + 3$

$\Rightarrow \underbrace{f(f(f.......f(x))))}_{n-times} = (x-3)^{(3^n)} + 3 \in (3, \infty)$

Case (ii): $x \in [-3, 3]$

$\Rightarrow f(x) = x \in [-3, 3] \quad \Rightarrow \quad f(f(x)) = x$

$\Rightarrow \underbrace{f(f(f(f.......f(x)))))}_{n-times} = x \in [-3,3]$

Case (iii): $x \in [-3, -3)$

$\Rightarrow f(x) = -(x + 3)^2 - 3 \in [-7, -3)$

$\Rightarrow f(f(x)) = -(f(x) + 3)^2 - 3$

$\Rightarrow f(f(x)) = -[-(x + 3)^2]^2 - 3 = -(x + 3)^4 - 3 < -3$

$\Rightarrow f(f(f(x))) = -(f(f(x)) + 3)^2 - 3 = -[-(x + 3)^4]^2 - 3 = -(x + 3)^8 - 3 < -3$

$\Rightarrow \underbrace{f(f(f(......(f(x))))).......}_{n-times} = -(x+3)^{(2^n)} - 3 < -3$

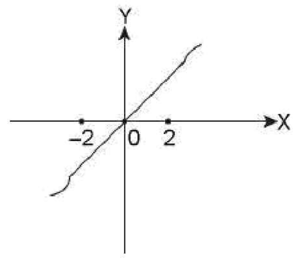
$\therefore$ From above, we observe that the composition of n – $f(x)$'s takes value $\in (-\infty, 3) \cup (3, \infty)$

$\Rightarrow$ Can take all 0, 4 and 8 values.

15. **(d)** $y = \begin{cases} (x-3)^3 + 3; \text{if } x \in (3,5] \\ x; \text{ if } x \in [-3,3] \\ -(x+3)^2 - 3; \text{if } x \in [-5,-3) \end{cases}$

$\therefore$ $f^{-1}(x)$ is the reflection of $f(x)$ on line $y = x$

$\therefore$ The graph of $f^{-1}(x)$ will be as shown below.



Passage E:

$f_1(x, y) = |x| + |y|$; $f_2(x, y) = \min.(x + y, x - y)$; $f_3(x, y) = [x] + [y]$; $[x]$ is gint function.

Now, $x + y = x - y$

$\Rightarrow$ $y = 0$, i.e., along x-axis

Now, $x + y > x - y \quad \Rightarrow \quad y > 0$ and $x + y < x - y$

$\Rightarrow$ $y < 0$

$\therefore$ $f_2(x, y) = \begin{cases} x + y; y < 0 \\ x - y; y > 0 \\ x\,; y = 0 \end{cases}$

16. **(b), (c)**

(a) $f_1(x, y) < f_2(x, y) \quad \Rightarrow \quad |x| + |y| < f_2(x, y)$

$\Rightarrow \begin{cases} |x| - y < x + y \text{ for } y < 0; \\ |x| + y < x + y \text{ for } y > 0; \\ |x| < x \text{ for } y = 0 \end{cases}$ which is not always true for instance for $y > 0$

$|x| + y < x - y$, for $x > 0$;

$\Rightarrow$ $x + y < x - y \quad \Rightarrow \quad y < -y$

$\Rightarrow$ $y < 0$; which is false.

(b) $f_2(x, y) < f_3(x, y)$

$\Rightarrow \begin{cases} x + y < [x] + [y] \text{ for } y < 0 \\ x - y < [x] + [y] \text{ for } y > 0 \\ x < [x] + [y] \text{ for } y = 0 \end{cases}$ but for $x, y \in \mathbb{Z}^+$

$\Rightarrow$ $x - y < [x] + [y]$ for $x, y \in \mathbb{Z}^+$

$\Rightarrow$ $x - [x] < y + [y]$ for $x, y \in \mathbb{Z}^+$

$\Rightarrow$ $\{x\} < y + [y]$

$\Rightarrow$ $\{x\} < 2y$, as $y \in \mathbb{Z}^+ \quad \Rightarrow \quad [y] = y$

$\Rightarrow$ $0 < 2y$ as $x \in \mathbb{Z}^+$ $\quad\Rightarrow$ $\{x\} = 0$

$\Rightarrow$ $0 < y$ which is true as $y \in \mathbb{Z}^+$

(c) $x, y \in \mathbb{Z}^-$

$\therefore$ $f_1(x, y) = -f_3(x, y)$

$\Rightarrow$ $|x| + |y| = -([x] + [y])$

$\Rightarrow$ $-x - y = -x - y$, which is true

17. (c) $h_1(x) = f_1(\sin\{x\}, \sin \pi x)\ \forall\ x \in \mathbb{R}$,

$h_2(x) = f_2(x, \sin x)\ \forall\ 0 < x < 1$,

$h_3(x) = f_3(x, -x)\ \forall\ x \in \mathbb{R}$

$\Rightarrow$ $h_1(x) = |\sin\{x\}| + |\sin \pi x|$

$\because$ $|\sin\{1 + x\}| = |\sin\{y\}| \Rightarrow$ period is 1 and $|\sin \pi x|$ has period $\dfrac{\pi}{\pi} = 1$

$\therefore$ $h_1(x)$ is periodic with period 1.

