Straight Lines and Curves

N. Vasilyev
V. Gutenmacher

Mir Publishers
Moscow

ИЗДАТЕЛЬСТВО · МИР ·

Н. Б. Васильев
В. Л. Гутенмахер

Прямые и кривые

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N. B. Vasilyev
V. L. Gutenmacher

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На английском языке

Contents

Preface

The main characters of this book are various geometric figures or, as they are frequently called here, "sets of points". The simplest figures in their different combinations appear first. They move, reveal new properties, intersect, combine, form entire families and change their appearance, sometimes to such an extent that they become unrecognizable. However, it is interesting to see old acquaintances in unusual situations surrounded by the new figures which appear at the end.

The book consists of approximately two hundred problems, most of them given with solutions or comments. There is a whole variety of problems, ranging from traditional problems in which one has to find and make use of some set of points, to simple investigations touching important mathematical concepts and theories (for instance "the cheese", "motor-boat", "bus" problems). Apart from ordinary geometric theorems on straight lines, circles and triangles, the book makes use of the method of coordinates, vectors and geometric transformations, and especially often the language of motion. A list of useful geometric facts and formulas is given in Appendices I and II. Some of the tedious finer points in the logic of the solutions are left to the reader. The symbol ⟨?⟩ replaces the words "Exercise", "Verify", "Is it clear to you?", "Think, why", etc., depending upon where it is. The beginning and the end of solutions are marked with the symbol □ while ↓ means that the solution or the answer to the problem is given at the end of the book. The problems at the beginning of each section are not usually difficult or else are analysed in detail in the book. The rest of the problems do not have to be solved in succession. One can, while reading the book, choose those which seem more attractive. It is useful to verify much of what is discussed in the problems through experiment: it is best to draw a diagram or—even better—several, with the figures in different positions. This experimental approach not only helps one to guess the answer and formulate a hypothesis but also often leads one to a mathematical proof. In drawing the diagrams in the margins the authors were convinced that almost behind every problem there is hidden an auxiliary problem of constructing the points or lines which are stated in the problem. The preliminary problem often appears to be more simple but it is no less interesting than the problem itself!

The authors are deeply thankful to I. M. Gelfand whose advice helped the entire work on the book, to I. M Yaglom, V. G. Boltyansky and J. M. Rabbot, who read the manuscript, for their significant remarks. Since the publication of the first edition (1970) of this book, it has been used in the work of the Moscow University Correspondence mathematics school. The experience which the teachers of this school shared with us and also the experience of our friends and colleagues has been taken into consideration in the detailed revision undertaken for the second edition.

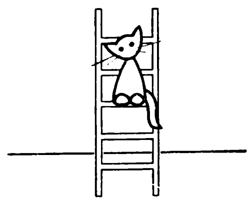
We thought it necessary to furnish the book with an additional appendix, Appendix III. This will assist in systematic study of the book, and will help to reveal relationships between different sections of the book which are not immediately apparent.

N. B. Vasilyev, V. L. Gutenmacher

Introduction

Introductory Problems

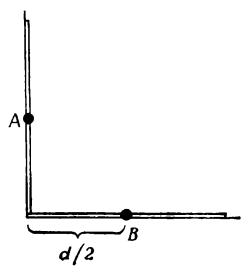
0.1. A ladder standing on a smooth floor against a wall slides down. Along what line does a cat sitting at the middle of the ladder move?



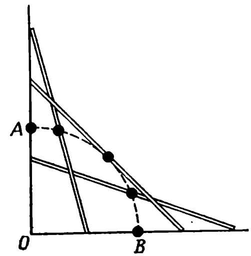
Let us suppose our cat is calm and sits quietly. Then, we can see behind this picturesque formulation the following mathematical problem.

A right angle is given. Find the midpoints of all the possible segments of given length d, which have their end-points lying on the sides of the given angle.

Let us try to guess what sort of a set this is. Obviously, when the segment rotates with its end-points sliding along the sides of the angle, its centre describes a certain line. (This is obvious from the first picturesque statement of the problem.) First of all, let us determine where the end-points of this line lie. They correspond to the extreme positions of the segment when it is vertical or



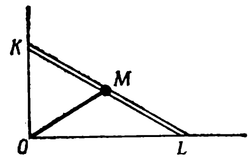
horizontal. This means that the end-points A and B of the line lie on the sides of the angle at a distance $d/2$ from its vertex.



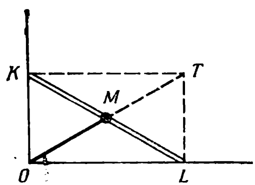
Let us plot a few intermediate points of this line. If you do this accurately enough, you will see that all of them lie at the same distance from the vertex O of the given angle. Thus, we can say that

The unknown line is an arc of a circle of radius $d/2$ with centre at O.

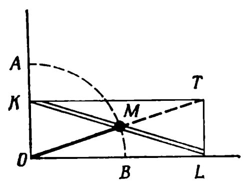
Now we must prove this.



□ We shall first prove that the midpoint M of the given segment KL ($|KL| = d$) always lies at a distance $d/2$ from the point O. This follows from the fact that the length of the median OM of the right-angled triangle KOL is equal to half the length of the hypotenuse KL. (One can easily convince oneself of the validity of this fact by extending the triangle KOL up to the rectangle $KOLT$ and recalling that the diagonals KL and OT of the rectangle are equal in length and are bisected by the point of intersection M.)



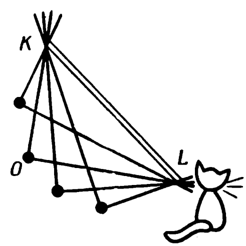
Thus, we have proved that the midpoint of the segment KL always lies on the arc $\overset{\frown}{AB}$ of a circle with centre O. This arc is the set of points we were looking for.



Strictly speaking, we have to prove also that an arbitrary point M of the arc $\widehat{AB}$ belongs to the unknown set. It is easy to do this. Through any point M of the arc $\widehat{AB}$ we may draw a ray OM, mark off the segment $|MT| = |OM|$ along it, drop perpendiculars TL and TK from the point T to the sides of the angle and the required segment KL with its midpoint at M is constructed. □

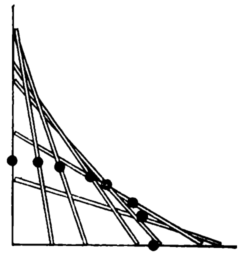
The second half of the proof might appear to be unnecessary: It is quite clear that the midpoint of the segment KL describes a "continuous line" with end-points A and B; it means that the point M passes through the whole of the arc $\widehat{AB}$ and not just through parts of it. This analysis is perfectly convincing, but it is not easy to give it a strict mathematical form.

Let us now consider the motion of the ladder (from problem **0.1**) from another point of view. Suppose that the segment KL (the "ladder") is fixed and the straight lines KO and LO ("the wall" and "the floor") rotate correspondingly about the points K and L so that the angle between them is always a right angle. The fact that the distance from the centre of the segment to the vertex O of the right angle always remains the same, reduces to a well-known theorem: *if two points K and L are given in a plane, then the set of points O for which the*

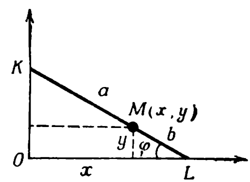


angle $\widehat{KOL}$ *equals* 90° *is a circle with diameter KL.* This theorem and also its generalization, which will be given in the proposition **E** of Sec. 2, will frequently help us in the solution of problems. Let us return to problem **0.1** and put a more general question.

0.2. Along what line does the cat move if it does not sit at the middle of the ladder?



In the figure a few points on one such line are plotted. It can be seen that it is neither a straight line nor a circle, i.e. it is a new curve for us. The coordinate method will help us to find out what sort of curve it is.



□ We introduce a coordinate system regarding the sides of the angle as the axes Ox and Oy. Suppose the cat sits at some point $M(x; y)$ at a distance a from the end-point K of the ladder and at a distance b from L ($a + b = d$). We shall find the equation connecting the x and y coordinates of the point M.

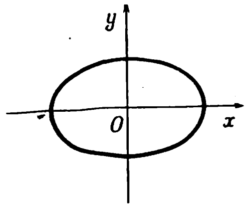
If the segment KL is inclined to the axis Ox at an angle φ, then $y = b \sin \varphi$ and $x = a \cos \varphi$; hence, for any arbitrary φ $\left(0 \leqslant \varphi \leqslant \frac{\pi}{2}\right)$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{1}$$

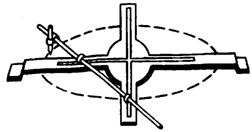
The set of points whose coordinates satisfy this equation is an *ellipse*.

Hence, the cat will move along an ellipse. □

Note that when $a = b = d/2$, then if the cat sits as above at the middle of the ladder, and equation (1) becomes the equation of a circle $x^2 + y^2 = (d/2)^2$. Thus, we get one more solution of problem **0.1**, an analytical solution.

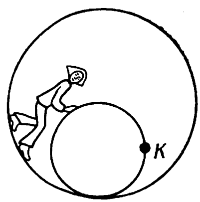


The result of problem **0.2** explains the construction of a mechanism for drawing ellipses. This mechanism shown in the figure is called *Leonardo da Vinci's ellipsograph.*

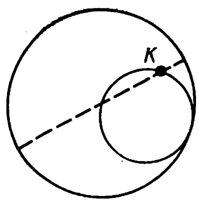


Copernicus' Theorem

0.3. Inside a stationary circle, another circle whose diameter is half the diameter of the first circle and which touches it from inside rolls without sliding. What line does the point K of the moving circle describe?

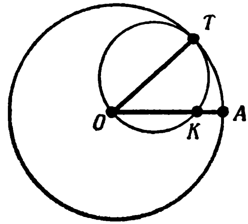


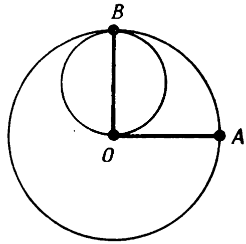
The answer to the problem is astonishingly simple: the point K moves along a *straight* line—more correctly along the diameter of the stationary circle. This result is called *Copernicus' theorem*.



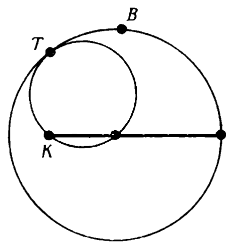
Try to convince yourself of the validity of this theorem by experiment. (It is important here that the inner circle rolls without sliding, i.e. the lengths of the arcs rolling against each other are equal). It is not difficult to prove, we need only to recall the theorem on the inscribed angle.

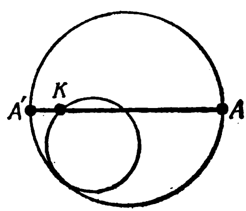
☐ Suppose that the point of the moving circle, which occupies position A on the stationary circle at the initial instant, has come to the position K, and that T is the point of contact of the circle at the present moment of time. Since the lengths of the arcs $\overset{\frown}{KT}$ and $\overset{\frown}{AT}$ are equal and the radius of the movable circle is half as large, the angular size of the arc $\overset{\frown}{KT}$ in degrees is double that of the arc $\overset{\frown}{AT}$. Therefore, if O is the centre of the stationary circle, then according to the *theorem on the inscribed angle* (see p. 24), $\widehat{AOT} = \widehat{KOT}$. Hence, the point K lies on the radius AO.





This argument holds until the moment when the moving circle has rolled around one quarter of the bigger circle (the circles then touch at the point B of the bigger circle, for which $\widehat{BOA} = 90°$ and K coincides with O). After this, the motion will be continued in exactly the same way—the whole picture will be simply reflected symmetrically about the straight line BO and then, after the point K reaches the opposite end A' of the diameter AA', the circle will roll along the lower half of the stationary

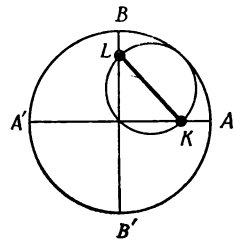


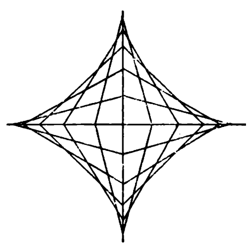


circle and the point K will return to A. □

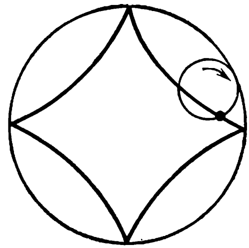
Let us compare the results of problems **0.1** and **0.3.** They are attractive probably for the following reason. Both problems deal with the motion of figures (the first with the motion of a segment, the second with the motion of a circle). The motion is quite complicated, but the paths of certain points appear to be unexpectedly simple. These two problems turn out to be not only related in appearance, but the motions themselves, discussed in the problems also coincide with each other.

Indeed, suppose a circle of radius $d/2$ rolls along the inside of another circle of radius d, and suppose KL is a diameter of the moving circle rigidly fixed to it. According to Copernicus' theorem the points K and L move along stationary straight lines (along the diameters AA' and BB' of the bigger circle, respectively). Thus, the diameter KL slides with its endpoints along two mutually perpendicular straight lines, i.e. it moves just like the segment in the problem **0.1.**





One more interesting problem connected with the motion of the segment KL: what set of points is covered by this segment, or what is the union of all the possible positions of the segment KL during its motion? The curve bounding this set is called the *astroid*. It is possible to construct this curve in the following way: make a circle of diameter $d/2$ roll inside another circle of diameter $2d$ and draw the trajectory of any particular point of the rolling circle. This trajectory will be the astroid. We shall discuss this curve and its close relatives in Sec. 7 of our book where the reader will make a more detailed acquaintance with the interconnection between the problems which we have discussed.



However, before discussing such intricate problems and curves, let us pay thorough attention to the problems dealing with straight lines and circles. Other types of lines will not appear in the first five paragraphs.

1 Set of Points

In this paragraph we shall discuss and illustrate with a number of examples the basic statements of the problems which the book deals with and also provide an arsenal of concepts and methods used for solving them. The paragraph ends with a set of various geometric problems.

We shall first discuss the term which is most often used in the book and which is at the head of the paragraph.

The concept of a "*set of points*" is very general. A set of points could be any figure, one point or several, a line or a domain in a plane.

In many of the problems of our book, it is required to find a set of points which satisfy a certain condition. Answers to such problems are, as a rule, figures known from school geometry (straight lines, circles, sometimes pieces into which these lines divide a plane, etc.). The main task is to guess what sort of a figure the answer is. Thus, in problem **0.1** about the cat, we have guessed the answer—it was a circle, and in problem **0.3** the answer turned out to be a straight line.

In solutions of some problems we have to carry out a thorough investigation. One has to establish the following:

(a) *all the points satisfying the given condition belong to the figure*;

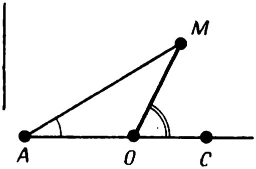
(b) *all the points of the figure satisfy the given condition.*

Sometimes both of these statements are obvious, the direct statement as well as its converse, sometimes only one of them. Sometimes it is even difficult to guess the answer.

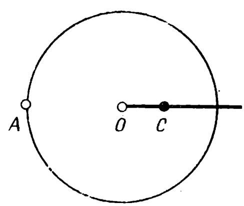
Let us investigate a few typical problems.

1.1. A point O lies on a segment AC. Find the set of points M for which

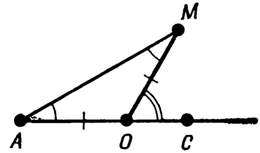
$$\widehat{MOC} = 2\widehat{MAC}.$$



□ *Answer*: The union of the circle with centre O and radius $|AO|$ (omitting the point A) and the ray OC (omitting the point O).



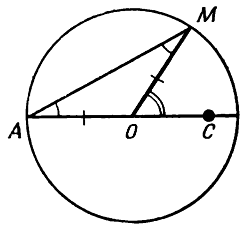
Let us establish this. Suppose, the point M of the unknown set does not belong to the straight line AO. We shall prove that the distance $|MO|$ from the point M to the point O is equal to $|AO|$. Let us construct the triangle OAM. According to the theorem on the exterior angle of a triangle, the angle MOC is equal in magnitude to the sum of the two interior angles not adjacent to it at A and M.



$$\widehat{OAM}+\widehat{AMO}=\widehat{MOC}=2\widehat{MAO}.$$

From the condition of the problem, it follows immediately that $\widehat{OAM} = \widehat{AMO}$. Hence, AMO is an isosceles triangle, i.e. $|OM| = |AO|$.

We shall now prove the validity of the converse statement: any point M of the circle described in the answer satisfies the condition.

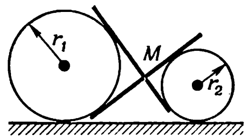


The triangle AMO is indeed isosceles, the values of its angles A and M are equal, and by the same theorem concerning the exterior angle, $\widehat{MOC} = 2\widehat{MAC}$.

Suppose now the point M belongs to the ray OC, $M \neq O$. Then, $\widehat{MOC} = 2\widehat{MAC} = 0$, and the condition is satisfied.

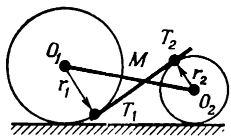
The remaining points of the straight line AO do not belong to the unknown set. For them one of the angles MOC and MAC is a straight angle while the other is zero (about the point O one can say nothing). □

1.2. Two wheels of radii r_1 and r_2 ($r_1 > r_2$) roll along a straight line l. Find the set of points of intersection M of their interior common tangents (see the figure).

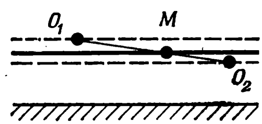


□ *Answer*: A straight line, parallel to l.

Note that the point M lies on the axis of symmetry of the two circles, i.e. on the straight line O_1O_2, where O_1 and O_2 are the centres of the circles. Therefore, we can look for the set of points of intersection of the straight line O_1O_2 with one of the tangents T_1T_2.



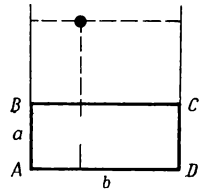
Let us consider an arbitrary composition of two circles and let us draw their radii O_1T_1 and O_2T_2 to the points of tangency. We see that the point M divides the segment O_1O_2 in the ratio r_1/r_2 (the right-angled triangles MO_1T_1 and MO_2T_2 are similar). It is clear that the set of centres O_1 and the set of centres O_2 are straight lines, parallel to the straight line l. The set of points M which divide the segments O_1O_2, with end-points on these straight lines, in the fixed proportion r_1/r_2, is itself a straight line parallel to l.



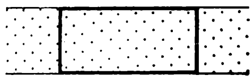
Thus, the set of points of intersection of the tangents is a straight line parallel to the line l and placed at a distance $2r_1r_2/(r_1 + r_2)$ from this line ⟨?⟩. □

The next problem demands a more thorough investigation. We have to divide the plane into several parts and carry out a separate argument in each of them.

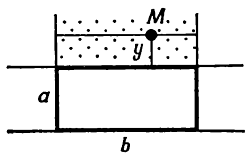
1.3. Given a rectangle $ABCD$. Find all points in the plane such that the sum of the distances from each point to two straight lines AB and CD is equal to the sum of the distances to the straight lines BC and AD.



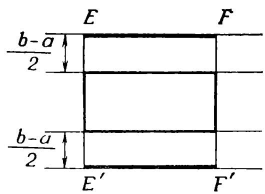
□ Let us denote the lengths of the sides of the rectangle by a and b. We consider first a rectangle which is not a square: let $a < b$.



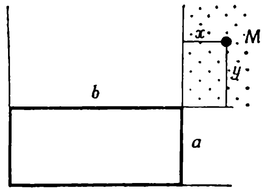
The points lying inside the rectangle and also between the extensions of its larger sides do not satisfy the requirements of the problem, since one sum of the distances is equal to a and the other is not less than b.



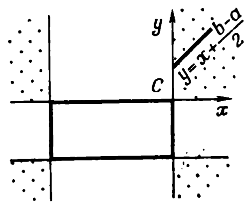
Let the point M now lie between the extensions of the smaller sides of the rectangle. Let us denote by y its distance from the nearest of the larger sides of the rectangle. Then its distance from the opposite side is equal to $y + a$. For the point to satisfy the requirement of the problem, the equality $y + (y + a) = b$ must hold, from which it follows that $y = (b - a)/2$. Therefore, among the points located between the extensions of the smaller sides of the rectangle those and only those which lie at a distance $(b - a)/2$ from the closer larger sides of the rectangle satisfy the condition. The set of points in this domain is the



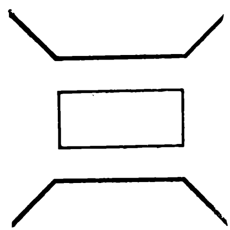
union of two segments EF and $E'F'$.



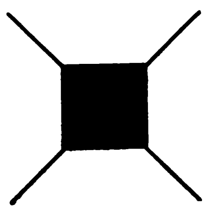
Finally, we shall consider an arbitrary point M lying in the angle between the extensions of the two neighbouring sides BC and DC of the rectangle. Let us denote by x and y the distances from the point M to the straight lines CD and BC, respectively. Then one can express the requirement of the problem as $x + (x + b) = y + (y + a)$ or $y = x + (b - a)/2$.



Note that the numbers x and y can be regarded as coordinates of the point M in the coordinate system with the axes Cx and Cy. In this coordinate system the equation $y = x + (b - a)/2$ defines a straight line parallel to the bisector of the angle xCy. Thus we have proved that among the points lying in the angle under consideration, those and only those which lie on the straight line $y = x + (b - a)/2$ satisfy the requirement of the problem.



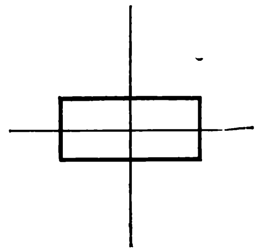
We can use the same argument for the remaining three angles. We have thus analysed all the points of the plane. The set of all the points which satisfy the stated requirement is plotted in the Figure.

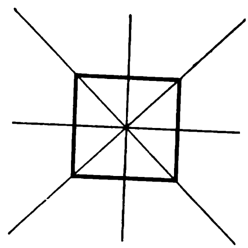


We also have to consider the case when the rectangle is a *square*, i.e. when $a = b$, and to determine what set the required set of points reduces to.

It can easily be seen that it will be the union of the square and the extensions of its diagonals ⟨?⟩. □

Note that *since a rectangle has two axes of symmetry and the pairs of its symmetrical sides in the given conditions are totally identical, the unknown set of points must also have those two axes of symmetry*. Therefore, in the solution it was sufficient to consider only any one of the quadrants into which the plane is divided by these axes, and not the whole plane.





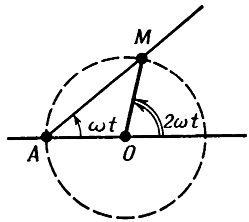
In the case of a square all four axes of symmetry of the square are also axes of symmetry of the set we are looking for.

A Family of Lines and Motion: Together with sets of *points* we shall also consider sets of lines or, as they are frequently called, *families of lines.*

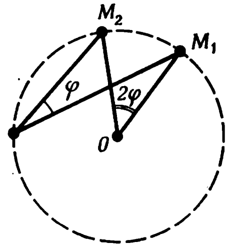
In a geometrical problem when we have to deal with a family of circles or straight lines, *it is convenient to imagine the family as a moving circle or a straight line*. We have already formulated and solved our first problems in the language of motion, and we shall use this language repeatedly in what follows, since many problems and theorems can be explained more vividly using it..

We don't have to look far for an example. Let us return to problem **1.1.** The result we found there can be given as follows:

Suppose the straight line AM rotates about the point A with constant angular velocity ω (i.e., it turns through an angle of magnitude ω in unit time) and the straight line OM rotates about the point O with angular velocity 2ω; at the initial point of time both lines coincide with the straight line AO). Then the point of intersection M of straight lines moves along a circle with centre O.

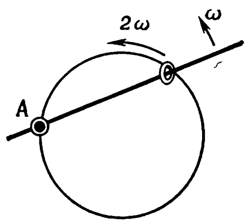


From this we can obtain a well-known *theorem on the inscribed angle.* If in time t the straight line AM rotates from the position AM_1 to the position AM_2 through an angle ωt, then the straight line OM rotates through an angle $2\omega t$ or, in other words, the *magnitude of the inscribed angle* M_1AM_2 *is half the magnitude of the corresponding central angle* M_1OM_2.



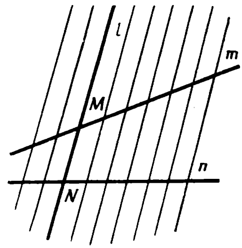
One can formulate this theorem more vividly as follows.

A Theorem About the Tiny Ring on a Circle. A small ring is put on a wire circle. A rod which passes through this ring rotates around the point A of the circle. *If the rod rotates uniformly with an angular velocity* ω, *the ring in this case moves uniformly around the circle with an angular velocity* 2ω.

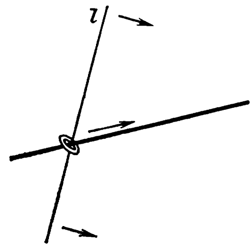


Let us give one more example of a theorem which may be formulated in the language of motion.

Suppose the straight line l describes a *uniform translation* in a plane, i.e. it moves in such a way that its direction remains unchanged and its point of intersection M with a certain stationary straight line m moves uniformly along the line m. Then, *the point of intersection N of the line l with any other stationary straight line n also moves uniformly.*[1] This is, in fact, a reformulation of the geometrical theorem which states that *parallel straight lines cut off proportional segments on the sides of an angle.* To make an analogy with the theorem about the ring on a circle, we can express this in the following way.

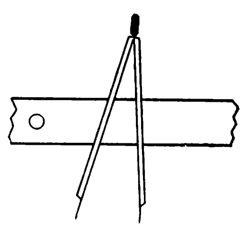


A Theorem About the Tiny Ring on a Straight Line. A small ring is placed at the point of intersection of two straight lines. *If one of the lines is fixed and the other describes a uniform translation* (parallel to itself), *the ring also moves uniformly.* We shall encounter various families of straight lines later on.



When one has to deal with a family of straight lines passing through a single point or parallel to a fixed direction, one or the other of these theorems about tiny rings may be useful.

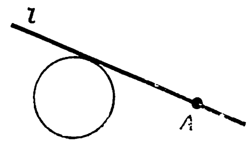
Construction Problems. In classical construction problems (how to "construct a triangle", "mark off a segment", "draw a secant", "find a point", etc.), it is usually meant that the construction should be done with "ruler and compasses" only. In other words, we can draw a straight line through any two points, draw a circle of a given



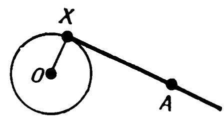
radius and similarly find points of intersection of lines constructed.

For the solution of such problems, *it is convenient to consider circles and straight lines as sets of points satisfying a certain condition.*

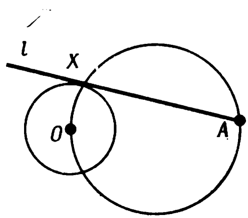
1.4. A circle is given with a point A outside it. Draw a straight line l through the point A touching the circle.



□ If X is the point where the straight line l touches the circle, then the angle OXA is rectangular. The set of points M for which the angle OMA is a right angle is, as we know, a circle with the diameter OA.

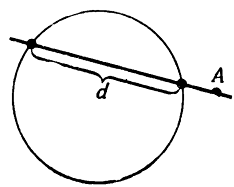


Thus, one can carry out the construction of the straight line l as follows. Draw a circle with the segment OA as diameter.

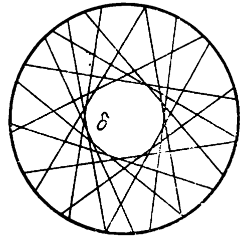


Find a point of intersection X of this circle with the given one (there are two such points, they are symmetrical relative to the straight line OA). Finally, draw a straight line l through the points A and X. □

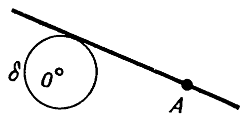
1.5. A point A and a circle are given. Draw a straight line through the point A so that the chord cut off by the circle along this straight line has a given length d.



□ Let us look at the set of all straight lines on which the circle marks off a chord of the given length d. These straight lines are tangents to a certain circle δ whose centre coincides with the centre O of the given circle and whose radius is equal to $\sqrt{r^2 - d^2/4}$, where r is the radius of the given circle ⟨?⟩. The problem thus reduces to the previous one: draw through the point A a tangent to the circle δ with centre O.



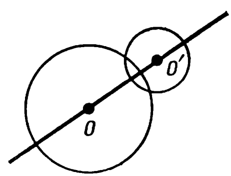
The problem has two solutions if the point A lies outside the circle δ, a unique solution, if it lies on the circle δ and no solution at all, if it lies inside the circle δ. □



Often, it is possible to find the unknown set from the known one with the help of some simple transformations such as a *rotation*, *symmetry*, *parallel displacement* (or translation) or *similarity transformation*. (This method is especially useful in construction problems.) Let us recall how we construct the image of a straight line or a circle under a translation or a similarity transformation.

For the straight line it is sufficient to plot two points A' and B', the images of two points A and B on the line, and to draw a straight line through the points A' and B'.

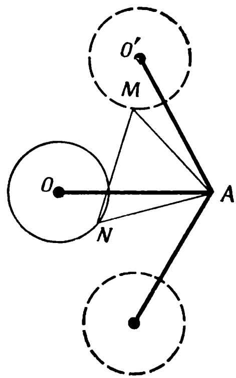
For a circle of radius r, it is sufficient to plot the point O', the image of its centre O, and to draw a circle with centre O' and the same radius (if the transformation is a translation) or of radius kr (if k is the ratio of



magnification of a similarity transformation). We shall give some typical examples of problems where transformations are used.

1.6. A point A and a circle are given. Find a set of vertices M of the equilateral triangles ANM which have vertex N lying on the given circle.

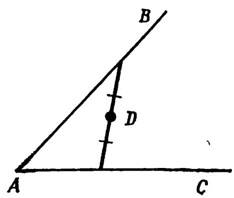
☐ Let N be an arbitrary point on the given circle. If we rotate the segment AN through 60° relative to the point A, then the point N comes to the vertex M of the equilateral triangle ANM. Hence, it is obvious that if we rotate the circle as a rigid figure about the point A through an angle of 60°, then each point N of the circle will come to the corresponding third vertex M of the equilateral triangle ANM.



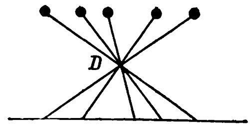
Thus, all the points M lie on one of the two circles obtained from the given one by a clockwise or anticlockwise rotation about the point A through an angle of 60°.

In exactly the same way we can show that each point M of the union of the two circles (we obtained) is the vertex of some equilateral triangle ANM. ☐

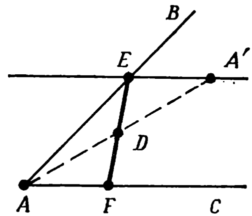
1.7a. An angle and a point D lying inside it are given. Construct a segment with its midpoint at the point D and its end-points on the sides of the given angle.



□ Let us consider a set of segments which have one end lying on the side AC of the given angle (with vertex A) and their midpoint at the given point D. The other ends of these segments are obviously contained in the ray symmetric to the side AC of the angle with respect to the point D.



The construction reduces to the following: construct the point A' symmetric about the point A with respect to D, then draw through A' a straight line parallel to AC up to the point of intersection E with the straight line AB to obtain the required segment EF with its midpoint at D. The problem always has a unique solution.

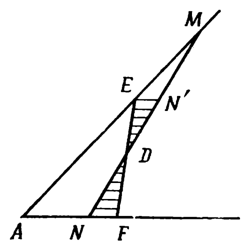


It is interesting to note that this very construction solves the following problem.

1.7b. An angle and a point D lying inside it are given. Draw through the point D a straight line which cuts off from the given angle a triangle with minimum possible area.

□ We shall prove that the unknown straight line is the same straight line EF which we constructed in the previous problem, i.e. that segment between the sides of the angle which is bisected by the point D.

Let us draw through the point D a straight line MN different from EF,



and prove that

$$S_{MAN} > S_{EAF}. \qquad (1)$$

We can assume that the point M on the side AB lies at a greater distance from the vertex of the angle A than E (the case when M lies closer to A than to E is analysed similarly, interchanging the roles of the sides AB and AC). It is sufficient to verify that

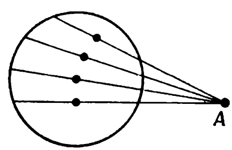
$$S_{EDM} > S_{FDN}, \qquad (2)$$

as inequality (1) follows readily from this. But inequality (2) is immediate, since the triangle EDM completely contains the triangle EDN' symmetric to the triangle FDN relative to the point D. □

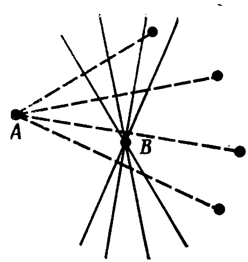
Set of Problems

1.8. Two points A and B are given. Find the set of feet of the perpendiculars dropped from the point A onto all possible straight lines passing through the point B.

1.9. Given a circle and a point A in a plane. Find the set of midpoints of the chords cut off by the given circle on straight lines passing through the point A. (Obviously, we should consider all the possible cases; when the point A lies inside the circle, outside the circle and on the circle.)



1.10. Given two points A and B. Find the set of points, each of which is symmetric about the point A with respect to some straight line passing through the point B.

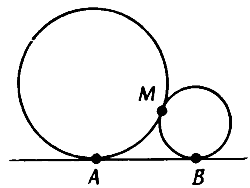


1.11. Construct a circle touching two given parallel straight lines and passing through a given point which lies in between the straight lines.

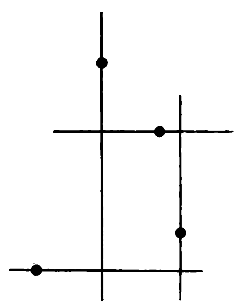
1.12. Construct a circle of a given radius r touching a given straight line and a given circle.

