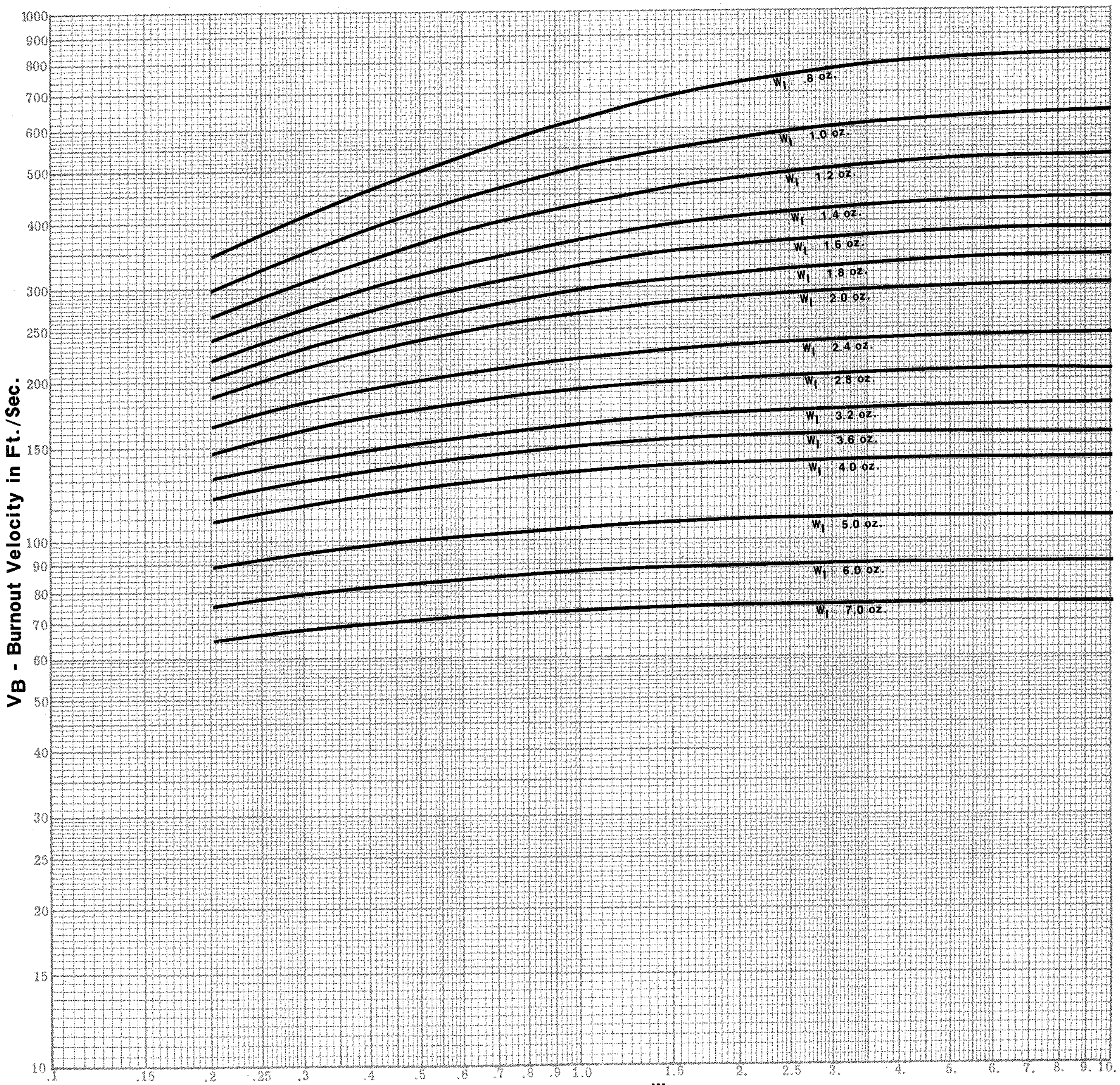
B14

B14
Burn Time t_b = .35 Sec.
Propellent Weight W_P = .220 Oz.
1/2 W_P = .110 Oz.
Average Thrust T = 51 Oz.

FIGURE 8B

Burnout Velocity (V_B) as a function of Initial Weight (W_I) and Ballistic Coefficient (β_t).



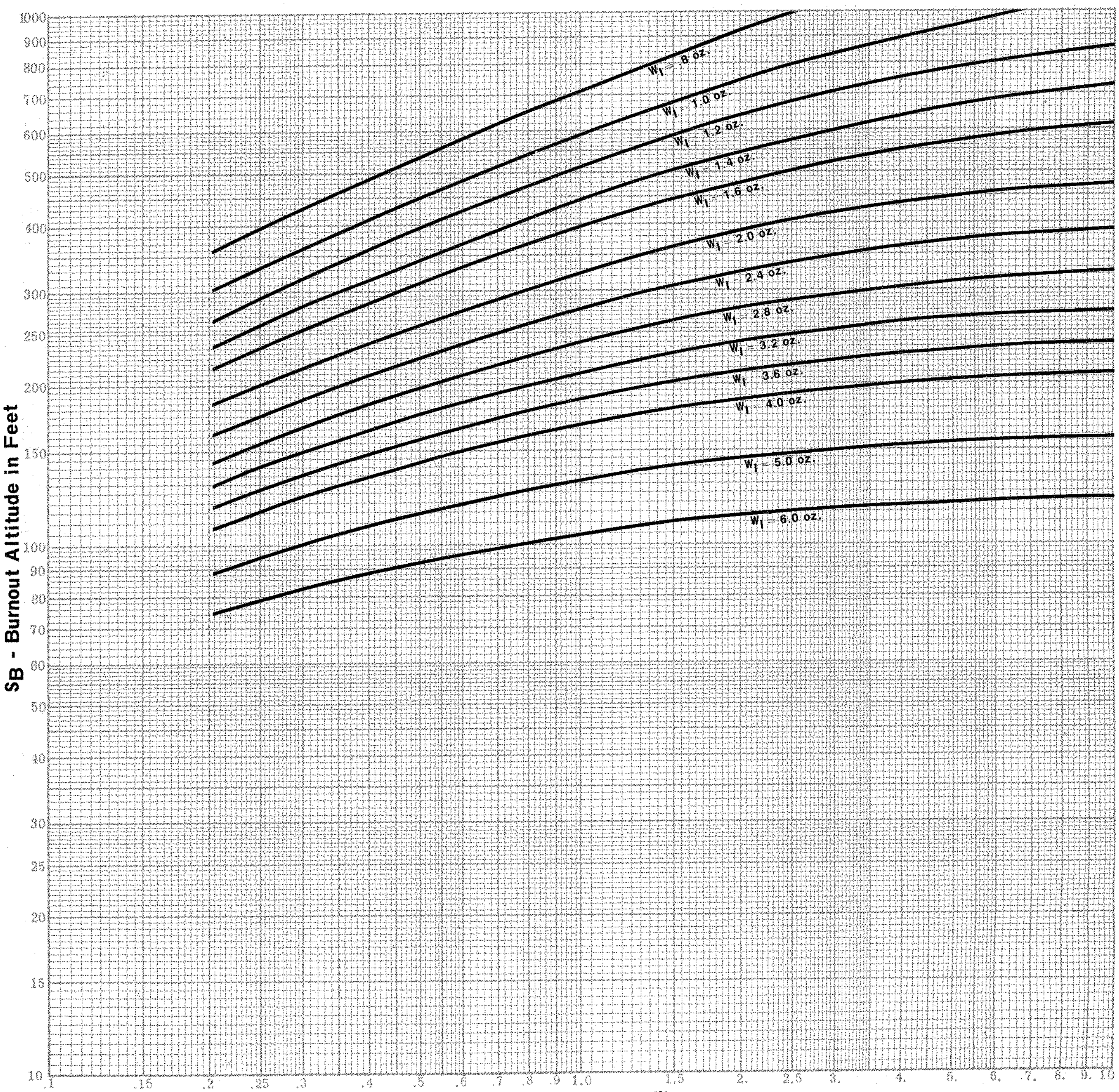
$$\beta_t = \text{Ballistic Coefficient} = \frac{W}{C_D A} = \frac{\text{ounces}}{\text{inch}^2}$$

C6

C6
Burn Time t_b = 1.7 Sec.
Propellent Weight W_P = .440 Oz.
1/2 W_P = .220 Oz.
Average Thrust T = 21 Oz.

FIGURE 9A

Burnout Altitude (S_B) as a function of Initial Weight (W_I) and Ballistic Coefficient (β_t).



β_t = **Ballistic Coefficient** $= \frac{W}{C_D A} = \frac{\text{ounces}}{\text{inch}^2}$

A8

A8

Burn Time t_b = .32 Sec.

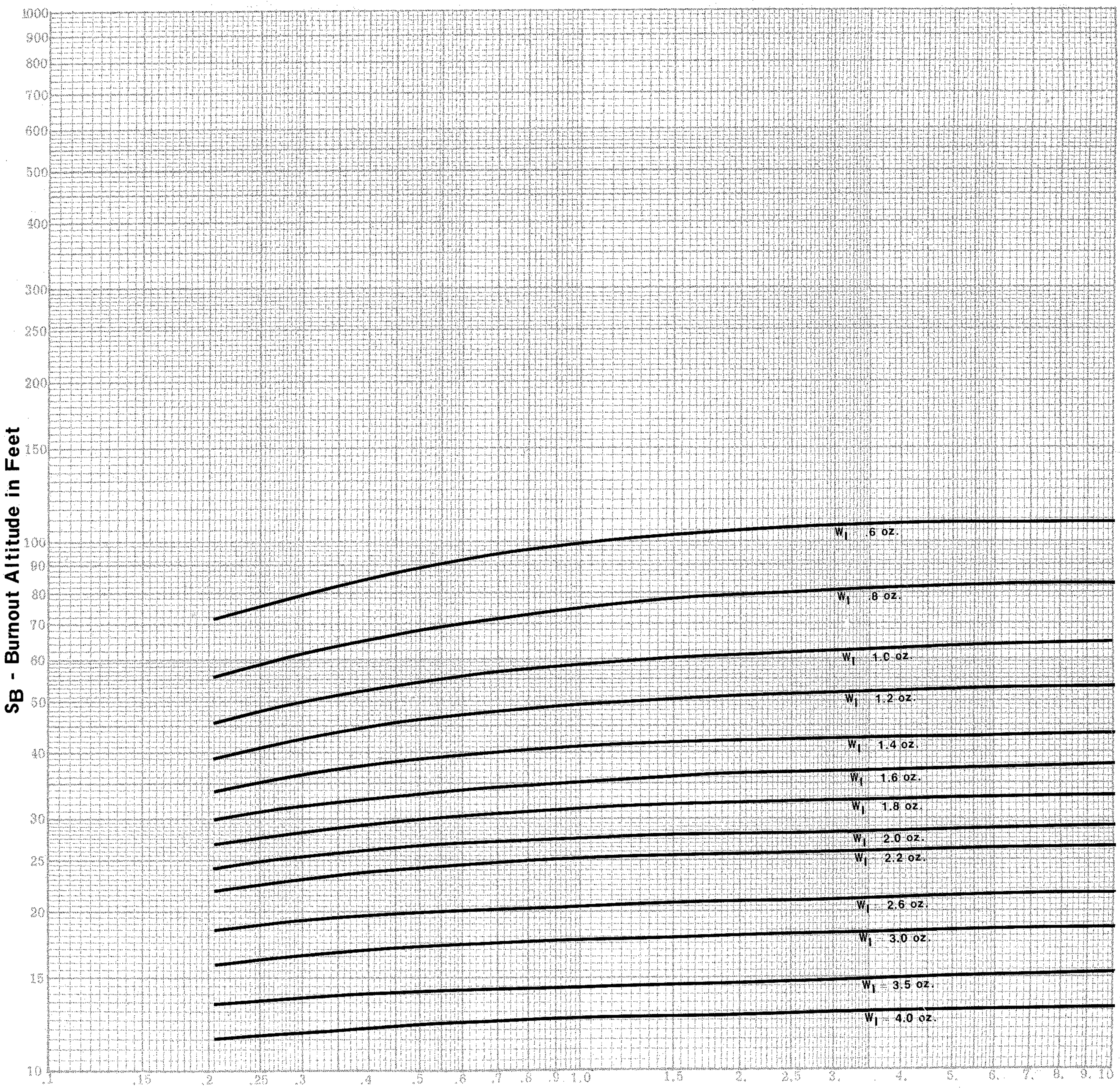
Propellent Weight W_P = .110 Oz.

$1/2\ W_P$ = .055 Oz.

Average Thrust T = 28 Oz.