Clearly, $h_1(1 + x) = |\sin\{1 + x\}| + |\sin \pi(1 + x)| = |\sin\{x\}| + |-\sin \pi x| = |\sin\{x\}| + |\sin \pi x| = h_1(x)$

(b) $h_2(x) = f_2(x, \sin x) = \min.\{x + \sin x, x - \sin x\}$

Now h_2: $(0, 1) \to (0, 1)$

$\Rightarrow$ $x \in (0, 1)$ $\quad\Rightarrow$ $\sin x > 0$

$\Rightarrow$ $\sin x > -\sin x$ $\quad\Rightarrow$ $x + \sin x > x - \sin x$

$\Rightarrow$ $h_2(x) = x - \sin x$ $\quad\Rightarrow$ $h'_2(x) = 1 - \cos x = 2\sin^2\dfrac{x}{2} > 0$

$\Rightarrow$ $h_2(x)$ is increasing function

$\Rightarrow$ $h_2(x)$ is injective

Also $h_2(x)$ is continuous and injective

$\Rightarrow$ Range of $h_2(x) = (h_2(0), h_2(1)) = (0, |-\sin|) \subset (0, 1)$ (property)

$\Rightarrow$ h_2: $(0, 1) \to (0, 1)$ is not subjective

$\Rightarrow$ h_2 is not bijective

(c) $h_3(x) = f_3(x, -x) = [x] + [-x] = -1$

$\Rightarrow$ $h_3(x) = -1$

$\Rightarrow$ $h_3(x)$ is a constant function, and hence is periodic.

18. (a) $f_1(x, y) > f_3(x, y)$

$\Rightarrow$ $|x| + |y| > [x] + [y]$

It is true for $x, y < 0$

(b) $f_2(x, y) = x + y;\ x, y < 0$

$\Rightarrow$ $\min.\{x + y, x - y\} = x + y;\ x, y < 0$

$\because$ $y < -y$ as $y < 0$

$\Rightarrow$ $x + y < x - y$ $\quad\Rightarrow$ $\min\{x + y, x - y\} = (x + y)$

(c) $f_3(x, y) = f_3(-x, -y)$ $\quad\Rightarrow$ $[x] + [y] = [-x] + [-y]$

$\Rightarrow$ $[x] + [y] = (-1 - [x]) + [-1 - [y]]$ for $x, y \notin \mathbb{Z}$

$\Rightarrow$ $2[x] + 2[y] = -2$ for $x, y \notin \mathbb{Z}$

$\Rightarrow$ $[x] + [y] = -1$ for $x, y \notin \mathbb{Z}$

$\Rightarrow$ $[y] = -1 - [x]$ $\quad\Rightarrow$ $[y] = [-x]$

$\Rightarrow$ $y = -x$

$\Rightarrow$ (c) can hold for $y = -x$ and $x, y \notin \mathbb{Z}$

(d) $f_1(x, y) = f_3(x, y)$; $x, y \in \mathbb{R} - \mathbb{Z}$

$\Rightarrow$ $|x| + |y| = [x] + [y]$; $x, y \in \mathbb{R} - \mathbb{Z}$

Case (i): If $x > 0, y > 0$

$\Rightarrow$ $x + y = [x] + [y]$ $\quad\Rightarrow$ $(x - [x]) + (y - [y]) = 0$

$\Rightarrow$ $\{x\} + \{y\} = 0$ $\quad\Rightarrow$ $\{y\} = \{y\} = 0$

$\Rightarrow$ $x, y, \in \mathbb{Z}$ but here $x, y \in \mathbb{R} - \mathbb{Z}$

$\therefore$ Equality does not hold.

Case (ii): If $x > 0, y < 0$

$\Rightarrow$ $x - y = [x] + [y]$

$\Rightarrow$ $\{x\} = y + [y]$ impossible, as L.H.S > 0 and R.H.S < 0

Case (iii): If $x < 0, y < 0$

$\Rightarrow$ $-x - y = [x] + [y]$ impossible as L.H.S > 0 and R.H.S < 0

Case (iv): if $x < 0, y > 0$ $\Rightarrow$ $-x + y = [x] + [y]$

$\Rightarrow$ $y - [y] = x + [x]$

$\Rightarrow$ $\{y\} = x + [x]$ impossible as L.H.S > 0; R.H.S < 0

Passage F:

H_1: $x^2 - 4y^2 + 4$; H_2: $x^2 - 9y^2 - \dfrac{(1 - a)}{(1 + a)}$

$H_1 = 0$ $\quad\Rightarrow$ $x^2 - 4y^2 = -4$

$$\Rightarrow \frac{y^2}{(1)} - \frac{x^2}{4} = 1 \qquad \ldots(i)$$

Which is a hyperbola of the form $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$, having asymptotes given by $y = \dfrac{\pm a}{b}x$, i.e., $y = \pm\dfrac{1}{2}x$

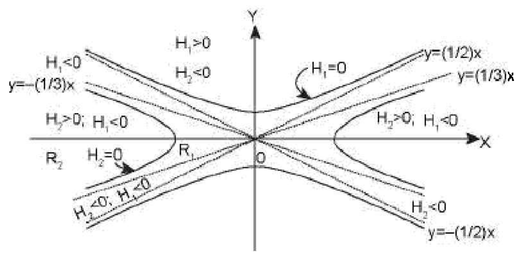
$$H_2 = 0 \quad\Rightarrow\quad x^2 - 9y^2 = \frac{(1 - a)}{(1 + a)} \qquad \ldots(ii)$$

