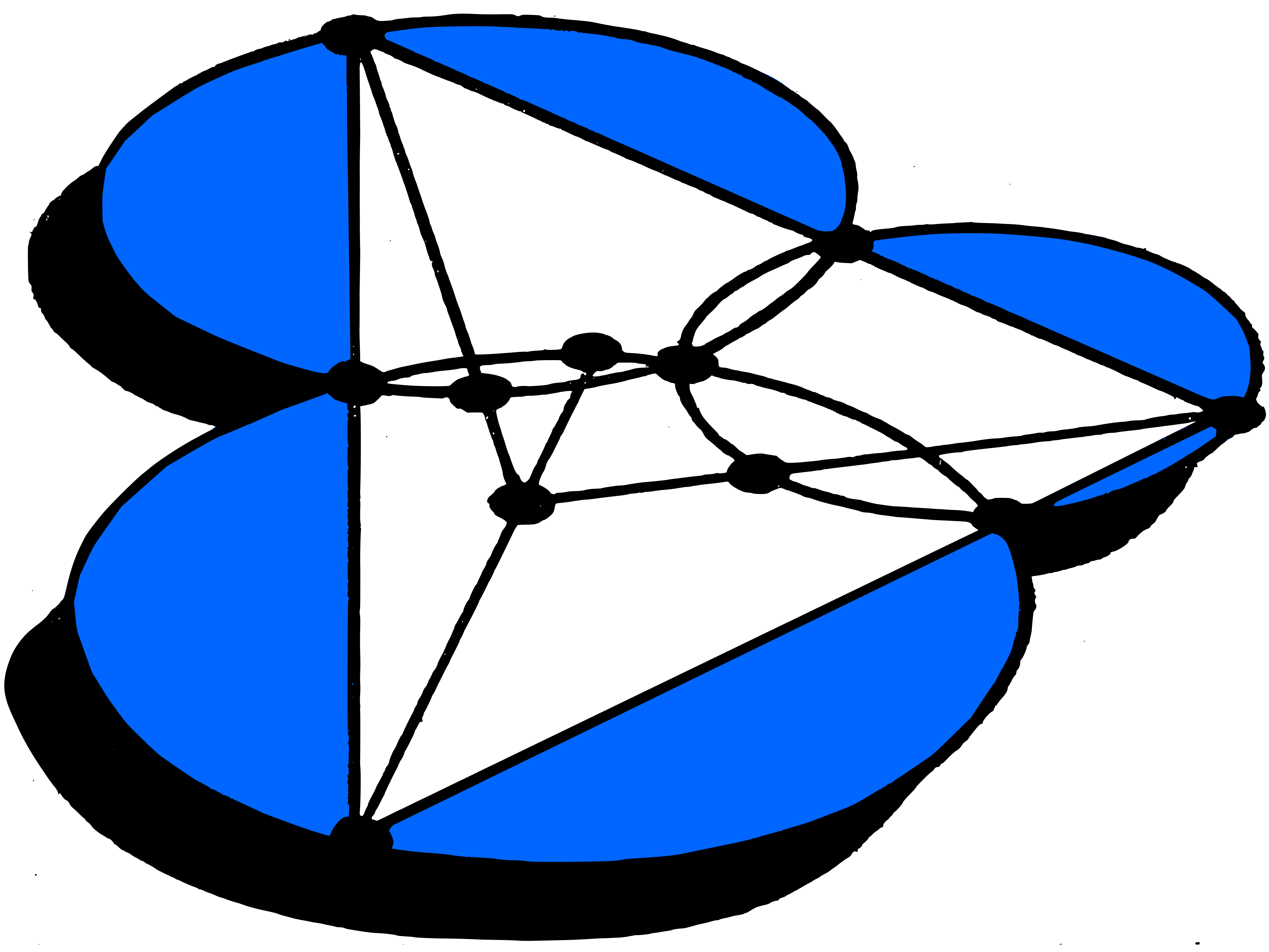
SCIENCE FOR EVERYONE

I. F. SHARYGIN

PROBLEMS IN PLANE GEOMETRY



MIR

SCIENCE FOR EVERYONE

This is a translation from the revised edition of the Russian book which was issued in 1982. It is actually the first in a two-volume work on solving problems in geometry, the second volume "Problems in Solid Geometry" having been published in English first by Mir Publishers in 1986.
Both volumes are designed for schoolchildren and teachers.
This is a unique collection of interesting and elegant problems presenting the subject in a manner comprehensible to a youthful mind making it both interesting and useful through a wide range of practical applications.
Although the problems in this collection vary in "age" (some of them can be found in old books and journals, others were offered at mathematical olympiads or published in the journal "Quant" (Moscow)), we still hope that some of the problems in the collection will be of interest to experienced geometers.
Almost every problem is non-standard (as compared with routine exercises on solving equations, inequalities, etc.): one has to think of what additional constructions must be made, or which formulas and theorems must be used. Therefore, this collection cannot be regarded as a problem book in geometry; it is rather a collection of geometrical puzzles aimed at demonstrating the elegance of elementary geometrical techniques of proof and methods of computation (without using vector algebra and with a minimal use of the method of coordinates, geometrical transformations, though a somewhat wider use of trigonometry).

Science
for Everyone

И.Ф. Шарыгин

Задачи по геометрии
Планиметрия

Издательство «Наука», Москва

I.F. Sharygin

Problems in Plane Geometry

Mir
Publishers
Moscow

Translated from Russian
by Leonid Levant

First published 1988
Revised from the 1986 Russian edition

Ш $\frac{1702040000-304}{056(01)-88}$ 21—88, ч. 2

На английском языке

Printed in the Union of Soviet Socialist Republics

ISBN 5-03-000180-8

Contents

Preface to the English Edition

This is a translation from the revised edition of the Russian book which was issued in 1982. It is actually the first in a two-volume work on solving problems in geometry, the second volume "Problems in Solid Geometry" having been published in English first by Mir Publishers in 1986.

Both volumes are designed for schoolchildren and teachers.

This volume contains over 600 problems in plane geometry and consists of two parts. The first part contains rather simple problems to be solved in classes and at home. The second part also contains hints and detailed solutions. Over 200 new problems have been added to the 1982 edition, the simpler problems in the first addition having been eliminated, and a number of new sections (circles and tangents, polygons, combinations of figures, etc.) having been introduced. The general structure of the book has been changed somewhat to accord with the new, more detailed, classification of the problems. As a result, all the problems in this volume have been rearranged.

Although the problems in this collection vary in "age" (some of them can be found in old books and journals, others were offered at mathematical olympiads or published in the journal "Quant" (Moscow)), I still hope that some of the problems in this collection will be of interest to experienced geometers.

Almost every geometrical problem is nonstandard (as compared with routine exercises on solving equations, inequalities, etc.): one has to think of what additional constructions must be made, or which formulas and theorems must be used. Therefore, this collection cannot be regarded as a problem-book in geometry; it is rather a collection of geometrical puzzles aimed at demonstrating the elegance of elementary geometrical techniques of proof and methods of computation (without using vector algebra and with a minimal use of the method of coordinates, geometrical transformations, though a somewhat wider use of trigonometry).

In conclusion, I should like to thank A.Z. Bershtein who assisted me in preparing the first section of the book for print. I am also grateful to A.A. Yagubiants who let me know several elegant geometrical facts.

The Author

Section 1

Fundamental Geometrical Facts and Theorems. Computational Problems

1. Prove that the medians in a triangle intersect at one point (the median point) and are divided by this point in the ratio 1 : 2.

2. Prove that the medians separate the triangle into six equivalent parts.

3. Prove that the diameter of the circle circumscribed about a triangle is equal to the ratio of its side to the sine of the opposite angle.

4. Let the vertex of an angle be located outside a circle, and let the sides of the angle intersect the circle. Prove that the angle is measured by the half-difference of the arcs inside the angle which are cut out by its sides on the circle.

5. Let the vertex of an angle lie inside a circle. Prove that the angle is measured by the half-sum of the arcs one of which is enclosed between its sides and the other between their extensions.

6. Let AB denote a chord of a circle, and l the tangent to the circle at the point A. Prove that either of the two angles between AB and l is measured by the half-arc of the

circle enclosed inside the angle under consideration.

7. Through the point M located at a distance a from the centre of a circle of radius R ($a > R$), a secant is drawn intersecting the circle at points A and B. Prove that the product $|MA| \cdot |MB|$ is constant for all the secants and equals $a^2 - R^2$ (which is the squared length of the tangent).

8. A chord AB is drawn through the point M situated at a distance a from the centre of a circle of radius R ($a < R$). Prove that $|AM| \cdot |MB|$ is constant for all the chords and equals $R^2 - a^2$.

9. Let AM be an angle bisector in the triangle ABC. Prove that $|BM| : |CM| = |AB| \quad |AC|$. The same is true for the bisector of the exterior angle of the triangle. (In this case the point M lies on the extension of the side BC.)

10. Prove that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of its sides.

11. Given the sides of a triangle (a, b, and c). Prove that the median m_a drawn to the side a can be computed by the formula

$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}.$$

12. Given two triangles having one vertex A in common, the other vertices being situated on two straight lines passing through A. Prove that the ratio of the areas of these triangles is equal to the ratio of the products of the two sides of each triangle emanating from the vertex A.

13. Prove that the area of the circumscribed polygon is equal to rp, where r is the radius of the inscribed circle and p its half-perimeter (in particular, this formula holds true for a triangle).

14. Prove that the area of a quadrilateral is equal to half the product of its diagonals and the sine of the angle between them.

15. Prove the validity of the following formulas for the area of a triangle:

$$S=\frac{a^2 \sin B \sin C}{2 \sin A}, \quad S=2R^2 \sin A \sin B \sin C,$$

where A, B, C are its angles, a is the side lying opposite the angle A, and R is the radius of the circumscribed circle.

16. Prove that the radius of the circle inscribed in a right triangle can be computed by the formula $r=\frac{a+b-c}{2}$, where a and b are the legs and c is the hypotenuse.

17. Prove that if a and b are two sides of a triangle, α the angle between them, and l the bisector of this angle, then

$$l=\frac{2ab \cos \frac{\alpha}{2}}{a+b}.$$

18. Prove that the distances from the vertex A of the triangle ABC to the points of tangency of the inscribed circle with the sides AB and AC are equal to $p - a$ (each), where p is the half-perimeter of the triangle ABC, $a = |BC|$.

19. Prove that if in a convex quadrilateral $ABCD$ $|AB| + |CD| = |AD| + |BC|$, then there is a circle touching all of its sides.

20. (a) Prove that the altitudes in a triangle are concurrent (that is intersect at one point). (b) Prove that the distance from any vertex of a triangle to the point of intersection of the altitudes is twice the distance from the centre of the circumscribed circle to the opposite side.

* * *

21. Points A and B are taken on one side of a right angle with vertex O and $|OA| = a$, $|OB| = b$. Find the radius of the circle passing through the points A and B and touching the other side of the angle.

22. The hypotenuse of a right triangle is equal to c, one of the acute angles being 30°. Find the radius of the circle with centre at the vertex of the angle of 30° which separates the triangle into two equivalent parts.

23. The legs of a right triangle are a and b. Find the distance from the vertex of the right angle to the nearest point of the inscribed circle.

24. One of the medians of a right triangle is equal to m and divides the right angle in the ratio $1:2$. Find the area of the triangle.

25. Given in a triangle ABC are three sides: $|BC| = a$, $|CA| = b$, $|AB| = c$. Find the ratio in which the point of intersection of the angle bisectors divides the bisector of the angle B.

26. Prove that the sum of the distances from any point of the base of an isosceles triangle to its sides is equal to the altitude drawn to either of the sides.

27. Prove that the sum of distances from any point inside an equilateral triangle to its sides is equal to the altitude of this triangle.

28. In an isosceles triangle ABC, taken on the base AC is a point M such that $|AM| = a$, $|MC| = b$. Circles are inscribed in the triangles ABM and CBM. Find the distance between the points at which these circles touch the side BM.

29. Find the area of the quadrilateral bounded by the angle bisectors of a parallelogram with sides a and b and angle α.

30. A circle is inscribed in a rhombus with altitude h and acute angle α. Find the radius of the greatest of two possible circles each of which touches the given circle and two sides of the rhombus.

31. Determine the acute angle of the rhombus whose side is the geometric mean of its diagonals.

32. The diagonals of a convex quadrilateral are equal to a and b, the line segments joining the midpoints of the opposite sides are congruent. Find the area of the quadrilateral.

33. The side AD of the rectangle $ABCD$ is three times the side AB; points M and N divide AD into three equal parts. Find $\angle AMB + \angle ANB + \angle ADB$.

34. Two circles intersect at points A and B. Chords AC and AD touching the given circles are drawn through the point A. Prove that $|AC|^2 \cdot |BD| = |AD|^2 \cdot |BC|$.

35. Prove that the bisector of the right angle in a right triangle bisects the angle between the median and the altitude drawn to the hypotenuse.

36. On a circle of radius r, three points are chosen so that the circle is divided into three arcs in the ratio $3 : 4 : 5$. At the division points, tangents are drawn to the circle. Find the area of the triangle formed by the tangents.

37. An equilateral trapezoid is circumscribed about a circle, the lateral side of the trapezoid is l, one of its bases is equal to a. Find the area of the trapezoid.

38. Two straight lines parallel to the bases of a trapezoid divide each lateral

side into three equal parts. The entire trapezoid is separated by the lines into three parts. Find the area of the middle part if the areas of the upper and lower parts are S_1 and S_2, respectively.

39. In the trapezoid $ABCD$ $|AB| = a$, $|BC| = b$ $(a \neq b)$. The bisector of the angle A intersects either the base BC or the lateral side CD. Find out which of them?

40. Find the length of the line segment parallel to the bases of a trapezoid and passing through the point of intersection of its diagonals if the bases of the trapezoid are a and b.

41. In an equilateral trapezoid circumscribed about a circle, the ratio of the parallel sides is k. Find the angle at the base.

42. In a trapezoid $ABCD$, the base AB is equal to a, and the base CD to b. Find the area of the trapezoid if the diagonals of the trapezoid are known to be the bisectors of the angles DAB and ABC.

43. In an equilateral trapezoid, the midline is equal to a, and the diagonals are mutually perpendicular. Find the area of the trapezoid.

44. The area of an equilateral trapezoid circumscribed about a circle is equal to S, and the altitude of the trapezoid is half its lateral side. Determine the radius of the circle inscribed in the trapezoid.

45. The areas of the triangles formed by

the segments of the diagonals of a trapezoid and its bases are equal to S_1 and S_2. Find the area of the trapezoid.

46. In a triangle ABC, the angle ABC is α. Find the angle AOC, where O is the centre of the inscribed circle.

47. The bisector of the right angle is drawn in a right triangle. Find the distance between the points of intersection of the altitudes of the triangles thus obtained, if the legs of the given triangle are a and b.

48. A straight line perpendicular to two sides of a parallelogram divides the latter into two trapezoids in each of which a circle can be inscribed. Find the acute angle of the parallelogram if its sides are a and b $(a < b)$.

49. Given a half-disc with diameter AB. Two straight lines are drawn through the midpoint of the semicircle which divide the half-disc into three equivalent areas. In what ratio is the diameter AB divided by these lines?

50. A square $ABCD$ with side a and two circles are constructed. The first circle is entirely inside the square touching the side AB at a point E and also the side BC and diagonal AC. The second circle with centre at A passes through the point E. Find the area of the common part of the two discs bounded by these circles.

51. The vertices of a regular hexagon with side a are the centres of the circles

with radius $a/\sqrt{2}$. Find the area of the part of the hexagon not enclosed by these circles.

52. A point A is taken outside a circle of radius R. Two secants are drawn from this point: one passes through the centre, the other at a distance of $R/2$ from the centre. Find the area of the region enclosed between these secants.

53. In a quadrilateral $ABCD$: $\angle DAB = 90°$, $\angle DBC = 90°$. $|DB| = a$, and $|DC| = b$. Find the distance between the centres of two circles one of which passes through the points D, A and B, the other through the points B, C, and D.

54. On the sides AB and AD of the rhombus $ABCD$ points M and N are taken such that the straight lines MC and NC separate the rhombus into three equivalent parts. Find $|MN|$ if $|BD| = d$.

55. Points M and N are taken on the side AB of a triangle ABC such that $|AM| : |MN| : |NB| = 1 : 2 : 3$. Through the points M and N straight lines are drawn parallel to the side AC. Find the area of the part of the triangle enclosed between these lines if the area of the triangle ABC is equal to S.

56. Given a circle and a point A located outside of this circle, straight lines AB and AC are tangent to it (B and C points of tangency). Prove that the centre of the

circle inscribed in the triangle ABC lies on the given circle.

57. A circle is circumscribed about an equilateral triangle ABC, and an arbitrary point M is taken on the arc BC. Prove that $|AM| = |BM| + |CM|$.

58. Let H be the point of intersection of the altitudes in a triangle ABC. Find the interior angles of the triangle ABC if $\angle BAH = \alpha$, $\angle ABH = \beta$.

59. The area of a rhombus is equal to S, the sum of its diagonals is m. Find the side of the rhombus.

60. A square with side a is inscribed in a circle. Find the side of the square inscribed in one of the segments thus obtained.

61. In a $120°$ segment of a circle with altitude h a rectangle $ABCD$ is inscribed so that $|AB| : |BC| = 1\ \ 4$ (BC lies on the chord). Find the area of the rectangle.

62. The area of an annulus is equal to S. The radius of the larger circle is equal to the circumference of the smaller. Find the radius of the smaller circle.

63. Express the side of a regular decagon in terms of the radius R of the circumscribed circle.

64. Tangents MA and MB are drawn from an exterior point M to a circle of radius R forming an angle α. Determine

the area of the figure bounded by the tangents and the minor arc of the circle.

65. Given a square $ABCD$ with side a. Find the centre of the circle passing through the following points: the midpoint of the side AB, the centre of the square, and the vertex C.

66. Given a rhombus with side a and acute angle α. Find the radius of the circle passing through two neighbouring vertices of the rhombus and touching the opposite side of the rhombus or its extension.

67. Given three pairwise tangent circles of radius r. Find the area of the triangle formed by three lines each of which touches two circles and does not intersect the third one.

68. A circle of radius r touches a straight line at a point M. Two points A and B are chosen on this line on opposite sides of M such that $|MA| = |MB| = a$. Find the radius of the circle passing through A and B and touching the given circle.

69. Given a square $ABCD$ with side a. Taken on the side BC is a point M such that $|BM| = 3|MC|$ and on the side CD a point N such that $2|CN| = |ND|$. Find the radius of the circle inscribed in the triangle AMN.

70. Given a square $ABCD$ with side a. Determine the distance between the midpoint of the line segment AM, where M is

the midpoint of BC, and a point N on the side CD such that $| CN | \quad | ND | = 3 \quad 1$.

71. A straight line emanating from the vertex A in a triangle ABC bisects the median BD (the point D lies on the side AC). What is the ratio in which this line divides the side BC?

72. In a right triangle ABC the leg CA is equal to b, the leg CB is equal to a, CH is the altitude, and AM is the median. Find the area of the triangle BMH.

73. Given an isosceles triangle ABC whose $\angle A = \alpha > 90^\circ$ and $| BC | = a$. Find the distance between the point of intersection of the altitudes and the centre of the circumscribed circle.

74. A circle is circumscribed about a triangle ABC where $| BC | = a$, $\angle B = \alpha$, $\angle C = \beta$. The bisector of the angle A meets the circle at a point K. Find $| AK |$.

75. In a circle of radius R, a diameter is drawn with a point A taken at a distance a from the centre. Find the radius of another circle which is tangent to the diameter at the point A and touches internally the given circle.

76. In a circle, three pairwise intersecting chords are drawn. Each chord is divided into three equal parts by the points of intersection. Find the radius of the circle if one of the chords is equal to a.

77. One regular hexagon is inscribed in a circle, the other is circumscribed about it. Find the radius of the circle if the difference between the perimeters of these hexagons is equal to a.

78. In an equilateral triangle ABC whose side is equal to a, the altitude BK is drawn. A circle is inscribed in each of the triangles ABK and BCK, and a common external tangent, different from the side AC, is drawn to them. Find the area of the triangle cut off by this tangent from the triangle ABC.

79. Given in an inscribed quadrilateral $ABCD$ are the angles: $\angle DAB = \alpha$, $\angle ABC = \beta$, $\angle BKC = \gamma$, where K is the point of intersection of the diagonals. Find the angle ACD.

80. In an inscribed quadrilateral $ABCD$ whose diagonals intersect at a point K, $|AB| = a$, $|BK| = b$, $|AK| = c$, $|CD| = d$. Find $|AC|$.

81. A circle is circumscribed about a trapezoid. The angle between one of the bases of the trapezoid and a lateral side is equal to α and the angle between this base and one of the diagonals is equal to β. Find the ratio of the area of the circle to the area of the trapezoid.

82. In an equilateral trapezoid $ABCD$, the base AD is equal to a, the base BC is equal to b, $|AB| = d$. Drawn through the vertex B is a straight line bisecting the diagonal AC and intersecting AD at a point K. Find the area of the triangle BDK.

83. Find the sum of the squares of the

distances from the point M taken on a diameter of a circle to the end points of any chord parallel to this diameter if the radius of the circle is R, and the distance from M to the centre of the circle is a.

84. A common chord of two intersecting circles can be observed from their centres at angles of 90° and 60°. Find the radii of the circles if the distance between their centres is equal to a.

85. Given a regular triangle ABC. A point K divides the side AC in the ratio $2:1$, and a point M divides the side AB in the ratio 1 2 (as measured from the vertex A in both cases). Prove that the length of the line segment KM is equal to the radius of the circle circumscribed about the triangle ABC.

86. Two circles of radii R and $R/2$ touch each other externally. One of the end points of the line segment of length $2R$ forming an angle of 30° with the centre line coincides with the centre of the circle of the smaller radius. What part of the line segment lies outside both circles? (The line segment intersects both circles.)

87. A median BK, an angle bisector BE, and an altitude AD are drawn in a triangle ABC. Find the side AC if it is known that the lines BK and BE divide the line segment AD into three equal parts and $|AB| = 4$.

88. The ratio of the radius of the circle inscribed in an isosceles triangle to the radius of the circle circumscribed about this triangle is equal to k. Find the base angle of the triangle.

89. Find the cosine of the angle at the base of an isosceles triangle if the point of intersection of its altitudes lies on the circle inscribed in the triangle.

90. Find the area of the pentagon bounded by the lines BC, CD, AN, AM, and BD, where A, B, and D are the vertices of a square $ABCD$, N the midpoint of the side BC, and M divides the side CD in the ratio $2:1$ (counting from the vertex C) if the side of the square $ABCD$ is equal to a.

91. Given in a triangle ABC: $\angle BAC = \alpha$, $\angle ABC = \beta$. A circle centred at B passes through A and intersects the line AC at a point K different from A, and the line BC at points E and F. Find the angles of the triangle EKF.

92. Given a square with side a. Find the area of the regular triangle one of whose vertices coincides with the midpoint of one of the sides of the square, the other two lying on the diagonals of the square.

93. Points M, N, and K are taken on the sides of a square $ABCD$, where M is the midpoint of AB, N lies on the side BC $(2\,|BN| = |NC|)$. K lies on the side DA $(2\,|DK| = |KA|)$. Find the sine of the angle between the lines MC and NK.

94. A circle of radius r passes through the vertices A and B of the triangle ABC and intersects the side BC at a point D. Find the radius of the circle passing through the points A, D, and C if $|AB| = c$, $|AC| = b$.

95. In a triangle ABC, the side AB is equal to 3, and the altitude CD dropped on the side AB is equal to $\sqrt{3}$. The foot D of the altitude CD lies on the side AB, and the line segment AD is equal to the side BC. Find $|AC|$.

96. A regular hexagon $ABCDEF$ is inscribed in a circle of radius R. Find the radius of the circle inscribed in the triangle ACD.

97. The side AB of a square $ABCD$ is equal to 1 and is a chord of a circle, the rest of the sides of the square lying outside this circle. The length of the tangent CK drawn from the vertex C to the circle is equal to 2. Find the diameter of the circle.

98. In a right triangle, the smaller angle is equal to α. A straight line drawn perpendicularly to the hypotenuse divides the triangle into two equivalent parts. Determine the ratio in which this line divides the hypotenuse.

99. Drawn inside a regular triangle with side equal to 1 are two circles touching each other. Each of the circles touches two sides of the triangle (each side of the triangle touches at least one of the circles). Prove that the sum of the radii of these circles is not less than $(\sqrt{3} - 1)/2$.