1.13. A circle and two points A, B lying inside it are given. It is required to inscribe a right-angled triangle in the circle so that the two given points lie on the sides forming the right angle.↓

1.14. The points A and B are given. The straight line AB touches two circles, one at the point A, the other at the point B, and the circles touch each other at the point M. Find the set of such points M. ↓



1.15. Four points are given in a plane. Find the set of centres of the rectangles formed by four straight lines passing respectively through the given points. ↓



1.16. The sides OP and OQ of the rectangle $OPMQ$ lie on the sides of a given right angle. Find the set of points M in the three following cases:

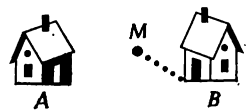
(a) the length of the diagonal PQ is equal to a given value d;

(b) the sum of the lengths of the sides OP and OQ is equal to a given value d;

(c) the sum of squares of the lengths of the sides OP and OQ is equal to a given value d.

1.17. Find the set of points, the sum of the squares of the distances from which to the four sides (or their extensions) of a given rectangle is equal to the square of the diagonal of the rectangle.

1.18. A and B are two different cities. Find the set of points M having the following property: if one travels in a straight line from M to B, then the distance from M to A always goes on increasing.



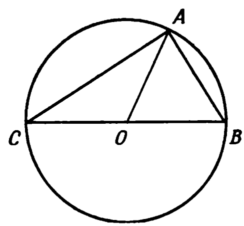
1.19. Suppose we know that in the triangle ABC the length of the median AO is:

(a) equal to half the length of the side BC;

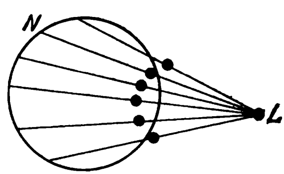
(b) greater than half the length of the side BC;

(c) less than half the length of the side BC.

Prove that the angle A is respectively: (a) a right angle, (b) an acute angle, (c) an obtuse angle.



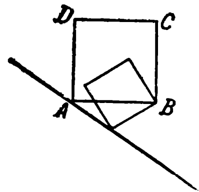
1.20. A circle and a point L are given in a plane. Find the set of midpoints of the segment LN, where N is an arbitrary point on the given circle.



1.21. Given a circle and a point lying outside it. Draw through this point a secant such that the length of the segment of the secant outside the circle is equal to the length of the segment inside it.

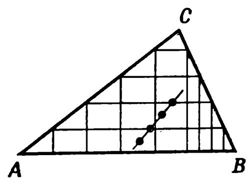
1.22. Through a point of intersection of two given circles draw a straight line on which these circles cut off chords of equal length.

1.23. Find the set of vertices C of the squares $ABCD$, where vertex A lies on a given straight line and vertex B is at a given point.



1.24. (a) Where can the fourth vertex of a square be, if two of its vertices lie on one of the sides of a given acute angle and the third vertex on the other side?

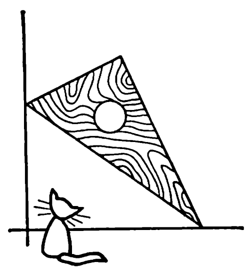
(b) Inscribe in a given acute triangle ABC a square, two vertices of which lie on the side AB.



1.25*. What line does the midpoint of the segment between two pedestrians walking uniformly along straight roads describe? ↓

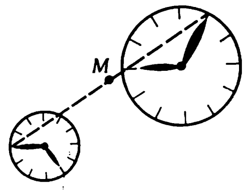
1.26*. Inside a given triangle ABC, all possible rectangles are inscribed, one side of which is on the straight line AB. Find the set of centres of all such rectangles.

1.27. A wooden right-angled triangle moves on a plane so that the ver-

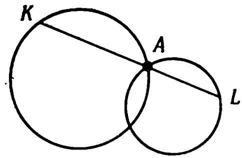


tices of its acute angles move along the sides of a given right angle. How does the vertex at the right angle of this triangle move?

1.28*. Two flat watches lie on a table. Both of them run accuratley. Along what path does the midpoint M of the segment connecting the end-points of their minute hands move? ↓



1.29*. Through the point of intersection A of two given circles, a straight line is drawn which crosses these circles once more at the points K and L respectively. Find the set of mid-points of the segment KL. ↓



2 The Alphabet

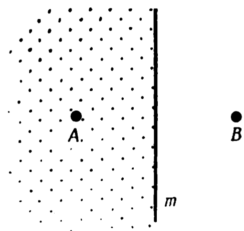
This section is a summary of theorems on sets of points satisfying various geometric conditions. We shall gradually compile a whole list of such theorems and conditions which can be used in the solution of problems of different types.

One can draw an analogy between the geometric problems of finding a set of points and the usual algebraic problems of solving an equation (a system of equations, an inequality). Solving an equation or an inequality means finding the set of numbers satisfying a certain condition. Just as in the algebra course at school when different equations (for example, trigonometric, logarithmic) are usually reduced to linear or quadratic equations, often even complicated geometric conditions turn out to be only new properties of the straight line or the circle.

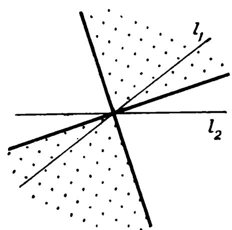
The analogy between algebraic problems and problems on finding sets of points is not only a superficial one. Using the method of coordinates one type of problems can be converted into the other. Using this method we shall see that geometric conditions, seemingly different at first sight, are covered by general theorems.

We start our geometric alphabet with the most simple assertions.

A. *The set of points equidistant from the two given points A and B is a straight line perpendicular to the segment AB and passing through its midpoint.* We shall call this straight line m the *perpendicular bisector* of the segment AB. It divides the plane into two half planes. The points in one of the half planes are closer to A than to B and in the other closer to B than to A. The points A and B are symmetric relative to m.

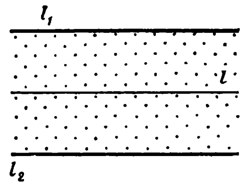


B. *The set of points equidistant from two given intersecting straight lines l_1 and l_2 is two mutually perpendicular straight lines which bisect the angles formed by the straight lines l_1 and l_2.*



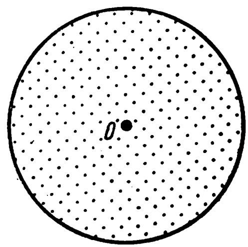
These straight lines are the axes of symmetry of the figure formed by the straight lines l_1 and l_2. This set—"*the cross bisector*"—divides the plane into four regions. In the figure two right angles—the set of points closer to the straight line l_1 than to the line l_2—are shown.

C. *The set of points whose distance from the given straight line l is equal to a given number h ($h > 0$) is a pair of straight lines l_1, l_2, parallel to l and lying on opposite sides of l.*



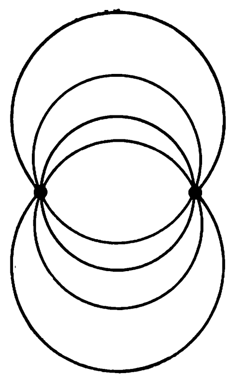
The belt between the straight lines l_1 and l_2 is the set of points which are at a distance less than h from the straight line l.

D. *The set of points whose distance from the given point O is equal to a given number r ($r > 0$) is a circle of radius r with centre O.*



(This is the definition of a circle.) The circle divides the plane into two parts: the internal and the external. For points inside the circle, the distance from the centre is less than r and for points outside the circle it is greater than r.

We shall give a few simple reformulations of the conditions **A**, **B**, **C**, **D** in the form of the following four problems.

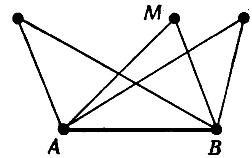


2.1. Find the set of centres of the circles passing through the two given points.

2.2. Find the set of centres of the circles touching two given intersecting straight lines.

2.3. Find the set of centres of circles of radius r touching a given straight line.

2.4. Given two points A and B. Find the set of such points M for which the area S_{AMB} of the triangle AMB is equal to a given number $c > 0$.



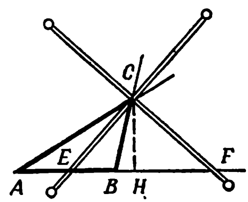
We shall illustrate the proposition **B** with a less trivial example—we shall prove the theorem on the bisector of a triangle.

2.5. Let the "cross bisector" of the straight lines AC and BC intersect the straight line AB at the points E and F. Prove that

$$\frac{|AE|}{|EB|} = \frac{|AF|}{|FB|} = \frac{|AC|}{|CB|}.$$

□ Let M be one of the points E or F. Note that

$$\frac{|AM|}{|MB|} = \frac{S_{ACM}}{S_{MCB}}.$$



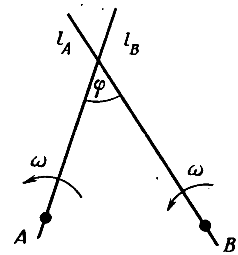
(The triangles ACM and MCB have the same height CH.)

The relation between the areas can be expressed in a different way; since the point M belongs to the cross bisector, it is equidistant from the straight lines AC and BC, hence,

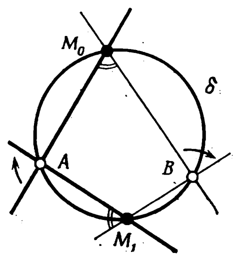
$$\frac{S_{ACM}}{S_{MCB}} = \frac{|AC|}{|CB|}. \quad □$$

A Circle and a Pair of Arcs. The next step of our "alphabet" is one more variant of the theorem on the inscribed angle and on the ring on a c icre which we have discussed in Sec. 1.

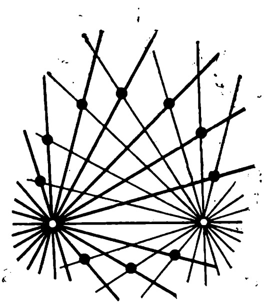
E°. *Two intersectiog straight lines l_A and l_B rotate, lyin8 on a plane, about two of thir points A and B with the same angular velocity* ω (here, the value of the angle between them obviously remains constant). *The trajectory of the point of intersection of these straight lines is a circle.*



□ Construct a circle δ passing through three points: A, B and a particular position M_0 of the point of intersection of the straight lines l_A and l_B. According to the theorem "on the ring on a circle" given in Sec. 1, the point of intersection of the straight line l_A and the circle δ moves uniformly along the circle δ with angular velocity 2ω. The point of intersection of l_B with the circle δ moves in exactly the same way. As they are coincident at a particular instant (at the position M_0), they also coincide at any other instant of time. □



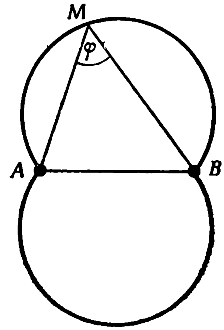
We shall give an alternative variant of theorem E, without using the language of motion.



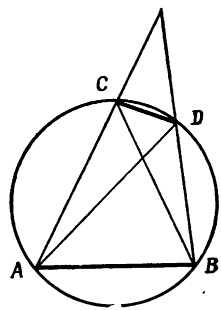
E. *The set of points at which the given segment AB subtends an angle of given value* φ (i.e. the set of points M for which $\widehat{AMB} = \varphi$) *is a pair of arcs with their end-points at A and B which* *are symmetric about the straight line AB.*

The region bounded by these two arcs is a set of points M for which $\widehat{AMB} > \varphi$.

Note that if $\varphi = 90°$, then the set **E** will be a circle with diameter AB. We have already mentioned this following problem **0.1.**

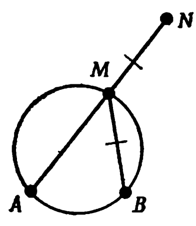


2.6. The chord AB of a given circle is fixed, and the chord CD is displaced without altering its length. Along what path does the point of intersection of the lines (a) AD and BC, (b) AC and BD move?



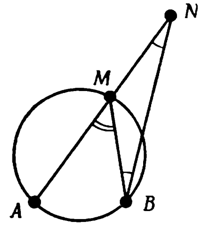
2.7. Given two points A and B, find the set of vertices M and N of parallelograms $AMBN$ with the given angle $\widehat{MAN} = \varphi$.

2.8a. A circle and two points A and B on it are given. Let M be an arbitrary point on this particular circle. A segment MN equal to the segment BM in length is marked off from the point M on the segment AM produced. Find the set of points N.

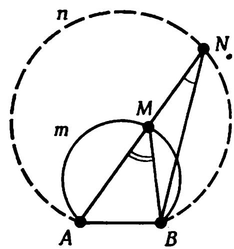


□ Let N be some point plotted in the same way as in the previous problem. Then $|BM| = |NM|$ and $\widehat{NBM} = \widehat{MNB}$. But since $\widehat{AMB} = \widehat{MBN} + \widehat{MNB}$, then $\widehat{ANB} =$

$= \widehat{AMB}/2$. The value of the angle AMB for all points M lying on one of the arcs $\overset{\frown}{AB}$ is constant (see E): $\widehat{AMB} = \varphi$. Hence, $\widehat{ANB} = \varphi/2$, i.e. all these points lie on the arc $\overset{\frown}{AnB}$ containing the angle $\varphi/2$. (The centre of the arc lies at the midpoint of the arc $\overset{\frown}{AmB}$ of the given circle ⟨?⟩.)

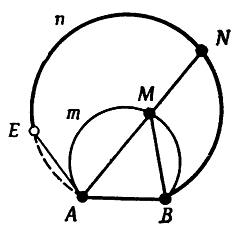


Do all the points of the arc $\overset{\frown}{AnB}$ satisfy our requirements? No, not all of them.



Note that when the point M runs along the arc $\overset{\frown}{AnB}$ from the point B to the point A, the chord AM rotates about the point A from the straight line AB up to the tangent to the given circle at the point A. Hence, only a part of the arc $\overset{\frown}{AnB}$ and in particular the arc $\overset{\frown}{EnB}$ (where E is the point of intersection of the arc $\overset{\frown}{AnB}$ with the tangent at the point A) belongs to the set we are looking for.

Note that we can take the point B as belonging to our set (when M coincides with B "the length of the segment MB is equal to zero"). Strictly speaking, the point E does not belong to our set; when the point M coincides with the point A, the direction of the straight line AM has no meaning.

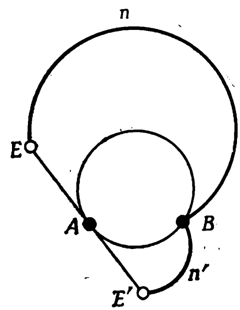


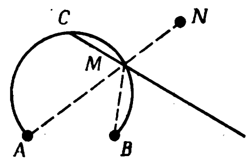
The points lying on the other side of the line AB are treated in a similar way.

Thus, the unknown set of points consists of two arcs $\overset{\frown}{EnB}$ and $\overset{\frown}{E'n'B}$. □

We may solve problem **2.8a** in a different way, if we notice that the points N and B are symmetric about the straight line CM, where C is the midpoint of the arc $\overset{\frown}{AmB}$. From this it follows that the set of points N reduces to the set of points found in problem **1.10** for the points A and C.

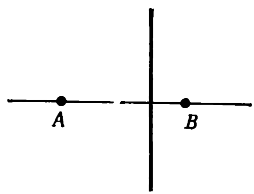
We shall present a problem similar to **2.8a** for the reader to investigate in the same way.





2.8b. The situation is the same as in problem **2.8a**; but the segment MN is marked off in the opposite direction on the ray MA.

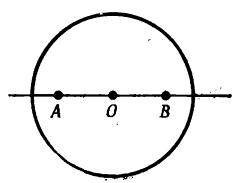
Squares of Distances. We shall consider two points A and B in a plane and an arbitrary number c.



F. *The set of points* M, *for which*

$$|AM|^2 - |BM|^2 = c,$$

is a straight line perpendicular to the segment AB (in particular, when $c = 0$, we get the perpendicular bisector, see proposition **A**).



G. Suppose $|AB| = 2a$. The *set of points* M, *for which*

$$|AM|^2 + |BM|^2 = c$$

is:

(a) a *circle with its centre* at the midpoint O of the segment AB and of radius $r = \sqrt{(c - 2a^2)/2}$, when $c > 2a^2$;

(b) a *point* O, when $c = 2a^2$;

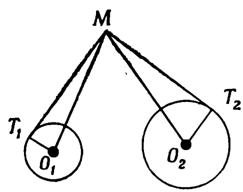
(c) the *empty set*, when $c < 2a^2$.

It is not difficult to prove propositions **F** and **G** using the Pythagorean theorem or by the method of coordinates ⟨?⟩.

We shall not present a separate proof for each statement, but deduce them both as corollaries of a more general theorem. But first we shall illustrate them with a few examples.

2.9. Find the set of points for which the tangents drawn to two given circles are equal in length.

□ Let O_1 and O_2 be the centres of the given circles, r_1 and r_2 their radii ($r_2 \geqslant r_1$), and let MT_1 and MT_2 be the tangents to them drawn from the point M. Using the Pythagorean theorem, the condition $|MT_1|^2 = |MT_2|^2$ may be written as:



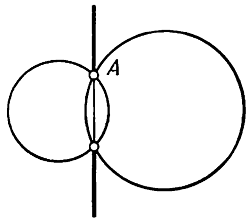
$$|MO_1|^2 - |O_1T_1|^2 = |MO_2|^2 - |O_2T_2|^2,$$

or

$$|MO_2|^2 - |MO_1|^2 = r_2^2 - r_1^2.$$

According to proposition **F**, the set of points M belongs to the straight line perpendicular to O_1O_2.

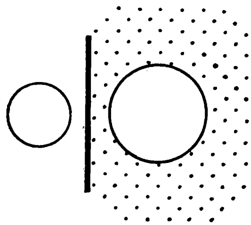
If the circles intersect, this straight line will pass through their points of intersection. For, if A is one of these points, then



$$|O_2A|^2 - |O_1A|^2 = r_2^2 - r_1^2$$

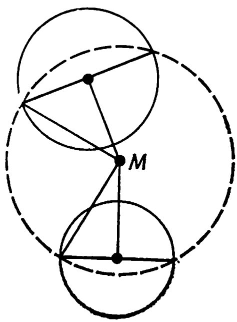
and, consequently, the point A lies on this straight line. The required set of points in this case is shown in the figure; it is the union of two rays.

If the circles are concentric (and $r_2 > r_1$), the required set is empty. If the circles coincide, all the points are outside the circle. If the circles are non-intersecting and non-concentric, the answer will be a straight line. □



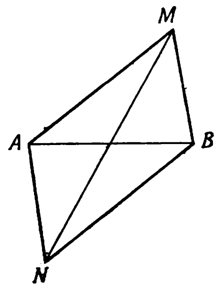
The straight line discussed in problem **2.9** is called the *radical axis of the two circles*. Suppose two non-intersecting circles are given. Then their radical axis divides the complement of the two circles into two regions: the set of points M for which $|MT_1| > |MT_2|$ and the set of points M for which $|MT_1| < |MT_2|$.

2.10. Find the set of centres of the circles which intersect each of two given circles at diametrically opposite points.

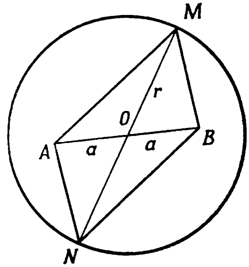


2.11. (a) The sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of squares of the lengths of its sides. Prove this.

(b) If the diagonals of a convex quadrilateral $AMBN$ are mutually perpendicular, then $|AM|^2 + |BN|^2 = |AN|^2 + |BM|^2$. Prove this.↓



□ (a) Let the vertices A and B of the parallelogram $AMBN$ be at a distance a from its centre O, the vertices M and N, at a distance r from O, and $c = 2(a^2 + r^2)$. As $|OM| = \sqrt{(c - 2a^2)/2}$, then according to proposition **G** the sum of the squares of the distances from the point M to the points A and B is equal to c. In the same way $|AN|^2 + |BN|^2 = c$, hence



$$|AM|^2 + |BM|^2 + |AN|^2 + |BN|^2 = 2c = 4(a^2 + r^2) = |MN|^2 + |AB|^2. \square$$

We shall present now the general theorem which contains propositions **F**, **G**, **A**, **D** of our alphabet.

Theorem on the Squares of the Distances. The set of points M for which the condition

$$\lambda_1 |MA_1|^2 + \lambda_2 |MA_2|^2 + \ldots + \lambda_n |MA_n|^2 = \mu, \quad (1)$$

is satisfied, where $A_1, A_2, \ldots, A_n$ are given points, $\lambda_1, \lambda_2, \ldots, \lambda_n, \mu$ are given numbers, is one of the following simple geometric figures:

1°. *If $\lambda_1 + \lambda_2 + \ldots + \lambda_n \neq 0$, it may be a circle, a point or the empty set.*

2°. *If* $\lambda_1 + \lambda_2 + \ldots + \lambda_n = 0$, *it may be a straight line, the entire plane or the empty set.*

We shall give a proof of the theorem using the method of coordinates.

□ The square of the distances between the points $M(x;\ y)$ and $A_k(x_k;\ y_k)$ is calculated according to the formula

$$|MA|^2 = (x - x_k)^2 + (y - y_k)^2 =$$
$$= x^2 + y^2 - 2x_k x - 2y_k y + x_k^2 + y_k^2.$$

Consider the expression

$$\lambda_1 |MA_1|^2 + \lambda_2 |MA_2|^2 + \ldots +$$
$$+ \lambda_n |MA_n|^2.$$

In order to write it in coordinates, it is necessary to add several expressions of the form

$$\lambda (x^2 + y^2 - 2px - 2qy + p^2 + q^2).$$

As a result, condition (1) may be written in the form of the equation

$$dx^2 + dy^2 + ax + by + c = 0, \qquad (2)$$

where $d = \lambda_1 + \lambda_2 + \ldots + \lambda_n$.

We shall now prove that *equation* (2) *gives one of the figures enumerated above.*

1°. If $d \neq 0$, we can transform (2) in the following manner:

$$x^2 + y^2 + \frac{a}{d}x + \frac{b}{d}y + \frac{c}{d} = 0$$

or

$$\left(x+\frac{a}{2d}\right)^2+\left(y+\frac{b}{2d}\right)^2=\frac{b^2+a^2-4dc}{4d^2}. \qquad (2')$$

We can see that this gives us:

a circle with centre at the point $C(-a/2d; -b/2d)$, if the right-hand side (2′) is positive;

a single point $C(-a/2d; -b/2d)$, if the right-hand side equals zero;

the empty set, if the right-hand side is negative.

2°. If $d = 0$, equation (2) takes the form

$$ax + by + c = 0.$$

This will be:

a straight line, if $a^2 + b^2 \neq 0$,

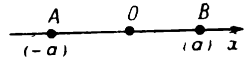
the entire plane, if $a = b = c = 0$,

the empty set, if $a = b = 0$, $c \neq 0$. □

As a rule, in a particular example, it is easy to determine which of these cases is involved. Let us return again to propositions **F** and **G** of our "alphabet" which have not been proved yet.

Proof of F. The condition $|MA|^2 - |MB|^2 = c$ is a particular case of (1), where $n = 2$, $\lambda_1 = 1$, $\lambda_2 = -1$, from which $d = 0$, and hence it determines either a straight line or a plane, or the empty set.

Since the equation $(x + a)^2 - (x - a)^2 = c$ always has a single solution, $x = c/4a$, one point of the set is on the straight line AB. Therefore, the required set is a straight line. It is clear from symmetry considerations that this straight line is perpendicular to the straight line AB. □

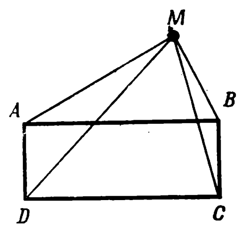


Proof of G. The condition $|MA|^2 + |MB|^2 = c$ is a particular case of (1). Here $\lambda_1 = 1$, $\lambda_2 = 1$, $d \neq 0$, and, therefore, the unknown set would be either the empty set, a point, or a circle. Since the points A and B appear in the condition symmetrically, the centre of the circle lies at the midpoint of the segment AB.

In order to find when the unknown set is a circle and to determine its radius, we find the points on the straight line AB which satisfy the condition $|AM|^2 + |BM|^2 = c$. To do this, note that the equation $(x - a)^2 + (x + a)^2 = c$ has a solution when $c \geqslant 2a^2$, and

$$x| = r = \sqrt{(c - 2a^2)/2}.$$ □

2.12. Find the set of points, the sum of the squares of the distances of which from two opposite vertices of a given rectangle is equal to the sum of the squares of the distances from the two other vertices.



□ *Answer*: The entire plane. Let us prove this. Let $ABCD$ be the given rectangle. Then we seek the set of points M, for which $|MA|^2 + |MC|^2 - |MB|^2 - |MD|^2 = 0$.

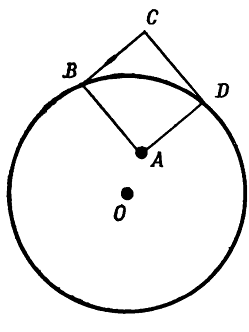
In condition (1) put $n = 4$, $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \lambda_4 = -1$ and $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$. According to the theorem, the required set is either a straight line, or the empty set or the entire plane.

We note that the vertices A, B, C, D of the rectangle itself satisfy the condition. For example, the following equality $|AA|^2 + |AC|^2 - |AB|^2 - |AD|^2 = 0$ (the Pythagorean theorem) is valid for the point A. Thus, the required set is neither the empty set nor a straight line. Hence, it follows that the required set is the entire plane. □

From the result of problem **2.12**, it follows that if $ABCD$ is a rectangle, then for any point M of the plane the following equality holds

$$|MA|^2 + |MC|^2 = |MB|^2 + |MD|^2$$

Solve the following problem using this fact.



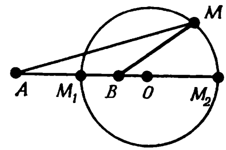
2.13. A circle and a point A inside it are given. Find the set of the fourth vertices C of the rectangles $ABCD$, whose vertices B and D belong to the given circle.

2.14. Prove that $|MA|^2 - |MB|^2 = 2|AB|\rho(M, m)$, where m is the perpendicular bisector of the segment AB, and, $|MA| > |MB|$.

We add to our alphabet one more proposition which is frequently used in geometry and is also a corollary from the theorem on the squares of the distances.

H. *The set of points M for which*

$|MA|/|MB| = k,\ k > 0,\ k \neq 1,$



is a circle whose diameter belongs to the straight line AB.

This set of points, the ratio of the distances of which from the two given points A and B is a constant, is called the *circle of Apollonius.*

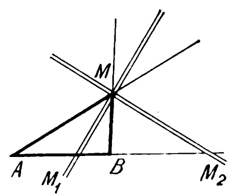
□ Let us rewrite condition **H** in the form

$|MA|^2 - k^2 |MB|^2 = 0.$

This condition is a particular case of the condition (1) where $n = 2$, $\lambda_1 = 1$, $\lambda_2 = -k^2$ and hence if $1 - k^2 \neq 0$, the required set will be either a circle, or a point, or the empty set. Since the equation

$(x + a)^2 = k^2 (x - a)^2$

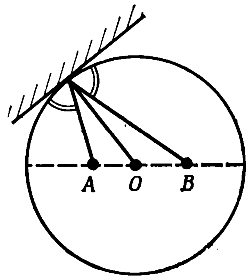
always has two solutions, when $k^2 \neq 1$, there exist two points M_1 and M_2 of this set on the straight line AB and hence the unknown set is a circle. As the condition is symmetric relative to the straight line AB, the diameter of this circle is the segment M_1M_2. □



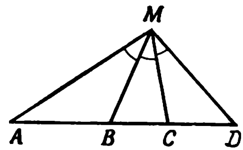
Incidentally note that if M is a point of the circle of Apollonius, then the cross bisector of the straight lines AM and MB intersects the line AB in the points M_1 and M_2. (This follows from the theorem on the cross bisector in **2.5**, since $|AM_1|/|BM_1| = |AM_2|/|BM_2| = |AM|/|BM|$.)

This argument is used in the next problem.

2.15. Two billiard balls A and B are placed on the diameter of a circular billiard table. Ball B is hit in such a way that after one rebound from the side of the table it strikes ball A. Find the trajectory of ball B, if the stroke is not directed along the diameter.



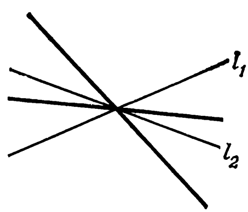
2.16. The points A, B, C, D are on a given straight line. Construct a point M in the plane, from which the segments AB, BC and CD are seen at one and the same angle (i.e. subtend the same angle at M).



Distances from Straight Lines. So far in this chapter various properties defining a circle are mainly used. In the next two propositions of our alphabet only straight lines (which will appear in pairs) will appear.

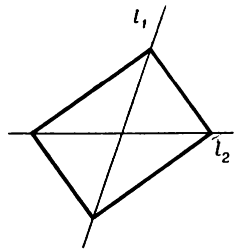
We shall consider two intersecting straight lines l_1 and l_2 in a plane and a positive number c.

I. *The set of points M, the ratio $\rho(M, l_1)/\rho(M, l_2)$ of whose distances from the straight lines l_1 and l_2 is equal to c is a pair of straight lines passing through the point of intersection of the straight lines l_1 and l_2.*



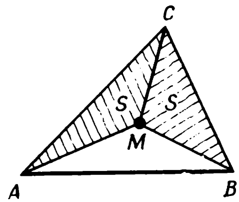
J. *The set of points M, the sum*

$\rho(M, l_1) + \rho(M, l_2)$ *of w ose distances from the straight lines* l_1 *and* l_2 *is equal to* c *is the boundary of a rectangle with diagonals lying on the lines* l_1 *and* l_2.



Before proving these theorems let us illustrate them by two examples.

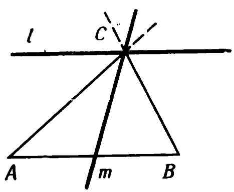
2.17. Given a triangle ABC, find the set of all points M for which $S_{AMC} = S_{BMC}$.



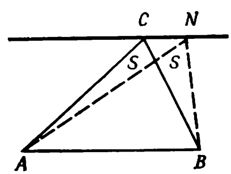
□ Let h_a and h_b be respectively the distances of the point M from the straight lines AC and BC. Then,

$$S_{AMC} = \frac{|AC| \cdot h_b}{2}, \quad S_{BMC} = \frac{|BC| \cdot h_a}{2},$$

consequently $h_a/h_b = |AC|/|BC|$.



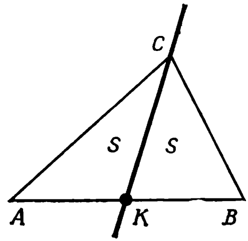
Hence the required set of points M is the set given in proposition I for the lines AC and BC and $c = |AC|/|BC|$. Thus it represents a pair of straight lines passing through the point C. We shall show that one of the straight lines m contains the median of the triangle, and the other, l, is parallel to the straight line AB. For this, it is sufficient to take a single point on each of the straight lines and verify that the condition stated is fulfilled for them.



Let us denote by h the altitude of the triangle drawn from the vertex C. Let N be a point on the straight line l, then

$$S_{ACN} = \frac{|CN| \cdot h}{2} \text{ and } S_{BCN} = \frac{|CN| \cdot h}{2}$$

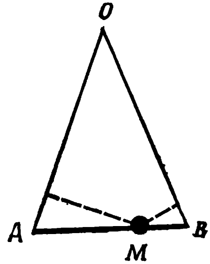
Hence $S_{ACN} = S_{BCN}$ a d the straight line l belongs to the required set.



Let K be the midpoint of the side AB, i.e. $|AK| = |KB|$. Then $S_{AKC} = |AK|\cdot h/2 = |BK|\cdot h/2 = S_{BKC}$, and, consequently, the whole line m belongs to the unknown set. □

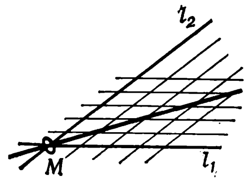
In analogy with the cross bisector one may call the pair of straight lines m and l "the cross median" of the vertex C of the triangle.

Proposition **J** can in essence be reduced to the following problem.



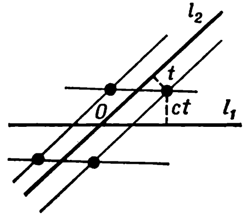
2.18. Given an isosceles triangle AOB. Prove that the sum of the distances from the point M on its base AB to the straight lines AO and BO is equal to the length of the altitude dropped onto a lateral side.

We shall not give geometrical proofs of propositions **I** and **J**, although they are not at all difficult. But we shall give proofs using the language of motion. (As was done above in proposition **E°** "A circle and a pair of arcs".) Let us first formulate a lemma which generalizes the theorem on a tiny ring on a straight line (page 24).

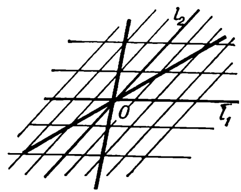


Lemma. A tiny ring M is placed on two straight lines l_1 and l_2 at their point of intersection. If each straight line describes a uniform translatory motion, then the ring M moves uniformly along some straight line.

□ This straight line can be constructed by marking two different positions M_1 and M_2 of the ring. The points of intersection of the moving straight lines with the stationary line M_1M_2 move uniformly. Since these points coincide with each other at two different points of time (when the ring M passes through M_1 and M_2), they must always coincide. □



Proof of I. The set of points lying at a distance t from l_2 and a distance ct from l_1 for some positive number t, is the four vertices of a parallelogram with its centre at the point O of intersection of l_1 and l_2. For, the set of points lying at a distance t from l_2 is a pair of parallel lines (see **C**) and the set of points lying at a distance ct from l_1 is also a pair of parallel lines and their points of intersection are the four vertices of the parallelogram. These four points satisfy the condition stated in **I**, since

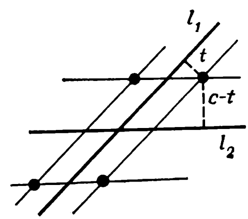


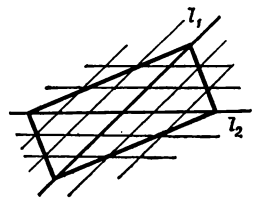
$$ct/t = c.$$

By varying the number t from zero to infinity, we get all the points of the required set.