FIGURE 5A

Burnout Altitude (S_B) as a function of Initial Weight (W_I) and Ballistic Coefficient (β_t).



$$\beta_t = \text{Ballistic Coefficient} = \frac{W}{C_D A} = \frac{\text{ounces}}{\text{inch}^2}$$

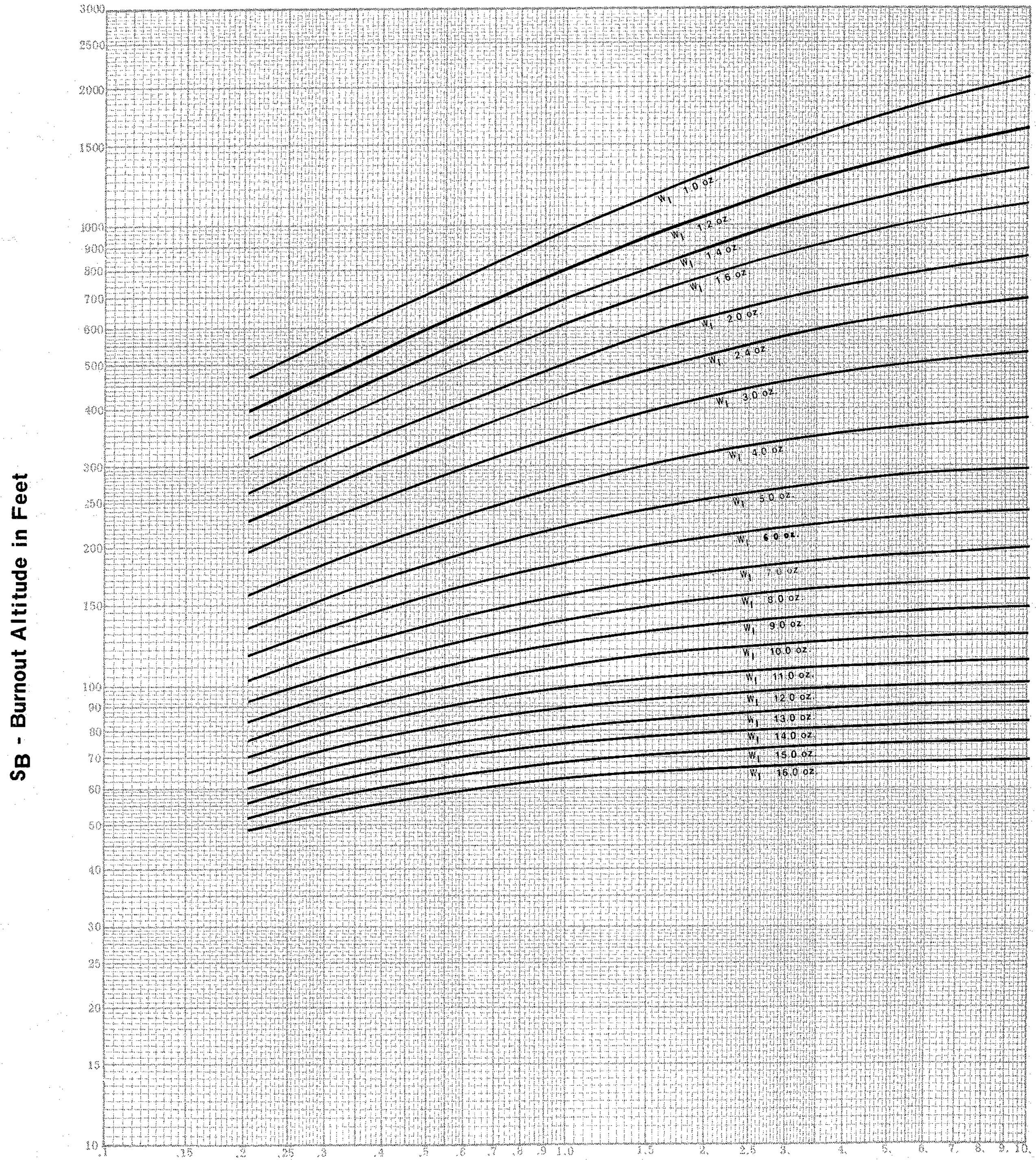
ESTES

D12

D12
Burn Time $t_b = 1.48$ Sec.
Propellent Weight $W_P = .879$ Oz.
$1/2\ W_P = .4395$ Oz.
Average Thrust T = 48 Oz.

FIGURE 10A

Burnout Altitude (S_B) as a function of Initial Weight (W_I) and Ballistic Coefficient (β_t).



$$\beta_t = \text{Ballistic Coefficient} = \frac{W}{C_D A} = \frac{\text{ounces}}{\text{inch}^2}$$

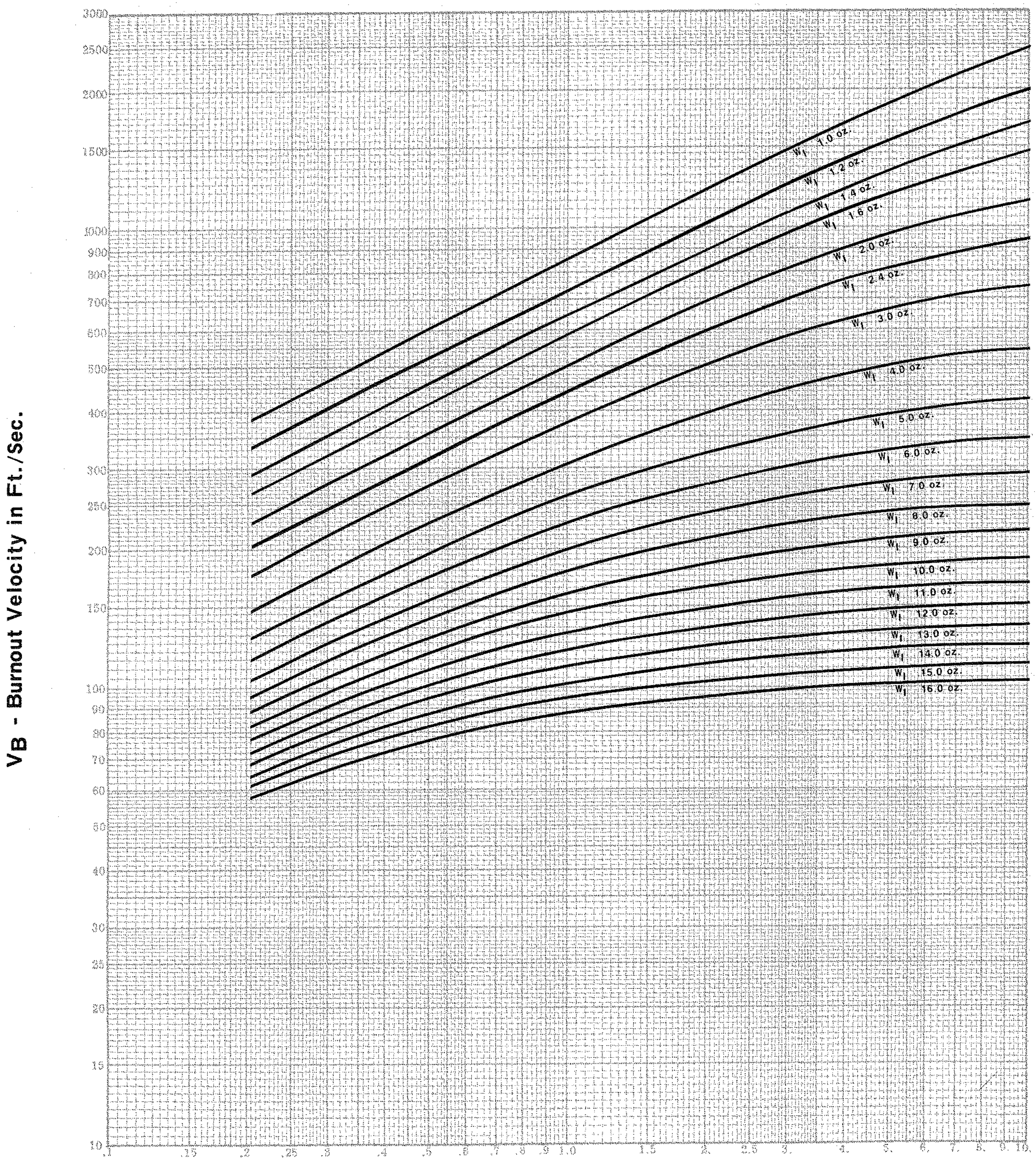
ESTES

D12

D12
Burn Time t_b = 1.48 Sec.
Propellent Weight W_P=.879 Oz.
1/2 W_P=.4395 Oz.
Average Thrust T = 48 Oz.

FIGURE 10B

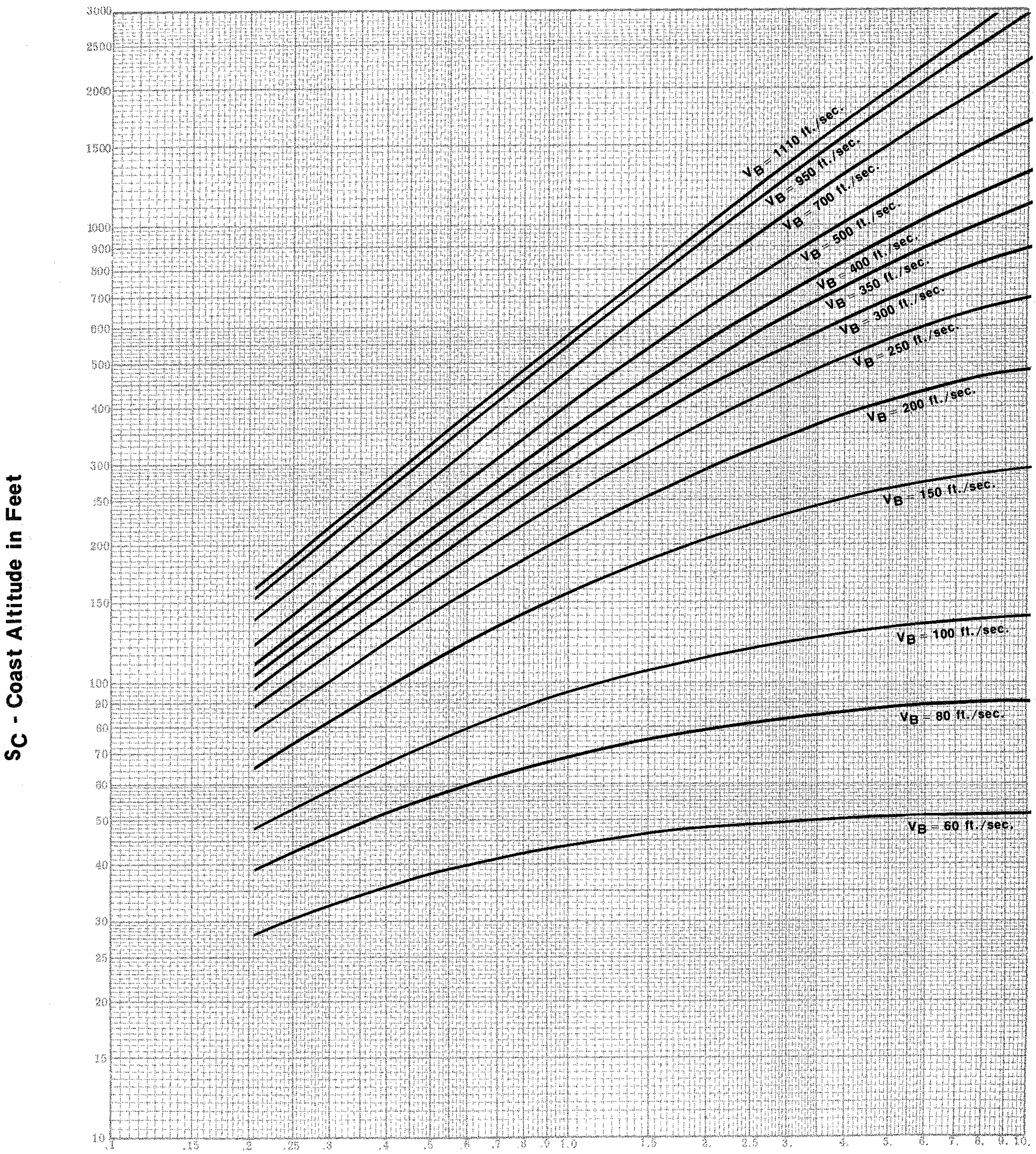
Burnout Velocity (V_B) as a function of Initial Weight (W_I) and Ballistic Coefficient (β_t).



β_t = **Ballistic Coefficient** $= \frac{W}{C_D A} = \frac{\text{ounces}}{\text{inch}^2}$

COAST ALTITUDES

FIGURE 11A
Altitude (S_C) Gained During Coast Phase as a Function of Burnout Velocity (V_B) and Ballistic Coefficient (β_c).

