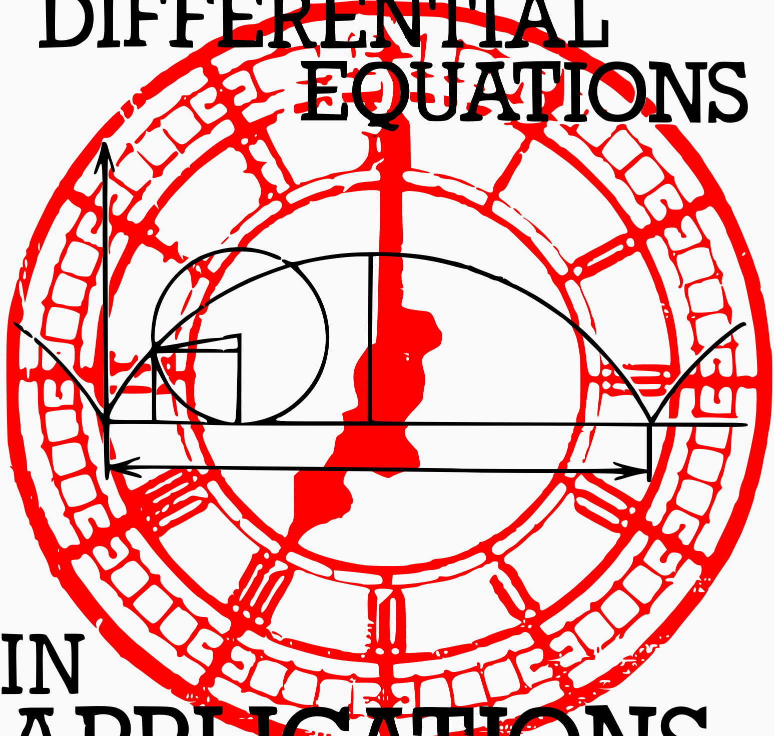
SCIENCE FOR EVERYONE

V. V. AMEL'KIN

DIFFERENTIAL EQUATIONS IN APPLICATIONS



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Science
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В. В. Амелькин

Дифференциальные уравнения в приложениях

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Contents

Preface

Differential equations belong to one of the main mathematical concepts. They are equations for finding functions whose derivatives (or differentials) satisfy given conditions. The differential equations arrived at in the process of studying a real phenomenon or process are called the *differential model* of this phenomenon or process. It is clear that differential models constitute a particular case of the numerous mathematical models that can be built as a result of studies of the world that surrounds us. It must be emphasized that there are different types of differential models. This book considers none but models described by what is known as *ordinary differential equations*, one characteristic of which is that the unknown functions in these equations depend on a single variable.

In constructing ordinary differential models it is important to know the laws of the branch of science relating to the nature of the problem being studied. For instance, in mechanics these may be Newton's laws, in the theory of electric circuits Kirchhoff's laws, in the theory of chemical reaction rates the law of mass action.

Of course, in practical life we often have to deal with cases where the laws that enable building a differentia lequation (or several differential equations) are not known, and we must resort to various assumptions (hypotheses) concerning the course of the process at small variations of the parameters, the variables. Passage to the limit will then lead to a differential equation, and if it so happens that the results of investigation of the differential equation as the mathematical model agree with the experimental data, this will mean that the hypothesis underlying the model reflects the true situation.*

When working on this book, I had two goals in mind. The first was to use examples, rich in content rather than purely illustrative, taken from various fields of knowledge so as to demonstrate the possibilities of using ordinary differential equations in gaining an understanding of the world about us. Of course, the examples far from exhaust the scope of problems solvable by ordinary differential equations. They give an idea of the role that ordinary differential equations play in solving practical problems.

The second goal was to acquaint the read-

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* If the reader wishes to know more about mathematical models, he can turn to the fascinating books by A.N. Tikhonov and D.P. Kostomarov, *Stories About Applied Mathematics* (Moscow: Nauka, 1979) (in Russian), and N.N. Moiseev, *Mathematics Stages an Experiment* (Moscow: Nauka, 1979) (in Russian).

er with the simplest tools and methods used in studying ordinary differential equations and characteristic of the *qualitative theory of differential equations*. The fact is that only in rare cases are we able to solve a differential equation in the so-called *closed form*, that is, represent the solution as a formula that employs a finite number of the simplest operations involving elementary functions, even when it is known that the differential equation has a solution. In other words, we can say that the great variety of solutions to differential equations is such that for their representation in closed form a finite number of analytical operations is insufficient. A similar situation exists in the theory of algebraic equations: while for first- and second-order algebraic equations the solutions can always be easily expressed in terms of radicals and for third- and fourth-order equations the solutions can still be expressed in terms of radicals (although the formulas become very complicated), for a general algebraic equation of an order higher than the fourth the solution cannot be expressed in radicals.

To return to differential equations. If an infinite series of this or that form is used to represent the solutions, then the scope of solvable equations broadens considerably. Unfortunately, it often happens that the most essential and interesting properties of the solutions cannot be revealed by studying the form of the series. More than that, even if a differential equation can be solved in closed form, more often than not it is impossible to analyze such a solution since the relationship between the various parameters of the solution often proves to be extremely complicated.

This shows how important it is to develop methods that make it possible to acquire the data on the various properties of the solutions without solving the differential equations themselves. And indeed, such methods do exist. They constitute the essence of the qualitative theory of differential equations, which is based on the general theorems regarding the existence and uniqueness of solutions and the continuous dependence of solutions on the initial data and parameters. The role of existence and uniqueness theorems is partially discussed in Section 2.2. As for the general qualitative theory of ordinary differential equations, ever since J.H. Poincaré and A.M. Lyapunov laid the foundations at the end of the 19th century, the theory has been intensively developing and its methods are widely used when studying the world about us.

I am indebted to Professors Yu.S. Bogdanov and M.V. Fedoryuk for the constructive remarks and comments expressed in the process of preparing the book for publication.

V.V. Amel'kin

Chapter 1

Construction of Differential Models and Their Solutions

1.1 Whose Coffee Was Hotter?

When Tom and Dick ordered coffee and cream in a lunch room, they were given both simultaneously and proceeded as follows. Tom poured some of his cream into the coffee, covered the cup with a paper napkin, and went to make a phone call. Dick covered his cup with a napkin and poured the same amount of cream into his coffee only after 10 minutes, when Tom returned. The two started drinking their coffee at the same time. Whose was hotter?

We will solve this problem on the natural assumption that according to the laws of physics heat transfer through the surface of the table and the paper napkin is much less than through the sides of the cups and that the temperature of the vapor above the surface of the coffee in the cups equals the temperature of the coffee.

We start by deriving a relationship indicating the time dependence of the temperature of the coffee in Dick's cup before the cream is added.

In accordance with our assumption on the basis of a law of physics, the amount of heat transferred to the air from Dick's cup is determined by the formula

$$dQ = \eta \frac{T-\theta}{l} s\, dt, \qquad (1)$$

where T is the coffee's temperature at time t, θ the temperature of the air in the lunch room, η the thermal conductivity of the material of the cup, l the thickness of the cup, and s the area of the cup's lateral surface. The amount of heat given off by the coffee is

$$dQ = -cm\, dT, \qquad (2)$$

where c is the specific heat capacity of the coffee, and m the mass (or amount) of coffee in the cup. If we now consider Eqs. (1) and (2) together, we arrive at the following equation:

$$\eta \frac{T-\theta}{l} s\, dt = -cm\, dT,$$

which, after variable separation, can be rewritten as follows:

$$\frac{dT}{T-\theta} = -\frac{\eta s}{lcm}\, dt. \qquad (3)$$

Denoting the initial temperature of the coffee by T_0 and integrating the differential equation (3), we find that

$$T = \theta + (T_0 - \theta) \exp\left(-\frac{\eta s}{lcm} t\right). \qquad (4)$$

This formula is the analytical expression of the law whereby the temperature of the coffee in Dick's cup varied prior to the addition of the cream.

■ Now let us establish the law whereby the temperature of the coffee in Dick's cup changed after Dick poured in cream. We use the heat balance equation, which here can be written as

$$cm\,(T - \theta_D) = c_1 m_1\,(\theta_D - T_1), \tag{5}$$

where θ_D is the temperature of the Dick's coffee with cream at time t, T_1 the temperature of the cream, c_1 the specific heat capacity of the cream, and m_1 the mass of cream added to the coffee.

Equation (5) yields

$$\theta_D = \frac{c_1 m_1}{cm + c_1 m_1} T_1 + \frac{cm}{cm + c_1 m_1} T. \tag{6}$$

Bearing in mind formula (4), we can rewrite (6) as follows:

$$\theta_D = \frac{c_1 m_1}{cm + c_1 m_1} T_1 + \frac{cm}{cm + c_1 m_1} \times \left[\theta + (T_0 - \theta) \exp\left(-\frac{\eta s}{lcm} t\right)\right], \tag{7}$$

which constitutes the law whereby the temperature of the coffee in Dick's cup varies after cream is added.

To derive the law for temperature variation of the coffee in Tom's cup we again employ the heat balance equation, which now assumes the form

$$cm\,(T_0 - \theta_0) = c_1 m_1\,(\theta_0 - T_1), \tag{8}$$

where θ_0 is the temperature of the mixture. If we solve (8) for θ_0, we get

$$\theta_0 = \frac{c_1 m_1}{cm + c_1 m_1} T_1 + \frac{cm}{cm + c_1 m_1} T_0.$$

Then, using Eq. (4) with θ_0 serving as the initial temperature and $cm + c_1 m_1$ substituted for cm, we arrive at the law for the temperature (θ_T) variation of the coffee in Tom's cup as analytically given by the following formula:

$$\theta_T = \theta + \left[\frac{c_1 m_1}{cm + c_1 m_1} T_1 + \frac{cm}{cm + c_1 m_1} T_0 - \theta\right] \times \exp\left(-\frac{\eta s}{l(cm + c_1 m_1)} t\right). \tag{9}$$

Thus, to answer the question posed in the problem we need only turn to formulas (7) and (9) and carry out the necessary calculations, bearing in mind that $c_1 \approx 3.9 \times 10^3$ J/kg·K, $c \approx 4.1 \times 10^3$ J/kg·K, and $\eta \approx 0.6$ V/m·K and assuming, for the sake of definiteness, that $m_1 = 2 \times 10^{-2}$ kg, $m = 8 \times 10^{-2}$ kg, $T_1 = 20$ °C, $\theta = 20$ °C, $T_0 = 80$ °C, $s = 11 \times 10^{-3}$ m², and $l = 2 \times 10^{-3}$ m. The calculations show that Tom's coffee was hotter.

1.2 Steady-State Heat Flow

The reader will recall that a steady-state heat flow is one in which the object's tem-

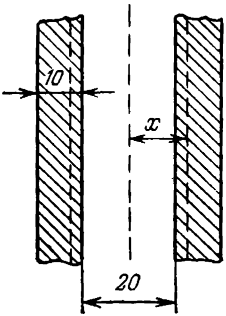

Fig. 1

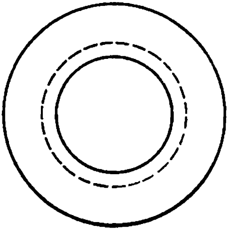

Fig. 2

perature at each point does not vary with time.

In problems whose physical content is related to the effects of heat flows, an important role is played by the so-called isothermal surfaces. To clarify this statement, let us consider a heat-conducting pipe (Figure 1) 20 cm in diameter, made of a homogeneous material, and protected by a layer of magnesium oxide 10 cm thick. We assume that the temperature of the pipe is 160 °C and the outer surface of the protective covering has a temperature of 30 °C. It is intuitively clear then that there is a surface, designated by a dashed curve in Figure 2, at each point of which the temperature is the same, say 95 °C. The dashed curve in Figure 2 is known as an *isotherm*, while the surface corresponding to this curve is known as an *isothermal surface.* In general, isotherms may have various shapes, depending, for one, on the nonsteady-state nature of the heat flow and on the nonhomogeneity of the material. In the case at hand the isotherms (isothermal surfaces) are represented by concentric circles (cylinders).

We wish to derive the law of temperature distribution inside the protective coating and find the amount of heat released by the pipe over a section 1 m long in the course of 24 hours, assuming that the thermal conductivity coefficient k is equal to 1.7×10^{-4}.

To this end we turn to the Fourier law, according to which *the amount of heat released per unit time by an object that is in stable thermal state and whose temperature T at each point is solely a function of coordinate x can be found according to the formula*

$$Q = -kF(x)\,\frac{dT}{dx} = \text{const}, \tag{10}$$

where $F(x)$ is the cross-sectional area normal to the direction of heat flow, and k the thermal conductivity coefficient.

The statement of the problem implies that $F(x) = 2\pi x l$, where l is the length of the pipe (cm), and x the radius of the base of the cylindrical surface lying inside the outer cylinder. Then on the basis of (10) we get

$$\int_{160}^{30} dT = -\frac{Q}{0.00017 \times 2\pi l} \int_{10}^{20} \frac{dx}{x}, \qquad (11)$$

$$\int_{160}^{T} d\tau = -\frac{Q}{0.00017 \times 2\pi l} \int_{10}^{x} \frac{d\xi}{\xi}. \qquad (12)$$

Integrating (11) and (12) yields

$$\frac{160-T}{130} = \frac{\ln 0.1x}{\ln 2} = \frac{\log 0.1x}{\log 2}.$$

Hence

$$T = 591.8 - 431.8 \log x.$$

This formula expresses the law of temperature distribution inside the protective coating. We see that the length of the pipe plays no role in this law.

To answer the second question, we turn to Eq. (11). Then for $l = 100$ cm we find that

$$Q = \frac{130 \times 0.00017 \times 2 \quad \times 100}{\ln 2}$$

$$= \frac{200\pi \times 130 \times 0.00017}{0.69315},$$

and, hence, the amount of heat released by the pipe in the course of 24 hours is $24 \times 60 \times 60Q = 726\,852$ J.

1.3 An Incident in a National Park

While patrolling a national park, two forest rangers found the carcass of a slain wild boar. Inspection showed that the poacher's shot had been precise and the boar had died on the spot. Reasoning that the poacher should return for the kill, the rangers decided to wait for him and hid nearby. Soon two men appeared, heading directly for the dead boar. When confronted by the rangers, the men denied having anything to do with the poaching. But by this time the rangers had collected indirect evidence of their guilt. It only remained to establish the exact time of the kill. This they did by using the law of heat emission. Let us see what reasoning this involved.

According to the law of heat emission, the rate at which an object cools off in air is proportional to the difference between the temperature of the object and that of the air,

$$\frac{dx}{dt} = -k(x-a), \tag{13}$$

where x is the temperature of the object at time t, a the temperature of the air, and k a positive proportionality coefficient.

Solution of the problem lies in an analysis of the relationship that results from integrating the differential equation (13). Here one must bear in mind that after the boar was killed, the temperature of the air could have remained constant but also could have varied with time. In the first case integration of the differential equation (13) with variables separable leads to

$$\ln \frac{x-a}{x_0-a} = -kt, \quad x \neq a, \tag{14}$$

where x_0 is the temperature of the object at time $t = 0$. Then, if at the time when the strangers were confronted by the rangers the temperature of the carcass, x, was 31 °C and after an hour 29 °C and if when the shot was fired the temperature of the boar was $x = 37$ °C and the temperature of the air $a = 21$ °C, we can establish when the shot was fired by putting $t = 0$ as the time when the strangers were detained. Using these data and Eq. (14), we find that

$$k = \ln \frac{31-21}{29-21} = \ln 1.25 = 0.22314. \tag{15}$$

Now, substituting this value of k and $x = 37$ into (14), we get

$$t = -\frac{1}{0.22314} \ln \frac{37-21}{31-21}$$

$$= -\frac{1}{0.22314} \ln 1.6 = -2.10630.$$

In other words, roughly 2 hours and 6 minutes passed between the time the boar was killed and the time the strangers were detained.

In the second case, that is, when the temperature of the air varies in time, the cooling off of the carcass is expressed by the following nonhomogeneous linear differential equation

$$\frac{dx}{dt} + kx = ka(t), \tag{16}$$

where $a(t)$ is the temperature of the air at time t.

To illustrate one of the methods for determining the time when the boar was killed, let us assume that the temperature of the carcass was 30 °C when the strangers were detained. Let us also suppose that it is known that on the day of the kill the temperature of the air dropped by 1 °C every hour in the afternoon and was 0 °C when the carcass was discovered. We will also assume that after an hour had passed after the discovery the temperature of the carcass was 25 °C and that of the air was down to —1° C. If we now assume that the shot was fired at $t = 0$ and that $x_0 = 37$ °C at $t = 0$, we get $a(t) = t^* - t$, where $t = t^*$ is the time when the rangers discovered the carcass.

Integrating Eq. (16), we get

$$x = (37 - t - k^{-1}) e^{-kt} + t^* - t + k^{-1}.$$

If we now bear in mind that $x = 30$ °C at $t = t^*$ and $x = 25$ °C at $t = t^* + 1$, the last formula yields

$$(37 - t^* - k^{-1}) \exp(-kt^*) + k^{-1} = 30,$$
$$(37 - t^* - k^{-1}) \exp[-k(t^* + 1)] + k^{-1} = 26.$$

These two equations can be used to derive an equation for k, namely,

$$(30 - k^{-1})e^{-k} - 26 + k^{-1} = 0. \qquad (17)$$

We can arrive at the same equation starting from different assumptions. Indeed, suppose that at $t = 0$ the carcass of the slain boar was found. Then $a(t) = -t$ and we arrive at the differential equation

$$\frac{dx}{dt} + kx = -kt \qquad (18)$$

(with the initial data $x_0 = 30$ at $t = 0$), from which we must find x as an explicit function of t.

Solving Eq. (18), we get

$$x = (30 - k^{-1}) \exp(-kt) - t + k^{-1}. \qquad (19)$$

Setting $x = 25$ and $t = 1$ in this relationship, we again arrive at Eq. (17), which enables solving the initial problem numerically.

Indeed, as is known, Eq. (17) cannot be solved algebraically for k. But it is easily solved by numerical methods for finding the roots of transcendental equations, for one, by Newton's method of approximation. This method, as well as other methods of successive approximations, is a way of using a rough estimate of the true value of a root used to obtain more exact values of the root. The process can be continued until the required accuracy is achieved.

To show how Newton's method is used, we transform Eq. (17) to the form

$$30k - 1 + (1 - 26k)\exp(k) = 0, \qquad (20)$$

and Eq. (19), setting $x = 37$, to the form

$$(37k - 1 + kt)\exp(kt) - 30k + 1 = 0. \qquad (21)$$

Both equations, (20) and (21), are of the type

$$(ax + b)\exp(\lambda x) + cx + d = 0. \qquad (22)$$

If we denote the left-hand side of Eq. (22) by $\varphi(x)$, differentiation with respect to x yields

$$\varphi'(x) = (\lambda ax + \lambda b + a)\exp(\lambda x) + c,$$
$$\varphi''(x) = (\lambda^2 ax + \lambda^2 b + 2\lambda a)\exp(\lambda x).$$

Then according to Newton's method for finding a root of Eq. (22), if for the ith approximation x_i we have the inequality

$$\varphi(x_i)\,\varphi''(x_i) > 0,$$

the next approximation, x_{i+1}, can be found via the formula

$$x_{i+1} = x_i - \frac{\varphi(x_i)}{\varphi'(x_i)}.$$

To directly calculate the root (to within, say, one part in a million), we compile the following program using BASIC*:

```
10 CLS:PRINT "Solution of equation by New-
     ton's method"
15 INPUT "lambda=", L;
20 INPUT "a=", A: INPUT "b=", B;
30 INPUT "c=", C: INPUT "d=", D
40 INPUT "approximate value of root=", X
50 PRINT "X", "f", "f'", "f''"
100 E = EXP (L*X)
110 F = (A*X + B)*E + C*X + D
120 F1 = (L*(A*X + B) + A)*E + C
130 F2 = L* (L* (A*X + B) + 2*A)*E
150 PRINT X, F, F1, F2
151 IF F = 0 THEN END
155 IF F1 = 0 THEN PRINT "Newton's method
    is divergent": END
170 IF F*F2 < 0 THEN PRINT "Newton's
    method is divergent": END
190 X = X - F/F1
200 GOTO 100
```

In this program X, f, f′ and f″ stand for k_n, $\varphi(k_n)$, $\varphi'(k_n)$, and $\varphi''(k_n)$, respectively.

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* *Editor's note.* Some BASIC lines, e.g. line 10, have been split due to the printed format; they should be input as one line.

After starting the program enter the requested values of the coefficients of the equation and the initial value of the root. The results can be listed in a table of the approximate values of the root and the respective values of the function and its first and second derivatives.

Employing this general procedure, let us turn to Eq. (20). Differentiating its left-hand side $\varphi(k)$ with respect to k, we arrive at

$$\varphi'(k) = 30 - (25 + 26k)\exp(k).$$

It can then easily be verified that $\varphi(0) = 0$, $\varphi(1) < 0$, and $\varphi'(0) > 0$. Thus, the function φ increases in a small neighborhood of the origin and then decreases to a negative value at $k = 1$. This implies that in the interval (0, 1) there is a root of the equation $\varphi(k) = 0$. To find this root we run the program. Below we give the protocol of the program:

Solution of equation by Newton's method
lambda= 1

a = —26 b = 1 c = 30 d = — 1

Approximate value of root = 0.5

X	f	f′	f″
.5	—5.784655	—32.65141	—105.5182
.322836	—1.525956	—16.11805	—82.02506
.2281622	—.3514252	—8.859806	—71.52333
.1884971	—5.519438E—02	—6.103379	—67.49665
.1794539	—2.748013E—03	—5.497021	—66.60768

.178954	—7.629395E—06	—5.46373	—66.55884
.1789526	—4.768372E—07	—5.463642	—66.5587
.1789525	0	—5.463635	—66.5587

OK

In this protocol X, f, f', and f" stand for k_n, $\varphi(k_n)$, $\varphi'(k_n)$, and $\varphi''(k_n)$, respectively.

The final step in solving the problem consists in substituting the calculated value $k_6 \approx k \approx 0.178952$ into Eq. (21) and solving the latter for t (the time when the wild boar was killed). To employ the above scheme, we denote the left-hand side of Eq. (21) by $g(t)$. Then, selecting —1 for the value of t_0 and bearing in mind that in this case $a = k$, $b = 37k - 1$, $c = 0$, $d = -30k + 1$, and $\lambda = k$, we can find the time when the boar was killed. To this end we find the coefficients b and d, which prove to be equal to 5.621243 and —4.368575, respectively, and then running the above program, find the sought time t. The protocol of the program is given below:

Solution of equation by Newton's method
lambda = 0.1789525
a = 0.1789525 b = 5.621243 c = 0
d = —4.368575
Approximate value of root = —1

X	f	f'	f''
—1	.181972	.9639622	.1992802
—1.188775	3.506184E—03	.9270549	.1917861

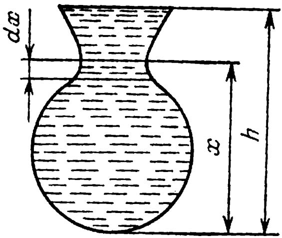

Fig. 3

—1.192557	1.430512E—06	.9263298	.1916388
—1.192559	0	.9263295	.1916387

OK

In this protocol X, f, f′, and f″ stand for t_n, $g(t_n)$, $g'(t_n)$, and $g''(t_n)$, respectively.

These results imply that the boar was killed approximately 1 hour and 12 minutes before the rangers discovered the carcass.

1.4 Liquid Flow Out of Vessels. The Water Clock

The two problems that we now discuss illustrate the relationship between the physical content of a problem and geometry. But first let us examine some general theoretical conclusions.

We take a vessel (Figure 3) whose horizontal cross section has an area that is a function of the distance from the bottom of the vessel to the cross section. Suppose that initially at time $t = 0$ the level of the liquid in the vessel is at a height of h meters. We will also suppose that the area of the vessel's cross section at height x is denoted by $S(x)$ and that the area of the opening in the bottom of the vessel is s.

As is known, *the rate v at which the liquid flows out of the vessel at the moment when the liquid's level is at height x is given by the formula* $v = k\sqrt{2gx}$, *where* $g = 9.8\ \mathrm{m/s^2}$, *and k is the rate constant of the outflow process.*

In the course of an infinitesimal time interval $\mathrm{d}t$ the outflow of the liquid can be assumed uniform, whereby during $\mathrm{d}t$ a column of liquid with a height of $v\,\mathrm{d}t$ and a cross-sectional area of s will flow out of the vessel, which causes the level of the liquid to change by $-\mathrm{d}x$ (the "minus" because the level lowers).

The above reasoning leads us to the following differential equation

$$ks\sqrt{2gx}\,\mathrm{d}t = -S(x)\,\mathrm{d}x,$$

which can be rewritten as

$$\mathrm{d}t = -\frac{S(x)}{ks\sqrt{2gx}}\,\mathrm{d}x. \tag{23}$$

Let us now solve the following problem. A cylindrical vessel with a vertical axis six meters high and four meters in diameter has a circular opening in the bottom. The radius of this opening is 1/12 m. Find how the level of water in the vessel depends on time t and the time it takes all the water to flow out. T

By hypothesis, $S(x) = 4\pi$ and $s = 1/144$. Since for water $k = 0.6$, Eq. (23) assumes the form

$$dt = -\frac{217.152}{\sqrt{x}}\, dx.$$

Integrating this differential equation yields

$$t = 434.304\,[\sqrt{6} - \sqrt{x}], \quad 0 \leqslant x \leqslant 6,$$

which is the sought dependence of the level of water in the vessel on time t. If we put $x = 0$ in the last formula, we find that it takes approximately 18 minutes for all the water to flow out of the vessel.

Now a second problem. An ancient water clock consists of a bowl with a small hole in the bottom through which water flows out of the bowl (Figure 4). Such clocks were used in ancient Greek and Roman courts to time the lawyers' speeches, so as to avoid prolonged speeches. We wish to determine the shape of the water clock that would ensure that the water level lowered at a constant rate.

This problem can easily be solved via Eq. (23). We rewrite this equation in the form

$$\sqrt{x} = -\frac{S(x)}{ks\sqrt{2g}}\,\frac{dx}{dt}. \tag{24}$$

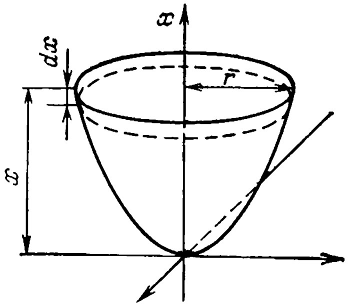

Fig. 4

Precisely, if we suppose that the bowl has the shape of a surface of revolution, in accordance with the notations used in Figure 4, Eq. (24) yields the following result:

$$\sqrt{x} = -\frac{\pi r^2}{ks\sqrt{2g}}\, a, \tag{25}$$

where $a = v_x = \mathrm{d}x/\mathrm{d}t$ is the projection on the x axis of the rate of motion of the water's free surface, which is constant by hypothesis. Squaring both sides of Eq. (25), we arrive at the equation

$$x = cr^4, \tag{26}$$

with $c = a^2\pi^2/2gk^2s^2$. The latter means that the sought shape of the water clock is obtained by rotating curve (26) about the x axis.

1.5 Effectiveness of Advertising

Suppose that a retail chain is selling commodities of a certain type, say B, about which only x buyers out of N potential buyers know at time t. Let us also assume that to speed up sales the chain has placed promotion materials in the local TV and radio network. All further information about the commodities is distributed among the buyers via personal contact between them. We may assume with a high probability that after the TV and radio network have released the information about B, the rate of change in the number of persons knowing about B is proportional both to the number of buyers knowing about B and to the number of buyers not knowing about B.

If we suppose that time is reckoned from the moment when the promotion materials and advertisements were released and N/γ persons have learned about the commodities, we arrive at the following differential equation:

$$\frac{dx}{dt} = kx(N - x), \tag{27}$$

with the initial condition that $x = N/\gamma$ at $t = 0$. In this equation k is a positive proportionality factor. Integrating, we find that

$$\frac{1}{N} \ln \frac{x}{N - x} = kt + C.$$

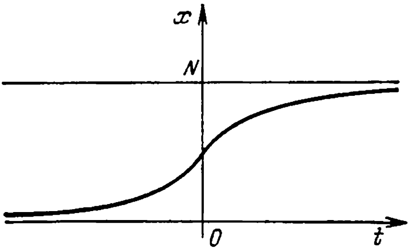

Fig. 5

Assuming that $NC = C_1$, we arrive at the equation

$$\frac{x}{N-x} = Ae^{Nkt},$$

with $A = e^{C_1}$. When solved for x, this equation yields

$$x = N\frac{Ae^{Nkt}}{Ae^{Nkt}+1} = \frac{N}{1+Pe^{-Nkt}}, \qquad (28)$$

with $P = 1/A$.

In economics, Eq. (28) is commonly known as the *logistic-curve equation.*

If we allow for the initial condition, Eq. (28) becomes

$$x = \frac{N}{1+(\gamma-1)\,e^{-Nkt}}.$$

Figure 5 provides a schematic of a logistic curve for $\gamma = 2$. In conclusion we note that the problem of dissemination of technological innovations can also be reduced to Eq. (27).

1.6 Supply and Demand

As is known, supply and demand constitute economic categories in a commodity economy, categories that emerge and function on the market, that is, in the sphere of trade. The first category represents the commodities that exist on the market or can be delivered to it, while the second represents the demand for commodities on the market. One of the main laws of a market economy is the law of supply and demand, which can be formulated simply by saying that for each commodity some price must exist that will cause the supply and the demand to be just equal. Such a price establishes an "equilibrium" on the market.

Let us consider the following problem. Suppose that in the course of a (relatively long) time interval a farmer sells his produce (say, apples) on the market, and that he does this immediately after the apple harvest has been taken in and then once each following week (i.e. with a week's interval). The farmer's stock of apples being fixed after he has collected his harvest, the week's supply will depend on both the expected price of apples in the coming week and the expected change in price in the following weeks. If it is assumed that in the coming week the price of apples will fall and in the following weeks it will grow, then supply will be restrained provided that the expected rise in price exceeds storage costs. In these conditions the greater the expected rise in price in the following weeks, the lower the supply of apples on the market. On the other hand, if the price of apples in the coming week is high and then a fall is expected in the following weeks, then the greater the expected fall in price in the following weeks, the higher the supply of apples on the market in the coming week.

If we denote the price of apples in the coming week by p and the time derivative of price (the *tendency of price formation*) by p', both supply and demand are functions of these quantities. As practice has shown, depending on various factors the supply and demand of a commodity may be represented by different functions of price and tendency of price formation. For example, one function is given by a linear dependence expressed mathematically in the form $y = ap' + bp + c$, where a, b, and c are real numbers (constants). Say, in our example, the price of apples in the coming week is taken at 1 rouble for 1 kg, in t weeks it was $p(t)$ roubles for 1 kg, and the supply s and demand q are given by the functions

$$s = 44p' + 2p - 1, \quad q = 4p' - 2p + 39.$$

Then, for the supply and demand to be in equilibrium, we must require that

$$44p' + 2p - 1 = 4p' - 2p + 39.$$

This condition leads us to the differential equation

$$\frac{dp}{p-10} = -10dt.$$

Integration yields $p = Ce^{-10t} + 10$. If we allow for the initial condition that $p = 1$ at $t = 0$, the equilibrium price is given by the formula

$$p = -9e^{-10t} + 10. \qquad (29)$$

Thus, if we want the supply and demand to be in equilibrium all the time, the price must vary according to formula (29).

1.7 Chemical Reactions

A chemical equation shows how the interaction of substances produces a new substance. Take, for instance, the equation

$$2H_2 + O_2 \rightarrow 2H_2O.$$

It shows how as a result of the interaction of two molecules of hydrogen and one molecule of oxygen two molecules of water are formed.

Generally a chemical reaction can be written in the form

$$aA + bB + cC + \quad \rightarrow mM + nN + pP +$$

where $A, B, C,$ are molecules of the interacting substances, $M, N, P,$ . . molecules of the reaction products, and the constants $a, b, c,$ . ., $m, n, p,$. positive integers that stand for the number of molecules participating in the reaction.

The rate at which a new substance is formed in a reaction is called the *reaction rate*, and the active mass or concentration of a reacting substance is given by the number of moles of this substance per unit volume.

One of the basic laws of the theory of chemical reaction rates is the law of mass action, according to which *the rate of a chemical reaction proceeding at a constant temperature is proportional to the product of the concentrations of the substances taking part in the reaction at a given moment.*

Let us solve the following problem. Two liquid chemical substances A and B occupying a volume of 10 and 20 liters, respectively, form as a result of a chemical reaction a new liquid chemical substance C. Assuming that the temperature does not change in the process of the reaction and that every two volumes of substance A and one volume of substance B form three volumes of substance C, find the amount of substance C at an arbitrary moment of time t if it takes 20 min to form 6 liters of C.

Let us denote by x the volume (in liters) of substance C that has formed by the time t (in hours). By hypothesis, by that time $2x/3$ liters of substance A and $x/3$ liters of substance B will have reacted. This means that we are left with $10 - 2x/3$ liters of A and $20 - x/3$ liters of B. Thus, in accordance with the law of mass action, we arrive at the following differential equation

$$\frac{dx}{dt} = K\left(10 - \frac{2x}{3}\right)\left(20 - \frac{x}{3}\right)$$

which can be rewritten as

$$\frac{dx}{dt} = k(15 - x)(60 - x),$$

where k is the proportionality factor ($k = 2K/9$). We must also bear in mind that since initially ($t = 0$) there was no substance C, we may assume that $x = 0$ at $t = 0$. As for the moment $t = 1/3$, we have $x = 6$.

Thus, the solution of the initial problem has been reduced to the solution of a so-called boundary value problem:

$$\frac{dx}{dt} = k(15 - x)(60 - x), \quad x(0) = 0,$$

$$x(1/3) = 6.$$

To solve it, we first integrate the differential equation and allow for the initial condition $x(0) = 0$. As a result we arrive at the relationship

$$\frac{60-x}{15-x} = 4e^{45kt}.$$

Since $x = 6$ at $t = 1/3$, substituting these values into the relationship yields $e^{15k} = 3/2$. Hence,

$$\frac{60-x}{15-x} = 4\,(e^{15k})^{3t} = 4\,(3/2)^{3t},$$

that is,

$$x = 15\,\frac{1-(2/3)^{3t}}{1-(1/4)\,(2/3)^{3t}}.$$

This gives the amount of substance C that has been formed in the reaction by time t.

A remark is in order. From practical considerations it is clear that only a finite volume of substance C can be formed when 10 liters of A interact chemically with 20 liters of B. However, a formal study of the above relationship between x and t shows that for a finite t, namely at $(2/3)^{3t} = 4$, the variable x becomes infinite. But this fact does not contradict practical considerations because it is realized only for a negative value of t, while the chemical reaction is considered only for t nonnegative.

1.8 Differential Models in Ecology

Ecology studies the interaction of man and, in general, living organisms with the environment. The basic object in ecology is the evolution of populations. Below we describe differential models of populations that deal with their reproduction or extinction and with the coexistence of various species of animals in the predator vs. prey situation.*

Let $x(t)$ be the number of individuals in a population at time t. If A is the number of individuals in the population that are born per unit time and B the number of individuals that die off per unit time, then there is reason to say that the rate at which x varies in time is given by the formula

$$\frac{dx}{dt} = A - B. \tag{30}$$

The problem consists in finding the dependence of A and B on x. The simplest situation is the one in which

$$A = ax, \quad B = bx, \tag{31}$$

where a and b are the coefficients of births and deaths of individuals per unit time, respectively. Allowing for (31), we can rewrite the differential equation (30) as

$$\frac{dx}{dt} = (a - b)\, x. \tag{32}$$

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* For more detail see J.D. Murray, "Some simple mathematical models in ecology," *Math. Spectrum* **16**, No. 2: 48-54 (1983/84).

Assuming that at $t = t_0$ the number of individuals in the population is $x = x_0$ and solving Eq. (32), we get

$$x(t) = x_0 \exp[(a - b)(t - t_0)].$$

This implies that if $a > b$, then the number of individuals x tends to infinity as $t \to \infty$, but if $a < b$, then $x \to 0$ as $t \to \infty$ and the population becomes extinct.

Although the above model is a simplified one, it still reflects the actual situation in some cases. However, practically all models describing real phenomena and processes are nonlinear, and instead of the differential equation (32) we are forced to consider an equation of the type

$$\frac{dx}{dt} = f(x),$$

where $f(x)$ is a nonlinear function, say the equation

$$\frac{dx}{dt} = f(x) = ax - bx^2,$$

where

$$a > 0,\ b > 0.$$

Assuming that $x = x_0$ at $t = t_0$ and solving the last equation, we get

$$x(t) = \frac{x_0 a/b}{x_0 + (a/b - x_0)\exp[-a(t - t_0)]}. \tag{33}$$

We see that as $t \to \infty$ the number of individuals in the population, $x(t)$, tends to

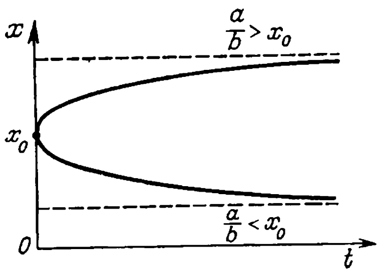

Fig. 6

a/b. Two cases are possible here: $a/b > x_0$ and $a/b < x_0$. The difference between the two is clearly seen in Figure 6. Note that formula (33) describes, for one thing, the populations of fruit pests and some types of bacteria.