By regarding t as "time", we see that the four straight lines constructed above move uniformly (remaining parallel to l_1 and l_2). By the lemma, their points of intersection, the rings, move along a straight line passing through the point O. □

Proof of J. Draw two straight lines at a distance t from l_1 and two, at a distance $c - t$ from l_2 $(0 \leqslant t \leqslant c)$. The four points of intersection of these straight lines belong to the required set. When the "time" t varies from zero to c, the straight lines move uniformly and each of the four points of intersection, by the lemma, moves through a segment. The end-points of these segments which correspond to $t = 0$ and $t = c$ lie on the straight lines l_1 and l_2 and are the vertices of a rectangle. □



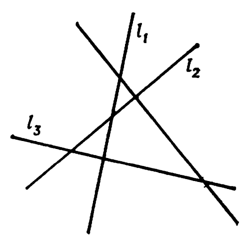


We shall now state a general theorem which includes propositions **B**, **C**, **I**, **J** of the alphabet. Consider *the set of points M for which*

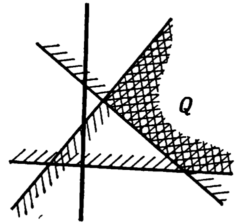
$$\lambda_1\rho(M, l_1) + \lambda_2\rho(M, l_2) + \dots + \lambda_n\rho(M, l_n) = \mu. \quad (3)$$

Here $l_1, l_2, \dots, l_n$ are given straight lines, and $\lambda_1, \lambda_2, \dots, \lambda_n, \mu$ are given numbers.

It is difficult to give an immediate description of this set on the entire plane. However, as we shall now see, in each of the pieces into which the straight lines $l_1, l_2, \dots, l_n$ divide the plane, set (3) is, as a rule, simply a part of some straight line. Let us denote one of these pieces by Q.



Theorem on the Distances from the Straight Lines. The set of points which *satisfy condition (3), belonging to Q, is either (1) the intersection of Q with a straight line, i.e. a ray, a segment or even a whole straight line, or (2) the whole of Q, or (3) the empty set.*

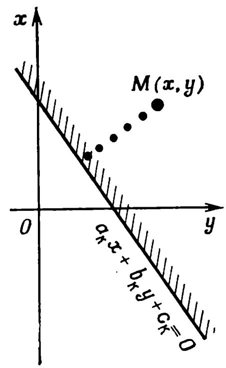


By finding the set on each of the pieces, we shall find the entire required set (as in **1.3**). We shall give a proof of the theorem using the method of coordinates.

□ Suppose we want to find the set of points on one of the pieces Q of the plane into which the lines $l_1, l_2, \ldots, l_n$ divide the plane. The piece Q of the plane can be imagined as the intersection of n half planes with boundary lines $l_1, l_2, \ldots, l_n$.

The equation $a_k x + b_k y + c_k = 0$ of the straight line l_k can be selected such that inside the required half plane, $a_k x + b_k y + c_k \geqslant 0$ and $a_k^2 + b_k^2 = 1$ ⟨?⟩; then for the point $M(x; y)$ in this half plane, $\rho(M, l_k) = a_k x + b_k y + c_k$.

In order to write the quantity $\lambda_1 \rho(M, l_1) + \lambda_2 \rho(M, l_2) + \ldots + \lambda_n \rho(M, l_n)$ in coordinates, we have to add several linear expressions of the form $\lambda_k a_k x + \lambda_k b_k y + \lambda_k c_k$. As a result, condition (3) is expressed by a linear equation



$$ax + by + c = 0.$$

If $a^2 + b^2 \neq 0$, this equation represents a straight line. If $a = b = 0$, it represents either the entire plane or the empty set. □

An alternative proof of this theorem can be found by reducing it using problem **2.14** to the theorem on the squares of the distances ⟨?⟩.

2.19. (a) A right triangle ABC is given. Find the set of points for which the sum of the distances from the straight lines AB, BC and CA is equal to a given number $\mu > 0$. ↓

(b) Given a rectangle $ABCD$. Find the set of points for which the sum of the distances from the straight lines AB, BC, CD, DA is equal to a given number μ.

2.20.* (a) Three straight lines l_0, l_1, l_2 intersect at a single point. The value of the angle between each two of them is equal to 60°. Find the set of points M for which

$$\rho(M, l_0) = \rho(M, l_1) + \rho(M, l_2).$$

(b) An equilateral triangle ABC is given. Find the set of points M for which the distance from one of the straight lines AB, BC, CA is half the sum of its distances from the remaining two lines. ↓

The Entire "Alphabet". The set of points satisfying a certain condition is denoted as follows: inside the braces a letter is first written to denote an arbitrary point of the set (in our case, it is, as a rule, the letter M, but it can be any letter); then there is a colon which is followed by the condition which specifies the required set of points.

Let us now summarize the sets of our "alphabet":

A. $\{M: |MA| = |MB|\}$.

B. $\{M: \rho(M, l_1) = \rho(M, l_2)\}$.

C. $\{M: \rho(M, l) = h\}$.

D. $\{M: |MO| = r\}$.

E. $\{M: \widehat{AMB} = \varphi\}$.

F. $\{M: |AM|^2 - |MB|^2 = c\}$.

G. $\{M: |AM|^2 + |MB|^2 = c\}$.

H. $\{M: |AM|/|MB| = k\}$.

I. $\{M: \rho(M, l_1)/\rho(M, l_2) = k\}$.

J. $\{M: \rho(M, l_1) + \rho(M, l_2) = c\}$.

Recall that we have separated the propositions of our "alphabet" with the exception of **E** into two groups:

A, D, F, G, H and **B, C, I, J** .

The sets in the first group are particular cases of the set

$$\{M: \lambda_1 |MA|^2 + \lambda_2 |MA_2|^2 + \ldots + \lambda_n |MA_n|^2 = \mu\},$$

and the sets in the second group are particular cases of the set

$$\{M: \lambda_1\rho(M, l_1) + \lambda_2\rho(M, l_2) + \ldots + \lambda_n\rho(M, l_n) = \mu\}.$$

In Sec. 6 we shall add four more "letters" to our "alphabet"

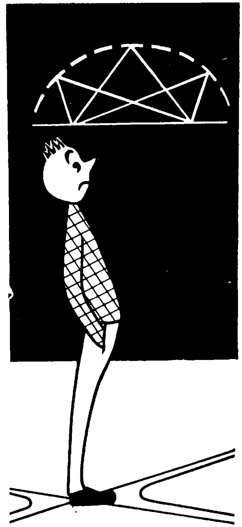
K. $\{M\colon |MA| + |MB| = c\}$.

L. $\{M\colon ||MA| - |MB|| = c\}$.

M. $\{M\colon |MA| = \rho(M, l)\}$.

N. $\{M\colon |MA|/\rho(M, l) = c\}$.

These sets are ellipses, hyperbolas, parabolas. These curves also fall naturally into a single group, the quadratic curves.

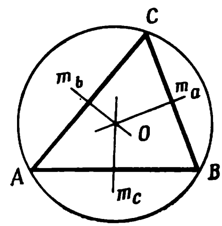


3 Logical Combinations

In this section we have collected various problems in which as a rule combinations of several geometric conditions are involved. In solving these problems we shall learn to classify the points and to consider logical relations between conditions as operations on sets.

Through a Single Point. In the first problems we shall touch on the traditional subject matter of geometry. We shall prove some theorems on the special points of a triangle with the help of simple manipulations using the sets of our "alphabet". The whole logic of the reasoning will as a rule reduce to the use of transitivity of equality: if $a = b$ and $b = c$, then $a = c$.

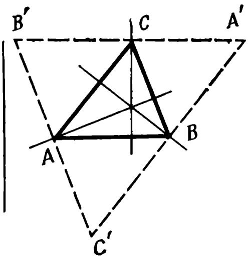
3.1. In a triangle ABC the midperpendiculars (perpendiculars at the midpoints of the sides) intersect at a single point (the centre of the circumscribed



circle of the triangle also known as the *circumcircle*).

□ The midperpendiculars m_c and m_a of the sides AB and BC must intersect at some point O. Since the point O belongs to the midperpendicular m_c, then according to A (Sec. 2), the equality $|OA| = |OB|$ holds true. In exactly the same way from the fact that O belongs to the midperpendicular m_a, it follows that $|OB| = |OC|$. Hence $|OA| = |OC|$ and consequently the point O belongs to the midperpendicular m_b of the side AC.

We have thus proved that all the three midperpendiculars intersect at the point O. □

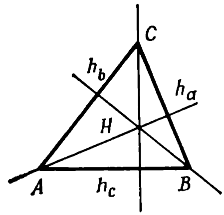


3.2. The three altitudes of a triangle ABC intersect at a single point. (This point is called the *orthocenter* of the triangle.) (The line through a vertex perpendicular to the opposite side is called an altitude.)

□ Draw through each of the vertices of the triangle a straight line parallel to the side opposite the vertex. These straight lines form a new triangle $A'B'C'$, in which the points A, B, C are the midpoints of its sides while the altitudes of the triangle ABC belong to perpendicular bisectors of the sides $A'B'$, $B'C'$, $C'A'$. Hence, by **3.1** they are concurrent. □

We shall give a second proof of **3.2**, similar to that of **3.1**.

□ Let us consider each of the altitudes as a set of points satisfying a certain condition. For this we shall use proposition **F** of the "alphabet".



We know that the set

$$\{M: |MA|^2 - |MB|^2 = d\}$$

is a straight line perpendicular to AB. Choose d such that this straight line contains the vertex C. To do this, we must take $d = |CA|^2 - |CB|^2$. Thus, the straight line

$$h_c = \{M: |MA|^2 - |MB|^2 = |CA|^2 - |CB|^2\},$$

contains the altitude of the triangle dropped from the vertex C.

One can consider the straight lines containing two other altitudes of the triangle in a similar way.

$$h_a = \{M: |MB|^2 - |MC|^2 = |AB|^2 - |AC|^2\};$$

$$h_b = \{M: |MC|^2 - |MA|^2 = |BC|^2 - |BA|^2\}.$$

Suppose the first two straight lines h_c and h_a intersect at the point H. Then when M coincides with this point both hold

$$|HA|^2 - |HB|^2 = |CA|^2 - |CB|^2,$$

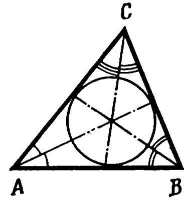
$$|HB|^2 - |HC|^2 = |AB|^2 - |CA|^2.$$

Adding these two equalities, we obtain

$$| HA |^2 - | HC |^2 = | AB |^2 - | CB |^2.$$

Hence, the point H also belongs to the third straight line h_b. □

3.3. Three bisectors of the angles of a triangle ABC intersect at a single point. (*At the centre of the inscribed circle of the triangle.*) (This circle is also known as the incircle of the triangle.)

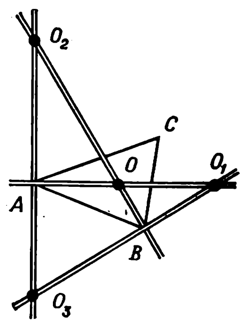


□ Let a, b and c be the straight lines to which the sides of the triangle belong. The bisectors l_a and l_b of the angles A and B must intersect at some point O (inside the triangle). For this point O the following equalities hold

$$\rho (O, b) = \rho (O, c) \quad \text{and}$$

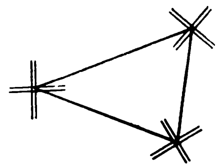
$$\rho (O, a) = \rho (O, c).$$

Hence, $\rho (O, b) = \rho (O, a)$ and point O belongs to the bisector l_c of angle C of the triangle. □

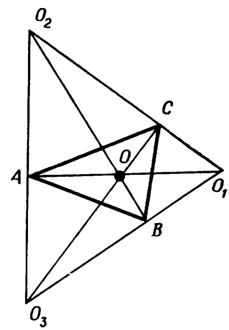


Note. The set of points M of the plane for which $\rho (M, c) = \rho (M, b)$ and $\rho (M, a) = \rho (M, c)$ consists of four points: O, O_1, O_2 and O_3, the points of intersection of the two "cross bisectors". Reasoning similarly as in the solution of **3.3**, we find that the third "cross" (the cross bisector of the straight lines a and b) also passes through these points.

From here it follows that the six bisectors of the internal and external angles of the triangle intersect in threes at four points. One of these points is the centre of the inscribed circle and the other three are the centres of the so-called *escribed circles*.



Note that, if in an arbitrary acute-angled triangle $O_1O_2O_3$ the points A, B, C are the feet of its altitudes, then O_1, O_2 and O_3 are the centres of the escribed circles (or excircles) of the triangle ABC. The altitudes of the triangle $O_1O_2O_3$ are therefore the bisectors of the angles of the triangle ABC.

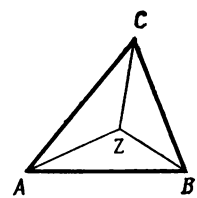


3.4. The medians of a triangle intersect at a single point, called the centroid of the triangle (or the "*centre of gravity*" *of the triangle*).

This theorem can be proved by different methods.

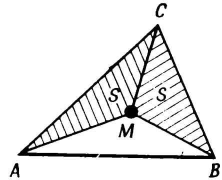
The first proof, which we give here, explains the term "the centre of gravity" of the triangle.

□ Let us place three weights W_A, W_B, W_C of the same mass, say 1 g, at the vertices of the triangle ABC, and find the position of their centre of gravity. The centre of gravity of the two weights W_A and W_B lies at the midpoint of the segment AB; hence, the centre of gravity Z lies on the corresponding median. We can show in the same way that Z belongs



to other two medians. Hence, all the three medians intersect at the point Z. □

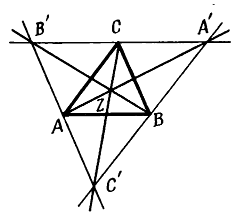
We shall also give a proof along the same lines as the three previous proofs.



□ Suppose we are given a triangle ABC. The points of the medians of the triangle drawn from the vertices A, B, C satisfy the following conditions (respectively) (see **2.17**):

$$S_{AMB} = S_{CMA}, \quad S_{AMB} = S_{BMC},$$

$$S_{BMC} = S_{CMA}. \qquad (1)$$

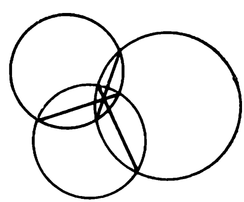


It is clear that the third condition follows from the first two, and so the medians intersect at a single point Z. □

Note. The set of points which satisfy the conditions (1) is, according to **2.17**, a pair of straight lines which we could call the "cross median". Thus, three such sets intersect at four points: Z, A', B', C'. Note that the triangle $A'B'C'$ is just the triangle considered in the first proof of the theorem on the altitudes in **3.2.**

3.5. (a) Prove that for any three circles the three radical axes of the pairs of circles pass through a single point or are parallel (see **2.9**).

(b) Prove that, if three circles intersect in pairs, then the three common chords of each pair of circles



(or their continuations) pass through a single point or are parallel. ↓

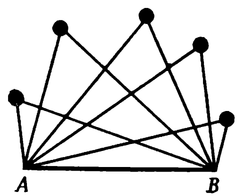
3.6. (Torricelli's point). Prove that in an acute-angled triangle ABC, there exists a point T (Torricelli's point) at which all the sides subtend the same angle (i.e. such that $\widehat{ATB} = \widehat{BTC} = \widehat{CTA}$).

3.7. Consider all the possible triangles with a given base AB with the vertex angle equal to φ. Find the set of:

(a) points of intersection of the medians,

(b) points of intersection of the bisectors, ↓

(c) points of intersection of the altitudes. ↓



3.8. (a) Three straight lines a, b, c (intersecting in pairs) pass through three given points A, B, C respectively. The lines rotate with angular velocity ω. Prove that at some moment of time these straight lines pass through a single point. ↓

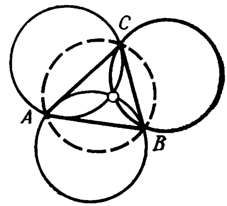
(b) Prove that three circles symmetric to the circumcircle of the triangle ABC relative to the straight lines AB, BC and CA pass through a single point, the orthocentre of the triangle ABC. ↓

3.9. (Ceva's Theorem). Points C_1, A_1, B_1 are selected on the sides AB,

BC, CA of the triangle. Prove that the segments AA_1, BB_1 and CC_1 are concurrent (intersecting at a single point) if and only if the condition:

$$\frac{|AC_1|}{|C_1B|} \cdot \frac{|BA_1|}{|A_1C|} \cdot \frac{|CB_1|}{|B_1A|} = 1$$

is satisfied. ↓

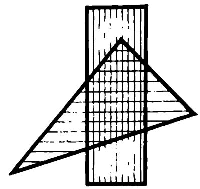


3.10. At the points C_1, A_1, B_1 lying respectively on the sides AB, BC, CA of a given triangle ABC, perpendiculars to the sides are erected.

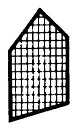
Prove that these three perpendiculars are concurrent if and only if the condition

$$|AC_1|^2 + |BA_1|^2 + |CB_1|^2 = \\ = |AB_1|^2 + |BC_1|^2 + |CA_1|^2$$

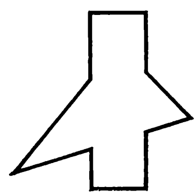
is satisfied. ↓



Intersection and Union. We now single out the basic operations which we are making constant use of.



Suppose two or more sets of points are given. The set of all points belonging simultaneously to all the given sets is called the *intersection* of the sets. The set of all points belonging to at least one of the given sets is called the *union* of these sets.



When it is required in a problem to find those points which *simultaneously* satisfy *several conditions*, we find the set of points satisfying each of the conditions separately and then take the intersection of these sets. We meet a similar situation in algebraic problems also. The set of solutions of the system of equations

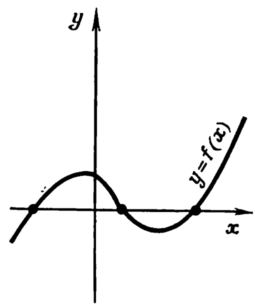
$$\begin{cases} f_1(x)=0 \\ f_2(x)=0 \end{cases}$$

is in fact the intersection of the solution sets of the individual equations making up this system.

If it is required in a problem to find those points which satisfy *at least one* of several conditions, we find the sets of points satisfying separately each of the conditions and then take the union of these sets. This is what we do when solving the equation $f(x) = 0$, when the left-hand side may be factorized,

$$f(x) = f_1(x) f_2(x).$$

We find the solution set of each of the equations $f_1(x) = 0$, $f_2(x) = 0$ and then take their union.



There is another concept which gives rise to an algebraic association, namely the partition (or subdivision) of a domain. In order to solve the inequality $f(x) > 0$ or $f(x) < 0$, it is usually sufficient to solve the corresponding equation $f(x) = 0$. The points obtained divide the domain of definition of the function f (an interval or the whole line) into pieces, in each of which the function does not change a sign. In exactly the same way, the sets of points of a plane for which various inequalities hold are usually domains bounded by lines on which the corresponding equalities are satisfied. We have already seen many simple examples of this type in Sec. 2.

We shall encounter more complicated partitions and combinations of sets in the next problem.

3.11. Let two points A and B be given in a plane. Find the set of points for which the triangle AMB is:

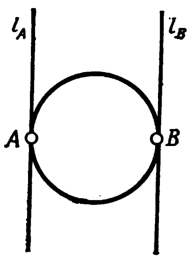
(a) a right-angled triangle,

(b) an acute-angled triangle,

(c) an obtuse-angled triangle.

□ (a) The triangle AMB is a right-angled triangle if one of the following three conditions is met: (1) $\widehat{AMB} = 90°$, (2) $\widehat{BAM} = 90°$, (3) $\widehat{ABM} = 90°$.

The unknown set is, therefore, the union of the following three sets: (1) a circle with $|AB|$ as diameter, (2) a straight line l_A passing through the point A and perpendicular to the segment AB, (3) a straight line l_B passing through the point B and perpendicular to the segment AB.

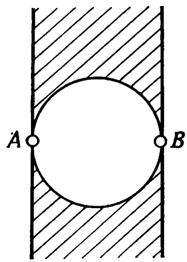


We must exclude from this union the points A and B on the line AB (they give rise to a "degenerate" triangle AMB). □

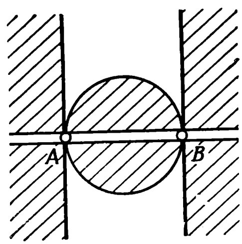
□ (b) The triangle AMB is an acute-angled triangle, if the following three conditions are simultaneously satisfied: (1) $\widehat{AMB} < 90°$, (2) $\widehat{BAM} < 90°$, (3) $\widehat{ABM} < 90°$.

The required set is therefore the intersection of the following three sets: (1) the exterior of a circle with the diameter AB (see Sec. 2, proposition **D**); (2) the half plane bounded by l_A containing the point B, with the boundary line l_A removed; (3) the half plane bounded by l_B containing the point A with the boundary line l_B removed.

The intersection is the strip between the lines l_A and l_B from which the circle with diameter AB is removed. □



□ (c) Note that every point M of the plane (not lying on the straight line AB) satisfies one of the following three conditions: either (a) $\triangle AMB$ is a right-angled triangle, or (b) $\triangle AMB$ is an acute-angled triangle or (c) $\triangle AMB$ is an obtuse-angled triangle. Note moreover that these conditions are, however, mutually exclusive. Hence, all the points of the plane which belong neither to (a) nor to (b) must belong to the set (c). This set is the union of a circle and two half planes (with the line AB removed). □

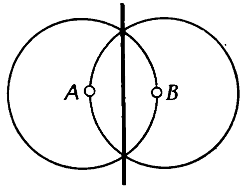


3.12. In a plane two points A and B are given. Find the set of points M such that:

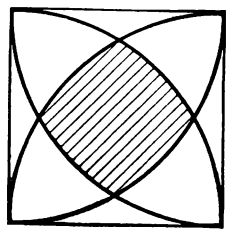
(a) the triangle AMB is an isosceles triangle,

(b) the side AB is the largest side of the triangle AMB,

(c) the side AM is the largest side of the triangle AMB.



3.13. A square with sides of unit length is given on a plane. Prove that if a point of the plane lies at a distance of not more than 1 from each of the vertices of this square, then it lies at a distance of not less than 1/8 from each side of the square.

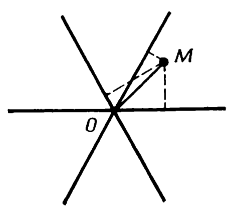


□ The set of points M at a distance of not more than 1 from each of the four vertices is the intersection of four circles of unit radius, with centres at the vertices of the square. It is a quadrilateral bounded by four arcs. Each of its vertices lies at a distance of $1-\frac{\sqrt{3}}{2}$ from the nearest side. Let us check that this number is greater than 1/8:

$$1-\frac{\sqrt{3}}{2}>\frac{1}{8} \Leftrightarrow \frac{7}{8}>\frac{\sqrt{3}}{2} \Leftrightarrow \frac{49}{16}>3.$$

It is thus clear that all the points of our set are at a distance of more than 1/8 from the sides of the square. □

3.14. Three straight lines passing through a point O of the plane divide the plane into six congruent angles. Prove this if the distance of the point M from each of the straight

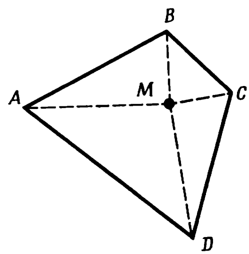


lines is less than 1 then the distance $|OM|$ is less than 7/6.

3.15. Given a square *ABCD*, find the set of points which are nearer to the straight line *AB* than to the lines *BC*, *CD* and *DA*.

3.16. Given a triangle *ABC*, find in the plane the set of points such that the area of each of the triangles *AMB*, *BMC*, *CMA* is less than that of the triangle *ABC*.

3.17. Circles are drawn with the sides of an arbitrary convex quadrilateral *ABCD* as diameters. Prove that they cover the whole quadrilateral.



□ Assume that inside the quadrilateral there exists a point *M* lying outside the circles. Then according to Sec. 2 proposition **E**, all the angles *AMB*, *BMC*, *CMD* and *DMA* are acute and their sum is less than 360°, which is impossible. □

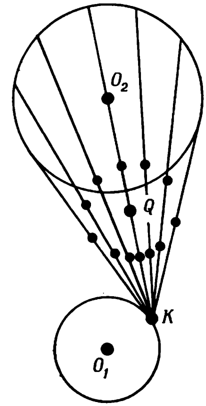
3.18*. A portion of a forest has the form of a convex polygon of area *S* and perimeter *p*. Prove that we can find a point in the forest distant more than S/p from the edge of the forest.

3.19*. A square *ABCD* is given in a plane. Find the set of points *M* such that $\widehat{AMB} = \widehat{CMD}$.

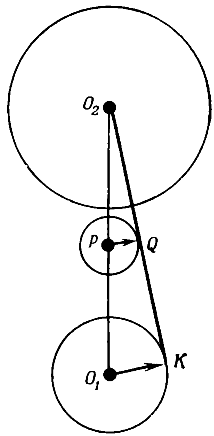
In problems that follow we have to deal with the union of an infinite number of sets.

3.20. (a) A point O is given. Consider the family of circles of radius 3 cm whose centres are located at a distance 5 cm from the point O, and the family of circles of radius 5 cm whose centres are located at a distance 3 cm from the point O. Prove that the union of the first family of circles coincides with the union of the second one.

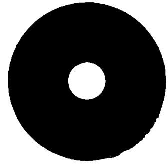
(b) Find the set of midpoints of the segments which have one end lying on one given circle and the other end on another given circle.



□ (b) Denote the radii of the given circles by r_1 and r_2 and their centres by O_1 and O_2, respectively. Let us first fix some point K of the first circle and find the set of midpoints of the segments which have one end at the point K. This set will obviously be a circle of radius $r_2/2$ with its centre Q at the midpoint of the segment KO_2. (This circle is the result of the similarity transformation of the circle (O_2, r_2) with coefficient $1/2$ and centre K.) Note that the point Q lies at a distance $r_1/2$ from the point P, the midpoint of the segment O_1O_2.



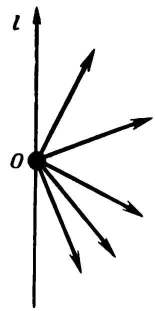
If we move the point K around the circle (O_1, r_1), the point Q will move around the circle of radius $r_1/2$, with centre at the point P. Thus, the required set is the union of all circles of radius $r_2/2$ which have their centres



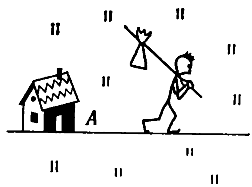
lying on a circle of radius $r_1/2$ with its centre at the point P.

What this union of an infinite number of circles turns out can be seen in the figure.

Consequently, the set of all points satisfying the condition of the problem, is a ring with external radius $(r_1 + r_2)/2$ and internal radius $|r_1 - r_2|/2$. When $r_1 = r_2$ this set becomes a circle. □



3.21. A point O is located on a straight line l, the boundary line of a half plane. In this half plane n vectors of unit length are drawn from the point O. Prove that if n is odd, the length of the sum of these vectors is not less than 1. ↓



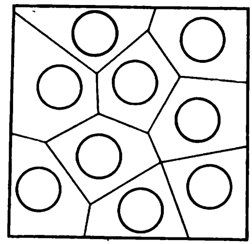
3.22. A straight road passes through a village A surrounded by meadows on all sides. A man can walk at the speed of 5 km/h along the road and at 2 km/h through the meadows. Find out the set of points to which he can walk from A in an hour.

The "Cheese" Problem

3.23. Is it always possible to cut a square piece of cheese with cavities into convex pieces so that there is only a single cavity in each piece?

Formulated mathematically, this problem is as follows.

Several pairwise non-intersecting circles are located inside a square. Is it possible to divide this square into convex polygons such that in each of them there is exactly one circle?

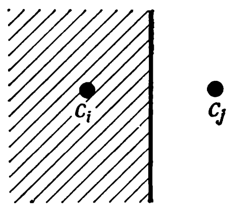


□ The answer turns out to be always positive. In any particular example when the number of circles is not large one can easily divide the square into convex polygons. But to give a general proof, we must give a method of partitioning the square, which can be used for any number and positioning of the circles.

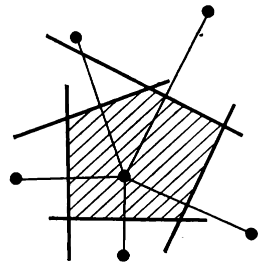
Let us first consider a more simple problem: *the radii of all the circles will be taken to be equal.* We propose the following method of partitioning the square. We shall at first describe it briefly in a single sentence.

Adjoin to each of the circles those points of the square which are nearer to this circle than to the other circles; these sets will be the required convex polygons ⟨?⟩ .

We shall explain this in more detail. Denote the centres of the given circles by $C_1, C_2, \ldots, C_n$. Let C_i be one of these centres. Let us find the set of points whose distance from C_i is not greater than the distance from the other centres C_j. The set of points of the plane which are nearer to C_i than to C_j (for a fixed j) is a half plane bounded by the perpendicular



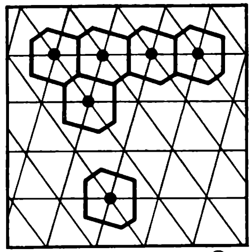
bisector of the segment C_iC_j (see **A**). We are interested in the points which are nearer to C_i than to the other centres, i.e. the points belonging to all such half planes corresponding to the different C_j $(j \neq i)$. This set of points which must be the intersection of all these $(n - 1)$ half planes, will clearly be a convex polygon. ⟨?⟩ Since each half plane contains the point C_i and the entire circle with its centre at C_i (The circles with centres C_i and C_j do not intersect and have equal radii), the intersection also contains the circle with its centre at C_i. There is such a polygon



$$\{M: \mid MC_i \mid \leqslant \mid MC_j \mid \text{ for all } j \neq i\}$$

for every centre C_i. It is clear that these polygons cover the entire square and have no interior points in common. In order to determine to which particular polygon the point M belongs, it is sufficient to answer the question "which of the centres C_i is closest to the point M?" If there are two or more such centres "closest to M" then M lies on one of the perpendicular bisectors, i.e. on a boundary line or line of partition of the polygons. Thus, the square is divided into convex polygons each of which contains exactly one circle.

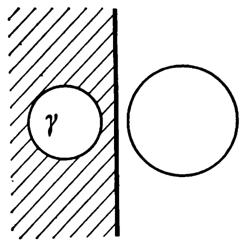
As a good example let us consider the case when *the centres of the circles* *are located at the nodes of a net formed by similar parallelograms.*



Our method of partition can be simply described in the following way.

Draw the minor diagonals in all the parallelograms of the net. This will yield a net with the same nodes and made from similar acute-angled triangles. Inside each triangle draw the midperpendiculars. The hexagons thus obtained, form the required partition of the square. Thus, we have analysed the case in problem **3.23** when all the circles have equal radii.

In a general case, when the *radii of the circles are different*, the square can be divided in the following manner. From each point located outside the given circles draw tangents to all the circles. The set corresponding to the circle γ will consist of the points of the circle γ and those points for which the length of the tangent to the circle γ is less than the length of the tangents to the remaining circles. This set is the intersection of several half planes containing the circle γ. The boundary lines of these half planes will be the radical axes of the circle γ and each of the other circles (see problems **2.9** and **3.5**). In this way the whole square will be represented as the union of convex polygons, with no interior points in common such that each polygon contains its own circle. □

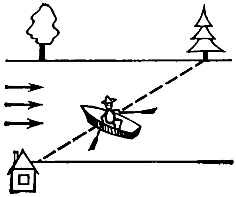


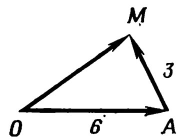
4 Maximum and Minimum

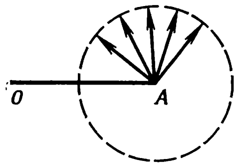
This section starts with very simple problems in which it is required to find the greatest and the least possible value of some quantity and ends with complicated research problems. Maximum and minimum problems can usually be reduced to the investigation of some function which is given analytically. But here we have collected problems where geometric considerations prove to be more effecve. You will see how in the solution of similar problems different sets of points are used.

4.1. At what angle to the bank of the river should one direct a boat so that it is taken by the current as little as possible while crossing through the river, if the speed of the current is 6 km/h and the speed of the boat in still water is 3 km/h?

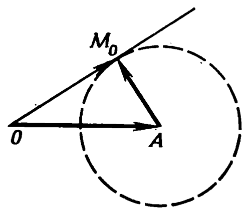
□ *Answer*: at an angle of 60°. We have to direct the boat so that its absolute velocity (the velocity relative to the bank) makes the largest possible angle with the bank ⟨?⟩ (see the Figure). Let the vector $\overrightarrow{OA}$ be the velocity of the river current and $\overrightarrow{AM}$ be the velocity of the boat relative to the water. The sum $\overrightarrow{OA} + \overrightarrow{AM} = \overrightarrow{OM}$ represents the absolute velocity of the boat. The length of the vector $\overrightarrow{AM}$ is equal to 3 and we can direct this vector arbitrarily. The set of possible positions of the point M is a circle of radius 3, with centre at the point A. It is clear that among all the vectors $\overrightarrow{OM}$, only $\overrightarrow{OM_0}$, which is directed along the tangent to the circle, makes the largest angle with the bank.





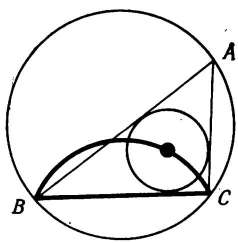


We obtain a right-angled triangle one leg of which is equal to half the hypotenuse. Such a triangle has one angle equal to 60°. □

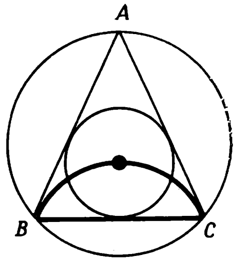


4.2. From the triangles with given base BC and $\hat{A} = \varphi$ select the one having the radius of its inscribed circle, the largest.