100. In a right triangle ABC with an acute angle A equal to 30°, the bisector of the other acute angle is drawn. Find the distance between the centres of the two circles inscribed in the triangles ABD and CBD if the smaller leg is equal to 1.

101. In a trapezoid $ABCD$, the angles A and D at the base AD are equal to 60° and 30°, respectively. A point N lies on the base BC, and $|BN| : |NC| = 2$. A point M lies on the base AD; the straight line MN is perpendicular to the bases of the trapezoid and divides its area into two equal parts. Find $|AM| : |MD|$.

102. Given in a triangle ABC: $|BC| = a$, $\angle A = \alpha$, $\angle B = \beta$. Find the radius of the circle touching both the side AC at a point A and the side BC.

103. Given in a triangle ABC: $|AB| = c$, $|BC| = a$, $\angle B = \beta$. On the side AB, a point M is taken such that $2|AM| = 3|MB|$. Find the distance from M to the midpoint of the side AC.

104. In a triangle ABC, a point M is taken on the side AB and a point N on the side AC such that $|AM| = 3|MB|$ and $2|AN| = |NC|$. Find the area of the quadrilateral $MBCN$ if the area of the triangle ABC is equal to S.

105. Given two concentric circles of radii R and r ($R > r$) with a common centre O. A third circle touches both of them. Find the tangent of the angle between the tangent lines to the third circle emanating from the point O.

106. Given in a parallelogram $ABCD$: $|AB| = a$, $|AD| = b$ $(b > a)$, $\angle BAD = \alpha$ $(\alpha < 90°)$. On the sides AD and BC, points K and M are taken such that $BKDM$ is a rhombus. Find the side of the rhombus.

107. In a right triangle, the hypotenuse is equal to c. The centres of three circles of radius $c/5$ are found at its vertices. Find the radius of a fourth circle which touches the three given circles and does not enclose them.

108. Find the radius of the circle which cuts on both sides of an angle α chords of length a if the distance between the nearest end points of these chords is known to be equal to b.

109. A circle is constructed on the side BC of a triangle ABC as diameter. This circle intersects the sides AB and AC at points M and N, respectively. Find the area of the triangle AMN if the area of the triangle ABC is equal to S, and $\angle BAC = \alpha$.

110. In a circle of radius R two mutually perpendicular chords MN and PQ are drawn. Find the distance between the points M and P if $|NQ| = a$.

111. In a triangle ABC, on the largest side BC equal to b, a point M is chosen. Find the shortest distance between the centres of the circles circumscribed about the triangles BAM and ACM.

112. Given in a parallelogram $ABCD$: $|AB| = a$, $|BC| = b$, $\angle ABC = \alpha$. Find the distance between the centres of the circles circumscribed about the triangles BCD and DAB.

113. In a triangle ABC, $\angle A = \alpha$, $|BA| = a$, $|AC| = b$. On the sides AC and AB, points M and N are taken, M being the midpoint of AC. Find the length of the line segment MN if the area of the triangle AMN is 1/3 of the area of the triangle ABC.

114. Find the angles of a rhombus if the area of the circle inscribed in it is half the area of the rhombus.

115. Find the common area of two equal squares of side a if one can be obtained from the other by rotating through an angle of 45° about its vertex.

116. In a quadrilateral inscribed in a circle, two opposite sides are mutually perpendicular, one of them being equal to a, the adjacent acute angle is divided by one of the diagonals into α and β. Determine the diagonals of the quadrilateral (the angle α is adjacent to the given side).

117. Given a parallelogram $ABCD$ with an acute angle DAB equal to α in which $|AB| = a$, $|AD| = b$ $(a < b)$. Let K denote the foot of the perpendicular dropped from the vertex B on AD, and M the foot of the perpendicular dropped from the point K on the extension of the side CD. Find the area of the triangle BKM.

118. In a triangle ABC, drawn from the vertex C are two rays dividing the angle ACB into three equal parts. Find the ratio of the segments of these rays enclosed inside the triangle if $|BC| = 3|AC|$, $\angle ACB = \alpha$.

119. In an isosceles triangle ABC ($|AB| = |BC|$) the angle bisector AD is drawn. The areas of the triangles ABD and ADC are equal to S_1 and S_2, respectively. Find $|AC|$.

120. A circle of radius R_1 is inscribed in an angle α. Another circle of radius R_2 touches one of the sides of the angle at the same point as the first one and intersects the other side of the angle at points A and B. Find $|AB|$.

121. On a straight line passing through the centre O of the circle of radius 12, points A and B are taken such that $|OA| = 15$, $|AB| = 5$. From the points A and B, tangents are drawn to the circle whose points of tangency lie on one side of the line OAB. Find the area of the triangle ABC, where C is the point of intersection of these tangents.

122. Given in a triangle ABC: $|BC| = a$, $\angle A = \alpha$, $\angle B = \beta$. Find the radius of the circle intersecting all of its sides and cutting off on each of them a chord of length d.

123. In a convex quadrilateral, the line segments joining the midpoints of the opposite sides are equal to a and b and intersect at an angle of 60° Find the diagonals of the quadrilateral.

124. In a triangle ABC, taken on the side BC is a point M such that the distance from the vertex B to the centre of gravity of the triangle AMC is equal to the distance from the vertex C to the centre of gravity of the triangle AMB. Prove that $|BM| = |DC|$ where D is the foot of the altitude dropped from the vertex A to BC.

125. In a right triangle ABC, the bisector BE of the right angle B is divided by the centre O of the inscribed circle so that $|BO| \quad |OE| = \sqrt{3} \quad \sqrt{2}$. Find the acute angles of the triangle.

126. A circle is constructed on a line segment AB of length R as diameter. A second circle of the same radius is centred at the point A. A third circle touches the first circle internally and the second circle externally; it also touches the line segment AB. Find the radius of the third circle.

127. Given a triangle ABC. It is known that $|AB| = 4$, $|AC| = 2$, and $|BC| = 3$. The bisector of the angle A intersects the side BC at a point K. The straight line passing through the point B and being parallel to AC intersects the extension of the angle bisector AK at the point M. Find $|KM|$.

128. A circle centred inside a right angle touches one of the sides of the angle, intersects the other side at points A and B and intersects the bisector of the angle at points C and D. The chord AB is equal to $\sqrt{6}$, the chord CD to $\sqrt{7}$. Find the radius of the circle.

129. Two circles of radius 1 lie in a parallelogram, each circle touching the other circle and three sides of the parallelogram. One of the segments of the side from the vertex to the point of tangency is equal to $\sqrt{3}$. Find the area of the parallelogram.

130. A circle of radius R passes through the vertices A and B of the triangle ABC and touches the line AC at A. Find the area of the triangle ABC if $\angle B = \alpha$, $\angle A = \beta$.

131. In a triangle ABC, the angle bisector AK is perpendicular to the median BM, and the angle B is equal to $120°$. Find the ratio of the area of the triangle ABC to the area of the circle circumscribed about this triangle.

132. In a right triangle ABC, a circle touching the side BC is drawn through the midpoints of AB and AC. Find the part of the hypotenuse AC which lies inside this circle if $|AB| = 3$, $|BC| = 4$.

133. Given a line segment a. Three circles of radius R are centred at the end points and midpoint of the line segment. Find the radius of the fourth circle which touches the three given circles.

134. Find the angle between the common external and internal tangents to two circles of radii R and r if the distance between their centres equals $\sqrt{2(R^2+r^2)}$ (the centres of the circles are on the same side of the common external tangent and on both sides of the common internal tangent).

135. The line segment AB is the diameter of a circle, and the point C lies outside this circle. The line segments AC and BC intersect the circle at points D and E, respectively. Find the angle CBD if the ratio of the areas of the triangles DCE and ABC is $1 \quad 4$.

136. In a rhombus $ABCD$ of side a, the angle at the vertex A is equal to $120°$. Points E and F lie on the sides BC and AD, respectively, the line segment EF and the diagonal AC of the rhombus intersect at M. The ratio of the areas of the quadrilaterals $BEFA$ and $ECDF$ is $1:2$. Find $|EM|$ if $|AM| \quad |MC| = 1 \quad 3$.

137. Given a circle of radius R centred at O. A tangent AK is drawn to the circle from the end point A of the line segment OA, which meets the circle at M. Find the radius of the circle touching the line segments AK, AM, and the arc MK if $\angle OAK = 60°$.

138. Inscribed in a circle is an isosceles triangle ABC in which $|AB| = |BC|$ and $\angle B = \beta$. The midline of the triangle is extended to intersect the circle at points D and E ($DE \parallel AC$). Find the ratio of the areas of the triangles ABC and DBE.

139. Given an angle α with vertex O. A point M is taken on one of its sides and a perpendicular is erected at this point to intersect the other side of the angle at a point N. Just in the same way, at a point K taken on the other side of the angle a perpendicular is erected to intersect the first side at a point P. Let B denote the point of intersection of the lines MN and KP, and A the point of intersection of the lines OB and NB. Find $|OA|$ if $|OM| = a$ and $|OP| = b$.

140. Two circles of radii R and r touch the sides of a given angle and each other. Find the radius of a third circle touching the sides of the same angle and whose centre is found at the point at which the given circles touch each other.

141. The distance between the centres of two non-intersecting circles is equal to a. Prove that the four points of intersection of common external and internal tangents lie on one circle. Find the radius of this circle.

142. Prove that the segment of a common external tangent to two circles which is enclosed between common internal tangents is equal to the length of a common internal tangent.

143. Two mutually perpendicular radii OA and OB are drawn in a circle centred at O. A point C is on the arc AB such that $\angle AOC = 60°$ ($\angle BOC = 30°$). A circle of radius AB centred at A intersects the extension of OC beyond the point C at D. Prove that the line segment CD is equal to the side of a regular decagon inscribed in the circle.

Let us now take a point M diametrically opposite to the point C. The line segment MD, increased by 1/5 of its length, is assumed to be approximately equal to half the circumference. Estimate the error of this approximation.

144. Given a rectangle 7×8. One vertex of a regular triangle coincides with one of the vertices of the rectangle, the two other vertices lying on its sides not containing this vertex. Find the side of the regular triangle.

145. Find the radius of the minimal circle containing an equilateral trapezoid with bases of 15 and 4 and lateral side of 9.

146. $ABCD$ is a rectangle in which $|AB| = 9$, $|BC| = 7$. A point M is taken on the side CD such that $|CM| = 3$, and point N on the side AD such that $|AN| = 2.5$. Find the greatest radius of the circle which goes inside the pentagon $ABCMN$.

147. Find the greatest angle of a triangle if the radius of the circle inscribed in the triangle with vertices at the feet of the altitudes of the given triangle is half the least altitude of the given triangle.

148. In a triangle ABC, the bisector of the angle C is perpendicular to the median emanating from the vertex B. The centre of the inscribed circle lies on the circle passing through the points A and C and the centre of the circumscribed circle. Find $|AB|$ if $|BC| = 1$.

149. A point M is at distances of 2, 3 and 6 from the sides of a regular triangle (that is, from the lines on which its sides are situated). Find the side of the regular triangle if its area is less than 14.

150. A point M is at distances of $\sqrt{3}$ and $3\sqrt{3}$ from the sides of an angle of $60°$ (the feet of the perpendiculars dropped from M on the sides of the angle lie on the sides themselves, but not on their extensions). A straight line passing through the point M intersects the sides of the angle and cuts off a triangle whose perimeter is 12. Find the area of this triangle.

151. Given a rectangle $ABCD$ in which $|AB| = 4$, $|BC| = 3$. Find the side of the rhombus one vertex of which coincides with A, and three others lie on the line segments AB, BC and BD (one vertex on each segment).

152. Given a square $ABCD$ with a side equal to 1. Find the side of the rhombus one vertex of which coincides with A, the opposite vertex lies on the line BD, and the two remaining vertices on the lines BC and CD.

153. In a parallelogram $ABCD$ the acute angle is equal to α. A circle of radius r passes through the vertices A, B, and C and intersects the lines AD and CD at points M and N. Find the area of the triangle BMN.

154. A circle passing through the vertices A, B, and C of the parallelogram $ABCD$ intersects the lines AD and CD at points M and N. The point M is at distances of 4, 3 and 2 from the vertices B, C, and D, respectively. Find $|MN|$.

155. Given a triangle ABC in which $\angle BAC = \pi/6$. The circle centred at A with radius equal to the altitude dropped on BC separates the triangle into two equal areas. Find the greatest angle of the triangle ABC.

156. In an isosceles triangle ABC $\angle B = 120°$. Find the common chord of two circles: one is circumscribed about ABC, the other passes through the centre of the inscribed circle and the feet of the bisectors of the angles A and C if $|AC| = 1$.

157. In a triangle ABC the side BC is equal to a, the radius of the inscribed circle is equal to r. Determine the radii of two equal circles tangent to each other, one of them touching the sides BC and BA, the other—the sides BC and CA.

158. A trapezoid is inscribed in a circle

of radius R. Straight lines passing through the end points of one of the bases of the trapezoid parallel to the lateral sides intersect at the centre of the circle. The lateral side can be observed from the centre at an angle α. Find the area of the trapezoid.

159. The hypotenuse of a right triangle is equal to c. What are the limits of change of the distance between the centre of the inscribed circle and the point of intersection of the medians?

160. The sides of a parallelogram are equal to a and b $(a \neq b)$. What are the limits of change of the cosine of the acute angle between the diagonals?

161. Three straight lines are drawn through a point M inside a triangle ABC parallel to its sides. The segments of the lines enclosed inside the triangle are equal to one another. Find their length if the sides of the triangle are a, b, and c.

162. Three equal circles are drawn inside a triangle ABC each of which touches two of its sides. The three circles have a common point. Find their radii if the radii of the circles inscribed in and circumscribed about the triangle ABC are equal to r and R, respectively.

163. In a triangle ABC, a median AD is drawn, $\angle DAC + \angle ABC = 90°$ Find $\angle BAC$ if $|AB| \neq |AC|$.

164. Three circles of radii 1, 2, and 3 touch one another externally. Find the radius of the circle passing through the points of tangency of these circles.

165. A square of unit area is inscribed in an isosceles triangle, one of the sides of the square lies on the base of the triangle. Find the area of the triangle if the centres of gravity of the triangle and square are known to coincide.

166. In an equilateral triangle ABC, the side is equal to a. Taken on the side BC is a point D, and on the side AB a point E such that $|BD| = a/3$, $|AE| = |DE|$. Find $|CE|$.

167. Given a right triangle ABC. The angle bisector CL ($|CL| = a$) and the median CM ($|CM| = b$) are drawn from the vertex of the right angle C. Find the area of the triangle ABC.

168. A circle is inscribed in a trapezoid. Find the area of the trapezoid given the length a of one of the bases and the line segments b and d into which one of the lateral sides is divided by the point of tangency (the segment b adjoins the base a).

169. The diagonals of a trapezoid are equal to 3 and 5, and the line segment joining the midpoints of the bases is equal to 2. Find the area of the trapezoid.

170. A circle of radius 1 is inscribed in a triangle ABC for which $\cos B = 0.8$. This circle touches the midline of the triangle ABC parallel to the side AC. Find AC.

171. Given a regular triangle ABC of area S. Drawn parallel to its sides at equal distances from them are three straight lines intersecting inside the triangle to form a triangle $A_1B_1C_1$ whose area is Q. Find the distance between the parallel sides of the triangles ABC and $A_1B_1C_1$.

172. The sides AB and CD of a quadrilateral $ABCD$ are mutually perpendicular; they are the diameters of two equal circles of radius r which touch each other. Find the area of the quadrilateral $ABCD$ if $|BC| : |AD| = k$.

173. Two circles touching each other are inscribed in an angle whose size is α. Determine the ratio of the radius of the smaller circle to the radius of a third circle touching both the circles and one of the sides of the angle.

174. In a triangle ABC, circle intersecting the sides AC and BC at points M and N, respectively, is constructed on the midline DE, parallel to AB, as on the diameter. Find $|MN|$ if $|BC| = a$, $|AC| = b$, $|AB| = c$.

175. The distance between the centres of two circles is equal to a. Find the side of a rhombus two opposite vertices of which lie on one circle, and the other two on the other if the radii of the circles are R and r.

176. Find the area of the rhombus $ABCD$ if the radii of the circles circumscribed about the triangles ABC and ABD are R and r, respectively.

177. Given an angle of size α with vertex at A and a point B at distances a and b from the sides of the angle. Find $|AB|$.

178. In a triangle ABC, the altitudes h_a and h_b drawn from the vertices A and B, respectively, and the length l of the bisector of the angle C are given. Find $\angle C$.

179. A circle is circumscribed about a right triangle. Another circle of the same radius touches the legs of this triangle, one of the vertices of the triangle being one of the points of tangency. Find the ratio of the area of the triangle to the area of the common part of the two given circles.

180. Given in a trapezoid $ABCD$: $|AB| = |BC| = |CD| = a$, $|DA| = 2a$. Taken respectively on the straight lines AB and AD are points E and F, other than the vertices of the trapezoid, so that the point of intersection of the altitudes of the triangle CEF coincides with the point of intersection of the diagonals of the trapezoid $ABCD$. Find the area of the triangle CEF.

* * *

181. The altitude of a right triangle ABC drawn to the hypotenuse AB is h, D being its foot; M and N are the midpoints of the line segments AD and DB, respectively.

Find the distance from the vertex C to the point of intersection of the altitudes of the triangle CMN.

182. Given an equilateral trapezoid with bases AD and BC: $|AB| = |CD| = a$, $|AC| = |BD| = b$, $|BC| = c$, M an arbitrary point of the arc BC of the circle circumscribed about $ABCD$. Find the ratio $\frac{|BM| + |MC|}{|AM| + |MD|}$.

183. Each lateral side of an isosceles triangle is equal to 1, the base being equal to a. A circle is circumscribed about the triangle. Find the chord intersecting the lateral sides of the triangle and divided by the points of intersection into three equal segments.

184. MN is a diameter of a circle, $|MN| = 1$, A and B are points on the circle situated on one side from MN, C is a point on the other semicircle. Given: A is the midpoint of semicircle, $|MB| = 3/5$, the length of the line segment formed by the intersection of the diameter MN with the chords AC and BC is equal to a. What is the greatest value of a?

185. $ABCD$ is a convex quadrilateral. M the midpoint of AB, N the midpoint of CD. The areas of triangles ABN and CDM are known to be equal, and the area of their common part is $1/k$ of the area of each of them. Find the ratio of the sides BC and AD.

186. Given an equilateral trapezoid $ABCD$ $(AD \parallel BC)$ whose acute angle at the larger base is equal to $60°$, the diagonal being equal to $\sqrt{3}$. The point M is found at distances 1 and 3 from the vertices A and D, respectively. Find $|MC|$.

187. The bisector of each angle of a triangle intersects the opposite side at a point equidistant from the midpoints of the two other sides of the triangle. Does it, in fact, mean that the triangle is regular?

188. Given in a triangle are two sides: a and b $(a > b)$. Find the third side if it is known that $a + h_a \leqslant b + h_b$, where h_a and h_b are the altitudes dropped on these sides (h_a the altitude drawn to the side a).

189. Given a convex quadrilateral $ABCD$ circumscribed about a circle of diameter 1. Inside $ABCD$, there is a point M such that $|MA|^2 + |MB|^2 + |MC|^2 + |MD|^2 = 2$. Find the area of $ABCD$.

190. Given in a quadrilateral $ABCD$: $|AB| = a$, $|BC| = b$, $|CD| = c$, $|DA| = d$; $a^2 + c^2 \neq b^2 + d^2$, $c \neq d$, M is a point on BD equidistant from A and C. Find the ratio $|BM| \quad |MD|$.

191. The smaller side of the rectangle $ABCD$ is equal to 1. Consider four concentric circles centred at A and passing, respectively, through B, C, D, and the intersection point of the diagonals of the rectangle $ABCD$. There also exists a rectangle with vertices on the constructed circles (one vertice per circle). Prove that there is a square whose vertices lie on the constructed circles. Find its side.

192. Given a triangle ABC. The perpendiculars erected to AB and BC at their midpoints intersect the line AC at points M and N such that $|MN| = |AC|$. The perpendiculars erected to AB and AC at their midpoints intersect BC at points K and L such that $|KL| = \frac{1}{2}|BC|$. Find the smallest angle of the triangle ABC.

193. A point M is taken on the side AB of a triangle ABC such that the straight line joining the centre of the circle circumscribed about the triangle ABC to the median point of the triangle BCM is perpendicular to CM. Find the ratio $|BM|$ $|BA|$ if $|BC|$ $|BA| = k$.

194. In an inscribed quadrilateral $ABCD$ where $|AB| = |BC|$, K is the intersection point of the diagonals. Find $|AB|$ if $|BK| = b$, $|KD| = d$.

195. Give the geometrical interpretations of equation (1) and systems (2), (3), and (4). Solve equation (1) and systems (2) and (3). In system (4) find $x + y + z$:

$$(1)\ \sqrt{x^2+a^2-ax\sqrt{3}}$$
$$+\sqrt{y^2+b^2-by\sqrt{3}}$$

$$+\sqrt{x^2+y^2-xy\sqrt{3}}$$
$$=\sqrt{a^2+b^2}\quad (a>0,\ b>0).$$

(2) $$\begin{cases} x=\sqrt{z^2-a^2}+\sqrt{y^2-a^2}, \\ y=\sqrt{x^2-b^2}+\sqrt{z^2-b^2}, \\ z=\sqrt{y^2-c^2}+\sqrt{x^2-c^2}. \end{cases}$$

(3) $x^2+y^2=(a-x^2)+b^2=a^2+(b-y^2.)$

(4) $$\begin{cases} x^2+xy+y^2=a^2, \\ y^2+yz+z^2=b^2, \\ z^2+zx+x^2=a^2+b^2. \end{cases}$$

196. The side of a square is equal to a and the products of the distances from the opposite vertices to a line l are equal to each other. Find the distance from the centre of the square to the line l if it is known that neither of the sides of the square is parallel to l.

197. One of the sides in a triangle ABC is twice the length of the other and $\angle B = 2\angle C$. Find the angles of the triangle.

198. A circle touches the sides AB and AC of an isosceles triangle ABC. Let M be the point of tangency with the side AB and N the point of intersection of the circle and the base BC. Find $|AN|$ if $|AM| = a$, $|BM| = b$.

199. Given a parallelogram $ABCD$ in which $|AB| = k\,|BC|$, K and L are points on the line CD (K on the side CD), and M is a point on BC, AD being the bisector of the angle KAL, AM the bisector of the angle KAB, $|BM| = a$, $|DL| = b$. Find $|AL|$.

200. Given a parallelogram $ABCD$. A straight line passing through the vertex C intersects the lines AB and AD at points K and L, respectively. The areas of the triangles KBC and CDL are equal to p and q, respectively. Find the area of the parallelogram $ABCD$.

201. Given a circle of radius R and two points A and B on it such that $|AB| = a$. Two circles of radii x and y touch the given circle at points A and B. Find: (a) the length of the common external tangent to the last circles if both of them touch the given circle in the same way (either internally or externally); (b) the length of the common internal tangent if the circle of radius x touches the given circle externally, while the circle of radius y touches the given circle internally.

202. Given in a triangle ABC: $|AB| = 12$, $|BC| = 13$, $|CA| = 15$. Taken on the side AC is a point M such that the radii of the circles inscribed in the triangles ABM and BCM are equal. Find the ratio $|AM| : |MC|$.

203. The radii of the circles inscribed in and circumscribed about a triangle are equal to r and R, respectively. Find the area of the triangle if the circle passing through the centres of the inscribed and circumscribed circles and the intersection point of the altitudes of the triangle is known to pass at least through one of the vertices of the triangle.

204. Given a rectangle $ABCD$ where $|AB| = 2a$, $|BC| = a\sqrt{2}$. On the side AB, as on diameter, a semicircle is constructed externally. Let M be an arbitrary point on the semicircle, the line MD intersect AB at N, and the line MC at L. Find $|AL|^2 + |BN|^2$ (Fermat's* problem).

205. Circles of radii R and r touch each other internally. Find the side of the regular triangle, one vertex of which coincides with the point of tangency, and the other two, lying on the given circles.

206. Two circles of radii R and r ($R > r$) touch each other externally at a point A. Through a point B taken on the larger circle a straight line is drawn touching the smaller circle at C. Find $|BC|$ if $|AB| = a$.

207. In a parallelogram $ABCD$ there are three pairwise tangent circles**; one of them also touches the sides AB and BC, the second the sides AB and AD, and the third the sides BC and AD. Find the radius of the third circle if the distance between the

---

* Fermat, Pierre de (1601-1665), a French amateur mathematician.