If we consider several coexisting species, say, big and small fish, where the small fish serve as prey for the big, then by setting up differential equations for each species we arrive at a system of equations

$$\frac{dx_i}{dt} = f_i(x_1, \quad x_n), \quad i = 1, 2, \quad n.$$

Let us study in greater detail the two-species predator vs. prey model, first introduced by the Italian mathematician Vito Volterra (1860-1940) to explain the oscillations in catch volumes in the Adriatic Sea, which have the same period but differ in phase.

Let x be the number of big fish (predators) that feed on small fish (prey), whose number we denote by y. Then the number of predator fish will grow as long as they have sufficient food, that is, prey fish. Finally, a situation will emerge in which there will not be enough food and the number of big fish will diminish. As a result, starting from a certain moment the number of small fish begins to increase. This, in turn, assists a new growth in the number of the big species, and the cycle is repeated. The model constructed by Volterra has the form

$$\frac{dx}{dt} = -ax + bxy, \tag{34}$$

$$\frac{dy}{dt} = cx - dxy, \tag{35}$$

where a, b, c, and d are positive constants.

In Eq. (34) for the big fish the term bxy reflects the dependence of the increase in the number of big fish on the number of small fish, while in Eq. (35) the term $-dxy$ expresses the decrease in the number of small fish as a function of the number of big fish.

To make the study of these two equations more convenient we introduce dimensionless variables:

$$u(\tau) = \frac{d}{c}x, \quad v(\tau) = \frac{b}{a}y, \quad \tau = ct, \quad \alpha = a/c.$$

As a result the differential equations (34) and (35) assume the form

$$u' = \alpha u (v - 1), \quad v' = v (1 - u), \qquad (36)$$

with α positive and the prime standing for differentiation with respect to τ.

Let us assume that at time $\tau = \tau_0$ the number of individuals of both species is known, that is,

$$u (\tau_0) = u_0, \quad v (\tau_0) = v_0. \qquad (37)$$

Note that we are interested only in positive solutions. Let us establish the relationship between u and v. To this end we divide the first equation in system (36) by the second and integrate the resulting differential equation. We get

$$\alpha v + u - \ln v^{\alpha} u = \alpha v_0 + u_0 - \ln v_0^{\alpha} u_0$$
$$= H,$$

where H is a constant determined by the initial conditions (37) and parameter α.

Figure 7 depicts the dependence of u on v for different values of H. We see that the (u, v)-plane contains only closed curves. Let us now assume that the initial values u_0 and v_0 are specified by point A on the trajectory that corresponds to the value $H = H_3$. Since $u_0 > 1$ and $v_0 < 1$, the first equation in (36) shows that at first variable u decreases. The same is true of variable v. Then, as u approaches a value close to unity, v' vanishes, which is followed by a prolonged period of time τ in the

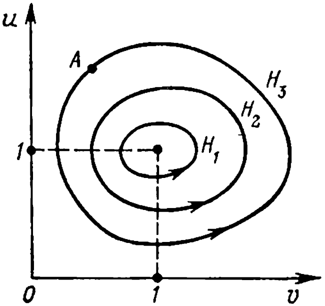

Fig. 7

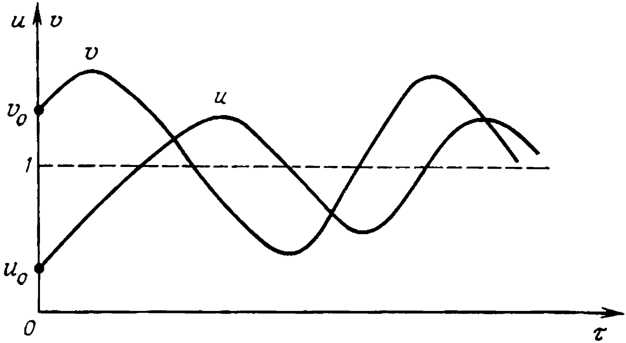

Fig. 8

course of which variable v increases. When v becomes equal to unity, u' vanishes and variable u begins to increase. Thus, both u and v traverse a closed trajectory. This means that the solutions are functions periodic in time. The variable u does not achieve its maximum when v does, that is, oscillations in the populations occur in different phases. A typical graph of u and v as functions of time is shown in Figure 8 (for the case where $v_0 > 1$ and $u_0 < 1$).

In conclusion we note that the study of communities that interact in a more complex manner provides results that are more interesting from the practical standpoint than those obtained above. For example, if two populations fight for the same source of food (the third population), it can be demonstrated that one of them will become extinct. Clearly, if this is the third population (the source of food), then the other two will also become extinct.

1.9 A Problem from the Mathematical Theory of Epidemics

Let us consider a differential model encountered in the theory of epidemics. Suppose that a population consisting of N individuals is split into three groups. The first include individuals that are susceptible to a certain disease but are healthy. We denote the number of such individuals at time t by $S(t)$. The second group incorporates individuals that are infected, that is, they are ill and serve as a source of infection. We denote the number of such idividuals in the population at time t by $I(t)$. Finally, the third group consists of individuals who are healthy and immune to the disease. We denote the number of such individuals at time t by $R(t)$. Thus,

$$S(t) + I(t) + R(t) = N. \tag{38}$$

Let us also assume that when the number of infected individuals exceeds a certain fixed number I^*, the rate of change in the number of individuals susceptible to the disease is proportional to the number of such individuals. As for the rate of change in the number of infected individuals that eventually recover, we assume that it is proportional to the number of infected individuals. Clearly, these assumptions simplify matters and in a number of cases they reflect the real situation. Because of the first assumption, we suppose that when the number of infected individuals $I(t)$ is greater than I^*, they can infect the individuals susceptible to the disease. This means that we have allowed for the fact that the infected individuals have been isolated for a certain time interval (as a result of quarantine or because they have been far from individuals susceptible to the disease). We, therefore, arrive at the following differential equation

$$\frac{dS}{dt} = \begin{cases} -\alpha S & \text{if } I(t) > I^*, \\ 0 & \text{if } I(t) \leqslant I^*. \end{cases} \tag{39}$$

Now, since each individual susceptible to the disease eventually falls ill and becomes a source of infection, the rate of change in the number of infected individuals is the difference, per unit time, between the newly infected individuals and those that are getting well. Hence,

$$\frac{dI}{dt} = \begin{cases} \alpha S - \beta I & \text{if } I(t) > I^*, \\ -\beta I & \text{if } I(t) \leqslant I^*. \end{cases} \qquad (40)$$

We will call the proportionality factors α and β the *coefficients of illness and recovery*.

Finally, the rate of change of the number of recovering individuals is given by the equation $dR/dt = \beta I$.

For the solutions of the respective equations to be unique, we must fix the initial conditions. For the sake of simplicity we assume that at time $t = 0$ no individuals in the population are immune to the disease, that is, $R(0) = 0$, and that initially the number of infected individuals was $I(0)$. Next we assume that the illness and recovery coefficients are equal, that is, $\alpha = \beta$ (the reader is advised to study the case where $\alpha \neq \beta$). Hence, we find ourselves in a situation in which we must consider two cases.

Case 1. $I(0) \leqslant I^*$. With the passage of time the individuals in the population will not become infected because in this case $dS/dt = 0$, and, hence, in accordance with Eq. (38) and the condition $R(0) = 0$, we have an equation valid for all values

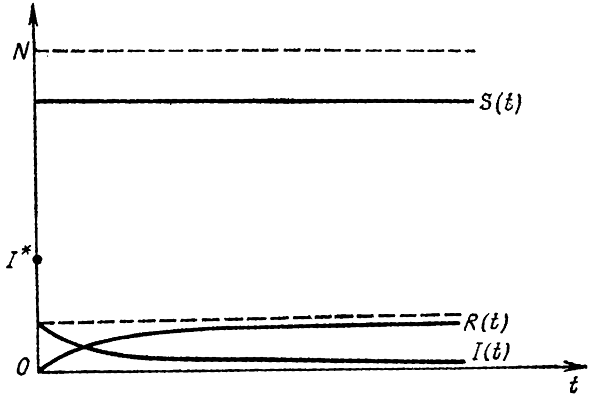

Fig. 9

of t:

$$S(t) = S(0) = N - I(0).$$

The case considered here corresponds to the situation when a fairly large number of infected individuals are placed in quarantine. Equation (40) then leads us to the following differential equation

$$\frac{dI}{dt} = -\alpha I.$$

This means that $I(t) = I(0)\, e^{-\alpha t}$ and, hence,

$$R(t) = N - S(t) - I(t)$$
$$= I(0)\,[1 - e^{-\alpha t}].$$

Figure 9 provides diagrams that illustrate the changes with the passage of time in the number of individuals in each of the three groups.

Case 2. $I(0) > I^*$. In this case there must exist a time interval $0 \leqslant t < T$ in which $I(t) > I^*$ for all values of t, since by its very meaning the function $I = I(t)$ is continuous. This implies that for all t's belonging to the interval $[0, T]$ the disease spreads to the individuals susceptible to it. Equation (39), therefore, implies that

$$S(t) = S(0)e^{-\alpha t}$$

for $0 \leqslant t < T$. Substituting this into Eq. (40), we arrive at the differential equation

$$\frac{dI}{dt} + \alpha I = \alpha S(0)\, e^{-\alpha t}. \tag{41}$$

If we now multiply both sides of Eq. (41) by $e^{\alpha t}$, we get the equation

$$\frac{d}{dt}(Ie^{\alpha t}) = \alpha S(0).$$

Hence, $Ie^{\alpha t} = \alpha S(0)\, t + C$ and, therefore, the set of all solutions to Eq. (41) is given by the formula

$$I(t) = Ce^{-\alpha t} + \alpha S(0)\, te^{-\alpha t}. \tag{42}$$

Assuming that $t = 0$, we get $C = I(0)$ and, hence, Eq. (42) takes the form

$$I(t) = [I(0) + \alpha S(0)\, t]\, e^{-\alpha t}, \tag{43}$$

with $0 \leqslant t < T$.

We devote our further investigations to the problem of finding the specific value of T and the moment of time t_{max} at which the number of infected individuals proves maximal.

The answer to the first question is important because starting from T the individuals susceptible to the disease cease to become infected. If we turn to Eq. (43), the aforesaid implies that at $t = T$ its right-hand side assumes the value I^*, that is,

$$I^* = [I(0) + \alpha S(0) T] e^{-\alpha T}. \qquad (44)$$

But

$$S(T) = \lim_{t \to \infty} S(t) = S(\infty)$$

is the number of individuals that are susceptible to the disease and yet avoid falling ill; for such individuals the following chain of equalities holds true:

$$S(T) = S(\infty) = S(0) e^{-\alpha T}.$$

Hence,

$$T = \frac{1}{\alpha} \ln \frac{S(0)}{S(\infty)}. \qquad (45)$$

Thus, if we can point to a definite value of $S(\infty)$, we can use Eq. (45) to predict the time when the epidemic will stop. Substituting T given by (45) into Eq. (44), we arrive at the equation

$$I^* = \left[I(0) + S(0) \ln \frac{S(0)}{S(\infty)} \right] \frac{S(\infty)}{S(0)},$$

or

$$\frac{I^*}{S(\infty)} = \frac{I(0)}{S(0)} + \ln \frac{S(0)}{S(\infty)},$$

which can be rewritten in the form

$$\frac{I^*}{S(\infty)} + \ln S(\infty) = \frac{I(0)}{S(0)} + \ln S(0). \qquad (46)$$

Since I^* and all the terms on the right-hand side of (46) are known, we can use this equation to determine $S(\infty)$.

To answer the second question, we turn to Eq. (43). In accordance with the posed question, this equation yields

$$\frac{dI}{dt} = [\alpha S(0) - \alpha I(0) - \alpha^2 S(0)\, t]\, e^{-\alpha t} = 0.$$

The moment in time at which I attains its maximal value is given by the following formula:

$$t_{max} = \frac{1}{\alpha}\left[1 - \frac{I(0)}{S(0)}\right].$$

If we now substitute this value into Eq. (43), we get

$$I_{max} = S(0) \exp\{-[1 - I(0)/S(0)]\}$$
$$= S(t_{max}).$$

This relationship shows, for one, that at time t_{max} the number of individuals susceptible to the disease coincides with the number of infected individuals.

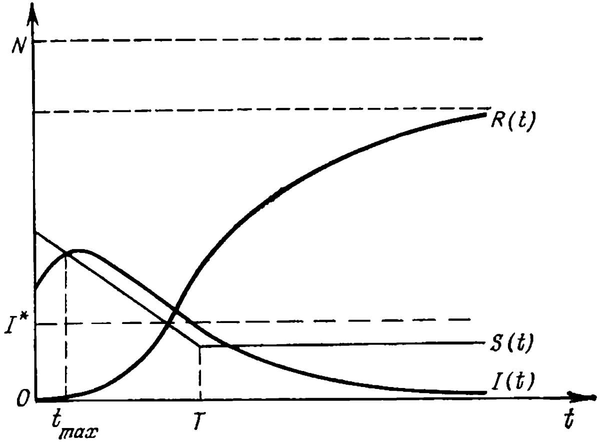

Fig. 10

But if $t > T$, the individuals susceptible to the disease cease to be infected and

$$I(t) = I^* e^{-\alpha(t-T)}.$$

Figure 10 gives a rough sketch of the diagrams that reflect the changes with the passage of time of the number of individuals in each of the three groups considered.

1.10 The Pursuit Curve

Let us examine an example in which differential equations are used to choose a correct strategy in pursuit problems.

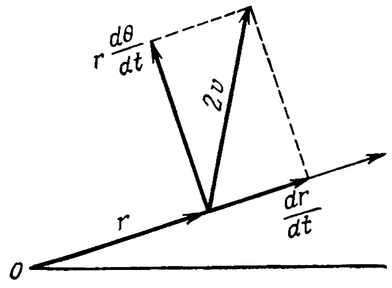

Fig. 11

We assume that a destroyer is in pursuit of a submarine in a dense fog. At a certain moment in time the fog lifts and the submarine is spotted floating on the surface three miles from the destroyer. The destroyer's speed is twice the submarine's speed. We wish to find the trajectory (the pursuit curve) that the destroyer must follow to pass exactly over the submarine if the latter immediately dives after being detected and proceeds at top speed and in a straight line in an unknown direction.

To solve the problem we first introduce polar coordinates, r and θ, in such a manner that the pole O is located at the point at which the submarine was located when discovered and that the point at which the destroyer was located when the submarine was discovered lies on the polar axis r (Figure 11). Further reasoning is based on the following considerations. First, the destroyer must take up a position in which it will be at the same distance from pole O as the submarine. Then it must move about O in such a way that both moving objects remain at the same distance from point O all the time. Only in this case will the destroyer eventually pass over the submarine when circling pole O. The aforesaid implies that the destroyer must first go straight to point O until it finds itself at the same distance x from O as the submarine.

Obviously, distance x can be found either from the equation

$$\frac{x}{v}=\frac{3-x}{2v}$$

or from the equation

$$\frac{x}{v}=\frac{3+x}{2v},$$

where v is the submarine's speed and $2v$ the destroyer's speed. Solving these equations, we find that this distance is either one mile or three miles.

Now, if the two still have not met, the destroyer must circle pole O (clockwise or counterclockwise) and head away from the pole at a speed equal to that of the submarine, v. Let us decompose the destroyer's velocity vector (of length $2v$) into two components, the radial component v_r and the tangential component v_t (Figure 11).

The radial component is the speed at which the destroyer moves away from pole O, that is

$$v_r = \frac{dr}{dt},$$

while the tangential component is the linear speed at which the destroyer circles the pole. The latter component, as is known, is equal to the product of the angular velocity $d\theta/dt$ and radius r, that is,

$$v_t = r\frac{d\theta}{dt}.$$

But since v_r must be equal to v, we have

$$v_t = \sqrt{(2v)^2 - v^2} = \sqrt{3}\,v.$$

Thus, the solution of the initial problem is reduced to the solution of a system of two differential equations,

$$\frac{dr}{dt} = v, \quad r\frac{d\theta}{dt} = \sqrt{3}\,v,$$

which, in turn, can be reduced to a single equation, $dr/r = d\theta/\sqrt{3}$, by excluding the variable t.

Solving the last differential equation, we find that

$$r = Ce^{\theta/\sqrt{3}},$$

with C an arbitrary constant.

If we now allow for the fact that destroyer begins its motion about pole O starting from the polar axis r at a distance of x miles away from O, that is, $r = 1$ at $\theta = 0$ and $r = 3$ at $\theta = -\pi$, we conclude that in the first case $C = 1$ and in the second $C = 3e^{\pi/\sqrt{3}}$. Thus, to fulfill its mission, the destroyer must move two or six miles along a straight line toward the point where the submarine was discovered and then move in the spiral $r = e^{\theta/\sqrt{3}}$ or the spiral $r = 3e^{(\theta+\pi)/\sqrt{3}}$.

1.11 Combat Models

During the First World War the British engineer and mathematician Frederick W. Lanchester (1868-1946) constructed several mathematical models of air battles. Later these models were generalized so as to describe battles involving regular troops or guerilla forces or the two simultaneously. Below we consider these three models.

Suppose that two opposing forces, x and y, are in combat. We denote the number of personnel at time t measured in days starting from the first day of combat operations by $x(t)$ and $y(t)$, respectively. It is the number of personnel that will play a decisive part in the construction of these models, since it is difficult, practically speaking, to specify the criteria that would allow taking into account, when comparing the opposing sides, not only the number of personnel but also combat readiness, level of military equipment, preparedness and experience of the officers, morale, and a great many other factors.

We also assume that $x(t)$ and $y(t)$ change continuously and, more than that, are differentiable as functions of time. Of course, these assumptions simplify the real situation because $x(t)$ and $y(t)$ are integers. But at the same time it is clear that if the number of personnel on each side is great, an increase by one or two persons will have an infinitesimal effect from the practical standpoint if compared to the effect produced by the entire force. Therefore, we may assume that during small time intervals the change in the number of personnel is also small (and does not constitute an integral number). The above reasoning is, of course, insufficient to specify concrete formulas for $x(t)$ and $y(t)$ as functions of t, but we can already point to a number of factors that enable describing the rate of change in the number of personnel on the two sides. Precisely, we denote by OLR the rate at which side x suffers losses from disease and other factors not related directly to combat operations, and by CLR the rate at which it suffers losses directly in combat operations involving side y. Finally by RR we denote the rate at which reinforcements are supplied to side x. It is then clear that the total rate of change of $x(t)$ is given by the equation

$$\frac{dx(t)}{dt} = -(OLR + CLR) + RR. \qquad (47)$$

A similar equation holds true for $y(t)$. The problem now is to find the appropriate formulas for the quantities *OLR*, *CLR*, and *RR* and then examine the resulting differential equations. The results should indicate the probable winner.

We use the following notation: a, b, c, d, g, and h are nonnegative constants characterizing the effect of various factors on personnel losses on both sides (x and y), $P(t)$ and $Q(t)$ are terms that allow for the possibility of reinforcements being supplied to x and y in the course of one day, and x_0 and y_0 are the personnel in x and y prior to combat. We are now ready to set up the three combat models suggested by Lanchester.* The first refers to combat operations involving regular troops:

$$\frac{dx(t)}{dt} = -ax(t) - by(t) + P(t),$$

$$\frac{dy(t)}{dt} = -cx(t) - dy(t) + Q(t).$$

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* Examples of combat operations are given in the article by C.S. Coleman, "Combat models" (see *Differential Equation Models*, eds. M. Braun, C.S. Coleman, and D.A. Drew, New York: Springer, 1983, pp. 109-131). The article shows how the examples agree with the models considered.

In what follows we call this system the A-type differential system (or simply the A-type system).

The second model, specified by the equations

$$\frac{dx(t)}{dt} = -ax(t) - gx(t)\,y(t) + P(t),$$

$$\frac{dy(t)}{dt} = -dy(t) - hx(t)\,y(t) + Q(t)$$

describes combat operations involving only guerilla forces. We will call this system the B-type system. Finally, the third model, which we call the C-type system, has the form

$$\frac{dx(t)}{dt} = -ax(t) - gx(t)\,y(t) + P(t),$$

$$\frac{dy(t)}{dt} = -cx(t) - dy(t) + Q(t)$$

and describes combat operations involving both regular troops and guerilla forces.

Each of these differential equations describes the rate of change in the number of personnel on the opposing sides as a function of various factors and has the form (47). Losses in personnel that are not directly related to combat operations and are determined by the terms $-ax(t)$ and $-dy(t)$ make it possible to describe the constant fractional loss rates (in the absence of combat operations and reinforcements) via the equations

$$\frac{1}{x}\frac{dx}{dt} = -a, \quad \frac{1}{y}\frac{dy}{dt} = -d.$$

If the Lanchester models contain only terms corresponding to reinforcements and losses not associated with combat operations, this means that there are no combat operations, while the presence of the terms $-by(t)$, $-cx(t)$, $-gx(t)\,y(t)$, and $-hx(t)\,y(t)$ means that combat operations take place.

In considering the A-type system, we assume, first, that each side is within the range of the fire weapons of the other and, second, that only the personnel directly involved in combat come under fire. Under such assumptions Lanchester introduced the term $-by(t)$ for the regular troops of side x to reflect combat losses. The factor b characterizes the effectiveness of side y in combat. Thus, the equation

$$\frac{1}{y}\frac{dx}{dt} = -b$$

shows that constant b measures the average effectiveness of each man on side y. The same interpretation can be given to the term $-cx(t)$. Of course, there is no simple way in which we can calculate the effectiveness coefficients b and c. One way is to write these coefficients in the form

$$b = r_y p_y, \quad c = r_x p_x, \tag{48}$$

where r_y and r_x are the coefficients of firepower of sides y and x, respectively, and p_y and p_x are the probabilities that each shot fired by sides y and x, respectively, proves accurate.

We note further that the terms corresponding to combat losses in the A-type system are linear, whereas in the B-type system they are nonlinear. The explanation lies in the following. Suppose that the guerilla forces amounting to $x(t)$ personnel occupy a certain territory R and remain undetected by the opposing side. And although the latter controls this territory by firepower, it cannot know the effectiveness of its actions. It is also highly probable that the losses suffered by the guerilla forces x are, on the one hand, proportional to the number of personnel $x(t)$ on R and, on the other, to the number $y(t)$ of personnel of the opposing side. Hence, the term corresponding to the losses suffered by the guerilla forces x has the form $-gx(t)\,y(t)$, where the coefficient reflecting the effectiveness of combat operations of side y is, in general, more difficult to estimate than the coeffiient b in the first relationship in (48). Nevertheless, to find g we can use the firepower coefficient r_y and also allow for the ideas expounded by Lanchester, according to which the probability of an accurate shot fired by side y is directly proportional to the so-called territorial effectiveness A_{ry}

of a single shot fired by y and inversely proportional to the area A_x of the territory R occupied by side x. Here by A_{ry} we denote the area occupied by a single guerilla fighter. Thus, with a high probability we can assume that the formulas for finding g and h are

$$g = r_y \frac{A_{ry}}{A_x}, \quad h = r_x \frac{A_{rx}}{A_y}. \tag{49}$$

Below we discuss in greater detail each of the three differential models.

Case A (differential systems of the A-type and the quadratic law). Let us assume that the regular troops of the two opposing sides are in combat in the simple situation in which the losses not associated directly with combat operations are nil. If, in addition, neither side receives any reinforcement, the mathematical model is reduced to the following differential system:

$$\frac{dx}{dt} = -by, \quad \frac{dy}{dt} = -cx. \tag{50}$$

Dividing the second equation by the first, we find that

$$\frac{dy}{dx} = \frac{cx}{by}. \tag{51}$$

Integrating, we arrive at the following relationship:

$$b\,[y^2(t) - y_0^2] = c\,[x^2(t) - x_0^2]. \tag{52}$$

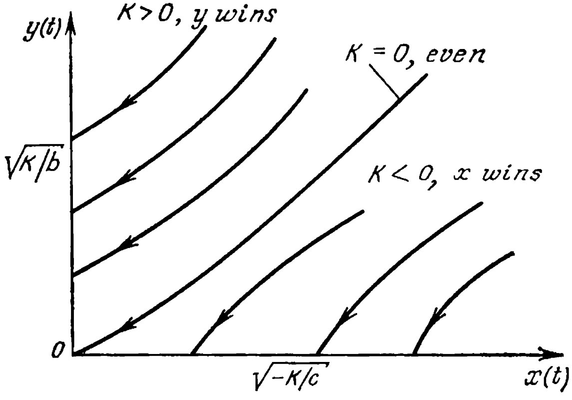

Fig. 12

This relationship explains why system (50) corresponds to a model with a quadratic law. If by K we denote the constant quantity $by_0^2 - cx_0^2$, the equation

$$by^2 - cx^2 = K \tag{53}$$

obtained from Eq. (52) specifies a hyperbola (or a pair of straight lines if $K = 0$) and we can classify system (50) with great precision. We can call it a differential system with a hyperbolic law.

Figure 12 depicts the hyperbolas for different values of K; for obvious reasons we consider only the first quadrant ($x \geqslant 0$, $y \geqslant 0$). The arrows on the curves point in the direction in which the number of personnel changes with passage of time.

To answer the question of who wins in the constructed model (50), we agree to say that side y (or x) wins if it is the first to wipe out the other side x (or y). For example, in our case the winner is side y if $K > 0$ since in accordance with Eq. (53) the variable y can never vanish, while the variable x vanishes at $y(t) = \sqrt{K/b}$. Thus, for y to win a victory, it must strive for a situation in which K is positive, that is,

$$by_0^2 > cx_0^2. \tag{54}$$

Using (48), we can rewrite (54) in the form

$$\left(\frac{y_0}{x_0}\right)^2 > \frac{r_x}{r_y}\frac{p_x}{p_y}. \tag{55}$$

The left-hand side of (55) demonstrates that changes in the personnel ratio y_0/x_0 give an advantage to one side, in accordance with the quadratic law. For example, a change in the y_0/x_0 from one to two gives y a four-fold advantage. Note also that Eq. (53) denotes the relation between the personnel numbers of the opposing sides but does not depend explicitly on time. To derive formulas that would provide us with a time dependence, we proceed as follows. We differentiate the first equation in (50) with respect to time and then use the second equation in this system. As a result we

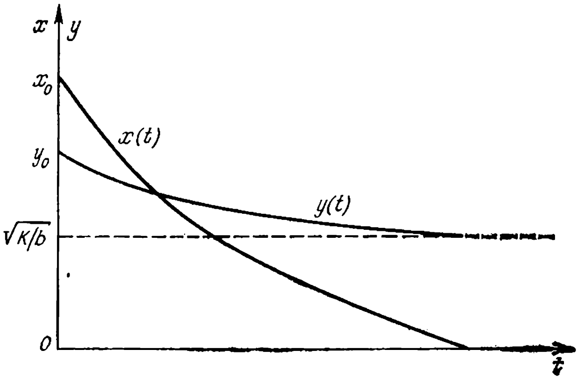

Fig. 13

arrive at the following differential equation:

$$\frac{d^2x}{dt^2} - bcx = 0. \tag{56}$$

If for the initial conditions we take

$$x(0) = x_0, \quad \left.\frac{dx}{dt}\right|_{t=0} = -by_0,$$

then the solution to Eq. (56) can be obtained in the form

$$x(t) = x_0 \cos \beta t - \gamma y_0 \sin \beta t, \tag{57}$$

where $\beta = \sqrt{bc}$ and $\gamma = \sqrt{b/c}$.

In a similar manner we can show that

$$y(t) = y_0 \cos \beta t - \frac{x_0}{\gamma} \sin \beta t. \tag{58}$$

Figure 13 depicts the graphs of the functions specified by Eqs. (57) and (58) in the case where $K > 0$ (i.e. $by_0^2 > cx_0^2$, or $\gamma y_0 > x_0$).

We note in conclusion that for side y to win a victory it is not necessary that y_0 be greater than x_0. The only requirement is that γy_0 be greater than x_0.

Case B (differential systems of the B-type and the linear law). The dynamical equations that model combat operations between two opposing sides can easily be solved, as in the previous case, if we exclude the possibility of losses not associated with combat and if neither side receives reinforcements. Under these limitations the B-type differential system assumes the form

$$\frac{dx}{dt} = -gxy, \quad \frac{dy}{dt} = -hxy. \tag{59}$$

Dividing the second equation in (59) by the first, we get the equation

$$\frac{dy}{dx} = \frac{h}{g},$$

which after being integrated yields

$$g\,[y\,(t) - y_0] = h\,[x\,(t) - x_0]. \tag{60}$$

This linear relationship explains why the nonlinear system (59) corresponds to a model with a linear law for conducting combat operations. Equation (60) can be rewritten in the form

$$gy - hx = L, \tag{61}$$

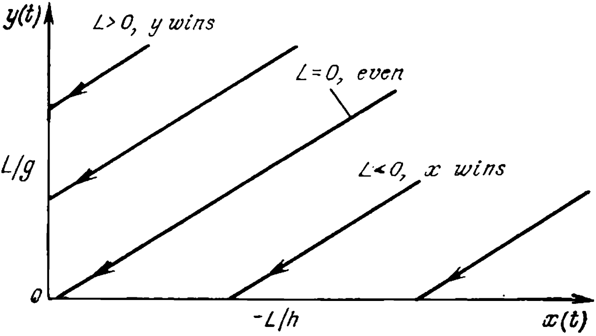

Fig. 14

with $L = gy_0 - hx_0$. This implies, for one, that if L is positive, side y wins as a result of combat operations, while if L is negative, side x wins.

Figure 14 provides a geometrical interpretation of the linear law (61) for different values of L.

Let us now study in greater detail the situation where one of the sides wins. We assume that this is side y. Then, as we already know, $gy_0 - hx_0$ must be positive, or

$$\frac{y_0}{x_0} > \frac{h}{g}.$$

If we now turn to (49), we see that side y wins if

$$\frac{y_0}{x_0} > \frac{r_x A_{rx} A_x}{r_y A_{ry} A_y}. \tag{62}$$

Thus, the strategy of y is to make the ratio y_0/x_0 as large as possible and the ratio A_x/A_y as small as possible. From the practical standpoint it is more convenient to write inequality (62) in the form

$$\frac{A_y y_0}{A_x x_0} > \frac{r_x A_{rx}}{r_y A_{ry}}.$$

We see that in a certain sense the products $A_y y_0$ and $A_x x_0$ constitute critical values.

We note, finally, that by combining (61) and (59) we can easily derive formulas that give the time dependence of changes in the personnel numbers of both sides.

Case C (differential systems of the C-type and the parabolic law). In the C-model the guerilla forces face regular troops. We introduce the simplifying assumptions that neither is supplied with reinforcements and that the losses associated directly with combat operations are nil. In this case we have the following system of differential equations:

$$\frac{dx}{dt} = -gxy, \qquad \frac{dy}{dt} = -cx, \tag{63}$$

where x (t) is the number of personnel in the guerilla forces, and y (t) the number of personnel in the regular troops. Dividing the second equation in (63) by the first, we arrive at the differential equation

$$\frac{dy}{dx} = \frac{c}{gy}.$$

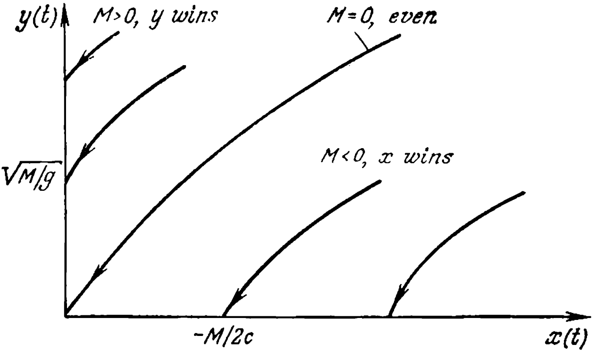

Fig. 15

Integrating with the appropriate limits yields the following relationship:

$$gy^2(t) = 2cx(t) + M, \tag{64}$$

where $M = gy_0^2 - 2cx_0$. Thus, the system of differential equations (63) corresponds to a model with a parabolic law for conducting combat operations. The guerilla forces win if M is negative; they are defeated if M is positive.

Figure 15 depicts the parabolas defined via Eq. (64) for different values of M. Experience shows that regular troops can defeat guerilla forces only if the ratio y_0/x_0 considerably exceeds unity. Basing our reasoning on the parabolic law for conducting combat operations and assuming M positive, we conclude that the victory of regular troops is guaranteed if $(y_0/x_0)^2$ is greater than $(2c/g)\, x_0^{-1}$. If we allow for (48) and (49), we can rewrite this condition in the form

$$\left(\frac{y_0}{x_0}\right)^2 > 2\,\frac{r_x}{r_y}\,\frac{A_x p_x''}{A_{ry}}\,\frac{1}{x_0}.$$

1.12 Why Are Pendulum Clocks Inaccurate?

To answer this question, let us consder an idealized model of a pendulum clock consisting of a rod of length l and a load of mass m attached to the lower end of the rod (the rod's mass is assumed so small that it can be ignored in comparison to m) (Figure 16). If the load is deflected by an angle α

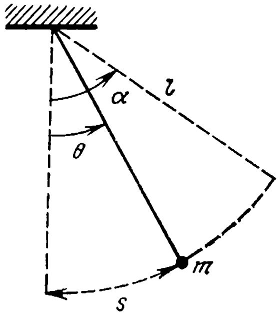

Fig. 16

and then released, in accordance with energy conservation we obtain

$$\frac{mv^2}{2} = mg\,(l\cos\theta - l\cos\alpha), \tag{65}$$

where v is the speed at which the load moves, and g is the acceleration of gravity.

Since $s = l\theta$, we have $v = ds/dt = l\,(d\theta/dt)$ and (65) leads us to the following differential equation:

$$\frac{l}{2}\left(\frac{d\theta}{dt}\right)^2 = g\,(\cos\theta - \cos\alpha). \tag{66}$$

If we now allow for the fact that θ decreases with the passage of time t (for small t's), we can rewrite Eq. (66) in the form

$$dt = -\sqrt{\frac{l}{2g}}\,\frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}.$$

If we denote the period of the pendulum by T, we have

$$\frac{T}{4} = -\sqrt{\frac{l}{2g}}\int_{\alpha}^{0}\frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}},$$

or

$$T = 4\sqrt{\frac{l}{2g}}\int_{0}^{\alpha}\frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}. \tag{67}$$

The last formula shows that the period of the pendulum depends on α. This is the main reason why pendulum clocks are inaccurate since, practically speaking, every time that the load is deflected to the extreme position the deflection angle differs from α.

We note that formula (67) can be written in a simpler form. Indeed, since

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}, \quad \cos\alpha = 1 - 2\sin^2\frac{\alpha}{2},$$

we have

$$T = 2\sqrt{\frac{l}{g}}\int_0^\alpha \frac{d\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}}$$

$$= 2\sqrt{\frac{l}{g}}\int_0^\alpha \frac{d\theta}{\sqrt{k^2 - \sin^2\frac{\theta}{2}}}, \qquad (68)$$

with $k = \sin(\alpha/2)$.

Now instead of variable θ we introduce a new variable φ via the formula $\sin(\theta/2) = k\sin\varphi$. This implies that when θ increases from 0 to α, the variable φ grows from 0 to $\pi/2$, with

$$\frac{1}{2}\cos\frac{\theta}{2}\,d\theta = k\cos\varphi\,d\varphi,$$

or

$$d\theta = \frac{2k\cos\varphi\,d\varphi}{\cos\frac{\theta}{2}} = \frac{2\sqrt{k^2 - \sin^2\frac{\theta}{2}}}{\sqrt{1 - k^2\sin^2\varphi}}\,d\varphi.$$

The last relationship enables us to rewrite formula (68) in the form

$$T=4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

$$=4\sqrt{\frac{l}{g}}F(k,\ \pi/2),$$

where the function

$$F(k,\ \psi)=\int_0^{\psi}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

is known as the elliptic integral of the first kind, differing from the elliptic integral of the second kind

$$E(k,\ \psi)=\int_0^{\psi}\sqrt{1-k^2\sin^2\varphi}\,d\varphi.$$

Elliptic integrals cannot be expressed in terms of elementary functions. Therefore, all further discussion of the pendulum problem will be related to an approach considered when conservative systems are studied in mechanics. Here we only note that the starting point in our studies will be the differential equation

$$\frac{d^2\theta}{dt^2}+k\sin\theta=0,\quad k=\sqrt{g/l},$$

which can be obtained from Eq. (60) by differentiating with respect to t.

1.13 The Cycloidal Clock

We have established that clocks with ordinary (circular) pendulums are inaccurate. Is there any pendulum whose period is independent of the swing? This problem was first formulated and solved as early as the 17th century. Below we give its solution, but first let us turn to the derivation of the equation of a remarkable curve, which Galileo Galilei called the *cycloid* (from the Greek word for "circular"). This is a plane curve generated by a point on the circumference of a circle (called the generating circle) as it rolls along a straight line without slippage.