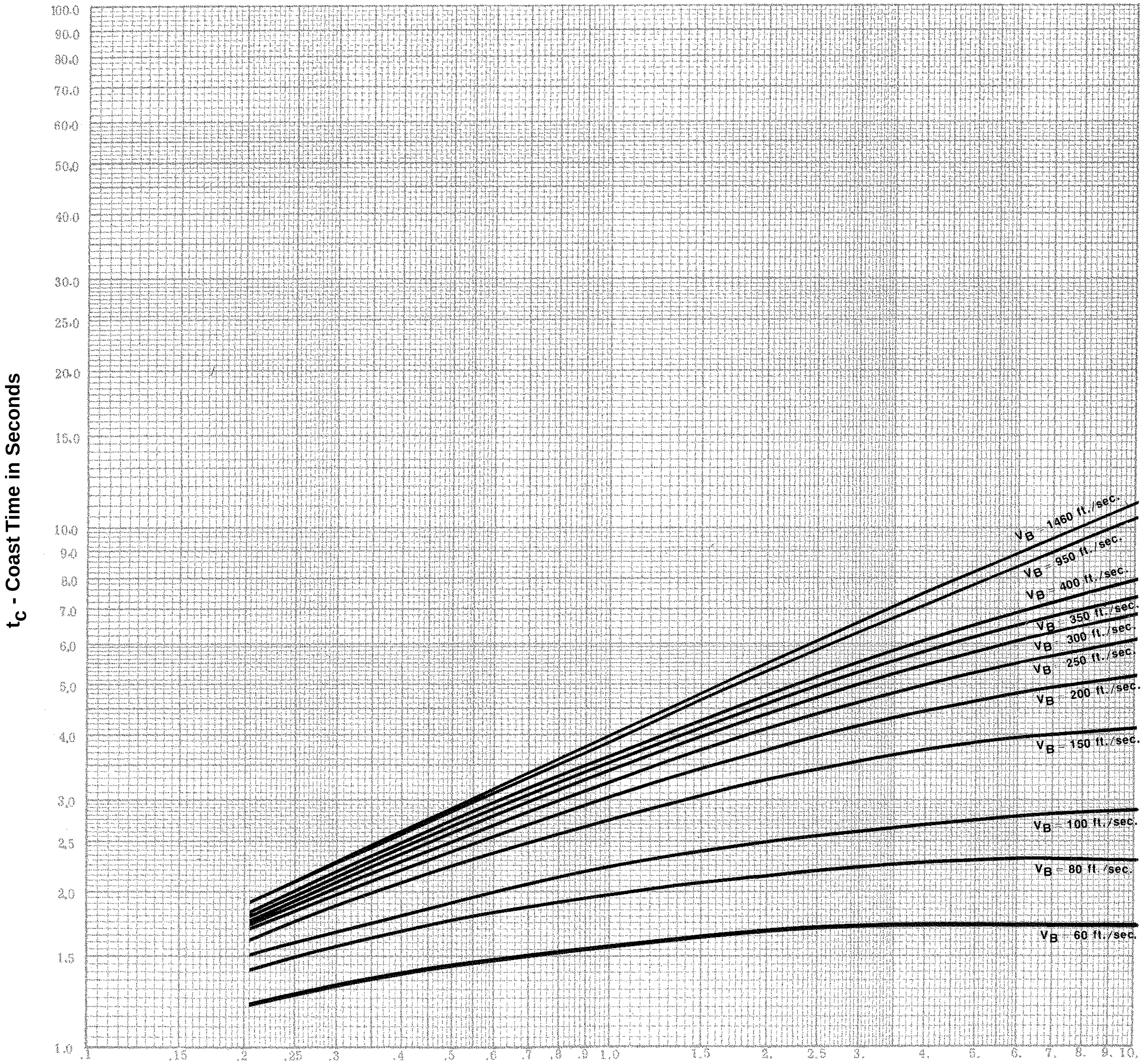


β_c = **Ballistic Coefficient** $= \frac{W}{C_D A} = \frac{\text{ounces}}{\text{inch}^2}$

COAST TIMES

FIGURE 11B

Time (t_c) in Seconds from Burnout to Peak Altitude as a function of Burnout Velocity (V_B) and Ballistic Coefficient (β_c).



β_c = Ballistic Coefficient $= \frac{W}{C_D A} = \frac{\text{ounces}}{\text{inch}^2}$

1A Astron Alpha (no longer available)

WORK AREA

Now let's try a simple problem with a real model rocket, using only the necessary information. If you do not understand a step, refer back to the instructions and the sample problem.

1. Body diameter =
   Rocket empty weight =
   Engine = A8-3
   Weight =
   Propellant weight =

   W_I =
   =
   =

2. C_DA =

3. Thrusting β = ____________________
   = ____________________
   = ____________________
   = ____________________
   =

4. Burnout Altitude from Figure 5A =
   Burnout Velocity from Figure 5B =

5. Coasting β = ________________
   = ________________
   = ________________
   =

6. Coasting Altitude =

7. Apogee point =
   =
   = 379 ft.

8. Coasting Time = 4.0 sec.

Since our A8 engine also comes with a 4-second delay, and the coasting time is 4 seconds rather than 3 seconds, the A8-5 (no longer available) engine should be used.

Refer to page 28 for a complete solution to this problem.

2A BIG BERTHA®

WORK AREA

To gain experience in understanding and using this system, calculate the performance of a Big Bertha®. After completing you calculations, check your results with the results shown on the next page.

1. Body diameter =
   Rocket empty weight =
   Engine = B4-2
   Weight =
   Propellant weight =

   W_I =
   =
   =

2. C_DA =

3. Thrusting β = ________________
   = ________________
   = ________________
   = ________________
   =

4. Burnout Altitude from Figure 6A =
   Burnout Velocity from Figure 6B =

5. Coasting β = ____________
   = ____________
   = ____________
   =

6. Coasting Altitude =

7. Apogee point =
   =
   =

8. Coasting time =

Check your answers with those on the next page.

1B ASTRON ALPHA (no longer available)

1. Body diameter = BT - 50 - 0.976 in.
   Rocket empty weight = 0.8 oz.
   Engine = A8-3
   Weight = 0.57 oz.
   Propellant weight = 0.110 oz.

   W_I = Rocket empty weight + Engine weight

   = 0.8 oz. + 0.57 oz.
   = 1.37 oz.

2. $C_DA = 0.55 \text{ in.}^2$
3. Thrusting $\beta = \frac{W_I - 1/2\, Wp}{C_DA}$

   $= \frac{1.37 \text{ oz.} - 1/2(0.110 \text{ oz.})}{0.55 \text{ in.}^2}$

   $= \frac{1.37 \text{ oz.} - 0.055 \text{ oz.}}{0.55 \text{ in.}^2}$

   $= \frac{1.315 \text{ oz.}}{0.55 \text{ in.}^2}$

   $= 2.39 \text{ oz./in.}^2$

4. Burnout Altitude from Figure 5A = 44 ft.
   Burnout Velocity from Figure 5B = 210 ft./sec.

5. Coasting $\beta = \frac{W_I - Wp}{C_DA}$

   $= \frac{1.37 \text{ oz.} - 0.110 \text{ oz.}}{0.55 \text{ in.}^2}$

   $= \frac{1.26 \text{ oz.}}{0.55 \text{ in.}^2}$

   $= 2.29 \text{ oz./in.}^2$

6. Coasting Altitude = 335 ft.

7. Apogee point = Burnout Altitude + Coasting Altitude

   = 44 ft. + 335 ft.
   = 379 ft.

8. Coasting Time = 4.0 sec.

2B BIG BERTHA®

1. Body diameter = BT-60-1.637 in.
   Rocket empty weight = 2.2 oz.
   Engine = B4-2
   Weight = 0.70 oz.
   Propellant weight = 0.294 oz.

   W_I = Rocket empty weight + Engine weight
   = 2.2 oz. + 0.70 oz.
   = 2.9 oz.

2. $C_DA = 1.55 \text{ in.}^2$

3. Thrusting $\beta = \frac{W_I - 1/2\, Wp}{C_DA}$

   $= \frac{2.90 \text{ oz.} - 1/2(0.294 \text{ oz.})}{1.55 \text{ in.}^2}$

   $= \frac{2.90 \text{ oz.} - 0.147\text{oz.}}{0.55 \text{ in.}^2}$

   $= \frac{2.753 \text{ oz.}}{1.55 \text{ in.}^2}$

   $= 1.78 \text{ oz./in.}^2$

4. Burnout Altitude from Figure 6A = 98 ft.
   Burnout Velocity from Figure 6B = 145 ft./sec.

5. Coasting $\beta = \frac{W_I - Wp}{C_DA}$

   $= \frac{2.90 \text{ oz.} - 0.294 \text{ oz.}}{1.55 \text{ in.}^2}$

   $= \frac{2.606\text{oz.}}{1.55 \text{ in.}^2}$

   $= 1.68 \text{ oz./in.}^2$

6. Coasting Altitude = 185 ft.

7. Apogee point = Burnout Altitude + Coasting Altitude
   = 98 ft. + 185 ft.
   = 283 ft.

8. Coasting time = 2.9 sec.

   The B4 engine comes with a 4 second delay. According to our figures, the B4-4 engine would be preferable to the B4-2 engine.

3A ASTRON STREAK (no longer available)

WORK AREA

Just to be certain you have mastered this system, calculate the apogee for an Astron Streak launched with an A5-4 engine (no longer available).

1. Body diameter =
   Rocket empty weight =
   Engine = A5-4
   Weight =
   Propellant weight =

W_I =
=
=

2. C_DA =

3. Thrusting β = ____________
   = ____________
   = ____________
   = ____________
   =

4. Burnout Altitude from Figure 4A =
   Burnout Velocity from Figure 4B =

5. Coasting B = ____________
   = ____________
   = ____________

6. Coasting Altitude =

7. Apogee point =
   =
   =

8. Coasting time =

Check your answers with those on the next page.

3B ASTRON STREAK (no longer available)

1. Body diameter = BT-10-0.720 in.
   Rocket empty weight = 0.1 oz.
   Engine = A5-4 (no longer available)
   Weight = 0.64 oz.
   Propellant weight = 0.110 oz.

   W_I = Rocket empty weight + Engine weight
   = 0.1 oz. + 0.64 oz.
   = 0.74 oz.

2. C_DA = 0.30 in.2

3. Thrusting $\beta = \frac{W_I - 1/2\, W_p}{C_DA}$

   $= \frac{0.74 \text{ oz.} - 1/2(0.110 \text{ oz.})}{0.30 \text{ in.}^2}$

   $= \frac{0.74 \text{ oz.} - 0.055 \text{ oz.}}{0.30 \text{ in.}^2}$

   $= \frac{0.685 \text{ oz.}}{0.30 \text{ in.}^2}$

   $= 2.28 \text{ oz./in.}^2$

4. Burnout Altitude from Figure 4A = 93 ft.
   Burnout Velocity from Figure 4B = 370 ft./sec.

5. Coasting $\beta = \frac{W_I - W_p}{C_DA}$

   $= \frac{0.74 \text{ oz.} - 0.110 \text{ oz.}}{0.30 \text{ in.}^2}$

   $= \frac{0.630 \text{ oz.}}{0.30 \text{ in.}^2}$

   $= 2.10 \text{ oz./in.}^2$

6. Coasting Altitude = 550 ft.

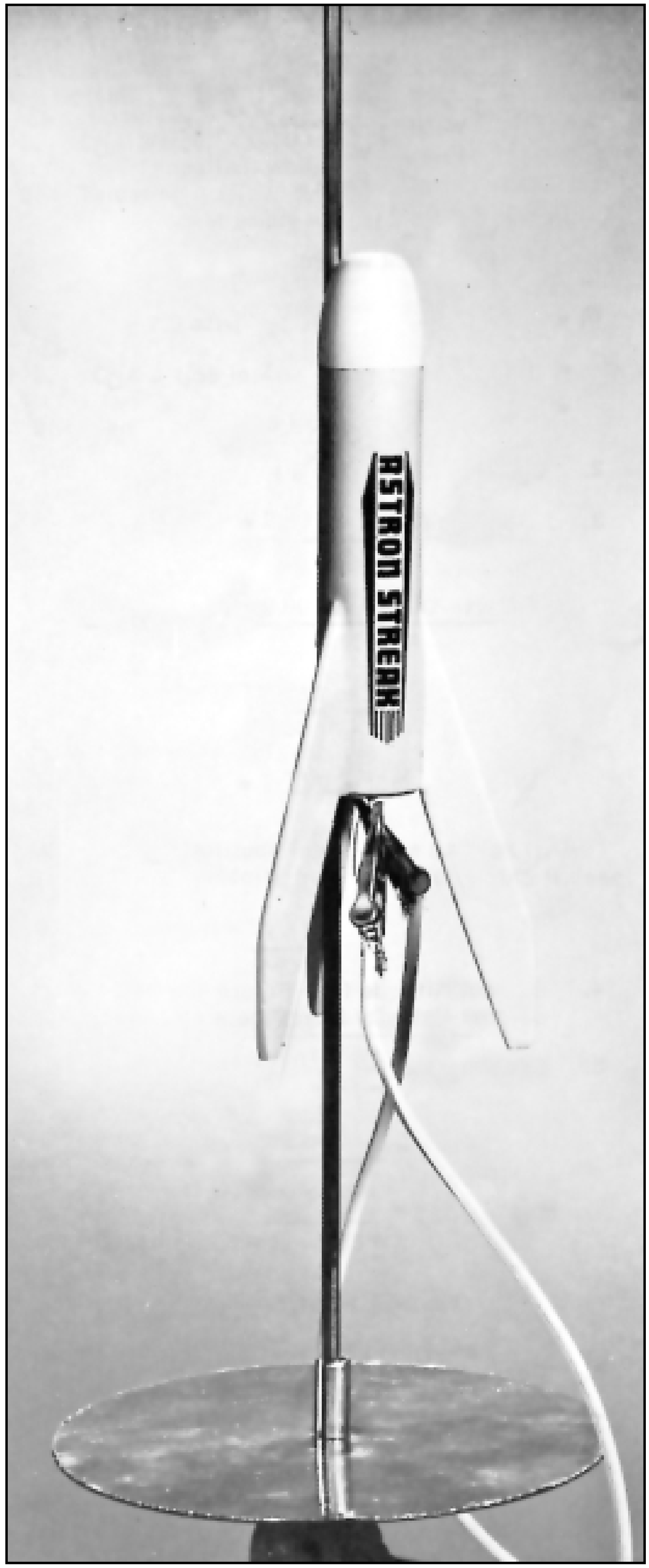
7. Apogee point = Burnout Altitude + Coasting Altitude