** Any two of them have a point of tangency.

points of tangency on the side AB is equal to a.

208. The diagonals of the quadrilateral $ABCD$ intersect at a point M, the angle between them equalling α. Let O_1, O_2, O_3, O_4 denote the centres of the circles circumscribed about the triangles ABM, BCM, CDM, DAM, respectively. Determine the ratio of the areas of the quadrilaterals $ABCD$ and $O_1O_2O_3O_4$.

209. In a parallelogram whose area is S, the bisectors of its interior angles are drawn to intersect one another. The area of the quadrilateral thus obtained is equal to Q. Find the ratio of the sides of the parallelogram.

210. In a triangle ABC, a point M is taken on the side AC and a point N on the side BC. The line segments AN and BM intersect at a point O. Find the area of the triangle CMN if the areas of the triangles OMA, OAB, and OBM are equal to S_1, S_2, and S_3, respectively.

211. The median point of a right triangle lies on the circle inscribed in this triangle. Find the acute angles of the triangle.

212. The circle inscribed in a triangle ABC divides the median BM into three equal parts. Find the ratio $|BC|$ $|CA|$ $|AB|$.

213. In a triangle ABC, the midperpendicular to the side AB intersects the line AC at M, and the midperpendicular to the side

AC intersects the line AB at N. It is known that $|MN| = |BC|$ and the line MN is perpendicular to the line BC. Determine the angles of the triangle ABC.

214. The area of a trapezoid with bases AD and BC is S, $|AD| : |BC| = 3$; situated on the straight line intersecting the extension of the base AD beyond the point D there is a line segment EF such that $AE \parallel DF$, $BE \parallel CF$ and $|AE| : |DF| = |CF| \quad |BE| = 2$. Determine the area of the triangle EFD.

215. In a triangle ABC the side BC is equal to a, and the radius of the inscribed circle is r. Find the area of the triangle if the inscribed circle touches the circle constructed on BC as diameter.

216. Given an equilateral triangle ABC with side a, BD being its altitude. A second equilateral triangle BDC_1 is constructed on BD, and a third equilateral triangle BD_1C_2 is constructed on the altitude BD_1 of this triangle. Find the radius of the circle circumscribed about the triangle CC_1C_2. Prove that its centre is found on one of the sides of the triangle ABC (C_2 is situated outside the triangle ABC).

217. The sides of a parallelogram are equal to a and b ($a \neq b$). Straight lines are drawn through the vertices of the obtuse angles of this parallelogram perpendicular to its sides. When intersecting, these lines form a parallelogram similar to the given one. Find the cosine of the acute angle of the given parallelogram.

218. Two angle bisectors KN and LP intersecting at a point Q are drawn in a triangle KLM. The line segment PN has a length of 1, and the vertex M lies on the circle passing through the points N, P, and Q. Find the sides and angles of the triangle PNQ.

219. The centre of a circle of radius r touching the sides AB, AD, and BC is located on the diagonal AC of a convex quadrilateral $ABCD$. The centre of a circle of the same radius r touching the sides BC, CD, and AD is found on the diagonal BD. Find the area of the quadrilateral $ABCD$ if the indicated circles touch each other externally.

220. The radius of the circle circumscribed about an acute-angled triangle ABC is equal to 1. The centre of the circle passing through the vertices A, C, and the intersection point of the altitudes of the triangle ABC is known to lie on this circle. Find $|AC|$.

221. Given a triangle ABC in which points M, N, and P are taken: M and N on the sides AC and BC, respectively, P on the line segment MN such that $|AM| : |MC| = |CN| : |NB| = |MP| : |PN|$. Find the area of the triangle ABC if the areas of the triangles AMP and BNP are T and Q, respectively.

222. Given a circle of radius R and a point A at a distance a from its centre ($a > R$). Let K denote the point of the circle nearest to the point A. A secant line passing through A intersects the circle at points M and N. Find $|MN|$ if the area of the triangle KMN is S.

223. In an isosceles triangle ABC ($|AB| = |BC|$), a perpendicular to AE is drawn through the end point E of the angle bisector AE to intersect the extension of the side AC at a point F (C lies bet'veen A and F). It is known that $|AC| = 2m$, $|FC| = m/4$. Find the area of the triangle ABC.

224. Two congruent regular triangles ABC and CDE with side 1 are arranged on a plane so that they have only one common point C, and the angle BCD is less than $\pi/3$. K denotes the midpoint of the side AC, L the midpoint of CE, and M the midpoint of BD. The area of the triangle KLM is equal to $\sqrt{3}/5$. Find $|BD|$.

225. From a point K situated outside a circle with centre O, two tangents KM and KN (M and N points of tangency) are drawn. A point C ($|MC| < |CN|$) is taken on the chord MN. Drawn through the point C perpendicular to the line segment OC is a straight line intersecting the line segment NK at B. The radius of the circle is known to be equal to R, $\angle MKN = \alpha$, $|MC| = b$. Find $|CB|$.

226. A pentagon $ABCDE$ is inscribed in a circle. The points M, Q, N, and P are the feet of the perpendiculars dropped from the vertex E of the sides AB, BC, CD (or their extensions), and the diagonal AD, respectively. It is known that $|EP| = d$, and the ratio of the areas of the triangles MQE and PNE is k. Find $|EM|$.

227. Given a right trapezoid. A straight line, parallel to the bases of the trapezoid separates the latter into two trapezoids such that a circle can be inscribed in each of them. Determine the bases of the original trapezoid if its lateral sides are equal to c and d $(d > c)$.

228. Points P and Q are chosen on the lateral sides KL and MN of an equilateral trapezoid $KLMN$, respectively, such that the line segment PQ is parallel to the bases of the trapezoid. A circle can be inscribed in each of the trapezoids $KPQN$ and $PLMQ$, the radii of these circles being equal to R and r, respectively. Determine the bases $|LM|$ and $|KN|$.

229. In a triangle ABC, the bisector of the angle A intersects the side BC at a point D. It is known that $|AB| - |BD| = a$, $|AC| + |CD| = b$. Find $|AD|$.

230. Using the result of the preceding problem, prove that the square of the bisector of the triangle is equal to the product of the sides enclosing this bisector minus the product of the line segments of the third side into which the latter is divided by the bisector.

231. Given a circle of diameter AB. A second circle centred at A intersects the first circle at points C and D and its diameter at E. A point M distinct from the points C and E is taken on the arc CE that does not include the point D. The ray BM intersects the first circle at a point N. It is known that $|CN| = a$, $|DN| = b$. Find $|MN|$.

232. In a triangle ABC, the angle B is $\pi/4$, the angle C is $\pi/6$. Constructed on the medians BN and CN as diameters are circles intersecting each other at points P and Q. The chord PQ intersects the side BC at a point D. Find the ratio $|BD|$ $|DC|$.

233. Let AB denote the diameter of a circle, O its centre, $AB = 2R$, C a point on the circle, M a point on the chord AC. From the point M, a perpendicular MN is dropped on AB and another one is erected to AC intersecting the circle at L (the line segment CL intersects AB). Find the distance between the midpoints of AO and CL if $|AN| = a$.

234. A circle is circumscribed about a triangle ABC. A tangent to the circle passing through the point B intersects the line AC at M. Find the ratio $|AM|$ $|MC|$ if $|AB| : |BC| = k$.

235. Points A, B, C, and D are situated in consecutive order on a straight line, where $|AC| = \alpha |AB|$, $|AD| = \beta |AB|$. An arbitrary circle is described through A and B, CM and DN being two tangents to this circle (M and N are points on the circle lying on opposite sides of the line AB). In what ratio is the line segment AB divided by the line MN?

236. In a circumscribed quadrilateral $ABCD$, each line segment from A to the points of tangency is equal to a, and each line segment from C to the points of tangency is b. What is the ratio in which the diagonal AC is divided by the diagonal BD?

237. A point K lies on the base AD of the trapezoid $ABCD$. such that $|AK| = \lambda |AD|$. Find the ratio $|AM| : |MD|$, where M is the point of intersection of the base AD and the line passing through the intersection points of the lines AB and CD and the lines BK and AC.

Setting $\lambda = 1/n$ ($n = 1, 2, 3, \ldots$), divide a given line segment into n equal parts using a straight edge only given a straight line parallel to this segment.

238. In a right triangle ABC with the hypotenuse AB equal to c, a circle is constructed on the altitude CD as diameter. Two tangents to this circle passing through the points A and B touch the circle at points M and N, respectively, and, when extended, intersect at a point K. Find $|MK|$.

239. Taken on the sides AB, BC and CA

of a triangle ABC are points C_1, A_1 and B_1 such that $|AC_1| : |C_1B| = |BA_1| : |A_1C| = |CB_1| : |B_1A| = k$. Taken on the sides A_1B_1, B_1C_1, and C_1A_1 are points C_2, A_2, and B_2, such that $|A_1C_2|$ $|C_2B_1| = |B_1A_2|$ $|A_2C_1| = |C_1B_2|$ $|B_2A_1| = 1/k$. Prove that the triangle $A_2B_2C_2$ is similar to the triangle ABC and find the ratio of similitude.

240. Given in a triangle ABC are the radii of the circumscribed (R) and inscribed (r) circles. Let A_1, B_1, C_1 denote the points of intersection of the angle bisectors of the triangle ABC and the circumscribed circle. Find the ratio of the areas of the triangles ABC and $A_1B_1C_1$.

241. There are two triangles with correspondingly parallel sides and areas S_1 and S_2, one of them being inscribed in a triangle ABC, the other circumscribed about this triangle. Find the area of the triangle ABC.

242. Determine the angle A of the triangle ABC if the bisector of this angle is perpendicular to the straight line passing through the intersection point of the altitudes of this triangle and the centre of the circumscribed circle.

243. Find the angles of a triangle if the distance between the centre of the circumscribed circle and the intersection point of the altitudes is one-half the length of the largest side and equals the smallest side.

244. Given a triangle ABC. A point D

is taken on the ray BA such that $|BD| = |BA| + |AC|$. Let K and M denote two points on the rays BA and BC, respectively, such that the area of the triangle BDM is equal to the area of the triangle BCK. Find $\angle BKM$ if $\angle BAC = \alpha$.

245. In a trapezoid $ABCD$, the lateral side AB is perpendicular to AD and BC, and $|AB| = \sqrt{|AD| \cdot |BC|}$. Let E denote the point of intersection of the nonparallel sides of the trapezoid, O the intersection point of the diagonals and M the midpoint of AB. Find $\angle EOM$.

246. Two points A and B and two straight lines intersecting at O are given in a plane. Let us denote the feet of the perpendiculars dropped from the point A on the given lines by M and N, and the feet of the perpendiculars dropped from B by K and L, respectively. Find the angle between the lines MN and KL if $\angle AOB = \alpha \leqslant 90°$.

247. Two circles touch each other internally at a point A. A radius OB touching the smaller circle at C is drawn from the centre O of the larger circle. Find the angle BAC.

248. Taken inside a square $ABCD$ is a point M such that $\angle MAB = 60°$, $\angle MCD = 15°$. Find $\angle MBC$.

249. Given in a triangle ABC are two angles: $\angle A = 45°$ and $\angle B = 15°$ Taken on the extension of the side AC beyond the

point C is a point M such that $|CM| = 2|AC|$. Find $\angle AMB$.

250. In a triangle ABC, $\angle B = 60°$ and the bisector of the angle A intersects BC at M. A point K is taken on the side AC such that $\angle AMK = 30°$. Find $\angle OKC$, where O is the centre of the circle circumscribed about the triangle AMC.

251. Given a triangle ABC in which $|AB| = |AC|$, $\angle A = 80°$. (a) A point M is taken inside the triangle such that $\angle MBC = 30°$, $\angle MCB = 10°$. Find $\angle AMC$. (b) A point P is taken outside the triangle such that $\angle PBC = \angle PCA = 30°$, and the line segment BP intersects the side AC. Find $\angle PAC$.

252. In a triangle ABC, $\angle B = 100°$, $\angle C = 65°$; a point M is taken on AB such that $\angle MCB = 55°$, and a point N is taken on AC such that $\angle NBC = 80°$. Find $\angle NMC$.

253. In a triangle ABC, $|AB| = |BC|$, $\angle B = 20°$. A point M is taken on the side AB such that $\angle MCA = 60°$, and a point N on the side CB such that $\angle NAC = 50°$. Find $\angle NMC$.

254. In a triangle ABC, $\angle B = 70°$, $\angle C = 50°$. A point M is taken on the side AB such that $\angle MCB = 40°$, and a point N on the side AC such that $\angle NBC = 50°$. Find $\angle NMC$.

255. Let M and N denote the points of tangency of the inscribed circle with the sides BC and BA of a triangle ABC, K the intersection point of the bisector of the angle A and the line MN. Prove that $\angle AKC = 90°$.

256. Let P and Q be points of the circle circumscribed about a triangle ABC such that $|PA|^2 = |PB|\cdot|PC|$, $|QA|^2 = |QB|\cdot|QC|$ (one of the points is on the arc AB, the other on the arc AC). Find $\angle PAB - \angle QAC$ if the difference between the angles B and C of the triangle ABC is α.

257. Two fixed points A and B are taken on a given circle and $\smile AB = \alpha$. An arbitrary circle passes through the points A and B. An arbitrary line l is also drawn through the point A and intersects the circles at points C and D different from B (the point C is on the given circle). The tangents to the circles at the points C and D (C and D the points of tangency) intersect at M; N is a point on the line l such that $|CN| = |AD|$, $|DN| = |CA|$. What are the values the $\angle CMN$ can assume?

258. Prove that if one angle of a triangle is equal to 120°, then the triangle formed by the feet of its angle bisectors is right-angled.

259. Given in a quadrilateral $ABCD$: $\angle DAB = 150°$, $\angle DAC + \angle ABD = 120°$, $\angle DBC - \angle ABD = 60°$. Find $\angle BDC$.

* * *

260. Given in a triangle ABC: $|AB| = 1$. $|AC| = 2$. Find $|BC|$ if the bisectors of the exterior angles A and C are known to be congruent (i.e., the line segment of the bisectors from the vertex to the intersection point with the straight line including the side of the triangle opposite to the angle).

261. A point D is taken on the side CB of a triangle ABC such that $|CD| = \alpha |AC|$. The radius of the circle circumscribed about the triangle ABC is R. Find the distance between the centres of the circles circumscribed about the triangles ABC and ADB.

262. A circle is circumscribed about a right triangle ABC ($\angle C = 90°$). Let CD denote the altitude of the triangle. A circle centred at D passes through the midpoint of the arc AB and intersects AB at M. Find $|CM|$ if $|AB| = c$.

263. Find the perimeter of the triangle ABC if $|BC| = a$ and the segment of the straight line tangent to the inscribed circle and parallel to BC which is enclosed inside the triangle is b.

264. Three straight lines parallel to the sides of a triangle and tangent to the inscribed circle are drawn. These lines cut off three triangles from the given one. The radii of the circles circumscribed about them

are equal to R_1, R_2, and R_3. Find the radius of the circle circumscribed about the given triangle.

265. Chords AB and AC are drawn in a circle of radius R. A point M is taken on AB or on its extension beyond the point B, the distance from M to the line AC being equal to $|AC|$. Analogously a point N is taken on AC or on its extension beyond the point C, the distance from N to the line AB being equal to $|AB|$. Find MN.

266. Given a circle of radius R centred at O. Two other circles touch the given circle internally and intersect at points A and B. Find the sum of the radii of these two circles if $\angle OAB = 90°$.

267. Two mutually perpendicular intersecting chords are drawn in a circle of radius R. Find (a) the sum of the squares of the four segments of these chords into which they are divided by the point of intersection; (b) the sum of the squares of the chords if the distance from the centre of the circle to the point of their intersection is equal to a.

268. Given two concentric circles of radii r and R ($r < R$). A straight line is drawn through a point P on the smaller circle to intersect the larger circle at points B and C. The perpendicular to BC at the point P intersects the smaller circle at A. Find $|PA|^2 + |PB|^2 + |PC|^2$.

269. In a semicircle, two intersecting

chords are drawn from the end points of the diameter. Prove that the sum of the products of each chord segment that adjoins the diameter by the entire chord is equal to the square of the diameter.

270. Let a, b, c and d be the sides of an inscribed quadrilateral (a be opposite to c), h_a, h_b, h_c, and h_d the distances from the centre of the circumscribed circle to the corresponding sides. Prove that if the centre of the circle is inside the quadrilateral, then $ah_c + ch_a = bh_d + dh_b$.

271. Two opposite sides of a quadrilateral inscribed in a circle intersect at points P and Q. Find $|PQ|$ if the tangents to the circle drawn from P and Q are equal to a and b, respectively.

272. A quadrilateral is inscribed in a circle of radius R. Let P, Q, and M denote the points of intersection of the diagonals of this quadrilateral with the extensions of the opposite sides, respectively. Find the sides of the triangle PQM if the distances from P, Q, and M to the centre of the circle are a, b, and c, respectively.

273. A quadrilateral $ABCD$ is circumscribed about a circle of radius r. The point of tangency of the circle with the side AB divides the latter into segments a and b, and the point at which the circle touches the side AD divides that side into segments a and c. What are the limits of change of r?

274. A circle of radius r touches internally a circle of radius R, A being the point of tangency. A straight line perpendicular to the centre line intersects one of the circles at B, the other at C. Find the radius of the circle circumscribed about the triangle ABC.

275. Two circles of radii R and r intersect each other, A being one of the points of intersection, BC a common tangent (B and C points of tangency). Find the radius of the circle circumscribed about the triangle ABC.

276. Given in a quadrilateral $ABCD$: $|AB| = a$, $|AD| = b$; the sides BC, CD, and AD touch a circle whose centre is in the middle of AB. Find $|BC|$.

277. Given in an inscribed quadrilateral $ABCD$: $|AB| = a$, $|AD| = b$ $(a > b)$. Find $|BC|$ if BC, CD, and AD touch a circle whose centre lies on AB.

* * *

278. In a convex quadrilateral $ABCD$, $|AB| = |AD|$. Inside the triangle ABC, a point M is taken such that $\angle MBA = \angle ADC$, $\angle MCA = \angle ACD$. Find $\angle MAC$ if $\angle BAC = \alpha$, $\angle ADC - \angle ACD = \varphi$, $|AM| < |AB|$.

279. Two intersecting circles are inscribed in the same angle, A being the vertex of the angle, B one of the intersection points of the circles, C the midpoint of the chord whose end points are the points of tangency of the first circle with the sides of the angle. Find the angle ABC if the common chord can be observed from the centre of the second circle at an angle α.

280. In an isosceles triangle ABC, $|AC| = |BC|$, BD is an angle bisector, $BDEF$ is a rectangle. Find $\angle BAF$ if $\angle BAE = 120°$

281. A circle centred at O is circumscribed about a triangle ABC. A tangent to the circle at point C intersects the line bisecting the angle B at a point K, the angle BKC being one-half the difference between the triple angle A and the angle C of the triangle. The sum of the sides AC and AB is equal to $2 + \sqrt{3}$ and the sum of the distances from the point O to the sides AC and AB equals 2. Find the radius of the circle.

282. The points symmetric to the vertices of a triangle with respect to the opposite sides represent the vertices of the triangle with sides $\sqrt{8}$, $\sqrt{8}$, $\sqrt{14}$. Determine the sides of the original triangle if their lengths are different.

283. In a triangle ABC, the angle between the median and altitude emanating from the angle A is α, and the angle between the median and altitude emanating from B is β. Find the angle between the median and altitude emanating from the angle C.

284. The radius of the circle circumscribed about a triangle is R. The distance from the centre of the circle to the median point of the triangle is d. Find the product of the area of the given triangle and the triangle formed by the lines passing through its vertices perpendicular to the medians emanating from those vertices.

285. The points A_1, A_3 and A_5 are situated on one straight line, and the points A_2, A_4, and A_6 on the other intersecting the first line. Find the angles between these lines if it is known that the sides of the hexagon $A_1A_2A_3A_4A_5A_6$ (possibly, a self-intersecting one) are equal to one another.

286. Two circles with centres O_1 and O_2 touch internally a circle of radius R centred at O. It is known that $|O_1O_2| = a$. A straight line touching the first two circles and intersecting the line segment O_1O_2 intersects their common external tangents at points M and N and the larger circle at points A and B. Find the ratio $|AB|$ $|MN|$ if (a) the line segment O_1O_2 contains the point O; (b) the circles with centres O_1 and O_2 touch each other.

287. The circle inscribed in a triangle ABC touches the side AC at a point M and the side BC at N; the bisector of the angle A intersects the line MN at K, and the bisector of the angle B intersects the line MN at L. Prove that the line segments MK, NL, and KL can form a triangle. Find the area of this triangle if the area of the triangle ABC is S, and the angle C is α.

288. Taken on the sides AB and BC of a square are two points M and N such that $|BM| + |BN| = |AB|$. Prove that the lines DM and DN divide the diagonal AC into three line segments which can form a triangle, one angle of this triangle being equal to 60°.

289. Given an isosceles triangle ABC, $|AB| = |BC|$, AD being an angle bisector. The perpendicular erected to AD at D intersects the extension of the side AC at a point E; the feet of the perpendiculars dropped from B and D on AC are points M and N, respectively. Find $|MN|$ if $|AE| = a$.

290. Two rays emanate from a point A at an angle α. Two points B and B_1 are taken on one ray and two points C and C_1 on the other. Find the common chord of the circles circumscribed about the triangles ABC and AB_1C_1 if $|AB| - |AC| = |AB_1| - |AC_1| = a$.

291. Let O be the centre of a circle, C a point on this circle, M the midpoint of OC. Points A and B lie on the circle on the same side of the line OC so that $\angle AMO = \angle BMC$. Find $|AB|$ if $|AM| - |BM| = a$.

292. Let A, B, and C be three points lying on the same line. Constructed on AB, BC, and AC as diameters are three semicircles located on the same side of the line. The centre of a circle touching the three semicircles is found at a distance d from the line AC. Find the radius of this circle.

293. A chord AB is drawn in a circle of radius R. Let M denote an arbitrary point of the circle. A line segment MN ($|MN| = R$) is laid off on the ray MA and on the ray MB a line segment MK equal to the distance from M to the intersection point of the altitudes of the triangle MAB. Find $|NK|$ if the smaller of the arcs subtended by AB is equal to 2α.

294. The altitude dropped from the right angle of a triangle on the hypotenuse separates the triangle into two triangles in each of which a circle is inscribed. Determine the angles and the area of the triangle formed by the legs of the original triangle and the line passing through the centres of the circles if the altitude of the original triangle is h.

295. The altitude of a right triangle drawn to the hypotenuse is equal to h. Prove that the vertices of the acute angles of the triangle and the projections of the foot of the altitude on the legs all lie on the same circle. Determine the length of the chord cut by this circle on the line containing the altitude and the segments of the chord into which it is divided by the hypotenuse.

296. A circle of radius R touches a line l at a point A, AB is a diameter of this circle, BC is an arbitrary chord. Let D denote the foot of the perpendicular dropped from C on AB. A point E lies on the extension of CD beyond the point D, and $|ED| = |BC|$ The tangents to the circle, passing through E, intersect the line l at points K and N. Find $|KN|$

297. Given in a convex quadrilateral $ABCD$: $|AB| = a$, $|AD| = b$, $|BC| = p - a$, $|DC| = p - b$. Let O be the point of intersection of the diagonals. Let us denote the angle BAC by α. What does $|AO|$ tend to as $\alpha \to 0$?

Section 2

Selected Problems and Theorems of Plane Geometry

Carnot's Theorem

1. Given points A and B. Prove that the locus of points M such that $|AM|^2 - |MB|^2 = k$ (where k is a given number) is a straight line perpendicular to AB.

2. Let the distances from a point M to the vertices A, B, and C of a triangle ABC be a, b, and c, respectively. Prove that there is no $d \neq 0$ and no point on the plane for which the distances to the vertices in the same order can be expressed by the numbers $\sqrt{a^2 + d}$, $\sqrt{b^2 + d}$, $\sqrt{c^2 + d}$.

3. Prove that for the perpendiculars dropped from the points A_1, B_1, and C_1 on the sides BC, CA, and AB of a triangle ABC to intersect at the same point, it is necessary and sufficient that

$$|A_1B|^2 - |BC_1|^2 + |C_1A|^2 - |AB_1|^2 + |B_1C|^2 - |CA_1|^2 = 0 \text{ (\textit{Carnot's theorem})}.$$

4. Prove that if the perpendiculars dropped from the points A_1, B_1, and C_1 on the sides BC, CA, and AB of the triangle ABC, respectively, intersect at the same point, then the perpendiculars dropped from the

points A, B, and C on the lines B_1C_1, C_1A_1, and A_1B_1 also intersect at one point.