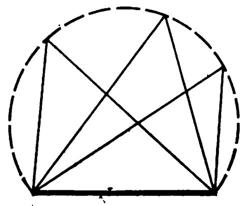
□ Let us consider the points A lying on one side of the straight line BC, for which $\widehat{BAC} = \varphi$. The set of centres of the inscribed circles of the triangle ABC is the arc of a circle



with end-points B and C (see **3.7b**). It is obvious that the isosceles triangle will have the largest radius of the inscribed circle. □

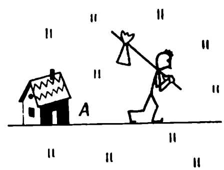


4.3. From all the triangles with a given base and a given vertex angle select the triangle with the largest area.



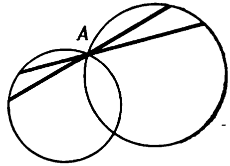
4.4. Two pedestrians walk along two mutually perpendicular roads, one at a speed of u and the other at a speed of v. When the first pedestrian crossed the second pedestrian's road, the second pedestrian still had d kilometers to go to reach the crossing. What will be the minimum distance between them? ↓

4.5. A straight road passes through a village A surrounded by meadows on all sides. A man can walk at a speed of 5 km/h along the road and at 2 km/h through the meadows (in any direction).



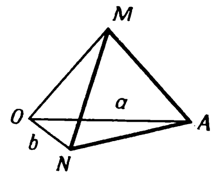
Along what route should the man walk to go in as fast a way as possible from village A to cottage B, which is situated at a distance of 13 km from the village and at a distance of 5 km from the road?

4.6. Two intersecting circles are given. Draw a straight line through the point of their intersection A such that the distance between the points of intersection (other than A) of the

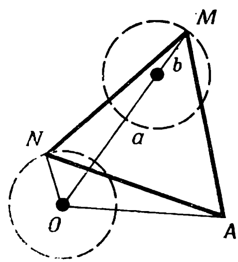


line and the circles is as large as possible. ↓

4.7. A point O is given in a plane. It is required that one of the vertices of an equilateral triangle lies at a distance a from the point O and a second vertex at a distance b. What is the maximum distance from O at which the third vertex can be situated?



□ *Answer*: $a + b$. Let AMN be an equilateral triangle for which $|OA| = a$ and $|ON| = b$. In order to answer the question we may restrict ourselves to triangles having a vertex fixed at a definite point A; for, when the triangle is rotated as a rigid body about the point O, none of the distances alter. Thus, we consider the point A fixed at a distance a from O while N runs around the circle of radius b with centre O. What position may the point M occupy? The answer has already been obtained in problem **1.9**: M lies on the circle obtained from the given circle by rotating it through 60° about the point A*. The centre O' of the rotated circle obviously lies at a distance a from the point O (for, $\triangle OO'A$ is equilateral). The radius of the rotated circle, as for the given one, is equal to b. Therefore, the



* We may take any of the circles, obtained by clockwise or anticlockwise rotation—they will lie at the same distance from O.

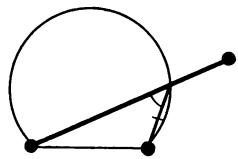
maximum distance from O to the third vertex M is equal to $a + b$. □

The following interesting corollary follows from this problem: the distance from an arbitrary point of the plane to one of the vertices of an equilateral triangle is not greater than the sum of the distances from it to the other two vertices.

4.8. What is the maximum distance at which the vertex M of a square $AKMN$ may lie from the point O, if it is known that

(a) $|OA| = |ON| = 1$;

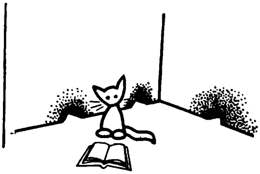
(b) $|OA| = a$, $|ON| = b$?



4.9. From all the triangles with a given base and a given vertex angle, select the one having the largest perimeter. ↓

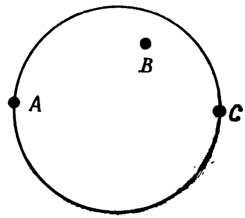
Where to Put the Point?

4.10. A cat knows the three exits A, B, C of a mouse's hole. Where should the cat sit so that its distance to the furthest exit is a minimum?



□ Let us consider circles of equal radius r with their centres at the points A, B and C. The required point K — the position of our cat—is determined as follows. We must find the minimum radius r_0 for which these circles have a common point. This is the required point K. For, if M is any other point, then it lies outside one of the circles and hence its distance from one of the vertices is greater than r_0.

In the case of an acute-angled triangle ABC, the point K is the centre of the circumscribed circle, and in the case of a right-angled or an obtuse-angled triangle ABC the point K is the midpoint of the largest side. □



□ The point K can also be found in the following way ⟨?⟩. Consider the circle of minimum radius, containing all three points. Then the point K is its centre. □

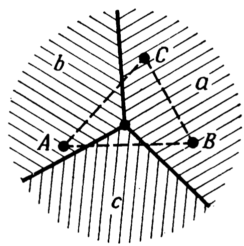
We shall give another approach to the solution of problem **4.10.**

□ Divide the plane into three sets:

(a) $\{M: |MA| \geqslant |MB|$ and $|MA| \geqslant |MC|\}$,

(b) $\{M: |MB| \geqslant |MA|$ and $|MB| \geqslant |MC|\}$,

(c) $\{M: |MC| \geqslant |MB|$ and $|MC| \geqslant |MA|\}$.



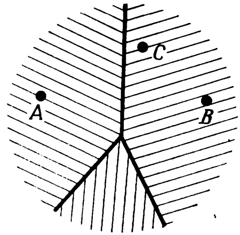
These are three angles, with their sides lying on the perpendicular bisectors of the sides of the triangle ABC. If the cat M sits in the angle (a), then the furthermost vertex from it will be A, if it sits in the angle (b) then the furthermost vertex is B, while, if it sits in (c), it is C.

If ABC is an acute-angled triangle, then in each of the three cases the

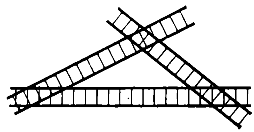
*

best thing for the cat to do is to sit at the vertex of the corresponding angle ((a), (b) or (c)), i.e. it should sit at the centre of the circumcircle.

If ABC is a right-angled or an obtuse-angled triangle, then obviously the best thing for the cat to do is to sit at the midpoint of the largest side of the triangle. □



4.11. A bear lives in a part of a forest surrounded by three straight railway lines. At which point of the forest should he build his den so that the distance from the nearest railway line is a maximum?

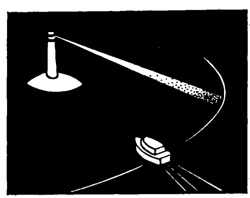


4.12*. (a) Three crocodiles live in a circular lake. Where should they lie so that the maximum distance from any point of the lake to the nearest crocodile is as small as possible?

(b) The same problem when there are four crocodiles.

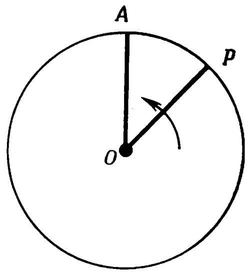
The "Motor-Boat" Problem

4.13*. A searchlight is located on a small island. Its beam lights up the sea surface to a distance of $a =$ $= 1$ km. The search light rotates uniformly about a vertical axis at a speed of one revolution in time interval $T = 1$ min. A motor-boat which moves at a speed v must reach



the island not being caught by the searchlight beam. What is the minimum value of v for which this is possible?

☐ Let us call the circle of radius a which is illuminated by the searchlight beam as the "detection circle". It is clear that for the motor-boat the best thing to do is to enter this circle at a point A through which the beam of the searchlight has just passed.



Detection circle

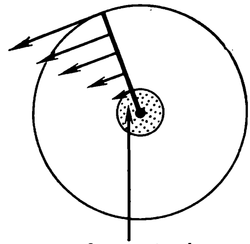
If the motor-boat heads straight for the island, it will reach the island in time a/v. In order that the beam of the searchlight does not catch it in this time, it is essential that the beam does not complete a full revolution within this time, i.e. that the inequality $a/v < T$ should hold, from which

$$v > a/T = 60 \text{ km/h}.$$

Thus, we have shown that the motor-boat may reach the island unnoticed, when $v > 60$ km/h. But, of course, it does not follow that 60 km/h is the minimum value of the speed of the boat for which this is possible, i.e. that moving along the segment AO is the best possible course which the captain of the motor-boat can select.

Indeed, as we shall see, this is not the case at all.*

Note that the linear velocity of the beam OP of the search light is different at different points: the nearer the point is to the centre, the smaller its velocity. The angular velocity of the beam is equal to $2\pi/T$. The motor-boat can easily travel ahead of the beam, around a circle of radius $r = vT/2\pi$, since the velocity of the boat here is equal to the linear velocity of the corresponding point of the beam. Outside the circle of radius r, with centre O, the speed of the beam is greater, and inside this circle (we shall call it the "safe circle") the speed of the beam is less than v.



If the motor-boat is able to reach some point of the safe circle without hindrance, then it can clearly reach the island unnoticed.

One of the possible courses inside the safe circle is a circle of radius $r/2$. If the motor-boat K moves around this circle with a speed v, then the segment KO will rotate about O with the same angular velocity with which the boat would have moved around a circle of radius r, i.e. with the same angular velocity as the beam of the search light (see problem **0.3**).

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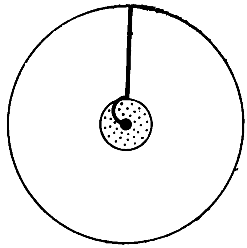
* Before reading the solution further, try to guess a route for the motor-boat to reach the island with a smaller value of v.

Hence, the boat will not be caught by the beam.



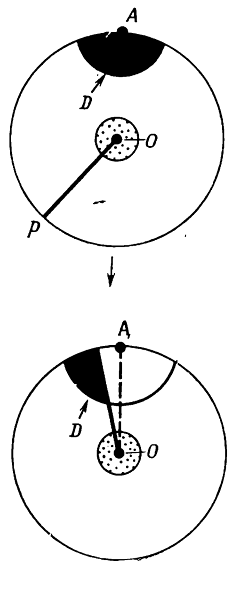
Thus, the aim of the motor-boat is to reach the safe circle!

If the motor-boat heads straight to the search light along the radius AO then it will be able to reach the safe circle without being detected by the beam of the search light, if

$$v > \frac{1}{1+(1/2\pi)}\frac{a}{T} \cong 0.862\frac{a}{T} =$$

$$= 51.7\ \mathrm{km/h}.$$

We have been able to improve our previous estimate of the minimum speed of the motor-boat. But, we shall see that even this is not the limit!



Now let us find the *minimum* value of the speed v for which the motor-boat can reach the island unnoticed.

The set of points in the detection circle which the motor-boat can reach in time t is the region bounded by an arc of radius vt, with centre at the point A. Of these points the motor-boat may reach those unnoticed which are located to the left of the beam OP.

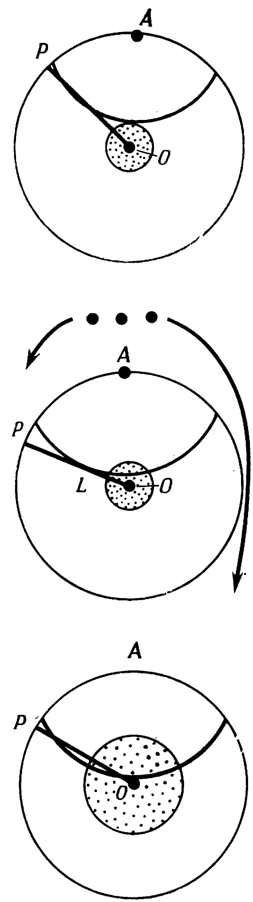
Denote the set of these 'reachable' points by D. The diagrams show how this set is changed with time until the moment, when ... here two different cases are possible.

(1) If the speed v is not sufficiently high, then at some instant t, the set D will be totally exhausted without the

safe-circle being reached: this means that in the time t, the boat will be spotted, i.e. for this speed-value the motor-boat will not be able to reach the island. Note that at the last moment $t = t_0$ the beam OP will touch the arc of radius vt_0 with centre A at some point L. Clearly the point L is located outside the safe circle (otherwise the motor-boat would be able to reach the island). Moreover, the greater the speed v, the longer is the detection time t_0 and the nearer the point L is located to the island.

(2) If the speed v is greater than some value v_0, then the set D extends to the safe circle at some point of time. This means that the motor-boat can reach the island when $v > v_0$.

The minimum value of the speed v_0 corresponds to the case, when the beam OP touches the arc of radius vt_0 right on the circumference of the safe circle. To find the value v_0, denote the value of the angle NOA by β and use the following equalities:



$$|NO| = r = \frac{v_0 T}{2\pi}, \quad |AN| = v_0 t_0,$$

$$\frac{|AN|}{|NO|} = \tan\beta, \quad \frac{2\pi + \beta}{t_0} = \frac{2\pi}{T},$$

$$|NO| = a\cos\beta.$$

From the first and last equations we find that

$$v_0 = (2\pi a \cos\beta)/T,$$

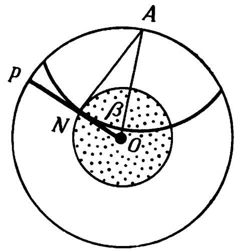
and from the first four equations we obtain an equation for β:

$$2\pi+\beta=\tan\beta.$$

We can only solve this equation approximately, for instance with the help of a computer. We get the value of β to be approximately equal to $0.92\pi/2$, hence

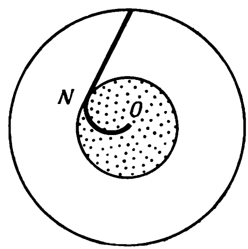
$$v_0 \cong 0.8a/T \cong 48\,\text{km/h}.$$

When the speed is greater than v_0, the motor-boat is able to reach the safe circle. □



4.14.* (a) A boy is swimming in the middle of a circular swimming pool. His father, who is standing at the edge of the swimming pool, does not know how to swim, but can run four times faster than his son can swim. The boy can run faster than his father. The boy wants to run away. Is it possible for him to do so?

(b) At what ratio between the speeds v and u (v is the speed at which the boy swims, u is the speed at which his father runs) will the boy be unable to run away?

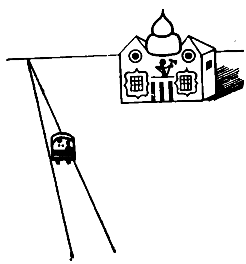


5 Level Curves

In this section the problems and the theorems of the previous section are discussed, using a new terminology. The concepts we are going to meet in this section are *functions defined on a plane and their level curves*. These are useful especially in the solutions of the maximum and minimum problems.

The "Bus" Problem

5.1. A tourist bus is travelling along a straight highway. A palace is situated by the side of the highway, at some angle to the highway. At what point on the highway should the bus stop for the tourists to be able to see the facade of the palace from the bus in the best possible way?



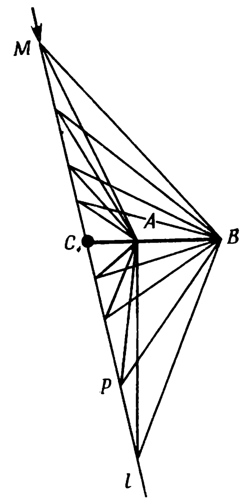
Mathematically the problem may be formulated as follows.

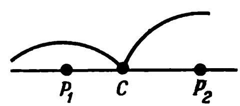
A straight line l and a segment AB, which does not intersect it are given. Find on the straight line l a point P

for which the angle APB assumes its maximum value.

Let us first have a look at how the angle AMB changes, when the point M moves along the straight line l. In other words, let us look at the behaviour of the function f which relates each point M of the line to the size of the corresponding angle $\widehat{AMB}$.

It is easy to draw a rough graph of this function. (Remember that a graph is drawn in the following way: above each point M of our straight line a point at a distance of $f(M) = \widehat{AMB}$ is plotted.)

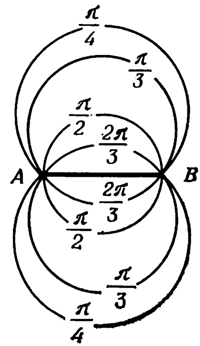




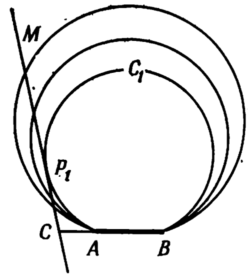
The problem may be solved analytically: introduce coordinates on the straight line l, express the value of the angle AMB in terms of the x-coordinate of the point M and find for what value of x, the function obtained reaches its maximum. However, the formula for $f(x)$ is quite complicated.

We shall give a more elementary and instructive solution. But to do this we have to study how the value of the angle AMB depends on the position of the point M in the whole plane (and not only on the straight line l).

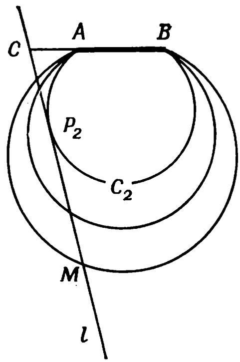
□ A set of points M in the plane, for which the angle AMB assumes a given value φ is a pair of symmetric arcs with their end-points at A and B (see Sec. 2, proposition **E**). If these arcs are drawn for different values of φ (where $0 < \varphi < \pi$), we get a family of arcs which cover the whole plane except the straight line AB. In the figure a few of these arcs are drawn and on each of them is marked its corresponding value of φ. For example, a circle with diameter AB corresponds to the value $\varphi = \pi/2$.



We shall now consider only the points M on the straight line l. From them we have to select that point for which the angle AMB assumes its maximum value. Through each point there passes some arc of our family: if $\widehat{AMB} = \varphi$, the point M lies on the arc corresponding to the value φ. Thus, the problem is reduced to the following: from all the arcs crossing the line l, select the one which corresponds to the maximum value of $\widehat{AMB} = \varphi$.

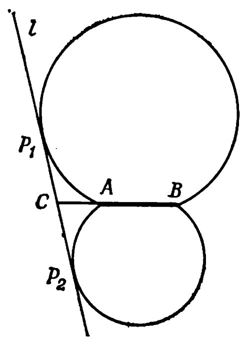


We shall examine the part of the straight line l located to one side of the point C, the point of intersection of the straight line AB with l. (We shall not consider the case, when the segment AB is parallel to the line l—we leave that to the reader.) We shall draw the arc c_1 touching this part of the straight line and prove that the segment AB subtends the maximum angle at the point of tangency P_1. Any point M of the straight line l, except P_1, lies outside the segment cut off by the arc c_1. As we know (proposition E, page 40), from this it follows that $\widehat{AMB} < \widehat{AP_1B}$.



It is obvious that on the other side of the point C everything will be exactly the same: the point P_2, at which the angle subtended by the segment AB is a maximum, is also the point of tangency of the straight line with one of the arcs of our family.

We have thus proved that the required point P of our problem coincides with one of the points P_1 or P_2 at which the circles passing through the points A and B touch the straight line l.



We should select as P the point for which the angle PCA is an acute angle. If the segment AB is perpendicular to the line l, then from symmetry considerations it is immediately obvious that the points P_1 and P_2 are completely equivalent; hence the number of points, solving the problem, in this case is two. (However the tourists, in any case must select that point P_1 or P_2 from which the facade of the palace is visible.)

Functions on a Plane. The main idea of the solution of problem **5.1** is to investigate over the whole plane the function f which relates each point to the corresponding angle value $\widehat{AMB}$, i.e. $f(M) = \widehat{AMB}$.

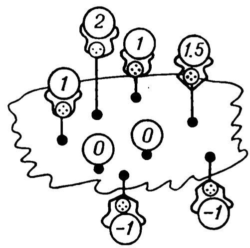
In the previous sections we have already encountered various types of function. Apart from the most simple functions on a plane such as $f(M) = |OM|$, $f(M) = \rho(l, M)$, $f(M) = \widehat{ABM}$ (where O, A, B are given points and l is a given straight line), we considered the sums, the differences, and the ratios of such functions, as well as other combinations of them.

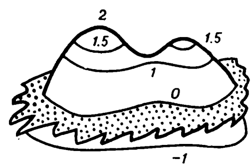
Level Curves. Most of the conditions by which our sets of points were defined, can be represented in the following way. On a plane (or on some region of it) a function f is given and it is required to find the set of points M for which this function assumes a given value h, i.e.

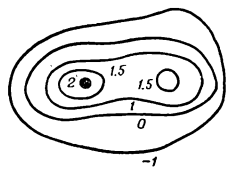
$\{M: f(M) = h\}$.

As a rule, for every fixed number h this set is some line; thus the plane is divided by the lines which are called the *level curves* of the function f. So, by solving problem 5.1 we have drawn the level curves of the function $f(M) = \widehat{AMB}$.

Graph of a Function. Let us now explain where the term "level curves" comes from. For the functions defined on a plane, we may draw graphs in exactly the same way as is done for the functions $y = f(x)$, defined on a straight line, now the graph has to be drawn in space. Let us suppose that the plane on which our function f is defined is horizontal, and for each point M of this plane let us plot the point located at a distance $|f(M)|$ above the point M, if $f(M) > 0$ and at a distance $|f(M)|$ below the point M, if $f(M) < 0$. The points plotted in such a manner usually form some surface, which is called the *graph of the function* f. In other words, if we introduce a coordinate system Oxy on the horizontal plane and direct an axis Oz vertically upwards, then the graph of the function will be the set of points with coordinates (x, y, z), where $z = f(M)$ and $(x; y)$ are the coordinates of the point M on the plane. (If the function is not defined for all the points of the plane, but only in some region, then the graph will be located only above the points of this domain of definition.)

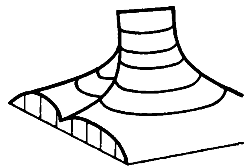






Hence, the level curve $\{M: f(M) = h\}$ consists of those points M above which the points of the graph are located at the same level, namely, at the height h.

On pages 98-99 we have shown the graphs of the functions, whose level curves represent the sets of our alphabet. Thus, we can see that the graph of the function $f(M) = \widehat{AMB}$ is a "mountain range" of height π above the segment AB, from which the graph gradually comes down to zero. (Remember that we have constructed the graph of this function at the very beginning of the solution of problem **5.1**, but only above a particular straight line l.)



A function f of the form

$$f(M) = \lambda_1 \rho(M, l_1) + \lambda_2 \rho(M, l_2) + \dots + \lambda_n \rho(M, l_n),$$

as mentioned in Sec. 2 (the theorem on the distances from straight lines), may be written as a linear expression

$$f(x, y) = ax + by + c$$

on each of the pieces Q, into which the plane is divided by the straight lines $l_1, l_2, \ldots, l_n$.

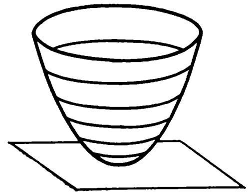
Its graph will thus consist of pieces of planes, either inclined or horizontal (if $a = b = 0$). This can be seen in the examples of sets given in propositions **C**, **I**, **J** of the "alphabet".

The level curves of such a function consist of pieces of straight lines, while if the graph has a horizontal plane, then one of the level curves includes the whole piece Q of the plane.

A function f of the form

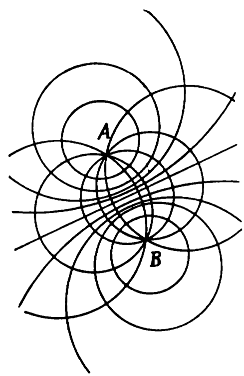
$$f(M) = \lambda_1 |MA_1|^2 + \lambda_2 |MA_2|^2 + \ldots + \lambda_n |MA_n|^2$$

when $\lambda_1 + \lambda_2 + \ldots + \lambda_n = 0$, also reduces to a linear function on the whole plane (e.g. proposition F) and in the general case, when $\lambda_1 + \lambda_2 + \ldots + \lambda_n \neq 0$, to a function of the form



$$f(M) = d\,|MA|^2,$$

where A is some point in the plane. Its level curves are circles (see the theorem on the squares of the distances in Sec. 2), and the graph is the surface of a *paraboloid of rotation.*



The functions $f(M) = \widehat{AMB}$ and $f(M) = |AM|/|BM|$ have perhaps the most complicated graph of our "alphabet". Note that there is an interesting relation between the maps of the level curves of these functions; if they are drawn on a single diagram, then we get two different families of circles, however, any circle of one family crosses any circle of the other family at right angles ⟨?⟩ Hence these families are said to be *orthogonal.*

We shall give one more example of a simple function, whose level curves are rays issuing from a single point and whose graph is a quite complicated surface. The function is $f(M) =$

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Here the graphs of functions corresponding to the propositions of "alphabet" are depicted and under each of them there is a map of the level curves.

C. $f(M) = \rho(M, l)$. The graph is a two-sided angle, the level curves are pairs of parallel lines.

D. $f(M) = |MO|$. The graph is a cone, the level curves are concentric circles.

E. $f(M) = \widehat{AMB}$. The graph is a mountain with its peak in the form of a horizontal segment, at the ends of which there are vertical drops.

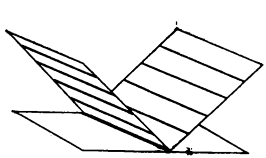
F. $f(M) = |MA|^2 - |MB|^2$. The graph is a plane, the level curves are parallel straight lines.

G. $f(M) = |MA|^2 + |MB|^2$. The graph is a paraboloid of rotation, the level curves are concentric circles.

H. $f(M) = |MA|/|MB|$. The graph has a depression near the point A and, near B it rises to infinity. The level curves are nonintersecting circles, whose centres lie on the straight line AB, each pair of which however has the same straight line, the perpendicular bisector of the segment AB, as radical axis.

I. $f(M) = \rho(M, l_1)/\rho(M, l_2)$. The graph is obtained in the following manner: consider a saddle-shaped surface—the "hyperbolic paraboloid" passing through the straight line l_1 and the vertical straight line passing through the point of intersection O of l_1 and l_2. The part of this surface lying below the given plane is reflected symmetrically relative to it. The level curves are pairs of straight lines passing through the point O.

J. $f(M) = \rho(M, l_1) + \rho(M, l_2)$. The graph is a four-sided angle. The level curves are rectangles with their diagonals belonging to l_1 and l_2.



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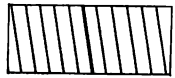

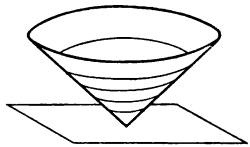

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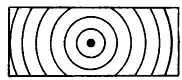

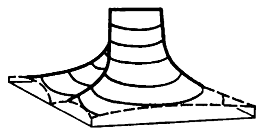

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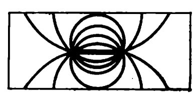

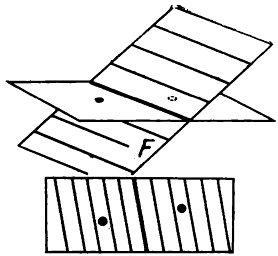
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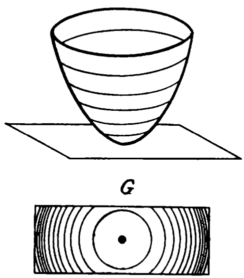
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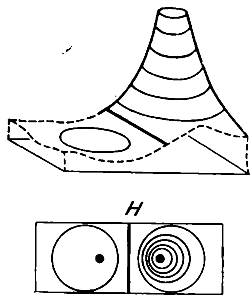
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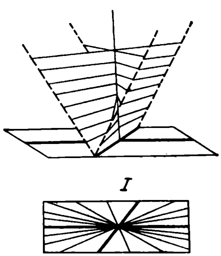
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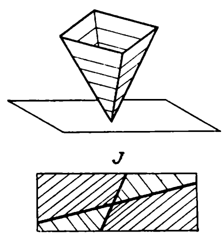
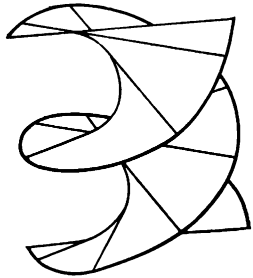
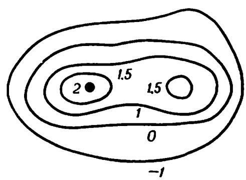

J

$= \widehat{MAB}$ (where A and B are given points of a plane). Its graph above each of the half planes into which the straight line AB divides the plane, is a *spiral surface* like the surface of a screw which is called the *helicoid*.



The Map of a Function. As we see, for many functions, it is difficult to draw their spatial graphs. It is easier, as a rule, to visualize the behaviour of the function on a plane by drawing the map of its level curves.

The physico-geographical maps are made in the following manner. Let $f(M)$ be the height of the surface above the sea-level, at the point M. Then the level curves $\{M: f(M) = 200 \text{ m}\}$, $\{M: f(M) = 400 \text{ m}\}$, etc. are depicted. The regions between these level curves are coloured with different colours: for instance, the region $\{M: 0 < f(M) < 200 \text{ m}\}$ is coloured green, the region $\{M: f(M) > 200 \text{ m}\}$—brown and the region $\{M: f(M) < 0\}$—various shades of blue.



To make the map of a function one must draw several level curves, as many as are needed to be able to judge from them where the other curves are and mark each of them with the value of the function they correspond to (i.e. the value of h).

If we decide to depict the level curves at equal intervals of the function values $0, \pm d, \pm 2d, \ldots$, then we can estimate the inclination of the graph from the density of the level curves: where there are more lines the inclination of the graph to the horizontal plane is greater.

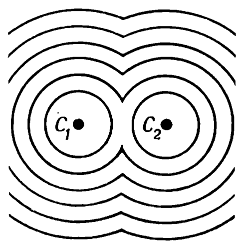
Boundary Lines. In the solution of problem **3.23** (on "the cheese") we considered a quite complicated function

$f(M) = \min\{|MC_1|, |MC_2|, \ldots, |MC_n|\}$,

which gives for every point M of the plane its minimum distance from the given points $C_1, C_2, \ldots, C_n$. Strictly speaking, in the solution of problem 3.23, we did not need this particular function so much, as the boundary lines connected with it partitioning the plane into polygonal regions. Let us try to visualize the map of the level curves and the graph of this function. We shall start with the simplest cases, $n = 2$ and $n = 3$.

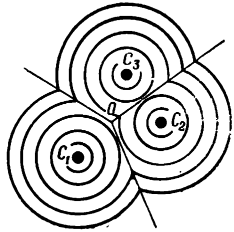
5.2. (a) Two points C_1 and C_2 are given in a plane. Draw the map of the level curves of the function $f(M) = \min\{|MC_1|, |MC_2|\}$.

(b) Three points C_1, C_2, C_3 are given in a plane. Draw the map of the level curves of the function $f(M) = \min\{|MC_1|, |MC_2|, |MC_3|\}$.



□ (a) Let us consider the set of points M for which $|MC_1| = |MC_2|$. This is, as we know, the perpendicular bisector of the segment C_1C_2. This perpendicular bisector divides the plane into two half planes. The points of one half plane are closer to C_1 and the points of the other, closer to C_2.

Thus, in one half plane $f(M) = |MC_1|$, and in the other $f(M) = |MC_2|$. Hence, in the first half plane, we must draw the curves of the function $f(M) = |MC_1|$, which are circles and reflect this map symmetrically in the perpendicular bisector.

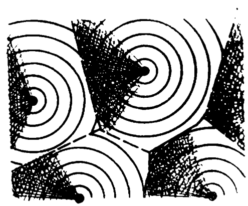


(b) Consider the sets of points, where $|MC_1| = |MC_2|$, where $|MC_2| = |MC_3|$ and where $|MC_1| = |MC_3|$. We looked at them in problem 3.1. They are the three mid-perpendiculars of the triangle $C_1C_2C_3$, which intersect at a single point O. The three rays formed by the mid-perpendiculars with their initial point at O, partition the plane into three regions. Clearly in the region containing the point C_1, $f(M) = |MC_1|$, in the region containing the point C_2, $f(M) = |MC_2|$, and in the region containing the point C_3, $f(M) = |MC_3|$. Thus, the map of the function $f(M) = \min\{|MC_1|, |MC_2|, |MC_3|\}$ is the union of three maps, joined along the lines of partition, i.e. along the three rays. □

The graph of the function

$$f(M) = \min\{|MC_1|, |MC_2|, \ldots, |MC_n|\}$$

may be visualized in the following manner. If a uniform layer of sand is placed in a box and holes are made in the bottom of the box at the points C_1, C_2, . . ., C_n through which the sand comes out, then around each hole a "funnel" is formed. The surface of all these "funnels" forms the graph of the function f. (We must of course use sand such that the angle of its natural slope is equal to 45° and we have to use a sufficiently thick layer of it.)

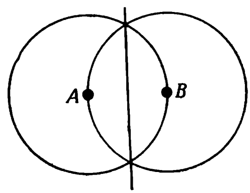


Let us now return to problems **3.11** and **3.12.** We can find in them also functions defined on a plane.

5.3. Let the points A and B be given in a plane. Draw the map of the level curves of the functions

(a) $f(M) = \max\{\widehat{AMB}, \widehat{BAM}, \widehat{MBA}\}$,
(b) $f(M) = \min\{|AM|, |MB|, |AB|\}$,

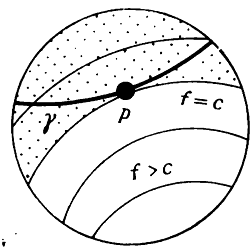
and describe their graphs.



Extrema of Functions. Let f be a given function defined on a plane. Imagine its graph as a hilly area. The maximum values of $f(M)$ correspond to the heights of the "hill tops" of its graph, and the minimum values to the depths of the valleys or depressions. On the map of the level curves of a function, the hill tops and the depressions are, as a rule, circled by level curves. For instance, for the function $f(M) = |MA|^2 + |MB|^2$, the minimum point M_0 is the midpoint of the segment AB, and the level curves are concentric circles with their centres at the point M_0.

We get a more complicated picture for the function $f(M) = \widehat{AMB}$. This function assumes its maximum value π at all the points of the segment AB, and its minimum value 0, at the remaining points of the straight line AB. The transition from the maximum to the minimum value at the points A and B is not gradual (f is not defined at these points): here the graph has vertical drops.