Case (i): when $\dfrac{1 - a}{1 + a} > 0$, i.e., a $\in (-1, 1)$

In this case (2) will be a hyperbola of the form $\dfrac{x^2}{\left(\sqrt{\dfrac{1 - a}{1 + a}}\right)^2} - \dfrac{y^2}{\left(\dfrac{1}{3}\sqrt{\dfrac{1 - a}{1 + a}}\right)^2} = 1$ having its asymptotes,

$y = \pm\dfrac{b}{a}x$ $\quad$ i.e., $y = \pm\dfrac{1}{3}x$

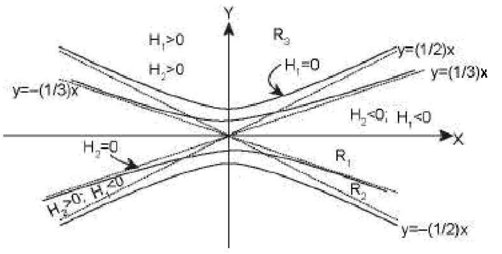
Thus, $H_1 = 0$ and $H_2 = 0$ will be hyperbolas as shown below



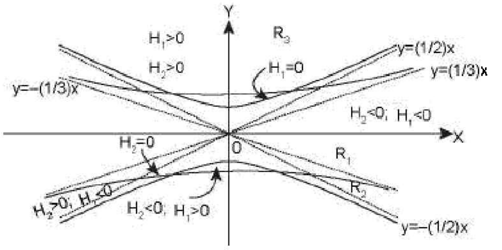
Case (ii): When $\dfrac{1 - a}{1 + a} < 0$, i.e., a $\in (-\infty, -1) \cup (1, \infty)$

In this case (2) will be a hyperbolas of the form $ay^2 - x^2 = \dfrac{(a - 1)}{(a + 1)}$ or $\dfrac{y^2}{\left(\dfrac{1}{3}\sqrt{\dfrac{a - 1}{a + 1}}\right)^2} - \dfrac{x^2}{\left(\sqrt{\dfrac{a - 1}{a + 1}}\right)^2} = 1$ with its asymptotes given by $y = \pm\dfrac{1}{3}x$

Thus, $H_1 = 0$ and $H_2 = 0$ will be hyperbola as shown below.



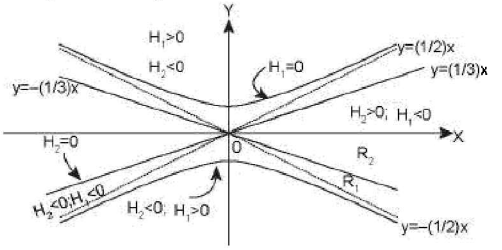
or



Case (iii): a = 1, In this case (2) becomes, H_2: $x^2 - 9y^2$

∴ $H_2 = 0$ is a pair of straight lines given by $y = \pm\frac{1}{3}x$

∴ $H_1 = 0$ and $H_2 = 0$ will be represented as shown below.



19. (d) From above we conclude that , $H_1 \le 0$ and $H_2 \le 0$ is the region R_1 exists for each a ∈ ℝ – {–1}.

20. (a) $H_1 \le 0$ and $H_2 \ge 0$ is represented by the region R_2 exists for each a ∈ ℝ – {–1}.

21. (d) $H_1 \ge 0$ and $H_2 \ge 0$ is the region R_3.

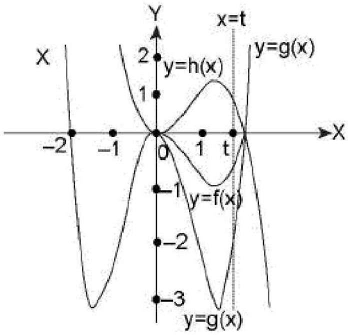
∴ $a = 0$ and $a = 1$ are two integer values for which R_3 does not exist.

Passage G:

$x^4 - 4x^2 \le f(x) \le 2x^2 - x^3\ \forall\ x \in [0, 2]$

Let $g(x) = x^4 - 4x^2 = x^2(x^2 - 4) = x^2(x - 2)(x + 2)$ and $h(x) = 2x^2 - x^3 = x^2(2 - x)$

The graphs of $y = g(x)$ and $y = h(x)$ are as shown below.



Now, $g(x) \le f(x) \le h(x)\ \forall\ x \in [0, 2]$

A.T.Q. $\int_0^t (f(x) - [x^4 - 4x^2]dx = n\int_0^t [(2x^2 - x^3) - f(x)]dx;$ $0 \le t \le 2$

$$\Rightarrow \int_0^t f(x)\,dx - \left[\frac{t^5}{5}-\frac{4t^3}{3}\right] = n\left[\frac{2t^3}{3}-\frac{t^4}{4}\right] - n\int_0^t f(x)dx; 0 \le t \le 2$$

$$\Rightarrow (n+1)\int_0^t f(x)dx = n\left(\frac{2t^3}{3}-\frac{t^4}{4}\right)+\left(\frac{t^5}{5}-\frac{4t^3}{3}\right);\ 0 \le t \le 2$$

$$\Rightarrow \int_0^t f(x)dx = \left(\frac{n}{n+1}\right)\left[\frac{2t^3}{3}-\frac{t^4}{4}\right]+\frac{1}{(n+1)}\left[\frac{t^5}{5}-\frac{4t^3}{3}\right]$$

$$\Rightarrow f(t) = \frac{n}{(n+1)}\left[\frac{2}{3}(3t^2)-\frac{1}{4}(4t^3)\right]+\frac{1}{(n+1)}\left[t^4-4t^2\right]$$

$$\Rightarrow f(t) = \frac{n}{(n+1)}(2t^2-t^3)+\frac{1}{(n+1)}(t^4-4t^2)$$

$$\Rightarrow f' = \frac{n}{(n+1)}\left[4t-3t^2\right]+\frac{1}{(n+1)}(4t^3-8t]$$