5. Given a quadrilateral $ABCD$. Let A_1, B_1, and C_1 denote the intersection points of the altitudes of the triangles BCD, ACD, and ABD. Prove that the perpendiculars dropped from A, B, and C on the lines B_1C_1, C_1A_1, and A_1B_1, respectively, intersect at the same point.

6. Given points A and B. Prove that the locus of points M such that $k\,|AM|^2 + l\,|BM|^2 = d$ (k, l, d given numbers, $k + l \neq 0$) is either a circle with centre on the line AB or a point or an empty set.

7. Let $A_1, A_2, \ldots, A_n$ be fixed points and $k_1, k_2, \ldots, k_n$ be given numbers. Then the locus of points M such that the sum $k_1\,|A_1M|^2 + k_2\,|A_2M|^2 + \ldots + k_n\,|A_nM|^2$ is constant is: (a) a circle, a point, or an empty set if $k_1 + k_2 + \ldots + k_n \neq 0$; (b) a straight line, an empty set, or the entire plane if $k_1 + k_2 + \ldots + k_n = 0$.

8. Given a circle and a point A outside the circle. Let a circle passing through A touch the given circle at an arbitrary point B, and the tangents to the second circle which are drawn through the points A and B intersect at a point M. Find the locus of points M.

9. Given points A and B. Find the locus of points M such that $|AM| : |MB| = k \neq 1$.

10. Points A, B, and C lie on a straight line (B between A and C). Let us take an arbitrary circle centred at B and denote by M the intersection point of the tangents drawn from A and C to that circle. Find the locus of points M such that the points of tangency of straight lines AM and CM with the circle belong to the open intervals AM and CM.

11. Given two circles. Find the locus of points M such that the ratio of the lengths of the tangents drawn from M to the given circles is a constant k.

12. Let a straight line intersect one circle at points A and B and the other at points C and D. Prove that the intersection points of the tangents to the first circle which are drawn at points A and B and the tangents drawn to the second circle at points C and D (under consideration are the intersection points of tangents to distinct circles) lie on a circle whose centre is found on the straight line passing through the centres of the given circles.

13. Let us take three circles each of which touch one side of a triangle and the extensions of two other sides. Prove that the perpendiculars erected to the sides of the triangle at the points of tangency of these circles intersect at the same point.

14. Given a triangle ABC. Consider all possible pairs of points M_1 and M_2 such that $|AM_1| : |BM_1| : |CM_1| = |AM_2| : |BM_2| \quad |CM_2|$. Prove that the lines

M_1M_2 pass through the same fixed point in the plane.

15. The distances from a point M to the vertices A, B, and C of a triangle are equal to 1, 2, and 3, respectively, and from a point M_1 to the same vertices to 3, $\sqrt{15}$, 5, respectively. Prove that the straight line MM_1 passes through the centre of the circle circumscribed about the triangle ABC.

16. Let A_1, B_1, C_1 denote the feet of the perpendiculars dropped from the vertices A, B, and C of a triangle ABC on the line l. Prove that the perpendiculars dropped from A_1, B_1, and C_1 on BC, CA, and AB, respectively, intersect at the same point.

17. Given a quadrilateral triangle ABC and an arbitrary point D. Let A_1, B_1, and C_1 denote the centres of the circles inscribed in the triangles BCD, CAD, and ABD, respectively. Prove that the perpendiculars dropped from the vertices A, B, and C on B_1C_1, C_1A_1, and A_1B_1, respectively, intersect at the same point.

18. Given three pairwise intersecting circles. Prove that the three common chords of these circles pass through the same point.

19. Points M and N are taken on lines AB and AC, respectively. Prove that the common chord of two circles with diameters CM and BN passes through the intersection point of the altitudes of the triangle ABC.

20. A circle and a point N are given in a

plane. Let AB be an arbitrary chord of the circle. Let M denote the point of intersection of the line AB and the tangent at the point N to the circle circumscribed about the triangle ABN. Find the locus of points M.

21. A point A is taken inside a circle. Find the locus of the points of intersection of the tangents to the circle at the end points of all possible chords passing through the point A.

22. Given numbers α, β, γ, and k. Let x, y, z denote the distances from a point M taken inside a triangle to its sides. Prove that the locus of points M such that $\alpha x + \beta y + \gamma z = k$ is either an empty set or a line segment or coincides with the set of all points of the triangle.

23. Find the locus of points M situated inside a given triangle and such that the distances from M to the sides of the given triangle can serve as sides of a certain triangle.

24. Let A_1, B_1, and C_1 be the midpoints of the sides BC, CA, and AB of a triangle ABC, respectively. Points A_2, B_2, and C_2 are taken on the perpendiculars dropped from a point M on the sides BC, CA, and AB, respectively. Prove that the perpendiculars dropped from A_1, B_1, and C_1 on the lines B_2C_2, C_2A_2, and A_2B_2, respectively, intersect at the same point.

25. Given a straight line l and three lines l_1, l_2, and l_3 perpendicular to l. Let A, B, and C denote three fixed points on the line l, A_1 an arbitrary point on l_1, B_1 an arbitrary point on l_2, C_1 an arbitrary point on l_3. Prove that if at a certain arrangement of the points A_1, B_1, and C_1 the perpendiculars dropped from A, B, and C on the lines B_1C_1, C_1A_1, and A_1B_1, respectively, intersect at one and the same point, then these perpendiculars meet in the same point at any arrangement of A_1, B_1, C_1.

26. Let AA_1, BB_1, CC_1 be the altitudes of a triangle ABC, A_2, B_2, and C_2 be the projections of A, B, and C on B_1C_1, C_1A_1, and A_1B_1, respectively. Prove that the perpendiculars dropped from A_2, B_2, and C_2 on BC, CA, and AB, respectively, intersect at the same point.

Ceva's* and Menelaus'** Theorems. Affine Problems

27. Prove that the area of a triangle whose sides are equal to the medians of a given triangle amounts to 3/4 of the area of the latter.

---

* Ceva, Giovanni (1647-1734). An Italian mathematician who gave static and geometric proofs for concurrency of straight lines through vertices of triangles.

** Menelaus of Alexandria (first cent. A.D.). A geometer who wrote several books on plane and spherical triangles, and circles.

28. Given a parallelogram $ABCD$. A straight line parallel to BC intersects AB and CD at points E and F, respectively, and a straight line parallel to AB intersects BC and DA at points G and H, respectively. Prove that the lines EH, GF, and BD either intersect at the same point or are parallel.

29. Given four fixed points on a straight line l : A, B, C, and D. Two parallel lines are drawn arbitrarily through the points A and B, another two through C and D. The lines thus drawn form a parallelogram. Prove that the diagonals of that parallelogram intersect l at two fixed points.

30. Given a quadrilateral $ABCD$. Let O be the point of intersection of the diagonals AC and BD, M a point on AC such that $|AM| = |OC|$, N a point on BD such that $|BN| = |OD|$, K and L the midpoints of AC and BD. Prove that the lines ML, NK, and the line joining the median points of the triangles ABC and ACD intersect at the same point.

31. Taken on the side BC of a triangle ABC are points A_1 and A_2 which are symmetric with respect to the midpoint of BC. In similar fashion taken on the side AC are points B_1 and B_2, and on the side AB points C_1 and C_2. Prove that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are equivalent, and the centres of gravity of the triangles $A_1B_1C_1$, $A_2B_2C_2$, and ABC are collinear.

32. Drawn through the intersection point

M of medians of a triangle ABC is a straight line intersecting the sides AB and AC at points K and L, respectively, and the extension of the side BC at a point P (C lying between P and B). Prove that $\frac{1}{|MK|} = \frac{1}{|ML|} + \frac{1}{|MP|}$.

33. Drawn through the intersection point of the diagonals of a quadrilateral $ABCD$ is a straight line intersecting AB at a point M and CD at a point N. Drawn through the points M and N are lines parallel to CD and AB, respectively, intersecting AC and BD at points E and F. Prove that BE is parallel to CF.

34. Given a quadrilateral $ABCD$. Taken on the lines AC and BD are points K and M, respectively, such that $BK \parallel AD$ and $AM \parallel BC$. Prove that $KM \parallel CD$.

35. Let E be an arbitrary point taken on the side AC of a triangle ABC. Drawn through the vertex B of the triangle is an arbitrary line l. The line passing through the point E parallel to BC intersects the line l at a point N, and the line parallel to AB at a point M. Prove that AN is parallel to CM.

* * *

36. Each of the sides of a convex quadrilateral is divided into $(2n + 1)$ equal parts. The division points on the opposite sides

are joined correspondingly. Prove that the area of the central quadrilateral amounts to $1/(2n+1)^2$ of the area of the entire quadrilateral.

37. A straight line passing through the midpoints of the diagonals AC and BC of a quadrilateral $ABCD$ intersects its sides AB and DC at points M an N, respectively. Prove that $S_{DCM} = S_{ABN}$*.

38. In a parallelogram $ABCD$, the vertices A, B, C, and D are joined to the midpoints of the sides CD, AD, AB, and BC, respectively. Prove that the area of the quadrilateral formed by these line segments is 1/5 of the area of the parallelogram.

39. Prove that the area of the octagon formed by the lines joining the vertices of a parallelogram to the midpoints of the opposite sides is 1/6 of the area of the parallelogram.

40. Two parallelograms $ACDE$ and $BCFG$ are constructed externally on the sides AC and BC of a triangle ABC. When extended, DE and FD intersect at a point H. Constructed on the side AB is a parallelogram $ABML$, whose sides AL and BM are equal and parallel to HC. Prove that the parallelogram $ABML$ is equivalent to the sum of the parallelograms constructed on AC and BC.

---

* Here and elsewhere, such a notation symbolizes the area of the figure denoted by the subscript.

41. Two parallel lines intersecting the larger base are drawn through the end points of the smaller base of a trapezoid. Those lines and the diagonals of the trapezoid separate the trapezoid into seven triangles and one pentagon. Prove that the sum of the areas of the triangles adjoining the lateral sides and the smaller base of the trapezoid is equal to the area of the pentagon.

42. In a parallelogram $ABCD$, a point E lies on the line AB, a point F on the line AD (B on the line segment AE, D on AF), K being the point of intersection of the lines ED and FB. Prove that the quadrilaterals $ABKD$ and $CEKF$ are equivalent.

* * *

43. Consider an arbitrary triangle ABC. Let A_1, B_1, and C_1 be three points on the lines BC, CA, and AB, respectively. Using the following notation

$$R = \frac{|AC_1|}{|C_1B|} \cdot \frac{|BA_1|}{|A_1C|} \cdot \frac{|CB_1|}{|B_1A|},$$

$$R^* = \frac{\sin \angle ACC_1}{\sin \angle C_1CB} \cdot \frac{\sin \angle BAA_1}{\sin \angle A_1AC} \cdot \frac{\sin \angle CBB_1}{\sin \angle B_1BA},$$

prove that $R = R^*$.

44. For the lines AA_1, BB_1, CC_1 to meet in the same point (or for all the three to be parallel), it is necessary and sufficient that $R = 1$ (see the preceding problem), and of three points A_1, B_1, C_1 the one or all the

three lie on the sides of the triangle ABC, and not on their extensions (*Ceva's theorem*).

45. For the points A_1, B_1, C_1 to lie on the same straight line, it is necessary and sufficient that $R = 1$ (see Problem 43, Sec. 2), and of three points A_1, B_1, C_1 no points or two lie on the sides of the triangle ABC, and not on their extensions (*Menelaus' theorem*).

Remark. Instead of the ratio $\frac{|AC_1|}{|C_1B|}$ and the other two, it is possible to consider the ratios of directed line segments which are denoted by $\frac{AC_1}{C_1B}$ and defined as follows: $\left|\frac{AC_1}{C_1B}\right| = \frac{|AC_1|}{|C_1B|}$, $\frac{AC_1}{C_1B}$ is positive when the vectors $\overrightarrow{AC_1}$ and $\overrightarrow{C_1B}$ are in the same direction and $\frac{AC_1}{C_1B}$ negative if these vectors are in opposite directions. $\left(\frac{AC_1}{C_1B}\right.$ has sense only for points situated on the same straight line.$\left.\right)$ It is easily seen that the ratio $\frac{AC_1}{C_1B}$ is positive if the point C_1 lies on the line segment AB and the ratio is negative if C_1 is outside AB. Accordingly, instead of R, we shall consider the product of the ratios of directed line segments which is denoted by $\tilde{R}$. Further, we introduce the notion of directed angles.

For instance, by $\measuredangle ACC_1$ we shall understand the angle through which we have to rotate CA about C anticlockwise to bring the ray CA into coincidence with the ray CC_1. Now, instead of R^* we shall consider the product of the ratios of the sines of directed angles $\tilde{R}^*$.

Now, we have to reformulate Problems 43, 44, and 45 of this Section in the following way:

43*. Prove that $\tilde{R} = \tilde{R}^*$.

44*. For the lines AA_1, BB_1, CC_1 to meet in the same point (or to be parallel), it is necessary and sufficient that $\tilde{R} = 1$ (*Ceva's theorem*).

45*. For the points A_1, B_1, C_1 to be collinear, it is necessary and sufficient that $\tilde{R} = -1$ (*Menelaus' theorem*).

46. Prove that if three straight lines, passing through the vertices of a triangle, meet in the same point, then the lines symmetric to them with respect to the corresponding angle bisectors of the triangle also intersect at one point or are parallel.

47. Let O denote an arbitrary point in a plane, M and N the feet of the perpendiculars dropped from O on the bisectors of the interior and exterior angle A of a triangle ABC; P and Q are defined in a similar manner for the angle B; R and T for the angle C. Prove that the lines MN, PQ, and RT

intersect at the same point or are parallel.

48. Let O be the centre of the circle inscribed in a triangle ABC, A_0, B_0, C_0 the points of tangency of this circle with the sides BC, CA, AB, respectively. Taken on the rays OA_0, OB_0, OC_0 are points L, M, K, respectively, equidistant from the point O. (a) Prove that the lines AL, BM, and CK meet in the same point. (b) Let A_1, B_1, C_1 be the projections of A, B, C, respectively, on an arbitrary line l passing through O. Prove that the lines A_1L, B_1M, and C_1K are concurrent (that is, intersect at a common point).

49. For the diagonals AD, BE, and CF of the hexagon $ABCDEF$ inscribed in a circle to meet in the same point, it is necessary and sufficient that the equality $|AB|\times |CD|\cdot|EF| = |BC|\cdot|DE|\ |FA|$ be fulfilled.

50. Prove that: (a) the bisectors of the exterior angles of a triangle intersect the extensions of its opposite sides at three points lying on the same straight line; (b) the tangents drawn from the vertices of the triangle to the circle circumscribed about it intersect its opposite sides at three collinear points.

51. A circle intersects the side AB of a triangle ABC at points C_1 and C_2, the side CA at points B_1 and B_2, the side BC at points A_1 and A_2. Prove that if the lines AA_1,

BB_1, and CC_1 meet in the same point, then the lines AA_2, BB_2, and CC_2 also intersect at the same point.

52. Taken on the sides AB, BC, and CA of a triangle ABC are points C_1, A_1, and B_1. Let C_2 be the intersection point of the lines AB and A_1B_1, A_2 the intersection point of the lines BC and B_1C_1, B_2 the intersection point of the lines AC and A_1C_1. Prove that if the lines AA_1, BB_1, and CC_1 meet in the same point, then the points A_2, B_2, and C_2 lie on a straight line.

53. A straight line intersects the sides AB, BC, and the extension of the side AC of a triangle ABC at points D, E, and F, respectively. Prove that the midpoints of the line segments DC, AE, and BF lie on a straight line (*Gaussian* line*).

54. Given a triangle ABC. Let us define a point A_1 on the side BC in the following way: A_1 is the midpoint of the side KL of a regular pentagon $MKLNP$ whose vertices K and L lie on BC, and the vertices M and N on AB and AC, respectively. Defined in a similar way on the sides AB and AC are points C_1 and B_1. Prove that the lines AA_1, BB_1, and CC_1 intersect at the same point.

---

* Gauss, Carl Friedrich (1777-1855). A German mathematician.

55. Given three pairwise* nonintersecting circles. Let us denote by A_1, A_2, A_3 the three points of intersection of common internal tangents to any two of them and by B_1, B_2, B_3 the corresponding points of intersection of the external tangents. Prove that these points are situated on four lines, three on each of them (A_1, A_2, B_3; A_1, B_2, A_3; B_1, A_2, A_3; B_1, B_2, B_3).

56. Prove that if the straight lines passing through the vertices A, B, and C of a triangle ABC parallel to the lines B_1C_1, C_1A_1, and A_1B_1 meet in the same point, then the straight lines passing through A_1, B_1, and C_1 parallel to the lines BC, CA, and AB also intersect at the same point (or are parallel).

57. Given a triangle ABC, M being an arbitrary point in its plane. The bisectors of two angles formed by the lines AM and BM intersect the line AB at points C_1 and C_2 (C_1 lying on the line segment AB), determined similarly on BC and CA are points A_1, A_2, and B_1, B_2, respectively. Prove that the points A_1, A_2, B_1, B_2, C_1, C_2 are situated on four lines, three on each of them.

58. Points A_1, B_1, C_1 are taken on the sides BC, CA, and AB of a triangle ABC, respectively, and points A_2, B_2, C_2 on the sides B_1C_1, C_1A_1, A_1B_1 of the triangle $A_1B_1C_1$. The lines AA_1, BB_1, CC_1 meet in

* No two of which intersect.

the same point, and the lines A_1A_2, B_1B_2, C_1C_2, also intersect at one point. Prove that the lines AA_2, BB_2, CC_2 are either concurrent or parallel.

59. In a quadrilateral $ABCD$, P is the intersection point of BC and AD, Q that of CA and BD, and R that of AB and CD. Prove that the intersection points of BC and QR, CA and RP, AB and PQ are collinear.

60. Given an angle with vertex O. Points A_1, A_2, A_3, A_4 are taken on one side of the angle and points B_1, B_2, B_3, B_4 on the other side. The lines A_1B_1 and A_2B_2 intersect at a point N, and the lines A_3B_3 and A_4B_4 at a point M. Prove that for the points O, N and M to be collinear, it is necessary and sufficient that the following equality be fulfilled:

$$\frac{OB_1}{OB_3} \cdot \frac{OB_2}{OB_4} \cdot \frac{B_3B_4}{B_1B_2} = \frac{OA_1}{OA_3} \cdot \frac{OA_2}{OA_4} \cdot \frac{A_3A_4}{A_1A_2} .$$

(See Remark to Problems 43-45).

61. Given a triangle ABC. Pairs of points A_1 and A_2, B_1 and B_2, C_1 and C_2 are taken on the sides BC, CA, and AB, respectively, such that AA_1, BB_1, and CC_1 meet in the same point, and AA_2, BB_2, and CC_2 also intersect at one point. Prove that: (a) the points of intersection of the lines A_1B_1 and AB, B_1C_1 and BC, C_1A_1 and CA lie on a straight line l_1. Just in the same way, the points A_2, B_2, and C_2 determine a straight line l_2, (b) the point A, the intersection point

of the lines l_1 and l_2 and the intersection point of the lines B_1C_1 and B_2C_2 lie on one straight line; (c) the intersection points of the lines BC and B_2C_1, CA and C_2A_2, AB and A_1B_1 are collinear.

62. An arbitrary straight line intersects the lines AB, BC, and CA at points K, M, and L, respectively, and the lines A_1B_1, B_1C_1, and C_1A_1 at points K_1, M_1, and L_1. Prove that if the lines A_1M, B_1L, and C_1K meet in the same point, then the lines AM_1, BL_1, and CK_1 are also concurrent.

63. Given a triangle ABC and a point D. Points E, F, and G are situated on the lines AD, BD, and CD, respectively, K is the intersection point of AF and BE, L the intersection point of BG and CF, M the intersection point of CE and AG, P, Q, and R are the intersection points of DK and AB, DL and BC, DM and AC. Prove that all the six lines AL, EQ, BM, FR, CK, and GP meet in the same point.

64. The points A and A_1 are symmetric with respect to a line l, as are the pairs B and B_1, C and C_1, and N is an arbitrary point on l. Prove that lines AN, BN, CN intersect, respectively, the lines B_1C_1, C_1A_1, and A_1B_1 at three points lying on a straight line.

65. Let A_1, A_3, A_5 be three points situated on one straight line, and A_2, A_4, A_6 on the other. Prove that the three points at which the pairs of lines A_1A_2 and A_4A_5, A_2A_3

and A_5A_6, A_3A_4 and A_6A_1 intersect lie on a straight line.

Loci of Points

66. Drawn through the intersection point of two circles is a straight line intersecting the circles for the second time at points A and B. Find the locus of the midpoints of AB.

67. Given a point A and a straight line l, B being an arbitrary point on l. Find the locus of points M such that ABM is a regular triangle.

68. Given a regular triangle ABC. Points D and E are taken on the extensions of its sides AB and AC beyond the points B and C, respectively, such that $|BD|\ |CE| = |BC|^2$. Find the locus of the points of intersection of the lines DC and BE.

69. Given three points A, B, and C on a straight line, and an arbitrary point D in a plane not on the line. Straight lines parallel to AD and BD intersecting the lines BD and AD at points P and Q are drawn through the point C. Find the locus of the feet M of perpendiculars dropped from C on PQ, and find all the points D for which M is a fixed point.

70. A point K is taken on the side AC of a triangle ABC and point P on the median BD such that the area of the triangle APK is equal to the area of the triangle

BPC. Find the locus of the intersection points of the lines AP and BK.

71. Two rays forming a given angle α are passing through a given point O inside a given angle. Let one ray intersect one side of the angle at a point A, and the other ray the other side of the angle at a point B. Find the locus of the feet of the perpendiculars dropped from O on the line AB.

72. Two mutually perpendicular diameters AC and BD are drawn in a circle. Let P be an arbitrary point of the circle, and let PA intersect BD at a point E. The straight line passing through E parallel to AC intersects the line PB at a point M. Find the locus of points M.

73. Given an angle with vertex at A and a point B. An arbitrary circle passing through the points A and B intersects the sides of the angle at points C and D (different from A). Find the locus of the centres of mass of triangles ACD.

74. One vertex of a rectangle is found at a given point, two other vertices, not belonging to the same side, lie on two given mutually perpendicular straight lines. Find the locus of fourth vertices of such rectangles.

75. Let A be one of the two intersection points of two given circles; drawn through the other point of intersection is an arbitrary line intersecting one circle at a point B and the other at a point C, both points different from common points of these circles. Find the locus of: (a) the centres of the circles circumscribed about the triangle ABC; (b) the centres of mass of the triangles ABC; (c) the intersection points of the altitudes of the triangle ABC.

76. Let B and C be two fixed points of a given circle and A a variable point of this circle. Find the locus of the feet of the perpendiculars dropped from the midpoint of AB on AC.

77. Find the locus of the intersection points of the diagonals of rectangles whose sides (or their extensions) pass through four given points in the plane.

78. Given two circles touching each other internally at a point A. A tangent to the smaller circle intersects the larger one at points B and C. Find the locus of centres of circles inscribed in triangles ABC.

79. Given two intersecting circles. Find the locus of centres of rectangles with vertices lying on these circles.

80. An elastic ball whose dimensions may be neglected is found inside a round billiard table at a point A different from the centre. Indicate the locus of points A from which this ball can be directed so that after three successive boundary reflections, bypassing the centre of the billiard table, it finds itself at the point A.

81. Through a point equidistant from two given parallel lines a straight line is drawn

intersecting these lines at points M and N. Find the locus of vertices P of equilateral triangles MNP.

82. Given two points A and B and a straight line l. Find the locus of the centres of circles passing through A and B and intersecting the line l.

83. Given two points O and M. Determine: (a) the locus of points in the plane which can serve as one of the vertices of a triangle with the centre of the circumscribed circle at the point O and the centre of mass at the point M; (b) the locus of points in the plane which can serve as one of the vertices of an obtuse triangle with the centre of the circumscribed circle at the point O and the centre of mass at the point M.

84. An equilateral triangle is inscribed in a circle. Find the locus of intersection points of the altitudes of all possible triangles inscribed in the circle if two sides of the triangles are parallel to those of the given one.

85. Find the locus of the centres of all possible rectangles circumscribed about a given triangle. (A rectangle will be called *circumscribed* if one of the vertices of the triangle coincides with a vertex of the rectangle, and two others lie on two sides of the rectangle not including this vertex.)

86. Given two squares whose sides are respectively parallel. Determine the locus of points M such that for any point P of the

first square there is a point Q of the second one such that the triangle MPQ is equilateral. Let the side of the first square be a and that of the second square is b. For what relationship between a and b is the desired locus not empty?