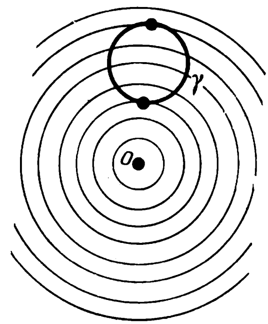
At the beginning of the section we used a map of level curves for the solution of problem 5.1. This is also a problem of finding a maximum, but of a different type. The problem may be stated generally in the following way: *Find the maximum and minimum values assumed on some curve γ by a function defined on a plane* (in the problem we looked at, γ was a straight line). The observation we made in problem 5.1 also holds for these similar problems: the *maximum (and minimum) values are usually assumed at the points where γ touches the level curves of the function f.**



* Or at the point where the function f itself reaches a maximum, if the curve γ passes through such a point.

Let us assume that the maximum value of the function f on the curve γ is attained at the point P and is equal to $f(P) = c$. Then the curve γ cannot enter the region $\{M: f(M) > c\}$—it must entirely belong to the complementary region $\{M: f(M) \leqslant c\}$. The point P lies on the line separating these regions, i.e. on the level curve $\{M: f(M) = c\}$. Thus, the curve γ cannot cross the level curve $\{M: f(M) = c\}$, i.e. it must touch this line at the point P.

You have seen how this "tangency principle" for finding an extremum arose in the problems in Sec. 4.



In these problems we looked for the maximum or minimum of the simple functions:

$$f(M) = \rho(M, l), \quad f(M) = \widehat{MOA},$$

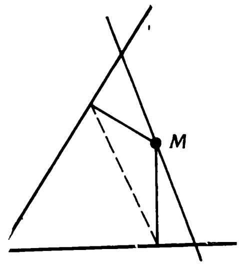
$$f(M) = |MA|$$

on a given curve γ. The level curves corresponding to the extreme value were touched by the curve γ. As a rule, this curve γ was a circle.

Some of the following problems also reduce to problems of finding the maximum (or minimum) of a function on a given circle or straight line.

5.4. (a) On the hypotenuse of a given right-angled triangle, find the point for which the distance between ts projections on the sides containing the right angle is a minimum.

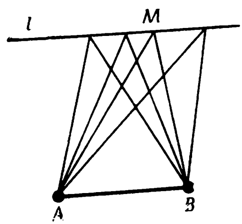
(b)* On a given straight line find a point M such that the distance between its projections on the sides containing the given angle is a minimum. ↓



5.5. Given a circle with centre O and a point A inside it, find a point M on the circle, for which the value of the angle AMO is a maximum.

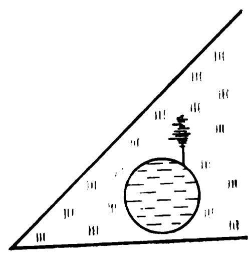
5.6. A and B are given points. Find on a given circle γ (a) a point M such that the sum of the squares of the distances from it to the points A and B is a minimum.

(b) a point M such that the difference between the squares of its distances from the points A and B is a minimum.



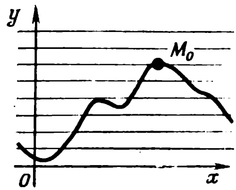
5.7. Given a straight line l and a segment AB parallel to it. Find the positions of the point M on the straight line l, for which the quantity $|AM|/|MB|$ assumes its maximum or minimum value. ↓

5.8. A lake is situated between two straight roads. Where should a sanatorium be built on the edge of the lake, so that the sum of the distances from it to the two roads is a minimum? Consider the cases when the lake is (a) circular, (b) rectangular.



Note that in finding the maximum

of a function of a single variable, $y = f(x)$, we are guided by the "tangency principle". Suppose we draw the graph of the function f on a plane. It will be some curve. To find the maximum of the function f, we must find the highest point of the graph. It is clear that, to do this we must draw a straight line parallel to the axis Ox tangent to the graph. Moreover, the tangent should be drawn in such a way that the whole graph lies below this straight line.



6 Quadratic Curves

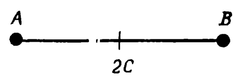
Ellipses, Hyperbolas, Parabolas. So far, we have limited ourselves to the lines which are thoroughly studied at school, namely, straight lines and circles. All the propositions of our "alphabet" from A to J involved them alone. In this section we are going to meet some other curves: ellipses, hyperbolas and parabolas. These curves, taken together, are called the "conic sections" or simply "conics", since they may all be obtained as the intersection of a plane with the surface of a cone, as is shown in the figure on pages 122-123.

Ellipses, hyperbolas and parabolas will first be defined here geometrically, as a continuation of our "alphabet" from Sec. 2. They will appear later as envelopes of families of lines. Finally, using the method of coordinates we shall find that these curves may be defined by algebraic equations of the second order. The proof of the equivalence of these definitions is not simple. However, they are all useful, each new definition allows us to solve a new class of problems with less difficulty.

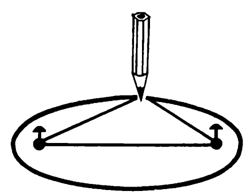
Thus, let us continue our "alphabet" with the new propositions, **K**, **L**, **M** and, a little later, **N**.

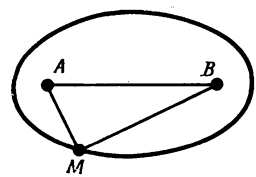
K. *The Ellipse.* Let us consider the *set of points M in a plane, the sum of whose distances from two given points A and B is equal to a constant.*

Denote this constant (as is the custom) by $2a$ and the distance $|AB|$ between the points A and B by $2c$. Note that when $a \leqslant c$ this set is of little interest: if $a < c$, then the required set is empty, as there is not a single point M on the plane for which $|AM| + |MB| < |AB|$; when $a = c$ the set is the segment AB.



To see what happens when $a > c$, proceed as follows. Fix two nails at A and B, put a loop of thread of length $2(a + c)$ around them, stretch the thread taut with a pencil, and draw a line with the pencil keeping the thread taut all the time. You will get a closed curve. This curve is called an "ellipse". The points A and B are called the *foci* of the ellipse. From the definition of ellipse it is clear that it has two axes of symmetry, the straight line AB and the straight line perpendicular to it pas-

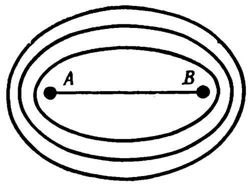




sing through the midpoint O of the segment AB. The segments of these straight lines lying inside the ellipse are called its *axes* and the point O is called the *centre* of the ellipse.

By altering the length of the thread, we can draw a whole family of ellipses having the same foci, in other words, we draw the map of the level curves of the function

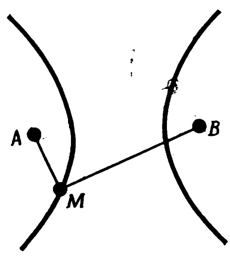
$$f(M) = |MA| + |MB|.$$



L. *The Hyperbola.* Let us consider the *set of points, the difference of whose distances from two given points A and B is equal in modulus to a constant value* $2a$ $(a > 0)$.

Let, $|AB| = 2c$ as before. If $a > c$, then the set **L** is empty, as there is not a single point M in the plane, for which $|AM| - |MB| > |AB|$ or $|MB| - |MA| > |AB|$. When $a = c$, the set **L** is a pair of rays of the straight line AB—we must exclude the segment $[AB]$ from the straight line AB.

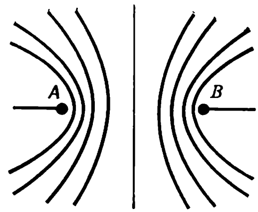
In the case when $a < c$, the set **L** consists of the two lines (branches) shown in the figure (one is the set $\{M: |MA| - |MB| = 2a\}$ and the other—$\{M: |MB| - |MA| = 2a\}$. This set is called a *hyperbola* and the points A and B are called its *foci*.



From the definition of the set **L**, it is clear that a hyperbola has two axes of symmetry. The midpoint of the segment AB is called the *centre* of the hyperbola.

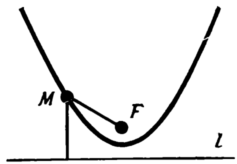
In order to get the whole map of the level curves of the function

$$f(M) = ||MA| - |MB||$$



we should also include the midperpendicular of the segment AB (it corresponds to the value $f(M) = 0$) in the family of hyperbolas with foci at A and B.

M. *The Parabola.* The *set of points* M, *equally distant from a given point* F *and a given straight line* l, *is called parabola.*



The point F is called its *focus*, and the straight line l— its *directrix*.

The parabola has a single axis of symmetry, which passes through the focus F and is perpendicular to the directrix.

Let us summarize our initial results. We have added to our "alphabet" the following sets:

K $\{M: |MA| + |MB| = 2a\}$,

L $\{M: ||MA| - |MB|| = 2a\}$,

M $\{M: |MF| = \rho(M, l)\}$.

Now we know that if a problem reduces to one of the sets **M**, **K**, or **L**, then the answer will be a parabola, an ellipse or a hyperbola respectively. Of course, in an answer not only the name of the curve but also its dimensions and its position should be indicated, for instance, by giving the foci and the number a.

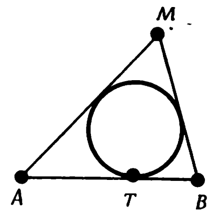
6.1. The points A and B are given in a plane. Find the set of points M for which:

(a) the perimeter of the triangle AMB is equal to a constant p,

(b) the perimeter of the triangle AMB is not greater than, p,

(c) the difference $|MA| - |MB|$ is not less than d.

6.2. A segment AB and a point T on it are given. Find the set of points M for which the circle inscribed in the triangle AMB touches the side AB at the point T.



6.3. Find the set of centres of the circles in the following cases. The circles touch:

(a) a given straight line and pass through a given point;

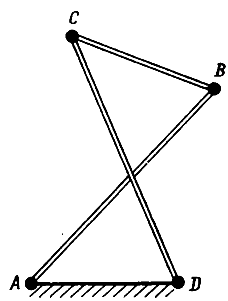
(b) a given circle and pass through a given point inside the circle;

(c) a given circle and pass through a given point outside the circle;

(d) a given circle and a given straight line;

(e)* two given circles. ↓

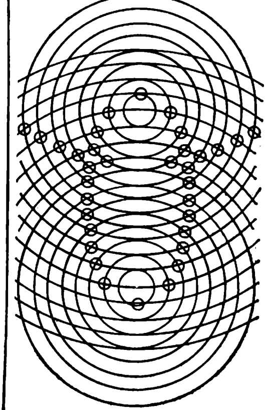
6.4. On a hinged closed polygon $ABCD$, for which $|AD| = |BC| = a$ and $|AB| = |CD| = b$, the link AD is fixed.



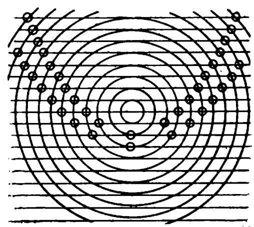
Find the set of points of intersection of the straight lines AB and CD,

(a) if $a < b$;
(b) if $a > b$.

6.5. (a) Two points A and B are given in a plane. The distance between them is an integer n (in the figure $n = 12$). All the circles with integer value radii with their centres at A and B are drawn. On the net of points obtained, a sequence of nodes (the points of intersection of the circles) is marked, in which any two neighbouring nodes are opposite vertices of a curvilinear quadrilateral. Prove that all the points of this sequence lie either on an ellipse or on a hyperbola.

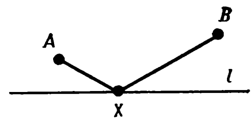


(b) A straight line l is given in a plane and on it a point F. All the circles of integer value radii with centre F and all the straight lines parallel to l and lying at some integer value distance from l are drawn. Prove that all the points of the sequence of nodes of the net, constructed as in problem (a), lie on a parabola with focus F.

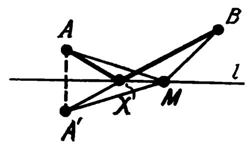


The surfaces obtained by rotating a parabola, an ellipse or a hyperbola in space about their axes of symmetry are called respectively a *paraboloid of rotation*, an *ellipsoid of rotation* or a *hyperboloid of rotation*.

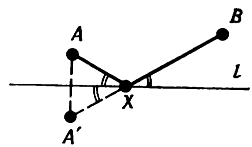
Foci and Tangents. Many interesting problems concerning ellipses, hyperbolas and parabolas are connected with the properties of the tangents to these curves. We shall obtain the basic property of tangents to the ellipse, by comparing two solutions of the following simple construction problem.



6.6. A straight line l is given together with two points A and B, one on each side of it. Find on the straight line l, a point X for which the sum of the distances $|AX| + |XB|$ from the points A and B is a minimum.

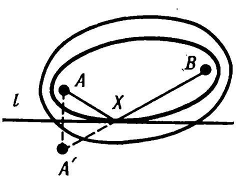


□ Consider the point A' symmetric to the point A relative to the straight line l. For any point M of this straight line $|A'M| = |AM|$. Hence the sum $|AM| + |MB| = |A'M| + |MB|$ assumes its minimum value $|A'B|$ at the point of intersection X of the segment $A'B$ with the line l. □



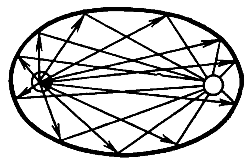
Note that the point X has the following property: the *segments* AX *and* BX *make equal angles with the straight line* l.

If we had solved problem **6.6** by the general scheme described in Sec. 5 using level curves, we would have proceeded as follows. Construct the family of ellipses corresponding to the parameter c with foci at A and B, $\{M: |AM| + |MB| = c\}$, and select from this family the particular ellipse which is touched by the straight line l.



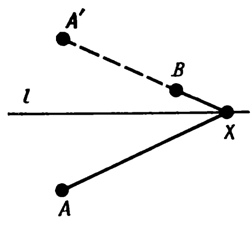
Thus, the *point* X *is a point of tangency of an ellipse* (with foci at A and B) *and the straight line* l. For, all other points M of the straight line apart from X are located outside the ellipse, i.e. for them the sum $|AM| + |MB|$ is greater.

Comparing the first solution with the second, we get the so called *focal property of an ellipse*: the *segments connecting the point X of an ellipse with its foci make angles of equal value with the tangent drawn to the ellipse at the point X*.



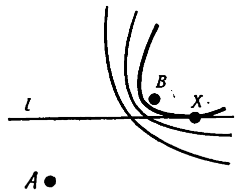
This property has an immediate physical interpretation. If the surface of a reflector (e.g. a head-light) is made in the form of a portion of an ellipsoid, and the lamp, taken to be a point source of light, is placed at one focus A, then after reflection the rays will converge at the other focus B (the word "focus" in Latin means "hearth").

The focal property of the hyperbola is completely analogous to that of the ellipse: the *segments connecting the point X of a hyperbola with its foci make angles of equal value with the tangent at the point X*. One can prove this property by solving the following problem in two different ways.

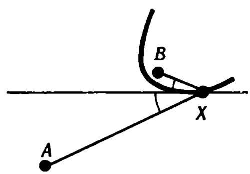


6.7. Suppose we are given a straight line l and two points A and B on opposite sides of it, where the point A, however, is located at a greater distance from l than the point B. Find the point X on the straight line for which the difference between the distances $|AX| - |BX|$ is a maximum.

One solution leads to the following answer: if we denote the point symmetric to the point A relative to the straight line l by A', then the required point X will be the point of intersection of the straight line $A'B$ with l ⟨?⟩. It is clear that for this point X, the segments AX and XB make angles of equal size with the straight line l.

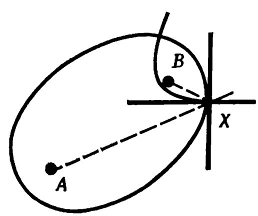


The other solution (got by the general scheme of Sec. 5) leads to the answer: X is a point of tangency of the straight line l with a hyperbola having its foci at A and B. Comparing these two answers we arrive at the focal property of a hyperbola.



There follows from the focal properties an interesting property which has to do with the families of all ellipses and hyperbolas having given foci A and B.

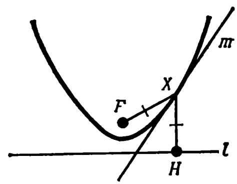
Consider an ellipse and a hyperbola passing through some point X. Draw through the point X the straight lines which make equal angles with the straight lines AX and BX. These straight lines are obviously perpendicular to each other.



From the focal properties it follows that one of the straight lines is a tangent to the ellipse and the other, a tangent to the hyperbola. Thus, the tangents to the ellipse and the hyperbola are perpendicular to each other, hence the families of ellipses and hyperbolas with foci A and B form two mutually orthogonal families (see page 97); each of the curves of one family intersects every curve of the other family at a right angle.

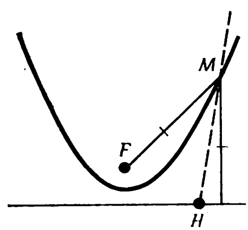
These two families can be clearly seen in the figure corresponding to problem **6.5a** if the "squares" are coloured in alternately as on a chessboard.

Focal Property of a Parabola. Suppose a parabola has focus F and directrix l, and suppose X is some point on it. Then the *straight line XF and the perpendicular dropped from X onto l make equal angles with the tangent to the parabola at the point X.*



Let us prove this. Suppose H is the foot of the perpendicular dropped from X onto l. By the definition of a parabola $|XF| = |XH|$. Therefore, the point X lies on the perpendicular bisector m of the segment FH.

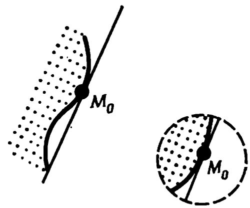
We shall prove that the straight line m is a tangent to the parabola. To do this we must show that it has only a single point in common with the parabola (namely the point X), and that the entire parabola is located on one side of m. The line m divides the plane into two half planes. One of them consists of the points M which are closer to F than to H.



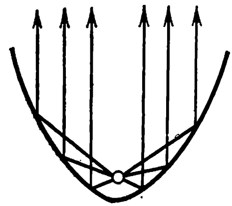
We shall show that the parabola is

located in this particular half plane, i.e. for any point M of the parabola (except the point X) $|MF| < |MH|$. This is immediate as $|MF| = \rho(M, l)$ and $\rho(M, l) < |MH|$ (the perpendicular is shorter than an inclined line).

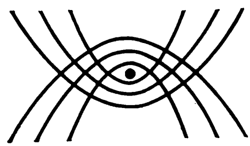
Note. For all the curves we met, the tangent was defined as follows: the tangent to the curve γ at the point M_0 is the straight line l passing through M_0 such that the curve γ (or at least a part of the curve contained in some circle with its centre at M_0) lies on one side of the straight line l.



The focal property of a parabola may be used in the following manner. If a reflector is made in the form of a paraboloid and a light source is placed at the focus F, then we have a projector: all the reflected rays will be parallel to the axis of the paraboloid.



6.8. Consider all the parabolas having a given focus and a given vertical axis. They naturally fall into two families: the parabolas of one family have their branches extending upwards and those of the other have their branches going downwards. Prove that any parabola of one family is orthogonal to any parabola of the other family.

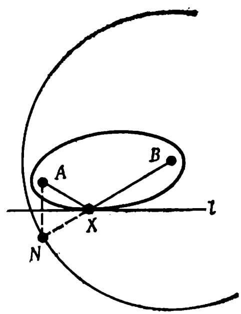


These two families of parabolas can be seen clearly if the "squares" in the figure of problem 6.5b are coloured in alternately as on a chess-board.

The solutions of the following problems depend only on the definitions of the curves and their focal properties.

6.9. (a) Suppose an ellipse with foci A and B is given. Prove that the set of points symmetric to the focus A relative to all the tangents to the ellipse is a circle.

(b) Prove that the set formed by the feet of the perpendiculars dropped from the focus A onto the tangents to the ellipse is a circle.

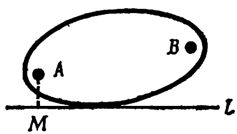


□ (a) Let l be a tangent to the ellipse at the point X and let N be a point symmetric to the focus A relative to l. Then, as we know (see problem 6.6), the point X lies on a straight line NB and the distance

$$|NB| = |AX| + |XB|$$

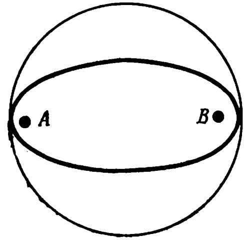
is constant. Denote this distance, as before, by $2a$. Thus, the distance between N and B is constant and the required set is a circle with centre at B and radius $2a$.

(b) Let M be the foot of the perpendicular dropped from the point A onto l. Clearly,



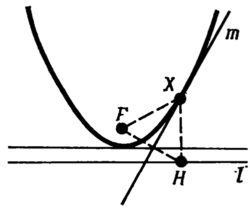
$$|AM| = \frac{1}{2}|AN|.$$

We know from problem 6.9(a) that the set of points N is a circle, so the problem reduces to the following problem. Suppose we are given a circle of radius $2a$, with centre at B and a point A inside it. Find the set of midpoints of the segments AN, where N is an arbitrary point of the circle. This set is a circle of radius a with its centre at the midpoint O of the segment AB. □



6.10. (a), (b). Prove the statements of (a) and (b) of problem **6.9** for a hyperbola.

6.11. Let there be given a parabola with focus F and directrix l.



(a) Find the set of all points symmetric to the focus F with respect to the tangents to the parabola.

(b) Prove that the set formed by the feet of the perpendiculars dropped from the focus F onto the tangents to the parabola, is a straight line parallel to l.

6.12*. (a) Prove that the product of the distances from the foci of an ellipse to any tangent is a constant (not depending on the particular tangent). ↓

(b) Find the set of points from which an ellipse subtends a right angle (i.e. the set of points where the pairs of tangents to the ellipse meet at right angles).

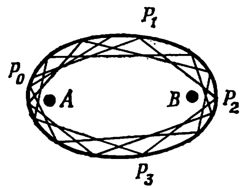
6.13*. Solve problem **6.12** (a) for a hyperbola.

6.14*. Solve problem **6.12** (b) for a parabola.

6.15*. Suppose, the trajectory $P_0P_1P_2P_3$ . . . of a ray of light inside an elliptic mirror does not pass through the foci A and B (P_0, P_1, P_2 . . . are points on the ellipse). Prove that:

(a) if the segment P_0P_1 does not intersect the segment AB, then all the following segments: P_1P_2, P_2P_3, P_3P_4, . . . also do not intersect the segment AB, and touch a single ellipse with foci at A and B; ↓

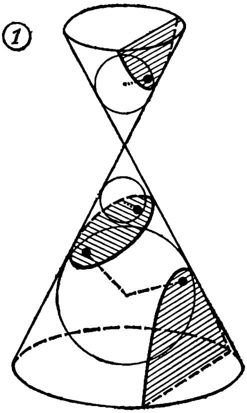
(b) if the segment P_0P_1 intersects AB, then all the following segments P_1P_2, P_2P_3, P_3P_4 . . . intersect the segment AB, and the straight lines P_0P_1, P_1P_2, P_2P_3 are all tangent to a single hyperbola with foci at A and B. ↓



Curves as the Envelopes of Straight Lines. So far, all the curves we have met—circles, ellipses, hyperbolas, parabolas—arose as sets of points satisfying certain conditions. In the following problems, these curves are generated in a different way: as envelopes of families of straight lines. The word "envelope" simply means that the curve is touched by all the straight lines of this family.

6.16. Suppose we are given a circle with centre at O and a point A. Through each point M on the circle a straight line perpendicular to the segment MA is drawn. Prove that the envelope of this family will be:

①



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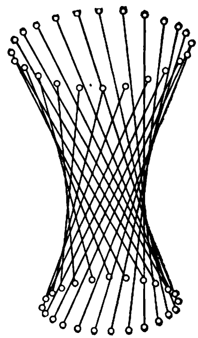
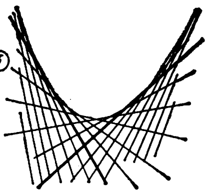


The section of a cone by an arbitrary plane (called a secant plane) not passing through its vertex is an ellipse, a hyperbola or a parab-olan(Fig. 1). If a sphere touching the secant pla e is inscribed in a cone, then the point of tangency will be the focus of the corres-ponding section and the directrix will be the line of intersection of the secant plane with the plane of the circle along which the sphere touches the cone.

The union of all straight lines which are at an equal distance from a given straight line l in space and which make a given acute angle with l is a surface known as a *one-sheet hyperboloid of rotation* (Fig. 2). The same surface can be obtained by rotating a hyperbola around its axis of symmetry l. The tangent plane to the hyperboloid at an arbitrary point intersects the hyperboloid along two straight lines. The remaining plane sections of this surface, as of a cone, are ellipses, hyperbolas and parabolas.

If the points P and N move uniformly along two intersecting straight lines, then



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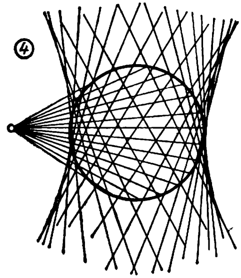

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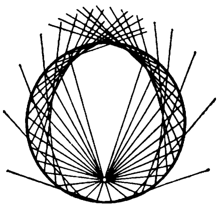

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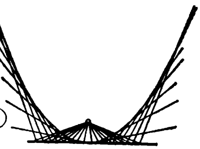

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the lines PN are either parallel to each other or (in the general case) touch a single parabola (Fig. 3). If the points P and N move uniformly along two skew lines in space, then the union of all the lines PN will be the surface of a hyperbolic paraboloid (saddle-shaped). The tangent plane to the saddle at any point on it intersects it along two straight lines; the remaining plane sections of the saddle are hyperbolas or parabolas. The saddle-shaped surface can also be obtained as the union of all the straight lines intersecting two given skew lines l_1 and l_2 and parallel to a given plane (crossing the lines l_1 and l_2).

Figs. 4-6 illustrate problems **6.16** and **6.17**. Note that on our diagrams, only the families of straight lines are drawn, however, the illusion is created that their envelopes: a hyperbola, an ellipse or a parabola, as the case may be are also drawn on them.

(a) a circle, if A coincides with the centre O;

(b) an ellipse, if A is located inside the circle;

(c) a hyperbola, if A is located outside the circle. ↓

6.17. A straight line l and a point A are given. Through each point M of the given line l, a straight line perpendicular to the segment MA is drawn. Prove that the envelope of this family of straight lines will be a parabola. ↓

These families of straight lines are depicted on pages 122-123. It is not accidental that all of them have an envelope: it can be proved that any "sufficiently nice" family of straight lines is either a set of parallel lines, or a set of straight lines passing through a single point, or in the general case, a set of tangents to some curve (the envelope of this family).

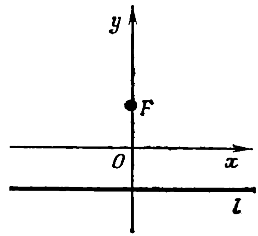
Equations of Curves. At the beginning of this section we gave geometrical definitions of an ellipse, a hyperbola and a parabola. We can obtain much more information about these curves, if the method of coordinates is used.

Let us start with the parabola. The analytical definition of a parabola as a graph of the function

$$y = ax^2 \qquad (1)$$

is well known.

We shall show how the geometric definition of a parabola given above results in this equation.



Let the distance from the point F to the straight line l be equal to $2h$. Let us choose our coordinate system Oxy so that the axis Ox is parallel to l and distant equally from F and l, and the axis Oy passes through the point F (the axis Oy will obviously be the axis of symmetry of the parabola). The equation obtained from the geometric definition of a parabola is easily transformed into (1):

$$\sqrt{x^2+(y-h)^2}=|y+h|,$$

$$\Updownarrow$$

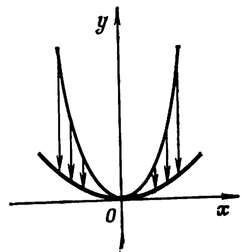
$$x^2+y^2-2yh+h^2=y^2+2yh+h^2,$$

$$\Updownarrow$$

$$y=x^2/(4h)$$

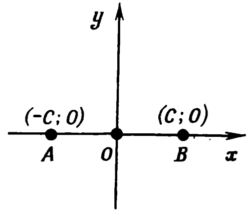
(it is sufficient to put $a = 1/(4h)$).

The graph of any function of the form $y = ax^2 + bx + c$ is also a parabola. It can be obtained from the parabola $y = ax^2$ by a parallel displacement.



In a similarity transformation $(x;\ y) \to (ax;\ ay)$ with coefficient a, the parabola $y = x^2$ becomes the parabola $y = ax^2$. Thus all the parabolas are similar to one another. But parabolas with different values of the parameter a, are of course not congruent: the larger the value of a, the "sharper the vertex" of the parabola. Note that one can obtain the parabola $y = ax^2$ from the parabola $y = x^2$ by a contraction (or extension) of one of the coordinate axes, i.e. by the transformation $(x; y) \to (x\sqrt{a}; y)$ or by the transformation $(x; y) \to (x; y/a)$.

Let us now consider the case of an *ellipse* and a *hyperbola* with foci at A and B. If their axes of symmetry are regarded as the axes Ox and Oy of a rectangular coordinate system, then the points A and B will have coordinates $A\,(-c;\ 0)$ and $B\,(c;\ 0)$, and we shall get the following equation for an ellipse:



$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=$$
$$=2a \text{ (where } a>c). \qquad (2')$$

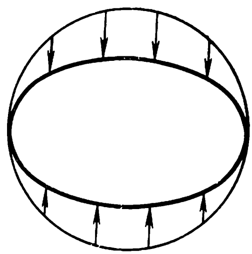
By eliminating the radicals, we can express this equation in a more convenient form:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{ where } b=\sqrt{a^2-c^2}. \qquad (2)$$

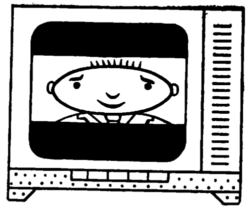
We shall briefly look at how we can get (2) from (2′).

It can be seen from equation (2) that an ellipse can also be obtained in the following way: take a circle of radius a

$$x^2 + y^2 = a^2$$



and contract it by the ratio a/b towards the axis Ox. Under this contraction, the point $(x;\ y)$ will be transformed to the point $(x;\ y')$, where $y' = yb/a$. (Substituting $y = y'a/b$ in the equation of the circle, we get the equation of an ellipse: $\frac{x^2}{a^2} + \frac{(y')^2}{b^2} = 1$.) Thus, if you have a television set you can get an ellipse without using a thread and nails; you have only to switch on the television set when the test card is transmitted and to turn the "Vertical control knob" for all the circles to be converted into ellipses. We can even do without a television set: the shadow cast by a plate, held at some angle on the top of a table, is an ellipse.



Two ellipses are similar to each other, if they have the same ratio b/a.

Taking the same coordinate system as in the case of the ellipse, we get the equation of a hyperbola

$$\left|\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}\right| = 2a \text{ where } a<c, \qquad (3')$$

or after simplification,

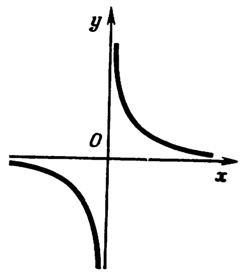
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ where } b=\sqrt{c^2-a^2}. \qquad (3)$$

In order to study the behaviour of a hyperbola in the first quadrant $x \geqslant 0,\ y \geqslant 0$, let us plot the graph of the function

$$y = \frac{b}{a}\sqrt{x^2-a^2}.$$

It is clear that this function is defined when $x \geqslant a$ and increases monotonically. It is not so clear that as x increases the hyperbola gets closer and closer to the straight line $y = \frac{b}{a} x$, i.e., that it has this straight line as an *asymptote**.

In fact, the hyperbola has two asymptotes: $y = bx/a$ and $y = -bx/a$.



One often encounters another equation, whose solution set is referred to as a hyperbola, and namely, the equation

$$xy = d \qquad (4)$$

(where d is some number, $d \neq 0$).

We have to ask ourselves whether this is some other curve or the same one.

The curve is of course the same one. To be more precise the equation $xy = d$ describes a hyperbola with perpendicular asymptotes. The standard equation (3) for such a hyperbola has the form

$$\frac{x^2}{2d} - \frac{y^2}{2d} = 1,$$

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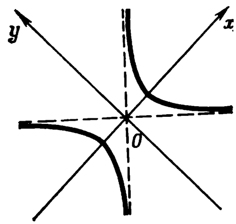
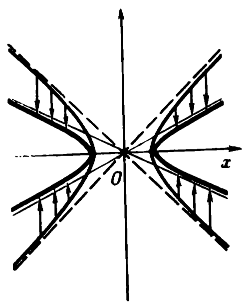
* More exactly, this means that for any arbitrary sequence x_n tending to infinity, the difference $\left| \frac{b}{a} \sqrt{x_n^2 - a^2} - \frac{b}{a} x_n \right|$ tends to zero. This can be readily proved by using the equality:

$$x - \sqrt{x^2 - a^2} = \frac{a^2}{\sqrt{x^2 - a^2} + x}.$$

but we get the equations of different types, if we use different coordinate systems. In one case we take the asymptotes of the hyperbola as the coordinate axes, in the other case, its axes of symmetry ⟨?⟩.



We have shown above how we can obtain an ellipse from the circle $x^2 + y^2 = a^2$ by contraction. In exactly the same way we can obtain the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (with arbitrary a and b) from the hyperbola with perpendicular asymptotes $x^2 - y^2 = a^2$ by a cotraction towards the axis Ox with coefficient a/b.



Two hyperbolas are similar, if they have the ratio b/a equal, or, equivalently, if they have the same angle 2γ between their asymptotes ($\tan\gamma = b/a$).

The Elimination of the Radicals. Suppose

$$z_1 = \left(\frac{\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2}}{2}\right)^2, \qquad (3'')$$

$$z_2 = \left(\frac{\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2}}{2}\right)^2 \qquad (2'')$$

Then $z_1 + z_2 = x^2 + y^2 + c^2$, $z_1 z_2 = c^2x^2$, i.e. z_1 and z_2 are the roots of the following quadratic equation:

$$z^2 - (x^2 + y^2 + c^2)\, z + c^2x^2 = 0. \qquad (5)$$

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The roots of this equation are always non-negative and $z_1 \leqslant c^2 \leqslant z_2$ because the quadratic trinomial on the left side of (5) is non-negative when $z = 0$ and non-positive when $z = c^2$.