22. (c) for $n = 1$; $f'(t)$

$$= 2t^2 - \frac{3t^2}{2} - 2t \Rightarrow f'(x) = 0 \Rightarrow \left(\frac{4t^3-3t^2-4t}{2}\right) = 0$$

⇒ $t(4t^2 - 3t - 4) = 0$

⇒ $t = 0$ or $t = \frac{3\pm\sqrt{73}}{2}$

Now $f''(t) = \frac{1}{2}(12t^2 - 6t - 4) = 6t^2 - 3t - 2$

$f''(0) = -2$

⇒ $t = 0$ is a point local maxima and $\frac{3+\sqrt{73}}{2} > 2$

∴ $f(x)$ has a local minima at $x = t = \frac{3-\sqrt{73}}{2}$

23. (c) $f(x) = \frac{n}{(n+1)}(2x^2-x^3)+\frac{1}{(n+1)}(x^4-4x^2)$

$$\Rightarrow f'(x) = \frac{n}{(n+1)}(4x-3x^2)+\frac{1}{(n+1)}(4x^3-8x)$$

$$\Rightarrow f''(x) = \frac{n}{(n+1)}(4-6x)+\frac{1}{n+1}(12x^2-8)$$

At point of inflexion, $f''(x) = 0$

⇒ $n(4 - 6x) + (12x^2 - 8) = 0$

⇒ $n(2 - 3x) + (6x^2 - 4) = 0$

For $n = 2$, $4 - 6x + 6x^2 - 4 = 0$

⇒ $6x^2 - 6x = 0$

⇒ $x(x - 1) = 0$

⇒ $x = 0$ or $x = 1$

∴ $f(x)$ attains a point of inflexion at $x = 0$ as $x = 1$

24. (b) $\int_{-1}^{1} f(x)dx = \int_{-1}^{1}\left[\frac{n}{n+1}\left(2x^2 - x^3\right) + \frac{1}{(n+1)}\left(x^4 - 4x^2\right)\right]dx$

$= 4\int_0^1 \frac{n}{(n+1)}(x^2)dx + \frac{2}{(n+1)}\int_0^1 x^4 dx - \frac{8}{(n+1)}\int_0^1 x^2 dx$

$= \frac{4n}{(n+1)}\cdot\left[\frac{x^3}{3}\right]_0^1 + \frac{2}{(n+1)}\left[\frac{x^5}{5}\right]_0^1 - \frac{8}{(n+1)}\left[\frac{x^3}{3}\right]_0^1$

$= \frac{4n}{3(n+1)} + \frac{2}{5(n+1)} - \frac{8}{3(n+1)}$

$= \frac{4n}{3(n+1)} + \frac{2}{5(n+1)} - \frac{8}{3(n+1)}$

$= \frac{20n+6-40}{15(n+1)} = \frac{20n-34}{15(n+1)} = \frac{2}{15(n+1)}(10n-17)$

SECTION–VII: COLUMN MATCHING

1. (i) → (d); (ii) →(a); (iii) →(d); (iv) →(b)

(i) $[x] = \cos^{-1} x$ ...(i)

$[x] \in \{0, \pm1, \pm2, \pm3, \ldots\}$ and $\cos^{-1} x \in [0, \pi]$

∴ (i) can holds for $\cos^{-1} x \in \{0, 1, 2, 3\}$

⇒ $[x] \in \{0, 1, 2, 3\}$

⇒ $x \in [0, 4)$

But for $y = \cos^{-1} x$; $x \in [-1, 1]$

∴ $x \in [-1, 1] \cap [0, 4]$

⇒ $x \in [0, 1]$

⇒ $[x] = 0$ or 1

If $[x] = 0$

⇒ $\cos^{-1}x = 0$

⇒ $x = 1$

But it $[x] = 1$ which is false.

If $[x] = 1$ $\cos^{-1} x = \cos 1 \Rightarrow [x] = 0$, which is false.

⇒ (i) has no solution.

(ii) $\sin^{-1} x = \text{sgn}(x)$ ...(i)

$\sin^{-1} x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$; sgn (x) $\{-1, 0, 1\}$

case (i): If sgn $(x) = -1$ ⇒ $\sin^{-1} x = -1$

⇒ $x = -\frac{\pi}{2}$, which satisfy (i)

case (ii): If sgn $(x) = 0$ $\sin^{-1} x = 0$

⇒ $x = 0$, which satisfy (i)

case (iii): If sgn $(x) = 1$

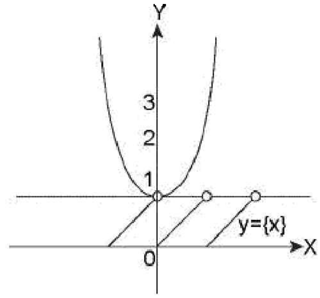
⇒ $\sin^{-1} x = 1$

⇒ $x = \frac{\pi}{2}$, which also satisfy (i)

∴ $x = \frac{-\pi}{2}, 0 \frac{\pi}{2}$, i.e., 3 solutions.