Suppose that the x axis is the straight line along which the generating circle rolls and that the radius of this circle is r (Figure 17). Let us also assume that initially the point that traces the cycloid is at the origin and that after the circle turns through an angle θ, the point occupies position M. Then, on the basis of geometrical reasoning we obtain

$$x = OS = OP - SP,$$

$$y = MS = CP - CN.$$

But

$$OP = \overset{\frown}{MP} = r\theta,\ SP = MN = r \sin \theta,$$

$$CP = r,\ CN = r \cos \theta,$$

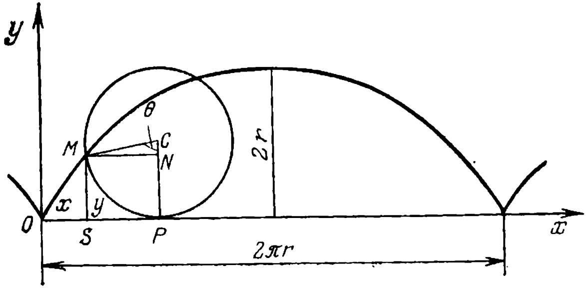

Fig. 17

Hence, parametrically the cycloid is specified by the following equations:

$$x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta). \quad (69)$$

If we exclude parameter θ from these equations, we arrive at the following equation of a cycloid in a rectangular Cartesian coordinate system Oxy:

$$x = r\cos^{-1}\left(\frac{r-y}{r}\right) - \sqrt{2ry - y^2}.$$

The very method of constructing a cycloid implies that the cycloid consists of congruent arcs each of which corresponds to a full revolution of the generating circle.*

* The reader may find many interesting facts about the cycloid and related curves in G.N. Berman's book *The Cycloid* (Moscow: Nauka, 1980) (in Russian).

The separate arcs are linked at points where they have a common vertical tangent. These points, known as *cusps*, correspond to the lowest possible positions occupied by the point on the generating circle that describes the cycloid. The highest possible positions occupied by the same point lie exactly in the middle of each arc and are known as the *vertices*. The distance along the straight line between successive cusps is $2\pi r$, and the segment of that straight line between successive cusps is known as the *base* of an arc of the cycloid.

The cycloid possesses the following properties:

(a) *the area bounded by an arc of a cycloid and the respective base is thrice the area of the generating circle (Galileo's theorem)*;

(b) *the length of one cycloid arc is four times the length of the diameter of the generating circle (Wren's theorem).*

The last result is quite unexpected, since to calculate the length of such a simple curve as the circumference of a circle it is necessary to introduce the irrational number π, whose calculation is not very simple, while the length of an arc of a cycloid is expressed as an integral multiple of the radius (or diameter) of the generating circle. The cycloid has many other interesting properties, which have proved to be extremely important for physics and engineering. For example, the profile of pinion teeth

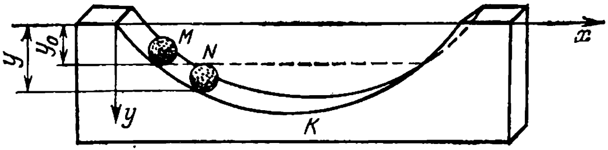

Fig. 18

and of many types of eccentrics, cams, and other mechanical parts of devices have the shape of a cycloid.

Let us now turn to the problem that enabled the Dutch physical scientist, astronomer, and mathematician Christian Huygens (1629-1695) to build an accurate clock in 1673. The problem consists of building in the vertical plane a curve for which the time of descent of a heavy particle sliding without friction to a fixed horizontal line is the same wherever the particle starts on the curve (it is assumed that initially, at time $t = t_0$, the particle is at rest). Huygens found that the cycloid possesses such an isochronous (from the Greek words for "equal" and "time") or tautochronous property (from tautos for "identical").

From the practical standpoint the problem can be solved in the following manner. Let us assume that a trough in the form of a cycloid is cut out of a piece of wood as shown in Figure 18. We take a small metal ball and send it rolling down the slope. Ignoring friction, let us try to determine the time it takes the ball to reach the lowest possible point, K, starting from point M, say.

Let x_0 and y_0 be the coordinates of the initial position of the ball, i.e. point M, and θ_0 the corresponding value of parameter θ. When the ball reaches a point $N(\theta)$, the distance of descent along the vertical will be h, which in view of Eq. (69) can be found in the following manner:

$$h = y - y_0 = r(1 - \cos\theta)$$
$$- r(1 - \cos\theta_0) = r(\cos\theta_0 - \cos\theta).$$

We know that the speed of a falling object is given by the formula

$$v = \sqrt{2gh},$$

where g is the acceleration of gravity. In our case the last formula assumes the form

$$v = \sqrt{2gr(\cos\theta_0 - \cos\theta)}$$

On the other hand, since speed is the derivative of distance s with respect to time T, we arrive at the formula

$$\frac{ds}{dT} = \sqrt{2gr(\cos\theta_0 - \cos\theta)}$$

Since for a cycloid $ds = 2r\sin(\theta/2)\,d\theta$, we can rewrite this formula (which is actually a differential equation) in the form

$$dT = \frac{2r\sin(\theta/2)\,d\theta}{\sqrt{2gr(\cos\theta_0 - \cos\theta)}}.$$

Integrating this equation within appropriate limits, we obtain

$$T = \int_0^{\pi} \frac{2r \sin(\theta/2)\, d\theta}{\sqrt{2gr(\cos\theta_0 - \cos\theta)}}$$

$$= -2\sqrt{\frac{r}{g}} \int_0^{\pi} \frac{d \cos(\theta/2)}{\sqrt{\cos^2 \frac{\theta_0}{2} - \cos^2 \frac{\theta}{2}}}$$

$$= \pi \sqrt{r/g}.$$

Thus, the time interval T in the course of which the ball rolls down from point M to point K is given by the formula

$$T = \pi \sqrt{r/g},$$

which shows that period T is independent of θ_0, that is, the period does not depend on the initial position of the ball, M. Obviously, two balls that begin their motion simultaneously from points M and N will roll down and find themselves at point K at the same moment of time.

Since we agreed to ignore friction, in its motion down the slope the ball passes point K and, by inertia, continues its motion up the slope to point M_1 lying at the same level as point M. Proceeding then in the opposite direction and traversing its path, the ball completes a full cycle. This constitutes the motion of a *cycloidal pendulum*

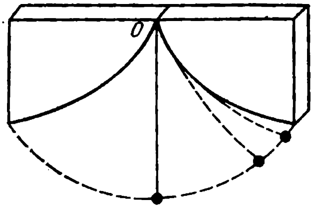

Fig. 19

with a period of oscillations

$$T_0 = 4\pi\sqrt{r/g}. \qquad (70)$$

A peculiar feature of the cycloidal pendulum, which sets it apart from the simple (circular) pendulum, is that its period does not depend on the amplitude (or swing).

Let us now show how an ordinary circular pendulum can be made to move in a tautochronous manner without resorting to trough and similar devices with considerable friction. To this end it is sufficient to make a template (say, out of wood) consisting of two semiarcs of a cycloid with a common cusp (Figure 19). The template is fixed in the vertical plane and a string with a ball is suspended from the cusp O. The length of the string must be twice the diameter of the generating circle of the cycloid.

If the ball on the string is deflected to an arbitrary point M, it begins to swing back and forth with a period that is independ-

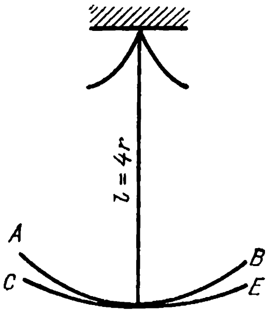

Fig. 20

ent of the position of point M. Even if due to friction and air drag the amplitude of the oscillations diminishes, the period remains unchanged. For a circular pendulum, which moves along an arc of a circumference, the isochronous property is satisfied approximately only for small amplitudes, when the arc differs little from the arc of a cycloid.

By way of an example let us study the small oscillations that a pendulum executes along the arc $\overset{\frown}{AB}$ of a cycloid (Figure 20). If these oscillations are very small, the effect of the guiding template is practically nil and the pendulum oscillates almost like an ordinary circular pendulum with a string (the "rod") whose length is $4r$. The path AB of a cycloidal pendulum will differ little from the path CE of a circular pendulum with a string length equal to $4r$. Hence, the period of small oscillations of a circular pendulum with a string length $l = 4r$ is practically the same as that of a cycloidal pendulum of the same length.

Now, if in (70) we put $r = l/4$, the period of small oscillations of a circular pendulum can be expressed in terms of the length of the string:

$$T = 2\pi \sqrt{l/g}.$$

This formula is derived (in a different way) in high school physics.

In conclusion we note that the cycloid is the only curve for which a particle moving along it performs isochronous oscillations.

1.14 The Brachistochrone Problem

The problem concerning the brachistochrone (from the Greek words for "shortest" and "time"), a curve of fastest descent, was proposed by the Swiss mathematician John Bernoulli (1667-1748) in 1696 as a challenge to mathematicians and consists of the following.

Take two points A and B lying in a vertical plane but not on a single vertical line (Figure 21). Among the various curves passing through these two points we must find

Fig. 21

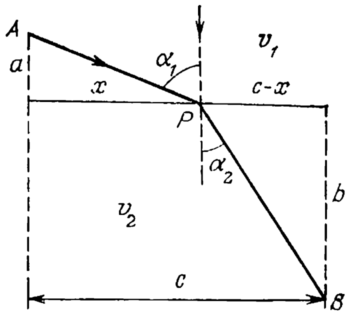

Fig. 22

the one for which the time required for a particle to fall from point A to point B along the curve under the force of gravity is minimal.

The problem was tackled by the best mathematicians. It was solved by John Bernoulli himself and also by Gottfried W. von Leibniz (1646-1716), Sir Isaac Newton (1642-1727), Guillaume L'Hospital (1661-1704), and Jakob Bernoulli (1654-1705). The problem is famous not only from the general scientific viewpoint but also for being the source of ideas in a completely new field of mathematics, the calculus of variations.

Solution of the brachistochrone problem can be linked with that of another problem originating in optics. Let us turn to Figure 22, in which a ray of light is depicted as propagating from point A to point P with a velocity v_1 and then from point P to point B in a denser medium with a lower velocity v_2. The total time T required for the ray to propagate from point A to point B can, obviously, be found from the following formula:

$$T=\frac{\sqrt{a^2+x^2}}{v_1}+\frac{\sqrt{b^2+(c-x)^2}}{v_2}.$$

If we assume that the ray of light propagates from point A to point B along this path in the shortest possible time T, the derivative $\mathrm{d}T/\mathrm{d}x$ must vanish. But then

$$\frac{x}{v_1\sqrt{a^2+x^2}}=\frac{c-x}{v_2\sqrt{b^2+(c-x)^2}},$$

or

$$\frac{\sin\alpha_1}{v_1}=\frac{\sin\alpha_2}{v_2}.$$

The last formula expresses Snell's well-known law of refraction, which initially was discovered in experiments in the form $\sin\alpha_1/\sin\alpha_2 = a$, with a constant.

The above assumption that light chooses a path from A to B that would take the shortest possible time to travel is known as *Fermat's principle*, or *the principle of least time*. The importance of this principle lies not only in the fact that it can be taken as a rational basis for deriving Snell's law but also, for one, in that it can be applied to finding the path of a ray of light in a medium of variable density, where generally

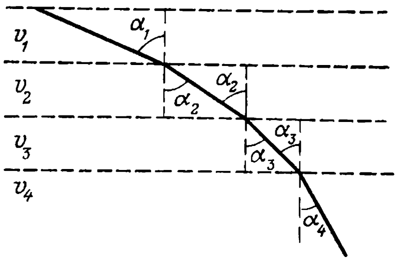

Fig. 23

the light travels not along straight-line segments.

For the sake of an example let us consider Figure 23, which depicts a ray of light propagating through a layered medium. In each layer the speed of light is constant, but it decreases when we pass from an upper layer to a lower layer. The incident ray is refracted more and more strongly as it passes from layer to layer and moves closer and closer to the vertical line. Applying Snell's law to the interfaces between the layers, we get

$$\frac{\sin \alpha_1}{v_1} = \frac{\sin \alpha_2}{v_2} = \frac{\sin \alpha_3}{v_3} = \frac{\sin \alpha_4}{v_4}.$$

Now let us assume that the layer thicknesses decrease without limit while the number of layers increases without limit. Then, in the limit, the velocity of light changes (decreases) continuously and we conclude

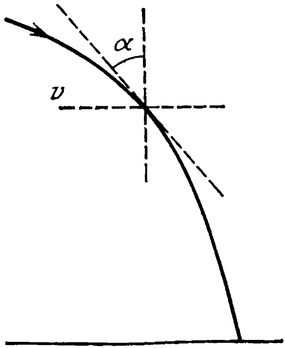

Fig. 24

(see Figure 24) that

$$\frac{\sin\alpha}{v} = a, \tag{71}$$

with a = const. A similar situation is observed (with certain reservations) when a ray of Sun light falls on Earth. As the ray travels through Earth's atmosphere of increasing density, its velocity decreases and the ray bends.

Let us go back to the brachistochrone problem. We introduce a system of coordinates in the vertical plane in the way shown in Figure 25. We imagine that a ball (like a ray of light propagating in media) is capable of choosing the path of descent from point A to point B with the shortest possible time of descent. Then, as fol-

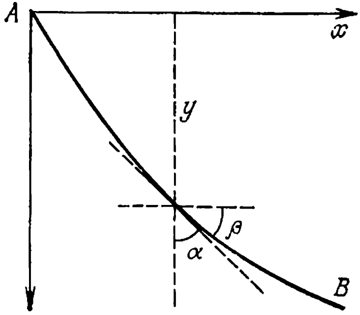

Fig. 25

lows from the above reasoning, formula (71) is valid.

The energy conservation principle implies that the speed gained by the ball at a given level depends only on the loss of potential energy as the ball reaches the level and not on the shape of the trajectory followed. This means that $v = \sqrt{2gy}$.

On the other hand, geometric construction enables us to show that

$$\sin\alpha = \cos\beta = \frac{1}{\sec\beta} = \frac{1}{\sqrt{1+\tan^2\beta}} = \frac{1}{\sqrt{1+(y')^2}}.$$

Combining the last two relationships with (71) yields

$$y\,[1+(y')^2] = C, \qquad (72)$$

This constitutes the brachistochrone equation. We wish to demonstrate that only a cycloid can represent a brachistochrone. Indeed, since $y' = dy/dx$, by dividing the variables in Eq. (72) we arrive at the equation

$$dx = \left(\frac{y}{C-y}\right)^{1/2} dy.$$

We introduce a new variable φ via the following relationship:

$$\left(\frac{y}{C-y}\right)^{1/2} = \tan\varphi.$$

Then

$$y = C\sin^2\varphi, \quad dy = 2C\sin\varphi\cos\varphi\, d\varphi,$$

$$dx = \tan\varphi\, dy = 2C\sin^2\varphi\, d\varphi$$

$$= C(1-\cos 2\varphi)\, d\varphi.$$

Integration of the last equation yields

$$x = \frac{C}{2}(2\varphi - \sin 2\varphi) + C_1,$$

where, in view of the initial conditions, $x = y = 0$ at $\varphi = 0$ and $C_1 = 0$. Thus,

$$x = \frac{C}{2}(2\varphi - \sin 2\varphi),$$

$$y = C\sin^2\varphi = \frac{C}{2}(1-\cos 2\varphi).$$

Assuming that $C/2 = r$ and $2\varphi = \theta$, we arrive at the standard parametric equations of the cycloid (69). The cycloid is, indeed, a remarkable curve: it is not only isochronous—it is brachistochronous.

1.15 The Arithmetic Mean, the Geometric Mean, and the Associated Differential Equation

Let us consider the following curious problem first suggested by the famous German mathematician Carl F. Gauss (1777-1855).

Let m_0 and n_0 be two arbitrary positive numbers ($m_0 > n_0$). Out of m_0 and n_0 we construct two new numbers m_1 and n_1 that are, respectively, the arithmetic mean and the geometric mean of m_0 and n_0. In other words, we put

$$m_1 = \frac{m_0 + n_0}{2}, \quad n_1 = \sqrt{m_0 n_0}.$$

Treating m_1 and n_1 in the same manner as m_0 and n_0, we put

$$m_2 = \frac{m_1 + n_1}{2}, \quad n_2 = \sqrt{m_1 n_1}.$$

Continuing this process indefinitely, we arrive at two sequences of real numbers, $\{m_k\}$ and $\{n_k\}$ ($k = 0, 1, 2, \ldots$), which, as can easily be demonstrated, are convergent. We wish to know the difference of the limits of these two sequences.

An elegant solution of this problem amounting to setting up a second-order linear differential equation is given below. It belongs to the German mathematician Carl W. Borchardt (1817-1880). Let a be the sought difference. It obviously depends on m_0 and n_0, a fact expressed analytically as follows: $a = f(m_0, n_0)$, where f is a function. The definition of a also implies that $a = f(m_1, n_1)$. Now, if we multiply m_0 and n_0 by the same number k, each of the numbers m_1, n_1, m_2, n_2, introduced above, including a, will be multiplied by k. This means that a is a first-order homogeneous function in m_0 and n_0 and, hence,

$$a = m_0 f(1, n_0/m_0) = m_1 f(1, n_1/m_1).$$

Introducing the notation $x = n_0/m_0$, $x_1 = n_1/m_1$, ., $y = 1/f(1, n_0/m_0)$, $y_1 = 1/f(1, n_1/m_1)$, we find that

$$y = y_1 \frac{m_0}{m_1} = \frac{2y_1}{1+x}. \tag{73}$$

Since x_1 is related to x via the equation

$$x_1 = \frac{2\sqrt{x}}{1+x},$$

we find that

$$\frac{dx_1}{dx} = \frac{1-x}{(1+x)^2\sqrt{x}} = \frac{(x_1 - x_1^3)(1+x)^2}{2(x-x^3)}.$$

On the other hand, Eq. (73) leads to the following relationship:

$$\frac{dy}{dx} = -\frac{2}{(1+x)^2} y_1 + \frac{2}{1+x} \frac{dy_1}{dx_1} \frac{dx_1}{dx}.$$

Replacing dx_1/dx with its value given by the previous relationship and factoring out $(x - x^3)$, we find that

$$(x-x^3)\frac{dy}{dx}=\frac{2x(x-1)}{1+x}y_1$$
$$+(1+x)(x_1-x_1^3)\frac{dy_1}{dx_1}.$$

Differentiating both sides of this equation with respect to x, we find that

$$\frac{d\left[(x-x^3)\frac{dy}{dx}\right]}{dx}$$
$$=2y_1\frac{d\left[\frac{x(x-1)}{1+x}\right]}{dx}+\frac{2x(x-1)}{1+x}\frac{dy_1}{dx_1}\frac{dx_1}{dx}$$
$$+(x_1-x_1^3)\frac{dy_1}{dx_1}$$
$$+(1+x)\frac{d}{dx_1}\left[(x_1-x_1^3)\frac{dy_1}{dx_1}\right]\frac{dx_1}{dx}.$$

By elementary transformation this equation can be reduced to

$$\frac{d}{dx}\left[(x-x^3)\frac{dy}{dx}\right]-xy$$
$$=\frac{1-x}{(1+x)\sqrt{x}}\left\{\frac{d}{dx_1}\left[(x_1-x_1^3)\frac{dy_1}{dx_1}\right]-x_1y_1\right\}.$$

If we now replace x with x_1, then x_1 will become x_2. If we then replace x_1 with x_2, we find that x_2 is transformed into x_3, etc. Hence, assuming that

$$\frac{d}{dx}\left[(x-x^3)\frac{dy}{dx}\right]-xy=a^*(y),$$

we arrive at the following formula:

$$a^*(y)=\frac{1-x}{(1+x)\sqrt{x}}\frac{1-x}{(1+x_1)\sqrt{x_1}}\frac{1-x_2}{(1+x_2)\sqrt{x_2}}$$
$$\times\qquad\times\frac{1-x_n}{(1+x_n)\sqrt{x_n}}a^*(y_n).$$

If we now send n to infinity, we find that $1-x_n$ tends to zero and, hence,

$$a^*(y)=0.$$

This means that y satisfies the differential equation

$$(x-x^3)\frac{d^2y}{dx^2}+(1-3x^2)\frac{dy}{dx}-xy=0. \quad (74)$$

If we note that

$$a=f(m_0, n_0)=y\frac{f^2(m_0, n_0)}{m_0},$$

we can easily find the value of this number. Indeed, since y must be the constant solution to Eq. (74), we find that only $y\equiv 0$ is such a solution. Thus, the difference of the limits of the sequences $\{m_k\}$ and $\{n_k\}$ is zero.

The differential equation (74) is remarkable not only because it enabled reducing the initial problem to an obvious one but also because it is linked directly to the solution of the problem of calculating the period of small oscillations of a circular pendulum.

As demonstrated earlier, the period of small oscillations of a circular pendulum can be found from the formula

$$T = 4\sqrt{l/g}\, F\ (k,\ \pi/2),$$

with

$$F(k,\ \pi/2) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}.$$

It has been found that if $0 \leqslant k < 1$, then

$$\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

$$= \frac{\pi}{2}\left(1+\sum_{n=1}^{\infty} \frac{1^2\times 3^2\times 5^2\times \ldots \times (2n-1)^2}{2^2\times 4^2\times 6^2\times \ldots \times (2n)^2}\, k^{2n}\right),$$

where

$$y = 1 + \sum_{n=1}^{\infty} \frac{1^2\times 3^2\times 5^2\times \ldots \times (2n-1)^2}{2^2\times 4^2\times 6^2\times \quad \times (2n)^2}\, x^{2n}$$

is a solution to the differential equation (74).

1.16 On the Flight of an Object Thrown at an Angle to the Horizon

Suppose that an object is thrown at an angle α to the horizon with an initial velocity v_0. We wish to derive the equation of the object's motion that ignores forces of friction (air drag).

Let us select the coordinate axes as shown in Figure 26. At an arbitrary point of the trajectory only the force of gravity P equal to mg, with m the mass of the object and g the acceleration of gravity, acts on the object. Hence, in accordance with Newton's second law, we can write the differential equations of the motion of the object as projected on the x and y axes as follows:

$$m\frac{d^2x}{dt^2}=0, \quad m\frac{d^2y}{dt^2}=-mg.$$

Factoring out m yields

$$\frac{d^2x}{dt^2}=0, \quad \frac{d^2y}{dt^2}=-g. \qquad (75)$$

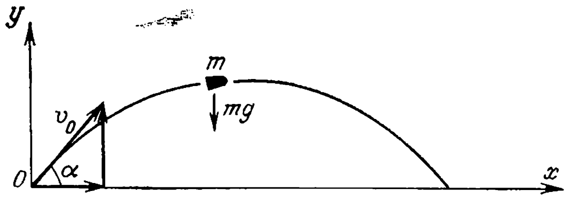

Fig. 26

The initial conditions imposed on the object's motion are

$$x=0, \quad y=0, \quad \frac{dx}{dt}=v_0 \cos\alpha, \\ \frac{dy}{dt}=v_0 \sin\alpha \text{ at } t=0. \tag{76}$$

Integrating Eqs. (75) and allowing for the initial conditions (76), we find that the equations of the object's motion are

$$x = (v_0 \cos \alpha)\, t, \quad y = (v_0 \sin \alpha)\, t - gt^2/2. \tag{77}$$

A number of conclusions concerning the character of the object's motion can be drawn from Eqs. (77). For example, we can find the time of the object's flight up to the moment when the object hits Earth, the range of the flight, the maximum height that the object reaches in its flight, and the shape of the trajectory.

The first problem can be solved by finding the value of time t at which $y = 0$. The second equation in (77) implies that this happens when

$$t\left[v_0 \sin\alpha - \frac{gt}{2}\right]=0,$$

that is, either $t = 0$ or $t = (2v_0 \sin \alpha)/g$. The second value provides the solution.

The second problem can be solved by calculating the value of x at a value of t equal to the time of flight. The first equation in (77) implies that the range of the flight is given by the formula

$$\frac{(v_0 \cos\alpha)(2v_0 \sin\alpha)}{g} = \frac{v_0^2 \sin 2\alpha}{g}.$$

This implies, for one thing, that the range is greatest when $2\alpha = 90°$, or $\alpha = 45°$ In this case the range is v_0^2/g.

The solution to the third problem can be obtained immediately by formulating the maximum condition for y. But this means that at the point where y is maximal the derivative $\mathrm{d}y/\mathrm{d}t$ vanishes. Noting that

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -gt + v_0 \sin\alpha,$$

we arrive at the equation $-gt + v_0 \sin\alpha = 0$, which yields

$$t = (v_0 \sin\alpha)/y.$$

Substituting this value of t into the second equation in (77), we find that the maximum height reached by the object is $v_0^2 \sin^2 \alpha/2g$.

The solution to the fourth problem has already been found. Namely, the trajectory is represented by a parabola since Eqs. (77) represent parametrically a parabola, which in rectangular Cartesian coordinates can be written as follows:

$$y = x \tan\alpha - \frac{gx^2}{2v_0^2} \sec^2 \alpha.$$

1.17 Weightlessness

The state of weightlessness (zero g) can be achieved in various ways, although it is associated (consciously or subconsciously) with the "floating" of astronauts in the cabin of a spacecraft.

Let us consider the following problem. Suppose that a person of weight P is standing in an elevator that is moving downward with an acceleration $\omega = \alpha g$, with $0<\alpha<1$ and g the acceleration of gravity. Let us determine the pressure that the person exerts on the cabin's floor and the acceleration that the elevator must undergo so that this pressure will vanish.

Two forces act on the person in the elevator (Figure 27): the force of gravity P and the force Q that the floor exerts on the per-

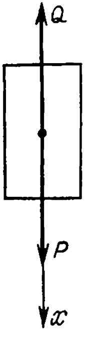

Fig. 27

son (equal numerically to the force of pressure of the person on the floor). The differential equation of the person's motion can be written in the form

$$m \frac{d^2x}{dt^2} = P - Q. \tag{78}$$

Since $d^2x/dt^2 = \omega = \alpha g$ and $m = P/g$, we can rewrite Eq. (78) as follows:

$$Q = P - m\frac{d^2x}{dt^2} = P(1 - \alpha). \tag{79}$$

Since $0 < \alpha < 1$, we conclude that $Q < P$. Thus, the pressure that the person exerts on the floor of the cabin of an elevator moving downward is determined by the force

$Q = P(1 - \alpha)$.

On the other hand, when the elevator is moving upward with an acceleration $\omega = \alpha g$, $0 < \alpha < 1$, the pressure that the person exerts on the floor of the cabin is determined by the force $Q = P(1 + \alpha)$. Let us now establish at what acceleration the pressure vanishes. For this it is sufficient to put $Q = 0$ in (79). We conclude that in this case $\alpha = 1$, that is, for Q to vanish the acceleration of the elevator must be equal to the acceleration of gravity.

Thus, when the cabin is falling freely with an acceleration equal to g, the pressure that the person exerts on the floor is nil. It is this state that is called *weightless-*

ness. In it the various parts of a person's body exert no pressure on each other, so that the person experiences extraordinary sensations. In the state of weightlessness all points of an object experience the same acceleration.

Of course, weightlessness is experienced not only during a free fall in an elevator. For illustration let us consider the following problem.

What must be the speed of a spacecraft moving around the Earth as an artificial satellite for a person inside it to be in the state of weightlessness?

One assumption in this problem is that the spacecraft follows a circular orbit of radius $r + h$, where r is Earth's radius, and h is the altitude at which the spacecraft travels (reckoned from Earth's surface). The previous problem implies that in the state of weightlessness the pressure on the walls of the spacecraft is zero, so that the force Q acting on an object inside the spacecraft is zero, too. Hence, $Q = 0$. Let us now turn our attention to Figure 28. The x axis is directed along the principal normal n to the circular trajectory of the spacecraft. We use the differential equation of the motion of a particle as projected on the principal normal:

$$\frac{mv^2}{\rho} = \sum_{k=1}^{n} F_{kn},$$

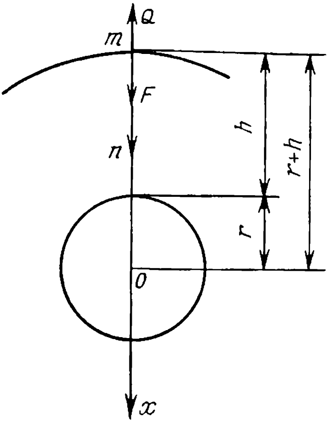

Fig. 28

where $\rho = r + h$, $\sum\limits_{k=1}^{n} F_{kn} = F$, and F is directed along the principal normal to the trajectory. We get

$$\frac{mv^2}{r+h} = F = m\frac{d^2x}{dt^2},$$

or the equation

$$\frac{d^2x}{dt^2} = \frac{v^2}{r+h}$$

Substituting this value of d^2x/dt^2 into Eq. (78), we find that

$$\frac{mv^2}{r+h} = P - Q. \qquad (80)$$

Here force P is equal to the force F of attraction to Earth, which, in accordance with Newton's law of gravitation, is inversely proportional to the square of the distance $r+h$ from the center of Earth, that is,

$$F=\frac{km}{(r+h)^2},$$

where m is the mass of the spacecraft, and constant k can be determined from the following considerations. At Earth's surface, where $h=0$, the force of gravity F is equal to mg. The above formula then yields $k=gr^2$. Hence,

$$P=F=\frac{mgr^2}{(r+h)^2},$$

where g is the acceleration of gravity at Earth's surface.

If we now substitute the obtained value of P into (80) and note that $Q=0$, we find that the required speed is given by the formula

$$v=r\sqrt{\frac{g}{r+h}}$$

1.18 Kepler's Laws of Planetary Motion

In accordance with Newton's law of gravitation, *any two objects separated by a distance r and having masses m and M are at-*

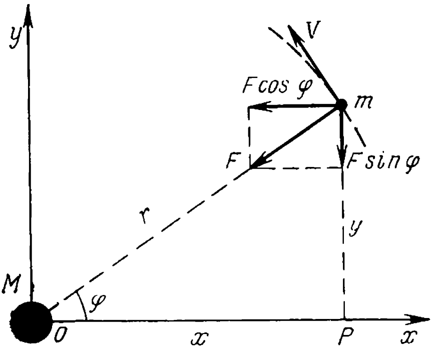

Fig. 29

tracted with a force

$$F=\frac{GmM}{r^2}, \tag{81}$$

where G is the gravitational constant.

Basing ourselves on this law, let us describe the motion of the planets in the solar system assuming that m is the mass of a planet orbiting the Sun and M is the Sun's mass. The effect of other planets on this motion will be ignored.

Let the Sun be at the origin of the coordinate system depicted in Figure 29 and the planet be at time t at a point with running coordinates x and y. The attractive force F acting on the planet can be decomposed into two components: one parallel to the x axis and equal to $F \cos \varphi$ and the other parallel to the y axis and equal to

$F \sin \varphi$. Using formula (81) and Newton's second law, we arrive at the following equations:

$$m\ddot{x} = -F \cos \varphi = -\frac{GmM}{r^2} \cos \varphi, \qquad (82)$$

$$m\ddot{y} = -F \sin \varphi = -\frac{GmM}{r^2} \sin \varphi. \qquad (83)$$

Bearing in mind that $\sin \varphi = y/r$ and $\cos \varphi = x/r$, we can rewrite Eqs. (82) and (83) as follows:

$$\ddot{x} = -\frac{kx}{r^3}, \quad \ddot{y} = -\frac{ky}{r^3},$$

where constant k is equal to GM.

Finally, allowing for the fact that $r = \sqrt{x^2 + y^2}$, we arrive at the differential equations

$$\ddot{x} = -\frac{kx}{(x^2+y^2)^{3/2}}, \quad \ddot{y} = -\frac{ky}{(x^2+y^2)^{3/2}}. \qquad (84)$$

Without loss of generality we can assume that

$$x = a,\ y = 0,\ \dot{x} = 0,\ \dot{y} = v_0 \text{ at } t = 0. \qquad (85)$$

We see that the problem has been reduced to the study of Eq. (84) under the initial conditions (85). The special features of Eqs. (84) suggest that the most convenient coordinate system here is the one using polar coordinates: $x = r \cos \varphi$ and

$y = r \sin \varphi$. Then

$$
\begin{aligned}
\dot{x} &= \dot{r} \cos \varphi - (r \sin \varphi)\, \dot{\varphi},\\
\dot{y} &= \dot{r} \sin \varphi + (r \cos \varphi)\, \dot{\varphi},\\
\ddot{x} &= \ddot{r} \cos \varphi - 2\, (\dot{r} \sin \varphi)\, \dot{\varphi} \qquad (86)\\
&\quad - (r \sin \varphi)\, \ddot{\varphi} - (r \cos \varphi)\, \dot{\varphi}^2,\\
\ddot{y} &= \ddot{r} \sin \varphi + 2\, (\dot{r} \cos \varphi)\, \dot{\varphi}\\
&\quad + (r \cos \varphi)\, \ddot{\varphi} - (r \sin \varphi)\, \dot{\varphi}^2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\ddot{x} &= (\ddot{r} - r\dot{\varphi}^2) \cos \varphi - (2\dot{r}\, \dot{\varphi} + r\, \ddot{\varphi}) \sin \varphi,\\
\ddot{y} &= (\ddot{r} - r\dot{\varphi}^2) \sin \varphi + (2\dot{r}\, \dot{\varphi} + r\, \ddot{\varphi}) \cos \varphi.
\end{aligned}
$$

Using the last two relationships, we can rewrite the differential equations (84) in the form

$$
(\ddot{r} - r\dot{\varphi}^2) \cos \varphi - (2\dot{r}\dot{\varphi} + r\, \ddot{\varphi}) \sin \varphi = -\frac{k \cos \varphi}{r^2}, \qquad (87)
$$

$$
(\ddot{r} - r\dot{\varphi}^2) \sin \varphi + (2\dot{r}\dot{\varphi} + r\, \ddot{\varphi}) \cos \varphi = -\frac{k \sin \varphi}{r^2}. \qquad (88)
$$

Multiplying both sides of Eq. (87) by $\cos \varphi$, both sides of Eq. (88) by $\sin \varphi$, and adding the products, we find that

$$\ddot{r} - r\dot{\varphi}^2 = -k/r^2. \tag{89}$$

Multiplying both sides of Eq. (87) by sin φ, both sides of Eq. (88) by cos φ, and subtracting the second product from the first, we arrive at the equation

$$2\dot{r}\dot{\varphi} + r\ddot{\varphi} = 0. \tag{90}$$

As for the initial conditions (85), in polar coordinates they assume the form

$$r = a, \ \varphi = 0, \ \dot{r} = 0, \ \dot{\varphi} = v_0/a \ \text{ at } \ t = 0. \tag{91}$$

Thus, we have reduced the problem of studying Eqs. (84) under the initial conditions (85) to that of studying Eqs. (89) and (90) under the initial conditions (91). We also note that Eq. (90) can be rewritten in the form

$$\frac{d}{dt}(r^2\dot{\varphi}) = 0. \tag{92}$$

But Eq. (92) yields

$$r^2\dot{\varphi} = C_1, \tag{93}$$

where C_1 is a constant possessing an interesting geometric interpretation. Precisely, suppose that an object moves from point P to point Q along the arc $\overset{\frown}{PQ}$ (Figure 30). Let S be the area of the sector limited by the segments OP and OQ and the

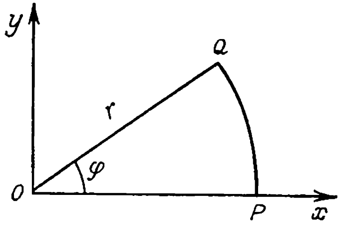

Fig. 30

arc $\widehat{PQ}$. From the calculus course we know that

$$S = \frac{1}{2} \int_0^{\varphi} r^2 \, d\varphi,$$

or $dS = (1/2)\, r^2 \, d\varphi$. Hence,

$$\frac{dS}{dt} = \frac{1}{2} r^2 \frac{d\varphi}{dt} = \frac{1}{2} r^2 \dot{\varphi}. \tag{94}$$

The derivative dS/dt constitutes what is known as the *areal velocity*, and since in view of (93) $r^2\dot{\varphi}$ is a constant, we conclude that the areal velocity is a constant, too. But this, in turn, means that the object moves in such a manner that the radius vector describes equal areas in equal time intervals. This *law of areas* constitutes one of the three Kepler laws. In full it can be formulated as follows: *each planet moves along a plane curve around the Sun in such a manner that the radius vector connecting* *the Sun with the planet describes equal areas in equal time intervals.*

To derive the next Kepler law, which deals with the shape of the planetary trajectories, we return to Eqs. (89) and (90) with the initial conditions (91) imposed on them. The initial conditions imply, for one, that $r = a$ and $\dot{\varphi} = v_0/a$ at $t = 0$. But then condition (93) implies that $C_1 = av_0$. Hence,

$$r^2\dot{\varphi} = av_0, \text{ or } \dot{\varphi} = av_0/r^2. \tag{95}$$

This transforms Eq. (89) into

$$\ddot{r} = \frac{a^2v_0^2}{r^3} - \frac{k}{r^2}.$$

Assuming that $\dot{r} = p$, we can rewrite this equation in the form

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}p}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t} = p\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{a^2v_0^2}{r^3} - \frac{k}{r^2},$$

or

$$p\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{a^2v_0^2}{r^3} - \frac{k}{r^2}$$

Separating the variables in the last differential equation and integrating, we arrive at the following relationship:

$$\frac{p^2}{2} = \frac{k}{r} - \frac{a^2v_0^2}{2r^2} + C_2.$$

Since $p = \dot{r} = 0$ at $r = a$, we find that

$$C_2 = \frac{v_0^2}{2} - \frac{k}{a}.$$

Thus, we arrive at the equation

$$\frac{\dot{r}^2}{2}=\frac{k}{r}-\frac{a^2v_0^2}{2r^2}+\frac{v_0^2}{2}-\frac{k}{a},$$

or, if we consider only the positive value of the square root,

$$\frac{dr}{dt}=\sqrt{\left(v_0-\frac{2k}{a}\right)+\frac{2k}{r}-\frac{a^2v_0^2}{r^2}} \qquad (96)$$

Dividing Eq. (96) by Eq. (95), we find that

$$\frac{dr}{d\varphi}=r\sqrt{\alpha r^2+2\beta r-1},$$

where

$$\alpha=\frac{1}{a^2}-\frac{2k}{a^3v_0^2}, \quad \beta=\frac{k}{a^2v_0^2}$$

The last equation can be integrated by substituting $1/u$ for r. The result is

$$r=\frac{a^2v_0^2/k}{1+e\cos(\varphi+C_3)},$$

where $e = \sqrt{\alpha+\beta^2}/\beta = av_0^2/k - 1$. The constant C_3 can be determined from the condition that $r = a$ at $\varphi = 0$. It is easy to verify that $C_3 = 0$. Thus, we finally have

$$r=\frac{a^2v_0^2/k}{1+e\cos\varphi} \qquad (97)$$

From analytic geometry we know that this is the equation of a conic in terms of polar coordinates, with e the eccentricity of the conic. The following cases are possible here:

(1) an ellipse if $e < 1$, or $v_0^2 < 2k/a$,
(2) a hyperbola if $e > 1$, or $v_0^2 > 2k/a$,
(3) a parabola if $e = 1$, or $v_0^2 = 2k/a$,
(4) a circle if $e = 0$, or $v_0^2 = k/a$.