Note that (if $z \neq 0$, $z \neq c^2$), equation (5) may be rewritten as follows:

$$\frac{x^2}{z} + \frac{y^2}{z - c^2} = 1.$$

Let $a^2 < c^2$, $a > 0$ and (3′) hold. Then $z = a^2$ is the smaller root of (5), $0 < z < c^2$, so, hence the equation

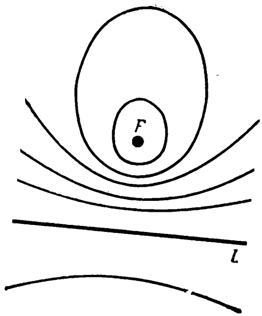
$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1 \qquad (6)$$

(provided that $0 < a < c$) is equivalent to (3′). Setting $b = \sqrt{c^2 - a^2}$, we see that (3) $\Leftrightarrow$ (3′).

Suppose $a^2 > c^2$, $a > 0$ and (2′) holds. Then $z = a^2$ is the larger root of (5), $z > c^2$. Hence equation (6) is equivalent to (2′) provided that $a > c$. Setting $b = \sqrt{a^2 - c^2}$, we obtain (2) $\Leftrightarrow$ (2′).

This proof illustrates a method which is frequently used for eliminating radicals: consider together with a given expression its conjugate expression which differs from it only in the sign before the radical.

The End of Our "Alphabet". Finally, let us consider one more function on the plane, whose map of level curves includes all the three types of curves appearing in this section. This will give us the last proposition of our "alphabet".



N. *Let there be given a point F and a straight line l, not containing the point F. The set of points, the ratio of whose distances from F and l is equal to a constant k, is an ellipse (when* $k < 1$), *a parabola (when* $k = 1$) or a *hyperbola (when* $k > 1$).

Let us prove this. Let us introduce a coordinate system as we did above in the section on the parabola. The equation of the required set

$$\frac{\sqrt{x^2+(y-h)^2}}{|y+h|}=k.$$

When $k = 1$, as we have already seen, this is equivalent to the equation of the parabola $y = ax^2$, where $a = 1/(4h)$. When $0 < k < 1$ it can be reduced to the form

$$\frac{x^2}{a^2}+\frac{(y-d)^2}{b^2}=1 \text{ (an ellipse)}, \qquad (7)$$

and when $k > 1$, to the form

$$-\frac{x^2}{a^2}+\frac{(y-d)^2}{b^2}=1 \text{ (a hyperbola)}, \qquad (8)$$

where in both cases

$$a=2kh/\sqrt{|k^2-1|}, \quad b=2kh/|k^2-1|,$$

and

$$d = h\,(k^2+1)/(k^2-1).$$

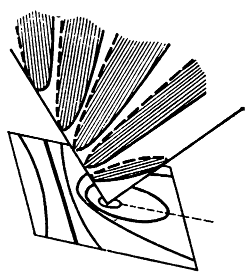
Equations (7) and (8) are obtained from the standard equations (2), (3) by a parallel displacement and an interchange of x and y. Now, the foci of the curves lie on the axis Oy, and the centres are displaced to the point $(0; d)$. It may be verified that the point F is the focus not only of the parabola but also of all the ellipses and hyperbolas. The straight line l is called their directrix.

Thus, we have seen that the set of level curves of the function

$$f(M) = \rho(M, F)/\rho(M, l)$$

consists of a single parabola, ellipses and hyperbolas.

We might have guessed that these curves must be "conic sections" (see pp. 108, 122) by reasoning as follows. Consider two functions on a plane: $f_1(M) = \rho(M, F)$ and $f_2(M) = k\rho(M, l)$. The graph of the first function (see p. 98) is the surface of a cone, the graph of the second consists of two inclined half planes (k is the tangent of the angle of inclination of these half planes to the horizontal). The intersection of these two graphs is an ellipse, a parabola or a hyperbola. The projections on the horizontal plane of these curves on an inclined plane, give the required sets:



$$\{M: f_1(M) = f_2(M)\} = \{M: \rho(M, F) = k\rho(M, l)\}$$

When projected the form of the curve changes, as if contracted towards the straight line l (in the ratio $\sqrt{k^2+1}$). Hence, our required curves are also ellipses, hyperbolas and a parabola.

As we have already repeatedly found the curves discussed in this section, the ellipse, the hyperbola and the parabola, possess many common or very similar properties. The relationship between these curves has a simple algebraic explanation: all of them are given by quadratic equations. Of course, the standard equations of these curves (1), (2), (3), (4), i.e.

$$y=ax^2,\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,\ xy=d,$$

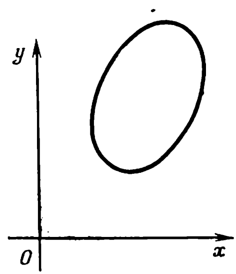
are obtained, only in specially selected coordinate systems. If the coordinate system is chosen in some other way, the equations may be more complicated. However, it is not difficult to prove that in any arbitrary coordinate system, the equations of these curves have the form

$$ax^2+bxy+cy^2+dx+ey+f=0 \quad (9)$$

(where a, b, c, d, e, f are certain numbers and $a^2+b^2+c^2\neq 0$).

It is remarkable that the converse is also true: any equation of the second degree $p(x, y) = 0$, i.e. any equation of the form (9), determines one of these curves. Let us formulate the theorem more exactly.

Equation (9) *defines an ellipse, a hyperbola or a parabola, only if the left-hand side does not decompose into factors* (if it did, we would get a pair of straight lines) *and assumes values of both signs* (if not, we would get a single point, a straight line or the empty set). The origin of the general name "quadratic curves" for ellipses, hyperbolas and parabolas becomes clear from this.



The important algebraic theorem on second-degree equations which we have formulated, is very helpful when looking for point-sets satisfying a geometrical condition: if we find that in some coordinate system this condition is expressed by a second-degree equation, then the required set is an ellipse, a hyperbola or a parabola. (Of course, in the case of degeneracy, we may get a pair of straight lines, a circle which is a particular case of an ellipse, a single point, etc.). One only has to determine their dimensions and the position in the plane (the foci, the centre, the asymptotes, etc.).

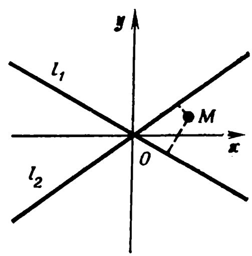
6.18. Find the set of points, the sum of whose distances from two given mutually perpendicular straight lines is a distance c greater than the distance from their point of intersection.

6.19. Given a straight line l and a point A in a plane, find the set of points:

(a) the sum of whose distances from A and l is equal to c;

(b) the difference of whose distances from A and l is equal (in modulus) to c;

(c) the ratio of whose distances from A and l is less than c, where c is a positive constant.



6.20. Find the set of points

(a) the sum,

(b) the difference

of the squares of whose distances from two given intersecting straight lines l_1, l_2 is equal to a constant d. Draw the map of the level curves of the corresponding functions:

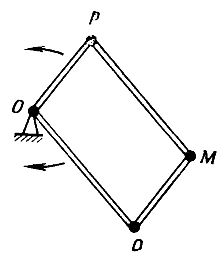
(a) $f(M) = \rho^2(M, l_1) + \rho^2(M, l_2)$,

(b) $f(M) = \rho^2(M, l_1) - \rho^2(M, l_2)$.

6.21. Given a point F and a straight line l in a plane, draw the map of the level curves of the functions:

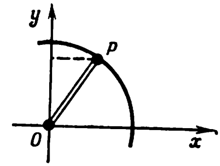
(a) $f(M) = \rho^2(M, F) + \rho^2(M, l)$,

(b) $f(M) = \rho^2(M, F) - \rho^2(M, l)$.



6.22. The vertex O of a hinged parallelogram $OPMQ$ is fixed while the sides OP and OQ rotate with an equal (in value) angular velocity, in opposite directions. Along what line does the vertex M move?

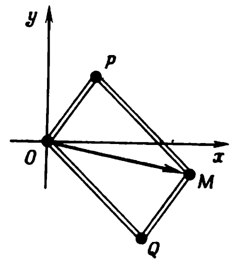
□ Let $|OP| = p$, $|OQ| = q$. Since OP and OQ rotate in opposite directions, they will coincide at some point of time. Take this as the initial point of time $t = 0$, and the coincident lines as the axis Ox (we take the origin of the coordinate system to be the point O).



Let the sides OP and OQ rotate with angular velocity ω. Then the coordinates of the points, P, Q at the time moment t will be equal to

$(p \cos \omega t;\ p \sin \omega t)$,

$(q \cos \omega t;\ -q \sin \omega t)$, respectively.



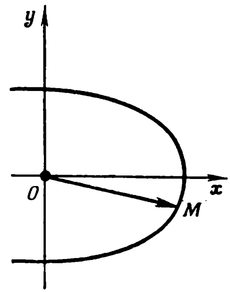
Hence, the coordinates of the point $M(x;\ y)$ will be

$$x = (p + q) \cos \omega t,$$

$$y = (p - q) \sin \omega t$$

(since $\overrightarrow{OM} = \overrightarrow{OP} + \overrightarrow{OQ}$). Therefore, the point M describes an ellipse

$$\frac{x^2}{(p+q)^2} + \frac{y^2}{(p-q)^2} = 1. \quad \square$$

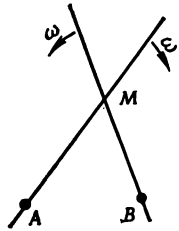


In the solution of this problem we obtained the ellipse as a set of points $(x,\ y)$ of the form

$$x = a \cos \omega t, \quad y = b \sin \omega t \qquad (10)$$

(where t—is an arbitrary real number). Equations of this type, expressing the coordinates $(x,\ y)$ in terms of an auxiliary parameter t are called *parametric* equations. In this particular case, the variable parameter t represents time.

6.23*. In a plane, two straight lines, which pass through two fixed points A and B, rotate about these points with equal angular velocities. What line does their point of intersection M describe, if the lines rotate in opposite directions? ↓



6.24*. Find the set of points M in a plane, for which $\widehat{MBA} = 2\widehat{MAB}$, where AB is a given segment in the plane. ↓

6.25*. (a) Consider all the segments cutting off a triangle of area S from a given angle. Prove that the midpoints of these segments lie on a hyperbola H whose asymptotes are the sides of the angle. ↓

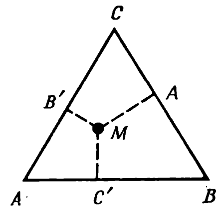
(b) Prove that all these segments touch the hyperbola H. ↓

(c) Prove that the segment of a tangent to the hyperbola cut off by the asymptotes is bisected at the point of tangency. ↓

6.26*. (a) Given an isosceles triangle ABC ($|AC| = |BC|$).

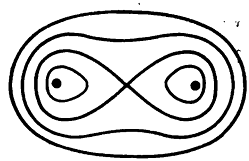
Find the set of points M in a plane such that the distance from M to the straight line AB is equal to the geometric mean of the distances from M to the lines AC and BC.

(b) Three straight lines intersecting each other form an equilateral triangle. Find the set of points M such that the distance from M to one of these straight lines is equal to the geometric mean of the distances from it to the other two.



6.27*. A rectangle $ABCD$ is given in a plane. Find the set of points M such that $\widehat{AMB} = \widehat{CMD}$.

Algebraic Curves. Obviously, the sets of points which one may meet in geometrical problems are not limited to straight lines and quadratic curves. Let us give two examples.



The set of points, the product of whose distances from two given points F_1 and F_2 is equal to a given positive number p, is called the *oval of Cassini*. A whole family of these curves—the family of level curves of the function

$$f(M) = \rho(M, F_1)\,\rho(M, F_2)$$

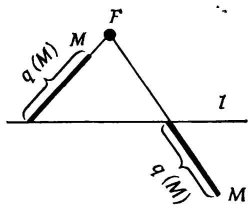
is shown in the figure.

Equations of these curves may be written as follows:

$$((x - c)^2 + y^2)((x + c)^2 + y^2) = p^2.$$

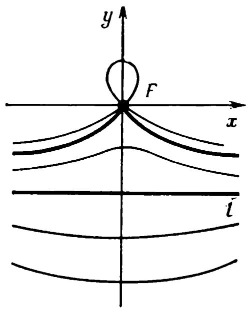
The oval of Cassini has the particularly interesting form of a "figure eight", when $p = c^2$. When $p < c^2$, the curve consists of two separate parts surrounding the points F_1 and F_2.

Here is the other example. Let a point F and a straight line l be given. Denote the distance of a point M from the point of intersection of the straight lines FM and l by $q(M)$. The set of points $\{M: q(M) = d\}$ is called the *conchoid of Nicomedes*. Its equation in the coordinate system where F is the origin and l is given by the equation $y + a = 0$ is expressed as follows:



$$(x^2 + y^2)(y + a)^2 - d^2y^2 = 0.$$

In general, the curve given by the equation $P(x, y) = 0$, where $P(x, y)$ is a polynomial in x and y, is called an *algebraic curve*. The degree of the polynomial P (provided that it does not factorize) is called the *order* of the curve. Thus, the oval of Cassini and the conchoid are the curves of the fourth order. It is already clear from these two examples that algebraic curves (of order higher than 2) may look somewhat peculiar, may possess singular points (the cusps, as the conchoid when $a = d$, or the points of self-intersection) and the form of these curves may change sharply when the parameters are changed. We shall meet some new curves in the next section.

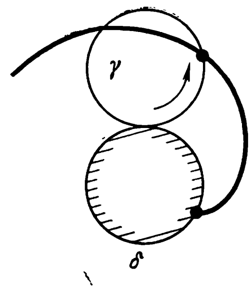


7 Rotations and Trajectories

In this last section we shall present the reader remarkable curves which are naturally generated as trajectories of points of a circle rolling along a straight line or a circle. Their most interesting properties are connected with tangents. At the beginning of the book, we said that the envelope of the family of segments in problem **0.1** about the cat is a curve with four cusps, the astroid. The reader will find the explanation of this fact here and will also see why the light spot in a cup formed by reflected rays has a characteristic singularity, a cusp. The devotee of classical geometry will find out about the connections between the nine-point circle of a triangle, its Simson's lines and their envelope, the cycloid with three cusps.

We shall first study one of the most simple cycloids in detail.

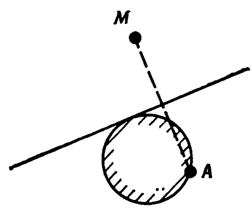
The Cardioid. Usually this curve is defined as the path of a point moving in the following way: a circle rolls without slipping around a stationary circle of the same radius. The locus of a point on the moving circle is called a *cardioid*.



It is possible to give other geometrical definitions of the cardioid. We shall give two of them in the form of a problem.

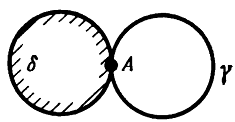
7.1. Prove that:

(a) the set of points symmetric to a particular point A of a given circle relative to all the possible tangents to this circle is a cardioid;



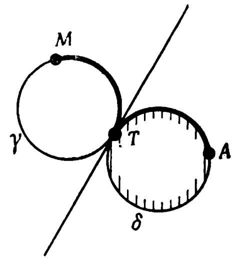
(b) the set consisting of the feet of the perpendiculars dropped from the point A of a given circle onto all the possible tangents to the circle is a cardioid.

☐ (a) Let us consider a circle γ which touches a given circle δ at the point A and has the same radius as δ. Let us roll the circle γ around the circle δ and let us follow the path of the point M of the moving circle which at the initial point of time coincides with the point A.



We assume that the circle rolls without slipping. This means that at every instant the lengths of arcs AT and MT are equal (T is the variable point of contact between the circles). Hence the point M is symmetric to the point A with respect to the tangent drawn through the point T.

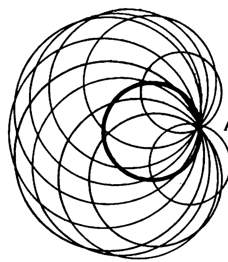
In a single revolution the point T runs around the whole of the circumference of the circle δ, and M around the entire cardioid.



(b) Clearly, we can obtain this set from the one mentioned in (a) by a similarity transformation with coefficient 1/2 and centre A. Hence it is also a cardioid of half the size of the cardioid in (a). □

Using problem **7.1** we can plot as many points of the cardioid as we please and so draw it quite accurately. This is a closed curve having at the point A a characteristic singularity, a "cusp". Its shape resembles the cross-section of an apple, somewhat in the shape of a heart, from which its name comes (Kardia—heart).

The next beautiful definition of a cardioid, in which it is generated as an "envelope of circles" also follows from problem **7.1**.

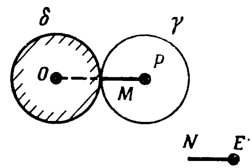


7.2*. A circle with a point A on it is given. Prove that the union of all the circles passing through the point A having their centres lying on a given circle is a region bounded by a cardioid. ↓

Addition of Rotations. We are now going to discuss ways of determining the geometric properties of curves with the help of kinematics. The cardioid will serve as an example. But before proceeding further, let us discuss the last sentence of the solution to problem 7.1 (a).

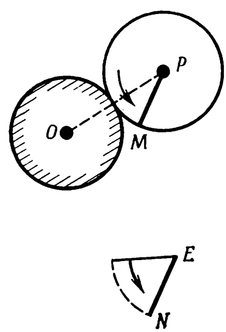
We said that the point T returns to the initial point A *after one revolution*. As we are dealing with several rotations this phrase needs to be made more precise: what is a "revolution", i.e. what rotation are we talking about?

What we meant was that the centre P of the moving circle γ (and therefore the point of tangency T) makes one revolution. But the circumference of the circle γ itself (we can visualize it better as a circular plate) rotates about its centre P quite quickly. Let us look at this question.

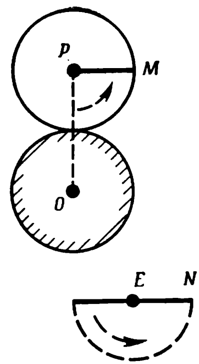


7.3. Suppose the centre P of the moving circle γ, rolling along a stationary circle δ of the same radius, makes one revolution. How many revolutions will the circle γ make about its centre P during this time?

□ In order to follow the rotation of the circle, let us draw some radius PM in it, fix somewhere in the plane a point E and consider a segment EN such that $\overrightarrow{EN} = \overrightarrow{PM}$. Our question is: how many revolutions will the segment EN make about its end-point E while the segment OP rotates through 360°? In other words, what is the ratio of the angular velocities of these segments?



To answer this question, it is sufficient to consider two different positions of the moving circle. One can see from the figure that when the radius OP turns through 90°, the segment EN turns through 180°. Continuing further in the same way we see that when the radius OP turns through 360°, the segment EN will turn through 720°, i.e. will make two complete revolutions (the ratio of angular velocities is equal to 2). This gives us the answer to problem **7.3.** □



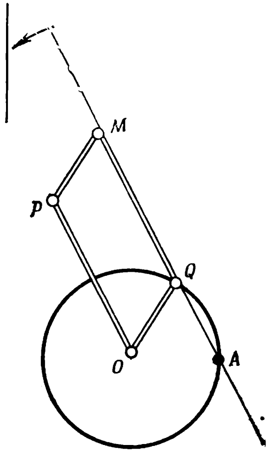
If we take the centre O of the stationary circle as the point E in the solution of problem **7.3** and mark off from it the segment $\overrightarrow{OQ} = \overrightarrow{PM}$, then we obtain the parallelogram $OPMQ$.

In the uniform rolling of the circle γ around δ, the vertex O is motionless and the sides OP and OQ rotate with the angular velocities ω and 2ω respectively (in the same direction). Thus, we obtain another definition of the cardioid using the convenient model of a hinged parallelogram:

If the sides OP and OQ ($|OP| = 2|OQ|$) rotate about the point O with angular velocities ω and 2ω, the locus of the fourth vertex M of the parallelogram $OPMQ$ is a cardioid.

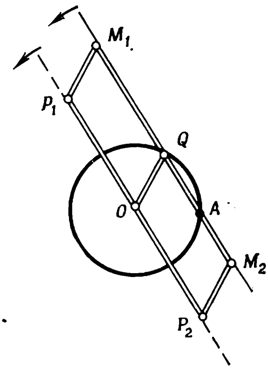
It is now easy to give one more method of constructing the points of a cardioid, and to obtain some more interesting properties of it.

7.4. If on every straight line l passing through the point A on a given circle δ of radius r, we mark off from the point of intersection Q of l and δ $(A \neq Q)$ the segment QM of length $2r$, then the set of all points M obtained will be a cardioid.



□ For every position of the straight line l we may construct a parallelogram $OPMQ$ where Q and M are as stated in the problem. Then, if the straight line l rotates about the point A with an angular velocity ω, the sides OP and OQ of the parallelogram will rotate with exactly the necessary velocities ω and 2ω (according to the theorem about the ring on a circle in Sec. 1), and so the point M will describe a cardioid. □

Try to construct a cardioid on a large sheet of paper using problems **7.1** and **7.4** and convince yourself that you obtain the same curve. Perhaps the second method is even more convenient. Note that in problem **7.4** we may mark off the segment QM of length $2r$ from the point Q in both directions. Doing this we obtain two points M_1 and M_2 of the cardioid. They correspond to two opposite positions of the hinged parallelogram (if the point Q makes one full revolution and returns to the initial point, then the side OM will turn through 180° and M_1 will coincide with the



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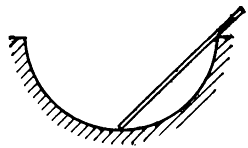
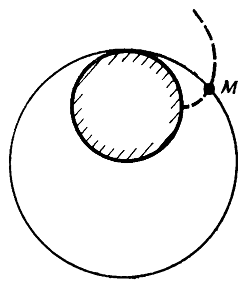
point M_2). This circumstance leads to the following property.

7.5. Prove that each chord M_1M_2 of the cardioid, passing through its cusp A has length $4r$, and that the midpoint of the chord lies on the stationary circle (of radius r) that generates the cardioid.

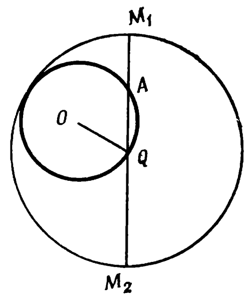
Here are two more problems, where the second method of constructing a cardioid is used.



7.6. A stick of length $2r$ moves in a vertical plane so that its lower end rests against the bottom of a hole in the ground whose vertical cross-section is a semicircle of radius r. The stick rests against the edge of the hole. Prove that the free upper end of the stick moves along a cardioid.

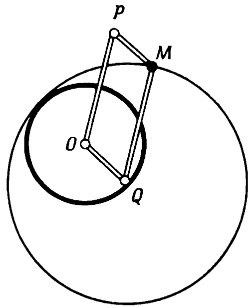


7.7. A hoop of radius $2r$ rolls around the outside of a stationary circle of radius r without slipping. Prove that the locus of a point on the hoop is a cardioid.

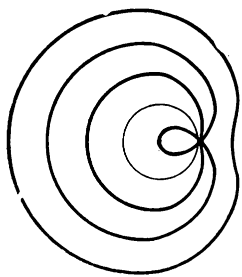


□ One of the solutions of this problem may be obtained if one compares it with Copernicus' theorem **0.3.** We are in fact dealing here with the same two circles, but the internal circle of radius r is fixed, and the external circle of radius $2r$ rolls around it. In this situation, Copernicus' theorem shows that if we fix a stick to the hoop along the diameter M_1M_2, then while rolling the stick passes through a fixed point A of the stationary circle. At the same time, the midpoint Q of the stick M_1M_2 moves around the stationary circle δ, and $|M_1Q| = |QM_2| = 2r$, so we arrive at problem 7.4 and see that the points M_1 and M_2 move on the same cardioid.

One can reason in a somewhat different way, bringing the problem to that of the hinged parallelogram. Let M be the point of the hoop we are following and Q its (variable) centre. We shall construct the parallelogram $OPMQ$. If the link OQ of the parallelogram is rotated with the angular velocity 2ω, then the hoop and with it the link QM rotates with the angular velocity ω. □



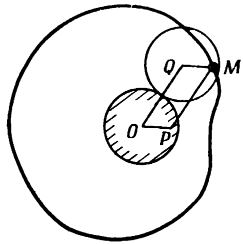
The curve which we have just been considering, the cardioid, is included in a natural way in the family of curves called *conchoids of a circle* or *Pascal's limaçon*. If in the statements of problem 7.4 we mark off on the straight line l, passing through the point A the segment QM of some constant length h (in either direction), then we get one of these curves for every $h > 0$. For $h = 2r$ the curve will be a cardioid. (Compare the definition of these curves with the definition of the conchoid in Sec. 6, page 139.) It turns out that we can give a kinematic definition of Pascal's limaçon for every h. We do this in the next problem.



7.8. (a) Prove that the vertex M of a hinged parallelogram, whose vertex O is fixed and whose sides OP and OQ rotate with angular velocities 2ω and ω, respectively, describes Pascal's limaçon.

(b) A circle of radius r is fixed in a plane. Around it rolls a circle of radius r with a moving plane rigidly fixed to it. Prove that every point of this plane describes a Pascal's limaçon.

(c) As in (b) but instead of a moving circle of radius r, we have a hoop of radius $2r$ encircling the stationary circle.

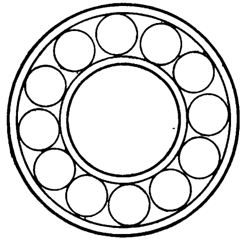


Now let us give some problems which require us to look at the addition of rotations where there is a different ratio between the velocities than we had in the case of the cardioid. We shall be reminded of some of the other cycloids shown in the figure on pp. 150-151.

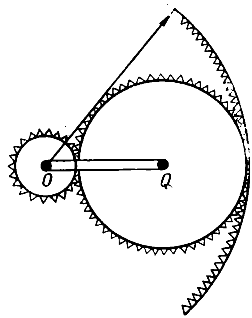
7.9. A circle of radius (a) $R/2$, (b) $R/3$, (c) $2R/3$ is rolling around the outside of a stationary circle of radius R. In each case how many revolutions will the circle make while its centre describes one revolution about the centre of the stationary circle? ↓

7.10. Solve the same problem but with the circle rolling around the inside.

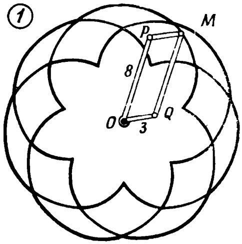
7.11. Between the axle of a bearing 6 mm in diameter and its stationary ball race of 10 mm in diameter, ball bearings of diameter 2 mm are located. When the axis rotates the ball bearings roll around the axle and the ball race without slipping. Find out with what angular velocity (a) the ball bearings rotate, (b) their centres run about the centre of the bearing if the axle rotates with an angular velocity of 100 revolutions per second.

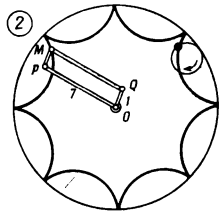


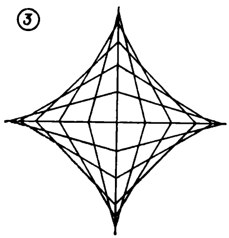
7.12. Gears setting in motion a grindstone are arranged as is shown in the diagram. Find the ratio of the radii of the moving wheels for which the smaller wheel (the grindstone) will revolve 12 times faster than the handle *OQ* which sets it in motion.

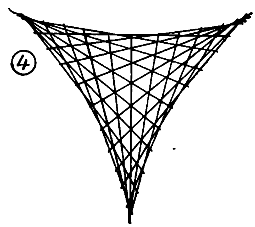


Consider two points on a circle rolling around another circle. It is clear that they must describe congruent paths. In particular, it is possible that these two paths coincide, the two points moving along the same line one following the other. This was the case, for instance, in the solution of problem 7.7 where we saw that the diametrically opposite points of a hoop described the same cardioid. We could have convinced ourselves of this by simply noticing that the paths of these points have their cusps at the same point of the stationary circle. We can use similar observations in the following problems.









A *k-cycloid* is the curve described by the vertex M of a hinged parallelogram $OPMQ$, whose vertex O is fixed and whose links OP and OQ rotate about O, and where the ratio ω_{OP}/ω_{OQ} of the angular velocities is equal to k and the ratio $|OP|/|OQ|$ of the lengths of the links is equal to $1/|k|$ $(k \neq 0, +1, -1)$.

If two points L and N move uniformly around a circle, so that the ratio ω_L/ω_N of their angular velocities is equal to k, then the envelope of the straight lines LN will be a k-cycloid **(7.19)**.

The shapes of a k-cycloid and a $(1/k)$-cycloid coincide **(7.14)**.

The k-cycloid may also be defined as the locus of a point of a circle of radius r which rolls around another circle of radius $|k-1|\cdot r$ without slipping, externally when $k > 1$ and internally when $k < 1$.

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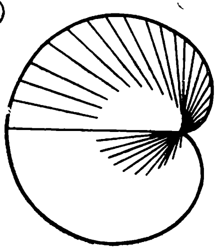

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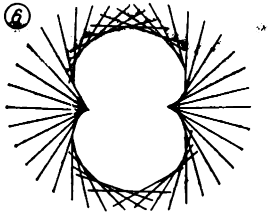

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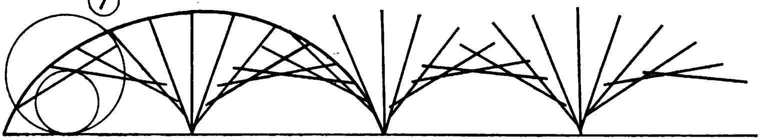
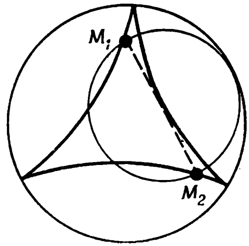


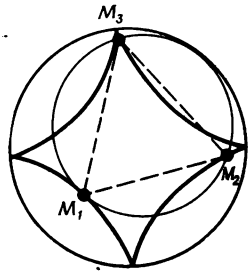
Usually k-cycloids are called *epicycloids*, when $k > 0$ and *hypocycloids*, when $k < 0$. In diagrams 1-6 k-cycloids are depicted for $k = 3/8$, $-1/7$, -3, -2, 2 and 3. The last four have special names: the *astroid*, *Steiner's curve*, the *cardioid* and *nephroid*. Several families of segments related to these curves are shown in diagrams 3-6. All the segments in each diagram have equal lengths (**7.4**, the theorem on two circles on page (154), **7.21**).

In the last diagram 7, the locus of a point of a circle rolling along a straight line is shown. This curve is known as the *cycloid*. The envelope of the diameters of the rolling circle is a cycloid of half the size (the theorem on two circles).

7.13. (a) Prove that the diametrically opposite points M_1 and M_2 of a circle of radius $2R/3$ rolling around the inside of a circle of radius R, describe one and the same *Steiner's curve.* ↓



(b) Prove that three points M_1, M_2 and M_3 lying on a circle of radius $3R/4$ at the vertices of an equilateral triangle will describe the same curve, an *astroid*, if the circle is rolled around the inside of a circle of radius R.



(c) The same problem, as in (b) where the radius is $3R/2$ instead of $3R/4$. In this case instead of an astroid we get a *nephroid* (and the movable circle encircles the stationary one like a hoop).

The three curves whose names we have just met, the *Steiner's curve* (also called a *deltoid*), the *astroid* (from astra—a "star") and the *nephroid* (from nephros—"kidney")—are obtained in these problems in a somewhat different way from the way they are defined on pp. 150-151.

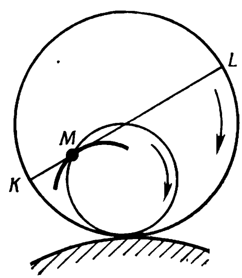
We have already seen from the example of the cardioid that a curve may be obtained as the paths of points on two different circles rolling around the one stationary circle (compare the first definition of the cardioid and problem 7.7: in the first case the centre of the moving circle is the vertex P of a hinged parallelogram $OPQM$, and in the second case the vertex Q). The following problem shows us what ratios between the radii of the circles we must take to obtain congruent paths.

7.14*. (a) Prove that a point on a circle of radius r, rolling around the outside of a stationary circle of radius R, and a point on a circle (hoop) of radius $R + r$ encircling the circle describe congruent paths.

(b) Prove that a point on a circle of radius r, rolling around the inside of a circle of radius R and a point on a circle of radius $R - r$ rolling inside the same fixed circle describe congruent paths. ↓

To solve these problems we have to learn how to calculate the ratios of the velocities of quite complicated rotations. We shall discuss how to do this below but now let us go on to the most interesting properties of cycloids, i.e. to the properties of their tangents.

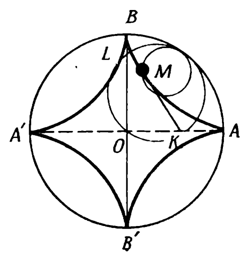
A Theorem on Two Circles. Let us formulate a curious rule, which allows us to describe the family of tangents to the trajectory of the point M of circle of radius r which rolls without slipping along a curve γ. Let us roll a circle of radius $2r$ along the same line γ, and suppose that a diameter KL of this circle (considered fixed to the circle), is chosen, in such a way that at some instant its end-point K and the point M coincide at the one



point A on the line γ. It so happens that in this case, at any point of time the *diameter KL touches the path of the point M*. In other words, the *path is the envelope of all the positions of the diameter KL*.

We have called this very convenient rule the "*theorem on two circles*". We shall discuss its proof later on but first let us make things a little clearer. If we roll the two circles mentioned in the theorem simultaneously, so that their points of tangency with the curve γ always coincide, then the smaller circle will roll around the bigger one without slipping. Then, from Copernicus' theorem, the point M will move along a fixed diameter KL of the bigger circle. Our theorem on two circles asserts that the straight line KL will be a tangent at the point M to the locus of this point M.