(iii) $\{x\} = e^{x^2}$

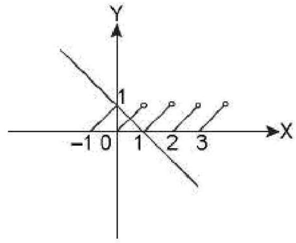
Drawing the graphs of $y = \{x\}$ and $\{y\}$ and $y = e^{x^2}$ on same reference frame as given below.



∴ There is no point of intersection, i.e., no solution.

(iv) $1 - x = \{x\}$

Draw the graph of $y = 1 - x$ and $y = \{x\}$ on same frame of reference as given below.



∴ There are two points of intersection, and hence, two solutions.

2. (i) →(d); (ii) →(c); (iii) →(a); (iv) →(b)

(i) $y = |f(x)| = \begin{cases} 1 \text{ for } -2 \le x \le 0 \\ 1 - x \text{ for } 0 \le x \le 1 \\ x - 1 \text{ for } 1 \le x \le 2 \end{cases}$

∴ **(i) → (s)**

(ii) $y = f(|x|) = \begin{cases} -x - 1 \text{ for } x \le 0 \\ x - 1 \text{ for } x > 0 \end{cases}$

∴ **(b)→(r)**

(iii) $y = f(-|x|) = \begin{cases} f(-x); x \ge 0 \\ f(x); x < 0 \end{cases} = \{-1\ \forall\, x \in \mathbb{R}$

∴ **(iii)→ (p)**

(iv) $y = \frac{1}{2}(|f(x)| - f(x)) = \begin{cases} 0 \text{ for } f(x) \ge 0 \\ -f(x) \text{for } f(x) < 0 \end{cases}$

$= \begin{cases} 1 \text{ for} -2 \le x \le 0 \\ 1 - x \text{ for} 0 < x \le 1 \\ 0 \text{ for } x > 1 \end{cases}$

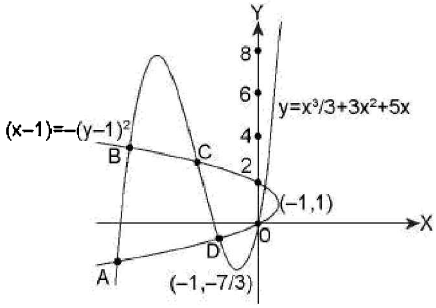
∴ **(iv)→ (b)**

3. (i) → (b); (ii) →(b); (iii) →(b); (iv) →(c);(v) →(d)

(i) $y = \frac{x^3}{3} + 3x^2 + 5x$ ...(i)

$x = 2y - y^2$ ...(ii)

Draw the graphs of $y = \frac{x^3}{3} + 3x^2 + 5x$ and $(x - 1) = -(y - 1)^2$ same frame of reference as shown below.

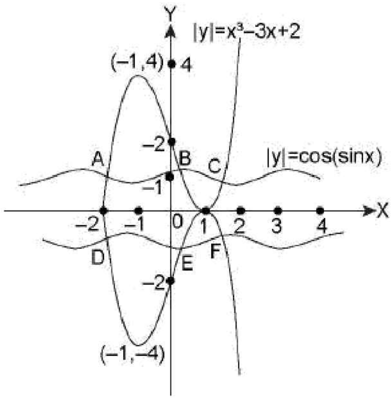


Clearly, the two graphs intersect each other at 6 points A, B, C, D, E and O.

$\therefore$ **(i)→ (b)**

(ii) $|y| = x^3 - 3x + 2$ and $|y| = \cos(\sin x)$.

The graphs of above relation drawn on same frame of reference are as shown below.



Clearly, there will be 6 point of intersection.

$\therefore$ **(ii)→ (b)**

(iii) $y = |\ln|x^2 - x||$ and $y = \left|\left|\dfrac{1}{x}\right| - 2\right|$

$|x^2 - x| \geq 1 \quad \Rightarrow \quad x^2 - x \leq -1$ or $x^2 - x \geq 1$

$\Rightarrow \; x^2 - x + 1 \leq 0$ or $x^2 - x - 1 \geq 0$

Disc. of $x^2 - x + 1 = -3 < 0$

$\Rightarrow \; x^2 - x + 1 > 0 \; \forall x \in \mathbb{R}$ and $x^2 - x - 1 \geq 0$

$$\Rightarrow \; x \in \left(-\infty, \frac{1-\sqrt{5}}{2}\right] \cup \left[\frac{1+\sqrt{5}}{2}, \infty\right)$$

$$\therefore \; y = \ln|x^2 - x| \text{ for } x \in \left(-\infty, \frac{1-\sqrt{5}}{2}, \infty\right] \cup \left[\frac{1+\sqrt{5}}{2}, \infty\right)$$