Astronomical observations have shown that for all the planets belonging to the solar system the value of v_0^2 is smaller han $2k/a$. We, therefore, arrive at another of Kepler's laws: *the planets describe ellipses with the Sun at one focus.*

Note that the orbits of the Moon and the artificial satellites of Earth are also ellipses, but in the majority of cases these ellipses are close to circles, that is, e differs little from zero.

As for recurrent comets, like, say, Halley's comet, their orbits resemble "prolate" ellipses whose eccentricity is smaller than unity but very close to it. Say, Halley's comet appears in Earth's neighborhood approximately every 76 years. Its latest apparition was in the period between the end of 1985 and the beginning of 1986.

Celestial bodies that move along parabolic and hyperbolic orbits may be observed only once, since they never return to the same place.

Let us now establish the physical meaning of eccentricity e. First we note that the components $\dot{x}$ and $\dot{y}$ of a planet's velocity vector V along the x and y axes, respectively, and the size v of vector V satisfy the relationship

$$v^2 = \dot{x}^2 + \dot{y}^2,$$

which when combined with (86) yields

$$v^2 = r^2\dot{\varphi}^2 + \dot{r}^2.$$

From this it follows that the kinetic energy of a planet of mass m is given by the formula

$$\frac{1}{2} mv^2 = \frac{1}{2} m (r^2\dot{\varphi}^2 + \dot{r}^2). \tag{98}$$

The potential energy of a system is minus one times the amount of work needed to move the planet to infinity (where the potential energy is zero by convention). Hence,

$$-\int_r^\infty \frac{km}{r^2}\, dr = \frac{km}{r}\Big|_r^\infty = -\frac{km}{r} \tag{99}$$

If by E we denote the total energy of the system, which in view of energy conservation must be constant, then formulas (98) and (99) yield

$$\frac{1}{2} m [r^2\dot{\varphi}^2 + \dot{r}^2] - \frac{km}{r} = E. \tag{100}$$

Assuming that $\varphi = 0$ and combining (97) with (100), we get

$$r = \frac{a^2v_0^2/k}{1+e}, \quad \frac{mr^2a^2v_0^2}{2r^4} - \frac{km}{r} = E.$$

Excluding r from the last two relationships, we find that

$$e=\sqrt{1+E\frac{2a^2v_0^2}{mk^2}}$$

Equation (97) for the shape of the orbit finally assumes the form

$$r=\frac{a^2v_0^2/k}{1+\sqrt{1+E\,(2a^2v_0^2/mk^2)}\cos\varphi}$$

This formula implies that the orbit is an ellipse, hyperboba, parabola, or circle if, respectively, $E<0$, $E>0$, $E=0$, or $E=-mk^2/2a^2v_0^2$. Thus, the shape of a planet's orbit is completely determined by the value of E. Say, if we could impart such a "blow" to Earth that it would increase Earth's total energy to a positive value, Earth would go over to a hyperbolic orbit and leave the solar system forever.

Let us now turn to Kepler's third law. This law deals with the period of revolution of planets around the Sun. Taking into account the results obtained in deriving the previous Kepler law, we naturally restrict our discussion to the case of elliptic orbits, whose equation in terms of Cartesian coordinates, as is known, is

$$\frac{x^2}{\xi^2}+\frac{y^2}{\eta^2}=1,$$

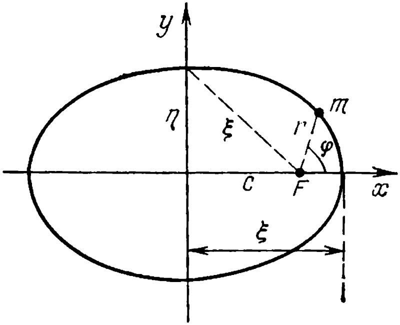

Fig. 31

where (Figure 31) the eccentricity $e = C/\xi$ with $C^2 = \xi^2 - \eta^2$, so that

$$e^2 = (\xi^2 - \eta^2)/\xi^2,$$

or

$$\eta^2 = \xi^2 (1 - e^2). \tag{101}$$

Combining this with (97) and allowing for the properties of an ellipse, we conclude that

$$\xi = \frac{1}{2}\left(\frac{a^2v_0^2/k}{1+e} + \frac{a^2v_0^2/k}{1-e}\right) = \frac{a^2v_0^2}{k(1-e^2)}$$

$$= \frac{a^2v_0^2\xi^2}{k\eta^2},$$

or

$$\eta^2 = \frac{a^2v_0^2\xi}{k}. \tag{102}$$

Let us denote the period of revolution of a planet by T. By definition, T is the time it takes the planet to complete one full orbit about the Sun. Then, since the area limited by an ellipse is $\pi\xi\eta$, we conclude, on the basis of (94) and (95), that $\pi\xi\eta = av_0T/2$. Finally, taking into account (102), we arrive at the following result:

$$T^2 = \frac{4\pi^2\xi^2\eta^2}{a^2v_0^2} = \frac{4\pi^2}{k}\xi^3.$$

This constitutes the analytical description of Kepler's third law: *the squares of the periods of revolution of the planets are proportional to the cubes of the major axes of the planets' orbits.*

1.19 Beam Deflection

Let us consider a horizontal beam AB (Figure 32) of a constant cross section and made of a homogeneous material. The symmetry axis of the beam is indicated in Figure 32 by a dashed line. Suppose that forces acting on the beam in the vertical plane containing the symmetry axis bend the beam as shown in Figure 33. These forces may be the weight of the beam itself or an external force or the two forces acting simultaneously. Clearly, the symmetry axis will also bend due to the action of these forces. Usually a bent symmetry axis is called the *elastic line*. The problem of determining the shape of this line plays an important role in elasticity theory.

Note that there can be various types of beam depending on the way in which beams are fixed or supported. For example, Figure 34 depicts a beam whose end A is rigidly fixed and end B is free. Such a beam is said to be a *cantilever*. Figure 35 depicts a beam lying freely on supports A and B. Another type of beam with supports is shown in Figure 36. There are also various ways in which loads can be applied to beams. For example, a uniformly distributed load is shown in Figure 34. Of course, the load can vary along the entire length of the beam or only a part of this length (Figure 35). Figure 36 illustrates the case of a concentrated load.

Let us consider a horizontal beam OA (Figure 37). Suppose that its symmetry axis (the dashed line in Figure 37) lies on the x axis, with the positive direction being to the right of point O, the origin. The positive direction of the y axis is downward from point O. External forces F_1, F_2, (and the weight of the beam if this is great) bend the symmetry axis, which becomes the elastic line (also depicted in Figure 38 by a curved dashed line). The deflection y of the elastic line from the x axis is known as the *sag* of the beam at point x. Thus, if we know the equation of the elastic line, we can always find the sag

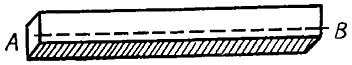

Fig. 32

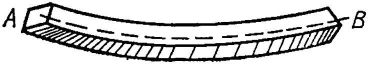

Fig. 33

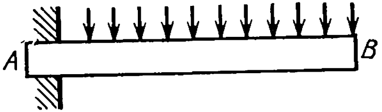

Fig. 34

of the beam. Below we show how this can be done in practical terms.

Let us denote by $M(x)$ the bending moment in the cross section of the beam at coordinate x. The bending moment is defined as the algebraic sum of the moments of the forces that act from one side of the beam at point x. In calculating the moments we assume that the forces acting on the beam upward result in negative moments while those acting downward result in positive moments.

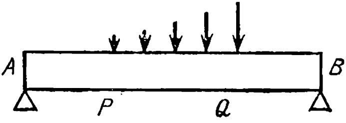

Fig. 35

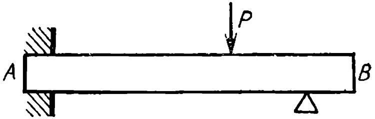

Fig. 36

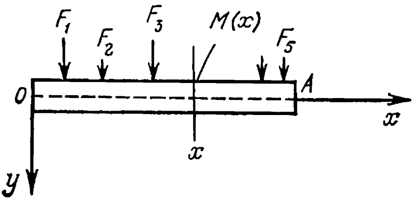

Fig. 37

The strength-of-materials course proves that the bending moment at point x is related to the curvature radius of the elastic line via the equation

$$EJ \frac{y''}{[1+(y')^2]^{3/2}} = M(x), \tag{103}$$

where E is Young's modulus, which de-

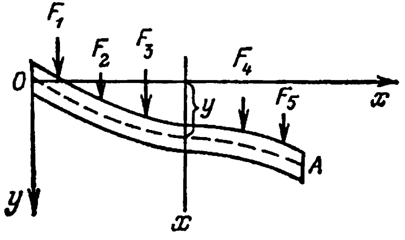

Fig. 38

pends on the type of material of the beam, J is the moment of inertia of the cross section of the beam at point x about the horizontal straight line passing through the center of mass of the cross section. The product EJ is commonly known as the *flexural rigidity*, in what follows we assume this product constant.

Now, if we suppose that the sag of the beam is small, which is usually the case in practice, the slope y' of the elastic line is extremely small and, therefore, instead of Eq. (103) we can take the approximate equation

$$EJy'' = M(x). \qquad (104)$$

To illustrate how Eq. (104) is used in practice, we consider the following problem. A horizontal homogeneous steel beam of length l lies freely on two supports and

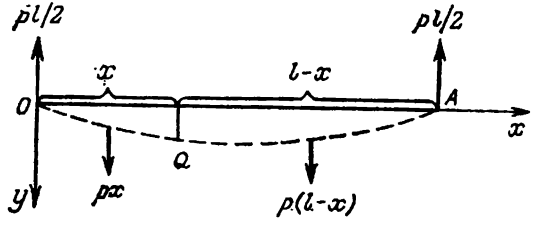

Fig. 39

sags under its own weight, which is p kgf per unit length. We wish to determine the equation of the elastic line and the maximal sag.

In Figure 39 the elastic line is depicted by the dashed curve. Since we are dealing with a two-support beam, each support acts on the beam with an upward reaction force equal to half the weight of the beam (or $pl/2$). The bending moment $M(x)$ is the algebraic sum of the moments of these forces acting on one side from point Q (Figure 39). Let us first consider the action of forces to the left of point Q. At a distance x from point Q a force equal to $pl/2$ acts on the beam upward and generates a negative moment. On the other hand, a force equal to px acts on the beam downward at a distance $x/2$ from point Q and generates a positive moment. Thus, the net bending moment at point Q is given by the formula

$$M(x) = -\frac{pl}{2}x + px\left(\frac{x}{2}\right) = \frac{px^2}{2} - \frac{plx}{2} \tag{105}$$

If we consider the forces acting to the right of point Q, we find that a force equal to $p(l - x)$ acts on the beam downward at a distance $(l - x)/2$ from point Q and generates a positive moment, while a force equal to $pl/2$ acts on the beam upward at a distance $l - x$ from point Q and generates a negative moment. The net bending moment in this case is calculated by the formula

$$M(x) = p(l - x)\frac{l - x}{2} - \frac{pl}{2}(l - x)$$

$$= \frac{px^2}{2} - \frac{plx}{2} \tag{106}$$

Formulas (105) and (106) show that the bending moments prove to be equal. Now, knowing how to find a bending moment, we can easily write the basic equation (104), which in our case assumes the form

$$EJy'' = \frac{px^2}{2} - \frac{plx}{2} \tag{107}$$

Since the beam does not bend at points O and A, we write the boundary conditions for Eq. (107) so as to find y:

$y = 0$ at $x = 0$ and $y = 0$ at $x = l$.

Integration of Eq. (107) then yields

$$y = \frac{p}{24EJ}(x^4 - 2lx^3 + l^3x). \qquad (108)$$

This constitutes the equation of the elastic line. It is used to calculate the maximal sag. For instance, in this example, basing our reasoning on symmetry considerations (the same can be done via direct calculations), we conclude that the maximal sag will occur at $x = l/2$ and is equal to $5pl^4/384EJ$, with $E = 21 \times 10^5$ kgf/cm^2 and $J = 3 \times 10^4$ cm^4.

1.20 Transportation of Logs

In transporting logs to saw mills, logging trucks move along forest roads some of the time. The width of the forest road is usually such that only one truck can travel along the road. Sections of the road are made wider so that trucks can pass each other. Ignoring the question of how traffic should be arranged that loaded and empty traffic trucks meet only at such sections, let us establish how wide the turns in the road must be and what trajectory the driver must try to follow so that, say, thirty-meter logs can be transported. It is assumed that the truck is sufficiently maneuverable to cope with a limited section of the road.

Usually a logging truck consists of a tractor unit and a trailer connected freely

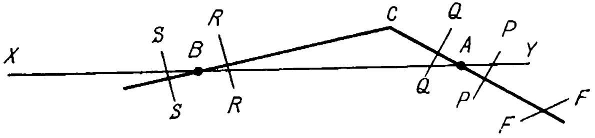

Fig. 40

to each other. The tractor unit has a front (driving) axle and two back axles above which a round platform carrying two posts and a rotating beam are fastened. This platform can rotate in the horizontal plane about a symmetrically positioned vertical axis. One end of each log is placed on this platform. The trailer has only two back axles, but also has a platform with two posts. The other ends of the logs are put on this platform. The trailer's chassis consists of two metal cylinders one of which can partly move inside the other. The chassis connects the back platform with the axis that links the trailer with the tractor unit. Thus, the length of the chassis can change during motion, which enables the tractor unit and the trailer to move independently to a certain extent. Schematically the logging truck is depicted in Figure 40. The points A and B correspond to the axes of the platforms a distance h apart. By XY we denote a log for which $AX = \lambda h$. Point C corresponds to a small axis that connects the tractor unit with the trailer, with $AC = ah$. Usually $a = 0.3$, but in the simplest case of log transportation $a = 0$. Next, FF is the front axle of the tractor unit, and PP and QQ are the back axles of the tractor unit, while RR and SS are the trailer axles. All axles have the same length $2L$, so that for the sake of simplicity we assume that the width of the logging truck is $2L$. As for the width of the load in its rear, we put it equal to $2W$. In what follows we will need the concept of the *sweep of the logging truck*, which is commonly understood to be the maximum deflection of the rear part of the logging truck (for the sake of simplicity we assume this part to be point X) from the trajectory along which the logging truck moves. Let us suppose that the road has a width of $2\beta h$ and that usually a turn in the road is simply an arc of a circle of radius h/α centered at point O (Figure 41). For the sake of simplicity we assume that a logging truck enters a turn in the road in such a way that the tractor unit and the trailer are positioned along a single straight line, the driver operates the truck in such a way that point A, corresponding to the axis of the front platform, is exactly above the road's center line. Point A in Figure 41 is determined by the angle χ that the truck AC makes with the initial direction. Here it is convenient to fix a coordinate system

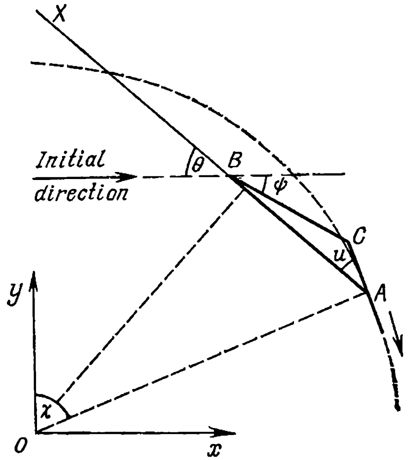

Fig. 41

Oxy in such a way that the horizontal axis points in the initial direction and the vertical axis is perpendicular to it. In a general situation the load carried by the truck will make an angle θ with the initial direction. As for angle *BAC* in Figure 41, denoting it by u we find that $u = \chi - \theta$. Usually this angle is the logging truck's *angle of lag*. The required halfwidth h of the road, which determines the sweep of the logging truck at a turn and is known as the *halfwidth* of the road at the outer curb of a turn, is defined as the algebraic sum $OX - OA + W$, while the halfwidth of the road at the inner curb of a turn is defined as the algebraic sum $OA + L - OP$, where OP is the perpendicular dropped from point O onto AB.

We stipulate that in its motion a logging truck's wheel either experiences no lateral skidding at all or the skidding is small. This requirement, for one thing, means that the center line AC of the tractor unit constitutes a tangent to the arc of a circle at point A, so that OA is perpendicular to AC, and angle χ is determined by the motion of point A along the arc of the circle. Note, further, that in building a road the curvature of a turn in the road is determined by an angle N° that corresponds to an arc length of a turn of approximately 30 m. In our notation,

$$N^\circ = \frac{180^\circ}{\pi} \frac{30\alpha}{h}, \tag{109}$$

where h is measured in meters. Thus, at $h = 9$ m and $\alpha = 0.1$ we have $N^\circ \approx 19^\circ$, while at $h = 12$ m and $\alpha = 1.0$ we have $N^\circ \approx 142^\circ$. For practical considerations we must consider only such α's that lie between 0 and 1, and the greater the value of α, the greater the maneuverability of the logging truck.

The log length λh will be greater than h, but again, reasoning on practical grounds, it must not be greater than $3h$. Thus, the value of λ varies between 1 and 3. As for constant a, it is assumed that $0 \leqslant a < 0.5$. Finally, we note that in each case the value of h is chosen differently, but it varies between 9 and 12 m.

Since the wheels of the tractor unit do not skid laterally, the coordinates of point A in Figure 41 are

$$x = \frac{h}{\alpha} \sin \chi, \quad y = \frac{h}{\alpha} \cos \chi.$$

The coordinates of point B are

$$X = \frac{h}{\alpha} \sin \chi - h \cos \theta,$$

$$Y = \frac{h}{\alpha} \cos \chi + h \sin \theta. \tag{110}$$

Since the trailer's wheels do not skid either, point B moves in the direction BC and

$$\frac{dY}{dX} = -\tan \psi, \tag{111}$$

where ψ is the angle which BC makes with the initial direction. Next, employing the fact that $\chi = u + \theta$ and studying triangle ABC, we arrive at the following chain of equalities:

$$\frac{\sin(\chi - \psi)}{h} = \frac{\sin(\theta - \psi)}{ah} = \frac{\sin u}{bh}, \tag{112}$$

where $0 < b < 1$, and ψ, θ, and u are functions of χ. If we combine (111) with (110), we find that

$$\left(-\frac{h}{\alpha}\sin\chi+h\,\frac{d\theta}{d\chi}\cos\theta\right)\cos\psi$$
$$+\left(\frac{h}{\alpha}\cos\chi+h\frac{d\theta}{d\chi}\sin\theta\right)\sin\psi=0.$$

Carrying out the necessary calculations, we arrive at

$$\sin(\chi-\psi)=\alpha\,\frac{d\theta}{d\chi}\cos(\theta-\psi). \qquad (113)$$

If we exclude variable ψ from this relationship for a fixed value of a by employing (112) and the fact that $\chi = u + \theta$, we arrive (since $\theta = 0$ at $\chi = 0$) at the differential equation

$$\frac{du}{d\chi}=1-\frac{\sin u}{\alpha\,(1-a\cos u)} \qquad (114)$$

with the initial condition $u(0) = 0$, where the angle of lag plays the role of the sought function.*

By substituting v for $\tan(u/2)$ in the differential equation (114), we can integrate the equation in closed form. However, the resulting u vs. χ dependence proves to be extremely complicated, which, of course, hinders an effective study. Nevertheless, Eq. (114) can easily be integrated and studied numerically. To this end we can

* See A.B. Tayler, "The sweep of a logging truck", *Math. Spectrum* 7, No. 1: 19-26 (1974/75).

employ, for example, the Runge-Kutta second-order numerical method the essence of which is the following.

Suppose that an integral curve $u = \varphi(\chi)$ of a differential equation $du/d\chi = f(\chi, u)$ passes through a point (χ_0, u_0). At equidistant points $\chi_0, \chi_1, \chi_2, \ldots, (\chi_{i+1} - \chi_i = \Delta\chi > 0)$ we select values $u_0, u_1, u_2, \ldots$ such that $u_i \approx \varphi_i(\chi)$, where the successive values $u_1, u_2, \ldots$ are specified by the formulas

$$u_{i+1} = u_i + (k_1 + k_2)/2,$$

with

$$k_1 = f(\chi_1, u_i)\,\Delta\chi,$$

$$k_2 = f(\chi_i + \Delta\chi, u_i + k_1)\,\Delta\chi.$$

To solve Eq. (114) numerically with the initial condition $u(0) = 0$, we compile the following program using BASIC:

```
10  REM Runge-Kutta second-order method
20  DEF FNF (X, U) =
     1 — SIN(U)/(AL*(1 — A*COS(U)))
30  CLS: PRINT "Solution of differential equa-
    tion"
40  PRINT "Runge-Kutta second-order method"
100 PRINT "Parameters:"
110 INPUT "Alpha=", AL
120 INPUT "a=", A
130 INPUT "Step of independent variable=", DX
140 PRINT "Initial values:"
150 INPUT "of independent variable=", X
```

```
160 INPUT "of function=", U
200 REM Next 20 values of function
210 CLS: PRINT "X", "U"
220 FOR I = 1 TO 20
230 PRINT X, U
240 K1 = DX*FNF (X,U)
250 X = X + DX
260 K2 = DX*FNF(X, U + K1)
270 U = U + (K1 + K2)/2
280 NEXT I
290 INPUT "Continue (Yes = 1, No = 0)"; I
300 IF I < > 0 GOTO 210
310 END
```

Note that to compile the table of values of the solution for concrete values of α and a, we must know the limiting value C of this solution. The number C can be found from the condition that

$$\alpha (1 - a \cos C) = \sin C,$$

which leads to the formula

$$C = \sin^{-1} \left(\frac{\alpha [1 - a \sqrt{1 - \alpha^2 + a^2\alpha^2}]}{1 + a^2\alpha^2} \right).$$

Here, if $\alpha = 1$, we have $C = \pi/2 - 2 \tan^{-1} a$, but if $\alpha \ll 1$, then $C \approx \alpha (1 - a)$. The limiting value C can be approximated by an exponential, namely

$$C - u \approx e^{-\gamma x},$$

where $\gamma = \alpha (\cos C - a)/\sin^2 C$. For small α's the value of γ is fairly great and is approximately given by the relationship

$$\gamma \approx \frac{1}{\alpha(1-a)}$$

But if $\alpha = 1$, we have

$$\gamma = \frac{a(1+a^2)}{1-a^2}.$$

For the step $\Delta\chi$ in the variation of the independent variable χ we must take a number that does not exceed C. This requirement is especially important for small α's and in the neighborhood of $\chi = 0$. Below we give the protocol for calculating the values of the function $u = u(\chi)$.

Solution of differential equation by
Runge-Kutta second-order method

Parameters:
alpha = 1.0
a = 0.3
Step of independent variable = 0.5
Initial values:
of independent variable = 0
of function = 0

χ	u
0	0
.2	.1437181
.4	.2299778
.6	.2824244
.8	.3146894
1	.3347111

1.2	.3472088
1.4	.3550403
1.6	.35996
1.8	.3630556
2	.3650054
2.2	.3662343
2.4	.3670091
2.600001	.3674978
2.800001	.367806
3.000001	.3680005
3.200001	.3681232
3.400001	.3682006
3.600001	.3682494
3.800001	.3682802

Continue (Yes = 1, No = 0)? 1

χ	u
4.000001	.3682997
4.200001	.368312
4.4	.3683197
4.6	.3683246
4.8	.3683276
5	.3683296
5.2	.3683308
5.399999	.3683316
5.599999	.3683321
5.799999	.3683324
5.999999	.3683326
6.199999	.3683327
6.399998	.3683328
6.599998	.3683328
6.799998	.3683329
6.999998	.3683329

7.199998	.3683329
7.399997	.3683329
7.599997	.3683329
7.799997	.3683329

Continue (Yes = 1, No = 0)? 1

χ	u
7.999997	.3683329
8.199997	.3683329
8.399997	.3683329
8.599997	.3683329
8.799996	.3683329
8.999996	.3683329
9.199996	.3683329
9.399996	.3683329
9.599996	.3683329
9.799996	.3683329
9.999995	.3683329
10.2	.3683329
10.4	.3683329
10.6	.3683329
10.8	.3683329
10.99999	.3683329
11.19999	.3683329
11.39999	.3683329
11.59999	.3683329
11.79999	.3683329

Continue (Yes = 1, No = 0)? 0
OK

The results of a numerical calculation at $a = 0.3$ are presented graphically in Figure 42. They show how an increase in χ

influences the angle of lag of the logging truck. For clarity of exposition the scales on the u and χ axes are chosen to be different.

Now let us determine the sweep of the logging truck by using the concept of halfwidth of the road at the outer curb of a turn (this quantity was defined earlier as the algebraic sum $OX - OA + W$ (Figure 41)). First we note that

$$OX^2 = \left(\frac{h}{\alpha}\sin\chi - \lambda h\cos\theta\right)^2$$

$$+\left(\frac{h}{\alpha}\cos\chi + \lambda n\sin\theta\right)^2$$

$$= h^2\left(\frac{1}{\alpha^2} + \lambda^2 - 2\frac{\lambda}{\alpha}\sin u\right).$$

This shows that the sweep decreases as χ grows since the angle of lag, u, increases. Thus, the maximal halfwidth βh of the road can be determined from the following formula for β:

$$\beta = \sqrt{\lambda^2 + \frac{1}{\alpha^2}} - \frac{1}{\alpha} + \frac{W}{h}.$$

In Figure 43 the solid curves reflect the relationship that exists between $\beta - W/h$ and α for different values of λ, while the dashed curves show what must be the value of $\beta - L/h$ to guarantee the necessary "margin" at the inner curb of the turn. For

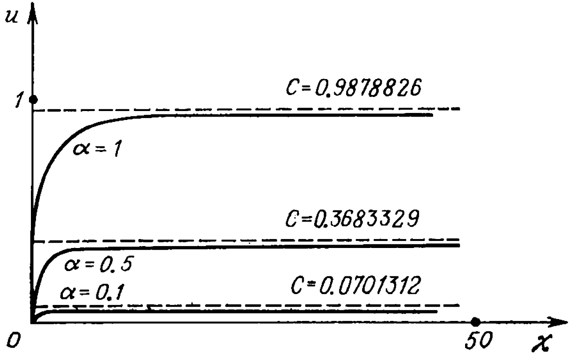
u
1
C=0.9878826
α = 1
C=0.3683329
α = 0.5
α = 0.1
C=0.0701312
0
50
x

Fig. 42

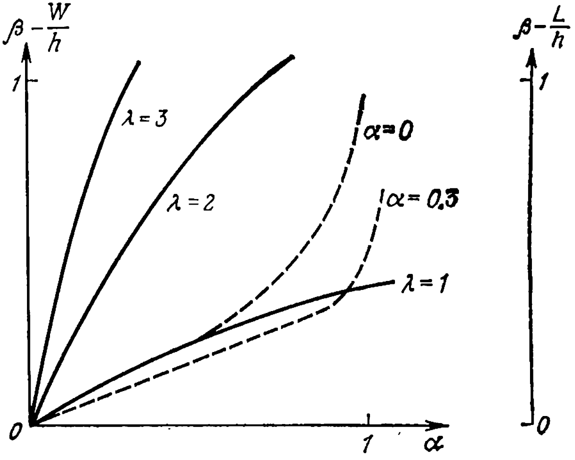
β - W/h
1
λ = 3
λ = 2
α=0
α= 0.3
λ = 1
0
1
α
β - L/h
1
0

Fig. 43

every position of the logging truck on the road we must have

$$\beta = \frac{1}{\alpha}(1-\cos u) + \frac{L}{h}$$
$$< \frac{1}{\alpha}(1-\cos C) + \frac{L}{h} \qquad (115)$$

Next, since the value of C decreases with a, the angle of lag u proves to be the greatest when $a = 0$, which corresponds to the simplest case of logging. Then (115) yields

$$\beta - \frac{L}{h} < \frac{1}{\alpha}(1-\cos C)\Big|_{a=0} = \frac{1-\sqrt{1-\alpha^2}}{\alpha}.$$

Now let us dwell on the results that follow from the above line of reasoning. Here is a typical example that illustrates these results.

If a logging truck in which the distance between the front and back platforms is 12 m follows a turn along an arc of a circle of radius 60 m, then $\alpha = 0.2$ and, in accordance with (109) $N°$ is approximately 28°. If the width of the logging truck is 2.4 m and width of the load in the rear is 1.2 m, then for logs of approximately 24 m in length (reckoning from the axis of the front platform) we have $\lambda = 2$, $W/h = 0.05$, and $L/h = 0.1$. Thus, from Figure 43 it follows that for all values of a we have $\beta = 0.45$ for the outer curb of the turn and $\beta = 0.2$ for the inner curb. Theoretically the necessary halfwidth of the road at the outer curb of the turn is equal to 5.4 m and that at the inner curb to 2.4 m. If the logging truck transports logs whose length is 14.4 m (reckoned from the axis of the front platform) and whose bunch width in the rear is 1.8 m, then in the case at hand $\lambda = 1.2$. The value of β then proves to be the same for the inner and outer curbs of the turn and equal to 0.22. Hence, as can easily be seen, the necessary halfwidth of the road at the outer curb of the turn is 2.64 m, while the same quantity at the inner curb, as in the previous case, is equal to 2.4 m. This reasoning shows that the longer the logs transported the wider the road at a turn must be. For one, if we compare the two cases considered here, we see that an increase in the length of the logs by 9.6 m requires widening the road at a turn by 2.76 m so that the driver can drive the truck along a curve whose length at the turn is approximately equal to the length of the road's center line. Practice has shown that an inexperienced driver is not able to drive his truck along such a curve and needs a road whose total width at a turn is at least 10.8 m (if the load transported is 24 m long) for a truck width of 2.4 m.

The theory developed above shows that the sweep of the logging truck is the greatest when the truck enters the turn, since in this case the angle of lag grows. This conclusion also holds true in the situation when the truck enters one section of a zigzag turn after completing the previous one at the point of inflection. The results represented graphically in Figure 43 correspond to the case where prior to entering a turn the tractor unit and the trailer are positioned on a single straight line. But if there exists a nonzero initial angle of lag C_0 caused by the zigzag nature of the turn, we must select the initial condition in the differential equation (114) in the form $u(0) = -C_0$. Then the required width of the road is determined from the following formula for β:

$$\beta = \sqrt{\lambda^2 + \frac{1}{\alpha^2} + 2\frac{\lambda}{\alpha}\sin C_0} - \frac{1}{\alpha} + \frac{W}{h}$$

We note that in the case of simple log transportation, that is, $a=0$, it is impossible to pass a turn with α greater than unity. However, for fairly large values of a the values $\alpha > 1$ become possible, they must obey the following relationship: $\alpha(1 - a\cos C) = \sin C$. Thus, the maximal value of α is $(1 - a^2)^{-1/2}$, with the practical extremal value of α being 1.25 at $a = 0.5$.

We note in conclusion that for $\alpha > 0.5$ considerable economy in the width of a road is achieved by increasing a (see Figure 43). If the load is such that λ is not much greater than unity, the value of α is chosen such that the required halfwidth of the road at the inner curb of any turn is always smaller than at the outer curb.

Chapter 2
Qualitative Methods of Studying Differential Models

In solving the problems discussed in Chapter 1 we constructed differential models and then sought answers by integrating the resulting differential equations. However, as noted in the Preface, the overwhelming majority of differential equations are not integrable in closed (analytical) form. Hence, to study differential models of real phenomena and processes we need methods that will enable us to extract the necessary information from the properties of the differential equation proper. Below with concrete examples we show how in solving practical problems one can use the simplest approaches and methods of the qualitative theory of ordinary differential equations.

2.1 Curves of Constant Direction of Magnetic Needle

Let us see how in qualitative integration, that is, the process of establishing the general nature of solutions to ordinary differential equations, one can use a general property of such equations whose analogue, for example, is the property of the magnetic field existing at Earth's surface. The reader will recall that curves at Earth's surface can be specified along which the direction of a magnetic needle is constant.

Thus, let us consider the first-order ordinary differential equation

$$\frac{dy}{dx}=f(x, y), \tag{116}$$

where the function $f(x, y)$ is assumed single-valued and continuous over the set of variables x and y within a certain domain D of the (x, y)-plane. To each point $M(x, y)$ belonging to the domain D of function $f(x, y)$ the differential equation assigns a value of dy/dx, the slope K of the tangent to the integral curve at point $M(x, y)$. Bearing this in mind, we say that at each point $M(x, y)$ of D the differential equation (116) defines a direction or a line element. The collection of all line elements in D is said to be the *field of directions* or the *line element field.* Graphically a linear element is depicted by a segment for which point $M(x, y)$ is an interior point and which makes an angle θ with the positive direction of the x axis such that $K = \tan\theta = f(x, y)$. From this it follows that geometrically the differential equation (116) expresses the fact that *the direction of a tangent at every point of an integral curve coincides with the direction of the field at this point.*

To construct the field of directions it has proved expedient to use the concept of isoclinals (derived from the Greek words for "equal" and "sloping"), that is, the set of points in the (x, y)-plane at which the direction of the field specified by the differential equation (116) is the same.

The isoclinals of the magnetic field at Earth's surface are curves at each point of which a magnetic needle points in the same direction. As for the differential equation (116), its isoclinals are given by the equation

$$f(x, y) = \nu,$$

where ν is a varying real parameter.

Knowing the isoclinals, we can approximately establish the behavior of the integral curves of a given differential equation. Let us consider, for example, the differential equation

$$\frac{dy}{dx} = x^2 + y^2,$$

which cannot be integrated in closed (analytical) form. The form of this equation suggests that the family of isoclinals is given by the equation

$$x^2 + y^2 = \nu, \quad \nu > 0,$$

that is, the isoclinals are concentric circles of radius $\sqrt{\nu}$ centered at the origin and lying in the (x, y)-plane. At each point of

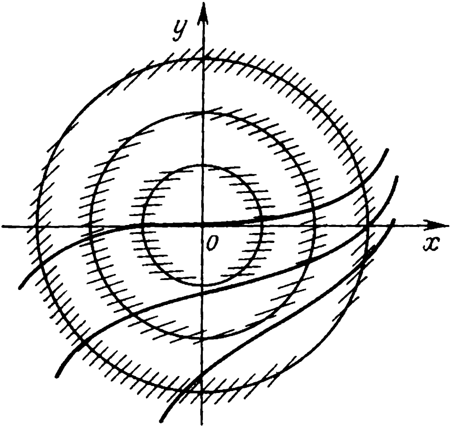

Fig. 44

such an isoclinal the slope of the tangent to the integral curve that passes through this point is equal to the squared radius of the corresponding circle. This information alone is sufficient to convey an idea of the behavior of the integral curves of the given differential equation (Figure 44).

We arrived at the final result quickly because the example was fairly simple. However, even with more complicated equations knowing the isoclinals may prove to be expedient in solving a specific problem.

Let us consider a geometric method of integration of differential equations of the type (116). The method is based on using the geometric properties of the curves given by the equations

$$f(x, y) = 0, \tag{I_0}$$

$$\frac{\partial f(x, y)}{\partial x} + \frac{\partial f(x, y)}{\partial y} f(x, y) = 0. \tag{L}$$

Equation (I_0), that is, the "zero" isoclinal equation, specifies curves at whose points $dy/dx = 0$. This means that the points of these curves may prove to be points of maxima or minima for the integral curves of the initial differential equation. This explains why out of the entire set of isoclinals we isolate the "zero" isoclinal.

For greater precision in constructing integral curves it is common to find the set of inflection points of these curves (provided that such points exist). As is known, points of inflection should be sought among the points at which y'' vanishes. Employing Eq. (116), we find that

$$y'' = \frac{\partial f(x, y)}{\partial x} + \frac{\partial f(x, y)}{\partial y} y'$$

$$= \frac{\partial f(x, y)}{\partial x} + \frac{\partial f(x, y)}{\partial y} f(x, y).$$

The curves specified by Eq. (L) are the possible point-of-inflection curves.* Note, for

* It is assumed here that integral curves that fill a certain domain possess the property that only one integral curve passes through each point of the domain.

one, that *a point of inflection of an integral curve is a point at which the integral curve touches an isoclinal.*

The curves consisting of extremum points (maxima and minima points) and points of inflection of integral curves break down the domain of f into such subdomains S_1, S_2, ..., S_m in which the first and second derivatives of the solution to the differential equation have definite signs. In each specific case these subdomains should be found. This enables giving a rough picture of the behavior of integral curves.