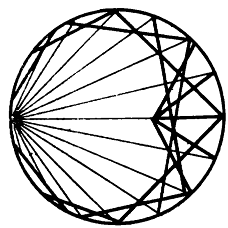
Let us pass on to the examples. Let us begin with the family of curves which we spoke about in the introduction to the book. Suppose a circle of radius r with the point M marked rolls around the inside of a circle of radius $R = 4r$. Roll together with it a circle of radius $2r$ along with its diameter KL (at the initial moment the points K and M coincide with the point A on the stationary circle). According to Copernicus' theorem the end-points of the diameter KL slide along two mutually perpendicular diameters AA'



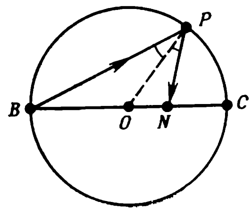
and BB' of the stationary circle. At the same time, according to the theorem on two circles, the diameter KL during its motion touches the trajectory of the point M, i.e. the *envelope of the straight lines* KL *is an astroid with cusps at the points* A, B, A', B'.

The next problem is about the cardioid.

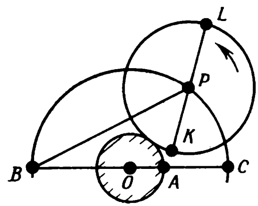
7.15*. A point B is given on a circle. From it a ray of light falls on any arbitrary point on the circle and is then reflected from the circle (the angle of incidence is equal to the angle of reflection). Prove that the envelope of the reflected rays is a cardioid.



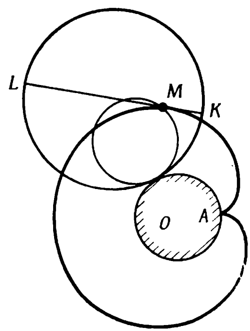
□ Let us denote the centre of the "reflecting" circle by O and the point diametrically opposite the point B by C. Suppose the ray BP after being reflected at the point P arrives at the point N of the segment BC (we consider for the time being that $\widehat{PBC} \leqslant 45°$). Then $\widehat{PNC} = \widehat{BPN} + \widehat{PBN} = 3\widehat{PBC}$. This means that, if we rotate the ray BP with an angular velocity ω, then the reflected ray will rotate with an angular velocity 3ω, and the point of reflection P will move around the reflecting circle with an angular velocity 2ω (according to



the "theorem about the tiny ring" from Sec. 1). Clearly, this will also be the ratio when $\widehat{PBC} > 45°$.

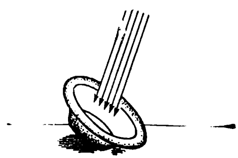


We can get the family of straight lines PN in which we are interested in the following way. Let us roll a circle of radius $2r$, together with its diameter KL, which at the initial moment lies along the straight line BC, around a fixed circle of radius $|OB|/3$ with its centre at O. If the centre P of the moving circle runs (around the circle of radius $3r$ with centre O) with an angular velocity 2ω, then the diameter KL will rotate with an angular velocity 3ω ⟨?⟩ — just as the reflected ray did.



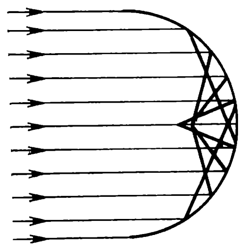
By the theorem about two circles, the envelope of the family of straight lines KL will be the trajectory of the point M of the circle of radius r rolling around a circle of the same radius r with centre O, i.e. a cardioid. At the initial moment the point M coincides with the point A dividing the segment BC in the ratio 2:1. This point will be the cusp of the cardioid. □

We often see this "cusp", in the form of a spot of light formed by reflected rays at the bottom of a cup or a sauce-pan inclined to incident rays from a lamp or the sun. However, in such cases it is more natural to consider the pencil of incident



rays as being parallel and not coming from a single point on the circle. We do not then get a cardioid, but another curve with a similar cusp also known to us.

7.16*. Prove that if a parallel pencil of rays falls on a mirror having the form of a semicircle as is shown in the diagram, then the reflected rays touch half of a nephroid.



If the mirror were parabolic then as we know from Sec. 6, the reflected rays would come together at a single point, the focus of the parabola. This comparison explains the other name of the nephroid: the *focal line of a circle.*

7.17. Find the set of points which cover the fixed diameter of a circle of radius r, rolling

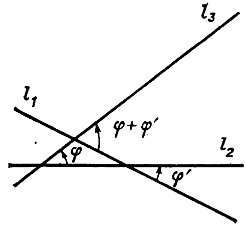
(a) around the outside of a circle of radius r;

(b) around the inside of a circle of radius $3r/2$.

A few more interesting problems about families of tangents appear below, after we discuss the kinematic concepts used in the solution of the last problems and in the proof of the theorem on two circles.

Velocities and Tangents. There are more convenient ways of determining the ratios of the angular velocities in the complicated rotations we have been considering than the quite primitive method we used in solving problem **7.4.** First of all there is the rule for adding angular velocities similar to the rule for the addition of linear velocities when changing to a new reference frame.

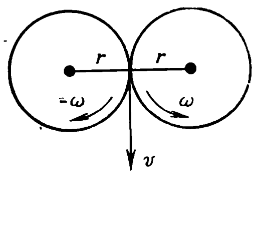
Let us take angles (and angular velocities) corresponding to anticlockwise rotations as being positive, and angles and rotations in a clockwise direction as being negative.



Then if the straight line l_2 is turned relative to the straight line l_1 through angle φ' and l_3 turned relative to l_2 through an angle φ, then l_3 turns with respect to l_1 through the angle $\varphi + \varphi'$.

Thus, if *the figure* γ_2 rotates with respect to the *"fixed" figure* γ_1 *with an angular velocity* ω', and γ_3 *with respect to* γ_2 *with an angular velocity* ω, *then* γ_3 *rotates with respect to* γ_1 *with the angular velocity* $\omega + \omega'$ (as we formally deal with rotations of circles, we shall suppose that some radius is marked on each of them in order to follow their rotations more easily).

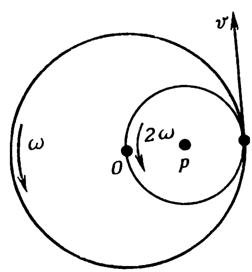
Let us show how we can apply this rule. Let us first consider two circles of radius r whose centres are fixed at a distance $2r$ from each other. If they rotate without slipping, then their angular velocities are equal in value and opposite in sign: the first has angular velocity $-\omega$ and the second has angular velocity ω. This



is because the linear velocities of the points of tangency of one and the other circles are equal (the fact that the circles rotate without slipping is used here). Since the value v of the linear velocity of the point M located at a distance r from the centre of the circle rotating with the angular velocity ω, is equal to $v = \omega r$, then from the equality of the *linear* velocities we get the equality of the *angular* velocities of the circles (in absolute value).

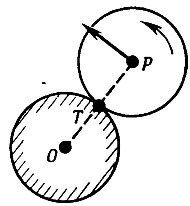
Now let us pass to a reference frame fixed to the first circle. We then have to add ω to all angular velocities: the angular velocity of the first circle will be 0 while the angular velocity of the second circle will be 2ω. We have already seen this in problem **7.4.**

And now another example. Suppose, the distance between the (fixed for the time being) centres O and P of two touching circles of radii $R = 2r$ and r is equal to r. Their angular velocities will be ω and 2ω respectively (the ratio of these values is inversely proportional to the ratio of the radii). In a reference frame fixed to the larger circle, their angular velocities are: $-\omega$ and 0 (this was the motion which we spoke about in Copernicus' theorem **0.3**). In a reference frame fixed to the smaller circle, their angular velocities are 0 and ω (see problem **7.7**).

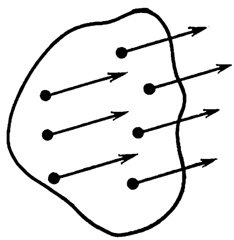


It is possible however, when determining the angular velocities, to avoid the introduction of a rotating reference frame. To do so, we must clarify how to find the (linear) velocities of the points on a rolling circle (a wheel). This question is of great importance in the next section, which deals with tangents to cycloids.

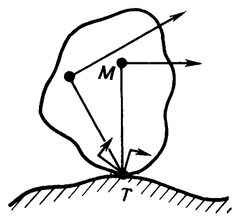
Thus, we return to the first example: let us consider some position of a circle of radius r, rolling around a circle of the same radius; denote by T the point on the moving circle coinciding at the moment considered with the point of tangency of the circles. Its velocity is equal to 0 (since the rotation is without slipping). How do we find the velocities of the other points?



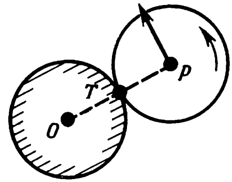
Let us use for this the following *theorem of Mozzi*:



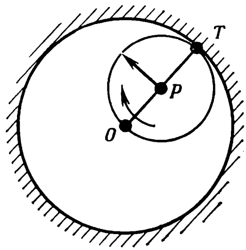
At any point of time, the *velocities of the points of a solid plate, which moves in a plane are either those of a body in translation* (i.e. are all equal in value and have the same direction) *or those of a rotating body*, i.e. the velocity of some point T is equal to zero and the velocity of every other point is equal in magnitude to $|MT|\,\omega$ (where ω is the angular velocity of the plate) and is perpendicular to the segment MT. This last case in particular takes place for a rolling circle and the point of tan-



gency plays the role of the point T ("the instantaneous centre of rotation"). (This will be true, even for an irregular wheel rolling on a bumpy road.) Making use of this, we can find the ratio of the angular velocity ω_1 of the rolling wheel to the angular velocity ω_2 with which its centre P rotates about the centre O of the stationary circle. To do this we express the linear velocity of the point P in two different ways: on the one hand, its value is equal to $2r\omega_2$, on the other hand it is equal to $r\omega_1$ since T is the instantaneous centre. Hence, $2r\omega_2 = r\omega_1$, and so $\omega_1 = 2\omega_2$.



The same argument for a circle of radius r, rolling around the inside of a circle of radius $2r$ so that its centre moves (around a circle of radius r) with the angular velocity $\omega_2 > 0$, gives us the following. Denote the angular velocity of the circle by ω_1 and note that $\omega_1 < 0$. Expressing the velocity of the point P in two different ways we get: $|\omega_1 r| = |\omega_2 r|$, giving $\omega_1 = -\omega_2$.



Similar reasoning helps us when studying other complex rotations.

But what is particularly important for us is that Mozzi's theorem allows us to find the *direction* of the velocity at every point of the figure: the velocity of the point M is directed perpendicularly to the segment MT join-

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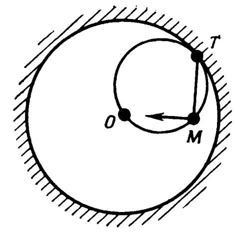
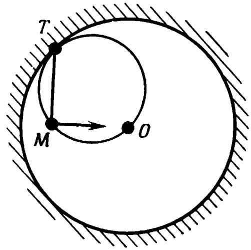
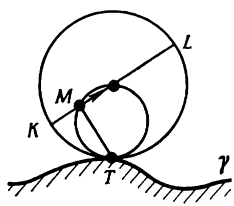
ing M with the instantaneous centre of rotation T.

We shall give one more proof of Copernicus' theorem. Let the point M be a point on a circle of radius r which rolls inside a circle of radius $2r$ with centre O. At any point of time, the velocity of the point M is perpendicular to the segment TM, where T is the point of contact of the circles (and the instantaneous centre of rotation of the smaller circle). Thus, the velocities of the point are always directed along the straight line MO (since T and O are diametrically opposite points on the smaller circle). Thus, the point M moves along a diameter of the larger circle, which is just what Copernicus' theorem asserts.



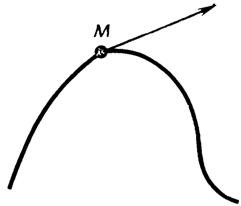


We now give the proof of the *theorem on two circles*. Let us simultaneously roll two circles of radii r and $2r$ along the curved (or straight) line γ. Let M and K be points on them which coincide at the initial moment with the point A of the line γ, and let T be the common instantaneous centre of rotation of the two circles (their point of contact with γ). The velocity of the point M is directed perpendicularly to the segment MT.

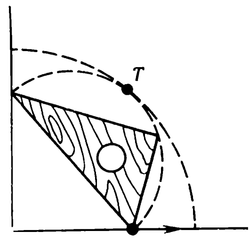


Thus, the *velocity* of the point M *is directed along the diameter of the larger circle*, that is, M lies on a certain diameter KL of this circle, and in its motion, the straight line KL touches the path of the point M. This is just the theorem on two circles.

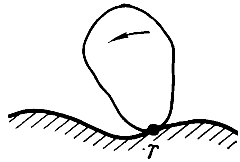
Note that here we have looked at the definition of the tangent to a curve in a new way. The *tangent at the point M to the path of a moving point is the straight line passing through the point M on the path whose direction coincides with the direction of the velocity at the given point M.*



We shall not give the proof of Mozzi's theorem, but we shall point out its geometrical analogue: any displacement of a plane which can be realized without turning the plane over onto the other side (moving it in any possible way in the plane), is either parallel displacement or rotation about some point T (Chasles's theorem). In connection with Mozzi's theorem we shall stress one more thing. In the case of the most general movement of a plate in a plane the instantaneous centre T changes its position not only in the stationary plane but also in the moving one (the plate) during the process of movement. In each case it describes some curve, one is called the *fixed centrode* and the other, the *moving centrode.* For instance, during the rolling of a wheel along a road, the fixed centrode would be the road and the moving centrode would be the rim of the wheel. In kinematics a theorem is



proved, which states that for every "smooth" enough motion of a plane, i.e. motion without "jerks", *the moving centrode rolls along the fixed one without slipping* and at each moment their point of contact is the instantaneous centre of rotation.

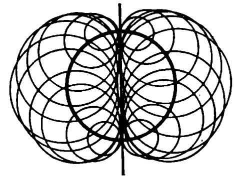


Thus the general motion of a plate in a plane reduces to the rolling of an irregular wheel on a bumpy road. From this point of view the subject matter of our section could be summarized as follows: the study of motions for which both centrodes are circles. With that we come to the end of our digression into kinematics*. We are now armed well enough to set about discovering some most remarkable properties of cycloids, those connected with the families of tangents to these curves.

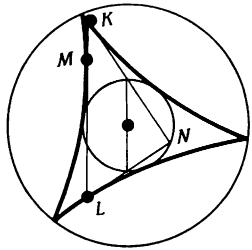
7.18. Prove that the tangents to a cardioid at the end-points of a chord passing through the cusp of the cardioid are mutually perpendicular, and that their point of intersection is located at a distance $3r$ from the centre of the stationary circle, where r is the radius of this circle. ↓

* One can find more detailed and precise explanations in any good text-book on theoretical mechanics, for example, in the wonderful *Leçons de Mécanique Analytique*, by the Belgian mathematician C. J. de La Vallée-Poussin.

7.19*. Two pedestrians L and N walk at a constant speed around a circle. The ratio of their angular velocities is k (k is not 0, 1 or -1). Find the envelope of all the straight lines LN. ↓



7.20*. A circle and a straight line passing through its centre are given. Prove that the union of all the circles whose centres lie on the given circle and which touch the given straight line is a nephroid.

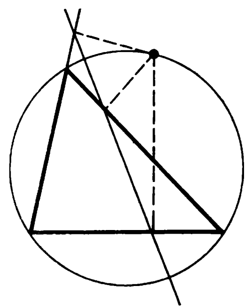


7.21*. Consider Steiner's curve described about a circle of radius $2r$ (the inscribed circle). Prove that an arbitrary tangent to this curve (at some point M) intersects the curve in two points K and L such that the segment KL has a constant length $4r$, and its midpoint lies on the given inscribed circle, the tangents to the curve at the points K and L are mutually perpendicular and intersect at the point N lying on the inscribed circle and the segments KN and LN are bisected by the inscribed circle. ↓

7.22*. Consider an astroid described about a circle of radius $2r$. Prove that from an arbitrary point of the inscribed circle P, it is possible to draw three straight lines PT_1, PT_2, PT_3 tangent to the astroid such that they form equal angles (of 60°) with each other and the three points of tangency T_1, T_2, T_3 are the vertices of a right-angled triangle inscribed in a circle of radius $3r$, which touches the circle described about the astroid.

The next and last problem in this series, which also may be solved using the language of motion, reveals an unexpected connection between the elementary geometry of a triangle and a cycloid. This curve is called after the geometer who discovered this connection.

7.23*. A triangle ABC is given. Prove that:



(a) the feet of the three perpendiculars from any point on the circumcircle of the triangle to lines AB, BC and AC lie on a single straight line (called *Simson's line*);

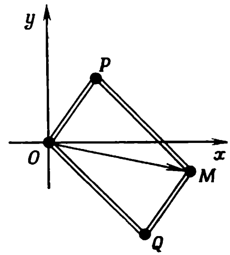
(b) the midpoints of the sides of a triangle, the feet of the altitudes and also the midpoints of the segments of the altitudes joining the orthocentre to the vertices lie on a single circle (called the *nine-point circle*);

(c) all the Simson's lines of the triangle ABC touch a single Steiner's curve, described about the nine-point circle. ↓

Parametric Equations. All the properties of cycloids may also be proved analytically. It is most convenient to write their equations in parametric form, expressing the coordinates $(x; y)$ of the point M through a parameter t

(the time). We have already come across these equations in problem **6.22.**

Consider the locus of the fourth vertex M of a hinged parallelogram $OPMQ$ whose vertex O is at the origin of coordinates. (Note that $\overrightarrow{OM} = \overrightarrow{OP} + \overrightarrow{OQ}$). If the point P moves around the circle of radius r_1 with its centre at the origin O of coordinates at an angular velocity ω_1 and the point Q moves around the circle of radius r_2 with centre O at an angular velocity ω_2, then at the moment t the coordinates of P will be $(r_1 \cos \omega_1 t, r_1 \sin \omega_1 t)$, the coordinates of Q will be $(r_2 \cos \omega_2 t, r_2 \sin \omega_2 t)$, and the coordinates of the fourth vertex M of the parallelogram $OPMQ$ will be:



$$x = r_1 \cos \omega_1 t + r_2 \cos \omega_2 t$$
$$y = r_1 \sin \omega_1 t + r_2 \sin \omega_2 t$$

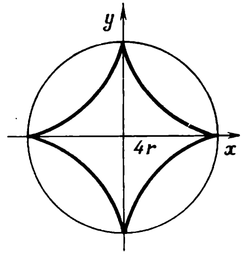
(at the initial point of time $t = 0$, the sides OP and OQ of the hinged parallelogram are both directed along the axis Ox).

In problem **6.22** we saw that when $\omega_2 = -\omega_1$ the point M describes an ellipse. In the general case, when we have the ratios:

$$\omega_1/\omega_2 = k, \quad r_2/r_1 = |k|$$

the point M describes a cycloid (which on page 150 we called the k-cycloid).

From the parametric equations by eliminating t, we get in some cases simple equations connecting the coordinates x and y. Consider as an example the astroid. For this curve we have $r_1 = 3r_2$, $\omega_2 = -3\omega_1$, we may take $\omega_1 = 1$, then $\omega_2 = -3$ and the parametric equations of the astroid will be (putting $r_2 = r$):

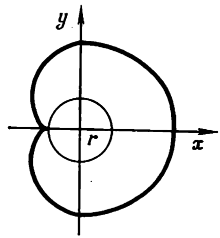


$$x = 3r \cos t + r \cos 3t$$

$$y = 3r \sin t - r \sin 3t.$$

or more simply ⟨?⟩:

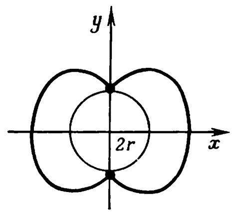
$$x = 4r \cos^3 t, \quad y = 4r \sin^3 t.$$



Hence we get the following equation of the astroid:

$$x^{2/3} + y^{2/3} = (4r)^{2/3}.$$

We can define the astroid and the other curves that we considered above by algebraic equations. Try to verify that the points $(x; y)$ of these curves satisfy the following equations:

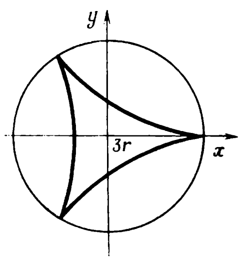


$$(x^2 + y^2 - 4r^2)^3 + 108r^2x^2y^2 = 0$$

(astroid)

$$(x^2 + y^2 - 2rx)^2 - 4r^2 (x^2 + y^2) = 0$$

(cardioid)



$$(x^2 + y^2 - 4r^2)^3 - 108x^2r^4 = 0$$

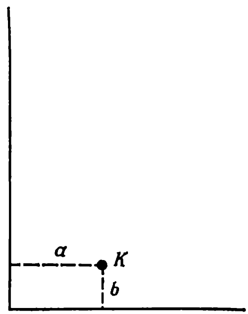
(nephroid)

$$(x^2 + y^2 + 9r^2)^2 + 8rx (3y^2 - x^2) -$$
$$- 108r^4 = 0 \quad \text{(Steiner's curve).}$$

Thus, the astroid and the nephroid are curves of the sixth order, and the cardioid and Steiner's curve are of the fourth order.

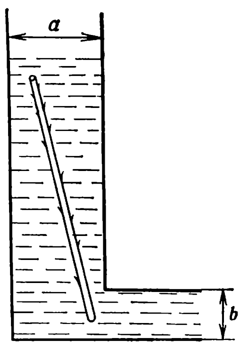
It can be proved that when $\frac{\omega_1}{\omega_2} = k$ is rational, cycloids are algebraic (and when k is irrational they are not; such curves pass arbitrarily close to any point of a ring with centre O and bounded by circles of radii $r_1 + r_2$ and $| r_1 - r_2 |$. They are said to "everywhere densely" cover this ring).

Comparing the equations of the curves with their geometric properties we get new and interesting corollaries. Here is an example where a property of the astroid is used.



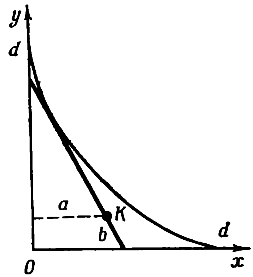
7.24. (a) Suppose we are given a right angle and inside it a point K which is distant a and b from its sides. Is it possible to draw through the point K a segment of length d with its end-points on the sides of the right angle?

(b) A canal, whose banks are parallel straight lines, has a right angled turn in it. Before the turn the width of the canal is a, and after the turn it is b. For what values of d can a thin log of length d pass round such a turn?



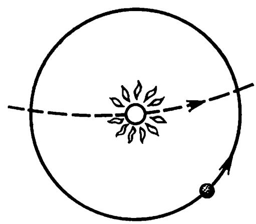
□ (a) Let us take the sides of the right angle as the coordinate axes. The segment of length d must touch an astroid whose cusps are at a distance d from the centre. The equation of such an astroid is: $x^{2/3} +$

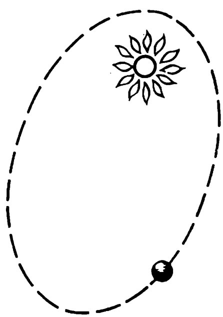
$+ y^{2/3} = d^{2/3}$. If the point K lies inside the region bounded by the astroid and the sides of the angle then the required segment exists (it is a segment of the tangent to the astroid passing through the point K). If K lies outside this region, it does not. Therefore, the necessary segment exists if and only if $a^{2/3} + b^{2/3} \leqslant d^{2/3}$. □



Note that though we have found how to "construct" the required segment when the condition $a^{2/3} + b^{2/3} \leqslant \leqslant d^{2/3}$ is satisfied using an astroid, this problem cannot be solved using ruler and compasses.

Conclusion. The remarkable curves with which we have acquainted ourselves in the last two paragraphs have been known for more than two thousand years. The basic properties of ellipses, hyperbolas, and parabolas were described in the work *On Conics* by the ancient Greek mathematician Apollonius of Perga, who lived at about the same time as Euclid (third century B.C.). Even in ancient times astronomers studied the paths of the complicated circular motions. This is not surprising. If in a very rough approximation the planets are considered to be rotating around the Sun along circles in a single plane, then the observed motions of another planet from the Earth will be some complicated circular motion. The description of planetary motions using





complicated cycloid curves was subject to more and more modifications as the number of astronomical observations increased until Johannes Kepler established that with high accuracy the trajectories of the planets are ellipses with the Sun located at one of the foci.

A wide range of problems from physics, mechanics and mathematics were connected with particular curves. These provided a "whet stone" for sharpening the powerful analytical tools invented in the seventeenth century by Descartes, Leibniz, Newton, Fermat and others. These methods enabled the transition from particular problems connected with specific curves to general laws possessed by whole classes of curves. Needless to say that we cannot do without analytical methods when designing complicated mechanisms and constructions. However, the intuitive representations to which this book is devoted sometimes prove useful, even in problems not at all connected with geometry. It is not without reason that research or computational results are frequently represented in the form of graphs or families of lines.

Answers, Hints, Solutions

1.13. Note that the vertices M of the right-angled triangles AMB with hypotenuse AB lie on the circle with the diameter AB.

1.14. Let us draw the common tangent through the point of contact M of the circles. Let it cross AB at the point O. Then $|AO| = |OB| = |OM|$ (the lengths of the tangents from the point O to the circles are equal).

1.15. *Answer*: The union of three circles. Let A, B, C and D be the given points. Draw a straight line l through the point A, a line parallel to l through the point C, and straight lines perpendicular to l through the points B and D. As a result we get a rectangle.

Let L be the midpoint of the segment AC and K the midpoint of the segment BD. Then it is easy to see that $\widehat{LMK} = 90°$, where M is the centre of the rectangle. Rotating l about the point A and rotating the other straight lines correspondingly, we find that the set of the centres M of the constructed rectangles is a circle with the diameter KL.

Since the four points A, B, C, D may be divided into two pairs in three different ways: (A, C) and (B, D); (A, B) and (C, D); (A, D) and (B, C), the entire set required consists of three circles.

1.25. *Answer*: Along a straight line. If the pedestrians P and Q move along parallel straight lines, then, clearly, the midpoint of the segment PQ also moves along a parallel straight line.

Suppose the straight lines intersect at the point O. Regard O as the origin. Then the velocities $\vec{v}_1$ and $\vec{v}_2$ of the pedestrians are vectors directed along straight lines, and their values are equal to the lengths of the paths walked by the pedestrians in unit time. Let the first pedestrian be situated at the point P at time t, and the second one at the point Q. Then $\vec{OP} = \vec{a} + t\vec{v_1}$ and $\vec{OQ} = \vec{b} + t\vec{v_2}$ (the vectors $\vec{a}$ and $\vec{b}$ define the initial positions of the pedestrians when $t = 0$).

The midpoint of the segment PQ is at the point M where

$$\vec{OM} = \frac{\vec{OP}+\vec{OQ}}{2} = \frac{\vec{a}+\vec{b}}{2} + t\,\frac{\vec{v_1}+\vec{v_2}}{2}$$

We find that it also moves along some straight line with a constant velocity $\frac{\vec{v_1}+\vec{v_2}}{2}$. In order to find this line, it is sufficient to mark the midpoint of the initial positions of the pedestrians and their positions after, say, unit time.

We may replace the vector calculations by the following geometric argument.

If P_0P_1 and Q_0Q_1 are any two (non-parallel) segments, then the segment M_0M_1, where M_0 and M_1 are the midpoints of the segments P_0Q_0 and P_1Q_1, is a median of the triangle $L_1M_0N_1$, where L_1 and N_1

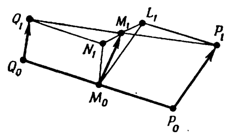


Fig. 1

are the fourth vertices of the parallelograms $P_1P_0M_0L_1$ and $Q_1Q_0M_0N_1$ (see Fig. 1; in the construction depicted, $P_1L_1Q_1N_1$ is a parallelogram, and P_1Q_1 and N_1L_1 are its diagonals).

It is now clear that if instead of P_1 and Q_1 we take points on the lines Q_0Q_1 and P_0P_1, such that $\vec{P_0P} = t\vec{P_0P_1}$ and $\vec{Q_0Q} = t\vec{Q_0Q_1}$, and the triangle LM_0N (with the median M_0M) is drawn, as before, then this triangle may be obtained simply by a similarity transformation with coefficient t and centre M_0 from the triangle $N_1M_1L_1$ (with the median M_0M_1), i.e. the point M will lie on the straight line M_0M_1 and $\vec{M_0M} = t\vec{M_0M_1}$.

1.28. Let us use Fig. 1 to problem 1.25. If the segments P_0P_1 and Q_0Q_1 rotate uniformly about the points P_0 and Q_0 with equal angular velocities (1 revolution per hour), then the triangle $N_1M_0L_1$ also rotates, along with its median M_0 as a rigid body about the point M_0 with the same angular velocity.

1.29. *Answer.* A circle. Let us translate this problem into the language of motion. Draw the radii O_1K and O_2L. Let the straight line KL rotate with a constant angular velocity ω.

Then according to the theorem "about the ring" the radii O_1K and O_2L will rotate uniformly with the same angular velocity 2ω, i.e. the size of the angle between the radii O_1K and O_2L remains constant. Thus the problem reduces to the previous one.

2.11. (b) Use proposition **F**.

2.19. *Answer*: If h is the height of the triangle ABC, then the required set is: empty when $\mu < h$, the entire triangle (Fig. 2) when $\mu = h$, the contour of a hexagon (Fig. 3) when $\mu > h$.

2.20. (b) See Fig. 4.

3.5. (b) The problem reduces to 3.5 (a) and is simply solved by "embedding in space": if three spheres are constructed with their centres in the plane α on the given circles (in the horizontal plane α)

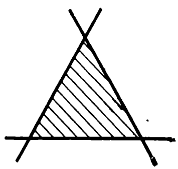


Fig. 2

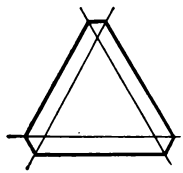


Fig. 3

and looked at from above, then we see three circles in which the spheres intersect (their projections on the horizontal plane are our three

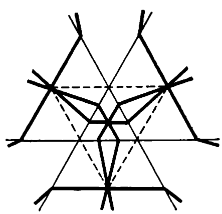


Fig. 4

chords) and also their point of intersection (its projection is the required point of intersection of the chords).

3.7. (b) Note that $\widehat{AMB} = 90° + \frac{\varphi}{2}$, where M is the centre of the inscribed circle of the triangle. According to E the set of points M is a pair of arcs together with their end-points A, B.

3.7. (c) *Answer*: The required set is a pair of arcs (see figures 5, *a*, *b*, *c*, of the corresponding cases: (a) $\varphi < 90°$, (b) $\varphi = 90°$, (c) $\varphi > 90°$).

Let l_A and l_B be two intersecting straight lines passing through the points A and B respectively and let k_A and k_B be the straight lines passing also through A and B, such that $k_A \perp l_B$, $k_B \perp l_A$. If the

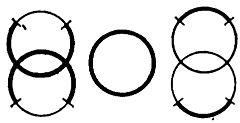


Fig. 5

lines l_A and l_B rotate about their points A and B, then k_A and k_B also rotate about their points A and B with the same constant angular velocity. According to proposition E°, the point of intersection of k_A and k_B moves in a circle.

Note that when the point of intersection of the straight lines l_A and l_B describes an arc of the circle γ, the point of intersection of the straight lines k_A and k_B also describes an arc of a circle symmetric to the circle γ relative to the straight line AB.

3.8. (a) Let a, b, c be straight lines passing through the points A, B, C respectively; K, L and M are the points of intersection of the straight lines a and b, b and c, a and c respectively. According to proposition E° of the "alphabet" the point K traces out a circle with chord AB, and the point L traces out a circle with chord BC. Let H be the point of intersection of these circles other than B.

When the straight line b (line KL) while rotating passes through the point H, then the points K and L coincide with M, hence the straight lines a and c also pass through H. (The particular cases when these two circles touch at the point B or coincide should be treated separately. In the first case the point M coincides with B, in the second the points K, L and M always coincide: "it is possible to put a single tiny ring around all three straight lines a, b and c").

By the way, note that during this rotation the triangle KLM remains similar to itself. When all the straight lines intersect at a single point H, it degenerates to a point and it attains its maximum size when a, b and c are perpendicular to the straight lines AH, BH and CH respectively. At this instant its vertices assume positions diametrically opposite to the point H on their paths (circles).

3.8. (b) Suppose the straight lines AH, BH and CH start to rotate with the same angular velocity about the points A, B and C (H is the orthocentre of the triangle ABC). Then the point of intersection of each pair of straight lines describes one of the circles mentioned in the statement of the problem.

3.9. Consider three sets of points M lying inside the triangle:

$$\left\{M: \frac{S_{AMB}}{S_{BMC}} = k_1\right\}, \quad \left\{M: \frac{S_{BMC}}{S_{AMC}} = k_2\right\}, \quad \left\{M: \frac{S_{AMC}}{S_{AMB}} = k_3\right\}$$

These three segments (see proposition **I**) are concurrent when and only when $k_1k_2k_3 = 1$.

3.10. Consider three sets:

$$\{M: |MA|^2-|MB|^2=h_1\}, \quad \{M: |MB|^2-|MC|^2=h_2\},$$
$$\{M: |MC|^2-|MA|^2=h_3\}$$

These three straight lines (see proposition **F**) are concurrent if and only if $h_1 + h_2 + h_3 = 0$.

3.21. Draw the set of end-points M of all the possible vectors

$$\overrightarrow{OM}=\overrightarrow{OE}_1+\overrightarrow{OE}_2+ \dots +\overrightarrow{OE}_n$$

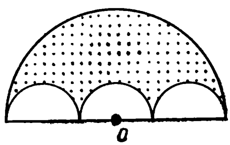


Fig. 6

(where $\overrightarrow{OE}_i$ are the unit vectors mentioned in the statement of the problem, first for $n = 1$, then for $n = 2$ and so on (Fig. 6).

4.4. *Answer*: The minimum distance between the pedestrians is equal to $du/\sqrt{u^2 + v^2}$.