Similarly, $|x^2 - x| \in (0, 1)$

$\Rightarrow \; -1 < x^2 - x < 1; \; x^2 - x \neq 0$

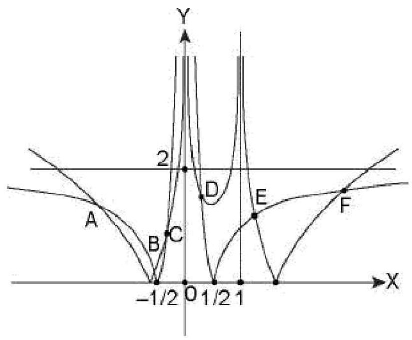
$\Rightarrow \; x^2 - x - 1 < 0$ and $x^2 - x + 1 > 0$

$$\Rightarrow \; x \in \left(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$$

$$\therefore \; y = \begin{cases} \ln|x^2 - x|; x \in \left(-\infty, \dfrac{1-\sqrt{5}}{2}\right] \cup \left[\dfrac{1+\sqrt{5}}{2}, \infty\right) \\ -\ln|x^2 - x|; x \in \left(\dfrac{1-\sqrt{5}}{2}, \dfrac{1+\sqrt{1+5}}{2}\right) \end{cases}$$

$$\Rightarrow \; y = \begin{cases} \ln\left(x^2 - x\right); x \in \left(-\infty, \dfrac{1-\sqrt{5}}{2}\right] \cup \left[\dfrac{1+\sqrt{5}}{2}, \infty\right) \\ -\ln\left(x^2 - x\right); x \in \left(\dfrac{1-\sqrt{5}}{2}, 0\right] \cup \left[1, \dfrac{1+\sqrt{5}}{2}\right) \\ -\ln(x - x^2)\,; x \in (0,1) \end{cases}$$

The graph of $y = |\ln|x^2 - x||$ and $y = \left|\left|\dfrac{1}{x}\right| - 2\right|$ will be as shown below.



Clearly, the two graphs intersect each at 6 points, i.e., A, B, C, D, E and F.

$\therefore$ (iii) → (b)

(iv) $y = \cos e^x$ and $y = \tan e^x \; \forall \; x \in \left(-\infty, \dfrac{\pi}{2}\right)$

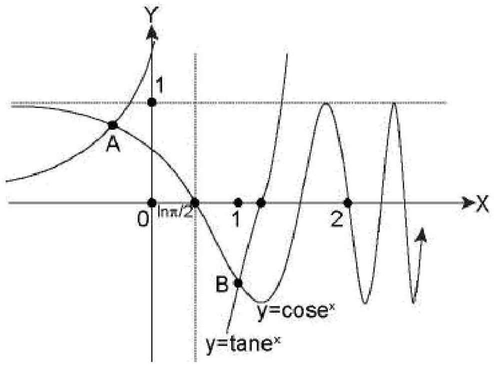
For $x \in \left(-\infty, \dfrac{\pi}{2}\right)$, $e^x \in \left(e^{-\infty}, e^{\frac{\pi}{2}}\right)$, i.e.,

$$\left(0, e^{\frac{\pi}{2}}\right) = \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right] \cup \left(\pi, e^{\frac{\pi}{2}}\right)$$

$$\Rightarrow \; \cos\; e^x = \begin{cases} \text{decrease from 1 to 0 for } e^x \in \left(0, \dfrac{\pi}{2}\right] \\ \text{decrease from 0 to} -1 \text{ for } e^x \in \left(\dfrac{\pi}{2}, \pi\right] \\ \text{increase from } -1, \cos\left(e^{\frac{\pi}{2}}\right) \text{ for } e^x \in \left(\pi, e^{\frac{\pi}{2}}\right) \end{cases}$$

$$\text{and } \tan e^x = \begin{cases} \text{increase from } 0 \text{ to } \infty \text{ for } e^x \in \left(0, \dfrac{\pi}{2}\right) \\ \text{increase from } -\infty \text{ to 0 for } e^x \in \left(\dfrac{\pi}{2}, \pi\right] \\ \text{increase from } 0 \text{ to tan for } e^x \in \left(\pi, e^{\frac{\pi}{2}}\right) \end{cases}$$

Thus, the graphs of $y = \cos e^x$ and $y = \tan e^x \; \forall \; x \in \left(-\infty, \dfrac{\pi}{2}\right)$ will be as shown below.



Clearly, the two functions intersect each other only at two points and B in $\left(-\infty, \frac{\pi}{2}\right)$ as shown above.

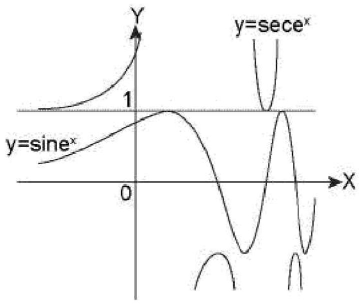
$\therefore$ There will be only 2 solutions.

$\therefore$ (iv) $\rightarrow$ (c)

(e) $y = \sin e^x$ and $y = \sec e^x$ $\forall\, x \in (-\infty, 2)$

For $x \in (-\infty, 2)$, $e^x \in (0, e^x) \subset \left(0, \frac{5\pi}{2}\right)$ and $(0, e^2) \supset (0, 2\pi)$

$y = \sin e^x$ and $y = \sec e^x$ drawn as shown below.



Clearly, there is no point of intersection.