   = 93 ft. + 550 ft.

   = 643 ft.

8. Coasting time = 4.6 sec.

The A5 engine does not come with a delay of over 4 seconds. We can go ahead and use the A5-4 with the understanding that the ejection discharge will occur before the rocket has reached the apogee point.



4A Scrambler (no longer available)

WORK AREA

1. Body diameter =
   Rocket empty weight =
   3 engines = C6-5
   Weight =
   Propellant weight =

As for the Scrambler, rockets powered by a cluster of engines employ modified formulas, since the graphs are designed for single engine calculations. Thus we work with the proportionate weight of the rocket being lifted by any one of the cluster of motors.

Weight actual =

W_{actual} =
=
=

W_I = ____________
= ____________
=

2. C_DA =

3. Thrusting β = ____________
= ____________
= ____________
= ____________
= ____________
=

4. Burnout Altitude from Figure 4A =
   Burnout Velocity from Figure 4B =

5. Coasting β = ____________
= ____________
= ____________
= ____________
= ____________

6. Coasting Altitude =

7. Apogee point =
=
=

8. Coasting time =

Check your answers with those on the next page.

4B SCRAMBLER (no longer available)

1. Body diameter = BT-65-1.796 in. (using the diameter of the payload section, since it is the largest diamter of the rocket)
   Rocket empty weight = 2.8 oz.
   3 engines = C6-5
   Weight = 0.91oz. each
   Propellant weight = 0.440 oz. each

   Weight actual = Rocket empty weight + Engine weight

   w_{actual} = 2.8 oz. + 3(0.91 oz.)
   = 2.8 oz. + 2.73 oz.
   = 5.53 oz.

   $W_I = \frac{W_{actual}}{N}$
   $= \frac{5.53 \text{ oz.}}{3}$
   = 1.84 oz.

2. $C_DA = 1.90 \text{ in.}^2$

3. Thrusting $\beta = \frac{W_{actual} - N(1/2\ Wp)}{C_DA}$
   $= \frac{5.53 \text{ oz.} - 3(\frac{0.440 \text{ oz.}}{2})}{1.90 \text{ in.}^2}$
   $= \frac{5.53 \text{ oz.} - 3(0.22 \text{ oz.})}{1.90 \text{ in.}^2}$
   $= \frac{5.53 \text{ oz.} - 0.660 \text{ oz.}}{1.90 \text{ in.}^2}$
   $= \frac{4.87 \text{ oz.}}{1.90 \text{ in.}^2}$

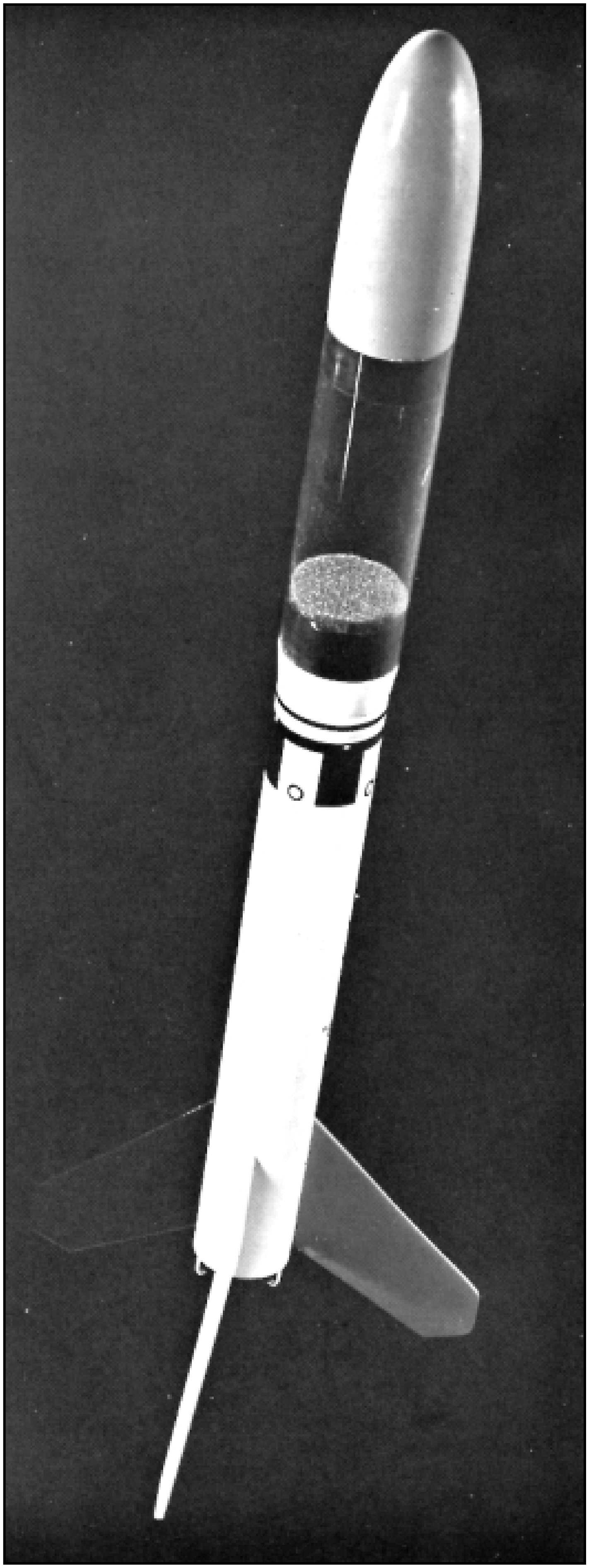
4. Burnout Altitude from Figure 9A = 440 ft.
   Burnout Velocity from Figure 9B = 420 ft./sec.

5. Coasting $\beta = \frac{W_{actual} - N(Wp)}{C_DA}$
   $= \frac{5.53 \text{ oz.} - 3(0.440 \text{ oz.})}{1.90 \text{ in.}^2}$
   $= \frac{5.53 \text{ oz.} - 1.32 \text{ oz.}}{1.90 \text{ in.}^2}$
   $= \frac{4.21 \text{ oz.}}{1.90 \text{ in.}^2}$
   $= 2.22 \text{ oz.} / \text{in.}^2$

6. Coasting Altitude = 640 ft.
7. Apogee point = Burnout Altitude + Coasting Altitude
   = 440 ft. + 640 ft.
   = 1080 ft.

8. Coasting time = 5.0 sec.

This choice of delay time was a good one.



5A CHEROKEE-D (no longer available)

WORK AREA

1. Body diameter =
   Rocket empty weight =
   Engine = D13-7 (no longer available)
      Weight =
      Propellant weight =

   W_I =

   =

   =

2. C_DA =

3. Thrusting β = ______________

   = ______________

   = ______________

   = ______________

   = ______________

4. Burnout Altitude from Figure 10A =
   Burnout Velocity from Figure 10B =

5. Coasting β = ______________

   = ______________

   = ______________

6. Coasting Altitude =

7. Apogee point =

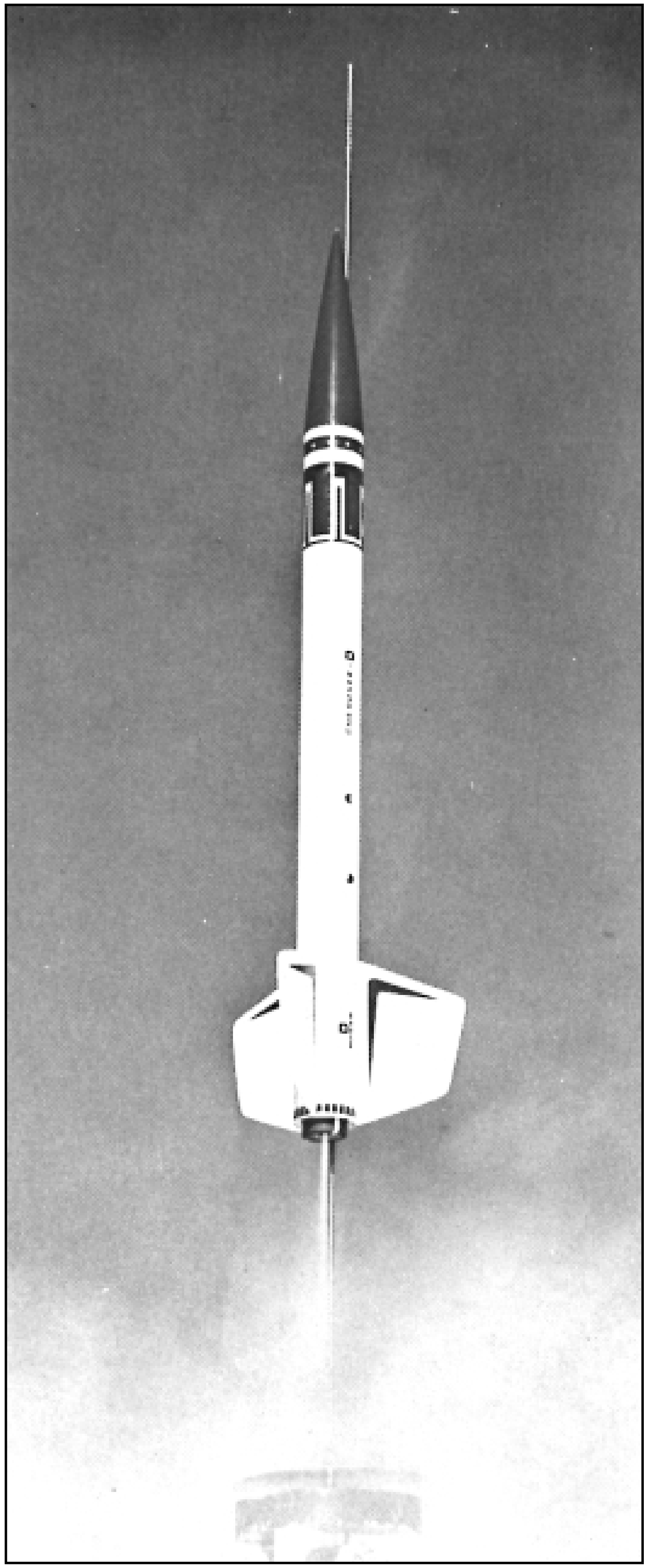
   =

   =

8. Coasting time =

Check your answers with those on the next page.

5B CHEROKEE-D (no longer available)



1. Body diameter = BT-55-1.325 in.
   Rocket empty weight = 2.75 oz.
   Engine = D13-7 (no longer available)
   Weight = 1.55 oz.
   Propellant weight = 0.879 oz.

   W_I = Rocket empty weight + Engine weight
   = 2.75 oz. + 1.55 oz.
   = 4.30 oz.