Suppose the first pedestrian P walks with velocity $\vec{u}$, the second, Q, with velocity $\vec{v}$ (the lengths u and v of these vectors are known).

Consider the relative motion of P in the pedestrian Q reference frame. This will be a uniform motion with a constant velocity $\vec{u} - \vec{v}$ (see **1.3**).

In the "initial" position, when P lies at the point P_0 where the roads cross, Q_0 is a distance $|Q_0P_0| = d$ from P_0 in the direction of the vector $-\vec{v}$. Thus, in order to find the answer, it is sufficient to draw through the point P_0 a straight line l parallel to the vector $\vec{u} - \vec{v}$ (it is the path of P in its relative motion in the reference frame of Q), and to determine the distance $|Q_0H|$ of the point Q_0 from the straight line l (H is the projection of Q_0 on l). Since the triangle Q_0P_0H is similar to the triangle formed by the vectors $\vec{u}$, $\vec{v}$ and $\vec{u} - \vec{v}$ $((Q_0P_0) \perp \vec{u}$, $(Q_0H) \perp (\vec{u} - \vec{v}))$, we have

$$|Q_0H|/|Q_0P_0| = |\vec{u}|/|\vec{u} - \vec{v}| = u/\sqrt{u^2 + v^2}.$$

4.6. From the centre O_1 of one of the circles, drop a perpendicular O_1N onto the secant l passing through the point A, and from the centre O_2 of the other circle drop a perpendicular O_2M onto the straight line O_1N. Then the length of $|O_2M|$ is half the distance between the points of intersection of the secant l and the circles (other than A).

4.9. *Answer*: An isosceles triangle. Use **2.8** (a).

5.4. (b) Prove that if the segment KL of constant length slides at its end-points along the sides of a given angle A, then the point M of intersection of the perpendiculars erected at the points K and L to the sides KA and LA of the angle moves around a circle with centre A (recall the discussion of Copernicus' theorem **0.3** in the introduction).

5.7. The following fact helps us to construct these points: the level curves of the function $f(M) = |AM|/|MB|$ are orthogonal to the circles passing through the points A and B (page 97).

6.3. (e) *Answer*: A hyperbola, if the given circles do not intersect one another (perhaps touching); the union of a hyperbola and an ellipse, if they intersect; an ellipse, if one circle lies inside the other (perhaps touching). The foci of the curves lie at the centres of the given circles.

In order to reduce the number of different cases that must be considered the position of the third circle, with respect to the first two, we can use the following general rule: circles of radii r and R with their centres a distance d apart, touch each other, if $r + R = d$ or $|R - r| = d$.

6.12. (a) For a given tangent construct the tangent symmetric to it with respect to the centre of the ellipse.

Use **6.9.** (b) and the theorem which states that the product of the segments of a chord drawn through a given point inside a circle is independent of the direction of the chord.

6.15. Construct in the case (a) an ellipse (in the case (b) a hyperbola) with foci at A and B, touching the first link P_0P_1, and prove that it also touches the second link P_1P_2. To do this use the fact that $\triangle A'P_1B \cong \triangle AP_1B'$, where A' is the point symmetric to A relative to P_0P_1 and B' the point symmetric to B relative to P_1P_2. The tangents will be the perpendicular bisectors of the segments AA' and BB' (**6.9** (a), **6.10** (a)).

6.16. (c) We construct the set of points N for which the midpoint of the segment AN lies on the given circle. This is a circle. Denote its centre by B, its radius by R. The set of points which are located nearer to the point A than to any point N of the constructed circle is the intersection of the half planes containing A which are bounded by the perpendicular bisectors of the segment AN. This set may be written as follows:

$$\{M: |MA| - |MB| \leqslant R\},$$

i.e. its boundary is a branch of a hyperbola.

6.17. Compare the hint to **6.16** with the proof of the focal property of a parabola.

6.23. Choose the origin at the midpoint of the segment AB, and the x-axis so that at some points of time both rotating straight lines are parallel to Ox. If we write the equations of the straight lines for time t, find the coordinates of their point of intersection and then eliminate t (as in the solution to **6.22**), then we obtain the equation of a hyperbola in the form (4) (page 128).

6.24. Imagine two straight lines rotating about the points A and B in different directions so that the second one has twice the angular velocity of the first. It is not difficult to guess that their point of intersection moves along a curve, like a hyperbola having asymptotes which form angles of 60° with the straight line AB and for which the point of intersection C with the segment AB divides AB in the ratio $|AC|/|BC| = 2$.

The answer to this problem is in fact a branch of a hyperbola. The following simple geometrical proof reduces the problem to proposition **N** of the "alphabet".

Construct the point M' symmetric to M with respect to the midperpendicular l of the segment AB, and note that the ray BM' is the bisector of the angle ABM, and $|MM'| = |MB|$, so that $|MB|/\rho(M, l) = 2$.

6.25. (a) If the coordinate system is selected in such a way that the sides of the angle are given by the equations $y = kx$ and $y = -kx$, $x \geqslant 0$, then the area of the triangle OPQ, where P and Q lie on the the sides of the angle, is $kx^2 - y^2/k$, where $(x; y)$ are the coordinates of the midpoint of the segment PQ.

(b) Use the result of problem **1.7** (b).

(c) Result follows from (a) and (b).

7.2. This union may be considered as the set of those points M for each of which there can be found a point P on the circle such that $|MP| \leqslant |PA|$ or as the set of points M for which the perpendicular bisector of the segment MA has a point in common with the circle. Compare this problem with **6.16-6.17**.

7.9. *Answer*: (a) 3; (b) 4; (c) 2.5. The ratio of the angular velocities may be found in the same way as was done in the examples on pages 158-161.

7.13. (a) The arc of a circle of radius R between two cusps of Steiner's curve (120°) has the same length as the circumference of a semicircle of radius $2R/3$.

7.14. (b) Both curves may be obtained as the paths of the vertex M of a hinged parallelogram, with side lengths $R - r$ and r and the ratio of the angular velocities ω_1/ω_2 is equal to $-r/(R - r)$ (the angular velocities have opposite signs, see page 158).

7.18. Use **7.7** and Mozzi's theorem.

7.19. *Answer*: a k-cycloid (see page 150).

7.21. Use **7.13** (a), Mozzi's theorem and the theorem on two circles.

7.23. Let M be a point on the circle described, moving around it with angular velocity ω. Then:

(1) the points M_1, M_2 an M_3, symmetric to the point M relative to the straight lines BC, CA and AB move around the circle (with angular velocity $-\omega$);

(2) these three circles intersect at a single point H, the orthocentre of the triangle ABC (**3.8**(b));

(3) each straight line M_iM ($i = 1$, 2 or 3) rotates with angular velocity $(-\omega/2)$ about H;

(4) three points M_1, M_2, M_3 lie on a single straight line l_M passing through H (i.e. the three straight lines M_iM are in fact a single line l_M);

(5) the midpoints of the segments MM_i ($i = 1, 2, 3$) and the midpoint K of the segment MH lie on a single straight line, the Simson line;

(6) the point K moves around the circle γ similar to the circle described with magnification ratio 1/2 and centre of similitude H;

(7) the circle γ passes through the 9 points mentioned in part (b) of problem **7.23**;

(8) the envelope of the straight lines l_M is a Steiner's curve touching the circle γ.

Appendix I

Method of Coordinates

As soon as a coordinate system Oxy is chosen on a plane, a pair of numbers is defined corresponding to each point in the plane—the coordinates of the point. The correspondence between the points of the plane and the pairs of numbers is one to one (to each point in the plane there corresponds a pair of numbers and vice versa).

1. The distance between the points $A(x_1; y_1)$ and $B(x_2; y_2)$ is determined by the formula

$$|AB| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

2. The set of points $(x; y)$, whose coordinates satisfy the equation $(x-a)^2+(y-b)^2=r^2$ (where a, b and r are given numbers, $r>0$) is a circle of radius r with its centre at the point $(a; b)$. In particular, $x^2+y^2=r^2$ is the equation of a circle of radius r with its centre at the origin.

3. The midpoint of the segment between the points $A(x_1; y_1)$ and $B(x_2; y_2)$ has the coordinates $\frac{x_1+x_2}{2}$, $\frac{y_1+y_2}{2}$. In general, the point dividing the segment AB in the ratio $p:q$ (where p and q are given positive numbers) has the coordinates $\frac{qx_1+px_2}{q+p}$, $\frac{qy_1+py_2}{q+p}$. These formulae assume a particularly simple form, if p and q are selected so that $q+p=1$.

4. The set of points whose coordinates satisfy the equation $ax+by+c=0$ (where a, b, c are numbers, and where a and b do not vanish simultaneously, i.e. $a^2+b^2\neq 0$) is a straight line. Conversely, each straight line may be defined by an equation of the form

$ax + by + c = 0$. In this case the numbers a, b and c are determined for the given straight line uniquely, apart from a constant of proportionality: if they are all multiplied by the same number k ($k \neq 0$), then the equation $kax + kby + kc = 0$ thus obtained also determines the same straight line.

The straight line divides the plane into two half planes: the set of points $(x; y)$ for which $ax + by + c > 0$, and the set of points $(x; y)$ for which $ax + by + c < 0$.

5. The distance $\rho(l, M)$ of the point $M(x_0; y_0)$ from the straight line l, given by the equation $ax + by + c = 0$, is given by the formula

$$\rho(M, l) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

This formula assumes a particularly simple form if $a^2 + b^2 = 1$.

Any equation $\alpha x + \beta y + \gamma = 0$ ($\alpha^2 + \beta^2 \neq 0$) of a straight line may be reduced to this particular form, by multiplying it by either of the numbers $\frac{1}{\sqrt{\alpha^2 + \beta^2}}$ or $-\frac{1}{\sqrt{\alpha^2 + \beta^2}}$.

Appendix II

A Few Facts from School Geometry

I. Proportional segments

1. *A theorem on proportional segments.* If several segments are marked off on a straight line l_1 and parallel straight lines intersecting l_2 are drawn through their end-points, the parallel lines then cut off on l_2 segments proportional to the segments marked off on l_1.

2. A straight line parallel to one side of a triangle and intersecting its other two sides, cuts off from the triangle a triangle similar to it.

3. *A theorem on the bisector of an angle of a triangle.* The bisector of an angle of a triangle divides the opposite side into segments which have the same ratio as the adjoining sides.

4. *A theorem on proportional segments in a circle.* If two chords AB and CD of a circle intersect at the point E, then

$$|AE|\cdot|BE|=|DE|\cdot|CE|$$

5. *A theorem on a tangent and a secant.* If through a point A outside a circle a tangent AT and a secant cutting the circle at the points B and C are drawn, then

$$|AT|^2=|AC|\cdot|BC|$$

Notes

1. The theorem on proportional segments is reformulated in the language of motion (pages 24-25) as the "theorem about the ring on a straight line". A more general assertion deduced from the theorem about the ring, is the lemma on page 53.

3. The theorem on the bisector of an angle of a triangle has been proved in problem 2.5 (page 38) in a more general form for the "cross-bisector" which is defined in proposition **B** of our alphabet (page 36).

5. The theorem on a tangent and a secant is not referred to anywhere in the book directly but it is closely related to the problems on the radical axis (page 44).

II. Distances. Perpendiculars

1. The distance from a point A to the foot of the perpendicular passing through A to the straight line l is less than the distance from the point A to any other point on l.

2. A line tangent to a circle is perpendicular to the radius drawn to the point of contact.

3. Of two oblique lines drawn from a given point to a given straight line l the one which has the larger projection on the straight line l is greater.

4. (a) If a point lies on the perpendicular bisector of a segment, then it is equidistant from the end-points of the segment.
(b) If a point is equidistant from the end-points of a segment, then it lies on the perpendicular bisector of the segment.
These two theorems may be combined in a single statement: the set of all points equidistant from the end-points of a segment is the perpendicular bisector of the segment.

5. (a) If a point lies on the bisector of an angle, then it is equidistant from the sides of the angle.
(b) If a point included in an angle (smaller than a straight angle) is equidistant from the sides of the angle, then it lies on the bisector of the angle.
From (a) and (b) it follows that: the set of all points contained in an angle (smaller than a straight angle) equidistant from the sides of the angle is the bisector of the angle.

6. One and only one circle can be inscribed in a triangle. This circle is called the *incircle*.

7. About a triangle one and only one circle can be circumscribed. This circle is called the *circumcircle*.

Notes

1-2. These statements may serve as simple illustrations of the tangency principle formulated in Sec. 5 (page 104). Let a straight line γ and a point A be given. Construct the family of concentric circles—

the level curves of the function $f(M) = |AM|$. The point on γ at which the function f attains its minimum value is the point of tangency of one of the circles of our family with the straight line γ.

3-4. The general statement of 4 is proposition **A** (page 36) of the "alphabet". The perpendicular bisector is often called the midperpendicular. Statement 3 is essentially contained in the statement of proposition **A** on the division of the plane into half planes.

5. A more general statement is formulated in proposition **B** of the "alphabet", where the term "cross bisector" is introduced (page 36).

6. The centre of an inscribed circle is determined in problem **3.3** (page 63).

7. The centre of a circumscribed circle is determined in problem **3.1** (page 60).

III. The circle

1. The radius perpendicular to a chord bisects the chord.

2. *A theorem on tangents.* If from a point A two tangents AT_1 and AT_2 are drawn to a circle (T_1 and T_2 are the points of contact), then $|AT_1| = |AT_2|$.

3. *A theorem on the circumscribed quadrilateral.* A circle can be inscribed in a convex quadrilateral if and only if the sum of the lengths of two opposite sides of the quadrilateral is equal to the sum of the lengths of the other two opposite sides.

4. The set of all the vertices of a right-angled triangle with a given hypotenuse AB is a circle of diameter AB (with the points A and B excluded).

5. *A theorem on an inscribed angle.* The magnitude of an inscribed angle (in degrees) is equal to half the magnitude (in degrees) of the intercepted arc. (In other words, the angle at the centre is double the angle at the circumference, when the rays forming the angles meet the circumference in the same two points.)

6. An angle formed by a tangent and a chord through the point of contact has half as many degrees as the arc intercepted by this angle.

7. An angle with its vertex inside a circle has half as many degrees as the sum of the two arcs, one of which is enclosed between the sides of this angle and the other between the sides produced.

An angle formed by two secants intersecting outside a circle is equal in degrees to one-half the difference between the intercepted arcs contained by the angle.

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8. *A theorem on the inscribed quadrilateral.* About a quadrilateral, a circle can be circumscribed if and only if the sum of its two opposite angles (in degrees) is equal to 180°.

Notes

4. This statement is discussed on page 11 in connection with the problem about the cat.

5. The theorem on the inscribed angle is reformulated in the language of motion (page 24) as the "theorem about the ring on a circle". A more general statement deduced from the theorem about the ring is proposition E° of the "alphabet".

6-7. Problem 2.6 touches on these theorems.

IV. **Triangles**

1. *A theorem on the exterior angle.* An exterior angle of a triangle is equal to the sum of the non-adjacent interior angles.

2. *A theorem on the midperpendicular.* The three medians of a triangle are concurrent in a point which divides each of them in the ratio 2 : 1 (measuring from the vertex).

3. *A theorem on the altitudes of a triangle.* The three altitudes of a triangle are concurrent.

4. *Pythagorean Theorem.* The square of the hypotenuse of a right triangle is equal to the sum of the squares of the legs.

5. The legs of a triangle are proportional to the sines of the opposite angles.

6. The area of a triangle is equal to one-half of the product of:
(a) the base and the altitude upon that base;
(b) two sides and the sine of the angle between them.

Notes

2-3. The proofs of these theorems are given on pages 61-65 in the solutions of problems 3.2 and 3.4 (the fact that a median divides the other median in the ratio 2 : 1 may be obtained from the solution of 3.4).

Appendix III

A Dozen Assignments

This appendix is intended for readers who after first going quickly through the book and trying to solve the problems that appealed to them find that they are not able to manage some of them, but still want to understand them, and are ready to work through the book systematically "with pencil and paper at hand".

The twelve assignments given below cover the contents of the book in different directions. They stress the relationships hidden at first glance between the various problems.

The assignments are constructed in the same way as is usual in the correspondence mathematical school of Moscow State University. First the subject-matter of an assignment explained and the pages of the book containing the theorems or exercises which should be carefully and thoroughly studied, are given. Then there is a series of exercises, in which the "obligatory" problems are differentiated from the supplementary by the sign ||. Some of the problems are provided with explanations. As far as the solutions go, we advise you to try to write them out concisely without unnecessary details, clearly stating the basic steps involved and any references to theorems from your geometry course. Do not forget about particular cases: sometimes they have to be analysed separately (as in problem **1.1**, when the point M lies on the straight line AC or in problem **1.3** in the case of a square). Although we are not suggesting that readers should give superfluous details when investigating and rigorously analysing all special cases, but we do advise them to precisely formulate the result in full, as is the custom among mathematicians.

1. Name the "Letters"

The aim of this exercise is to make a first acquaintance with our "alphabet", i.e. with the theorems dealing with sets of points which are useful in the solution of the later problems.

Go through Sec. 2 and make a list of the propositions from **A** to **J** of the "alphabet" on a separate sheet of paper. Against each letter write down the formula (see page 58) and draw the corresponding diagram.

2.1, **2.2**, **2.3**, **2.4**, **1.16** (a), (b), **5.4** (a), **1.11**, **1.12** || **2.13**, **2.15**, **2.16**, **3.6**.

Remarks

In the first five problems it is only required to state in the answer the appropriate letter of the "alphabet".

Problem **1.16** (a) helps one to solve problem **5.4** (a) without any calculations.

In "construction" problems, everything reduces to the construction of a certain point—the centre of a circle, etc. The required point is obtained as the intersection of two sets from the "alphabet" (see **1.4**). It is essential to name these sets (propositions of the "alphabet") and indicate how many answers the given problem has.

The short solution to **2.13** is based on the result of **2.12**.

2. Transformations and Constructions

In the solutions of the problems making up this assignment, you have to use the various geometric transformations of the circle and the straight line which are discussed on pages 27-28 and which often appear in the book (**6.9** (a), (b), **7.1** (a), (b)).

1.20, **1.21**, **1.22**, **1.23**, **1.24** (a), (b) || **3.7** (a), **4.8** (a).

Remarks

1.22. See the solution to problem **1.7** (a).

1.23. See the solution to problem **1.6**.

1.24 (a). Give the answer only.

3.7 (a). Use the fact that the centroid divides the median in the ratio 2 : 1, measuring from the vertex.

4.8. Read the solution to problem **4.7**.

In all these exercises we suggest that you make sketches of all necessary constructions. Write your solutions concisely, paying attention to the sets and transformations used. Indicate how many solutions a problem has.

3. Rotating Straight Lines

This assignment basically concerns the different variants of the theorem on the inscribed angle and its corollaries.

Go through the book in the following order: problem **0.1** (about the cat), problem **1.1**, the theorem about the ring on a circle (pages 23-24), the propositions **E°** and **E** of the "alphabet" (pages 39-40). Note that the theorem about the ring (and the problem about the cat) should not be understood literally: the imagined "ring" is simply the point of intersection of the straight line and the circle; if we make a wire model, then after a single rotation (in either direction) the ring would become stuck.

1.8, **1.9**, **1.10**, **1.13**, **1.18**, **2.6** (a), (b) || **1.27**, **2.7**, **2.8** (b), **4.6**, **7.5**, **7.6**.

Remarks

1.9. Draw diagrams for the different positions of the point A.

1.10. Draw a straight line through the point B, plot the point A' symmetric to the point A with respect to this straight line, and then draw the segment BA'.

Show the sets of points in the answers to problems **1.8** and **1.10**, in the one diagram. By what transformation can one get the set in **1.10** from the one in **1.8**?

1.13. State how many answers the problem has.

1.27. Carry out an experiment using an ordinary set square. Hint: circumscribe a circle about the wooden triangle, join the vertices of the right angles and use the theorem on the inscribed angle.

2.6. Imagine that the movable chord is moving uniformly around the circle.

2.8. (b) The solution is analogous to **2.8.** (a). Sort out the second variant of the solution of this problem, given on page 42.

4. Straight Lines and Linear Relations

This assignment deals with problems in which no curves but only straight lines appear.

Go through the book in the following order: problems **1.2** and **1.3** about the "bicycle" and the rectangle (pages 20-23), the theorem about the ring on a straight line (pages 24-25) and the important lemma (page 53) which extends it, and also propositions **F**, **I**, **J** of the "alphabet" and the general theorems on the distances to straight lines and the squares of distances (pages 42-56).*

1.24 (a), (b), **2.18**, **2.19** (b), **3.9**, **3.14**, **3.15**, **3.16** || **1.26**, **1.27**, **2.14**, **2.20** (a), **3.18**.

Remarks

2.18. See solutions to **2.5** and **2.17**.

2.19 (b). Find out how the answer depends on the dimensions of the rectangle $a \times b$ and the parameter μ (see the answer to problem **2.19** (a)).

3.14-3.16. See proposition C of the "alphabet".

1.27. Let a and b be the lengths of the legs of the wooden triangle. Find the ratio of the distances from its free vertex to the sides of the given right angle.

2.20. It is sufficient to give the answer and a diagram.

3.18. Read the solution to **3.17**.

5. The Tangency Principle (Conditional Extremum)

The assignment consists of problems on finding maxima and minima. Every problem may be reduced to one in which it is required to find the point on some line (as a rule one of the sets from the "alphabet"), at which some function reaches its maximum or minimum value. Read the solutions to problems **4.1**, **4.2** and **4.7** (pages 78-82), the solution to problem **5.1** and the rest of Sec. 5, particularly pages 103-105. Study (or redraw) the maps of the level curves on pages 98-99.

4.3, **4.9**, **5.4** (a); **5.5**, **5.6** (a), (b), **5.8** || **4.8**, **5.4** (b), **5.7**.

Remarks

5.4 (a) See problem **1.16** (a).

6. Partitions

In this assignment we find various sets of points satisfying inequality conditions and also the set operations (intersection, union), corresponding to the logical combinations of the conditions. Many propositions of our "alphabet" of Sec. 2 have conditions of the following type: the line consisting of the points M for which $f(M) = a$ divides the plane into two domains, one in which $f(M) < a$ and the other in which $f(M) > a$ (here f is some function on the plane, see page 93). In exactly the same way, if f and g are two functions on a plane, then the set of points M, where $f(M) = g(M)$ partitions the plane into regions, in some of these $f(M) > g(M)$ while in others $f(M) < g(M)$. Go through the text of Sec. 3 (pages 67-68), and the solutions to problems **3.11**, **3.23** (about the cheese).

1.19, 3.12, 3.14, 5.3 (a), (b), **3.15, 3.16** || **3.18, 3.19, 4.11, 4.12** (a), (b).

Remarks

1.19. Draw the segment BC and indicate the set of points of the vertices A of the triangles ABC, for which each of the conditions (a), (b), (c) is fulfilled; use second paragraphs of propositions **D** and **E** of the "alphabet".

3.14. Read the solution to **3.13.**

3.15-3.16. Use proposition **C**, and in problem **3.16** recall **2.4.**

3.18. Construct for each side of the polygon the strip as in proposition **C** corresponding to $h = S/p$. Can these sets cover the whole of our polygon of area S?

4.11-4.12. Read the solution to **4.10.**

7. Ellipses, Hyperbolas and Parabolas

The aim of this assignment is to acquaint ourselves with the first definitions of these curves, given in propositions **K, L, M** of our "alphabet". Go through Sec. 6 and list the propositions of "alphabet". For each 'letter' write down the formula and draw the corresponding diagram (problems **6.5** (a), (b) of this exercise will help you to do this).

6.1 (a), (b), (c), **6.2, 6.3** (a), (b), (c), (d), **6.4** (a), (b), **6.5** (a), (b), **6.10** (a), (b), **6.11** (a), (b) || **6.8, 6.12** (a), (b), **6.13** (a), **6.14, 6.24.**

Remarks

6.1 (a), (b), (c). Indicate how the answer depends on the parameter (put $|AB| = 2c$).

6.2. Use the theorem on the segments of the tangents to a circle.

6.4 (b). Consider positions of the quadrilateral $ABCD$ for which the link BC crosses AD.

The following problems deal with the focal properties of the curves.

6.10 (a). The proof is on the same lines as in the solution to **6.9** (a), and is also based upon problem **6.7.**

6.11 (a). Compare the definition of a parabola (proposition **M** of the "alphabet") and its focal property.

6.8. The proof is similar to the proof of the orthogonality of equifocal ellipses and hyperbolas (pages **116-117**).

8. Envelopes, Infinite Unions

In this assignment the problems are all quite complicated. In each problem an entire family of straight lines or circles is considered. If the union of the lines of this family is taken, a whole region of the plane is obtained. It often happens that the boundary of this region is the envelope of this family of lines—a curve (or a straight line) which touches all the lines of the family. (For example, in the solution to problem **1.5** on page 27, we used the fact that the envelope of the family of chords of equal length of the given circle is a circle concentric to the given circle.) We urge you to draw a diagram for each problem; it is not necessary, however, to draw the envelopes. If you draw a large enough number of lines of the family, then the envelopes appear automatically (as in the diagrams on pages 122-123).

Read the text on pages 121, page 15, the solutions to **3.20** (b), **6.6**, **6.7** and the proof of the focal property of a parabola (pages 117-118).

1.30, **3.20** (a), **3.22**, **4.5**, **6.16** (a), (b), **6.17** || **6.15** (a), **6.25** (a), (b), **7.2**, **7.20**.

Remarks

3.20. Imagine this union as the set of vertices M of a hinged parallelogram $OPMQ$ with sides 3 cm and 5 cm; compare this method with Sec. 7 (pages 152-153).

3.22. If in the first t minutes the man walks along the road and then $60 - t$ minutes through the meadow, where can he reach? Now take the union of the sets obtained for all t from 0 to 60.

4.5. What set provides the answer to problem **3.22**, if 1 hour is replaced by T hours? Find for what value of T this set contains the point B.

7.20. The family of tangents to the nephroid has been considered in problem **7.16**. Also recall here problems **7.1** (a), **7.2** on the cardioid, and the theorem about two circles (pages 153-155).

9. Tangents to Cycloids

This assignment includes a series of problems in which one has to prove that the envelope of some family of straight lines is a cycloid.

The solutions of most of them are based on the theorem about two circles. Read the statement and the examples of the application of this theorem on pages 153-157 and also analyse its proof (pages 158-162).

7.17(a), (b), **7.16**, **7.18**, **7.19** || **7.21**, **7.22**, **7.23**.

Remarks

7.17. Find along what curve the end-points of the diameters move, and what curve is their envelope. (Compare the result with the last diagram on page 151.)

7.16. Using the theorem about two circles, describe the family of tangents to the nephroid. Read the solution to problem **7.15**.

10. Equations of Curves

The method of coordinates allows one to formulate general theorems, which generalize particular geometric observations in a natural way (go through the general theorems in Sec. 2, pages 45-47, 55-56, Sec. 6, pages 124-134). The representation of curves in the form of equations gives us the possibility of solving geometric problems in the language of algebra. In this assignment there are exercises on the method of coordinates and problems in which it is used in a natural way. Most of the problems are related to second-order curves. In some problems it is necessary to change over from parametric equations to algebraic ones (see the solution to **0.2**, pages 12).

1.16 (c), **6.18**, **6.19** (a), (b), (c), **6.20** (a), (b), **7.24** (b) || **6.21** (a), (b), **6.23**, **6.25** (a), **6.26** (a), (b), **6.27**.

Remarks

In the problems on the distances to points and straight lines you must carefully investigate how the answer depends on the parameter. For each of these problems, you must draw the corresponding diagram — the family of curves. It is convenient to draw the ellipse according to the given equation, representing it as a compressed circle (page 126) and the hyperbola by drawing its asymptotes and marking its vertices (the points of the hyperbola closest to its centre).

In problem **6.26**, if we limit ourselves to points M lying inside the triangle, then a beautiful geometric solution may be given using the similarity of triangles, and also the theorems on the inscribed angle and the angle between a tangent and a chord.

11. Geometrical Practical Work

In this assignment you have to construct diagrams which illustrate the most interesting definitions and properties of curves. This will enable you to look at the book from a new point of view.

Problems can be considered in the light of the assertion: "geometry is the art of reasoning correctly on an incorrect diagram". But sometimes it is sensible to have the same approach to geometry as to physics: an exact diagram is a geometrical experiment. This point of view helps us analyse difficult statements about whole families of lines or complicated configurations or to discover some new regularity.

We advise you to repeat (sometimes with additions) those diagrams which depict interesting families of straight lines and circles. To make these illustrations, from the technical point of view, is a comparatively simple task, but all the same accuracy and a certain inventiveness are required to make them beautiful. On a large sheet of paper these drawings will look considerably more significant than our little diagrams in the margins.

1. The *astroid* (page 16). Try to make the midpoints of the segments be distributed uniformly around the circle on which they lie. The larger the number of segments are drawn, the better will their envelope, the astroid, be seen.

2. *Orthogonal families of circles* (page 97). The first family is the family of all possible circles passing through the points A and B (see **2.1**). The second family is the family of circles whose centres lie on the straight line AB; if M is the centre of one of them, then its radius is the segment of the tangent drawn from the point M to the circle with diameter AB.

3. *Ellipses, hyperbolas and parabolas* (page 113). The method of construction is mentioned in problems **65** (a), (b). Colour the "squares" obtained alternately with two different colours as on a chess-board (see page 16 and the remarks on problem **6.8**). Make another copy of each of the diagrams to problems **6.5** (a), (b), and mark on them with ink the families of ellipses, hyperbolas and parabolas.

4. *Second-order curves as the envelopes of straight lines* (pages 122-123, figures 4-6). The method of construction follows from **6.16** and **6.17**.

5. *Rotating straight lines.* Make your own diagram illustrating proposition **E°** of the "alphabet" (the lower diagram on page 39). Draw a circle and divide it into 12 equal parts. Draw straight lines through one of the points of division A and the other points of division and also the tangent to the circle at the point A: (one gets a bundle of 12 straight lines dividing the plane into 24 angles of equal size). By moving a pencil around the circle we can see that whenever we go from one point

of division M to the next, the straight line AM turns through the same angle. Choose another point of division B (say, the fourth point from A) and construct for it a bundle of 12 straight lines similar to the one for the point A. Mark for each point of division M, the acute angle between the straight lines AM and BM. (All these angles are equal!)

From theorem E°, it follows that if all the 23 straight lines constructed are produced to their points of intersection, then all the 110 points of intersection thus obtained (not counting the points A and B) lie on 11 circles, 10 on each circle ⟨?⟩

Colour the "squares" of the net obtained as on a chess-board. You will then see immediately the family of circles passing through the points A and B and the family of hyperbolas (better, take a bundle of 24 but not 12 straight lines). For, if straight lines passing through the points A and B rotate in opposite directions with equal angular velocities, then their point of intersection will move along a hyperbola (6.23).

6. *The conchoid of Nicomedes and the limaçon of Pascal* (pages 139 and 147). The conchoid of Nicomedes is obtained in the following manner. A straight line and a point are given. On straight lines passing through the given point, lay off from their points of intersection with the given straight line segments of constant length ***d***, in both directions. Draw the family of such conchoids (for various ***d***).

The limaçon of Pascal is obtained in a similar manner. Suppose we are given a circle and a point on it. On straight lines passing through the given point, lay off from their points of intersection with the circle segments of constant length, in both directions.

7. *The cardioid and the nephroid as the envelopes of the circles* (page 142, **7.2** and page 165, **7.20**).

8. *The cardioid and the nephroid as the envelopes of reflected rays* (the drawings on pages 155, 157). It is convenient to construct these drawings using the fact that the chord of the incident ray is equal in length to the chord of the reflected ray.

9. *Pedestrians on straight lines and circles.* Copy figure 3 on page 123. Sketch the cycloid curves (3-6 on pages 150-151) using problem **7.19**, for $k = -3, -2, 2, 3$.

Let us investigate it for the case $k = -2$. Divide a circle into, say, 24 equal parts. Let the point of division A be the initial position of pedestrians P and Q and suppose each of them moves uniformly in the circle. Since $k = -2$, they move in opposite directions and the velocity of Q is twice as great as the velocity of P. Mark their positions at equal intervals of time (when the point P passes through the respective point of division) and join them with the straight lines PQ (when the pedestrians arrive together at a point of division, draw the tangent to the circle). The envelope of these straight lines is Steiner's curve.

12. Small Investigations

Almost any problem on geometry is a subject for independent research demanding inventiveness and an original way of thinking. In this assignment we mark out four difficult problems in the solution of which a whole range of different arguments must be used.

4.12 (a), (b), **4.14** (a), (b), **6.15** (a), (b), **7.23** (a), (b), (c).

The solution to problem **4.14** (b) is very similar to that of the problem about the motor boat.

There are hints to the last two problems **6.15** and **7.23** at the end of the book. It is possible to draw beautiful diagrams for the last problem which depict the family of Simson's lines of a triangle (the envelope is a Steiner's curve).

Notation

$|AB| = \rho(A, B)$ — the length of the segment AB (the distance between the points A and B).

$\rho(A, l)$ — the distance from the point A to the straight line l.

$\widehat{ABC}$ — the value of the angle ABC (in degrees or radians)

$\overset{\frown}{AB}$ — the arc of a circle with endpoints A and B.

$\triangle ABC$ — triangle ABC.

S_{ABC} — the area of the triangle ABC.

$\{M: f(M) = c\}$ — the set of points M, which satisfies the condition $f(M) = c$.

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The authors N. B. Vasilyev and V. L. Gutenmacher are professors of mathematics at Moscow University. N. B. Vasilyev works in the field of the application of mathematical methods to biology, while V. I. Gutenmacher works in the field of mathematical methods used in the analysis of economic models.

In addition to their scientific work, they have both written many articles and books for high school and university students, and have worked with the Correspondence Mathematics School, which draws its pupils from all over the Soviet Union. They have worked on the committee organizing the "mathematical olympiad" problem competitions, which have greatly stimulated interest in mathematics among young people in the Soviet Union. They regularly contribute to the magazine "Kvant" ("Quantum"), a remarkable educational magazine devoted to mathematics and physics.

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