87. Inside a given triangle, find the locus of points M for each of which and for any point N on the boundary of the triangle there is a point P, inside the triangle or on its boundary, such that the area of the triangle MNP is not less than 1/6 of the area of the given triangle.

88. Given two points A and I. Find the locus of points B such that there exists a triangle ABC with the centre of the inscribed circle at the point I, all of whose angles are less than α ($60° < \alpha < 90°$).

89. Points A, B, and C lie on the same straight line (B is found between A and C). Find the locus of points M such that $\cot \angle AMB + \cot \angle BMC = k$.

90. Given two points A and Q. Find the locus of points B such that there exists an acute triangle ABC for which Q is the centre of mass.

91. Given two points A and H. Find the locus of points B such that there is a triangle ABC for which H is the point of intersection of its altitudes, and each of whose angles is greater than α ($\alpha < \pi/4$).

92. Given two rays in a plane. Find the locus of points in the plane equidistant

from these rays. (The distance from a point to a ray is equal to the distance from this point to the nearest point of the ray.)

93. Given an angle and a circle centred at O inscribed in this angle. An arbitrary line touches the circle and intersects the sides of the angle at points M and N. Find the locus of the centres of circles circumscribed about the triangle MON.

94. Given two circles and two points A and B (one on either circle) equidistant from the midpoint of the line segment joining their centres. Find the locus of the midpoints of line segments AB.

95. Given a line segment AB. Let us take an arbitrary point M on AB and consider two squares $AMCD$ and $MBEF$ situated on the same side of AB. We then circumscribe circles about these squares and denote the point of their intersection by N (N is different from M). Prove that: (a) AF and BC intersect at N; (b) MN passes through a fixed point in the plane. Find the locus of the midpoints of line segments joining the centres of the squares.

96. Given a circle and a point A. Let M denote an arbitrary point on the circle. Find the locus of points of intersection of the midperpendicular to the line segment AM and the tangent to the circle passing through the point M.

97. Two circles touch each other at a point A. One line passing through A intersects these circles for the second time at points B and C, the other line—at points B_1 and C_1 (B and B_1 lie on the same circle). Find the locus of points of intersection of the circles circumscribed about the triangles AB_1C and ABC_1.

98. Find the locus of the vertices of right angles of all possible right isosceles triangles the end points of whose hypotenuses lie on two given circles.

99. The sides of a given triangle serve as diagonals of three parallelograms. The sides of the parallelograms are parallel to two straight lines l and p. Prove that the three diagonals of these parallelograms, different from the sides of the triangle, intersect at a point M. Find the locus of points M if l and p are arbitrary and mutually perpendicular.

100. Let B and C denote two fixed points of a circle, A being an arbitrary point of the circle. Let H be the intersection point of the altitudes of the triangle ABC and M be the projection of H on the bisector of the angle BAC. Find the locus of points M.

101. Given a triangle ABC. Let D be an arbitrary point on the line BC. Straight lines passing through D parallel to AB and AC intersect AC and AB at points E and F, respectively. Find the locus of the centres of circles passing through the points D, E, and F.

102. Given a regular triangle ABC.

Find the locus of points M inside the triangle such that $\angle MAB + \angle MBC + \angle MCA = \pi/2$.

103. A point M is taken inside a triangle such that there is a straight line l passing through M and separating the triangle into two parts so that in the symmetric mapping with respect to l one part turns out to be inside, or at the boundary of, the other. Find the locus of points M.

Triangles. A Triangle and a Circle

104. From the vertex A of a triangle ABC, perpendiculars AM and AN are dropped on the bisectors of the exterior angles B and C of the triangle. Prove that the line segment MN is equal to half the perimeter of the triangle ABC.

105. An altitude BD is drawn in a triangle ABC, AN is perpendicular to AB, CM is perpendicular to BC, and $|AN| = |DC|$, $|CM| = |AD|$. Prove that M and N are equidistant from the vertex B.

106. Prove that for any right triangle the radius of the circle which touches internally the circumscribed circle and the legs is equal to the diameter of the inscribed circle.

107. Prove that if one of the sides of a triangle lies on a fixed line in a plane and if the point of intersection of the altitudes coincides with the fixed point, then the circle circumscribed about this triangle also passes through the fixed point.

108. Given a triangle ABC. Let A_1, B_1 and C_1 be the points of the circle circumscribed about ABC and diametrically opposite to the vertices A, B, and C, respectively. Straight lines parallel to BC, CA, and AB are drawn through A_1, B_1, and C_1, respectively. Prove that the triangle formed by these lines is homothetic to the triangle ABC, with the ratio of similitude 2 and centre at the intersection point of the altitudes of the triangle ABC.

109. Prove that the projections of the foot of the altitude of a triangle on the sides enclosing this altitude and on the two other altitudes lie on one straight line.

110. In a triangle ABC, a point D is taken on the side AB extended beyond the point B such that $|BD| = |CB|$ In the same manner, taken on the extension of the side CB beyond the point B is a point F such that $|BF| = |AB|$. Prove that the points A, C, D, and F lie on the same circle whose centre is found on the circle circumscribed about the triangle ABC.

111. Three equal circles pass through a point H. Prove that H is the intersection point of the altitudes of the triangle whose vertices coincide with three other points of pairwise intersection of the circles.

112. Let P denote an arbitrary point of the circle circumscribed about a rectangle. Two straight lines passing through the point P parallel to the sides of the rectangle intersect the sides of the rectangle or their extensions at points K, L, M, and N. Prove that N is the intersection point of the altitudes of the triangle KLM. Prove also that the feet of the altitudes of the triangle KLM, different from P, lie on the diagonals of the rectangle.

113. Drawn in a triangle ABC are the angle bisectors AD, BE, and CF. The straight line perpendicular to AD and passing through the midpoint of AD intersects AC at a point P. The straight line perpendicular to BE and passing through the midpoint of BE intersects AB at a point Q. Finally, the straight line perpendicular to CF and passing through the midpoint of CF intersects CB at a point R. Prove that the triangles DEF and PQR are equivalent.

114. In an isosceles triangle ABC ($|AB| = |BC|$), D is the midpoint of AC, E the projection of D on BC, F the midpoint of DE. Prove that the lines BF and AE are mutually perpendicular.

115. A circle inscribed in a triangle ABC touches the sides AB and AC at points C_1 and B_1, and the circle touching the side BC and the extensions of AB and AC touches the lines AB and AC at points C_2 and B_2. Let D be the midpoint of the side BC.

The line AD intersects the lines B_1C_1 and B_2C_2 at points E and F. Prove that $BECF$ is a parallelogram.

116. A bisector AD of an interior angle is drawn in a triangle ABC. Let us construct a tangent l to the circumscribed circle at a point A. Prove that the straight line drawn through D parallel to l touches the inscribed circle of the triangle ABC.

117. A straight line is drawn in a triangle ABC to intersect the sides AC and BC at points M and N such that $|MN| = |AM| + |BN|$ Prove that all such lines touch the same circle.

118. Prove that the points symmetric to the centre of the circle circumscribed about a triangle with respect to the midpoints of its medians lie on the altitudes of the triangle.

119. Prove that if the altitude of a triangle is $\sqrt{2}$ times the radius of the circumscribed circle, then the straight line joining the feet of the perpendiculars dropped from the foot of this altitude on the sides enclosing it passes through the centre of the circumscribed circle.

120. Let ABC be a right triangle ($\angle C = 90°$), CD its altitude, K a point in the plane such that $|AK| = |AC|$ Prove that the diameter of the circle circumscribed about the triangle ABK passing through the vertex A is perpendicular to the line DK.

121. In a triangle ABC a line is drawn through the vertex A parallel to BC; a point D is taken on this line such that $|AD| = |AC| + |AB|$; the line segment DB intersects the side AC at a point E. Prove that the line drawn through the point E parallel to BC passes through the centre of the circle inscribed in the triangle ABC.

122. Two circles pass through a vertex of an angle and a point lying on the angle bisector. Prove that the segments of the sides of the angle enclosed between the circles are congruent.

123. Given a triangle ABC and a point D. The line AD, BD, and CD for the second time intersect the circle circumscribed about the triangle ABC at points A_1, B_1, and C_1, respectively. Consider two circles: the first passes through A and A_1, the second through B and B_1. Prove that the end points of the common chord of these two circles and the points C and C_1 lie on the same circle.

124. Three parallel lines l_1, l_2, and l_3 are drawn through the vertices A, B, and C of a triangle ABC, respectively. Prove that the lines symmetric to l_1, l_2, and l_3 with respect to the bisectors of the angles A, B, and C, respectively, intersect at a point situated on the circle circumscribed about the triangle ABC.

125. Prove that if M is a point inside a triangle ABC and the lines AM, BM, and CM pass, respectively, through the centres of the circles circumscribed about the triangles BMC, CMA, and AMB, then M is the centre of the circle inscribed in the triangle ABC.

126. In a triangle ABC points A_1, B_1, and C_1 are taken on the sides BC, CA, and AB, respectively. Let M be an arbitrary point in the plane. The straight line BM intersects for the second time a circle passing through A_1, B, and C_1 at a point B_2, CM intersects the circle described through A_1, B_1, and C at a point C_2, and AM—the circle passing through A, B_1, and C_1 at a point A_2. Prove that the points A_2, B_2, C_2, and M lie on the same circle.

127. Let A_1 be a point symmetric to the point of tangency of the circle inscribed in a triangle ABC to the side BC with respect to the bisector of the angle A. Points B_1 and C_1 can be determined in a similar way. Prove that the lines AA_1, BB_1, CC_1, and the line passing through the centres of the circles inscribed in and circumscribed about the triangle ABC meet in the same point.

128. Let AA_1, BB_1, CC_1 be the altitudes of a triangle ABC. A straight line perpendicular to AB intersects AC and A_1C_1 at points K and L. Prove that the centre of the circle circumscribed about the triangle KLB_1 lies on the straight line BC.

129. Four circles of equal radius pass through a point A. Prove that three line segments whose end points are different from A and are the points of intersection of two circles (the opposite end points of each line segment do not belong to one circle) meet in the same point.

130. Given a right triangle ABC with a right angle C. Let O be the centre of the circumscribed circle, M the point of tangency of the inscribed circle and the hypotenuse. Let a circle centred at M passing through O intersect the bisectors of the angles A and B at points K and L different from O. Prove that K and L are the centres of the circles inscribed in the triangles ACD and BCD, respectively, where CD is the altitude of the triangle ABC.

131. Prove that in a triangle ABC the bisector of the angle A, the midline parallel to AC, and the straight line joining the points of tangency of the inscribed circle with the sides CB and CA intersect at the same point.

132. Given three straight lines. One of them passes through the feet of two altitudes of a triangle, the second line through the end points of two of its angle bisectors, and the third through two points at which the inscribed circle touches the triangle sides (all the points are situated on two sides of the triangle). Prove that the three straight lines intersect at one point.

133. In a triangle ABC points A_1, B_1, and C_1 are taken on the sides BC, CA, and AB, respectively, such that the lines AA_1, BB_1, and CC_1 meet in the same point. Prove that if AA_1 is the bisector of the angle $B_1A_1C_1$, then AA_1 is the altitude of the triangle ABC.

134. Taken on the sides BC, CA, and AB of a triangle ABC are points A_1, B_1, and C_1, respectively, such that $\angle AA_1C = \angle BB_1A = \angle CC_1B$ (the angles are measured in the same direction). Prove that the centre of the cirle circumscribed about the triangle bounded by the lines AA_1, BB_1, and CC_1 coincides with the intersection point of the altitudes of the triangle ABC.

135. The vertices of a triangle $A_1B_1C_1$ lie on the straight lines BC, CA, and AB (A_1 on BC, B_1 on CA, C_1 on AB). Prove that if the triangles ABC and $A_1B_1C_1$ are similar (the vertices A and A_1, B and B_1, C and C_1 are similar pairwise), then the intersection point of the altitudes of the triangle $A_1B_1C_1$ is the centre of the circle circumscribed about the triangle ABC. Is the converse true?

136. Two points are taken on each side of a triangle such that all the six line segments joining each point to the opposite vertex are congruent. Prove that the midpoints of the six segments lie on the same circle.

137. In a triangle ABC, line segments $|AM| = |CN| = p$ are laid off on the rays AB and CB, where p is the half-perimeter of the triangle (B lies between A and M, and between C and N). Let K be a point on the circle circumscribed about the triangle ABC and diametrically opposite to the point B. Prove that the perpendicular dropped from K on MN passes through the centre of the inscribed circle.

138. From a point on the circle circumscribed about an equilateral triangle ABC straight lines are drawn parallel to BC, CA, and AB which intersect CA, AB, and BC at points M, N, and Q, respectively. Prove that M, N, and Q lie on a straight line.

139. Prove that three lines which are symmetric to an arbitrary straight line passing through the intersection point of the altitudes of a triangle with respect to the sides of the triangle are concurrent.

140. Let M be an arbitrary point in the plane, G the centre of mass of a triangle ABC. Then the following equality is fulfilled: $3\,|MG|^2 = |MA|^2 + |MB|^2 + |MC|^2 - \frac{1}{3}(|AB|^2 + |BC|^2 + |CA|^2)$ (*Leibniz's theorem*).

141. Let ABC be a regular triangle with side a, and M some point in the plane found at a distance d from the centre of the triangle ABC. Prove that the area of the triangle whose sides are equal to the line segments MA, MB, and MC can be expressed by the formula $S = \frac{\sqrt{3}}{12}\,|a^2 - 3d^2|$.

142. Given two regular triangles: ABC and $A_1B_1C_1$. Find the locus of points M such that the two triangles formed by the line segments MA, MB, MC and MA_1, MB_1, MC_1 are equivalent.

143. Given a triangle ABC. Line segments AK and CM are laid off on the rays AB and CB, respectively, which are equal to AC. Prove that the radius of the circle circumscribed about the triangle BKM is equal to the distance between the centres of the circles circumscribed about and inscribed in the triangle ABC, and that the straight line KM is perpendicular to the line joining the centres of the inscribed and circumscribed circles.

144. A straight line is drawn through a vertex of a triangle perpendicular to the line joining the centres of the inscribed and circumscribed circles. Prove that this line and the sides of the given triangle form two triangles for which the difference between the radii of the circumscribed circles is equal to the distance between the centres of the circles inscribed in and circumscribed about the original triangle.

145. Prove that if the lengths of the sides of a triangle form an arithmetic progression, then: (a) the radius of the inscribed circle is equal to 1/3 of the altitude dropped on the middle-length side; (b) the line joining the centre of mass of the triangle and the centre of the inscribed circle is parallel to the middle-length side; (c) the bisector of the interior angle opposite to the middle-length side is perpendicular to the line joining the centres of the inscribed and circumscribed circles; (d) for all the points of this angle bisector, the sum of distances to the sides of the triangle is constant; (e) the centre of the inscribed circle, the midpoints of the largest and smallest sides, and the vertex of the angle formed by them lie on the same circle.

146. Let K denote the midpoint of the side BC of a triangle ABC, M the foot of the altitude dropped on BC. The circle inscribed in the triangle ABC touches the side BC at a point D; the escribed circle touching the extensions of AB and AC and the side BC touches BC at a point E. A common tangent to these circles, which is different from the sides of the triangle, intersects the circle passing through K and M at points F and G. Prove that the points D, E, F, and G lie on the same circle.

* * *

147. Prove that the centre of mass of a triangle, the intersection point of the altitudes, and the centre of the circumscribed circle lie on a straight line (*Euler's* line*).

---

* Euler, Leonhard (1707-1783). A Swiss mathematician.

148. What sides are intersected by Euler's line in an acute and an obtuse triangles?

149. Let K denote a point symmetric to the centre of the cirle circumscribed about a triangle ABC with respect to the side BC. Prove that the Euler line of the triangle ABC bisects the line segment AK.

150. Prove that there is a point P on the Euler line of a triangle ABC such that the distances from the centres of mass of the triangles ABP, BCP, and CAP to the vertices C, A, and B, respectively, are equal.

151. Let P be a point inside a triangle ABC such that each of the angles APB, BPC, and CPA is equal to 120° (any interior angle of the triangle ABC is assumed to be less than 120°). Prove that the Euler lines of the triangles APB, BPC, and CPA meet in the same point.

Remark. When solving this problem use the result of Problem 296 of this section.

152. Prove that the straight line joining the centres of the circles inscribed in and circumscribed about a given triangle is the Euler line of the triangle with vertices at the points of tangency of the inscribed circle with the sides of the given triangle.

* * *

153. Prove that the feet of the perpendiculars from an arbitrary point of the circle circumscribed about the triangle upon the sides of the triangle are collinear (*Simson's** *line*).

154. Prove that the angle between two Simson's lines corresponding to two points of a circle is measured by half the arc between these points.

155. Let M be a point on the circle circumscribed about a triangle ABC. A straight line passing through M and perpendicular to BC intersects the circle for the second time at a point N. Prove that the Simson line corresponding to the point M is parallel to the line AN.

156. Prove that the projection of the side AB of a triangle ABC on the Simson line, corresponding to the point M, is equal to the distance between the projections of the point M on the sides AC and BC.

157. Let AA_1, BB_1, CC_1 be the altitudes of a triangle ABC. The lines AA_1, BB_1, CC_1 intersect the circle circumscribed about the triangle ABC for the second time at points A_2, B_2, C_2, respectively. The Simson lines corresponding to the points A_2, B_2, C_2 form a triangle $A_3B_3C_3$ (A_3 is the intersection point of the Simson lines corresponding to the points B_2 and C_2, and so forth). Prove that the centres of mass of the triangles $A_1B_1C_1$ and $A_3B_3C_3$ coincide, while the lines A_2A_3, B_2B_3, and C_2C_3 meet in the same point.

---

* Robert Simson (1687-1768). A Scottish mathematician.

158. Let A_1, B_1, and C_1 be points on the circle circumscribed about a triangle ABC such that $\cup AA_1 + \cup BB_1 + \cup CC_1 = 2k\pi$ (all the arcs are measured in the same direction, k an integer). Prove that the Simson lines for the triangle ABC corresponding to the points A_1, B_1, and C_1 meet in the same point.

159. Prove that the tangent to a parabola at its vertex is a Simson line for a triangle formed by any three intersecting tangents to the same parabola.

* * *

160. Prove that the midpoints of the sides of a triangle, the feet of its altitudes, and the midpoints of the line segments between the vertices and the intersection point of the altitudes all lie on a circle called the nine-point circle.

161. Let H denote the intersection point of the altitudes of a triangle, D the midpoint of a side, and K one of the intersection points of the line HD and the circumscribed circle, D lying between H and K. Prove that D is the midpoint of the line segment HK.

162. Let M denote the median point of a triangle, E the foot of an altitude, F one of the points of intersection of the line ME and the circumscribed circle, M lying between E and F. Prove that $|FM| = 2|EM|$.

163. The altitude drawn to the side BC of a triangle ABC intersects the circumscribed circle at a point A_1. Prove that the distance from the centre of the nine-point circle to the side BC is equal to $\frac{1}{4}\,|AA_1|$

164. In a triangle ABC, AA_1 is an altitude, H is the intersection point of the altitudes. Let P denote an arbitrary point of the circle circumscribed about the triangle ABC, M a point on the line HP such that $|HP| \quad |HM| = |HA_1| \quad |HA|$ (H lies on the line segment MP if the triangle ABC is acute-angled and outside if it is obtuse-angled). Prove that M lies on the nine-point circle of the triangle ABC.

165. In a triangle ABC, BK is the altitude drawn from the vertex B to the side AC, BL the median drawn from the same vertex, M and N the projections of the points A and C on the bisector of the angle B. Prove that all the points K, L, M, and N lie on a circle whose centre is located on the nine-point circle of the triangle ABC.

166. Let H be the intersection point of the altitudes of a triangle, and F an arbitrary point of the circumscribed circle. Prove that the Simson line corresponding to the point F passes through one of the intersection points of the line FH and the nine-point circle (see Problems 153 and 159 of the section).

167. Let l denote an arbitrary line passing through the centre of the circle circumscribed about the triangle ABC, and let A_1, B_1, and C_1 be the projections of A, B, and C on l. Three straight lines are drawn: through A_1 a line perpendicular to BC, through B_1 a line perpendicular to AC, and through C_1 a line perpendicular to AB. Prove that these three lines meet in a point situated on the nine-point circle of the triangle ABC.

168. Given a triangle ABC. AA_1, BB_1, and CC_1 are its altitudes. Prove that Euler's lines of the triangles AB_1C_1, A_1BC_1, and A_1B_1C intersect at a point P of the nine-point circles such that one of the line segments PA_1, PB_1, PC_1 is equal to the sum of the two others (*Thebault's* problem*).

169. There are three circles, each of which passes through a vertex of a triangle and through the foot of the altitude drawn from this vertex and touches the radius of the circle circumscribed about the triangle which is drawn to this vertex. Prove that all the circles intersect at two points situated on Euler's line of the given triangle.

170. Consider three circles each of which passes through one of the vertices of a triangle and through the feet of two angle bisectors (interior and exterior) emanating

---

* Thebault, Victor. A modern French geometer.

from this vertex (these circles are called *Apollonius's* circles*). Prove that: (a) these three circles intersect at two points (M_1 and M_2); (b) the line M_1M_2 passes through the centre of the circle circumscribed about the triangle; (c) the feet of the perpendiculars from the points M_1 and M_2 upon the sides of the triangle serve as vertices of two regular triangles.

171. A straight line symmetric to a median of a triangle about the bisector of the angle opposite the median is called a *symedian*. Let the symedian emanating from the vertex B of a triangle ABC intersect AC at point K. Prove that $|AK| \ |KC| = |AB|^2 : |BC|^2$.

172. Let D be an arbitrary point on the side BC of a triangle ABC. Let E and F be points on the sides AC and AB such that DE is parallel to AB, and DF is parallel to AC. A circle passing through D, E, and F intersects for the second time BC, CA, and AB at points D_1, E_1, and F_1, respectively. Let M and N denote the intersection points of DE and F_1D_1, DF and D_1E_1, respectively. Prove that M and N lie on the symedian emanating from the vertex A. If D coincides with the foot of the symedian, then the circle passing through D, E, and F touches

---

* Apollonius of Perga (*circa* 255-170 B.C.). A great Greek geometer who carried on the work of Euclid.

the side BC. (This circle is called *Tucker's** *circle.*)

173. Prove that the common chords of the circle circumscribed about a given triangle and Apollonius' circles are the three symedians of this triangle (see Problems 170 and 171 of the section).

* * *

174. Given a trapezoid $ABCD$ whose lateral side CD is perpendicular to the bases AD and BC. A circle of diameter AB intersects AD at a point P (P is different from A). The tangent to the circle at the point P intersects CD at a point M. Another tangent is drawn from M to the circle touching it at a point Q. Prove that the straight line BQ bisects CD.

175. Let M and N denote the projections of the intersection point of the altitudes of a triangle ABC on the bisectors of the interior and exterior angles B. Prove that the line MN bisects the side AC.

176. Given a circle and two points A and B on it. The tangents to the circle which pass through A and B intersect each other at a point C. A circle passing through C touches the line AB at a point B and for the sec-

---

* Tucker, Howard Gregory (b. 1922). A modern American mathematician.

ond time intersects the given circle at a point M. Prove that the line AM bisects the line segment CB.

177. Drawn to a circle from a point A, situated outside this circle, are two tangents AM and AN (M and N the points of tangency) and a secant intersecting the circle at points K and L. An arbitrary straight line l is drawn parallel to AM. Let KM and LM intersect l at points P and Q, respectively. Prove that the line MN bisects the line segment PQ.

178. A circle is inscribed in a triangle ABC. Its diameter passes through the point of tangency with the side BC and intersects the chord joining two other points of tangency at a point N. Prove that AN bisects BC.

179. A circle is inscribed in a triangle ABC. Let M be the point at which the circle touches the side AC and MK be the diameter. The line BK intersects AC at a point N. Prove that $|AM| = |NC|$

180. A circle is inscribed in a triangle ABC and touches the side BC at a point M, MK being its diameter. The line AK intersects the circle at a point P. Prove that the tangent to the circle at the point P bisects the side BC.

181. A straight line l touches a circle at a point A. Let CD be a chord parallel to l and B an arbitrary point on the line l. The lines CB and DB for the second time intersect the circle at points L and K, respectively. Prove that the line LK bisects the line segment AB.

182. Given two intersecting circles. Let A be one of the points of their intersection. Drawn from an arbitrary point lying on the extension of the common chord of the given circles to one of them two tangents touching it at points M and N. Let P and Q denote the points of intersection (distinct from A) of the straight lines MA and NA and the second circle, respectively. Prove that the line MN bisects the line segment PQ.