2. $C_DA = 1.00 \text{ in.}^2$

3. Thrusting $\beta = \dfrac{W_I - 1/2\,Wp}{C_DA}$

   $= \dfrac{4.30 \text{ oz.} - 1/2(0.879 \text{ oz.})}{1.00 \text{ in.}^2}$

   $= \dfrac{4.30 \text{ oz.} - 0.440 \text{ oz.}}{1.00 \text{ in.}^2}$

   $= \dfrac{3.86 \text{ oz.}}{1.00 \text{ in.}^2}$

   $= 3.86 \text{ oz./in.}^2$

4. Burnout Altitude from Figure 10A = 320 ft.
   Burnout Velocity from Figure 10B = 430 ft./sec.

5. Coasting $\beta = \dfrac{W_I - Wp}{C_DA}$

   $= \dfrac{4.3 \text{ oz.} - 0.879 \text{ oz.}}{1.00 \text{ in.}^2}$

   $= \dfrac{3.421 \text{ oz.}}{1.00 \text{ in.}^2}$

   $= 3.421 \text{ oz./in.}^2$

6. Coasting Altitude = 820 ft.

7. Apogee point = Burnout Altitude + Coasting Altitude

   = 320 ft. + 820 ft.

   = 1140 ft.

8. Coasting time = 5.8 sec.

To minimize lateral drift of the rocket as it descends under its parachute during recovery, it is desirable to use an engine with a delay to permit the rocket to begin to descend before the parachute ejection occurs. Thus, the D13-7 (no longer available) would be an appropriate engine to use.

APPENDIX I - THE EQUATIONS ASSUMPTIONS

The assumptions used in deriving these equations are essentially the same as used in the June, 1964 Model Rocket News (reference 1) article. In review:

1. Weight is assumed to be constant during thrusting (an average value is used).

2. Drag is proportional to V^2

   $D = C_D A\ 1/2\ \rho V^2$ (which is quite reasonable up to 700 ft/sec.)

3. Thrust is constant during burning (again an average value is used):

   $$T = \frac{\text{Total Impulse}}{\text{Burn Time}}$$

4. The rocket is launched straight up and no cross-wind or turbulence exists, so the flight continues straight up at zero angle-of-attack.

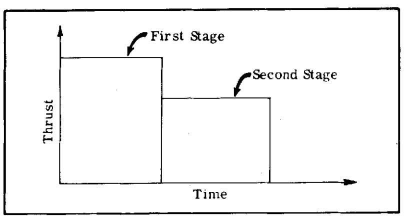
5. Thrust due to smoke delay burnoff is negligible.

6. The overall drag coefficient does not increase during the coast phase due to base drag (i.e. the smoke delay burnoff cancels it).

7. Additional weight losses due to smoke delay burnoff are neglected.

8. Atmospheric density is uniform and does not vary from the launch pad value.

9. Second and subsequent stages have no time delay for ignition and buildup to the average thrust value of the next stage. An example of the assumed time history is shown:



TERMINOLOGY

The terminology used is as follows:

sinh	=	hyperbolic sine
cosh	=	hyperbolic cosine
tanh	=	hyperbolic tangent
$\tanh^{-1}$	=	arc hyperbolic tangent
ln	=	natural logarithm
cos	=	cosine
tan	=	tangent
$\tan^{-1}$	=	arc tangent
W	=	weight of rocket in pounds (a) use average weight during thrusting (b) use final weight during coasting
C_D	=	aerodynamic drag coefficient
A	=	reference area in $ft.^2$ for professional rocketry or $in.^2$ for model rocketry
ρ	=	sea level density of air $= .002378\ \frac{\text{lb-sec}^2}{\text{ft}^4}$
T	=	average thrust in pounds for professional rocketry or ounces for model rocketry
g	=	earth's gravitational constant $= 32.174\ \text{ft/sec}^2$
t	=	time in seconds
S	=	altitude in feet
V	=	velocity in feet per second

Subscript terminology:

B	=	refers to burnout
C	=	refers to coasting
1	=	refers to first stage
2	=	refers to second stage

GENERAL EQUATIONS OF MODEL ROCKET MOTION, TERMINOLOGY AND ASSUMPTIONS

The equations of motion are presented here for those rocketeers who are interested in obtaining the complete time history of their rocket's motion, for those with an interest in mathematics, and for those who are curious as to the origin of the charts.

A. THRUSTING

1. Altitude as a function of time

$$S = \frac{1}{g}\frac{W}{C_D A\,1/2\rho}\ \ln\cosh\left\{g\sqrt{\left(\frac{T}{W}-1\right)\frac{C_D A\,1/2\rho}{W}}\,t\right\}$$

2. Velocity as a function of time

$$V = \sqrt{\frac{W}{C_D A\,1/2\rho}\left(\frac{T}{W}-1\right)}\ \tanh\left\{g\sqrt{\left(\frac{T}{W}-1\right)\frac{C_D A\,1/2\rho}{W}}\,t\right\}$$

Note that when time (t) is the burnout time (t_B) that the altitude (S) becomes the burnout altitude (S_B) and the velocity (V) becomes the burnout velocity (V_B).

B. COASTING

1. The coast altitude (S_C) is the distance gained between the time of burnout and the time when the rocket reaches its peak.

$$S_C = \frac{1}{2g}\frac{W}{C_D A\,1/2\rho}\ \ln\left\{1+\frac{C_D A\,1/2\rho\ V_B^2}{W}\right\}$$

2. The coast time (t_C) is the time it takes the rocket to slow down to zero velocity (V=0). That is, the peak is reached and the rocket will now begin to fall back to the ground.

$$t_c = \frac{1}{g}\sqrt{\frac{W}{C_D A\,1/2\rho}}\ \tan^{-1}\sqrt{\frac{C_D A\,1/2\rho\ V_B^2}{W}}$$

3. Additional altitude gained as a function of time from burnout.

$$S = S_C + \frac{1}{g}\frac{W}{C_D A\,1/2\rho}\ \ln\cos\left\{g\sqrt{\frac{C_D A\,1/2\rho}{W}}\,(t_c - t)\right\}$$

4. Velocity as a function of time from burnout

$$V = \sqrt{\frac{W}{C_D A\,1/2\rho}}\ \tan\left\{g\sqrt{\frac{C_D A\,1/2\rho}{W}}\,(t_c - t)\right\}$$

C. SECOND AND SUBSEQUENT STAGES

1. Altitude gained as a function of time from the previous stage's burnout.

$$S_2 = \frac{1}{g}\frac{W}{C_D A\,1/2\rho}\ \ln\left\{\cosh\left[g\sqrt{\frac{C_D A\,1/2\rho}{W}\left(\frac{T}{W}-1\right)}\,t\right] + V_{B_1}\sqrt{\frac{C_D A\,1/2\rho}{W}\frac{1}{\left(\frac{T}{W}-1\right)}}\ \sinh\left[g\sqrt{\frac{C_D A\,1/2\rho}{W}\left(\frac{T}{W}-1\right)}\,t\right]\right\}$$

2. Velocity as a function of time

$$V_2 = \sqrt{\frac{W}{C_D A\,1/2\rho}\left(\frac{T}{W}-1\right)}\ \tanh\left\{g\sqrt{\frac{C_D A\,1/2\rho}{W}\left(\frac{T}{W}-1\right)}\,t + \tanh^{-1}\sqrt{\frac{C_D A\,1/2\rho\ V_{B_1}^2}{W}\frac{1}{\left(\frac{T}{W}-1\right)}}\right\}$$

Again note that when time (t) is the burnout time (t_B) that the altitude (S_2) becomes the second (or third) stage altitude increment (S_{B2}) and the velocity (V_2) becomes the second (or third) stage burnout velocity (V_{B2}).

GENERAL EQUATIONS CONVERTED FOR MODEL ROCKETRY APPLICATIONS

The basic equations are converted for model rocketry calculation use by first substituting the known values for the constants ρ and g. Also since we deal primarily with weights in ounces instead of pounds and frontal areas in square inches instead of square feet we use the additional conversion factors:

1 pound = 16 ounces

1 square foot = 144 square inches

We can also shorten the repetitive calculations considerably by precomputing the ballistic coefficient (β)

$$\beta = \frac{W}{C_D A}\ \text{in}\ \frac{\text{ounces}}{\text{Square inches of frontal area}}$$

and the drag-free acceleration (a)

$$a = \left(\frac{T}{W}-1\right)\ \text{in g's of acceleration}$$

(Note that the drag-free acceleration (a) is non-dimensional so both thrust (T) and weight (W) must be in ounces.

With these modifications the A, B, and C equations for altitude and velocity reduce to the following:

A. THRUSTING

1. Altitude as a function of time

$$S = 235.26\,\beta\ \ln\cosh\left[.36981\frac{\sqrt{a}}{\sqrt{\beta}}\,t\right]$$

2. Velocity as a function of time

$$V = 87.0\sqrt{\beta}\sqrt{a}\ \tanh\left[.36981\frac{\sqrt{a}}{\sqrt{\beta}}\,t\right]$$

Again note that when time (t) is equal to the motor burnout time (t_B) that the altitude (S) equals the burnout altitude (S_B) and the velocity (V) equals the burnout velocity (V_B).

B. COASTING

1. The coast altitude (S_C) is the distance gained between the time of burnout and the time at which the rocket reaches its peak.

$$S_C = 117.63\,\beta\ \ln\left[1+\frac{V_B^2}{7569.386\beta}\right]$$

2. The coast time (t_C) is the time it takes the rocket to slow down to zero velocity (V = 0). That is, the peak is reached on the rocket and will now begin to fall back down.

$$t_c = 2.7041\sqrt{\beta}\ \tan^{-1}\left[\frac{V_B}{87.0\sqrt{\beta}}\right]$$

Express t^{-1} in radians.

3. Additional altitude gained as a general function of time from burnout.

$$S = S_C + 235.26\,\beta\ \ln\cos\left\{.36981\frac{(t_c - t)}{\sqrt{\beta}}\right\}$$

4. Velocity as a general function of time from burnout.

$$V = 87.0\sqrt{\beta}\ \tan\left\{.36981\frac{(t_c - t)}{\sqrt{\beta}}\right\}$$

C. SECOND AND SUBSEQUENT STAGES

1. Altitude gained as a function of time from burnout of the previous stage.

$$S_2 = 235.26\,\beta\ \ln\left\{\cosh\left[.36981\frac{\sqrt{a}}{\sqrt{\beta}}\,t\right] + \frac{V_{B1}}{87.0\sqrt{\beta}\sqrt{a}}\sinh\left[.36981\frac{\sqrt{a}}{\sqrt{\beta}}\,t\right]\right\}$$

2. Velocity of the second (or third) stage as a function of time from burnout of the previous stage.

$$V_{B_2} = 87.0\sqrt{\beta}\sqrt{a}\ \tanh\left\{.36981\frac{\sqrt{a}}{\sqrt{\beta}}\,t + \tanh^{-1}\frac{V_{B1}}{87.0\sqrt{\beta}\sqrt{a}}\right\}$$

Again note that when time (t) is equal to the burnout time (t_B) for this stage that the altitude (S_2) becomes the second (or third) stage altitude increment (S_{B2}) and the velocity (V_2) becomes the second (or third) stage burnout velocity (V_{B2}).

HOW TO ANALYZE TWO OR THREE STAGE ROCKETS

In order to predict altitudes for multiple stage rockets one must resort to suing equation (C1) and (C2). A book of mathematical tables (such as reference 5) must be obtained in order to evaluate the hyperbolic sines, cosines, tangents and arc tangents (perhaps one of your teachers can help you the first time through).

The first stage burnout velocity and altitude can be found using the charts in the usual way. The ballistic coefficient (β) and drag-free acceleration (a) are then calculated for the second stage. Equations (Cl) and (C2) use these value in conjunction with the burnout velocity of the first stage (V_{B_1}) and the motor burn time of second stage (t_B) to obtain the altitude gained during second stage thrusting (S_{B_2}) and the new velocity at second stage burnout (V_{B_2}).

If it is just a two-stage vehicle it will now coast to the peak; and so the coasting chart data can be utilized as usual.

A three-stage vehicle on the other hand has to make use of equations (C1) and (C2) again to find the third stage burnout velocity and the increment of altitude gained during third-stage thrusting. The coast altitude and time can then be found as a function of the ballistic coefficient (as based on the empty weight of the third stage) and the third stage burnout velocity.

Needless to say, it is important to use weight values in your calculations that properly reflect the effect of the booster stages falling away after they burn out.

The reason that the second and third stage data had to be calculated, instead of just simply read from a chart, is that there were too many variable factors.

You will notice that four variables affect the velocity and altitude gained during second or third stage thrusting: 1) type of rocket motor, 2) weight of rocket, 3) drag of rocket, and 4) initial velocity due to previous stage. Single-stage rockets only have three variables that effect velocity and altitude: 1) type of rocket motor, 2) weight of rocket, and 3) drag of rocket. Plotting each motor type as a separate graph for single-stage rockets essentially reduces the number of variables from 3 to 2, thus lending itself to plotting on two-dimensional graph paper. Using this same procedure for multiple-stage rockets still leaves us with trying to plot mathematical functions of three variables on two-dimensional graph paper.