As an example let us consider the differential equation $y' = x + y$. The equation of curve (I_0) in this case has the form $x + y = 0$, or $y = -x$. A direct check verifies that curve (I_0) is not an integral curve. As for curve (L), whose equation in this case is $y + x + 1 = 0$, we find that it is an integral curve and, hence, is not a point-of-inflection curve.

The straight lines (I_0) and (L) break down the (x, y)-plane into three subdomains (Figure 45): S_1 $(y' > 0,\ y'' > 0)$, to the right of the straight line (I_0), S_2 $(y' < 0,\ y'' > 0)$, between the straight lines (I_0) and (L), and S_3 $(y' < 0,\ y'' < 0)$, to the left of the straight line (L). The points of minima of the integral curves lie on the straight line (I_0). To the right of (I_0) the integral curves point upward, while to the left they point downward (left to right in Figure 45). There are

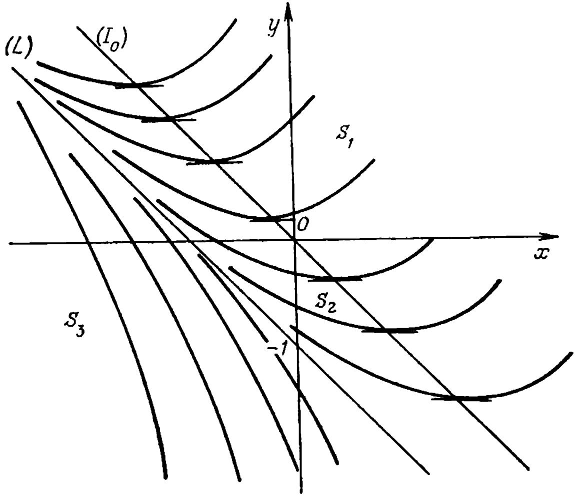

Fig. 45

no points of inflection. To the right of the straight line (L) the curves are convex downward and to the left convex upward. The behavior of the integral curves on the whole is shown in Figure 45.

Note that in the given case the integral curve (L) is a kind of "dividing" line, since it separates one family of integral curves from another. Such a curve is commonly known as a separatrix.

2.2 Why Must an Engineer Know Existence and Uniqueness Theorems?

When speaking of isoclinals and point-of-inflection curves in Section 2.1, we tacitly assumed that the differential equation in question had a solution. The problem of when a solution exists and of when it is unique is solved by the so-called existence and uniqueness theorems. These theorems are important for both theory and practice.

Existence and uniqueness theorems are highly important because they guarantee the legitimacy of using the qualitative methods of the theory of differential equations to solve problems that emerge in science and engineering. They serve as a basis for creating new methods and theories. Often their proof is constructive, that is, the methods by which the theorems are proved suggest methods of finding approximate solutions with any degree of accuracy. Thus, existence and uniqueness theorems lie at the base of not only the above-noted qualitative theory of differential equations but also the methods of numerical integration.

Many methods of numerical solution of differential equations have been developed, and although they have the common drawback that each provides only a concrete solution, which narrows their practical potential, they are widely used. It must be noted, however, that before numerically integrating a differential equation one must always turn to existence and uniqueness theorems. This is essential to avoid misunderstandings and incorrect conclusions.

To illustrate what has been said, let us take two simple examples,* but first let us formulate one variant of existence and uniqueness theorems.

Existence theorem *If in Eq.* (116) *function f is defined and continuous on a bounded domain D in the* (x, y)*-plane, then for every point* $(x_0, y_0) \in D$ *there exists a solution* $y(x)$ *to the initial-value problem* **

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0, \tag{117}$$

that is defined on a certain interval containing point x_0.

Existence and uniqueness theorem *If in Eq.* (116) *function f is defined and continuous on a bounded domain D in the* (x, y)-

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* See C.E. Roberts, Jr., "Why teach existence and uniqueness theorems in the first course of ordinary differential equations?", *Int. J. Math. Educ. Sci. Technol.* 7, No. 1: 41-44 (1976).

** If we wish to find a solution of a differential equation satisfying a certain *initial condition* (in our case the initial condition is $y(x_0) = y_0$), such a problem is said to be an *initial-value problem.*

plane and satisfies in D the Lipschitz condition in variable y, that is,

$$| f(x, y_2) - f(x, y_1) | < L | y_2 - y_1 |,$$

with L a positive constant, then for every point $(x_0, y_0) \in D$ *there exists a unique solution* $y(x)$ *to the initial-value problem* (117) *defined on a certain interval containing point* x_0.

Extension theorem *If the hypotheses of the existence theorem or the existence and uniqueness theorem are satisfied, then every solution to Eq.* (116) *with the initial data* $(x_0, y_0) \in D$ *can be extended to a point that lies as close to the boundary of D as desired. In the first case the extension is not necessarily unique while in the second it is.*

Let us consider the following problem. Using the numerical Euler integration method with the iteration scheme $y_{i+1} = y_i + hf(x_i, y_i)$ and step $h = 0.1$, solve the initial-value problem

$$y' = -x/y, \quad y(-1) = 0.21 \qquad (118)$$

on the interval $[-1, 3]$.

Note that the problem involving the equation $y' = -x/y$ emerges, for example, from the problem considered on p. 162 concerned with a conservative system consisting of an object oscillating horizontally in a vacuum under forces exerted by linear springs.

To numerically integrate the initial-value problem (117) on the [—1, 2.8] interval we compile a program for building the graph of the solution:

```
10   REM Numerical integration of differential
     equation
20   DEF FNF(X, Y) =
30   GOSUB 1110: REM Coordinate axes
40   REM Next value of function
1000 LINE —(FNX(X), FNY(Y)), 2
1010 IF X < 2.9 GOTO 100
1020 LOCATE 23, 1
1030 END
1100 REM Construction of coordinate axes
1110 SCREEN 1, 1, 0: KEY OFF: CLS
1120 DEF FNX(X) = 88 + 80*X
1130 DEF FNY(Y) = 96 — 80*Y
1140 REM Legends on coordinate axes
1150 LOCATE 1, 11, 0  PRINT "Y"
1160 LOCATE 3, 11: PRINT "1"
1170 LOCATE 13, 39: PRINT "X"
1180 LOCATE 23, 10: PRINT "—1"
1190 FOR I = —1 TO 2: LOCATE 13, 10*I +
     11: PRINT USING "# #"; I;: NEXT I
1200 REM Oy
1210 DRAW "BM88, ONM — 2, +8NM + 2,
     +8D16"
1220 FOR I = 1 TO 11: DRAW "NR2D16": NEXT
1230 REM Ox
```

```
1240 DRAW "BM319, 96NM — 7, —2NM — 7
     +2L23"
1250 FOR I = 1 TO 19: DRAW "NU2L16": NEXT
1260 REM Input of initial data
1270 LOCATE 25, 1: PRINT STRING$(40,    "),
1280 LOCATE 25, 1: INPUT, "xO =", X: INPUT;
     "dx =", DX: INPUT; "yO =", Y
1290 PSET (FNX(X), FNY(Y)), 2
1300 RETURN
```

Here in line 20 the right-hand side of the differential equation is specified, and lines 50 to 990 must hold the program of the numerical integration of the differential equation.

In our case (the initial-value problem (118)) the beginning of the program has the following form:

```
10  REM Euler's method
20  DEF FNF(X, Y) = —X/Y
30  GOSUB 1100: REM Coordinate axes
40  REM Next value of function
100 Y = Y + DX*FNF(X, Y)
110 X = X + DX
```

The results are presented graphically in Figure 46.

Let us now turn to the existence theorem. For the initial-value problem (118), the function $f(x, y) = -x/y$ is defined and continuous in the entire (x, y)-plane exclud-

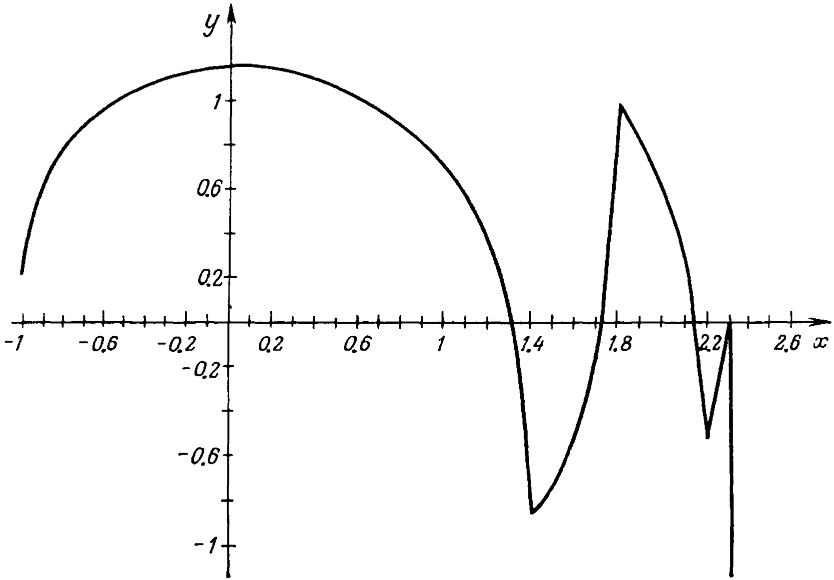

Fig. 46

ing the x axis. Thus, in accordance with the existence theorem, the initial-value problem (118) has a solution $y(x)$ defined on a certain interval containing point $x_0 = -1$. This solution, according to the extension theorem, can be extended to a value of $y(x)$ close to the value $y(x) = 0$. As a result of numerical integration we have arrived at a solution of (118) defined on an interval (a, b), with $a < -1$ and $1.3 < b < 1.4$. However, allowing for the concrete form of the differential equation, we can specify the true interval in which the solution to the initial-value problem (118) exists. Indeed, since in the initial equation the variables can be separated, we have

$$\int_{0.21}^{y} \eta \, d\eta = -\int_{-1}^{x} \xi \, d\xi.$$

Integrating, we get $y = \sqrt{1.0441 - x^2}$. Hence, a solution to the initial-value problem (118) exists only for $|x| < \sqrt{1.0441} \approx 1.0218$.

Thus, by resorting to the existence theorem (and to the extension theorem) we were able to "cut off" the segment on which there is certain to be no solution of the initial-value problem. If we employ only numerical integration, we arrive at an erroneous result. The fact is that as the solution $y = y(x)$ approaches the x axis, the angle of slope of the curve tends to 90°. Therefore, in the time that the independent variable x changes by 0.1, the value of y is able to "jump over" the x axis, and we find ourselves on an integral curve that differs from the original. This happens because the Euler method allows for the angle of slope only at the running point.

The following example is even more instructive. We wish to solve the initial-value problem

$$y' = 3x\sqrt[3]{y}, \quad y(-1) = -1 \tag{119}$$

on the segment [— 1, 1]. The approach here consists in first employing the Euler method and then an improved Euler method with a step $h = 0.1$ and an iteration scheme $y_{i+1} = y_i + hf(x_{i+1/2}, y_{i+1/2})$, with $y_{i+1/2} = y_i + hf(x_i, y_i)/2$.

We solve the initial-value problem (119) using the above program, whose beginning in the case at hand has the following form:

```
10 REM Euler's method
20 DEF FNF(X, Y) = 3*X*SGN(Y)*ABS(Y) ^
      (1/3)
30 GOSUB 1100: REM Coordinate axes
40 REM Next value of function
100 Y = Y + DX*FNF(X, Y)
110 X = X + DX
```

The results are presented graphically in Figure 47.

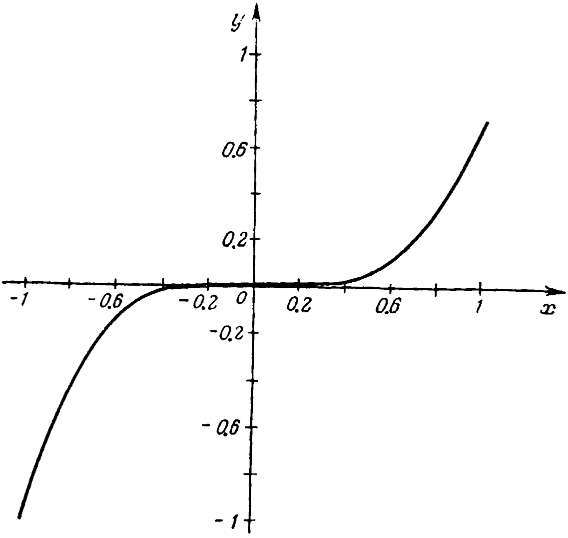

Fig. 47

As for the improved Euler method, the beginning of the program has the following form:

```
10 REM Improved Euler's method
20 DEF FNF(X, Y) = 3*X*SGN(Y)*ABS(Y) ^
   (1/3)
30 GOSUB 1100: REM Coordinate axes
40 REM Next value of function
50 D2 = DX/2
```

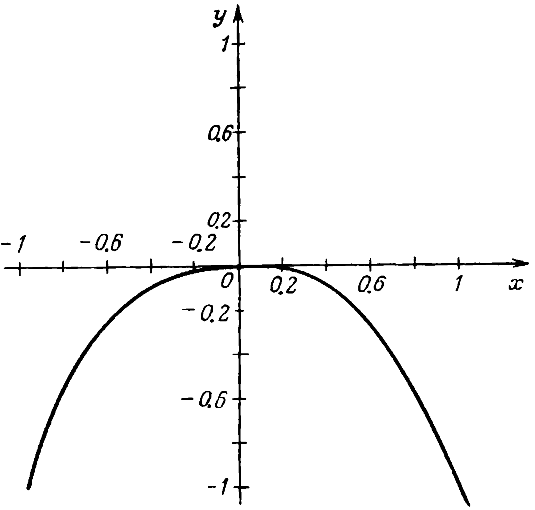

Fig. 48

```
100 Y1 = Y + D2*FNF(X, Y)
110 Y = Y + DX*FNF(X + D2, Y1)
120 X = X + DX
```

The results are presented graphically in Figure 48, and the diagram differs from that depicted in Figure 47.

To look into the reason for such striking discrepancy in the results, we integrate the initial-value problem. Separating the variables, we get

$$\int_{-1}^{y} \eta^{-1/3}\, d\eta = 3 \int_{-1}^{x} \xi\, d\xi,$$

or, finally, $y = \pm x^3$. This result already suggests that the solution via the Euler method gives the function $y_1(x) = x^3$ while the solution via the improved Euler method gives

$$y_2(x) = \begin{cases} x^3 & \text{if} \quad x \leqslant 0, \\ -x^3 & \text{if} \quad x > 0. \end{cases}$$

Both y_1 and y_2 are solutions to the initial-value problem (119), which means that the solution of the initial-value problem considered on the segment $[-1, 1]$ is not unique.

Let us now turn to the existence and uniqueness theorem in connection with this problem. First, we note that since the function $f(x, y) = 3x\sqrt[3]{y}$ is continuous in the entire (x, y)-plane, the existence theorem implies that the initial-value problem (119) has a solution defined on a segment containing point $x_0 = -1$, and, according to the extension theorem, this solution can be extended to any segment. Further, since $\partial f(x, y)/\partial y = xy^{-2/3}$, the function $f(x, y) =$

$3x\sqrt[3]{y}$ satisfies the Lipschitz condition in variable y in any domain not containing the x axis. If, however, a domain does contain points belonging to the x axis, it is easy to show that the function does not satisfy the Lipschitz condition. Hence, from the existence and uniqueness theorem (and the extension theorem) it follows that in this case the solution to the initial-value problem can be extended in a unique manner at least to the x axis. But since the straight line $y = 0$ constitutes a singular integral curve of the differential equation $y' = 3x\sqrt[3]{y}$, we already know that as soon as y vanishes, there is no way in which we can extend the solution to the initial-value problem (119) beyond point O (0, 0) in a unique manner.

Thus, by resorting to the existence and uniqueness theorem (and the extension theorem) we were able to understand the results of numerical integration, that is, if we are speaking of the uniqueness of the solution of the initial-value problem (119) on the $[-1, 1]$ segment, the solution exists and is defined only on the segment $[-1, 0]$. Generally, however, there can be several such solutions.

2.3 A Dynamical Interpretation of Second-Order Differential Equations

Let us consider the nonlinear differential equation

$$\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}\right) \tag{120}$$

whose particular case is the second-order differential equation obtained on p. 72 when we considered pendulum clocks. We take a simple dynamical system consisting of a particle of unit mass that moves along the x axis (Figure 49) and on which a force $f(x, dx/dt)$ acts. Then the differential equation (120) is the equation of motion of the particle. The values of x and dx/dt at each moment in time characterize the state of the system and correspond to a point in the $(x, dx/dt)$-plane (Figure 50), which is known as the *plane of states* or the *phase* $(x, dx/dt)$*-plane*. The phase plane depicts the set of all possible states of the dynamical system considered. Each new state of the system corresponds to a new point in the phase plane. Thus, the changes in the state of the system can be represented by the motion of a certain point in the phase plane. This point is called a *representative point*, the trajectory of the representative point is known as the *phase trajectory*,

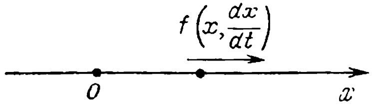

Fig. 49

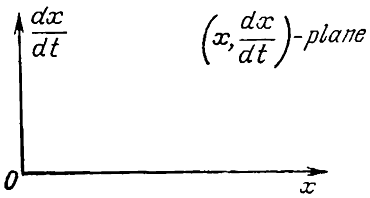

Fig. 50

and the rate of motion of this point as the *phase velocity*.

If we introduce the variable $y = \mathrm{d}x/\mathrm{d}t$, Eq. (120) can be reduced to a system of two differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = f(x, y). \tag{121}$$

If we take t as a parameter, then the solution to system (121) consists of two functions, $x(t)$ and $y(t)$, that in the phase (x, y)-plane define a curve (a *phase trajectory*).

It can be shown that system (121) and even a more general system

$$\frac{dx}{dt}=X(x, y), \quad \frac{dy}{dt}=Y(x, y), \qquad (122)$$

where the functions X and Y and their partial derivatives are continuous in a domain D, posseses the property that if $x(t)$ and $y(t)$ constitute a solution to the differential system, we can write

$$x = x(t + C), \ y = y(t + C), \qquad (123)$$

where C is an arbitrary real constant, and (123) also constitute a solution to the same differential system. All solutions (123) with different values of C correspond to a single phase trajectory in the phase (x, y)-plane. Further, if two phase trajectories have at least one common point, they coincide. Here the increase or decrease in parameter t corresponds to a certain direction of motion of the representative point along the trajectory. In other words, a phase trajectory is a directed, or oriented, curve. When we are interested in the direction of the curve, we depict the direction of the representative point along the trajectory by placing a small arrow on the curve.

Systems of the (122) type belong to the class of *autonomous systems* of differential equations, that is, systems of ordinary differential equations whose right-hand sides do not explicitly depend on time t. But if at least in one of the equations of the system the right-hand side depends explicitly on time t, then such a system is said to be *nonautonomous.*

In connection with this classification of differential equations the following remark is in order. If a solution $x(t)$ to Eq. (120) is a nonconstant periodic solution, then the phase trajectory of the representative point in the phase (x, y)-plane is a simple closed curve, that is, a closed curve without self-intersections. The converse is also true.

If differential systems of the (122) type are specified in the entire (x, y)-plane, then, generally speaking, phase trajectories will completely cover the phase plane without intersections. And if it so happens that

$$X(x_0, y_0) = Y(x_0, y_0) = 0$$

at a point $M_0(x_0, y_0)$, the trajectory degenerates into a point. Such points are called *singular.* In what follows we consider primarily only isolated singular points. A singular point $M_0(x_0, y_0)$ is said to be *isolated* if there exists a neighborhood of this point which contains no other singular points except $M_0(x_0, y_0)$.

From the viewpoint of a physical interpretation of Eq. (120), the point $M_0(x_0, 0)$ is a singular point. At this point $y = 0$ and $f(x_0, 0) = 0$. Thus, in this case the isolated singular point corresponds to the state of a particle of unit mass in which both the speed $\mathrm{d}x/\mathrm{d}t$ and the acceleration $\mathrm{d}y/\mathrm{d}t = \mathrm{d}^2x/\mathrm{d}t^2$ of the particle are simultaneously zero, which simply means that the particle is in the state of rest or in equilibrium. In view of this, singular points are also called *points of rest* or *points of equilibrium*.

The equilibrium states of a physical system constitute very special states of the system. Hence, a study of the types of singular points occupies an important place in the theory of differential equations.

The first to consider in detail the classification of singular points of differential systems of the (122) type was the distinguished Russian scientist Nikolai E. Zhukovsky (1847-1921). In his master's thesis "The kinematics of a liquid body", presented in 1876, this problem emerged in connection with the theory of velocities and accelerations of fluids. The modern names of various types of singular points were suggested by the great French mathematician Jules H. Poincaré (1854-1912).

Now let us try to answer the question of the physical meaning that can be attached to phase trajectories and singular points of differential systems of the (122) type. For the sake of clarity we introduce a two-dimensional vector field (Figure 51) defined by the function

$$\mathbf{V}(x, y) = X(x, y)\mathbf{i} + Y(x, y)\mathbf{j},$$

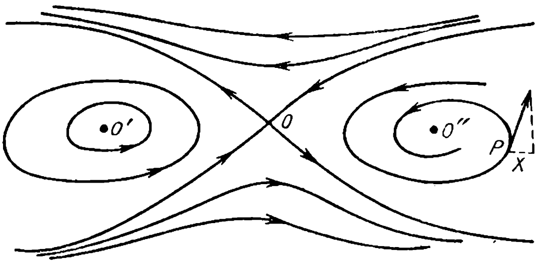

Fig. 51

where **i** and **j** are the unit vectors directed along the x and y axes, respectively, in a Cartesian system of coordinates. At every point $P(x, y)$ the field has two components, the horizontal $X(x, y)$ and the vertical $Y(x, y)$. Since $dx/dt = X(x, y)$ and $dy/dt = Y(x, y)$, the vector associated with each nonsingular point $P(x, y)$ is tangent at this point to a phase trajectory.

If variable t is interpreted as time, vector **V** can be thought of as the vector of velocity of a representative point moving along a trajectory. Thus, we can assume that the entire phase plane is filled with representative points and that each phase trajectory constitutes the trace of a moving representative point. As a result we arrive at an analogy with the two-dimensional motion of an incompressible fluid. Here, since system (122) is autonomous, vector **V** at each fixed point $P(x, y)$ is time-independent and, therefore, the motion of the fluid is steady-state. The phase trajectories in this case are simply the trajectories of the moving particles of fluid and the singular points O, O', and O'' (see Figure 51) are those where the fluid is at rest.

The most characteristic features of the fluid motion shown in Figure 51 are (1) the presence of singular points, (2) the different patterns of phase trajectories near singular points, (3) the stability or instability at singular points (i.e. two possibilities may realize themselves: the particles that are in the vicinity of singular points remain there with the passage of time or they leave the vicinity for other parts of the plane), and (4) the presence of closed trajectories, which in the given case correspond to periodic motion.

These features constitute the main part of the phase portrait, or the complete qualitative behavior pattern of the phase trajectories of a general-type system (122). Since, as noted earlier, differential equations cannot generally be solved analytically, the aim of the qualitative theory of ordinary differential equations of the (122) type is to build a phase portrait as complete as possible directly from the functions $X(x, y)$ and $Y(x, y)$.

2.4 Conservative Systems in Mechanics

Practice gives us ample examples of the fact that any real dynamical system *dissipates* energy. The *dissipation* usually occurs as a result of some form of friction. But in some cases it is so slow that it can be neglected if the system is studied over a fairly small time interval. The law of energy conservation, namely, that the sum of kinetic and potential energies remains constant, can be assumed to hold true for such systems. Systems of this kind are called *conservative.* For example, rotating Earth may be seen as a conservative system if we take a time interval of several centuries. But if we study Earth's motion over several million years, we must allow for energy dissipation related to tidal flows of water in seas and oceans.

A simple example of a conservative system is one consisting of an object moving horizontally in a vacuum under forces exerted by two springs (Figure 52). If x is the displacement of the object (mass m) from the state of equilibrium and the force with which the two springs act on the object (the *restoring force*) is proportional to x, the equation of motion has the form

$$m\frac{d^2x}{dt^2}+kx=0, \quad k>0.$$

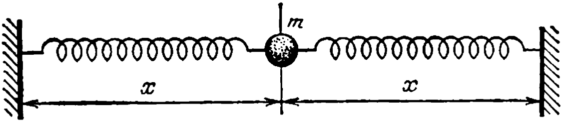

Fig. 52

Springs of this type are known as *linear*, since the restoring force exerted by them is a linear function of x.

If an object of mass m moves in a medium that exerts a drag on it (the *damping force*) proportional to the object's velocity, the equation of motion for such a nonconservative system is

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=0, \quad c>0. \tag{124}$$

Here we are dealing with linear damping, since the damping force is a linear function of velocity dx/dt.

If f and g are such arbitrary functions that $f(0)=0$ and $g(0)=0$, the more general equation,

$$m\frac{d^2x}{dt^2}+g\left(\frac{dx}{dt}\right)+f(x)=0, \tag{125}$$

can be interpreted as the equation of motion of an object of mass m under a restoring force $-f(x)$ and a damping force $-g(dx/dt)$. Generally, these forces are nonlinear; hence Eq. (125) can be considered the basic equation of nonlinear mechanics.

Let us briefly examine the special case of a nonlinear conservative system described by the equation

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+f(x)=0, \tag{126}$$

where the damping force is zero and, hence, energy is not dissipated. From Eq. (126) we can pass to the autonomous system

$$\frac{\mathrm{d}x}{\mathrm{d}t}=y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t}=-\frac{f(x)}{m}. \tag{127}$$

If we now exclude time t from Eqs. (127), we arrive at a differential equation for the trajectory of the system in the phase plane:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{f(x)}{my}. \tag{128}$$

This equation can be written as

$$my\,\mathrm{d}y=-f(x)\,\mathrm{d}x. \tag{129}$$

Then, assuming that $x = x_0$ at $t = t_0$ and $y = y_0$, we can integrate Eq. (129) from t_0 to t. The result is

$$\frac{1}{2}my^2-\frac{1}{2}my_0^2=-\int_{x_0}^{x} f(\xi)\,\mathrm{d}\xi,$$

which may be rewritten as

$$\frac{1}{2} my^2 + \int_0^x f(\xi)\, d\xi = \frac{1}{2} my_0^2 + \int_0^{x_0} f(x)\, dx. \tag{130}$$

Note that $my^2/2 = m\,(dx/dt)^2/2$ is the kinetic energy of a dynamical system and

$$V(x) = \int_0^x f(\xi)\, d\xi \tag{131}$$

is the system's potential energy. Thus, Eq. (130) expresses the law of energy conservation:

$$\frac{1}{2} my^2 + V(x) = E, \tag{132}$$

where $E = my_0^2/2 + V(x_0)$ is the total energy of the system. Clearly, Eq. (132) is the equation of the phase trajectories of system (127), since it is obtained by integrating Eq. (128). Thus, different values of E correspond to different curves of constant energy in the phase plane. The singular points of system (127) are the points $M_\nu (x_\nu, 0)$, where x_ν are the roots of the equation $f(x) = 0$. As noted earlier, the singular points are points of equilibrium of the dynamical system described by Eq. (126). Equation (128) implies that the phase trajectories of the system intersect the x axis at right angles, while the straight lines $x = x_\nu$ are parallel to the x axis. In addition, Eq. (132) shows that the phase trajectories are symmetric with respect to the x axis.

In this case, if we write Eq. (132) in the form

$$y = \pm\sqrt{\frac{2}{m}[E - V(x)]}, \tag{133}$$

we can easily plot the phase trajectories. Indeed, let us introduce the (x, z)-plane, the plane of energy balance (Figure 53), with the z axis lying on the same vertical line as the y axis of the phase plane. We then plot the graph of the function $z = V(x)$ and several straight lines $z = E$ in the (x, z)-plane (one such straight line is depicted in Figure 53). We mark a value of $E - V(x)$ on the graph. Then for a definite x we multiply $E - V(x)$ by $2/m$ and allow for formula (133). This enables us to mark the respective values of y in the phase plane. Note that since $dx/dt = y$, the positive direction along any trajectory is determined by the motion of the representative point from left to right above the x axis and from right to left below the x axis.

The above reasoning is fairly general and makes it possible to investigate the equation of motion of a pendulum in a medium without drag, which has the form (see p. 72)

$$\frac{d^2x}{dt^2} + k\sin x = 0, \tag{134}$$

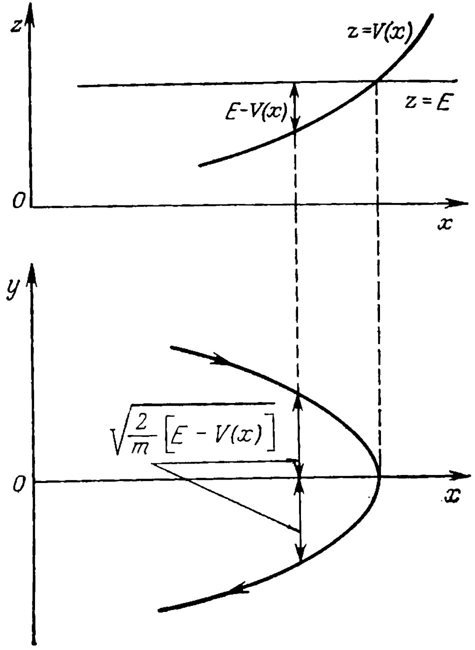

Fig. 53

with k a positive constant.

Since Eq. (134) constitutes a particular case of Eq. (126), it can be interpreted as an equation describing the frictionless motion in a straight line of an object of unit mass under a restoring force equal to

$-k \sin x$. In this case the autonomous system corresponding to Eq. (134) is

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -k \sin x. \qquad (135)$$

The singular points are (0, 0), ($\pm \pi$, 0), ($\pm 2\pi$, 0), . . ., and the differential equation of the phase trajectories of system (135) assumes the form

$$\frac{dy}{dx} = -\frac{k \sin x}{y}.$$

Separating the variables and integrating, we arrive at an equation for the phase trajectories,

$$\frac{1}{2} y^2 + k(1 - \cos x) = E.$$

This equation is a particular case of Eq. (132) with $m = 1$, where the potential energy determined by (131) is specified by the relationship

$$V(x) = \int_0^x f(\xi)\, d\xi = k(1 - \cos x).$$

In the (x, z)-plane we plot the function $z = V(x)$ as well as several straight lines $z = E$ (in Figure 54 only one such line, $z = E = 2k$, is shown). After determining a value of $E - V(x)$ we can draw a sketch

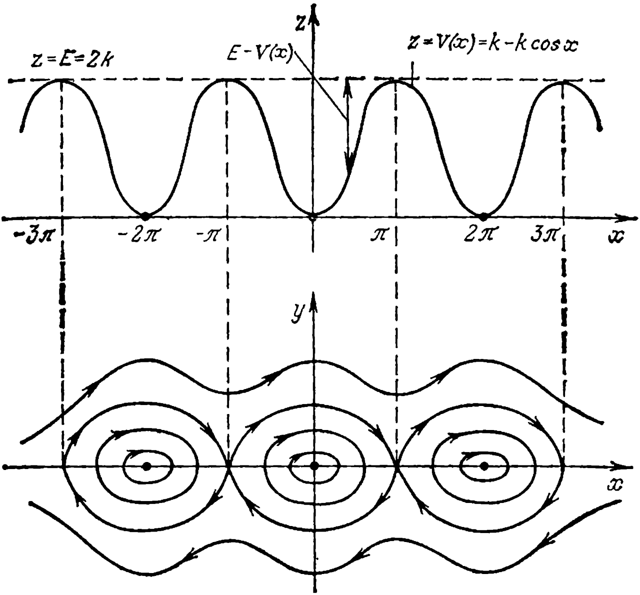

Fig. 54

of the trajectories in the phase plane by employing the relationship

$y = \pm\sqrt{2[E - V(x)]}$.

The resulting phase portrait shows (see Figure 54) that if the energy E varies from 0 to $2k$, the corresponding phase trajectories prove to be closed and Eq. (134) acquires periodic solutions. On the other hand, if $E > 2k$, the respective phase trajectories are not closed and Eq. (134) has no periodic solutions. Finally, the value $E = 2k$ corresponds to a phase trajectory in the phase plane that separates two types of motion, that is, is a *separatrix*. The wavy lines lying outside the separatrices correspond to rotations of a pendulum, while the closed trajectories lying inside the regions bounded by separatrices correspond to oscillations of the pendulum.

Figure 54 shows that in the vicinity of the singular points $(\pm 2\pi m, 0)$, $m = 0, 1, 2, \ldots$, the behavior of the phase trajectories differs from that of the phase trajectories in the vicinity of the singular points $(\pm \pi n, 0)$, where $n = 1, 2, \ldots$

There are different types of singular points. With some we will get acquainted shortly. As for the above example, the singular points $(\pm 2\pi m, 0)$, $m = 0, 1, 2, \ldots$, belong to the *vortex-point* type, while the singular points $(\pm \pi n, 0)$, $n = 1, 2, \ldots$, belong to the *saddle-point* type. A singular point of an autonomous differential system of the (122) type is said to be a *vortex point* if there exists a neighborhood of this point completely filled with nonintersecting phase trajectories surrounding the point. A *saddle point* is a singular point adjoined by a finite number of phase trajectories ("whiskers") separating a neighborhood of the singular point into regions where the trajectories behave like a family of hyperbolas defined by the equation $xy = \text{const}$.

Now let us establish the effect of linear friction on the behavior of the phase trajectories of a conservative system. The equation is

$$\frac{d^2x}{dt^2} + c\frac{dx}{dt} + k\sin x = 0, \quad c > 0.$$

If friction is low, that is, the pendulum is able to oscillate about the position of equilibrium, it can be shown that the phase trajectories are such as shown in Figure 55. But if friction is so high that oscillations become impossible, the pattern of phase trajectories resembles the one depicted in Figure 56.

If we now compare the phase portrait of a conservative system with the last two portraits of nonconservative systems, we see that saddle points have not changed (we consider only small neighborhoods of singular points), while in the neighborhood of the singular points $(\pm 2\pi m, 0)$, $m = 0, 1, 2, \ldots$, the closed phase trajectories have transformed into spirals (for low friction) or into trajectories that "enter" the singular points in certain directions (for high fric-

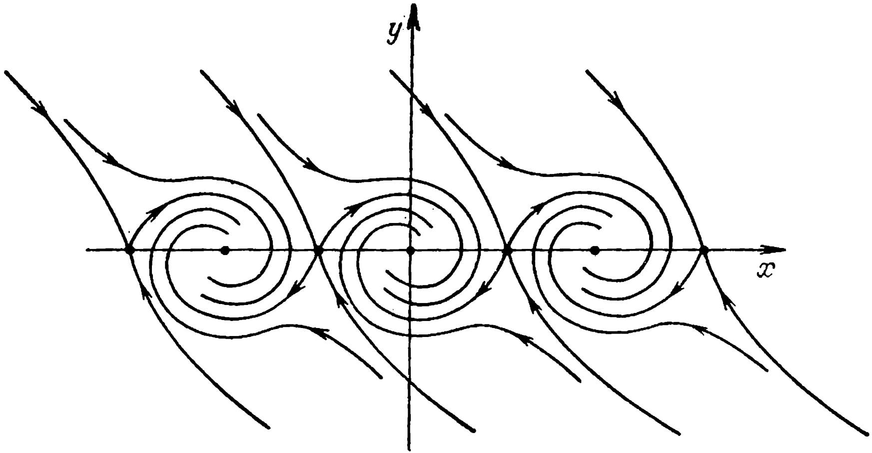

Fig. 55

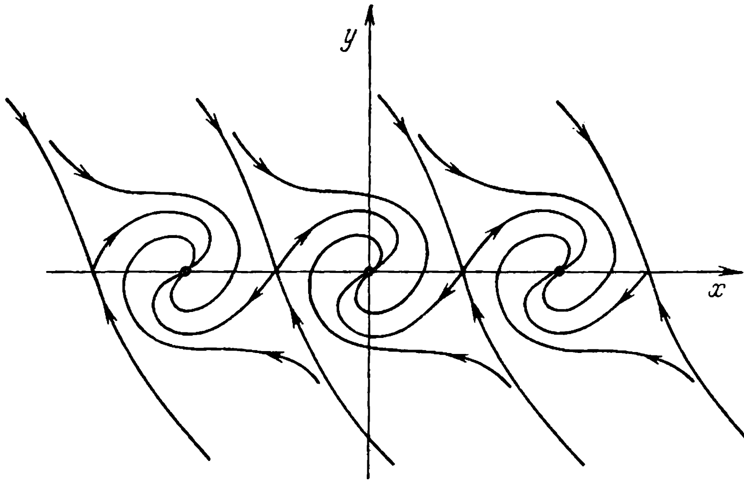

Fig. 56

tion). In the first case (spirals) we have singular points of the *focal-point* type and in the second, of the *nodal-point* type.

A singular point of a two-dimensional autonomous differential system of the general type (122) (if such a point exists) is said to be a *focal point* if there exists a neighborhood of this point that is completely filled with nonintersecting phase trajectories resembling spirals that "wind" onto the singular point either as $t \to +\infty$ or as $t \to -\infty$. A *nodal point* is a singular point in whose neighborhood each phase trajectory behaves like a branch of a parabola or a half-line adjoining the point along a certain direction.