183. In a triangle ABC, constructed on the altitude BD as diameter is a circle intersecting the sides AB and BC at points K and L, respectively. The lines touching the circle at points K and L intersect at a point M. Prove that the line BM bisects the side AC.

184. A straight line l is perpendicular to the line segment AB and passes through B. A circle centred on l passes through A and intersects l at points C and D. The tangents to the circle at the points A and C intersect at N. Prove that the line DN bisects the line segment AB.

185. A circle is circumscribed about a triangle ABC. Let N denote the intersection point of the tangents to the circle which pass through the points B and C. M is a point of the circle such that AM is parallel to BC and K is the intersection point of MN and the circle. Prove that KA bisects BC.

186. Let A denote the projection of the centre of a circle on a straight line l. Two points B and C are taken on this line such that $|AB| = |AC|$. Two arbitrary secants each intersecting the circle at pairs of points, P, Q and M, N, respectively are drawn through B and C. Let the lines NP and MQ intersect the line l at points R and S, respectively. Prove that $|RA| = |AS|$

187. Given a triangle ABC. A_1, B_1, C_1 are the midpoints of the sides BC, CA and AB; K and L are the feet of the perpendiculars from the vertices B and C on the straight lines A_1C_1 and A_1B_1, respectively; O is the centre of the nine-point circle. Prove that the line A_1O bisects the line segment KL.

* * *

188. Let the points A_1, B_1, C_1 be symmetric to a point P with respect to the sides BC, CA, and AB of a triangle ABC. Prove that (a) the circles circumscribed about the triangles A_1BC, AB_1C, and ABC_1 have a common point; (b) the circles circumscribed about the triangles A_1B_1C, A_1BC_1, and AB_1C_1 have a common point.

189. Let AB be the diameter of a semicircle and M a point on the diameter AB.

Points C, D, E, and F lie on the semicircle so that $\angle AMD = \angle EMB$, $\angle CMA = \angle FMB$. Let P denote the intersection point of the lines CD and EF. Prove that the line PM is perpendicular to AB.

190. In a triangle ABC, the perpendicular to the side AB at its midpoint D intersects the circle circumscribed about the triangle ABC at a point E (C and E lie on the same side of AB), F is the projection of E on AC. Prove that the line DF bisects the perimeter of the triangle ABC, and that three such lines constructed for each side of the triangle are concurrent.

191. Prove that a straight line dividing the perimeter and area of a triangle in the same ratio passes through the centre of the inscribed circle.

192. Prove that three lines passing through the vertices of a triangle and bisecting its perimeter intersect at one point (called *Nagell's* point*). Let M denote the centre of mass of the triangle, I the centre of the inscribed circle, S the centre of the circle inscribed in the triangle with vertices at the midpoints of the sides of the given triangle. Prove that the points N, M, I, and S lie on a straight line and $|MN| = 2|IM|$, $|IS| = |SN|$.

---

* Nagell, Trygve (1895-1958). A Norwegian mathematician.

* * *

193. Let a, b, and c denote the sides of a triangle and $a+b+c=2p$. Let G be the median point of the triangle and O, I and I_a the centres of the circumscribed, inscribed, and escribed circles, respectively (the escribed circle touches the side BC and the extensions of the sides AB and AC), R, r, and r_a being their radii, respectively. Prove that the following relationships are valid:

(a) $a^2+b^2+c^2=2p^2-2r^2-8Rr$;

(b) $|OG|^2=R^2-\frac{1}{9}(a^2+b^2+c^2)$;

(c) $|IG|^2=\frac{1}{9}(p^2+5r^2-16Rr)$;

(d) $|OI|^2=R^2-2Rr$ (*Euler*);

(e) $|OI_a|^2=R^2+2Rr_a$;

(f) $|II_a|^2=4R(r_a-r)$.

194. Let BB_1 and CC_1 denote the bisectors of the angles B and C, respectively, of a triangle ABC. Using the notation of the preceding problem, prove that $|B_1C_1|=\frac{abc}{(b+a)(c+a)R}|OI_a|$.

195. Prove that the points which are symmetric to the centres of the escribed circles with respect to the centre of the circumscribed circle lie on a circle which is concentric with the inscribed circle whose radius is equal to the diameter of the circumscribed circle.

196. Given a triangle ABC. Prove that the sum of the areas of the three triangles the vertices of each of which are the three points of tangency of the escribed circle with the corresponding side of the triangle ABC and the extensions of two other sides is equal to twice the area of the triangle ABC plus the area of the triangle with vertices at the points of tangency of the circle inscribed in $\triangle ABC$.

197. Find the sum of the squares of the distances from the points at which the circle inscribed in the given triangle touches its sides to the centre of the circumscribed circle if the radius of the inscribed circle is r, and that of the circumscribed circle is R.

198. A circle is described through the feet of the angle bisectors in a triangle ABC. Prove that one of the chords formed by intersection of the circle with the sides of the triangle is equal to the sum of the other two chords.

199. Let AA_1, BB_1, and CC_1 be the angle bisectors of a triangle ABC, L the point of intersection of the lines AA_1 and B_1C_1, and K the point of intersection of the lines CC_1 and A_1B_1. Prove that BB_1 is the bisector of the angle LBK.

200. In a triangle ABC, points K and L are taken on the sides AB and BC such that $|AK| = |KL| = |LC|$. Through the point of intersection of the lines AL and CK a straight line is drawn parallel to the bisector of the angle B to intersect the line AB at a point M. Prove that $|AM| = |BC|$.

201. In a triangle ABC, the bisector of the angle B intersects at a point M the line passing through the midpoint of AC and the midpoint of the altitude drawn to AC; N is the midpoint of the bisector of the angle B. Prove that the bisector of the angle C is also the bisector of the angle MCN.

202. (a) Prove that if a triangle has two equal angle bisectors then such a triangle is isosceles (*Steiner's theorem*).

(b) Prove that if in a triangle ABC, the bisectors of the angles adjacent to the angles A and C are equal and are either both inside or both outside the angle ABC, then $|AB| = |BC|$. Is it true that if a triangle has two equal exterior angle bisectors, then the triangle is isosceles?

203. Given a triangle. The triangle formed by the feet of its angle bisectors is known to be isosceles. Will the statement that the given triangle is also isosceles be true?

---

* Steiner, Jakob (1796-1863). A Swiss mathematician.

* * *

204. Let $ABCDEF$ be an inscribed hexagon. Let K denote the point of intersection of AC and BF, and L the point of intersection of CE and FD. Prove that the diagonals AD and BE and the line KL intersect at the same point (*Pascal's theorem*).

205. Given a triangle ABC and a point M. A straight line passing through the point M intersects the lines AB, BC, and CA at points C_1, A_1, and B_1, respectively. The lines AM, BM, and CM intersect the circle circumscribed about the triangle ABC at points A_2, B_2, and C_2, respectively. Prove that the lines A_1A_2, B_1B_2, and C_1C_2 intersect at a point situated on the circle circumscribed about the triangle ABC.

206. Two mutually perpendicular lines are drawn through the intersection point of the altitudes of a triangle. Prove that the midpoints of the line segments intercepted by these lines on the sides of the triangle (that is, on the lines forming the triangle) lie on a straight line.

* * *

207. Given a triangle ABC and an arbitrary point P. The feet of the perpendiculars dropped from the point P on the sides of the triangle ABC serve as the vertices of the triangle $A_1B_1C_1$. The vertices of the triangle $A_2B_2C_2$ are found at the intersection points (distinct from A, B, C) of the straight lines AP, BP, and CP with the circle circumscribed about the triangle ABC. Prove that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are similar. How many points P are there for a scalene triangle ABC such that the corresponding triangles $A_1B_1C_1$ and $A_2B_2C_2$ are similar to the triangle ABC?

208. Let A_1, B_1, C_1 denote the feet of the perpendiculars dropped from an arbitrary point M on the sides BC, CA, and AB of a triangle ABC, respectively. Prove that three straight lines passing through the midpoints of the line segments B_1C_1 and MA, C_1A_1 and MB, A_1B_1 and MC intersect at one point.

209. Let S be the area of a given triangle, and R the radius of the circle circumscribed about this triangle. Let, further, S_1 denote the area of the triangle formed by the feet of the perpendiculars dropped on the sides of the given triangle from a point located at a distance d from the centre of the circumscribed circle. Prove that $S_1 = \frac{S}{4}\left|1 - \frac{d^2}{R^2}\right|$ (*Euler's theorem*).

210. Prove that if A, B, C, and D are arbitrary points in the plane, then the four circles each of which passes through the midpoints either of the line segments AB, AC, and AD; or BA, BC, and BD;

or CA, CB, and CD; or DA, DB, and DC have a common point.

211. Given a triangle ABC and an arbitrary point D in the plane. The triangle formed by the feet of the perpendiculars dropped from D on the sides of the triangle ABC will be called the *pedal triangle of the point D with respect to the triangle ABC*, and the circle circumscribed about the pedal triangle, the *pedal circle*. Let D_1 denote the point of intersection of the lines symmetric to the lines AD, BD, and CD with respect to the bisectors of the angles A, B, and C, respectively, of the triangle ABC. Prove that the pedal circles of the points D and D_1 coincide.

212. Consider four points in the plane no three of which are collinear. Prove that the four pedal circles each of which corresponds to one of the points under consideration with respect to the triangle whose vertices are the remaining three points have a common point.

213. A straight line passing through the centre of the circle circumscribed about a triangle ABC intersects AB and AC at points C_1 and B_1, respectively. Prove that the circles constructed on BB_1 and CC_1 as diameters intersect at two points one of which lies on the circle circumscribed about the triangle ABC, the other on the nine-point circle of the triangle ABC.

Quadrilaterals

214. Given an inscribed quadrilateral $ABCD$. The circle diameter is AB. Prove that the projections of the sides AD and BC on the line CD are equal in length.

215. In a convex quadrilateral $ABCD$: O is the point of intersection of its diagonals, E, F, and G are the projections of B, C, and O on AD. Prove that the area of the quadrilateral is equal to $\frac{|AD|\cdot|BE|\cdot|CF|}{2|OG|}$.

216. Let $ABCD$ be a convex quadrilateral. Consider four circles each of which touches three sides of this quadrilateral.

(a) Prove that the centres of these circles lie on one circle.

(b) Let r_1, r_2, r_3, and r_4 denote the radii of these circles (r_1 does not touch the side DC, r_2 the side DA, r_3 the side AB, and r_4 the side BC). Prove that $\frac{|AB|}{r_1}+\frac{|CD|}{r_3}=\frac{|BC|}{r_2}+\frac{|AD|}{r_4}$.

217. Prove that for the area S of an inscribed quadrilateral the following formula holds true:
$S=\sqrt{(p-a)(p-b)(p-c)(p-d)}$,
where p is the semiperimeter, and a, b, c, and d are the sides of the quadrilateral.

218. Let 2φ be the sum of two opposite angles of a circumscribed quadrilateral, a, b, c, and d its sides, S its area. Prove that $S=\sqrt{abcd}\sin\varphi$.

219. Points M and N are taken on the sides AB and CD of a convex quadrilateral $ABCD$ to divide them in the same ratio (counting from the vertices A and C). Joining these points to all the vertices of the quadrilateral $ABCD$, we separate the latter into six triangles and a quadrilateral. Prove that the area of the quadrilateral thus obtained is equal to the sum of the areas of two triangles adjacent to the sides BC and AD.

220. A diameter AB and a chord CD which does not intersect that diameter are drawn in a circle. Let E and F denote the feet of the perpendiculars dropped from the points A and B on the line CD. Prove that the area of the quadrilateral $AEFB$ is equal to the sum of the areas of the triangles ACB and ADB.

221. Given a convex quadrilateral Q_1. Four straight lines perpendicular to its sides and passing through their midpoints form a quadrilateral Q_2. A quadrilateral Q_3 is formed in the same way for the quadrilateral Q_2. Prove that the quadrilateral Q_3 is similar to the original quadrilateral Q_1.

222. Points M and N are taken on opposite sides BC and DA of a convex quadrilateral such that $|BM| : |MC| = |AN|$ $|ND| = |AB| : |CD|$. Prove that the line MN is parallel to the bisector of the angle formed by the sides AB and CD.

223. A convex quadrilateral is separated by its diagonals into four triangles. The circles inscribed in these triangles are of the same radius. Prove that the given quadrilateral is a rhombus.

224. The diagonals of a quadrilateral separate the latter into four triangles having equal perimeters. Prove that the quadrilateral is a rhombus.

225. In a quadrilateral $ABCD$, the circles inscribed in the triangles ABC, BCD, CDA, DAB are of the same radius. Prove that the given quadrilateral is a rectangle.

226. A quadrilateral $ABCD$ is inscribed in a circle. Let M be the point of intersection of the tangents to the circle passing through A and C, N the point of intersection of the tangents drawn through B and D, K the intersection point of the bisectors of the angles A and C of the quadrilateral, L the intersection point of the angles B and D. Prove that if one of the four statements is true, i.e.: (a) M belongs to the straight line BD, (b) N belongs to the straight line AC, (c) K lies on BD, (d) L lies on AC, then the remaining three statements are also true.

227. Prove that four lines each of which passes through the feet of two perpendiculars dropped from a vertex of an inscribed quadrilateral on the sides not including this vertex intersect at one point.

228. Let AB and CD be two chords of a circle, M the point of intersection of two perpendiculars: one of them to AB at the point A and the other to CD at the point C. Let N be the point of intersection of the perpendiculars to AB and CD at the points B and D, respectively. Prove that the line MN passes through the point of intersection of BC and AD.

229. Given a parallelogram $ABCD$. A circle of radius R passes through the points A and B. Another circle of the same radius passes through the points B and C. Let M denote the second point of intersection of these circles. Prove that the radii of the circles circumscribed about the triangles AMD and CMD are R.

230. Let $ABCD$ be a parallelogram. A circle touches the straight lines AB and AD and intersects BD at points M and N. Prove that there is a circle passing through M and N and touching the lines CB and CD.

231. Let $ABCD$ be a parallelogram. Let us construct a circle on the diagonal AC as diameter and denote by M and N the points of intersection of the circle with the lines AB and AD, respectively. Prove that the lines BD and MN and the tangent to the circle at the point C intersect at the same point.

232. A quadrilateral $ABCD$ is inscribed in a circle. Let O_1, O_2, O_3, O_4 be the centres of the circles inscribed in the triangles

ABC, BCD, CDA, DAB, respectively, and H_1, H_2, H_3, and H_4 the intersection points of the altitudes of the same triangles. Prove that the quadrilateral $O_1O_2O_3O_4$ is a rectangle, and the quadrilateral $H_1H_2H_3H_4$ is equal to the quadrilateral $ABCD$.

233. Given a triangle ABC and an arbitrary point D in the plane. Prove that the intersection points of the altitudes of the triangles ABD, BCD, CAD are the vertices of the triangle equivalent to the given one.

234. Prove that if a circle can be inscribed in a quadrilateral, then: (a) the circles inscribed in the two triangles into which the given quadrilateral is separated by a diagonal touch each other, (b) the points of tangency of these circles with the sides of the quadrilateral are the vertices of the inscribed quadrilateral.

235. Prove that if $ABCD$ is an inscribed quadrilateral, then the sum of the radii of the circles inscribed in the triangles ABC and ACD is equal to the sum of the radii inscribed in triangles BCD and BDA.

* * *

236. Let a, b, c, and d be the sides of a quadrilateral, m and n its diagonals, A and C two opposite angles. Then the following relationship is fulfilled: $m^2n^2 = a^2c^2 + b^2d^2 - 2abcd \cos(A + C)$ *(Bretschneider's* *theorem* or the law of cosines for a quadrilateral).

237. Let a, b, c, and d denote the sides of an inscribed quadrilateral and m and n its diagonals. Prove that $mn = ac + bd$ (*Ptolemy's* theorem*).

238. Prove that if ABC is a regular triangle, M an arbitrary point in the plane not lying on the circle circumscribed about the triangle ABC, then there is a triangle whose sides are equal to $|MA|$, $|MB|$, and $|MC|$ (*Pompeiu's** theorem*). Find the angle of this triangle which is opposite the side equal to $|MB|$ if $\angle AMC = \alpha$.

239. Let $ABCD$ be an inscribed quadrilateral. Four circles, α, β, γ, and δ, touch the circle circumscribed about the quadrilateral $ABCD$ at points A, B, C, and D, respectively. Let $t_{\alpha\beta}$ denote the segment of the tangent to the circles α and β, $t_{\alpha\beta}$ being the segment of a common external tangent if α and β touch the given circle in the same manner (internally or externally), and the segment of a common internal tangent if α and β touch the given circle in a different way (the quantities $t_{\beta\gamma}$, $t_{\alpha\delta}$, etc. are defined in a similar way).

---

* Ptolemy (Caudius Ptolemaus) (*circa* A.D. 150). An Alexandrian geometer, astronomer, and geographer.

** Pompeiu, Dimitrie (1873-1954). A Rumanian mathematician.

Prove that

$$t_{\alpha\beta}t_{\gamma\delta} + t_{\beta\gamma}t_{\delta\alpha} = t_{\alpha\gamma}t_{\beta\delta} \qquad (*)$$

(*Ptolemy's generalized theorem*).

240. Let α, β, γ, and δ be four circles in the plane. Prove that if the following relationship is fulfilled:

$$t_{\alpha\beta}t_{\gamma\delta} + t_{\beta\gamma}t_{\delta\alpha} = t_{\alpha\gamma}t_{\beta\delta}, \qquad (*)$$

where $t_{\alpha\beta}$, etc. are line segments of common external or internal tangents to the circles α and β, etc. (for any three circles we take either three external tangents or one external and two internal, then the circles α, β, γ, and δ touch the same circle.

* * *

241. The extensions of the sides AB and DC of a convex quadrilateral $ABCD$ intersect at a point K, and the extensions of the sides AD and BC at a point L, the line segment BL intersecting DK. Prove that if one of the three relationships

$$|AB| + |CD| = |BC| + |AD|,$$
$$|BK| + |BL| = |DK| + |DL|,$$
$$|AK| + |CL| = |AL| + |CK|$$

is fulfilled, then the two others are also fulfilled.

242. The extensions of the sides AB and DC of a convex quadrilateral $ABCD$ intersect at a point K, and those of the sides

AD and BC at a point L, the line segment BL intersecting DK. Prove that if one of the three relationships $|AD| + |DC| = |AB| + |CB|$, $|AK| + |CK| = |AL| + |CL|$, $|BK| + |DK| = |BL| + |DL|$ is fulfilled, then the two others are also fulfilled.

243. Prove that if there exists a circle touching the straight lines AB, BC, CD, and DA, then its centre and the midpoints of AC and BD are collinear.

244. Let $ABCD$ be an inscribed quadrilateral. The perpendicular to BA erected at a point A intersects the line CD at a point M, the perpendicular to DA erected at A intersects the line BC at a point N. Prove that MN passes through the centre of the circle circumscribed about the quadrilateral $ABCD$.

245. Let $ABCD$ be an inscribed quadrilateral, E an arbitrary point on the straight line AB, and F an arbitrary point on the line DC. The straight line AF intersects the circle at a point M, and the line DE at a point N. Prove that the lines BC, EF, and MN are either concurrent or parallel.

246. Prove that the feet of the perpendiculars dropped from the intersection point of the diagonals of an inscribed quadrilateral on its sides are the vertices of a quadrilateral in which a circle can be inscribed. Find the radius of that circle if the diagonals of the inscribed quadrilateral are mutually perpendicular, the radius of the given circle is R, and the distance from its centre to the point of intersection of the diagonals is d.

247. The diagonals of an inscribed quadrilateral are mutually perpendicular. Prove that the midpoints of its sides and the feet of the perpendiculars dropped from the point of intersection of the diagonals on the sides lie on a circle. Find the radius of that circle if the radius of the given circle is R, and the distance from its centre to the point of intersection of the diagonals of the quadrilateral is d.

248. Prove that if a quadrilateral is both inscribed in a circle of radius R and circumscribed about a circle of radius r, the distance between the centres of those circles being d, then the relationship $\frac{1}{(R+d)^2}+\frac{1}{(R-d)^2}=1/r^2$ is true. In this case there are infinitely many quadrilaterals both inscribed in the larger circle and circumscribed about the smaller one (any point of the larger circle may be taken as one of the vertices).

249. A convex quadrilateral is separated by its diagonals into four triangles. Prove that the line joining the centres of mass of two opposite triangles is perpendicular to the straight line connecting the intersection points of the altitudes of two other triangles.

250. Let $ABCD$ be an inscribed quadrilateral, M and N the midpoints of AC and BD, respectively. Prove that if BD is the bisector of the angle ANC, then AC is the bisector of the angle BMD.

251. Let $ABCD$ be an inscribed quadrilateral. When extended, the opposite sides AB and CD intersect at a point K, and the sides BC and AD at a point L. Prove that the bisectors of the angles BKC and BLA are mutually perpendicular and their intersection point lies on the straight line joining the midpoints of AC and BD.

252. The diagonals of a quadrilateral are mutually perpendicular. Prove that the four straight lines each of which joins one of the vertices of the quadrilateral to the centre of the circle passing through that vertex and two adjacent ones of the quadrilateral intersect at one point.

253. Let P, Q, and M are the respective intersection points of the diagonals of an inscribed quadrilateral and the extensions of its opposite sides. Prove that the intersection point of the altitudes of the triangle PQM coincides with the centre of the circle circumscribed about the given quadrilateral (*Brodcard's theorem*).

254. Let $ABCD$ be a circumscribed quadrilateral, K the point of intersection of the straight lines AB and CD, L the point of intersection of AD and BC. Prove that the intersection point of the altitudes of the triangle formed by the lines KL, AC, and BD coincides with the centre of the circle inscribed in the quadrilateral $ABCD$.

255. Let $ABCD$ be a convex quadrilateral, $\angle ABC = \angle ADC$, M and N the feet of the perpendiculars dropped from A on BC and CD, respectively, K the point of intersection of the straight lines MD and NB. Prove that the straight lines AK and MN are mutually perpendicular.

* * *

256. Prove that four circles circumscribed about four triangles formed by four intersecting straight lines in the plane have a common point (*Michell's* point*).

257. Prove that the centres of four circles circumscribed about four triangles formed by four intersecting straight lines in the plane lie on a circle.

258. Given four pairwise intersecting lines. Let M denote the Michell's point corresponding to these lines (see Problem 256 of Sec. 2). Prove that if four of the six points of pairwise intersection of the given lines lie on a circle centred at O, then the straight line passing through the two re-

---

* Michell, John Henry (1879-1940). An Australian mathematician.

maining points contains the point M and is perpendicular to OM.

259. Four pairwise intersecting straight lines form four triangles. Prove that if one of the lines is parallel to Euler's line (see Problem 147 of Sec. 2) of the triangle formed by the three other lines then any other line possesses the same property.

260. Given a triangle ABC. A straight line intersects the straight lines AB, BC, and CA at points D, E, and F, respectively. The lines DC, AE, and BF form a triangle KLM. Prove that the circles constructed on DC, AE, and BF as diameters intersect at two points P and N (these circles are assumed to intersect pairwise), and the line PN passes through the centre of the circle circumscribed about the triangle KLM and also through the intersection points of the altitudes of the triangles ABC, BDE, DAF, and CEF.

261. Given a triangle ABC. An arbitrary line intersects the straight lines AB, BC, and CA at points D, E, and F, respectively. Prove that the intersection points of the altitudes of the triangles ABC, BDE, DAF, and CEF lie on one line perpendicular to the Gaussian line (see Problem 53 of Sec. 2).

262. Prove that the middle perpendiculars to the line segments joining the intersection points of the altitudes to the centres of the circumscribed circles of the four triangles formed by four arbitrary straight lines in the plane intersect at one point (*Herwey's point*).

263. Consider sixteen points serving as centres of all possible inscribed and escribed circles for four triangles formed by four intersecting lines in the plane. Prove that these sixteen points can be grouped into four quadruples in two ways so that each quadruple lies on one circle. When the first method is used the centres of these circles lie on one line, when the second—on the other line. These lines are mutually perpendicular and intersect at Michell's point, which is a common point of the circles circumscribed about four triangles.

Circles and Tangents. Feuerbach's Theorem

264. On a straight line, points A, B, C, and D are situated so that $|BC| = 2|AB|$, $|CD| = |AC|$. One circle passes through the points A and C, and the other through the points B and D. Prove that the common chord of these circles bisects the line segment AC.