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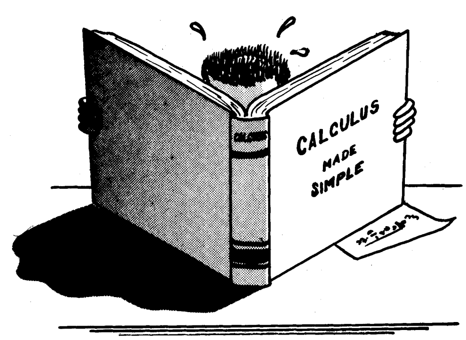
WORK AREA



DISCUSSION OF CALCULUS

Calculus is the branch of mathematics which allows one to analyze movement and change. Complex motions can essentially be broken into small increments, and the changes occurring at each instant can be investigated.

Prior to this report, model rocket altitudes with aerodynamic drag effects were frequently determined by using small time interval step-by-step computation methods. At a given velocity the drag can be computed and subtracted from the thrust to find the net force acting on the rocket. This, in turn, gives the net acceleration for that time period. The assumption is then made that this acceleration will be constant for the next short time period. Next the increase in velocity due to the assumed constant acceleration is computed and the entire process is ready to be repeated. We do know, however, that the velocity and drag do not take small jumps after each time period but in reality are smoothly increasing during thrusting, and are smoothly decreasing during coasting.

The step-by-step method thus introduces slight errors in the predicted altitude. Taking smaller time periods increases the accuracy. With calculus the time periods are infinitely small and as a result the velocity and drag become the perfectly smooth variables with time that we know them to be. A very well illustrated, interesting and simple to understand layman's description of calculus is given in Chapter 5 of the "Mathematics" volume of the Life Science Library (reference 6). This chapter helps very much to convey the importance of calculus as a fundamental tool in today's modern world of spacecraft and electronics.

The following two sections of the report can be bypassed with no loss in continuity. These detailed derivations of the thrusting and coasting equations for model rockets are presented primarily for those persons with the prerequisite background in the calculus operations of "differentiation" and "integration", who desire a more complete understanding of where the equations for the altitude prediction charts originate.

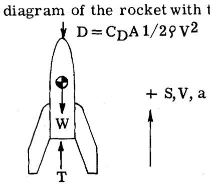
DERIVATION OF THE THRUSTING EQUATIONS

The equation of motion for the rocket during thrusting is obtained by applying Sir Isaac Newton's first law, which can be written as:

$$F = ma \qquad (1)$$

Where: F = The summation of all external forces applied to the rocket.
m = The mass of the rocket.
and: a = The acceleration of the rocket.

A free body diagram of the rocket with the proper forces is shown.



Let us, for convenience, choose the displacement S, the velocity V, and the acceleration a, as positive in the upward direction. Since we are interested only in the one-dimensional upward trajectory of the rocket we eliminate all possible forces acting in any other direction than S.

The following forces are acting on the rocket:

1. THRUST (T)
   The thrust is considered to be constant from ignition to burnout. It is easily seen that the thrust is acting in the positive direction and thus is a positive force.

2. WEIGHT (W)
   This is the earth's gravitational field acting on the mass of the rocket. We assume that the weight loss from ignition to burnout is small enough so that taking the average weight during this period gives sufficient accuracy. Thus, the weight will be a constant and, of course, can be seen to be acting in the negative direction.

3. DRAG (D) $\quad D = C_D A\, 1/2\, \rho\, V^2$
   The aerodynamic drag is assumed to hold to the V^2 law for all flight speeds. It too is a negative force.

Thus, equation (1) becomes:

$$T - D - W = ma \qquad (2)$$

From physics we know the mass (m) is equal to the weight (W) divided by the acceleration due to gravity (g)

$$m = \frac{W}{g}$$

Substituting the values of m and D into equation (2) we obtain:

$$T - C_D A\, 1/2\, \rho V^2 - W = \frac{W}{g} a$$

For a given rocket the term $C_D A\, 1/2\, \rho$ is a constant.

Instead of carrying all these terms through the derivation we call it the constant K.

$$K = C_D A\, 1/2\, \rho$$

Thus, our equation becomes:

$$T - W - KV^2 = \frac{W}{g} a$$

Now the acceleration (a) is the rate of change of velocity with respect to time. In calculus this is written as:

$$a = \frac{dV}{dt}$$ and is called the first derivative of velocity.

our equation now becomes:

$$T - W - KV^2 = \frac{W}{g}\frac{dV}{dt} \qquad (3)$$

This is called a first order differential equation. Luckily, with a bit of rearranging it can be integrated.

Multiplying both sides by $\frac{dt}{T-W-KV^2}$

We obtain:

$$dt = \frac{W}{g}\, \frac{dV}{T-W-KV^2} \qquad (4)$$

Next we integrate both sides:
dt from an initial time (t_o) to some time (t), and dV from an initial velocity (V_o) to velocity (V) at time (t).

thus:

$$\int_{t_o}^{t} dt = \frac{W}{g}\int_{V_o}^{V} \frac{dV}{T-W-KV^2}$$

The more difficult right-hand side can be reduced to the standard integral form given on page 29 of reference 5. This is accomplished by making the following substitutions:

$$a^2 = T - W$$
$$b^2 = K$$
$$U = V$$

Reference 5 gives the solution as:

$$\int \frac{dU}{a^2 - b^2U^2} = \frac{1}{ab}\tanh^{-1}\frac{bU}{a}$$

When applied to our equation we find that:

$$t\Big]_{t_o}^{t} = \frac{W}{g}\frac{1}{\sqrt{T-W}\sqrt{K}}\tanh^{-1}\left(\sqrt{\frac{K}{T-W}}\,V\right)\Big]_{V_o}^{V}$$

or:

$$t - t_o = \frac{W}{g}\frac{1}{\sqrt{(T-W)K}}\tanh^{-1}\left(\sqrt{\frac{K}{T-W}}\,V\right) - \frac{W}{g}\frac{1}{\sqrt{(T-W)K}}\tanh^{-1}\left(\sqrt{\frac{K}{T-W}}\,V_o\right)$$

Now at ignition time $t_o = 0$. We will have the rocket's velocity $V_o = 0$. Upon substitution of these values we get:

$$t = \frac{W}{g}\frac{1}{\sqrt{(T-W)K}}\tanh^{-1}\left(\sqrt{\frac{K}{T-W}}\,V\right) \qquad (5)$$

since the value of $\tanh^{-1}(0)$ is zero.

We now find it convenient to define a constant β_o called the density ballistic coefficient

$$\beta_o = \frac{W}{C_D A\, 1/2\rho} = \frac{W}{K}$$

Where K is the constant previously defined. Note that the "density ballistic coefficient" must be distinguished from the "ballistic coefficient" defined in the introduction and subsequently used throughout the report.

Seeing the "density ballistic coefficient" terms appearing in each of the final equations summarized earlier should give one a clue as to why the density correction factor due to temperature and launch altitude variations can be included right in the "ballistic coefficient". This correction factor is simply the ratio of density at any altitude and temperature ($\rho_{H,T}$) to the sea level density (ρ).

Thus, the actual density is simply: $\rho_{H,T} = \left(\frac{\rho_{H,T}}{\rho}\right)\rho$

where $\left(\frac{\rho_{H,T}}{\rho}\right)$ is the correction factor for air density presented in Figure 9.

For values of density other than sea level we will have

$$\beta_o = \frac{W}{C_D A \frac{1}{2}\rho_{HT}} = \frac{W}{C_D A \frac{1}{2}\left(\frac{\rho_{HT}}{\rho}\right)\rho} = \frac{\beta}{\frac{1}{2}\rho}$$

Where our regular β includes the density correction term in a mathematically acceptable manner.

$$\beta = \frac{W}{C_D A\left(\frac{\rho_{HT}}{\rho}\right)}$$

We also find it convenient to use the drag free acceleration in g's

$$a_o = \frac{T}{W} - 1$$

Note that we now use the subscript "o" for the drag free acceleration as a means for distinguishing it from the true rocket acceleration term (a) as used in Newton's basic motion equation $F = ma$.

a_o and β_o can be utilized by working with the constants of equation (5). These constants are re-arranged in the following manner:

$$\frac{W}{g}\frac{1}{\sqrt{(T-W)K}} = \frac{1}{g}\sqrt{\frac{W^2}{(T-W)K}}$$

$$= \frac{1}{g}\sqrt{\frac{W}{K}\frac{1}{\left(\frac{T-W}{W}\right)}}$$

$$= \frac{1}{g}\sqrt{\frac{W}{K}\frac{1}{\left(\frac{T}{W}-1\right)}}$$

$$= \frac{1}{g}\sqrt{\frac{\beta_o}{a_o}}$$

also:

$$\sqrt{\frac{K}{T-W}} = \sqrt{\frac{K\ W}{(T-W)\ W}}$$

$$= \sqrt{\frac{1}{\beta_o}\frac{1}{\left(\frac{T}{W}-1\right)}}$$

$$= \sqrt{\frac{1}{\beta_o a_o}}$$

$$= \frac{1}{\sqrt{\beta_o a_o}}$$

thus, equation (5) becomes:

$$t = \frac{1}{g}\sqrt{\frac{\beta_o}{a_o}}\ \tanh^{-1}\left(\frac{V}{\sqrt{\beta_o a_o}}\right) \qquad (6)$$

Noting that the identity form $M = \tanh^{-1}(N)$ means that $N = \tanh(M)$ gives us a method by which we can obtain velocity (V) as a function of time (t) from equation (6) so

$$\boxed{V = \sqrt{\beta_o a_o}\ \tanh\left[g\sqrt{\frac{a_o}{\beta_o}}\,t\right]} \qquad (7)$$

which is equation (A2) as can be verified by substituting in the terms represented by β_o and a_o.

This equation gives us the velocity of the rocket at any time (t) as long as the motor is burning. At the time (t_B) when the motor burns out, we then have the burnout velocity (V_B) of the rocket.

To find the height that the rocket will achieve as a function of time during thrusting, we note that velocity (V) is the rate of change of displacement (or distance if you like) with respect to time. In calculus this is written as

$V = \frac{dS}{dt}$ called the first derivative of displacement.

Thus for equation (7) we have:

$$\frac{dS}{dt} = \sqrt{\beta_o a_o}\ \tanh\left[g\sqrt{\frac{a_o}{\beta_o}}\,t\right]$$

This is another first order differential equation and can be solved as before;

$$dS = \sqrt{\beta_o a_o}\ \tanh\left[g\sqrt{\frac{a_o}{\beta_o}}\,t\right]dt$$

integrating both sides for "dS" from an initial displacement (S_o) to some displacement (S), and "dt" from the initial time (t_o) to the corresponding time (t) we obtain:

$$\int_{S_o}^{S} dS = \sqrt{\beta_o a_o}\int_{t_o}^{t}\tanh\left[g\sqrt{\frac{a_o}{\beta_o}}\,t\right]dt$$

A standard integral form on Page 155 of reference 5 is:

$$\int \tanh X\, dX = \ln\cosh X$$

In our case

$$X = g\sqrt{\frac{a_o}{\beta_o}}\,t$$

and the differential "dX" must have the value

$$dX = g\sqrt{\frac{a_o}{\beta_o}}\,dt$$

This value is inserted as follows:

$$\int_{S_o}^{S} dS = \sqrt{\beta_o a_o}\int_{t_o}^{t}\tanh\left[g\sqrt{\frac{a_o}{\beta_o}}\,t\right]\frac{g\sqrt{\frac{a_o}{\beta_o}}}{g\sqrt{\frac{a_o}{\beta_o}}}dt$$

$$\int_{S_o}^{S} dS = \frac{\sqrt{\beta_o a_o}}{g\sqrt{\frac{a_o}{\beta_o}}}\int_{t_o}^{t}\tanh\left[g\sqrt{\frac{a_o}{\beta_o}}\,t\right]\left(g\sqrt{\frac{a_o}{\beta_o}}\,dt\right)$$

and is now integrable. Performing the integration we obtain:

$$S - S_o = \frac{\beta_o}{g}\ln\cosh\left[g\sqrt{\frac{a_o}{\beta_o}}\,t\right] - \frac{\beta_o}{g}\ln\cosh\left[g\sqrt{\frac{a_o}{\beta_o}}\,t_o\right]$$

Note that at ignition time $t_o = 0$, we have zero distance $S_o = 0$ and zero velocity. Thus we obtain:

$$\boxed{S = \frac{\beta_o}{g}\ln\cosh\left(g\sqrt{\frac{a_o}{\beta_o}}\,t\right)}$$

Since the value of

$\cosh(0) = 1$

and $\ln(1) = 0$

or $\ln\cosh(0) = 0$

The above equation is identical to (A1) and again can be verified by substituting in all the terms represented by β_o and a_o. This equation gives the burnout altitude (S_B) by substituting the burnout time (t_B) for t.