Note that if a conservative system has a periodic solution, the solution cannot be isolated. More than that, if Γ is a closed phase trajectory corresponding to a periodic solution of the conservative system, there exists a certain neighborhood of Γ that is completely filled with closed phase trajectories.

Note, in addition, that the above definitions of types of singular points have a purely qualitative, descriptive nature. As for the analytical features of these types, there are no criteria, unfortunately, in the general case of systems of the (122) types, but for some classes of differential equations such criteria can be formulated.

A simple example is the linear system

$$\frac{dx}{dt} = a_1x + b_1y, \ \frac{dy}{dt} = a_2x + b_2y,$$

where a_1, b_1, a_2, and b_2 are real constants.

If the coefficient matrix of this sytem is nonsingular, that is, the determinant

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0,$$

the origin O (0, 0) of the phase plane is the only singular point of the differential system.

Assuming the last inequality valid, we denote the eigenvalues of the coefficient matrix by λ_1 and λ_2. It can then be demonstrated that

(1) if λ_1 and λ_2 are real and of the same sign, the singular point is a nodal point,

(2) if λ_1 and λ_2 are real and of opposite sign, the singular point is a saddle point,

(3) if λ_1 and λ_2 are not real and are not pure imaginary, the singular point is a focal point, and

(4) if λ_1 and λ_2 are pure imaginary, the singular point is a vortex point.

Note that the first three types of singular points belong to the so-called coarse singular points, that is, singular points whose nature is not affected by small perturbations of the right-hand sides of the initial differential system. On the other hand, a vortex point is a fine singular point; its nature changes even under small perturbations of the right-hand sides.

2.5 Stability of Equilibrium Points and of Periodic Motion

As we already know, singular points of different types are characterized by different patterns of the phase trajectories in sufficiently small neighborhoods of these points. There is also another characteristic, the stability of an equilibrium point, which provides additional information on the behavior of phase trajectories in the neighborhood of singular points. Consider the pendulum depicted in Figure 57. Two states of equilibrium are shown: (a) an object of mass m is in a state of equilibrium at the uppermost point, and (b) the same object is in a state of equilibrium at its lowest point. The first state is unstable and the second, stable. And this is why. If the object is in its uppermost state of equilibrium, a slight push is enough to start it moving with an ever increasing speed away from the equilibrium position and, hence, away from the initial position. But if the object is in the lowest possible state, a push makes it move away from the position of equilibrium with a decreasing speed, and the weaker the push the smaller the distance by which the

Fig. 57

object is displaced from the initial position.

The state of equilibrium of a physical system corresponds to a singular point in the phase plane. Small perturbations at an unstable point of equilibrium lead to large displacements from this point, while at a stable point of equilibrium small perturbations lead to small displacements. Starting from these pictorial ideas, let us consider an isolated singular point of system (122), assuming for the sake of simplicity that the point is at the origin O (0, 0) of the phase plane. We will say that this singular point is *stable* if for every positive R there exists a positive $r \leqslant R$ such that

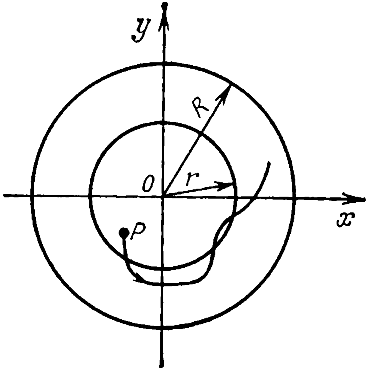

Fig. 58

every phase trajectory originating at the initial moment $t = t_0$ at a point P lying inside the circle $x^2 + y^2 = r^2$ will lie inside the circle $x^2 + y^2 = R^2$ for all $t > t_0$ (Figure 58). Without adhering to rigorous reasoning we can say that a singular point is stable if all phase trajectories that are near the point initially remain there with the passage of time. Also, a singular point is said to be *asymptotically stable* if it is stable and if there exists a circle $x^2 + y^2 = r_0^2$ such that each trajectory that at time $t = t_0$ lies inside the circle converges to the origin as $t \to +\infty$. Finally, if a singular point is not stable, it is said to be *unstable*.

A singular point of the vortex type is always stable (but not asymptotically). A saddle point is always unstable. In Figure 55, which illustrates the behavior of the phase trajectories for pendulum oscillations in a medium with low drag, the singular points, focal points, are asymptotically stable; in Figure 56 the singular points, which are nodal points, are also asymptotically stable.

The introduced concept of stability of an equilibrium point is purely qualitative, since no mention of properties referring to the behavior of phase trajectories has been made. As for the concept of asymptotic stability, if compared with the notion of simple stability, it is additionally necessary that every phase trajectory tend to the origin with the passage of time. However, in this case, too, no conditions are imposed on how the phase trajectory must approach point O (0, 0).

The concepts of stability and asymptotic stability play an important role in applications. The fact is that if a device is designed without due regard for stability considerations, when built it will be sensitive to the very smallest external perturbations, which in the final analysis may lead to extremely unpleasant consequences. Emphasizing the importance of the concept of stability, the well-known Soviet specialist in the field of mathematics and mechanics

Nikolai G. Chetaev (1902-1959) wrote: *

> ...If a passenger plane is being designed, a certain degree of stability must be provided for in the future movements of the plane so that it will be stable in flight and accident-free during take-off and landing. The crank-shaft must be so designed that it does not break from the vibrations that appear in real conditions of motor operation. To ensure that an artillery gun has the highest possible accuracy of aim and the smallest possible spread, the gun and the projectiles must be constructed in such a manner that the trajectories of projectile flight are stable and the projectiles fly correctly.
>
> Numerous examples can be added to this list, and all will prove that real movements require selecting out of the possible solutions of the equations of motion only those that correspond to stable states. Moreover, if we wish to avoid a certain solution, it is advisable to change the design of the corresponding device in such a way that the state of motion corresponding to this solution becomes unstable.

Returning to the pendulum depicted in Figure 57, we note the following curious and somewhat unexpected fact. Research has shown that the upper (unstable) position of equilibrium can be made stable by

* See N.G. Chetaev, *Stability of Motion* (Moscow: Nauka, 1965: pp. 8-9 (in Russian).

introducing vertical oscillations of the point of suspension. More than that, not only the upper (vertical) position of equilibrium can be made stable but also any other of the pendulum's positions (for one, a horizontal position) by properly vibrating the point of suspension. *

Now let us turn to a concept no less important than the stability of an equilibrium point, the concept of stability of periodic movements (solutions). Let us assume that we are studying a conservative system that has periodic solutions. In the phase plane these solutions are represented by closed trajectories that completely fill a certain region. Thus, to each periodic motion of a conservative system there corresponds a motion of the representative point along a fixed closed trajectory in the phase plane.

Generally, the period of traversal of different trajectories by representative points is different. In other words, the period of oscillations in a conservative system depends on the initial data. Geometrically this means that two closely spaced representative points that begin moving at a certain moment $t = t_0$ (say, at the x axis) will move apart to a certain finite distance

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* The reader can find many examples of stabilization of different types of pendulums in the book by T.G. Strizhak, *Methods of Investigating Dynamical Systems of the "Pendulum" Type* (Alma-Ata: Nauka, 1981) (in Russian).

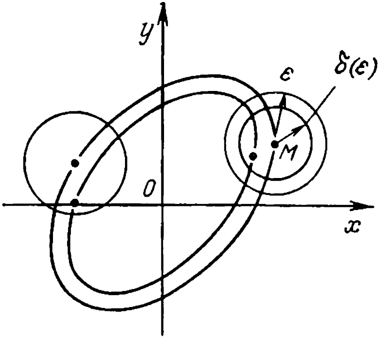

Fig. 59

with the passage of time. But it also may happen that these points do not separate. To distinguish between these two possibilities, the concept of stability in the sense of Lyapunov is introduced for periodic solutions. The essence of this concept lies in the following. If knowing an ε-neighborhood (with ε as small as desired) of a point M moving along a closed trajectory Γ (Figure 59) * ensures that we know a moving δ (ε)-neighborhood of the same point M such that every representative point that initially lies in the δ (ε)-neighborhood will never leave the ε-neighborhood with the passage of time, then the periodic solution

* An ε-neighborhood of a point M is understood to be a disk of radius ε centered at point M.

corresponding to Γ is said to be *stable in the sense of Lyapunov.* If a periodic solution is not stable in the sense of Lyapunov, it is said to be *unstable in the sense of Lyapunov.*

When it comes to periodic solutions that are unstable in the sense of Lyapunov, we must bear in mind that they still possess some sort of stability, *orbital stability*, which means that under small variations of initial data the representative point transfers from one phase trajectory to another lying as close as desired to the initially considered trajectory.

Examples of periodic solutions that are stable in the sense of Lyapunov are those that emerge, for instance, when we consider the differential equation that describes the horizontal movements of an object of mass m in a vacuum with two linear springs acting on the object (Figure 52). An example of periodic solutions that are unstable in the sense of Lyapunov but are orbitally stable is the solutions of the differential equation (134), which describes the motion of a circular pendulum in a medium without drag.

In the first case the oscillation period does not depend on the initial data and is found by using the formula $T = 2\pi\sqrt{m/k}$. In the second the oscillation period depends on the initial data and is expressed, as we know, in terms of an elliptic integral of the first kind taken from 0 to $\pi/2$, Finally, we note that the question of whether periodic movements are stable in the sense of Lyapunov is directly linked to the question of isochronous vibrations. *

2.6 Lyapunov Functions

Intuitively it is clear that if the total energy of a physical system is at its minimum at a point of equilibrium, the point is one of stable equilibrium. This idea lies at the base of one of two methods used in studying stability problems, both suggested by the famous Russian mathematician Aleksandr M. Lyapunov (1857-1918). This method is known as *Lyapunov's direct*, or *second, method* for stability investigations.**

We illustrate Lyapunov's direct method using the (122) type of system when the origin is a singular point.

Suppose that Γ is a phase trajectory of system (122). We consider a function $V = V(x, y)$ that is continuous together with its first partial derivatives $\partial V/\partial x$ and $\partial V/\partial y$

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* See, for example, the book by V. V. Amel'kin, N. A. Lukashevich, and A.P. Sadovskii, *Nonlinear Vibrations in Second-Order Systems* (Minsk: Belorussian Univ. Press, 1982) (in Russian).

** The reader will find many interesting examples of stability investigations involving differential models in the book by N. Rouche, P. Habets, and M. Laloy, *Stability Theory by Lyapunov's Direct Method* (New York: Springer, 1977).

in a domain containing Γ in the phase plane. If the representative point $(x(t), y(t))$ moves along curve Γ, then on this curve the function $V(x, y)$ may be considered a function of t, with the result that the rate at which $V(x, y)$ varies along Γ is given by the formula

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} = \frac{\partial V}{\partial x} X(x, y) + \frac{\partial V}{\partial y} Y(x, y), \quad (136)$$

where $X(x, y)$ and $Y(x, y)$ are the right-hand sides of system (122).

Formula (136) is essential in the realization of Lyapunov's direct method. The following concepts are important for the practical application of the method.

Suppose that $V = V(x, y)$ is continuous together with its first partial derivatives $\partial V/\partial x$ and $\partial V/\partial y$ in a domain G containing the origin in the phase plane, with $V(0, 0) = 0$. This function is said to be *positive* (*negative*) *definite* if at all points of G except the origin $V(x, y)$ it is positive (negative). But if at points of G we have $V(x, y) \geqslant 0$ ($\leqslant 0$), the function $V = V(x, y)$ is said to be *nonnegative* (*nonpositive*). For example, the function V defined by the formula $V(x, y) = x^2 + y^2$ and considered in the (x, y)-plane is positive definite, while the function $V(x, y) = x^2$ is nonnegative since it vanishes on the entire y axis.

If $V(x, y)$ is positive definite, we can require that at all points of the domain $G \setminus O$ the following inequality hold true: $\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 \neq 0$.

This means that the equation $z = V(x, y)$ can be interpreted as the equation of a surface resembling a paraboloid that touches the (x, y)-plane at point $O(0, 0)$ (Figure 60). Generally, the equation $z = V(x, y)$ with a positive definite V may specify a surface of a more complex structure. One such surface is shown in Figure 61, where the section of the surface with the plane $z = C$ results not in a curve but in a ring.

If a positive definite function $V(x, y)$ is such that

$$W(x, y) = \frac{\partial V(x, y)}{\partial x} X(x, y) + \frac{\partial V(x, y)}{\partial y} Y(x, y) \qquad (137)$$

is nonpositive, V is said to be the *Lyapunov function* of system (122). We note here that in view of (136) the requirement that W be nonpositive means that $dV/dt \leqslant 0$ and, hence, the function $V = V(x, y)$ does not increase along the trajectory Γ in the neighborhood of the origin.

Here is a result arrived at by Lyapunov: *if for system* (122) *there exists a Lyapunov function* $V(x, y)$, *then the origin, which is*

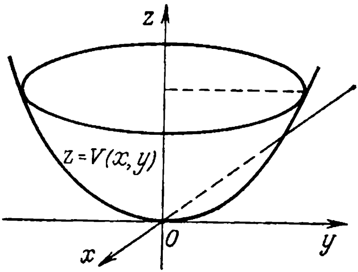

Fig. 60

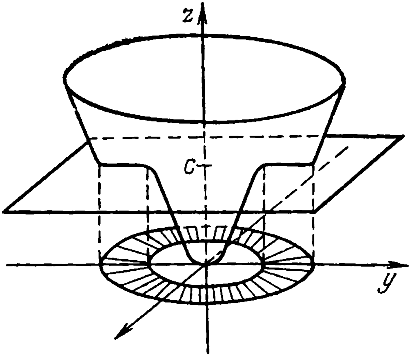

Fig. 61

a singular point, is stable. If the positive definite function $V = V(x, y)$ is such that function W defined via (137) *is negative definite, then the origin is asymptotically stable.*

We will show with an example how to

apply the above result. Let us consider the equation of motion of an object of unit mass under a force exerted by a spring, which in view of (124) can be written in the form

$$\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=0, \quad c>0. \tag{138}$$

The reader will recall that in this equation $c>0$ characterizes the drag of the medium in which the object moves and $k>0$ characterizes the properties of the spring (the spring constant). The autonomous system corresponding to Eq. (138) has the form

$$\frac{dx}{dt}=y, \quad \frac{dy}{dt}=-kx-cy. \tag{139}$$

For this system the origin of the phase (x, y)-plane is the only singular point. The kinetic energy of the object (of unit mass) is $y^2/2$ and the potential energy (i.e. the energy stored by the spring) is

$$\int_0^x k\xi\, d\xi=\frac{1}{2}kx^2.$$

This implies that the total energy of the system is

$$V(x, y)=\frac{1}{2}y^2+\frac{1}{2}kx^2. \tag{140}$$

It is easy to see that V specified by (140) is a positive definite function. And since in the given case

$$\frac{\partial V}{\partial x} X(x, y)+\frac{\partial V}{\partial y} Y(x, y)$$
$$=kxy+y(-kx-cy)=-cy^2\leqslant 0,$$

this function is the Lyapunov function of system (139), which means that the singular point O (0, 0) is stable.

In the above example the result was obtained fairly quickly. This is not always the case, however. The formulated Lyapunov criterion is purely qualitative and this does not provide a procedure for finding the Lyapunov function even if we know that such a function exists. This makes it much more difficult to determine whether a concrete system is stable or not.

The reader must bear in mind that the above criterion of Lyapunov must be seen as a device for finding effective indications of equilibrium. Many studies have been devoted to this problem and a number of interesting results have been obtained in recent years.*

2.7 Simple States of Equilibrium

The dynamical interpretation of second-order differential equations already implies

* The interested reader can refer to the book by E.A. Barbashin, *Lyapunov Functions* (Moscow: Nauka, 1970) (in Russian).

that investigation of the nature of equilibrium states or, which is the same, the singular points provides a key for establishing the behavior of integral curves.

It is also clear that, generally speaking, differential equations cannot be integrated in closed form. We need criteria that will enable us to determine the type of a singular point from the form of the initial differential equation. Unfortunately, as a rule it is extremely difficult to find such criteria, but it is possible to isolate certain classes of differential equations for which this can be done fairly easily. Below we show, using the example of an object of unit mass subjected to the action of linear springs and moving in a medium with linear drag, how some results of the qualitative theory of differential equations can be used to this end. But first let us discuss a system of the (122) type. It so happens that the simplest case in establishing the type of a singular point is when the Jacobian or functional determinant

$$J(x, y) = \begin{vmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} \end{vmatrix}$$

is nonzero at the point.

If (x^*, y^*) is a singular point of system (122) and if $J^* = J(x^*, y^*) \neq 0$, then the type of the singular point, which in the given case is called a *simple singular point*, depends largely on the sign of constant J^*. For instance, if J^* is negative, the singular point (x^*, y^*) is a saddle point, and if J^* is positive, the singular point may be a vortex point, a nodal point, or a focal point. The singular point may be a vortex point only if the *divergence*

$$D(x, y) = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}$$

vanishes at the singular point, that is, only if $D^* = D(x^*, y^*) = 0$. Note, however, that the condition $D^* = 0$ is generally insufficient for the singular point (x^*, y^*) to be a vortex point. For a vortex point to be present certain additional conditions must be met, conditions that include higher partial derivatives. And, generally, there can be an infinite number of such conditions. But if functions X and Y are linear in variables x and y, the condition $D^* = 0$ becomes sufficient for the singular point (x^*, y^*) to be a vortex point.

If $J^* > 0$ but the singular point (x^*, y^*) is not a vortex point, a sufficiently small neighborhood of this point is filled with trajectories that either spiral into this point or converge to it in certain directions. Here, if $D^* > 0$, the singular point is reached as $t \to -\infty$ and is unstable while if $D^* < 0$, the singular point is reached as $t \to +\infty$ and proves to be stable. If the phase trajectories that reach the singular point are spirals, we are dealing with a focal point, but if the integral curves converge to a singular point along a certain tangent, the point is a nodal point (Figure 62).

Irrespective of the sign of Jacobian J^*, the tangents to the trajectories of the differential system (122) at a singular point (x^*, y^*) can be found from what is known as the *characteristic equation*

$$\frac{\tilde{y}}{\tilde{x}}=\frac{\frac{\partial Y(x^*,\, y^*)}{\partial x}\tilde{x}+\frac{\partial Y(x^*,\, y^*)}{\partial y}\tilde{y}}{\frac{\partial X(x^*,\, y^*)}{\partial x}\tilde{x}+\frac{\partial X(x^*,\, y^*)}{\partial y}\tilde{y}}, \qquad (141)$$

where

$$\tilde{x}=x-x^*, \quad \tilde{y}=y-y^*. \qquad (142)$$

If X and Y contain linear terms, the partial derivatives in Eq. (141) act as coefficients of x and y in the system obtained from system (122) after introducing the substitution (142).

Equation (141) is homogeneous. Hence, if we introduce the slope $\lambda=\tilde{y}/\tilde{x}$ of what is known as *exceptional directions*, we have the following quadratic equation for finding λ:

$$X_y^*\lambda^2+(X_x^*-Y_y^*)\,\lambda-Y_x^*=0. \qquad (143)$$

The discriminant of this equation is

$$\Delta=(X_x^*+Y_y^*)^2-4J^*=D^{*2}-4J^*.$$

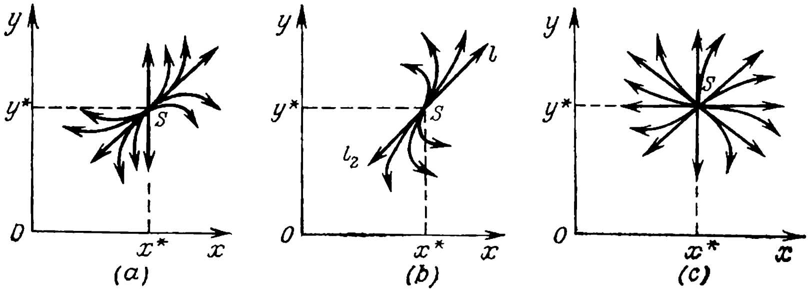

Fig. 62

Hence, if $J^* < 0$, which corresponds to a saddle point, Eq. (143) always gives two real exceptional directions. But if $J^* > 0$, there are no real exceptional directions, or there are two, or there is one 2-fold direction. In the first case the singular point is either a vortex point or a focal point.

The existence of real exceptional directions means (provided that J^* is positive) that there is a singular point of the nodal type. For one thing, if there are two real exceptional directions, it can be proved that there are exactly two trajectories (one on each side) whose tangent at the singular point is one of the exceptional straight lines (directions) while all the other trajectories "enter" the singular point touching the other exceptional straight line (Figure 62*a*).

If $\Delta = 0$ and Eq. (143) is not an identity, we have only one exceptional straight line. The pattern of the trajectories for this case is illustrated by Figure 62*b*. It can be obtained from the previous case when the two exceptional directions coincide. The singular point divides the exceptional straight line into two half-lines, l_1 and l_2, while the neighborhood of the singular point is divided into two sectors, one of which is completely filled with trajectories that "enter" the singular point and touch l_1 and the other is completely filled with trajectories that "enter" the singular point and touch l_2. The boundary between the sectors consists of two trajectories, one of which touches l_1 at the singular point and the other touches l_2 at this point.

If in Eq. (143) all coefficients vanish, we arrive at an identity, and then all the straight lines passing through the singular point are exceptional and there are exactly two trajectories (one on each side) that touch each of these straight lines at the singular point. This point (Figure 62*c*) is similar to the point with one 2-fold real exceptional direction.

2.8 Motion of a Unit-Mass Object Under the Action of Linear Springs in a Medium with Linear Drag

As demonstrated earlier, the differential equation describing the motion of a unit-mass object under the action of linear springs in a medium with linear drag has the form

$$\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=0. \tag{144}$$

So as not to restrict the differential model (144) to particular cases we will not fix the directions in which the forces $-c\,(dx/dt)$ and $-kx$ act. As shown earlier, with Eq. (144) we can associate an autonomous system of the form

$$\frac{dx}{dt}=y, \quad \frac{dy}{dt}=-kx-cy. \tag{145}$$

If we now exclude the trivial case with $k = 0$, which we assume is true for the meantime, the differential system (145) has an isolated singular point at the origin. System (145) is a particular case of the general system (122). In our concrete example

$$X(x, y) = y, \quad Y(x, y) = -kx - cy,$$

the Jacobian $J(x, y) = k$, and the divergence $D(x, y) = -c$. The characteristic equation assumes the form

$$\lambda^2 + c\lambda + k = 0,$$

where $\Delta = c^2 - 4k$ is the discriminant of this equation. In accordance with the results obtained in Section 2.7 we arrive at the following cases.

(1) If k is negative, the singular point is a saddle point with one positive and one negative exceptional direction. The phase trajectory pattern is illustrated in Figure 63, where we can distinguish between three different types of motion. When the initial conditions correspond to point a, at which the velocity vector is directed to the origin and the velocity is sufficiently great, the representative point moves along a trajectory toward the singular point at a decreasing speed; after passing the origin the representative point moves away from it at an increasing speed. If the initial velocity decreases to a critical value, which

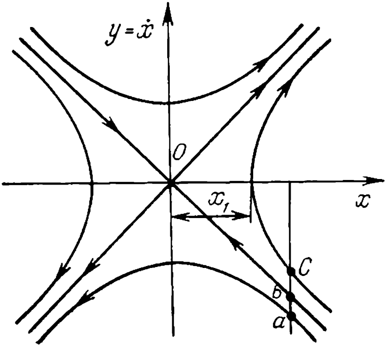

Fig. 63

corresponds to point b, the representative point approaches the singular point at a decreasing speed and "reaches" the origin in an infinitely long time interval. Finally, if the initial velocity is lower than the critical value and corresponds, say, to point C, the representative point approaches the origin at a decreasing speed, which vanishes at a certain distance x_1 from the origin. At point $(x_1, 0)$ the velocity vector reverses its direction and the representative point moves away from the origin.

If the phase point corresponding to the initial state of the dynamical system lies in either one of the other three quadrants, the interpretation of the motion is obvious.

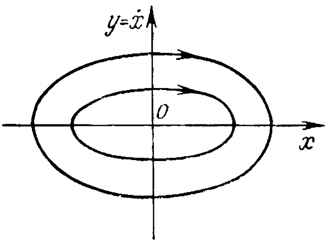

Fig. 64

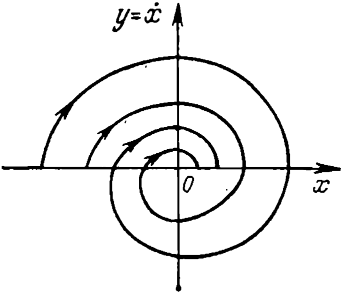

Fig. 65

(2) If $k > 0$, then J^* is positive and the type of singular point depends on the value of c. This leaves us with the following possibilities:

(2a) If $c = 0$, that is, drag is nil, the singular point is a vortex point (Figure 64). The movements are periodic and their amplitude depends on the initial conditions.

(2b) If $c > 0$, that is, damping is positive, the divergence $D = \partial X/\partial x + \partial Y/\partial y$ is negative and, hence, the representative point moves along a trajectory toward the origin and reaches it in an infinitely long time interval.

More precisely:

($2b_1$) If $\Delta < 0$, that is, $c^2 < 4k$, the singular point proves to be a focal point (Figure 65) and, hence, the dynamical system performs damped oscillations about the state of equilibrium with a decaying amplitude.

($2b_2$) If $\Delta = 0$, that is, $c^2 = 4k$, the singular point is a nodal point with a single negative exceptional direction (Figure 66). The motion in this case is aperiodic and corresponds to the so-called *critical damping*.

($2b_3$) If $\Delta > 0$, that is, $c^2 > 4k$, the singular point is a nodal point with two negative exceptional directions (Figure 67). Qualitatively the motion of the dynamical system is the same as in the previous case and corresponds to damped oscillations.

From the above results it follows that when $c > 0$ and $k > 0$, that is, drag is positive and the restoring force is attractive, the dynamical system tends to a state of equilibrium and its motion is stable.

(2c) If $c < 0$, that is, damping is negative, the qualitative pattern of the phase trajectories is the same as in the case (2b), the only difference being that here the

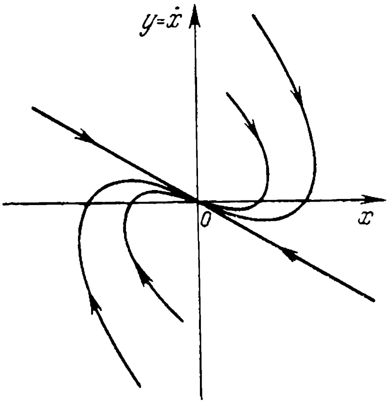

Fig. 66

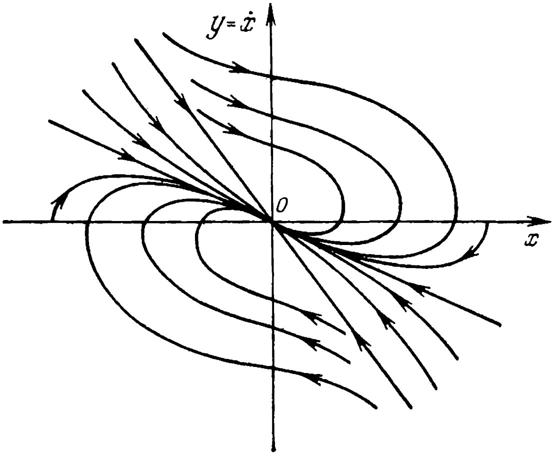

Fig. 67

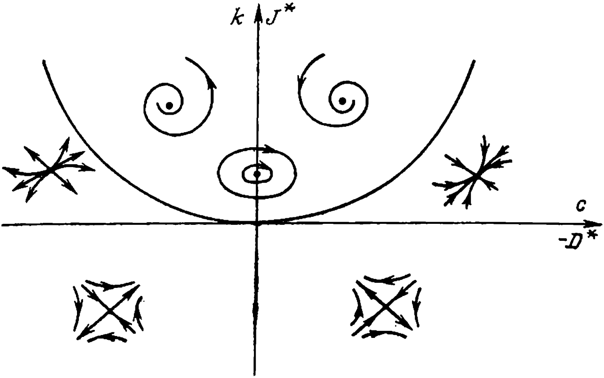

Fig. 68

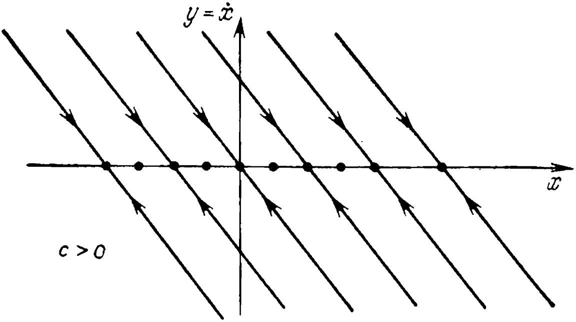

Fig. 69

dynamical system ceases to be stable.

Figure 68 contains all the above results and the dependence of the type of singular point on the values of parameters c and k. Note that the diagrams can also be interpreted as a summary of the results of studies of the types of singular points of system (122) when $J^* \neq 0$ at $c = -D^*$ and $k = J^*$. However, the fact that $c = 0$ does not generally mean that system (122) possesses a vortex point, and the fact that $k = 0$ does not mean that the system of a general type has no singular point. These cases belong to those of complex singular points, which we consider below.

Returning to the dynamical system considered in this section, we note that if $k = 0$ ($c \neq 0$), the autonomous system (145) corresponding to Eq. (144) assumes the form

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -cy.$$

This implies that the straight line $y = 0$ is densely populated by singular points, with the phase trajectory pattern shown in Figure 69.

Finally, if $k = c = 0$, Eq. (144) assumes the form

$$\frac{d^2x}{dt^2} = 0.$$

The respective autonomous system is

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = 0.$$

Here, as in the previous case, the x axis is densely populated by singular points. The respective phase trajectory pattern is shown in Figure 70.

2.9 Adiabatic Flow of a Perfect Gas Through a Nozzle of Varying Cross Section

A study of the flow of compressible viscous media is highly important from the practical viewpoint. For one thing, such flow emerges in the vicinity of a wing and fuselage of an airplane; it also influences the operation of steam and gas turbines, jet engines, the nuclear reactors.

Below we discuss the flow of a perfect gas through a nozzle with a varying cross section (Figure 71); the specific heat capacity of the gas is c_p and the nozzle's varying cross-sectional area is denoted by A. The flow is interpreted as one-dimensional, that is, all its properties are assumed to be uniform in a single cross section of the nozzle. Friction in the boundary layer is caused by the tangential stress τ given by the formula

$$\tau = q\rho v^2/2, \tag{146}$$

where q is the friction coefficient depending basically on the Reynolds number but assumed constant along the nozzle, ρ the

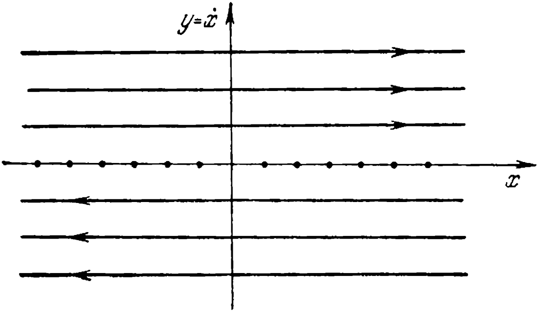

Fig. 70

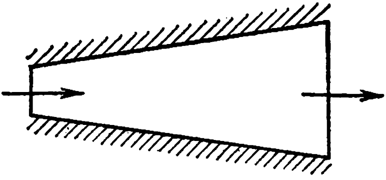

Fig. 71

flux density, and v the flow velocity. Finally, we assume that adiabaticity conditions are met, that is, drag, combustion, chemical transformations, evaporation, and condensation are excluded.

One of the basic equations describing this type of flow is the well-known continuity equation, which in the given case is written in the form

$$w = \rho A v, \tag{147}$$

where the flux variation rate w is assumed constant. From this equation it follows that

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dv}{v} = 0. \tag{148}$$

Let us now turn to the equation for the energy of steady-state flow. We note that generally such an equation links the external work done on the system and the action of external heat sources with the increase in enthalpy (heat content) in the flow and the kinetic and potential energies. In our case the flow is adiabatic; hence, the energy-balance equation can be written in the form

$$0 = w(h + dh) - wh + w[v^2/2 + d(v^2/2)] - wv^2/2,$$

or

$$dh + d(v^2/2) = 0, \tag{149}$$

where h is the enthalpy of the flow (the thermodynamic potential) at absolute temperature T. But in Eq. (149) $dh = c_p dT$ and, therefore, we can write the equation for the flow energy as

$$c_p dT + d(v^2/2) = 0. \tag{150}$$

Now let us derive the momentum equation for the flow. Note that here the common approach to problems involving a steady-state flow is to use Newton's second law. Assuming that the divergence angle of the nozzle wall is small, we can write the momentum equation in the form

$$pA + p\,\mathrm{d}A - (p + \mathrm{d}p)(A + \mathrm{d}A) - \tau\,\mathrm{d}A = w\,\mathrm{d}v,$$

or

$$-A\,\mathrm{d}p - \mathrm{d}A\,\mathrm{d}p - \tau\,\mathrm{d}A = w\,\mathrm{d}v, \qquad (151)$$

where p is the static pressure.

The term $\mathrm{d}A\,\mathrm{d}p$ in Eq. (151) is of a higher order than the other terms and, therefore, we can always assume that the momentum equation for the flow has the form

$$-A\,\mathrm{d}p - \tau\,\mathrm{d}A = w\,\mathrm{d}v. \qquad (152)$$

If we denote by D the hydraulic diameter, we note that its variation along the nozzle's axis is determined by a function F such that $D = F(x)$, where x is the coordinate along the nozzle's axis. From the definition of the hydraulic diameter it follows that

$$\frac{\mathrm{d}A}{A} = \frac{4\,\mathrm{d}x}{D}. \qquad (153)$$

Noting that $\rho v^2/2 = \gamma p \mathrm{M}^2$, where γ is the specific heat ratio of the medium, and M is the Mach number, we can write formula (146) thus:

$$\tau = q\gamma p\mathrm{M}^2. \qquad (154)$$

Combining this with Eqs. (147) and (153), we arrive at the following representation for the momentum equation (152):

$$\frac{dp}{p} + \gamma\mathrm{M}^2\left(4q\frac{dx}{D} + \frac{dv^2}{v^2}\right) = 0. \qquad (155)$$

Denoting the square of the Mach number by y, employing Eqs. (148), (150), and (155), and performing the necessary algebraic transformations, we arrive at the following differential equation:

$$\frac{dy}{dx} = \frac{4y\left(1+\frac{\gamma-1}{2}y\right)(\gamma q y - F'(x))}{(1-y)F(x)}, \qquad (156)$$

where the prime stands for derivative of the respective quantity.*

The denominator of the right-hand side of Eq. (156) vanishes at $y = 1$, that is, when the Mach number becomes equal to unity. This means that the integral curves of the last equation intersect what is known as the *sonic line* and have vertical tangents at the intersection points. Since the right-hand

* See J. Kestin and S.K. Zaremba, "One-dimensional high-speed flows. Flow patterns derived for the flow of gases through nozzles, including compressibility and viscosity effects," *Aircraft Engin.* 25, No. 292: 172-175, 179 (1953).

side of Eq. (156) reverses its sign in the process of intersection, the integral curves "flip over" and the possibility of inflection points is excluded. The physical meaning of this phenomenon implies that along integral curves the value of x must increase continuously. Hence, the section on which the integral curves intersect the sonic line with vertical tangents must be the exit section of the nozzle. Thus, the transition from subsonic flow to supersonic (and back) can occur inside the nozzle only through a singular point with real exceptional directions, that is, through a saddle point or a nodal point.

The coordinates of the singular points of Eq. (156) are specified by the equations

$$y^* = 1, \quad F'(x^*) = \gamma q,$$

which imply that these points are situated in the diverging part of the nozzle. A saddle point appears if J^* is negative, that is, $F''(x^*) > 0$. Since q is a sufficiently small constant, a saddle point appears near the throat of the nozzle. A nodal point, on the other hand, appears only if $F''(x^*)$ is negative. Thus, a nodal point emerges in the part of the nozzle that lies behind an inflection point of the nozzle's profile or, in practical terms, at a certain distance from the throat of the nozzle, provided that the profile contains an inflection point.

From the characteristic equation

$$F(x^*)\lambda^2 + 2q\gamma(\gamma+1)\lambda$$
$$-2(\gamma+1)F''(x^*) = 0$$

we can see that the slopes of the two exceptional directions are opposite in sign in the case of a saddle point and have the same sign (are negative) in the case of a nodal point. This means that only a saddle point allows for a transition from supersonic to subsonic velocities and from subsonic to supersonic velocities (Figure 72). The case of a nodal point (Figure 73) allows for a continuous transition only from supersonic to subsonic flow.