$\therefore$ **(v)→ (d)**

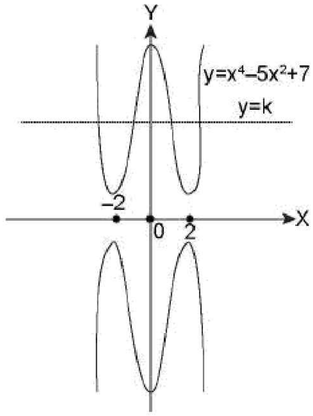
4. (i) →(d); (ii) →(c); (iii) →(a); (iv) →(b); (v) →(e)

$f(x) = x^4 - 5x^2 + 7$

$|y| = x^4 - 5x^2 + 7$; Clearly, Disc. of $(x^4 - 5x^2 + 7) < 0$

$\Rightarrow$ $x^4 - 5x^2 + 7 > 0$ $\forall\, x \in \mathbb{R}$

The graph of $y = x^4 - 5x^2 + 7$ and $|y| = x^4 - 5x^2 + 7$ drawn on same reference frame are as shown below.



(i) Clearly for $n = 0$, i.e., no point of intersection $k \in \left(\frac{-3}{4}, \frac{3}{4}\right)$ $\therefore$ **(i)→ (d)**

(ii) Clearly for $n = 1$, i.e., only one point of intersection is impossible $\therefore$ **(ii)→ (c)**

(iii) For $n = 2$, i.e., two point of intersection, $k \in (-\infty, -7) \cup (7, \infty) \cup \left\{\pm\frac{3}{4}\right\}$ $\therefore$ **(iii)→ (a)**

(iv) For $n = 3$, i.e., three points of intersection, $k \in \{-7, 7\}$ $\therefore$ **(iv)→ (b)**

(v) For $n = 4$, i.e., four points of intersection, $k \in \left(-7, \frac{-3}{4}\right) \cup \left(\frac{3}{4}, 7\right)$ $\therefore$ **(v)→ (e)**

SECTION–VIII: INTEGER TYPE

1. since $y = f(x)$ has 4 points of non-differentiability x_1, x_2, x_3, x_4 (say)

$y = f(|x|)$ being reflection of $y = f(x)$ on y-axis along with $y = f(x)$ has $-x_1, -x_2, -x_3, -x_4$ points of non-differentiability as well .

Also $f'(0^+) = k(\neq 0)$ $\Rightarrow$ $f(0^-) = -k(\neq 0)$

$\Rightarrow$ $x = 0$ is also a point of non-differentiability, hence, total 9 points of differentiability are there.

2. Since $y = f(x - 2)$ $\Rightarrow$ $2 \le x - 2 \le 10$

$\Rightarrow$ $4 \le x \le 12$

$\Rightarrow$ $y = g(x)$ would be having points of non-differentiability at $x = 5, 6, 7, 8$ and 9, i.e., 5 points in [2, 10), i.e., 5 points.

3. Clearly $x_1 = 2$, $x_2 = 6$, $y_1 = 4$, $y_2 = 10$, $z_1 = 2.5$, $z_2 = 7.2$, $z_3 = 7.8$, $z_4 = 10.5$ and $u_1 = 6$

$\therefore$ Required sum $= (2 + 6) + (4 + 10) - (2.5 + 7.2 + 7.8 + 10.5) + 6 = (22 + 6) - (28) = 0$

4. $y = f(|x|)$ is obtained by reflecting the graph of $y = f(x)$ on y-axis and $y = |f(|x|)$ will be obtained by reflecting the negative portion of $y = f(|x|)$ on x-axis and finally $|y| = |f(|x|)\,|$ is obtained by reflecting the $+ve$ portion of $y = |f(|x|)|$ on x-axis. Thus, the required area = 4(Area bounded by $y = f(x)$ with x-axis for $x \in [0, 10]$).

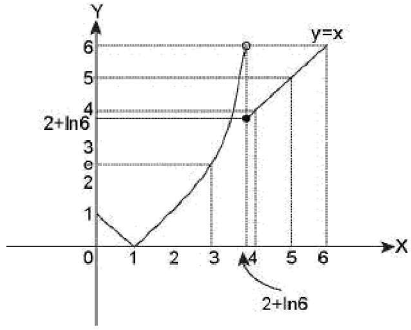
$$= 4\left[\frac{1}{2}(1)(1) + (2)(2) + \frac{1}{2}(2+1)(2) + 1\times\left(\frac{1}{2}\right) + 3\times\frac{1}{2} + \frac{1}{2}\pi\left(\frac{3}{2}\right)^2 + \frac{1}{2}\times\left(1\times\frac{1}{2}\right)\right]$$

$$= 4\left[\frac{1}{2} + 4 + 3 + \frac{1}{2} + \frac{3}{2} + \frac{9\pi}{8} + \frac{1}{4}\right] = 4\left[9.75 + \frac{9\pi}{8}\right] = 39 + \frac{9\pi}{2}$$

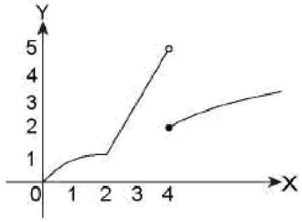
$= a + \frac{b\pi}{c}$ $a = 39$, $b = 9$, $c = 2$

$\Rightarrow$ $a + c - 4b = 41 - 36 = 5$

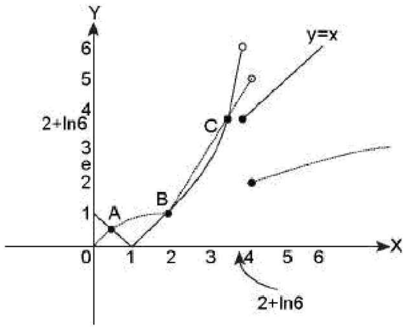
5. Graph of $y = f(x)$ is as shown below.