DERIVATION OF THE COASTING EQUATIONS

We now return to our initial equation.

$$T - D - W = ma = \frac{W}{g}\frac{dV}{dt}$$

It is easily seen that we have a coasting situation when thrust (T) = 0, and weight (W) equals the final weight of the rocket at burnout. Therefore, our coasting equation becomes:

$$-KV^2 - W = \frac{W}{g}\frac{dV}{dt} \qquad (8)$$

Multiplying both sides by $\frac{dt}{-KV^2-W}$ we get:

$$dt = -\frac{W}{g}\frac{dV}{(W+KV^2)} \qquad (9)$$

which will be integrated from the burnout time (t_B) to some time (t), and from the burnout velocity (V_B) to the velocity (V) which corresponds to time (t).

$$\int_{t_B}^{t} dt = -\frac{W}{g}\int_{V_B}^{V}\frac{dV}{W+KV^2} \qquad (10)$$

The right-hand side can be reduced to the standard integral form given on page 25 of reference 5 by the following substitutions:

$a^2 = W$

$b^2 = K$

$U = V$

The standard form is:

$$\int\frac{dU}{a^2+b^2U^2} = \frac{1}{ab}\tan^{-1}\frac{bU}{a}$$

Applying the standard form to equation (10) we obtain:

$$t\Big]_{t_B}^{t} = -\frac{W}{g}\frac{1}{\sqrt{WK}}\tan^{-1}\left(\sqrt{\frac{K}{W}}V\right)\Big]_{V_B}^{V}$$

or:

$$t - t_B = -\frac{W}{g}\frac{1}{\sqrt{WK}}\tan^{-1}\left(\sqrt{\frac{K}{W}}V\right) + \frac{W}{g}\frac{1}{\sqrt{WK}}\tan^{-1}\left(\sqrt{\frac{K}{W}}V_B\right)$$

In order to predict delay times easily we start counting time from the time of motor burnout (t_B). This is accomplished by setting t_B to zero. Thus:

$$t = -\frac{W}{g}\frac{1}{\sqrt{WK}}\tan^{-1}\left(\sqrt{\frac{K}{W}}V\right) + \frac{W}{g}\frac{1}{\sqrt{WK}}\tan^{-1}\left(\sqrt{\frac{K}{W}}V_B\right) \qquad (11)$$

where t is the time from motor burnout.

Now we again introduce the "density ballistic coefficient" in order to reduce the complexity of the equation.

$$\beta_o = \frac{W}{C_D A\, 1/2\rho} = \frac{W}{K}$$

now:

$$\frac{W}{g}\frac{1}{\sqrt{WK}} = \frac{1}{g}\sqrt{\frac{W^2}{WK}}$$

$$= \frac{1}{g}\sqrt{\frac{W}{K}}$$

$$= \frac{1}{g}\sqrt{\beta_o}$$

also:

$$\sqrt{\frac{K}{W}} = \sqrt{\frac{1}{\beta_o}}$$

$$= \frac{1}{\sqrt{\beta_o}}$$

Therefore, equation (11) becomes:

$$t = -\frac{1}{g}\sqrt{\beta_o}\ \tan^{-1}\left(\frac{V}{\sqrt{\beta_o}}\right) + \frac{1}{g}\sqrt{\beta_o}\ \tan^{-1}\left(\frac{V_B}{\sqrt{\beta_o}}\right) \qquad (12)$$

which can be rearranged as follows to solve for velocity (V) as a function of time (t).

$$\frac{gt}{\sqrt{\beta_o}} = -\tan^{-1}\left(\frac{V}{\sqrt{\beta_o}}\right) + \tan^{-1}\left(\frac{V_B}{\sqrt{\beta_o}}\right)$$

$$\tan^{-1}\left(\frac{V}{\sqrt{\beta_o}}\right) = \tan^{-1}\left(\frac{V_B}{\sqrt{\beta_o}}\right) - \frac{gt}{\sqrt{\beta_o}}$$

Noting that the following is true;

If $M = \tan^{-1}(N)$

then $N = \tan(M)$

we can proceed to solve for V. Thus:

$$\boxed{V = \sqrt{\beta_o}\ \tan\left\{\tan^{-1}\left(\frac{V_B}{\sqrt{\beta_o}}\right) - \frac{gt}{\sqrt{\beta_o}}\right\}} \qquad (13)$$

which is essentially equation (B4),* a general equation for rocket velocity after burnout as a function of time after burnout. (Further manipulations will get (13) into the exact form as presented in the report.) We can also use (13) to determine the coast time (t_c) of the rocket. This will be the time at which the rocket slows down to zero vertical velocity.

Substituting V=0 on the left side of (13) and noting that tan(0)=0 we obtain:

$$0 = \tan^{-1}\left(\frac{V_B}{\sqrt{\beta_o}}\right) - \frac{gt_c}{\sqrt{\beta_o}}$$

Therefore:

$$\boxed{t_c = \frac{1}{g}\sqrt{\beta_o}\ \tan^{-1}\left(\frac{V_B}{\sqrt{\beta_o}}\right)} \qquad (14)$$

which is equation (B2)* with the proper β_o substitution.

Next we want to find the distance the rocket coasts as a function of time from motor burnout. We can make a first order differential equation relating distance (S) and time (t) by again using the substitution:

$$V = \frac{dS}{dt}$$

thus, for equation (13) we get:

$$\frac{dS}{dt} = \sqrt{\beta_o}\ \tan\left\{\tan^{-1}\left(\frac{V_B}{\sqrt{\beta_o}}\right) - \frac{gt}{\sqrt{\beta_o}}\right\} \qquad (15)$$

A reduction in complexity can be accomplished by multiplying both sides of equation (14) by $\frac{g}{\sqrt{\beta_o}}$.

$$\frac{g[t_c]}{\sqrt{\beta_o}} = \frac{g}{\sqrt{\beta_o}}\left[\frac{1}{g}\sqrt{\beta_o}\ \tan^{-1}\frac{V_B}{\sqrt{\beta_o}}\right] = \tan^{-1}\frac{V_B}{\sqrt{\beta_o}} \qquad (16)$$

thus, (15) becomes:

$$\frac{dS}{dt} = \sqrt{\beta_o}\ \tan\left\{\frac{gt_c}{\sqrt{\beta_o}} - \frac{gt}{\sqrt{\beta_o}}\right\}$$

$$\frac{dS}{dt} = \boxed{V = \sqrt{\beta_o}\ \tan\left[\frac{g}{\sqrt{\beta_o}}(t_c - t)\right]} \qquad (17)$$

Note that the velocity (equation 17) is now in the form as presented in equation (B4). Now rearranging equation (17) and integrating dS from the burnout displacement (S_B) to some displacement (S) and dt from the burnout time (t_B) to the time (t) corresponding to the displacement (S), we obtain:

$$\int_{S_B}^{S} dS = \sqrt{\beta_o}\int_{t_B}^{t}\tan\left[\frac{g}{\sqrt{\beta_o}}(t_c - t)\right]dt$$

and can use the standard integral form on page 110 of reference 5 to solve the right-hand side;

$$\int \tan X\, dX = -\ln\cos X$$

in this situation our $X = \frac{g}{\sqrt{\beta_o}}(t_c - t)$

and so the differential dX must have the value

$$dX = -\frac{g}{\sqrt{\beta_o}}$$

The factor is included as before and the result is:

$$\int_{S_B}^{S} dS = \frac{\sqrt{\beta_o}}{\frac{-g}{\sqrt{\beta_o}}}\int_{t_B}^{t}\tan\left[\frac{g}{\sqrt{\beta_o}}(t_c - t)\right]\left(\frac{-g}{\sqrt{\beta_o}}\,dt\right)$$

which is now integrable. Thus:

$$S - S_B = -\frac{\beta_o}{-g}\ln\cos\left[\frac{g}{\sqrt{\beta_o}}(t_c - t)\right]$$

$$-(-)\frac{\beta_o}{-g}\ln\cos\left[\frac{g}{\sqrt{\beta_o}}(t_c - t_B)\right]$$

In order to have the burnout conditions as a reference point to measure coast altitude and delay time, we set SB=0 and tB=0, our equation becomes:

$$S = +\frac{\beta_o}{g}\ln\cos\left[\frac{g}{\sqrt{\beta_o}}(t_c - t)\right] - \frac{\beta_o}{g}\ln\cos\left(\frac{g}{\sqrt{\beta_o}}t_c\right) \qquad (18)$$

it can readily be seen that when (t) the flight time equals the coast time to peak (tc) the rocket will have reached its maximum altitude from burnout (S_C).

$$\boxed{S_c = \frac{-\beta_o}{g}\ln\cos\left(\frac{g}{\sqrt{\beta_o}}t_c\right)} \qquad (19)$$

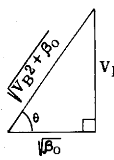
Since cos (0)=1 and In(1)=0 therefore In cos (0)=0. Note that equation (19) is an alternate form of equation (B1) and is useful when coast time (t_C) is also computed.

Substitution for the value of t_c from equation (14) into this latest equation yields:

$$S_c = \frac{-\beta_o}{g}\ln\cos\left[\frac{g}{\sqrt{\beta_o}}\left(\frac{\sqrt{\beta_o}}{g}\tan^{-1}\frac{V_B}{\sqrt{\beta_o}}\right)\right]$$

$$S_c = \frac{-\beta_o}{g}\ln\cos\tan^{-1}\left(\frac{V_B}{\sqrt{\beta_o}}\right) \qquad (20)$$

The dual trigonometric functions suggest some simplifying can be done. Let us look at the right triangle:



(The Value for the hypotenuse is found by using the Pythagorean Theorem.)

Now: $\tan\theta = \frac{V_B}{\sqrt{\beta_o}}$

or: $\theta = \tan^{-1}\left(\frac{V_B}{\sqrt{\beta_o}}\right)$

also: $\cos(\theta) = \frac{\sqrt{\beta_o}}{\sqrt{V_B^2+\beta_o}} = \cos\left(\tan^{-1}\frac{V_B}{\sqrt{\beta_o}}\right)$

Thus our coasting altitude formula (20) can be simplified to:

$$S_c = \frac{-\beta_o}{g}\ln\frac{\sqrt{\beta_o}}{\sqrt{V_B^2+\beta_o}}$$

$$= \frac{-\beta_o}{g}\ln\frac{1}{\sqrt{\frac{V_B^2}{\beta_o}+1}}$$

$$= \frac{-\beta_o}{g}\ln\left(\frac{V_B^2}{\beta_o}+1\right)^{-1/2}$$

$$= -(-1/2)\frac{\beta_o}{g}\ln\left(\frac{V_B^2}{\beta_o}+1\right)$$

$$\boxed{S_c = \frac{\beta_o}{2g}\ln\left[1+\frac{V_B^2}{\beta_o}\right]}$$

This is equation (B1)* for the coast altitude as can be verified by substitution of the terms comprising (βo).

The general equation for distance at any time between burnout and peak altitudes, which is presented in the report as equation (B3)*, is found by substituting the value of (S_C) found in equation (19), directly into equation (18).
Thus we obtain equation (B3) as:

$$\boxed{S = S_c + \frac{\beta_o}{g}\ln\cos\left\{\frac{g}{\sqrt{\beta_o}}(t_c - t)\right\}}$$

Some of you may be interested in verifying that the equations degenerate to the drag free projectile equations when drag equals zero. This can easily be done by applying L' hopital's limit rule for $C_D \rightarrow 0$.

NOTE: An easy method for finding natural logarithms of numbers less than 1.0 that should prove useful when doing hand computations involves noting:

that In(AB) = In A + In B

and In(A/B) = In A - In B

Next, assume we want to compute in(.572). This can easily be accomplished as follows:

$$\ln(.572) \quad \ln\left(\frac{5.72}{10}\right) = \ln(5.72) - \ln(10)$$
$$= 1.7440 - 2.3026$$

$$\ln(.572) = .5586$$

A remark or two concerning the use of the equations and graphs are in order at this point. No calculated value or burnout altitude, burnout velocity, coast altitude and coast time can be regarded as perfectly exact. The formulas used are based on certain assumptions as to the magnitude and repeatability of the average thrust durations for each type engine Also, the slight decrease in air density as the rocket climbs, the minor perturbations to the flight path that will surely occur and the true propellant weight burnoff time history are not taken into consideration.

The mathematical analysis, on the other hand, is exact and perhaps elaborate and impressive to the uninitiated. The disadvantage in the using of these allegedly "precise" formulas is the possibility of being misled into thinking that the results they yield correspond exactly to the real condition. It must be kept in mind that the results in reality are just close approximations and are limited by the basic assumptions made.