Since Eq. (156) cannot be integrated in closed form, we must employ numerical methods of integration in any further discussion. It is advisable in this connection to begin the construction of the four separatrices of a saddle point as integral curves by allowing for the fact that the singular point itself is a point, so to say, from which these integral curves emerge. Such a construction is indeed possible since the characteristic equation provides us with the direction of the two tangents at the singular point $S(x^*, 1)$. If this is ignored and the motion monitored as beginning at points a and b in Figure 72, which lie on different sides of a separatrix, the corresponding points move along curves α and β that strongly diverge and, hence, provide no in-

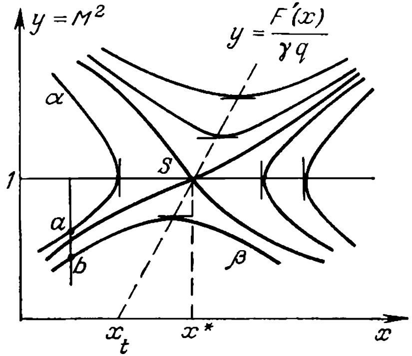

Fig. 72

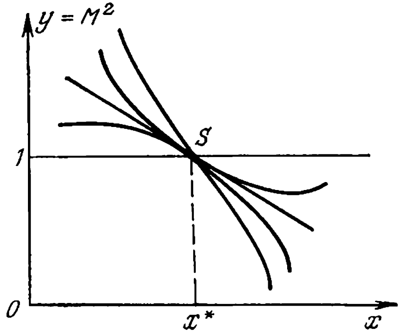

Fig. 73

formation about the integral curve (the separatrix) that "enters" point S. On the other hand, if we move along an integral curve that "emerges" from point S and we assume that the initial segment of the curve coincides with a segment of an exceptional straight line, the error may be minimized if we allow for the convergence of integral curves in the direction in which the values of x diminish.

Figure 72 illustrates the pattern of integral curves in the vicinity of a singular point. The straight line passing through point x_t (the throat of the nozzle) corresponds to values at which the numerator in the right-hand side of Eq. (156) vanishes, which points to the presence of extrema.

2.10 Higher-Order Points of Equilibrium

In previous sections we studied the types of singular points that emerge when the Jacobian J^* is nonzero. But suppose that all the partial derivatives of the functions X and Y in the right-hand sides of system (122) vanish up to the nth order inclusive. Then in the vicinity of a singular point there may be an infinitude of phase-trajectory patterns. However, if we exclude the possibility of equilibrium points of the vortex and focal types emerging in this picture, then it appears that the neighborhood of a singular point can be broken down into a finite number of sectors belonging to three standard types. These are the hyperbolic, parabolic, and elliptic. Below we describe these sectors in detail, but first let us make several assumptions to simplify matters.

We assume that the origin is shifted to the singular point, that is, $x^* = y^* = 0$; the right-hand sides of system (122) can be written in the form

$$\begin{aligned} X(x, y) &= X_n(x, y) + \Phi(x, y), \\ Y(x, y) &= Y_n(x, y) + \Psi(x, y), \end{aligned} \tag{157}$$

where X_n and Y_n are polynomials of degree n homogeneous in variables x and y (one of these polynomials may be identically zero), and the functions Φ and Ψ have in the neighborhood of the origin continuous first partial derivatives. In addition, we assume that the functions

$$\frac{\Phi(x, y)}{(x^2+y^2)^{(n+1)/2}}, \quad \frac{\Phi_x(x, y)}{(x^2+y^2)^{n/2}}, \quad \frac{\Phi_y(x, y)}{(x^2+y^2)^{n/2}},$$

$$\frac{\Psi(x, y)}{(x^2+y^2)^{(n+1)/2}}, \quad \frac{\Psi_x(x, y)}{(x^2+y^2)^{n/2}}, \quad \frac{\Psi_y(x, y)}{(x^2+y^2)^{n/2}}$$

are bounded in the neighborhood of the origin. Under these assumptions the following assertions hold true.

(1) Every trajectory of the system of equations (122) with right-hand sides of the (157) form that "enters" the origin along a certain tangent touches one of the ex-

ceptional straight lines specified by the equation

$$xY_n(x, y) - yX_n(x, y) = 0. \qquad (158)$$

Since the functions X_n and Y_n are homogeneous, we can rewrite Eq. (158) as an equation for the slope $\lambda = y/x$. Then the exceptional straight line is said to be singular if

$$X_n(x, y) = Y_n(x, y) = 0$$

on this straight line. Some examples of such straight lines are shown in Figure 62. The straight lines defined by Eq. (158) but not singular are said to be regular.

(2) The pattern of the phase trajectories of (122) in the vicinity of one of the two rays that "emerge" from the origin and together form an exceptional straight line can be studied by considering a small disk (centered at the origin) from which we select a sector limited by two radii lying sufficiently close to the ray on both sides of it. Such a sector is commonly known as a *standard domain*.

More than that, in the case of a regular exceptional straight line, which corresponds to a linear factor in Eq. (158), the standard domain considered in a disk of a sufficiently small radius belongs to either one of two types: attractive or repulsive.

(2a) The attractive standard domain is characterized by the fact that each tra-

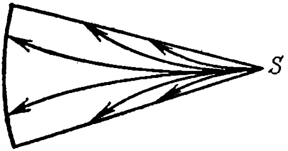

Fig. 74

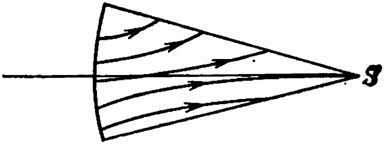

Fig. 75

jectory passing through it reaches the origin along the tangent that coincides with the exceptional straight line (Figure 74).

(2b) The repulsive standard domain is characterized by the fact that only one phase trajectory passing through it reaches the singular point along the tangent that coincides with the exceptional straight line. All other phase trajectories of (122) that enter the standard domain through the boundary of the disk leave the disk by crossing one of the radii that limit the domain (Figure 75).

Let us turn our attention to the following fact. If the disk centered at the origin is small enough, the two types of standard domains can be classified according to the behavior of the vector (X, Y) on the boundary of the domain. The behavior of vector (X, Y) can be identified here with the behavior of vector (X_n, Y_n). More than that, it can be demonstrated that if, as assumed, a fixed exceptional direction does not correspond to a multiple root of the characteristic equation, the vector considered on one of the radii that limit the domain is directed either inward or outward. Then if in the first case the vector considered on the part of the boundary of the standard domain that is the arc of the circle is also directed inward, and in the second case outward, the standard domain is attractive. But if the opposite situation is true, the standard domain is repulsive. It must be noted that in any case the vector considered on the part of the boundary that is the arc of the circle is always directed either inward or outward since it is almost parallel to the radius.

Standard domains corresponding to singular exceptional directions or multiple roots of the characteristic equation have a more complicated nature, but since to some extent they constitute a highly rare phenomenon, we will not describe them here.

If we now turn, for example, to a saddle point, we note that it allows for four repulsive standard domains. In the neighborhood of a singular point of the nodal type there are two attractive standard domains and two repulsive.

(3) If there exist real-valued exceptional-direction straight lines, the neighborhood of a singular point can be divided into a finite number of sectors each of which is bounded by the two phase trajectories of (122) that "enter" the origin along definite tangents. Each of such sectors belongs to one of the following three types.

(3a) The elliptic sector (Figure 76) contains an infinitude of phase trajectories in the form of loops passing through the origin and touching on each side of the boundary of the sector.

(3b) The parabolic sector (Figure 77) is filled with phase trajectories that connect the singular point with the boundary of the neighborhood.

(3c) The hyperbolic sector (Figure 78) is filled with phase trajectories that approach the boundary of the neighborhood in both directions.

More precisely:

(4a) Elliptic sectors are formed between two phase trajectories belonging to two successive standard domains, both of which are attractive.

(4b) Parabolic sectors are formed between two phase trajectories belonging to two

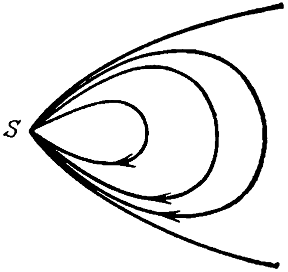

Fig. 76

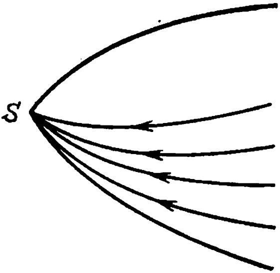

Fig. 77

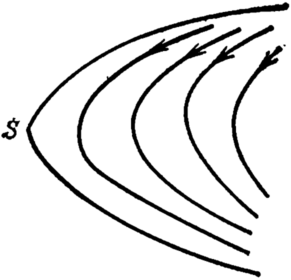

Fig. 78

successive standard domains one of which is attractive and the other repulsive. All phase trajectories that pass through the latter domain touch at the singular point of the exceptional straight line that defines the attractive domain.

(4c) Hyperbolic sectors are formed between two phase trajectories belonging to two successive repulsive standard domains.

For example, it is easy to distinguish the four hyperbolic sectors at a saddle point and the four parabolic sectors at a nodal point. Elliptic sectors do not appear in the case of simple singular points, where the Jacobian J^* is nonzero.

If a singular point does not allow for the existence of real exceptional directions, the phase trajectories in its neighborhood always possess the vortex or focal structure.*

2.11 Inversion with Respect to a Circle and Homogeneous Coordinates

Above we described methods for establishing the local behavior of phase trajectories of differential systems of the (122) type in the neighborhood of singular points. And although in many cases all required information can be extracted by following these methods, there may be a need to study the

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* Methods that make it possible to distinguish between a vortex point and a focal point are discussed, for example, in the book by V.V Amel'kin, N.A. Lukashevich, and A.P. Sadovskii, *Nonlinear Vibrations in Second-Order Systems* (Minsk: Belorussian Univ. Press, 1982) (in Russian).

trajectory behavior in infinitely distant parts of the phase plane, as $x^2 + y^2 \to \infty$. A simple way of studying the asymptotic behavior of the phase trajectories of the differential system (122) is to introduce the point at infinity by transforming the initial differential system in an appropriate manner, say by inversion, which is defined by the following formulas:

$$x = \frac{\xi}{\xi^2+\eta^2}, \quad y = \frac{\eta}{\xi^2+\eta^2}$$
$$\left(\xi = \frac{x}{x^2+y^2}, \quad \eta = \frac{y}{x^2+y^2}\right). \tag{159}$$

Geometrically this transformation constitutes what has become known as *inversion with respect to a circle* and maps the origin into the point at infinity and vice versa. Transformation (159) maps every finite point M (x, y) of the phase plane into point M' (ξ, η) of the same plane, with points M and M' lying on a single ray that emerges from the origin and obeying the condition $OM \times OM' = r^2$ (Figure 79). It is well known that such a transformation maps circles into circles (straight lines are considered circles passing through the point at infinity). For one thing, straight lines passing through the origin are invariant under transformation (159). Hence, the slopes of asymptotic directions are the slopes of tangents at the new origin $\xi = \eta = 0$. Note that in the majority of

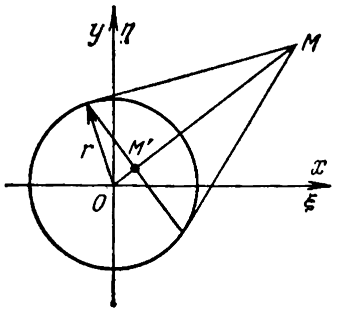

Fig. 79

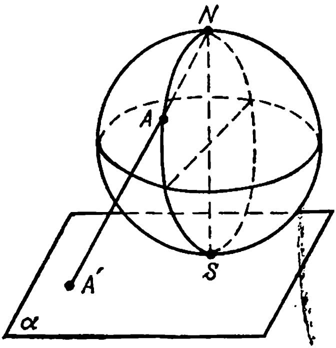

Fig. 80

cases the new origin serves as a singular point. The reasons for this are discussed below.

As for the question of how to construct in the old (x, y)-plane a curve that has a definite asymptotic direction, that is, has a definite tangent at the origin of the new (ξ, η)-plane, this can be started by considering the (ξ, η)-plane, more precisely, by considering this curve in, say, a unit circle in the (ξ, η)-plane. The fact is that since a unit circle is mapped, via transformation (159), into itself, we can always establish the point where the curve intersects the respective unit circle in the (x, y)-plane; any further investigations can be carried out in the usual manner.

We also note that completion of the (x, y)-plane with the point at infinity is topologically equivalent to inversion of the stereographic projection (Figure 80), in which the points on a sphere are mapped onto a plane that is tangent to the sphere at point S. The projection center N is the antipodal point of S. It is clear that the projection center N corresponds to the point at infinity in the (x, y)-plane. Conversely, if we map the plane onto the sphere, a vector field on the plane transforms into a vector field on the sphere and the point at infinity may prove to be a singular point on the sphere.

Although inversion with respect to a circle is useful, it proves cumbersome and inconvenient when the point at infinity has a complicated structure. In such cases another, more convenient, transformation of the (x, y)-plane is used by introducing *homogeneous coordinates*:

$$x = \xi/z, \quad y = \eta/z.$$

Under this transformation, each point of the (x, y)-plane is associated with a triple of real numbers (ξ, η, z) that are not simultaneously zero and no difference is made between the triples (ξ, η, z) and $(k\xi, k\eta, kz)$ for every real $k \neq 0$. If a point (x, y) is not at infinity, $z \neq 0$. But if $z = 0$, we have a straight line at infinity. The (x, y)-plane completed with the straight line at infinity is called the *projective plane*. Such a straight line may carry several singular points, and the nature of these points is usually simpler than that of a singular point introduced by inversion with respect to a circle.

If we now consider a pencil of lines and describe its center with a sphere of, say, unit radius, then each line of the pencil intersects the sphere at antipodal points. This implies that every point of the projective plane is mapped continuously and in a one-to-one manner onto a pair of antipodal points on the unit sphere. Thus, the projective plane may be interpreted as the set of all pairs of antipodal points on a unit sphere. To visualize the projective plane, we need only consider one-half of the sphere and assume its points to be the points of the

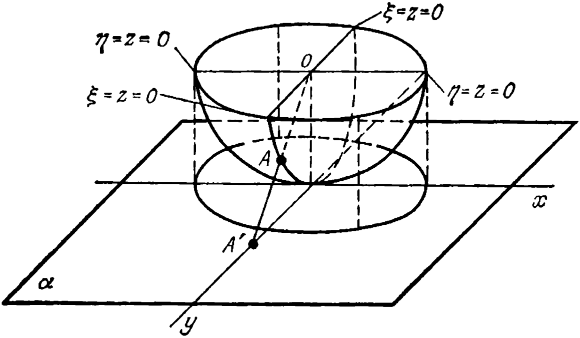

Fig. 81

projective plane. If we orthogonally project this hemisphere (say, the lower one) onto the α-plane, which touches it at pole S (Figure 81), the projective plane is mapped onto a unit disk whose antipodal points on the boundary are assumed identical. Each pair of the antipodal points of the boundary corresponds to a line at infinity, and the completion of the Euclidean plane with this line transforms the plane into a closed surface, the projective plane.

2.12 Flow of a Perfect Gas Through a Rotating Tube of Uniform Cross Section

In some types of turboprop helicopters and airplanes and in jet turbines, the gaseous fuel-air mixture is forced through rotating

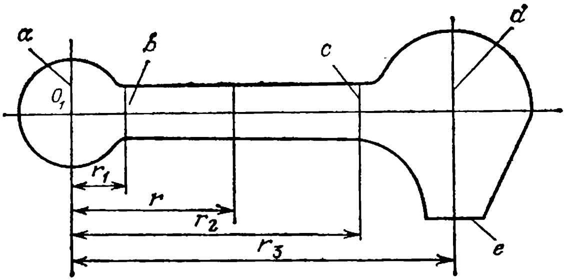

Fig. 82

tubes of uniform cross section installed in the compressor blades and linked by a hollow vertical axle. To establish the optimal conditions for rotation we must analyze the flow of the mixture through a rotating tube and link the solution with boundary conditions determined by the tube design. In a blade the gaseous mixture participates in a rotational motion with respect to the axis with constant angular velocity ω and moves relative to the tube with an acceleration $v\,(dv/dr)$, where v is the speed of a gas particle with respect to the tube, and r is the coordinate measured along the rotating compressor blades.

Figure 82 depicts schematically a single rotating tube of a compressor blade.* It is

* See J. Kestin and S.K. Zaremba, "Adiabatic one-dimensional flow of a perfect gas through a rotating tube of uniform cross section," *Aeronaut. Quart.* 4: 373-399 (1954).

assumed that the fuel-air mixture, whose initial state is known, is supplied along the hollow axle to a cavity on the axle in which the flow velocity may be assumed insignificant. The boundary of the cavity occupied by the gas is denoted by a, the gas is assumed perfect with a specific heat ratio γ, and all the processes that the gas mixture undergoes are assumed reversibly adiabatic (exceptions are noted below).

It is assumed that the gas expands through a nozzle with an outlet cross section b that at the same time is the inlet of a tube of uniform cross section. The expansion of the gas mixture from state a to state b is assumed to proceed isentropically; we denote the velocity of the gas after expansion by v_1 and the distance from the rotation axis O_1 to the cross section b by r_1.

When passing through the tube, whose uniform cross-sectional area will be denoted by A and whose hydraulic diameter by D, the gas is accelerated thanks to the combined action of the pressure drop and dynamical acceleration in the rotation compressor blade. We ignore here the effect produced by pressure variations in the tube (if they exist at all) and by variations in pressure drop acting on the cross section plane, both of which are the result of the Coriolis force. The last assumption, generally speaking, requires experimental verification, since the existence of a lateral pressure drop may serve as a cause of secondary flows. But if the diameter of the tube is small compared to the tube's length, such an assumption is justified.

It is now clear that the equations of momentum and energy balance for a compressible mixture traveling along a tube of uniform cross section must be modified so as to allow for forces of inertia that appear in a rotating reference frame. As for the continuity equation, it remains the same.

Now let us suppose that starting from cross section c at the right end of the tube the gas is compressed isentropically and passes through a diverging nozzle. In the process it transfers into a state of rest with respect to the compressor blade in the second cavity at a distance r_3 from the rotation axis and reaches a state with pressure P_d and temperature T_d.

From the second cavity the gaseous mixture expands isentropically into a converging or converging-diverging nozzle in such a manner that it leaves the cavity at right angles to the tube's axis. This produces a thrust force caused by the presence of a torque.

Below for the sake of simplicity we assume that the exit nozzle is a converging one and has an exit (throat) of area A^*. Denoting the external (atmospheric) pressure by P_a, we consider two modes of passage of the mixture through the nozzle.

The first applies to a situation in which the P_a-to-P_d ratio exceeds the critical value, or

$$P_a/P_d > (2/(\gamma + 1))^{\gamma/(\gamma-1)}.$$

In this case the flow at the nozzle's throat is subsonic and, hence, pressure P_3 at the throat is equal to the atmospheric pressure, that is, $P_3 = P_a$. The second case applies to a situation in which the P_a-to-P_d ratio is below the critical value, or

$$P_a/P_d < (2/(\gamma + 1)^{\gamma/(\gamma-1)}.$$

The pressure P_3 at the nozzle's throat has a fixed value that depends on P_d but not on P_a. Hence,

$$P_3 = (2/(\gamma + 1))^{\gamma/(\gamma-1)} P_d.$$

In the latter case the flow at the nozzle's throat has the speed of sound

$$v_3 = (2/(\gamma + 1))^{1/2} a_d,$$

where a_d depends only on temperature T_d.

In further analysis the flow is assumed adiabatic everywhere and isentropic everywhere except in the tube between cross sections b and c.

As in the case where we derive the differential equation that describes the adiabatic flow of a perfect gas through a nozzle of varying cross section, let us now consider the continuity equation, the equation of

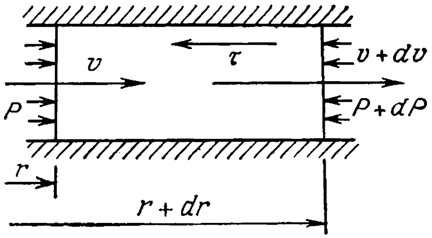

Fig. 83

momentum, and the equation of energy balance. For instance, the equation of continuity in this case has the form

$$\psi = \frac{v}{V} = \frac{v_3}{V_3}\,\frac{A^*}{A} = \text{const}, \tag{160}$$

with V the specific volume and ψ the flux density, or

$$\psi = m'/A,$$

where m' is the flux mass.

To derive the equation of motion, or the momentum equation, we turn to Figure 83. We note that the dynamical effect of the rotational motion of the compressor blade can be used to describe the flow with respect to the moving tube, where in accordance with the D'Alembert principle the force of inertia

$$dI = \frac{A}{V} w^2 r \, dr$$

is assumed to act in the positive direction of r. Hence, the element of mass $\mathrm{d}m = (A/V)\,\mathrm{d}r$ moves with an acceleration $v\,(\mathrm{d}v/\mathrm{d}r)$ caused by the combined action of the force of inertia $\mathrm{d}I$, the force of pressure $A\,\mathrm{d}P$, and the friction force $\mathrm{d}F = (4A/D)\,\mathrm{d}r$. Here $\tau = \lambda\,(v^2/2V)$, where λ depends on the Reynolds number R. As a first approximation we can assume that λ remains constant along the entire tube. With this in mind, we can write the momentum equation in the form

$$\frac{A}{V}\,\mathrm{d}r\,v\frac{\mathrm{d}v}{\mathrm{d}r} = -A\,\mathrm{d}P - \frac{2\lambda A v^2}{VD}\,\mathrm{d}r + \frac{A}{V}\,w^2 r\,\mathrm{d}r,$$

or after certain manipulations,

$$V\,\mathrm{d}P + v\,\mathrm{d}v + \frac{2\lambda}{D}\,v^2\mathrm{d}r - w^2 r\,\mathrm{d}r = 0. \qquad (161)$$

As for the energy-balance equation, it is simple to derive if we use the first law of thermodynamics for open systems and bear in mind that the amount of work performed by the system is $w^2 r\,\mathrm{d}r$. Thus

$$\mathrm{d}h + v\,\mathrm{d}v - w^2 r\,\mathrm{d}r = 0,$$

where h is the enthalpy. If we define the speed of sound a via the equation

$$h = \frac{a^2}{\gamma - 1},$$

we find that

$$a\,\mathrm{d}a + \frac{\gamma-1}{2}\,v\,\mathrm{d}v - \frac{\gamma-1}{2}\,w^2 r\,\mathrm{d}r = 0.$$

This leads us to the following energy-balance equation:

$$a_0^2 = a^2 + \frac{\gamma - 1}{2} v^2 - \frac{\gamma - 1}{2} w^2 r^2. \tag{162}$$

If we now introduce the dimensionless quantities

$$M_0 = v/a_0, \quad x = r/D, \quad G^2 = w^2 D^2/a_0^2, \tag{163}$$

we arrive at the equation

$$\frac{a_0^2}{a^2} = 1 - \frac{\gamma - 1}{2} M_0^2 + \frac{\gamma - 1}{2} G^2 x^2. \tag{164}$$

By excluding the pressure and specific volume from the basic equations (160), (161), and (164) we can derive an equation that links the dimensionless quantity M_0 with the dimensionless distance. This equation serves as the basic equation for solving our problem. The continuity equation (160) implies that

$$V = v/\psi. \tag{165}$$

Then, combining

$$a^2 = \gamma P V = \frac{\gamma P v}{\psi}$$

with Eq. (162), we arrive at the following relationships:

$$P = \left(\frac{a_0^2}{\gamma v} - \frac{\gamma - 1}{2\gamma} v + \frac{\gamma - 1}{2\gamma} \frac{w^2}{v} r^2 \right) \psi,$$

$$\frac{dP}{dr} = \frac{\gamma - 1}{\gamma} \frac{w^2 r}{v} - \left(\frac{\gamma - 1}{2\gamma} + \frac{a_0^2}{\gamma v^2} + \frac{\gamma - 1}{2\gamma} \frac{w^2 r^2}{v^2} \right) \frac{dv}{dr}. \tag{166}$$

Substituting the value of V from Eq. (165) and the value of $\mathrm{d}P/\mathrm{d}r$ from Eq. (166) into the momentum equation (161) and allowing for the relationship (163), we arrive at the differential equation

$$\frac{\mathrm{d}M_0^2}{\mathrm{d}x}=\frac{2M_0^2\,(2\lambda\gamma M_0^2-G^2x)}{1-\frac{\gamma+1}{2}M_0^2+\frac{\gamma-1}{2}G^2x^2}. \tag{167}$$

Assuming that

$$M_0^2=y,\quad m=2\lambda\gamma,\quad p=\frac{1}{2}(\gamma+1),$$

$$q=\frac{1}{2}(\gamma-1),$$

we can rewrite Eq. (167) as

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2y\,(my-G^2x)}{1-py+qG^2x^2}. \tag{168}$$

The differential equation (168) is the object of our further investigation. Introducing the variables (159), we reduce it to

$$\frac{\mathrm{d}\eta}{\mathrm{d}\xi}=\frac{2\xi\eta\,(\Xi^2-p\eta\Xi+qG^2\xi^2)+2\eta\,(\eta^2-\xi^2)\,\Omega}{\Lambda\,(\Xi^2-p\eta\Xi+qG^2\xi^2)+4\xi\eta^2\Omega}, \tag{169}$$

where $\Xi=\xi^2+\eta^2$, $\Omega=m\eta-G^2\xi$, $\Lambda=\xi^2-\eta^2$. Clearly, the origin constitutes a singular point for Eq. (169). The lowest-order terms in both numerator and denominator are quadratic. If we discard higher-order terms, as we did on p. 211 when discussing higher-order equilibrium points, we find that

$$Y_4(\xi, \eta) = 2qG^2\xi^3\eta + 2\eta(\eta^2 - \xi^2)(m\eta - G^2\xi),$$

$$X_4(\xi, \eta) = qG^2\xi^2(\xi^2 - \eta^2) + 4\xi\eta^2(m\eta - G^2\xi),$$

and, hence, the origin is a higher-order singular point.

The characteristic equation of the differential equation (169) assumes the following form after certain manipulations:

$$\xi\eta(\xi^2 + \eta^2)[(q + 2)G^2\xi - 2m\eta] = 0. \tag{170}$$

This leads us to three real-valued exceptional straight lines:

$$\text{(a) } \xi = 0, \quad \text{(b) } \eta = 0, \quad \text{(c) } (q + 2)G^2\xi - 2m\eta = 0. \tag{171}$$

Each is a regular straight line and corresponds to one of the factors in Eq. (170).

It is easy to see that for ξ and η positive the value of $X_4(\xi, \eta)$ is positive in the neighborhood of all three exceptional straight lines, with the result that the expression

$$\frac{Y_4(\xi, \eta)}{X_4(\xi, \eta)} - \frac{\eta}{\xi} = \frac{\xi\eta(\xi^2 + \eta^2)[(q+2)G^2\xi - 2m\eta]}{\xi[qG^2\xi^2(\xi^2 - \eta^2) + 4\xi\eta^2(m\eta - G^2\xi)]} \tag{172}$$

has the same sign in both the first quadrant and the neighborhood of the straight lines as the left-hand side of (171c) and, hence, is negative below the straight line (171c) and positive above. The explanation lies in the fact that (172) may change sign only when an exceptional straight line is crossed. Geometrically this means that the right-hand side of the differential equation

$$\frac{d\eta}{d\xi}=\frac{Y_4(\xi, \eta)}{X_4(\xi, \eta)}$$

determining the behavior of integral curves in a disk of sufficiently small radius and centered at the origin fixes an angle greater than the angle of inclination of the radii that lie in the first quadrant between the straight lines (171b) and (171c). As for the radii lying between the straight lines (171c) and (171a), on them the right-hand side of this differential equation fixes an angle that is smaller than the angle of inclination of these radii (Figure 84). Thus, the standard domain containing the straight line (171c) must be repulsive, while the domains containing the straight lines (171a) and (171b) must be attractive. In view of the symmetry of the field specified by vector (X_4, Y_4), the above facts remain valid for standard domains obtained from those mentioned earlier by a rotation about the singular point through an angle of 180°.

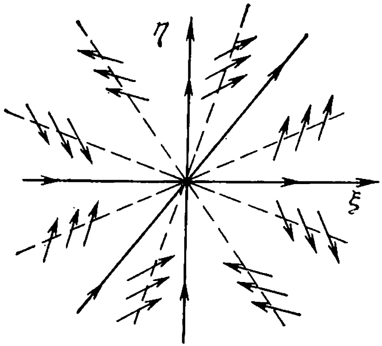

Fig. 84

Thus, there are exactly two integral curves that "enter" the singular point along the tangent (171c) and an infinitude of integral curves that touch the coordinate axes (171a) and (171b) at the point of rest.

We see then that the second and fourth quadrants contain elliptic sectors since they lie between two successive attractive standard domains (Figure 85). Each of the first and third quadrants is divided into two sectors by the integral curves that touch the exceptional straight line (171c) at the origin. These sectors are parabolic because they lie between two successive standard domains, one of which is attractive and the other repulsive. More than that, all the integral curves except those that touch the straight line (171c) touch the

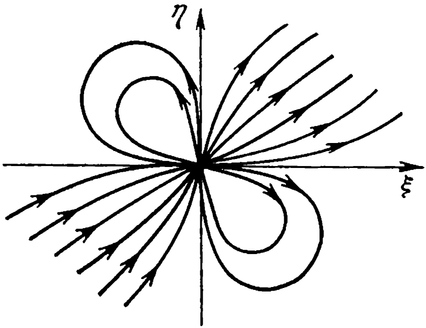

Fig. 85

coordinate axes (171a) and (171b) at the singular point.

Basing our reasoning on physical considerations, we can analyze the differential equation (168) solely in the first quadrant of the (x, y)-plane. Turning to this plane, we see that there exists exactly one integral curve having an asymptotic direction with a slope $(q + 2) G^2/2m$. All other integral curves allow for the asymptotic direction of one of the coordinate axes. Indeed, it is easy to prove that all these integral curves asymptotically approach one of the coordinate axes, that is, along each of them in the movement toward infinity not only does $y/x \to 0$ or $x/y \to 0$ but so does $y \to 0$ or $x \to 0$ (with $x \to \infty$ or $y \to \infty$, respec-

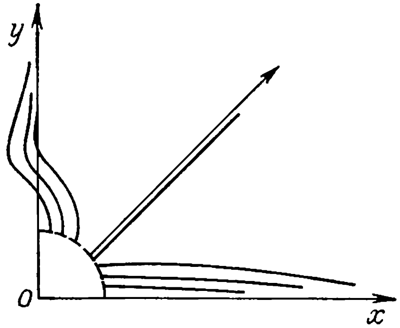

Fig. 86

tively). This is illustrated in Figure 86, where some integral curves have been plotted at great values of x and y. Note that the pattern of integral curves at a finite distance from the origin depends primarily on the position of the singular points, and a special investigation is required to clarify it. It can also be proved that the integral curves that approach the coordinate axes asymptotically (Figure 86) leave the first quadrant at a finite distance from the origin.

2.13 Isolated Closed Trajectories

We already know that in the case of a singular point of the vortex type a certain region of the phase plane is completely filled with closed trajectories. However, a more complicated situation may occur when there is an isolated closed trajectory, that is, a trajectory in a certain neighborhood of which there are no other closed trajectories. This case is directly linked with the existence of isolated periodic solutions. Interestingly, only nonlinear differential equations and systems can have isolated closed trajectories.

Isolated periodic solutions correspond to a broad spectrum of phenomena and processes occurring in biology, radiophysics, oscillation theory, astronomy, medicine, and the theory of device design. Such solutions emerge in differential models in economics, in various aspects of automatic control, in airplane design, and in other fields. Below we study the possibility of isolated periodic solutions emerging in processes that occur in electric circuits; we also consider as a model the nonlinear differential system

$$\begin{aligned} \frac{dx}{dt} &= -y + x(1 - x^2 - y^2), \\ \frac{dy}{dt} &= x + y(1 - x^2 - y^2). \end{aligned} \tag{173}$$

To solve this system, we introduce polar coordinates r and θ, where $x = r\cos\theta$ and $y = r\sin\theta$. Then, differentiating the relationships $x^2 + y^2 = r^2$ and $\theta =$

$\tan^{-1}(y/x)$ with respect to t, we arrive at

$$x\frac{dx}{dt}+y\frac{dy}{dt}=r\frac{dr}{dt},\quad x\frac{dy}{dt}-y\frac{dx}{dt}=r^2\frac{d\theta}{dt}. \tag{174}$$

Multiplying the first equation into x and the second into y, adding the products, and allowing for the first relationship in (174), we find that

$$r\frac{dr}{dt}=r^2(1-r^2). \tag{175}$$

Multiplying the second equation in (173) into x and the first into y, subtracting one product from another, and allowing for the second relationship in (174), we find that

$$r^2\frac{d\theta}{dt}=r^2. \tag{176}$$

System (173) has only one singular point, O (0, 0). Since at the moment we are only interested in constructing trajectories, we can assume that r is positive. Then Eqs. (175) and (176) imply that system (173) can be reduced to the form

$$\frac{dr}{dt}=r(1-r^2),\quad \frac{d\theta}{dt}=1. \tag{177}$$

Each of these equations can easily be integrated and the entire family of solutions, as can easily be seen, is given by the formulas

$$r=\frac{1}{\sqrt{1+Ce^{-2t}}}, \quad \theta=t+t_0, \tag{178}$$

or, in terms of the old variables x and y,

$$x=\frac{\cos(t+t_0)}{\sqrt{1+Ce^{-2t}}}, \quad y=\frac{\sin(t+t_0)}{\sqrt{1+Ce^{-2t}}}$$

If now in the first equation in (178) we put $C=0$, we get $r=1$ and $\theta=t+t_0$. These two relationships define a closed trajectory, a circle $x^2+y^2=1$. If C is negative, it is clear that r is greater than unity and tends to unity as $t\to+\infty$. But if C is positive, it is clear that r is less than unity and tends to unity as $t\to+\infty$. This means that there exists only one closed trajectory $r=1$ which all other trajectories approach along spirals with the passage of time (Figure 87).

Closed phase trajectories possessing such properties are known as *limit cycles* or, more precisely, (*orbitally*) *stable limit cycles.* In fact, there can be two additional types of limit cycles. A limit cycle is said to be (*orbitally*) *unstable* if all neighboring trajectories spiral away from it as $t\to+\infty$. And a limit cycle is said to be (*orbitally*) *half-stable* if all neighboring trajectories on one side (say, from the inner side) spiral on to it and all neighboring trajectories

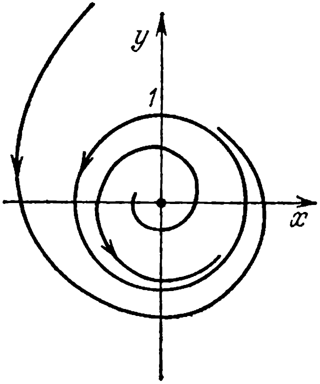

Fig. 87

on the other side (say, on the outer side) spiral away from it as $t \to +\infty$.

In the above example we were able to find in explicit form the equation of a closed phase trajectory, but generally, of course, this cannot be done. Hence the importance in the theory of ordinary differential equations of criteria that enable at least specifying the regions where a limit cycle may occur. Note that a closed trajectory of (122), if such a trajectory exists, contains within its interior at least one singular point of the system. This, for one thing, implies that if there are no singular points of a differential system within a region of the phase plane, there are no closed trajectories in the region either.

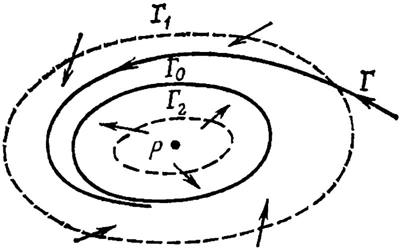

Fig. 88

Let D be a bounded domain that lies together with its boundary in the phase plane and does not contain any singular points of system (122). Then the *Poincaré-Bendixson criterion* holds true, namely, *if* Γ *is a trajectory of* (122) *that at the initial moment* $t = t_0$ *emerges from a point that lies in* D *and remains in* D *for all* $t \geqslant t_0$, *then* Γ *is either closed or approaches a closed trajectory along a spiral with the passage of time.*

We illustrate this in Figure 88. Here D consists of two closed curves Γ_1 and Γ_2 and the circular domain between them. With each boundary point of D we associate a vector

$$\mathbf{V}(x, y) = X(x, y)\,\mathbf{i} + Y(x, y)\,\mathbf{j}.$$

Then, if a trajectory Γ that emerges at the initial moment $t = t_0$ from a boundary point, enters D, and remains there at all moments $t \geqslant t_0$, then, according to the above criterion, it will along a spiral approach a closed trajectory Γ_0 that lies entirely in D. The curve Γ_0 must surround a singular point of the differential system, point P, not lying in D.

The differential system (173) provides a simple example illustrating the application of the above criterion in finding limit cycles. Indeed, system (173) has only one singular point, $O(0, 0)$, and, therefore, the domain D lying between the circles with radii $r = 1/2$ and $r = 2$ contains no singular points. The first equation in (177) implies that dr/dt is positive on the inner circle and negative on the outer. Vector $\mathbf{V}$, which is associated with the points on the boundary of D, is always directed into D. This means that the circular domain lying between circles with radii $r = 1/2$ and $r = 2$ must contain a closed trajectory of the differential system (173). Such a closed trajectory does indeed exist, it is a circle of radius $r = 1$.