The graph of $y = g(x)$ is as shown below.



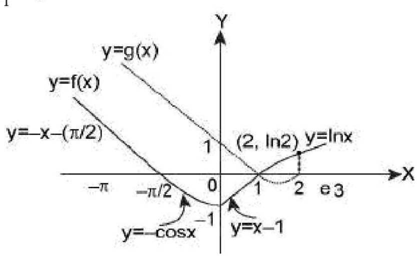
Let's draw the graph of $y = f(x)$ and $y = g(x)$ on same co- ordinate plane as shown below.



Clearly the two graphs intersect each other at 3 points A, B and C.

6. Clearly, $f(x) = g(x) + ln(2)$ has 3 points of intersection (by vertical transformation)

$\Rightarrow$ $k_1 = 3$



$x = 0$ is the only point of non-differentiability of $y = f(x)$

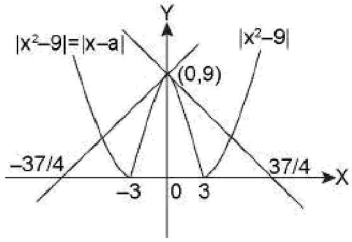
$\Rightarrow$ $k_2 = 1$ and $x = 2$ is the only point of non-differentiability of $y = g(x)$

$\Rightarrow$ $k_3 = 1$

$\therefore$ $\dfrac{k_1 + k_2}{k_3} = \dfrac{3+1}{1} = 4$

7. For tangency, $x^2 - 9 = x - a$

$\Rightarrow$ $x^2 - x + a - 9 = 0$



Put $D = 0$

$\Rightarrow$ $1 - 4a + 36 = 0$

$\Rightarrow$ $a = \dfrac{37}{4}$

Similarly, $a = \dfrac{-37}{4}$

$\therefore$ For 4 distinct solution,

$a \in \left(\dfrac{-37}{4}, -3\right) \cup (-3, 3) \cup \left(3, \dfrac{37}{4}\right)$

Hence, number of integers are 17.

8. Since f is injective, so, $\dfrac{p}{r} = \dfrac{p-q}{q-r}$

$\Rightarrow$ $pq - pr = rp - rq$

$\Rightarrow$ $2pr = q(p + r)$ ......(1)

Also, $\dfrac{q}{r} = \dfrac{r}{p}$

$\Rightarrow$ p, r, q are in G.P.

So, let $r = pa$, $q = pa^2$, where a is the common ratio of G.P. Therefore, from equation (1), we get $2 \cdot p \cdot pa = pa^2 (p + pa)$

$\Rightarrow$ $2 = a^2 + a$ $\Rightarrow$ $a^2 + a - 2 = 0$

$\Rightarrow$ $(a + 2)(a - 1) = 0$ $\Rightarrow$ $a = -2, 1$

So, $(p, -2p, 4p)$ and (p, p, p)

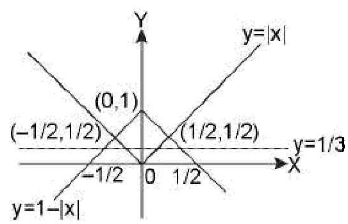
(But common ratio $= a = 1$ is not possible as p, q, r are non-zero distinct quantities.)

Also, $p + q + r = 6$ [As $g(x) = px^2 + qx + r$ passes through $M(1, 6)$]

$\Rightarrow$ $p + 4p - 2p = 6$ $\Rightarrow$ $p = 2$

Hence, $q = 4p = 4(2) = 8$

9. From the graph given below, clearly $R_f = \left(-\infty, -\dfrac{1}{2}\right]$



$\therefore$ $3f(x) - 1 = 0$ $\Rightarrow$ $f(x) = 1/3$

$\Rightarrow$ $y = f(x)$ and $y = 1/3$ will intersect at 4 points.

10. $y = ax + b$ is inverse of itself. $\Rightarrow a = \pm 1$

For $a = -1$, $y = -x + b$ is its own inverse $\forall\ b \in \mathbb{R}$

For $a = 1$, $y = x + b$ will be inverse of itself for $b = 0$, i.e., $y = x$

Given that it is passing through $(2, 3)$

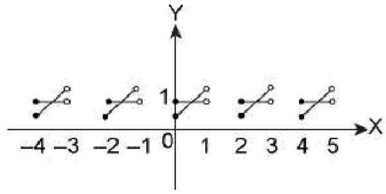
$\Rightarrow\ 3 = -2 + b$

$\Rightarrow\ b = 5$ and $a = -1$, i.e., $y = -x + 5$

$\Rightarrow\ 2a + b = -2 + 5 = 3$

11. Clearly domain of $f(x) = \bigcup\limits_{n \in \mathbb{Z}} [2n, 2n + 1)$

$\Rightarrow\ f(x) = 1,\ \forall\ x \in D_f$



Now, $2f(x) = 1 + 2\{x\} \Rightarrow f(x) = \frac{1}{2} + \{x\}$

Clearly, the two graphs $y = f(x)$ and $y = \frac{1}{2} + \{x\}$ intersect at 5 points. Thus, there will be 5 solutions.

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