In actuality for this work, as in altitude tracking, great precision in numerical work is not justified and slide rule calculations giving results to three significant figures are sufficient. Admittedly, though, the crudest results obtained using the methods of this report will be much more realistic than the grossly erroneous altitudes calculated without any consideration of aerodynamic drag effects.

Perhaps the main value of this paper lies in the fact that it is simple to use for all rocketeers and at the same time contains some scientific aspects which will keep the more advanced rocketeers busy investigating and eventually understand the more sophisticated principles involved.

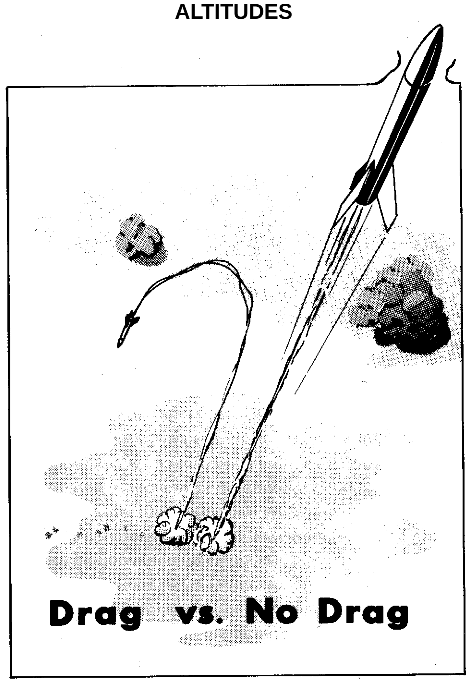
APPENDIX II

SUGGESTIONS FOR EXPERIMENTS

The following experiments will require precise altitude measurement. Setting up a two station tracking system as outlined in Estes Industries Technical Report TR-3, "Altitude Tracking", should be adequate. In addition, you will need a scale to weigh your rockets. A stopwatch to time the ascent will also be useful.

Experiment I

DRAG VERSUS NO-DRAG ALTITUDES



In this experiment you will perform some flight tests to compare the actual altitudes reached by one of your rockets to altitudes computed a) with drag effects, and b) without drag effects.* Use an aerodynamic drag coefficient of C_D=.75 to start with and be sure to use a motor that has a delay time slightly greater than the computed coast time (t_C). This will insure that your model will reach its peak altitude prior to nosecone ejection. Calculate your model's altitude in the same manner as outlined in the sample problem.

By performing this experiment with both large and small diameter rockets for both large and medium total impulse motors, you will have gained a real understanding of the importance of aerodynamic drag. It will also help you realize that neglecting the effects of drag gives completely ridiculous results.

It is convenient to calculate the no-drag altitude of any single stage rocket by the following formula.

$$S_{Total} = \frac{1}{2}\left(\frac{T}{W}\right)\left(\frac{T}{W} - 1\right) g t_B^2$$

This formula was derived in Model Rocket News, Volume 4, Number 2.

Experiment 2

DRAG COEFFICIENT MEASUREMENT

Up until now all we talked about was using Mr. G. H. Stine's aerodynamic drag coefficient of C_D=.75 as a standard for every rocket, inasmuch as that value was determined using an accurate wind tunnel. This experiment uses a method whereby a good value for the drag coefficient of a rocket can be determined without having to build an expensive wind tunnel. All you will have to do is fly your birds a few times and measure the peak altitudes. That's your favorite pastime anyway - we hope.

First calculate total altitudes for various assumed drag coefficient values for your rocket as shown in table 1. Plot the data as shown in Figure 11. Note that for ballistic coefficient (β) values greater than 10 we just use β = 10 during motor burning, and use the actual value of the coasting ballistic coefficient to obtain our coasting altitudes and times. The reason for this can be seen by looking at the thrusting graphs. The curves are almost flat at B=10 and higher values of β will give no significant increases in either burnout velocity (V_B) or burnout altitude (S_B). The .5 ounce and .75 ounce curves for the bigger motors seems to disagree with what has just been said, but one must keep in mind that these larger motors weight almost .75 ounce each, exclusive of the rocket. The .5 ounce and 75 ounce each, exclusive of the rocket. The .5 ounce and .75 ounce curves were included in these graphs for theoretical comparisons only.

Once the actual altitude reached by this rocket is measured you work backwards with this graph to determine the drag coefficient C_D. Find the point on the curve, which corresponds with the measured altitude, and mark it. The altitude reached will vary slightly from flight to flight so it is best to make at least three good vertical flights and then use the average drag coefficient value obtained. You can also measure the flight time with a stopwatch and compare that to the plotted values of total flight time versus drag coefficient. The total flight time (t_{Total}) is simply the sum of the motor burn time (t_B) plus the coast time (t_C)

$$t_{Total} = t_B + t_C$$

Measuring both flight time and altitude for each flight gives you two data points per flight to use instead of one per flight for your drag coefficient measurement experiment. As a result you save both time and money.

You might find it very interesting to repeat the experiment using different total impulse motors. By ballasting the rocket with say an NAR standard 1-ounce payload you can plot even more curves by which the coefficient value can be verified. The results of such experimentation will be surprisingly good as long as you don't change the external shape of paint finish between flights. The results of such a test should come out looking something like the graph of figure 12 and figure 13.

Experiment 3

DETAILED TRAJECTORY TIME HISTORIES

With natural larger logarithm, cosine, tangent, hyperbolic sine, hyperbolic cosines and hyperbolic tangent tables (check the math section of your library for books similar to reference 5). A complete time history of the motion for the rocket whose drag coefficient was previously determined can be calculated. Using equations (A1) and (A2) for a few time increments from lift off to burnout, we can calculate the corresponding velocity and distance. Similarly, after burnout we can use equations (B3) and (B4) in conjunction with (B1) and (B2) to calculate the velocity and altitude at various times.

To get the actual altitude time history we must add the altitude gained during coasting to the altitude at burnout and the corresponding coast flight times to the burnout time. Using the formula

$$D=C_D A 1/2 \rho V^2 = .0001321\ C_D A V^2$$

we can calculate the drag in ounces at any time. With the formula

$$a = \frac{T - D}{W} - 1$$

we obtain the net acceleration in g's on our rocket at any time during the upward flight.

We can verify the accuracy of our time history by using engines which will cause ejection to occur before the peak altitude is reached. The altitude at the instant of ejection should be close to the indicated time history value at that time. For other time points a stop watch will be required. A verbal signal to the trackers can be given for any desired time after liftoff. It should be noted, however, that the recorded altitudes are more susceptible to errors during the portions of the flight when the rocket is traveling at high velocity. A time history plot of the typical performance of an A8-4 powered 1-ounce rocket is given in *Model Rocket News*, Volume 4, Number 2 (A8-4 is no longer available).

Experiment 4

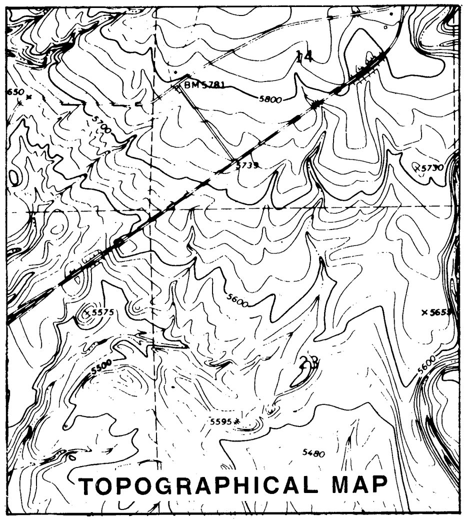
EFFECTS OF TEMPERATURE AND LAUNCH ALTITUDE ON ROCKET PERFORMANCE

From the basic drag equation $D=C_D A \frac{1}{2} \rho V^2$ we can see that lowering the air density (ρ) lowers the drag on a rocket traveling at a given velocity and raising the air density (ρ) increases the drag by a proportional amount. It turns out that atmospheric density is a well-defined function of temperature and altitude. Figure 9 presents a correction for the air density (ρ) which properly reflects the variations in our basic motion equations due to temperature and launch altitude.

It should be noted that rocket motor thrust is also a function of the temperature of the propellant before ignition and the air density. Some research has been done on these effects, but until more detailed information is available, we will just have to neglect it.

All our previous work has been based on standard sea level conditions, which means a temperature of 59˚F and sea level altitude (H=0 feet). To allow for this we must know the temperature at the time of launch and the altitude above sea level of our launch site. We can easily obtain temperature readings with a thermometer at the launch site just prior to lift-off. The best way to determine the altitude of your site is to obtain a topographical map for your area through the U.S. Government.

The topographical maps are inexpensive and are very interesting in themselves. Some libraries carry sets of them and some of the larger cities have Geological Survey offices at which you can browse through these maps at your leisure. Some book and office supply stores carry these items.



Each intermediate line means a rise or fall of 20' elevation of land surface in the sample of a topographical map shown above. Primary lines mark each 100', and exact figures are given for the high and low points in the topography.

Once you have this data you can try to verify the effects on peak altitudes. Since your launch altitude is fixed, it would probably be easiest to make flights for which only temperature is a variable. (Flights in early morning when it is cool and also during the warmer afternoon temperatures should prove to be adequate). By calculating total altitudes for our rocket at different temperatures, we can generate a theoretical graph as shown in Figure 14. This graph should contain all information relative to the rocket as included on this sample. Note that points with appropriate comments have been included to represent recorded flight test data.

If nothing else it should be enlightening to consider such facts as that the clubs in Denver, Colorado (altitude above sea level H=5000 feet) who try for altitude records on hot days have a definite advantage over the rest of us in the U.S.A. If a Denver club is out in 90-degree weather the peak theoretical altitude reached by our 1 ounce A8-4 (no longer available) sample problem rocket becomes 780 feet. This is a very good improvement over the 700-foot altitude obtained under the standard conditions of sea level altitude and 59˚F.

We anxiously await news of new altitude records obtained by those who drive up Pike's Peak in Colorado (elevation 14,110) to fly their rockets. Launched from the top of a mountain at mid-afternoon temperature of 40˚F our 1-ounce A8-4 (not available) sample problem rocket would reach 855 feet.

The drag coefficient measurement experiment can be refined by including temperature and launch altitude effects.

The assumption that air density is uniform through the entire flight of a model is not particularly valid for rockets, which reach higher altitudes. It might be wiser to use the average altitude upon which to base density corrections rather than the launch altitude. This would be similar to using the average weight during thrusting than the full or empty weight.

Using the average altitude of a 2000-foot flight would increase the ballistic coefficient by less than 3%. In view of the errors that arise in tracking at such altitudes, such calculations for most of us may not merit the time spent doing them.

Experiment 5

CLUSTERS

Clustered rockets are primarily used for heavier payloads. It would probably be very useful to know how high a cluster-powered rocket can go for various payload weights and what ejection delay times should be used. Actually, plotting this data as a function of weight will be quite informative. Super-imposing the corresponding data for a single motor booster of the same shape and weight will give a clear understanding of the performance improvements obtained by clustering.

Experiment 6

MULTIPLE STAGE ROCKETS

Perhaps one of the most interesting comparisons one can make involves comparing the altitude reached by a three-stage rocket using identical "type" motors (such as 1/2A6-0, and 1/2A6-4) to a two-stage rocket of identical initial total weight, size and shape, which has a cluster of two motors for the first stage (such as two 1/2A6-0 motors in the first stage and a 1/2A6-4 in the second stage). Both rockets have the same total impulse input, but which will go higher? Using the methods of this report you can predict the results with confidence before the firing button is pushed and actual tracking measurement is made. (1/2A6-0 engines are no longer available.)

As one becomes more familiar with the effects of drag on different shape and size rockets through the use of this report, one eventually will be able to follow the above procedures even before one starts building more complex original rockets in order to decide, in fact, what is the best way to accomplish a desired mission. This same type of pre-flight performance analysis would carry the name "optimization study" if done by the aerospace engineers and scientists who design our country's big rockets.

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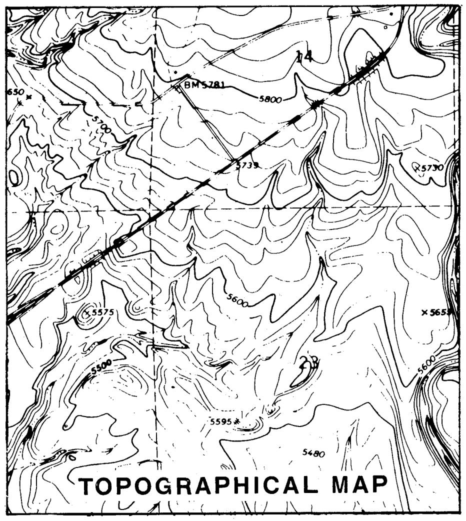

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