265. Let B denote a point belonging to the line segment AC. The figure bounded by the arcs of three semicircles of diameters AB, BC, and CA lying on the same side of the line AC is called the *shoemaker knife* or *Archimedean arbelos*. Prove that the radii of two circles each of which

touches both two semicircles and the line perpendicular to AC and passing through B are equal to each other (*Archimedean problem*).

266. Of three circles each passes through two given points in the plane. Let O_1, O_2, O_3 denote their centres. The straight line passing through one of the points common to all the three circles intersects them for the second time at points A_1, A_2, A_3, respectively. Prove that $|A_1A_2| \quad |A_2A_3| = |O_1O_2| : |O_2O_3|$.

267. Given two non-intersecting circles. Prove that the four points of tangency of common external tangents to these circles lie on a circle; in similar fashion, the four points of tangency of common internal tangents lie on a second circle, and the four points of intersection of the common internal tangents with the common external tangents lie on a third circle, all the three circles being concentric.

268. Given two non-intersecting circles. A third circle touches them externally and is centred on the line passing through the centres of the given circles. Prove that the third circle intersects the common internal tangents to the given circles at four points forming a quadrilateral two sides of which are parallel to the common external tangents to the given circles.

269. Given two circles. A straight line intersecting one circle at points A and C

and the other at points B and D is drawn through the centre of the first circle. Prove that if $|AB|\ \ |BC| = |AD| : |DC|$, then the circles are perpendicular, that is, the angle between the tangents to them at the point of their intersection is a right one.

270. Points A, B, C, and D lie on a circle or a straight line. Four circles are drawn through the points A and B, B and C, C and D, D and A. Let B_1, C_1, D_1, and A_1 denote the intersection points (distinct from A, B, C, and D) of the first and second, the second and third, the third and fourth, the fourth and first circles, respectively. Prove that the points A_1, B_1, C_1, and D_1 lie on a circle (or a straight line).

271. From a point A taken outside a circle, two tangents AM and AN (M and N points of tangency) and two secants are drawn. Let P and Q denote the intersection points of the circle with the first secant, and K and L with the second one, respectively. Prove that the straight lines PK, QL, and MN either intersect at a point or are parallel.

Try to develop the method of construction of a tangent to a given circle through a given point with a ruler alone.

272. Given a circle with centre O and a point A. Let B denote an arbitrary point of the circle. Find the locus of intersection points of tangents to the circle at the point

B with the straight line passing through O perpendicular to AB.

273. Given a circle and two points A and B on it. Let N be an arbitrary point on the line AB. We construct two circles, each passing through the point N and touching the given circle: one at a point A, the other at a point B. Let M denote a second point of intersection of those circles. Find the locus of points M.

274. Two arbitrary chords PQ and KL are drawn through a fixed point inside a circle. Find the locus of intersection points of the lines PK and QL.

275. Two circles intersect at points A and B. An arbitrary straight line passes through the point B and, for the second time, intersects the first circle at a point C, and the second at a point D. The tangents to the first circle at C and to the second at D intersect at a point M. Through the point of intersection of AM and CD, there passes a line parallel to CM and intersecting AC at a point K. Prove that KB touches the second circle.

276. Given a circle and a tangent l to it. Let N denote the point of tangency, and NM the diameter. On the line NM a fixed point A is taken. Consider an arbitrary circle passing through the point A with centre on l. Let C and D be the points of intersection of this circle with l, and P and Q the points of intersection of the straight lines MC and MD with the given circle. Prove that the chord PQ passes through the fixed point in the plane.

277. The points O_1 and O_2 are the centres of two intersecting circles, A being one of the points of their intersection. Two common tangents are drawn to the circles; BC and EF are the chords of those circles with ends at the points of tangency (C and F being most remote from A), M and N are the midpoints of BC and EF, respectively. Prove that $\angle O_1AO_2 = \angle MAN = 2\angle CAE$.

278. A diameter AB is drawn in a circle, CD being a chord perpendicular to AB. An arbitrary circle touches the chord CD and the arc CBD. Prove that a tangent to this circle drawn from the point A is equal to AC.

279. Given a segment of a circle. Two arbitrary circles touch the chord and the arc of the segment and intersect at points M and N. Prove that the straight line MN passes through a fixed point in the plane.

280. Given two equal non-intersecting circles. Two arbitrary points F and F' are taken on two common internal tangents. From both points one more tangent can be drawn to each of the circles. Let the tangents drawn from the points F and F' to one circle meet in a point A, to the other in a point B. It is required to prove that: (1) the line AB is parallel to the line joining the centres of the circles (in the case of unequal circles, it passes through the intersection point of the external tangents); (2) the line joining the midpoints of FF' and AB passes through the midpoint of the line segment joining the centres of the circles.

(This problem was suggested to the readers of "The Bulletin of Experimental Physics and Elementary Mathematics" by Professor V. Ermakov. This journal was issued in Russia last century. The problem was published in issue 14(2) of "The Bulletin" in 1887. A prize, some mathematical books, was offered to readers for the correct solution.)

281. Given three circles α, β, and γ. Let l_1 and l_2 denote the common internal tangents to the circles α and β, m_1 and m_2 the common internal tangents to the circles β and γ, and n_1 and n_2 to the circles γ and α. Prove that if the lines l_1, m_1, and n_1 are concurrent, then the lines l_2, m_2, and n_2 are also concurrent.

282. An arc AB of a circle is divided into three equal parts by the points C and D (C is nearest to A). When rotated about the point A through an angle of $\pi/3$, the points B, C, and D go into points B_1, C_1, and D_1; F is the point of intersection of the straight lines AB_1 and DC_1; E is a point on the bisector of the angle B_1BA such that $|BD| = |DE|$. Prove that the triangle CEF is regular (*Finlay's theorem*).

* * *

283. Given an angle with vertex A and a circle inscribed in it. An arbitrary straight line touching the given circle intersects the sides of the angle at points B and C. Prove that the circle circumscribed about the triangle ABC touches the circle inscribed in the given angle.

284. In a triangle ABC, a point D is taken on the side AC. Consider the circle touching the line segment AD at a point M, the line segment BD and the circle circumscribed about the triangle ABC. Prove that the straight line passing through the point M parallel to BD touches the circle inscribed in the triangle ABC.

285. In a triangle ABC, a point D is taken on the side AC. Let O_1 be the centre of the circle touching the line segments AD, BD, and the circle circumscribed about the triangle ABC, and let O_2 be the centre of the circle touching the line segments CD, BD, and the circumscribed circle. Prove that the line O_1O_2 passes through the centre O of the circle inscribed in the triangle ABC and $|O_1O| \ |OO_2| = \tan^2(\varphi/2)$, where $\varphi = \angle BDA$ (*Thebault's theorem*).

286. Each of four circles touches internally a given circle and two of its mutually intersecting chords. Prove that the diagonals of the quadrilateral with vertices at the centres of those four circles are mutually perpendicular.

* * *

287. Prove that the nine-point circle (see Problem 160 of Sec. 2) touches the circle inscribed in the triangle and all of the escribed circles (*Feuerbach's theorem*).

288. Let H denote the intersection point of the altitudes of a triangle ABC. Prove that the nine-point circle touches all of the inscribed and escribed circles of the triangles AHB, BHC, and CHA.

289. Prove that the intersection point of the diagonals of the quadrilateral with vertices at the points of tangency of the nine-point circle of a triangle ABC with the inscribed and escribed circles of the triangle lies on its midline.

290. Let F, F_a, F_b, and F_c denote the points of tangency of the nine-point circle of a triangle ABC with the inscribed and three escribed circles (F_a is the point of tangency with the circle centred at I_a and so on). Let further A_1 and A_2, B_1 and B_2, and C_1 and C_2 denote the intersection points of the bisectors of the interior and exterior angles A, B, and C with the opposite sides, respectively. Prove that the following triangles are similar pairwise: $\Delta F_aF_bF_c$ and $\Delta A_1B_1C_1$, ΔFF_bF_c and $\Delta A_1B_2C_2$, ΔFF_cF_a and $\Delta B_1C_2A_2$, ΔFF_aF_b and $\Delta C_1A_2B_2$ (*Thebault's theorem*).

Combinations of Figures. Displacements in the Plane. Polygons

291. Three squares $BCDE$, $ACFG$, and $BAHK$ are constructed externally on the sides BC, CA, and AB of a triangle ABC. Let $FCDQ$ and $EBKP$ be parallelograms. Prove that the triangle APQ is a right isosceles triangle.

292. Let $ABCD$ be a rectangle, E a point on BC, F a point on DC, E_1 the midpoint of AE_1, F_1 the midpoint of AF. Prove that if the triangle AEF is equilateral, then the triangles DE_1C and BF_1C are also equilateral.

293. Two squares $ACKL$ and $BCMN$ are constructed externally on the legs AC and BC of a right triangle. Prove that the quadrilateral bounded by the legs of the given triangle and the straight lines LB and NA is equivalent to the triangle formed by the lines LB, NA, and the hypotenuse AB.

294. Squares are constructed externally on the sides of a convex quadrilateral. Prove that if the diagonals of the quadrilateral are mutually perpendicular, then the line segments joining the centres of the opposite squares pass through the intersection point of the diagonals of the quadrilateral.

295. Prove that if the centres of the squares constructed externally on the sides of a given triangle serve as the vertices of the triangle whose area is twice the area of the given triangle, then the centres of the squares constructed internally on the sides of the triangle lie on a straight line.

296. Constructed externally on the sides BC, CA, and AB of a triangle ABC are triangles A_1BC, B_1CA, and C_1AB such that $\angle A_1BC = \angle C_1BA$, $\angle C_1AB = \angle B_1AC$, $\angle B_1CA = \angle A_1CB$. Prove that the lines AA_1, BB_1, CC_1 intersect at a point.

297. Let ABC be an isosceles triangle ($|AB| = |BC|$) and BD its altitude. A disc of radius BD rolls along the straight line AC. Prove that as long as the vertex B is inside the disc, the length of the circular arc inside the triangle is constant.

298. Two points move in two intersecting straight lines with equal velocities. Prove that there is a fixed point in the plane which is equidistant from the moving points at all instants of time.

299. Two cyclists ride round two intersecting circles, each running round his circle with a constant speed. Having started simultaneously from a point at which the circles intersect, the cyclists meet once again at this point after one circuit. Prove that there is a fixed point such that the distances from it to the cyclist are equal all the time if they ride: (a) in the same direction (clockwise); (b) in opposite directions.

300. Prove that: (a) the rotation about a point O through an angle α is equivalent to two successive axial symmetry mappings whose axes pass through the point O and the angle between the axes is $\alpha/2$; a translation is equivalent to two axial symmetry mappings with parallel axes; (b) two successive rotations in the same direction, one about the point O_1 through an angle α and the other about the point O_2 through an angle β ($0 \leqslant \alpha < 2\pi$, $0 \leqslant \beta < 2\pi$) are equivalent to one rotation through an angle $\alpha + \beta$ about a certain point O if $\alpha + \beta \neq 2\pi$. Find the angles of the triangle O_1O_2O.

301. Given an arbitrary triangle ABC. Three isosceles triangles AKB, BLC, and CMA with the vertex angles K, L, and M equal to α, β, and γ, respectively, $\alpha + \beta + \gamma = 2\pi$ are constructed on its sides as bases. All the triangles are located either outside the triangle ABC or inside it. Prove that the angles of the triangle KLM are equal to $\alpha/2$, $\beta/2$, $\gamma/2$.

302. Let $ABCDEF$ be an inscribed hexagon in which $|AB| = |CD| = |EF| = R$, where R is the radius of the circumscribed circle, O its centre. Prove that the points of pairwise intersections of the circles circumscribed about the triangles BOC, DOE, FOA, distinct from O, serve as the vertices of an equilateral triangle with side R.

303. Four rhombi each having an acute angle α are constructed externally on the sides of a convex quadrilateral. The angles of two rhombi adjacent to one vertex of the quadrilateral are equal. Prove that the line segments joining the centres of opposite rhombi are equal to each other, and the acute angle between those segments is α.

304. Given an arbitrary triangle. Constructed externally on its sides are equilateral triangles whose centres serve as vertices of the triangle Δ. The centres of the equilateral triangles constructed internally on the sides of the original triangle serve as vertices of another triangle δ. Prove that: (a) Δ and δ are equilateral triangles; (b) the centres of Δ and δ coincide with the centre of mass of the original triangle; (c) the difference between the areas of Δ and δ is equal to the area of the original triangle.

305. Three points are given in a plane. Through these points three lines are drawn forming a regular triangle. Find the locus of centres of those triangles.

306. Given a triangle ABC. On the line passing through the vertex A and perpendicular to the side BC, two points A_1 and A_2 are taken such that $|AA_1| = |AA_2| = |BC|$ (A_1 being nearer to the line BC than A_2). Similarly, on the line perpendicular to AC and passing through B points B_1 and B_2 are taken such that $|BB_1| =$

$|BB_2| = |AC|$. Prove that the line segments A_1B_2 and A_2B_1 are equal and mutually perpendicular.

* * *

307. Prove that a circumscribed polygon having equal sides is regular if the number of its sides is odd.

308. A straight line is drawn through the centre of a regular n-gon inscribed in a unit circle. Find the sum of the squares of the distances from the line to the vertices of the n-gon.

309. Prove that the sum of the distances from an arbitrary point inside a convex polygon to its sides is constant if: (a) all the sides of the polygon are equal; (b) all the angles of the polygon are equal.

310. A semicircle is divided by the points A_0, A_1, . ., A_{2n+1} into $2n+1$ equal arcs (A_0 and A_{2n+1} the end points of the semicircle), O is the centre of the semicircle. Prove that the straight lines A_1A_{2n}, A_2A_{2n-1}, . . ., A_nA_{n+1}, when intersecting the straight lines OA_n and OA_{n+1}, form line segments whose sum is equal to the radius of the circle.

311. Prove that if perpendiculars are drawn to the sides of an inscribed $2n$-gon form an arbitrary point of a circle, then the products of the lengths of the alternate perpendiculars are equal.

312. Let $A_1A_2 \dots A_n$ be an inscribed

polygon; the centre of the circle is found inside the polygon. A system of circles touch internally the given circle at points $A_1, A_2, \ldots, A_n$, one of the intersection points of two neighbouring circles lying on a side of the polygon. Prove that if n is odd, then all the circles have the same radius. The length of the outer boundary of the union of the inscribed circles is equal to the circumference of the given circle.

313. Consider the circle in which a regular $(2n + 1)$-gon $A_1A_2 \ldots A_{2n+1}$ is inscribed. Let A be an arbitrary point of the arc A_1A_{2n+1}.

(a) Prove that the sum of the distances from A to the even vertices is equal to the sum of the distances from A to the odd vertices.

(b) Let us construct equal circles touching the given circle in the same manner at points $A_1, A_2, \ldots, A_{2n+1}$. Prove that the sum of the tangents drawn from A to the circles touching the given circle at even vertices is equal to the sum of the tangents drawn to the circles touching the given circle at odd vertices.

314. (a) Two tangents are drawn to a given circle. Let A and B denote the points of tangency and C the point of intersection of the tangents. Let us draw an arbitrary straight line l which touches the given circle and does not pass through the points

A and B. Let u and v be the distances from A and B to l, respectively, w the distance from C to l. Find uv/w^2 if $\angle ACB = \alpha$.

(b) A polygon is circumscribed about a circle. Let l be an arbitrary line touching the circle and coinciding with no side of the polygon. Prove that the ratio of the product of the distances from the vertices of the polygon to the line l to the product of the distances from the points of tangency of the sides of the polygon with the circle to l is independent of the position of the line l.

(c) Let $A_1A_2 \ldots A_{2n}$ be a $2n$-gon circumscribed about a circle and l an arbitrary tangent to the circle. Prove that the product of the distances from the odd vertices to the line l and the product of the distances from the even vertices to the line l are in a constant ratio independent of l (the line l is assumed to contain no vertices of the polygon).

315. Drawn in an inscribed polygon are non-intersecting diagonals separating the polygon into triangles. Prove that the sum of the radii of the circles inscribed in those triangles is independent of the way the diagonals are drawn.

316. Let $A_1A_2 \ldots A_n$ be a polygon of perimeter $2p$ circumscribed about a circle of radius r, B_1, B_2, $\ldots$, B_n the points at which the circle touches the sides A_1A_2, A_2A_3, $\ldots$, A_nA_1, respectively, and M a point found at a distance d from the centre of the circle. Prove that $|MB_1|^2 \cdot |A_1A_2| + |MB_2|^2 \cdot |A_2A_3| + \ldots + |MB_n|^2 \cdot |A_nA_1| = 2p\,(r^2 + d^2)$.

317. Let $ABCD$ denote an inscribed quadrilateral, M an arbitrary point on the circle. Prove that the projections of the point M on Simson's lines (see Problem 153 of Sec. 2), corresponding to the point M with respect to the triangles ABC, BCD, CDA, and DAB, lie in a straight line (*Simson's line of a quadrilateral*).

Further, knowing Simson's line of an n-gon, let us determine Simson's line of an $(n + 1)$-gon by induction. Namely, for an arbitrary inscribed $(n + 1)$-gon and a point M on the circle, the projections of this point on all possible Simson's lines of this point with respect to all possible n-gons formed by n vertices of this $(n + 1)$-gon lie on a straight line which is Simson's line of an $(n + 1)$-gon.

318. A circle β is situated inside a circle α. On the circle α, two sequences of points are given: $A_1, A_2, A_3 \ldots$ and $B_1, B_2, B_3 \ldots$ following in the same direction and such that the straight lines $A_1A_2, A_2A_3, A_3A_4 \ldots$ and $B_1B_2, B_2B_3, B_3B_4 \ldots$ touch the circle β. Prove that the straight lines $A_1B_1, A_2B_2, A_3B_3 \ldots$ touch one and the same circle whose centre is found on the straight line passing through the centres of the circles α and β.

319. Using the result of the preceding problem, prove the following statement (*Poncelet's* theorem*). If there is one n-gon inscribed in a circle α and circumscribed about another circle β, then there are infinitely many n-gons inscribed in the circle α and circumscribed about the circle β and any point of the circle can be taken as one of the vertices of such an n-gon.

320. On the sides of a regular triangle PQR as bases, isosceles triangles PXQ, QYR, and RZP are constructed externally so that $\angle PXQ = \frac{1}{3}(\pi + 2\angle A)$, $\angle QYR = \frac{1}{3}(\pi + 2\angle B)$, $RZP = \frac{1}{3} \times (\pi + 2\angle C)$, where A, B, C are the angles of a certain triangle ABC. Let A_0 denote the intersection point of the straight lines ZP and YQ, B_0 the point of intersection of the lines XQ and ZR, and C_0 the point of intersection of YR and XP. Prove that the angles of the triangle $A_0B_0C_0$ are congruent to the corresponding angles of the triangle ABC.

Using the obtained result, prove the following *Morley's** theorem*: if the angles of an arbitrary triangle are divided into three equal parts each (or trisected, hence,

* Poncelet, Jean Victor (1788-1867). A French geometer and engineer.

** Morley, Frank (1860-1937). An English mathematician.

the relevant lines are called *trisectrices*), then the three points which are the intersection points of the pairs of trisectrices adjacent to the corresponding sides of the triangle are the vertices of a regular triangle.

321. We arrange the vertices of a triangle ABC in positive order (anticlockwise). For any two rays α and β the symbol $\widehat{(\alpha, \beta)}$ denotes the angle through which the ray α must be rotated anticlockwise to be brought into coincidence with the ray β. Let α_1 and α_1' denote two rays emanating from A for which $\widehat{(AB, \alpha_1)} = \widehat{(\alpha_1, \alpha_1')} = \widehat{(\alpha_1', AC)} = \frac{1}{3} \angle A$, α_2 and α_2' the rays for which $\widehat{(AB, \alpha_2)} = \widehat{(\alpha_2, \alpha_2')} = \widehat{(\alpha_2', AC)} = \frac{1}{3} (\angle A + 2\pi)$, and, finally, α_3 and α_3' the rays for which $\widehat{(AB, \alpha_3)} = \widehat{(\alpha_3, \alpha_3')} = \widehat{(\alpha_3', AC)} = \frac{1}{3} (\angle A + 4\pi)$ (α_i, α_i', where $i = 1, 2, 3$, are called *trisectrices of the first, second and third types*). In similar fashion, for the vertices B and C we determine β_j, β_j' and γ_k, γ_k' (j, $k = 1, 2, 3$). We denote by $\alpha_i\beta_j\gamma_k$ the triangle formed by respectively intersecting lines (not rays) α_i and β_j', β_j and γ_k', γ_k and α_i'. Prove that for all i, j, k such that $i + j + k - 1$ is not multiple of three, the triangles $\alpha_i\beta_j\gamma_k$ are regular, their corresponding sides are parallel, and the vertices lie on nine straight lines, six on each line (*Morley's complete theorem*).

Geometrical Inequalities. Problems on Extrema

322. At the beginning of the nineteenth century, the Italian geometer Malfatti* suggested the following problem: from a given triangle, cut out three circles such that the sum of their areas is the greatest. In later investigations, *Malfatti's circles* were understood as three circles touching pairwise each other, each of which also touches two sides of the given triangle. Prove that for a regular triangle Malfatti's circles yield no solution of the original problem. (Only in the middle of this century was it proved for any triangle that Malfatti's circles yield no solution of the original problem.)

323. Prove that $p \geqslant \frac{3}{2}\sqrt{6Rr}$, where p is the semiperimeter, r and R are the radii of the inscribed and circumscribed circles of a triangle, respectively.

---

* Malfatti, Giovanni Francesco Giuseppe (1731-1807). An Italian mathematician.

324. Prove that the perimeter of the triangle whose vertices are the feet of the altitudes of a given acute triangle does not exceed the semiperimeter of the given triangle.

325. Prove that if the triangle formed by the medians of another triangle is obtuse, then the smallest angle of the former triangle is less than 45°

326. Let $ABCD$ be a convex quadrilateral. Prove that at least one of the four angles BAC, DBC, ACD, BDA does not exceed $\pi/4$.

327. Prove that the median drawn to the largest side of a triangle forms with the sides enclosing this median angles each of which is not less than half the smallest angle of the triangle.

328. Prove that if in a triangle ABC the angle B is obtuse and $|AB| = |AC|/2$, then $\angle C > \angle A/2$.

329. Prove that the circle circumscribed about a triangle cannot pass through the centre of an escribed circle.

330. In a triangle, a median, a bisector, and an altitude emanate from the vertex A. Given the angle A, find out which of the angles is greater: between the median and bisector or between the bisector and the altitude.

331. Prove that if the medians drawn from the vertices B and C of a triangle

ABC are mutually perpendicular, then $\cot B + \cot C \geqslant 2/3$.

332. Given a triangle ABC, $|AB| < |BC|$. Prove that for an arbitrary point M on the median drawn from the vertex B, $\angle BAM > \angle BCM$.

333. Two tangents AB and AC are drawn to a circle from an exterior point A; the midpoints D and E of the tangents are joined by the straight line DE. Prove that this line does not intersect the circle.

334. Prove that if a straight line does not intersect a circle, then for any two points of the line the distance between them is enclosed between the sum and difference of the lengths of the tangents drawn from these points to the circle. Prove the converse: if for some two points on the straight line the assertion is not fulfilled, then the line intersects the circle.

335. In a triangle ABC, the angles are related by the inequality $3\angle A - \angle C < \pi$. The angle B is divided into four equal parts by the straight lines intersecting the side AC. Prove that the third of the line segments (counting from the vertex A) into which the side AC is divided is less than $|AC|/4$.

336. Let a, b, c, d be successive sides of a quadrilateral. Prove that if S is its area, then $S \leqslant (ac + bd)/2$, an equality occurring only for an inscribed quadrilateral whose diagonals are mutually perpendicular.

337. Prove that if the lengths of the angle bisectors of a triangle are less than 1, then its area is less than $\sqrt{3}/3$.

338. Prove that a triangle is either acute, or right, or obtuse accordingly as the expression $a^2 + b^2 + c^2 - 8R^2$ is, respectively, either positive, or zero, or negative (a, b, c the sides of the triangle, R the radius of the circumscribed circle).

339. Prove that a triangle is either acute, or right, or obtuse accordingly as its semiperimeter is, respectively, either greater than, or equal to, or less than the sum of the diameter of the circumscribed circle and the radius of the inscribed circle.

340. Prove that if the lengths of the sides of a triangle are related by the inequality $a^2 + b^2 > 5c^2$, then c is the smallest side.

341. In a triangle ABC, $\angle A < \angle B < \angle C$, I is the centre of the inscribed circle, O the centre of the circumscribed circle, and H the intersection point of the altitudes. Prove that I lies inside the triangle BOH.