Note, however, that great difficulties are generally encountered in a system of the (122) type when we wish to realize practically the Poincaré-Bendixson criterion, since no general methods exist for building the appropriate domains and, therefore, success depends both on the type of system and on the experience of the researcher. At the same time we must bear in mind that finding the conditions in which no limit cycles exist is no less important than establishing criteria for their existence. In this respect the most widespread condition is the *Dulac criterion: if there exists a function* $B(x, y)$ *that is continuous together with its first partial derivatives and is such that in a simply connected domain D of the phase plane the sum*

$$\frac{\partial (BX)}{\partial x} + \frac{\partial (BY)}{\partial y}$$

is a function of fixed sign, then no limit cycles of the differential system* (122) *can exist in D.* At $B(x, y) \equiv 1$ the criterion transforms into the *Bendixson criterion.*

If we turn to the differential equation (156), which constitutes a differential model for describing an adiabatic one-dimensional flow of a perfect gas of constant specific heat ratio through a nozzle with drag, then for this equation we have

$$X(x, y) = (1-y) F(x),$$
$$Y(x, y) = 4y\left(1 + \frac{\gamma-1}{2} y\right)(\gamma q y - F'(x)).$$

If we put

$$B(x, y) = \left\{4yF(x)\left[1 + \frac{\gamma-1}{2} y\right]\right\}^{-1}$$

* That is, positive definite or negative definite.

we find that

$$\frac{\partial (BX)}{\partial x} + \frac{\partial (BY)}{\partial y} \equiv \frac{\gamma q}{F(x)} > 0$$

and, hence, Eq. (156) has no closed integral curves.

Let us discuss one more concept that can be employed to establish the existence of limit cycles. The concept is that of the *index of a singular point.*

Let Γ be a simple closed curve (i.e. a curve without self-intersections) that is not necessarily a phase trajectory of system (122), lies in the phase plane, and does not pass through the singular points of this system. Then, if $P(x, y)$ is a point of Γ, the vector

$$\mathbf{V}(x, y) = X(x, y)\,\mathbf{i} + Y(x, y)\,\mathbf{j},$$

with $\mathbf{i}$ and $\mathbf{j}$ the unit vectors directed along the Cartesian axes, is a nonzero vector and, hence, is characterized by a certain direction specified by an angle θ (Figure 89). If point $P(x, y)$ moves along Γ, say, counterclockwise and completes a full cycle, vector $\mathbf{V}$ performs an integral number of cycles in the process, that is, angle θ acquires an increment $\Delta\theta = 2\pi n$, with n a positive integer, a negative integer, or zero. This number n is said to be the *index of the closed curve* Γ (or the *index of cycle* Γ).

If we begin to contract Γ in such a manner that under this deformation Γ does not

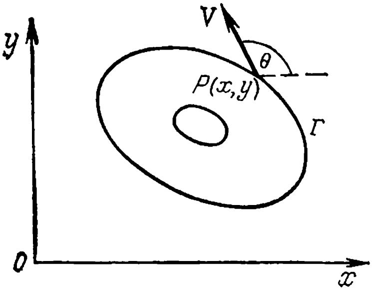

Fig. 89

pass through singular points of the given vector field, the index of the cycle must, on the one hand, vary continuously and, on the other, remain an integer. This means that under continuous deformation of the curve the index of the cycle does not change. This property leads to the notion of the *index of a singular point* as the index of a simple closed curve surrounding the singular point.

The index has the following properties:

(1) *the index of a closed trajectory of the differential system* (122) *is equal to* $+1$,

(2) *the index of a closed curve surrounding several singular points is equal to the sum of the indices of these points, and*

(3) *the index of a closed curve encompassing only ordinary points is zero.*

This implies, for one thing, that since the index of a closed trajectory of system (122) is always +1, a closed trajectory must enclose either one singular point with an index +1 or several singular points with a net index equal to +1. This fact is often used to prove the absence of limit cycles.

The index of a singular point is calculated by the formula

$$n = 1 + \frac{e-h}{2}, \tag{179}$$

where e is the number of elliptic sectors, and h is the number of hyperbolic sectors. For practical purposes the following simple method may be suggested. Suppose that L is a cycle that does not pass through singular points of (122) and is such that any trajectory of (122) has no more than a finite number of points common to L. The trajectories may intersect L or touch it. In the latter case only exterior points of contact (type A) or interior points of contact (type B) are taken into account, while the C-type points, points of inflection, are not (see Figure 90). We can still use formula (179) to calculate the index of a singular point, but now e is the number of interior points of contact and h the number of exterior points of contact of trajectories of (122) with cycle L.

Figure 91 depicts singular points with

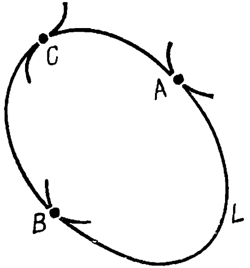

Fig. 90

indices 0, +2, +3, +1, and −2, respectively.

Earlier it was noted that constructing a complete picture of the behavior of the phase trajectories of the differential system (122) is facilitated by introducing the point at infinity via transformations (159). Topological considerations provide a very general theorem which states that when a continuous vector field with a finite number of singular points is specified on a sphere, the net index of the points is +2. Thus, if the net index of all the singular points of a differential system (possessing a finite number of such points) that lie in a finite phase-plane domain is distinct from +2, the point at infinity must be a singular point with a nonzero index.

But if instead of inversion we employ homogeneous coordinates, the net index of all the singular points is already +1. That

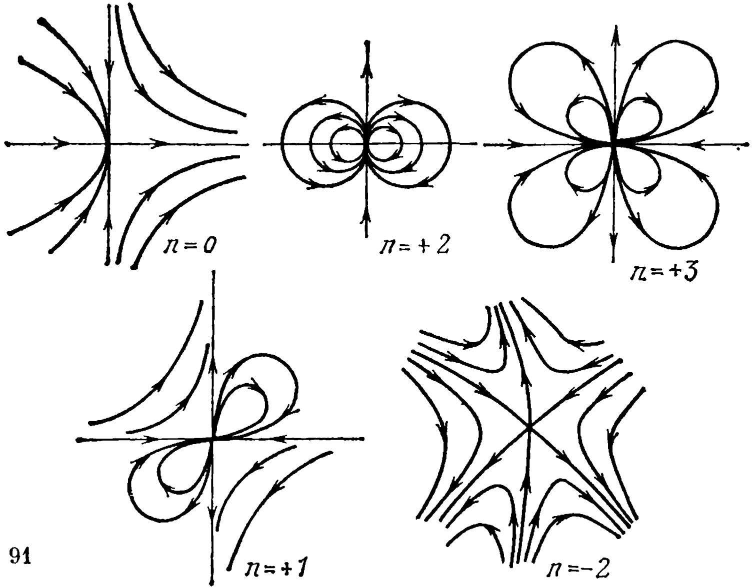

Fig. 91

this is so can be seen from the fact that if the plane is projected onto a sphere with the center of projection placed at the center of the sphere, two points on the sphere correspond to a single point on the projective plane, and the circumference of the great circle parallel to the plane corresponds to a straight line at infinity.

If we turn to Eq. (168), which describes adiabatic one-dimensional flow of a perfect gas through a rotating tube of uniform cross section, then, as shown in Figure 85, for the singular point at infinity we have $e = 2$ and $h = 0$. It follows from this that the index of this point is $+2$ and does not depend on the values of the constants in Eq. (168). This implies, for one thing, that the net index of finite singular points is zero. It can also be shown that, depending on whether the straight line specified by the equation $my - Gx^2 = 0$ has two points, one double point, or not a single point of intersection with the parabola fixed by the equation $1 - py + qG^2x^2 = 0$ (which, in turn, is equivalent to G being greater than G_0, equal to G_0, or less than G_0, with $G_0 = 2mq^{1/2}/p$), the following combinations of finite singular points occur.

(a) $G > G_0$. A saddle point and a nodal point.

(b) $G = G_0$. A higher-order singular point with two hyperbolic ($h = 2$) sectors and two parabolic ($e = 0$).

(c) $G < G_0$. Singular points are absent.

We see that in three cases the net index is zero, as it should be.

2.14 Periodic Modes in Electric Circuits

We will show how limit cycles emerge in a dynatron oscillator (Figure 92). An analysis of the operation of such an oscillator leads to what is known as the Van der Pol equation. Although phenomena linked with generation of limit cycles can be illustrated by examples from mechanics, biology, and economics,[1] we will show how such phenomena emerge in the study of electric circuits.

Figure 92 depicts schematically a dynatron oscillator, with the i_a-v_a characteristic shown by a solid curve in Figure 93. Here i_a is the current and v_a the voltage in the

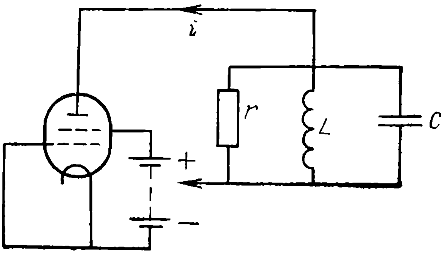

Fig. 92

screen-grid tube, the dynatron. The circuit consists of resistance r, inductance L, and capacitance C connected in parallel and known as the tank circuit, which is connected in series with the screen-grid tube. The real circuit can be replaced in this case with an equivalent circuit shown schematically in Figure 94. The characteristic of the tube may be approximated with a third-degree polynomial $i = \alpha v + \gamma v^3$, which is shown by a dashed curve in Figure 93. Here i and v stand for the coordinates in a system whose origin is shifted to the point of inflection O. As follows from Figure 93,

$$\alpha > 0, \quad \gamma > 0.$$

In accordance with one of Kirchhoff's laws,

$$i + i_r + i_L + i_C = 0,$$

with $i_r = v/r$, $L\,(di_L/dt) = v$, and $i_C = C\dot{v}$. As a result of simple manipulations we arrive at the following differential equation:

$$\ddot{v} + \left(\frac{\alpha}{C} + \frac{1}{rC} + \frac{3\gamma}{C}v^2\right)\dot{v} + \frac{1}{LC} = 0.$$

If we now put

$$\frac{\alpha}{C} + \frac{1}{rC} = a, \quad \frac{3\gamma}{C} = b, \quad \frac{1}{LC} = \omega_0^2,$$

we can write the previous equation in the form

$$\ddot{v} + (a + bv^2)\,\dot{v} + \omega_0^2 v = 0.$$

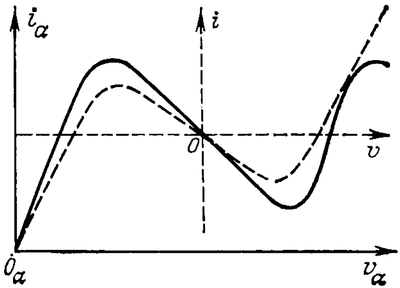

Fig. 93

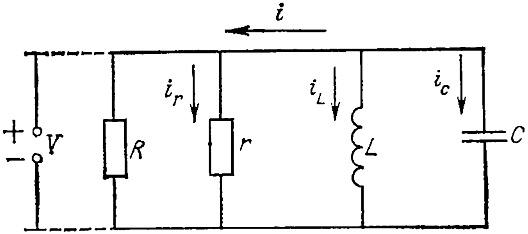

Fig. 94

This differential equation is known as the *Van der Pol equation.* If we introduce the transformations

$$\dot{v} = y, \quad \ddot{v} = y \frac{dy}{dv},$$

we can associate with the Van der Pol

equation the following first-order differential equation:

$$\frac{dy}{dv} = -\frac{(a+bv^2)\,y+\omega_0^2 v}{y} = \frac{Y(v,\ y)}{V(v,\ y)} \qquad (180)$$

The only finite singular point of this equation is the origin, and

$$J^*(0,\ 0) = \omega_0^2 > 0,$$

$$D(v,\ y) = -(a + bv^2).$$

Since $b > 0$, we assume that $a > 0$ and conclude that divergence D does not reverse its sign and, hence, Eq. (180) can have no closed integral curves. We, therefore, will consider only the case where a is negative, that is, $\alpha < -1/r$. This implies that $D(0,\ 0) = -a > 0$ and, hence, the singular point is either a nodal point or a focal point. If we now consider the differential system corresponding to the differential equation (180), that is,

$$\frac{dy}{dt} = Y(v,\ y), \quad \frac{dv}{dt} = V(v,\ y),$$

we see that as t grows, the representative point moves along a trajectory toward the singular point. Thus, a trajectory that emerges from the point at infinity cannot reach the singular point at the origin no matter what the value of t including the case where $t = +\infty$. This implies that if we can prove that a trajectory originating at the point at infinity resembles a spiral winding onto the origin as $t \to \infty$, this will guarantee that at least one limit cycle exists.

The existence proof for such a cycle can be obtained either numerically or analytically. A numerical method suggested by the Dutch physicist and mathematician Van der Pol (1889-1959) consists in building a trajectory that originates at a point positioned at a great distance from the origin and in checking whether this trajectory possesses the above-mentioned property. Such a procedure yields an approximation to the limit cycle, but it can be employed only in the case where concrete numerical values are known.

Below we give a proof procedure* based on analytical considerations and the investigation of the properties of singular points at infinity. Here, in contrast to the method by which Eq. (168) was studied, we employ the more convenient (in the present problem) homogeneous-coordinate transformations

$$v = \xi/z, \quad y = \eta/z. \tag{181}$$

The straight line at infinity is fixed by the equation $z = 0$. To reduce the number of

* See the paper by J. Kestin and S.K. Zaremba, "Geometrical methods in the analysis of ordinary differential equations," *Appl. Sci. Res.* **B3**: 144 189 (1953).

variables to two, we first exclude the point $\xi = z = 0$. This can be done if we assume that $\xi = 1$. Then

$$v = 1/z, \quad y = \eta/z.$$

In this case the differential system associated with Eq. (180) is transformed to

$$\frac{dz}{dt} = -\eta z, \quad \frac{d\eta}{dt} = -\frac{(az^2+b)\eta+\omega_0^2 z^2}{z^2} - \eta^2.$$

It is convenient to introduce a new parameter, θ, in the following manner:

$$dt = z^2\, d\theta. \tag{182}$$

Then this system of equations can be written as

$$\begin{aligned} \frac{d\eta}{d\theta} &= -(az^2+b)\,\eta - \omega_0^2 z^2 - \eta^2 z^2 = \mathcal{Y}, \\ \frac{dz}{d\theta} &= -\eta z^3 = \mathfrak{Z}. \end{aligned} \tag{183}$$

Note, first, that the straight line at infinity, $z = 0$, constitutes a trajectory of the differential system (183) and as t grows (hence, as θ grows) the representative point moves along this trajectory toward the only singular point $z = \eta = 0$. The characteristic equation, which in this case has the form $-b\eta z = 0$, specifies the regular exceptional direction $z = 0$, with two repulsive regions corresponding to this direction, which can be established by studying the appropriate

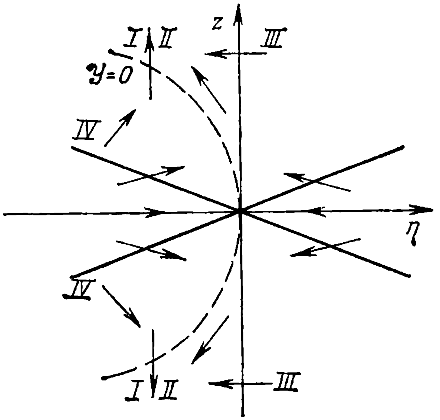

Fig. 95

vector field (Figure 95). The second exceptional direction $\eta = 0$ is singular and, therefore, additional reasoning is required.

The locus of points $\mathcal{Y} = 0$ is a curve that touches the straight line $\eta = 0$ at the origin and passes through the second and third quadrants (Figure 95), which makes it possible to fix three directions, *I*, *II*, and *III*, on different sides of the symmetry axis $z = 0$. The region lying between the curve fixed by the equation $\mathcal{Y} = 0$ and the axis $\eta = 0$ is topologically equivalent to two repulsive regions. Thus, on each side of the straight line $\eta = 0$ there is at least one trajectory that touches the straight line at the origin.

If we now consider the differential equation

$$\frac{d\eta}{dz}=\frac{a}{z}+\frac{b}{z^3}+\frac{\omega_0^2}{\eta z}+\frac{\eta}{z}=f(\eta,\ z),$$

we note that for small values of $|\eta|$

$$\frac{\partial f}{\partial\eta}=\frac{1}{z}\left(1-\frac{\omega_0^2}{\eta^2}\right)<0$$

in the second quadrant between curve $\mathcal{Y}=0$ and axis $\eta=0$. Hence, if we take two phase trajectories with the same value of η, it is easy to see that the representative points moving along these trajectories will also move apart as z decreases. This means that on each side of the straight line $\eta=0$ there can be only one phase trajectory that touches this straight line at the origin and belongs to the region considered. It is also clear that the qualitative picture is symmetric about the axis $z=0$. More than that, since for small values of z and positive η we have

$$|\mathcal{Y}/\mathcal{Z}|\geqslant\omega_0^2/|\eta z|,$$

there can be no curves that touch the axis $\eta=0$ at the origin and pass through the first or fourth quadrant. Such curves are also absent from the region that lies to the left of the curve fixed by the equation $\mathcal{Y}=0$ since in this case $\mathcal{Y}$ is positive and

$\mathcal{Z}$ has the same sign as z (Figure 95, arrows *IV*). This reasoning suggests that the singular point $\eta = z = 0$ is a saddle point. The two phase trajectories that reach this point as $t \to +\infty$ are segments of the straight line at infinity, $z = 0$, that link the previous point with the point $\xi = z = 0$. The other two phase trajectories reach the saddle point as $t \to -\infty$.

In our reasoning we did not discuss the point $\xi = z = 0$. To conclude our investigation, let us put $\eta = 1$ in (181). Then the differential system associated with the differential equation (180) can be written as follows:

$$\frac{d\xi}{d\theta} = z^2 + \xi\,(az^2 + b\xi^2 + \omega_0^2\xi z^2) = P,$$

$$\frac{dz}{d\theta} = z\,(az^2 + b\xi^2 + \omega_0^2\xi z^2) = Q,$$

where the variable θ is defined in (182). The point $\xi = z = 0$ proves to be singular, and the characteristic equation here is $z^3 = 0$. Hence, the exceptional direction $z = 0$ is singular, too. The curve fixed by the equation $P(\xi, z) = 0$, (Figure 96) touches the axis $z = 0$ at the origin and has a cuspidal point there, while the curve fixed by the equation $Q(\xi, z) = 0$ has a multiple point at the origin. By studying the signs of P and Q we can establish the pattern of the vector field (Figure 96) and the phase-tra-

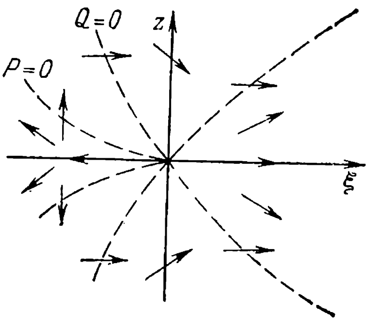

Fig. 96

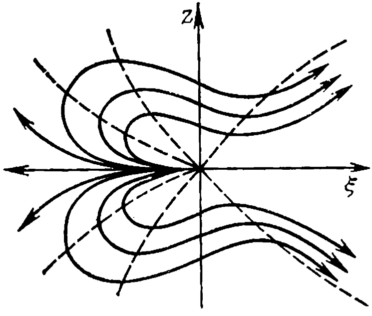

Fig. 97

jectory pattern in the neighborhood of the singular point $\xi = z = 0$ (Figure 97).

We note, for one thing, that far from the axis $\xi = 0$ there are no phase trajectories that enter the singular point from the right.

This follows from the fact that in the first and fourth quadrants

$$|z/\xi| > |Q/P|.$$

The above reasoning shows that the Van der Pol equation has no phase trajectories that tend to infinity as t grows but has an infinitude of phase curves that leave infinity as t grows. This proves the existence of at least one limit cycle for the Van der Pol differential equation.

2.15 Curves Without Contact

In fairly simple cases the complete pattern of the integral curves of a given differential equation or, which is the same, the phase-trajectory pattern of the corresponding differential system is determined by the type of singular points and closed integral curves (phase trajectories), if the latter exist. Sometimes the qualitative picture can be constructed if, in addition to establishing the types of singular points, we can find the curves, separatrices, that link the singular points. Unfortunately, no general methods exist for doing this. Hence, it is expedient in qualitative integration to employ the so-called curves without contact. The reader will recall that a curve or an arc of a curve with a continuous tangent is said to be a *curve* (*arc*) *without contact* if it touches the vector (X, Y) specified by the differential system (122) nowhere. The definition implies that vector (X, Y) must point in one direction on the entire curve. Thus, a curve without contact may be intersected by the phase trajectories of (122) only in one direction when t grows and in the opposite direction when t diminishes. Therefore, knowing the respective curve may provide information about the pattern of a particular part of a phase trajectory.

Various auxiliary inequalities can be used in qualitative integration of differential equations. For example, if two differential equations are known, say,

$$\frac{dy}{dx} = f(x, y), \quad \frac{dy}{dx} = g(x, y)$$

and it is known that $f(x, y) \leqslant g(x, y)$ in a domain D, then, denoting by $y_1(x)$ a solution of the first equation such that $y_1(x_0) = y_0$, with $(x_0, y_0) \in D$, and by $y_2(x)$ a solution of the second equation with the same initial data, we can prove that $y_1(x) \leqslant y_2(x)$ for $x \geqslant x_0$ in D. But if in D we have the strict inequality $f(x, y) < g(x, y)$, then $y_1(x) < y_2(x)$ for $x > x_0$ in D and the curve $y = y_2(x)$ is a curve without contact.

As an example let us consider the differential equation (168). We have already shown that if $G > G_0$, the equation allows for two finite singular points, a saddle point and a nodal point, which are points of intersection of the straight line $my - G^2x = 0$ and the parabola $1 - py + qG^2x^2 = 0$. The segments of the straight line and parabola that link these two finite singular points are curves without contact and they specify a region of the plane which we denote by Δ. If we put

$$X(x, y) = 1 - py + qG^2x^2,$$
$$Y(x, y) = 2y(my - G^2x),$$

it is easy to see that vector (X, Y) is directed on the boundary of Δ outward except at the singular points. Hence, if the representative point emerges from any interior point of Δ and moves along an integral curve with t decreasing, it cannot leave Δ without passing through one of the singular points. But since inside Δ we have $X(x, y) < 0$, the singular point that is attractive is the nodal point.

Finding the slopes of the exceptional directions for the saddle point, we can see that one of the exceptional straight lines passes through Δ. This implies that an integral curve that is tangent to this straight line at the saddle point must enter Δ and then proceed to the nodal point.

2.16 The Topographical System of Curves. The Contact Curve

Earlier we noted that in the qualitative solution of differential equations it is expedient to use curves without contact. It must be noted, however, that there is no general method of building such curves that would be applicable in every case. Hence, the importance of various particular methods and approaches in solving this problem. One approach is linked to the selection of an appropriate topographical system of curves. Here *a topographical system of curves defined by an equation* $\Phi(x, y) = C$, *with* C *a real-valued parameter, is understood to be a family of nonintersecting, enveloping each other, and continuously differentiable simple closed curves that completely fill a doubly connected domain* G *in the phase plane.**

If the topographical system is selected in such a manner that each value of parameter C is associated with a unique curve, then, assuming for the sake of definiteness that the curve corresponding to a definite value of C envelopes all the curves with smaller values of C (which means that the "size" of the curves grows with C) and that no singular points of the corresponding

* Other definitions of a topographical system also exist in the mathematical literature, but essentially they differ little from the one given here.

differential system lie on the curves belonging to the topographical system, we arrive at the following conclusion. If we consider the function $\Phi(\varphi(t), \psi(t))$ where $x = \varphi(t)$ and $y = \psi(t)$ are the parametric equations of the trajectories of the differential system of the (122) type and calculate the derivative with respect to t, that is,

$$\frac{d\Phi(\varphi(t), \psi(t))}{dt} = \frac{\partial\Phi(x, y)}{\partial x} X(x, y) + \frac{\partial\Phi(x, y)}{\partial y} Y(x, y),$$

then the curves (cycles in our case) without contact are the curves of the topographical system on which the derivative $d\Phi/dt$ is a function of fixed sign. If, in addition, $d\Phi/dt$ is positive on a certain curve belonging to the topographical system, then such a curve, being a cycle without contact, possesses the property that all phase trajectories intersecting it leave, with the passage of time t, the finite region bounded by it. And if $d\Phi/dt$ is negative, all the phase trajectories enter this region. This implies, for one thing, that if in an annular region completely filled with curves belonging to the topographical system the derivative $d\Phi/dt$ is a function of fixed sign, the region cannot contain any closed trajectories, say, the limit cycles of the differential system. Limit cycles may exist only in the annular regions where the derivative $d\Phi/dt$ is a function with alternating signs.

To illustrate this reasoning, we consider the differential system

$$\frac{dx}{dt}=y-x+x^3, \quad \frac{dy}{dt}=-x-y+y^3, \qquad (184)$$

which has only one singular point in the finite part of the plane, $O\ (0,\ 0)$, a stable focal point. For the topographical system of curves we select the family of concentric circles centered at point $O\ (0,\ 0)$, that is, the family of curves specified by the equation $x^2+y^2=C$, with C a positive parameter. In our case the derivative $d\Phi/dt$ is given by the relationship

$$\frac{d\Phi}{dt}=-2(x^2+y^2)+2(x^4+y^4),$$

which in polar coordinates $x=r\cos\theta$ and $y=r\sin\theta$ assumes the form

$$\frac{d\Phi}{dt}=-2r^2+2r^4(\cos^4\theta+\sin^4\theta)=-2r^2$$
$$+2r^4\left(\frac{3}{4}+\frac{1}{4}\cos 4\theta\right).$$

Noting that the greatest and smallest values of the expression in parentheses are 1 and 1/2, respectively, we arrive a the following conclusion. For r greater than $\sqrt{2}$ the value of the derivative $d\Phi/dt$ is positive while for r less than unity it is negative. On the basis of the Poincaré-Bendixson criterion we conclude (if we replace t with $-t$ in system (184)) that the annular region bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$ contains a limit cycle of system (184), and this limit cycle is unique.

To prove the uniqueness it is sufficient to employ the *Dulac criterion for a doubly connected domain: if there exists a function* $B(x, y)$ *that is continuous together with its first partial derivatives and is such that in a doubly connected domain G belonging to the domain of definition of system* (122) *the function*

$$\frac{\partial (BX)}{\partial x} + \frac{\partial (BY)}{\partial y}$$

is of fixed sign, then in G there can be no more than one simple closed curve consisting of trajectories of (122) *and containing within it the interior boundary of G.*

In our case, selecting $B(x, y) \equiv 1$ for system (184), we find that $\partial X/\partial x + \partial Y/\partial y = 3(x^2 + y^2) - 2$ and, as can easily be seen, in the annular region with boundaries $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$ the expression $3(x^2 + y^2) - 2$ always retains its sign. Allowing now for the type of the singular point $O(0, 0)$, we conclude that the cycle is an unstable limit cycle.

To return to the problem of building curves without contact in the general case, let us examine a somewhat different way of using the topographical system of curves. This approach is based on the notion of a contact curve. Introduction of this concept can be explained by the fact that if the derivative $d\Phi/dt$ vanishes on a set of points of the phase plane, this set constitutes the locus of points at which the trajectories of the differential system touch the curves belonging to the topographical system. Indeed, the slope of the tangent to a trajectory of the differential system is Y/X and the slope of the tangent to a curve belonging to the topographical system is $-(\partial\Phi/\partial x)/(\partial\Phi/\partial y)$. Thus, when

$$\frac{\partial\Phi}{\partial x}X+\frac{\partial\Phi}{\partial y}Y=0, \tag{185}$$

the slopes coincide, or

$$\frac{Y}{X}=-\frac{\partial\Phi}{\partial x}\Big/\frac{\partial\Phi}{\partial y}.$$

Hence, the locus of points at which the trajectories of the differential system of the (122) type touch the curves belonging to the topographical system (defined by the equation $\Phi(x, y) = C$) is called a *contact curve.* The equation of a contact curve has the (185) form. Of particular interest here is the case where the topographical system can be selected in such a manner that either the contact curve itself or a real branch of this curve proves to be a simple closed curve. Then the topographical system con-

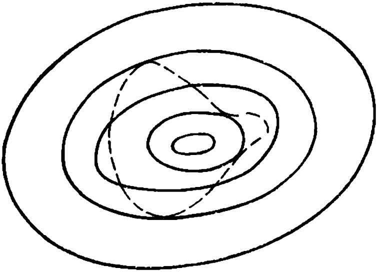

Fig. 98

tains the "greatest" and "smallest" curves that are touched either by a contact curve or by a real branch of this curve (in Figure 98 the contact curve is depicted by a dashed curve). If the derivative dΦ/dt is nonpositive (nonnegative) on the "greatest" curve specified by the equation $\Phi(x, y) = C_1$ and nonnegative (nonpositive) on the "smallest" curve specified by the equation $\Phi(x, y) = C_2$, then the annular region bounded by these curves contains at least one limit cycle of the differential system studied.

Specifically, in the last example (see (184)), the contact curve specified by the equation $x^2 + y^2 = x^4 + y^4$ is a closed curve (Figure 99), which enables us to find the "greatest" and "smallest" curves in the selected topographical system that are touched by the contact curve. Indeed, if

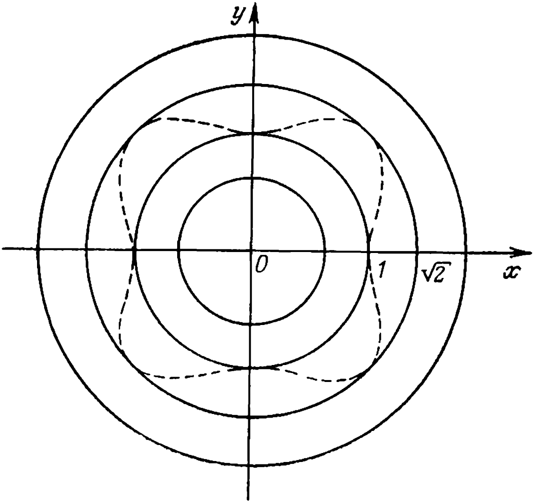

Fig. 99

we note that in polar coordinates the equation of the contact curve has the form $r^2 = (\cos^4\theta + \sin^4\theta)^{-1}$, we can easily find the parametric representation of this curve:

$$x = \frac{\cos\theta}{\sqrt{\cos^4\theta + \sin^4\theta}}, \quad y = \frac{\sin\theta}{\sqrt{\cos^4\theta + \sin^4\theta}}$$

Allowing now for the fact that the greatest and smallest values of r^2 are, respectively, 2 and 1, we conclude that the "greatest" and "smallest" curves of the topographical system $x^2 + y^2 = C$ are the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$, respectively. These circles, as we already know, specify an annular region containing the limit cycle of system (184).

2.17 The Divergence of a Vector Field and Limit Cycles

Returning once more to the general case, we note that in building the topographical system of curves we can sometimes employ the right-hand side of system (122). It has been found that the differential system may be such that the equation

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = \lambda, \qquad (186)$$

where λ is a real parameter, specifies the topographical system of curves.

For instance, if we turn to the differential system (184), Eq. (186) assumes the form $3(x^2 + y^2) - 2 = \lambda$. Then, assuming that $\Lambda = (\lambda + 2)/3$, with $\lambda \in (-2, +\infty)$, we arrive at the topographical system of curves $x^2 + y^2 = \Lambda$ used above.

A remark is in order. If the "greatest" and "smallest" curves merge, that is, the contact curve or a real branch of this curve coincides with one of the curves belonging to the topographical system, such a curve is a trajectory of the differential system, Let us consider, for example, the differential system (173):

$$\frac{dx}{dt} = -y + x(1 - x^2 - y^2),$$

$$\frac{dy}{dt} = x + y(1 - x^2 - y^2).$$

For this system

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 2 - 4(x^2 + y^2),$$

and Eq. (186) assumes the form $2 - 4(x^2 + y^2) = \lambda$. If instead of parameter λ we introduce a new parameter $\Lambda = (2 - \lambda)/4$, with $\lambda \in (-\infty, 2)$, then the latter equation, which assumes the form $x^2 + y^2 = \Lambda$, defines the topographical system of curves. The contact curve in this case is given by the equation $(x^2 + y^2) \times (x^2 + y^2 - 1) = 0$ and, as we see, its real branch $x^2 + y^2 = 1$ coincides with one of the curves belonging to the topographical system. This branch, as shown on pp. 237-239, proves to be a limit cycle of system (173).

The last two examples, of course, illustrate, a particular situation. At the same time the very idea of building a topographical system of curves by employing the concept of divergence proves to be fruitful and leads to results of a general nature. We will not dwell any further on these results, but only note that the following fact may serve as justification for what has been said: *if a differential system of the (122) type has a limit cycle L, there exists a real constant λ and a positive and continuously differentiable in the phase plane function B (x, y) such that the equation*

$$\frac{\partial (BX)}{\partial x}+\frac{\partial (BY)}{\partial y}=\lambda$$

specifies a curve that has a finite real branch coinciding with cycle L.

Let us consider, for example, the differential system

$$\frac{dx}{dt}=2x-2x^3+x^2y-3xy^2+y^3,$$

$$\frac{dy}{dt}=-x^3+x^2y+xy^2,$$

which has a limit cycle specified by the equation $x^2+y^2=1$. For this system the divergence of the vector field is

$$\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}=2-5x^2-3y^2.$$

We can easily verify that there is not a single real λ for which the equation $2-5x^2-3y^2=\lambda$ specifies a trajectory of the initial differential system. But if we take a function $B(x, y)=3x^2-4xy+7y^2+3$, then

$$\frac{\partial (BX)}{\partial x}+\frac{\partial (BY)}{\partial y}=(x^2+y^2-1)$$

$$\times(-23x^2+16xy-25y^2-20)-14,$$

and the equation

$$\frac{\partial (BX)}{\partial x} + \frac{\partial (BY)}{\partial y} = -14$$

specifies a curve whose finite real branch $x^2 + y^2 - 1 = 0$ proves to be the limit cycle.

In conclusion we note that in studying concrete differential models it often proves expedient to employ methods not discussed in this book. Everything depends on the complexity of the differential model, on how deep the appropriate mathematical tools are developed, and, of course, on the erudition and experience of the researcher.

Selected Readings

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Appendices

Appendix 1. Derivatives of Elementary Functions

Function	Derivative
C (constant)	0
x	1
x^n	nx^{n-1}
$\frac{1}{x}$	$-\frac{1}{x^2}$
$\frac{1}{x^n}$	$-\frac{n}{x^{n+1}}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\sqrt[n]{x}$	$\frac{1}{n\sqrt[n]{x^{n-1}}}$
e^x	e^x
a^x	$a^x \ln a$
$\ln x$	$\frac{1}{x}$
$\log_a x$	$\frac{1}{x} \log_a e = \frac{1}{x \ln a}$
$\log_{10} x$	$\frac{1}{x} \log_{10} e \approx \frac{0.4343}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$

Function	Derivative
$\tan x$	$\frac{1}{\cos^2 x} = \sec^2 x$
$\cot x$	$-\frac{1}{\sin^2 x} = -\csc^2 x$
$\sec x$	$\frac{\sin x}{\cos^2 x} = \tan x \sec x$
$\csc x$	$-\frac{\cos x}{\sin^2 x} = -\cot x \csc x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cot^{-1} x$	$-\frac{1}{1+x^2}$
$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$
$\csc^{-1} x$	$-\frac{1}{x\sqrt{x^2-1}}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\frac{1}{\cosh^2 x}$
$\coth x$	$-\frac{1}{\sinh^2 x}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{1+x^2}}$

Function	Derivative
$\cosh^{-1} x$	$\dfrac{1}{\sqrt{x^2-1}}$, $x > 1$
$\tanh^{-1} x$	$\dfrac{1}{1-x^2}$, $\|x\| < 1$
$\coth^{-1} x$	$\dfrac{1}{1-x^2}$, $\|x\| > 1$

Appendix 2.
Basic Integrals

Power-Law Functions

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} \quad (n \neq 1)$$

$$\int \frac{dx}{x} = \ln |x|$$

Trigonometric Functions

$$\int \sin x \, dx = -\cos x$$

$$\int \cos x \, dx = \sin x$$

$$\int \tan x \, dx = -\ln |\cos x|$$

$$\int \cot x \, dx = \ln |\sin x|$$

$$\int \frac{dx}{\cos^2 x} = \tan x$$

$$\int \frac{dx}{\sin^2 x} = -\cot x$$

Fractional Rational Functions

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right)$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \tanh^{-1} \left(\frac{x}{a} \right)$$

$$= \frac{1}{2a} \ln \frac{a+x}{a-x} \quad (|x| < a)$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{1}{2a}\ln\frac{x-a}{x+a} \quad (|x| > a)$$

Exponential Functions

$$\int e^x\,dx = e^x$$

$$\int a^x\,dx = \frac{a^x}{\ln a}$$

Hyperbolic Functions

$$\int \sinh x\,dx = \cosh x$$

$$\int \cosh x\,dx = \sinh x$$

$$\int \tanh x\,dx = \ln\,|\cosh x|$$

$$\int \coth x\,dx = \ln\,|\sinh x|$$

$$\int \frac{dx}{\cosh^2 x} = \tanh x$$

$$\int \frac{dx}{\sinh^2 x} = -\coth x$$

Irrational Functions

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sinh^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) = \ln|(x+\sqrt{x^2+a^2})|$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\,|(x+\sqrt{x^2-a^2})|$$

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SCIENCE FOR EVERYONE

This book is an easy-reading introduction to ordinary differential equations and their use in the study of real phenomena and processes.s Problems taken from various fields of knowledge illustrate the tools used in setting up differential equations and the methods employed in their qualitative investigation. The book should be useful to high-school students, teachers of science courses, college students, and specialists of, non-mathematical professions who use mathematics in their work.

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