342. The triangles ABC and AMC are arranged so that MC intersects AB at a point O, and $|AM| + |MC| = |AB| + |BC|$. Prove that if $|AB| = |BC|$, then $|OB| > |OM|$.

343. In a triangle ABC, a point M lies on the side BC. Prove that $(|AM| - |AC|)\,|BC| \leqslant (|AB| - |AC|)\,|MC|$.

344. Let a, b, c be the sides of a triangle ABC, and M an arbitrary point in the plane. Find the minimum of the sum $|MA|^2 + |MB|^2 + |MC|^2$.

345. The sides of an angle equal to α form the sides of a billiards. What maximum number of reflections from the sides can be done by a ball (the ball is assumed to be dimensionless)?

346. Four villages are situated at the vertices of a square of side 2 km. The villages are connected by roads so that each village is joined to any other. Is it possible for the total length of the roads to be less than 5.5 km?

347. A point A lies between two parallel lines at distances a and b from them. This point serves as a vertex of the angle α for all possible triangles, two other vertices of which lying on the given straight lines (one on either line). Find the area of the least triangle.

348. In a circle of radius R centred at O, AB is its diameter, a point M is on the radius OA such that $|AM| \quad |MO| = k$. An arbitrary chord CD is drawn through the point M. What is the maximal area of the quadrilateral $ABCD$?

349. Given an angle with vertex A and two points M and N inside this angle. Drawn through M is a straight line intersecting the sides of the angle at points B and C. Prove that for the area of the quadrilateral $ABNC$ to be minimal, it is necessary and sufficient that the straight line

BC intersects AN at a point P such that $|BP| = |MC|$. Give the method of construction of this line.

350. The vertex of an angle α is found at a point O, A is a fixed point inside the angle. On the sides of the angle, points M and N are taken such that $\angle MAN = \beta$ $(\alpha + \beta < \pi)$. Prove that if $|AM| = |AN|$, then the area of the quadrilateral $OMAN$ reaches its maximum (of all possible quadrilaterals resulting from change in M and N).

351. Bearing in mind the result of the preceding problem, solve the following. A point A is taken inside an angle with vertex O. The straight line OA forms angles φ and ψ with the sides of the angle. On the sides of the former angle, find points M and N such that $\angle MAN = \beta$ $(\varphi + \psi + \beta < \pi)$ and the area of the quadrilateral $OMAN$ is maximal.

352. Given a triangle OBC $(\angle BOC = \alpha)$. For each point A on the side BC we define points M and N on OB and OC, respectively, so that $\angle MAN = \beta$ $(\alpha + \beta < \pi)$ and the area of the quadrilateral $OMAN$ is maximal. Prove that this maximal area reaches its minimum for such points A, M, and N for which $|MA| = |AN|$, and the straight line MN is parallel to BC. (Such points exist if the angles B and C of the triangle ABC do not exceed $\frac{\pi}{2} + \frac{\beta}{2}$.)

353. Let $ABCD$ be an inscribed quadrilateral. The diagonal AC is equal to a and forms angles α and β with the sides AB and AD, respectively. Prove that the magnitude of the area of the quadrilateral lies between $\frac{a^2 \sin(\alpha+\beta)\sin\beta}{2\sin\alpha}$ and $\frac{a^2 \sin(\alpha+\beta)\sin\alpha}{2\sin\beta}$.

354. Given an angle α with vertex at a point O and a point A inside the angle. Consider all quadrilaterals $OMAN$ with vertices M and N on the sides of the angle and such that $\angle MAN = \beta$ $(\alpha + \beta > \pi)$. Prove that if among these quadrilaterals there is a convex one such that $|MA| = |AN|$, then it has the least area among all the quadrilaterals under consideration.

355. Consider a point A inside an angle with vertex O, OA forming angles φ and ψ with the sides of the given angle. On the sides of the angle, find points M and N such that $\angle MAN = \beta$ $(\varphi + \psi + \beta > \pi)$ with minimal area of the quadrilateral $OMAN$.

356. Given a triangle OBC, $\angle BOC = \alpha$. For any point A on the side BC we define points M and N on OB and OC, respectively, so that $\angle MAN = \beta$, and the area of the quadrilateral $OMAN$ is minimal. Prove that this minimal area is a maximum for such points A, M, and N for which $|MA| = |AN|$ and the straight line MN is parallel to BC. (If there is no such a point A, then the maximum is reached at the end of the side BC for a degenerate quadrilateral.)

357. Find the radius of the largest circle which can be overlapped by three circles of radius R. Solve the problem in the general case when the radii are R_1, R_2, R_3.

358. Is it possible to cover a square 5/4 on a side with three unit squares?

359. What is the greatest area of an equilateral triangle which can be covered with three equilateral triangles of side 1?

360. In a triangle ABC, on the sides AC and BC, points M and N are taken, respectively, and a point L on the line segment MN. Let the areas of the triangles ABC, AML, and BNL be equal to S, P, and Q, respectively. Prove that $\sqrt[3]{S} \geqslant \sqrt[3]{P} + \sqrt[3]{Q}$.

361. Let a, b, c, S denote, respectively, the sides and area of a triangle, and α, β, γ the angles of another triangle. Prove that $a^2 \cot \alpha + b^2 \cot \beta + c^2 \cot \gamma \geqslant 4S$, an equality occurring only in the case when the triangles are similar.

362. Prove the inequality $a^2 + b^2 + c^2 \geqslant 4S\sqrt{3} + (a-b)^2 + (b-c)^2 + (c-a)^2$, where a, b, c, S are the sides and area of the triangle, respectively (the *Finsler-Hadviger inequality*).

363. Given a triangle with sides a, b, and c. Determine the area of the greatest regular triangle circumscribed about the given

triangle and the area of the smallest regular triangle inscribed in it.

364. Let M be an arbitrary point inside a triangle ABC. A straight line AM intersects the circle circumscribed about the triangle ABC at a point A_1. Prove that $\frac{|BM|\cdot|CM|}{|A_1M|} \geqslant 2r$, where r is the radius of the inscribed circle, an equality being obtained when M coincides with the centre of the inscribed circle.

365. Let M be an arbitrary point inside a triangle ABC. Prove that $|AM| \sin \angle BMC + |BM| \sin \angle AMC + |CM| \sin \angle AMB \leqslant p$ (p the semiperimeter of the triangle ABC), an equality occurring when M coincides with the centre of the inscribed circle.

366. Let h_1, h_2, h_3 be the altitudes of a triangle ABC, and u, v, w the distances to the corresponding sides from a point M situated inside the triangle ABC. Prove the following inequalities:

(a) $\frac{h_1}{u}+\frac{h_2}{v}+\frac{h_3}{w} \geqslant 9$;

(b) $h_1h_2h_3 \geqslant 27uvw$;

(c) $(h_1-u)(h_2-v)(h_3-w) \geqslant 8uvw$.

367. Let h be the greatest altitude of a non-obtuse triangle and R and r the radii of the circumscribed and inscribed circles,

respectively. Prove that $R + r \leqslant h$ (the *Herdesh theorem*).

368. Prove that the radius of the circle circumscribed about the triangle formed by the medians of an acute triangle is greater than 5/6 of the radius of the circle circumscribed about the original triangle.

369. Prove that the sum of the squares of the distances from an arbitrary point in the plane to the sides of a triangle takes on the least value for such a point inside the triangle whose distances to the corresponding sides are proportional to these sides. Prove also that this point is the intersection point of the symedians of the given triangle (*Lemuan's point*).

370. Given a triangle each angle of which is less than 120° Prove that the sum of the distances from an arbitrary point inside it to the vertices of this triangle takes on the least value if each side of the triangle can be observed at an angle of 120° (*Torricelli's point*).

371. Prove that among all triangles inscribed in a given acute triangle the one whose vertices are the feet of the altitudes of the given triangle has the smallest perimeter.

372. Prove that the sum of the distances from a point inside a triangle to its vertices is not less than $6r$, where r is the radius of the inscribed circle.

373. For an arbitrary triangle, prove the inequality $\frac{bc \cos A}{b+c} + a < p < \frac{bc+a^2}{a}$, where a, b, and c are the sides of the triangle and p its semiperimeter.

374. Let K denote the intersection point of the diagonals of a convex quadrilateral $ABCD$, L a point on the side AD, N a point on the side BC, M a point on the diagonal AC, KL and MN being parallel to AB, LM parallel to DC. Prove that $KLMN$ is a parallelogram and its area is less than 8/27 of the area of the quadrilateral $ABCD$ (*Hattori's theorem*).

375. Two triangles have a common side. Prove that the distance between the centres of the circles inscribed in them is less than the distance between their noncoincident vertices (*Zalgaller's problem*).

376. Given a triangle ABC whose angles are equal to α, β, and γ. A triangle DEF is circumscribed about the triangle ABC so that the vertices A, B, and C are found on the sides EF, FD, and DE, respectively, and $\angle ECA = \angle DBC = \angle FAB = \varphi$. Determine the value of the angle φ for which the area of the triangle EFD reaches its maximum.

377. In a triangle ABC, points A_1, B_1, C_1 are taken on its sides BC, CA, and AB, respectively. Prove that the area of the triangle $A_1B_1C_1$ is no less than the area of at least one of the three triangles: AB_1C_1, A_1BC_1, A_1B_1C.

378. Let O, I, and H denote the centres of the circumscribed and inscribed circles of a triangle and the point of intersection of its altitudes, respectively. Prove that $|OH| \geqslant |IH| \sqrt{2}$.

379. Let M be an arbitrary point inside a triangle ABC; x, y, and z the distances from the point M to the vertices A, B, and C; u, v, and w the distances from the point M to the sides BC, CA, and AB, respectively; a, b, and c the sides of the triangle ABC; S its area; R and r are the radii of the circumscribed and inscribed circles, respectively. Prove the following inequalities:

(a) $ax + by + cz \geqslant 4S$;

(b) $x + y + z \geqslant 2(u + v + w)$ (*Herdesh's inequality*);

(c) $xu + yv + zw \geqslant 2(uv + vw + wu)$;

(d) $2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \leqslant \frac{1}{u}+\frac{1}{v}+\frac{1}{w}$;

(e) $xyz \geqslant \frac{R}{2r}(u+v)(v+w)(w+u)$;

(f) $xyz \geqslant \frac{4R}{r} uvw$;

(g) $xy + yz + zx \geqslant \frac{2R}{r}(uv + vw + wu)$.

380. In a given triangle, we draw the median to the greatest side. This median separates the triangle into two parts. In each of the triangles thus obtained, we also draw the median to the greatest side, and so forth. Prove that all the triangles thus constructed can be divided into a finite number of classes in such a manner that all the triangles belonging to the same class are similar. Also, prove that any angle of any newly obtained triangle is no less than half the smallest angle of the original triangle.

381. Find the triangle of the least area which can cover any triangle with sides not exceeding 1.

Answers, Hints, Solutions

Section 1

17. The angle bisector separates the given triangle into two parts whose areas are $\frac{al}{2}\sin\frac{\alpha}{2}$, $\frac{bl}{2}\sin\frac{\alpha}{2}$, and the area of the entire triangle is $\frac{ab}{2}\sin\alpha$; hence $\left(\frac{al}{2}+\frac{bl}{2}\right)\cdot\sin\frac{\alpha}{2}=\frac{ab}{2}\sin\alpha$,

$$l=\frac{2ab\cos\frac{\alpha}{2}}{a+b}.$$

19. Let us take a circle touching the sides AB, BC, and CA. If this circle does not touch the side DA, then drawing the tangent DA_1 (A_1 lies on AB), we obtain a triangle DAA_1 in which one side is equal to the sum of the two others.

20. Drawing through the vertices of the triangle straight lines parallel to the opposite sides, we get a triangle for which the altitudes of the original triangle are perpendiculars to the sides at their midpoints.

21. $\frac{a+b}{2}$. 22. $\frac{c}{2}\sqrt{\frac{3\sqrt{3}}{\pi}}$.

23. $\frac{\sqrt{2}-1}{2}\left(a+b-\sqrt{a^2+b^2}\right)$. 24. $\frac{m^2\sqrt{3}}{2}$.

25. $\frac{c+a}{b}$. 28. $\frac{|a-b|}{2}$. 29. $\frac{1}{2}(a-b)^2\sin\alpha$.

30. $\frac{h}{2}\tan^2\frac{\pi-\alpha}{4}$. 31. 30°. 32. $\frac{ab}{2}$.

33. 90°. 36. $r^2(2\sqrt{3}+3)$. 37. $l\sqrt{a(2l-a)}$.

38. $\frac{1}{2}(S_1+S_2)$.

39. If $a>b$, then the bisector intersects the lateral side CD; if $a<b$, then the base is BC.

40. $\frac{2ab}{a+b}$. 41. $\arccos\frac{1-k}{1+k}$.

42. $\frac{a+b}{4}\sqrt{3b^2+2ab-a^2}$. 43. a^2.

44. $\frac{1}{2}\sqrt{\frac{S}{2}}$.

45. $(\sqrt{S_1}+\sqrt{S_2})^3$. 46. $90°+\frac{\alpha}{2}$.

47. $\frac{|a-b|}{a+b}\sqrt{a^2+b^2}$. 48. $\arcsin\left(\frac{b}{a}-1\right)$.

49. $(6-\pi):2\pi:(6-\pi)$.

50. $\frac{a^2}{8}(\sqrt{2}-1)[(2\sqrt{2}-1)\pi-4]$.

51. $\frac{a^2}{4}(6\sqrt{3}-6-\pi)$. 52. $\frac{R^2}{2}\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)$.

53. $\frac{1}{2}\sqrt{b^2-a^2}$. 54. $\frac{d}{3}$. 55. $\frac{4}{9}S$.

58. If $\alpha<90°$, $\beta<90°$, then the angles of $\triangle ABC$ are equal to $90°-\alpha$, $90°-\beta$, $\alpha+\beta$; if $\alpha>90°$, $\beta<90°$, then they are $\alpha-90°$, $90°+\beta$, $180°-\alpha-\beta$; if $\alpha<90°$, $\beta>90°$, then they are $90°+\alpha$, $\beta-90°$, $180°-\alpha-\beta$.

59. $\frac{1}{2}\sqrt{m^2-4S}$. 60. $\frac{a}{5}$. 61. $\frac{36}{25}h^2$.

62. $\sqrt{\frac{S}{\pi(4\pi^2-1)}}$.

63. In an isosceles triangle with the vertex angle of $\pi/5$, the bisector of the base angle separates the triangle into two isosceles triangles one of which is similar to the original one.

Answer: $\frac{\sqrt{5}-1}{2}R$.

64. $R^2\left[\cot\frac{\alpha}{2}-\frac{1}{2}(\pi-\alpha)\right]$. 65. $\frac{a}{4}\sqrt{10}$.

66. $\frac{a(4\sin^2\alpha+1)}{8\sin\alpha}$. 67. $2r^2(2\sqrt{3}+3)$.

68. $\frac{a^2+4r^2}{4r}$. 69. $\frac{3a}{2(5+\sqrt{13})}$. 70. $\frac{a\sqrt{10}}{4}$.

71. 2. 72. $\frac{a^3b}{4(a^2+b^2)}$. 73. $\frac{a}{2}\left(\tan\frac{\alpha}{2}-\cot\alpha\right)$.

74. $\frac{a\cos\frac{\alpha-\beta}{2}}{\sin(\alpha+\beta)}$. 75. $\frac{R^2-a^2}{2R}$.

76. $\frac{a\sqrt{7}}{3\sqrt{3}}$. 77. $a\left(\frac{\sqrt{3}}{3}+\frac{1}{2}\right)$. 78. $\frac{a^2\sqrt{3}}{12}$.

79. $\frac{1}{2}(\beta+\gamma-\alpha)$. 80. $\frac{ac+bd}{a}$.

81. $\frac{\pi}{2\sin^2\alpha\sin 2\beta}$. 82. $\frac{|b-a|}{4}\sqrt{4d^2-(b-a)^2}$.

83. $2(R^2+a^2)$.

84. Two cases are possible: the two centres are on both sides of the common chord and on the same side of it. Accordingly, we have two pairs of answers: $a(\sqrt{3}-1)$, $a\frac{\sqrt{2}}{2}(\sqrt{3}-1)$ and $a(\sqrt{3}+1)$, $a\frac{\sqrt{2}}{2}(\sqrt{3}+1)$. 86. $\frac{3-\sqrt{7}}{4}$.

87. $\sqrt{13}$. 88. $\arccos\frac{1\pm\sqrt{1-2k}}{2}$.

89. $\frac{2}{3}$ 90. $\frac{3a^2}{8}$. 91. $\frac{\pi}{2}$, $\left|\alpha+\frac{\beta}{2}-\frac{\pi}{2}\right|$

$\frac{\pi}{2}-\left|\alpha+\frac{\beta}{2}-\frac{\pi}{2}\right|$. 92. $a^2\,\frac{2\sqrt{3}-3}{8}$. (Generally speaking, two triangles are possible, but in one of them two vertices lie on the extensions of the diagonals.)

93. $\frac{7\sqrt{2}}{10}$. 94. $\frac{br}{c}$. 95. $\sqrt{7}$.

96. $\frac{R}{2}(\sqrt{3}-1)$. 97. $\sqrt{10}$. 98. $\frac{\sqrt{2}}{\cos\alpha}-1$.

100. $\frac{1}{3}\sqrt{96-54\sqrt{3}}$. 101. $3:4$.

102. $a\,\frac{\sin\beta}{\sin\alpha}\cot\frac{\alpha+\beta}{2}$.

103. $\frac{1}{10}\sqrt{25a^2+c^2+10ac\cos\beta}$. 104. $\frac{3}{4}S$.

105. $\frac{4\sqrt{Rr}\,(R-r)}{6Rr-r^2-R^2}$. 106. $\frac{a^2+b^2-2ab\cos\alpha}{2(b-a\cos\alpha)}$.

107. $\frac{3}{10}c$. 108. $\frac{\sqrt{b^2+a^2+2ab\sin\frac{\alpha}{2}}}{2\cos\frac{\alpha}{2}}$.

109. $S\cos^2\alpha$. 110. $\sqrt{4R^2-a^2}$. 111. $\frac{b}{2}$.

112. $\sqrt{a^2+b^2+2ab\cos\alpha}\cdot|\cot\alpha|$.

113. $\sqrt{\frac{1}{4}b^2+\frac{4}{9}a^2-\frac{2}{3}ab\cos\alpha}$.

114. $\arcsin\frac{2}{\pi}$ and $\pi-\arcsin\frac{2}{\pi}$.

11*

115. $a^2(\sqrt{2}-1)$. 116. $\dfrac{a\cos(\alpha+\beta)}{\cos(2\alpha+\beta)}, \dfrac{a\sin(\alpha+\beta)}{\cos(2\alpha+\beta)}$.

117. $\dfrac{1}{2}a(b-a\cos\alpha)\sin^3\alpha$. 118. $\dfrac{2\cos\frac{\alpha}{3}+3}{6\cos\frac{\alpha}{3}+1}$.

119. $\dfrac{2\sqrt{S_2(S_1+S_2)}}{\sqrt[4]{4S_1^2-S_2^2}}$.

120. $4\cos\frac{\alpha}{2}\sqrt{(R_2-R_1)\left(R_2\sin^2\frac{\alpha}{2}+R_1\cos^2\frac{\alpha}{2}\right)}$.

121. $\dfrac{150}{7}$. 122. $\sqrt{\dfrac{d^2}{4}+\dfrac{\sin^2\frac{\beta}{2}\cos^2\frac{\alpha+\beta}{2}}{a^2\cos^2\frac{\alpha}{2}}}$.

123. $\sqrt{a^2+b^2-ab}$, $\sqrt{a^2+b^2+ab}$. 125. 15°, 75°.

126. $\dfrac{R\sqrt{3}}{8}$. 127. $2\sqrt{6}$. 128. $\sqrt{2}$.

129. $\dfrac{4}{3}(2\sqrt{3}+3)$. 130. $\dfrac{2R^2\sin^3\alpha\sin\beta}{\sin(\alpha+\beta)}$.

131. $\dfrac{3\sqrt{3}(\sqrt{13}-1)}{32\pi}$. 132. 1.1 133. If $a/4 < R < a/2$, there is only one solution: $a^2/(16R)$. If $0 < R \leqslant a/4$ or $R \geqslant 2$, we have two solutions: $a^2/(16R)$ and $a^2/(8R)$.

134. $\dfrac{\pi}{2}$ and $\arccos\dfrac{R^2-r^2}{R^2+r^2}$. 135. 30°.

136. $a\sqrt{7}/4$. 137. $R(3-2\sqrt{2})/3$.

138. $4\sqrt{\dfrac{1-\cos\beta}{3-\cos\beta}}$. 139. $\dfrac{ab\tan\alpha}{\sqrt{a^2\tan^2\alpha+(a-b^2)}}$.

(In the triangle *ONP*, *KP* and *NM* are altitudes, therefore *OA* is an altitude.)

140. $2Rr/(R+r)$. 141. $a/2$.

143. The error does not exceed 0.00005 of the radius of the circle.

144. $113-56\sqrt{3}$. **145.** 7.5. **146.** $3\frac{1}{12}$

147. $\frac{2\pi}{3}$. **148.** $\frac{\sqrt{3}+\sqrt{15}}{2}$. **149.** $\frac{2\sqrt{3}}{3}$.

150. $4\sqrt{3}$. **151.** $\frac{16}{9}(4-\sqrt{7})$. **152.** $\frac{\sqrt{5}}{2}$.

153. $2r^2\sin^2\alpha\sin 2\alpha$. **154.** $2\frac{2}{3}$.

155. $\frac{5}{12}\pi+\frac{1}{2}\arccos\left(\frac{3}{\pi}-\frac{\sqrt{3}}{2}\right)$.

156. $\sqrt[4]{12}(2-\sqrt{3})$. **157.** $ar/(a+2r)$.

158. If $\alpha<\frac{\pi}{3}$, then the problem has two solutions: $R^2\sin\alpha\left(1\pm\sin\frac{\alpha}{2}\right)$; if $\frac{\pi}{3}\leqslant\alpha<\pi$, the only one: $R^2\sin\alpha\left(1+\sin\frac{\alpha}{2}\right)$.

159. From $\frac{c}{6}(3\sqrt{2}-4)$ to $\frac{c}{3}$. **160.** From $\frac{|a^2-b^2|}{a^2+b^2}$ to 1. **161.** $\frac{2abc}{ab+bc+ca}$.
(Through an arbitrary point inside the triangle, we draw three straight lines parallel to its sides. Let the first line cut off the triangle which is similar to the original one with the ratio of similitude equal to λ, the second line, with the ratio equal to μ, and the third—with γ. Prove that $\lambda+\mu+\gamma=2$.)

162. $\frac{Rr}{R+r}$.

163.* Take on the line BA a point A_1 such that $|A_1B|=|A_1C|$. The points A_1, A, D and C

lie on a circle ($\angle DA_1C = 90° - \angle ABC = \angle DAC$). Consequently, $\angle A_1AC = \angle A_1DC = 90°$, and hence $\angle BAC = 90°$.

164. 1. 165. $2\frac{1}{4}$. 166. $\frac{13}{15}a$.

167. $\frac{a^2+a\sqrt{a^2+8b^2}}{4}$. 168. $\frac{a^2+a(d-b)}{a-b}\sqrt{bd}$.

169. 6. 170. 3.

171. If $Q \geqslant \frac{1}{4}S$, then the desired distance is $\frac{\sqrt[4]{3}}{3}(\sqrt{S}-\sqrt{Q})$. And if $Q < \frac{1}{4}S$, then two answers are possible: $\frac{\sqrt[4]{3}}{3}(\sqrt{S}\pm\sqrt{Q})$.

172. $3r^2\frac{|1-k^2|}{1+k^2}$. 173. $\frac{2\left(1+\cos\frac{\alpha}{2}\right)}{1+\sin\frac{\alpha}{2}}$.

174. $\frac{(a^2+b^2-c^2)c}{4ab}$.

175. Let A and B denote two adjacent vertices of the rhombus, M the point of intersection of its diagonals, O_1 and O_2 the centres of the circles (O_1 on AM, O_2 on BM). We have: $|AB|^2 = |AM|^2 + |BM|^2 = (|O_2A|^2 - |O_2M|^2) + (|O_1B|^2 - |O_1M|^2) = R^2 + r^2 - (|O_1M|^2 + |O_2M|^2) = R^2 + r^2 - a^2$.

Answer: $\sqrt{R^2+r^2-a^2}$,

176. $\frac{8R^3r^3}{(R^2+r^2)^2}$.

177. $|AB| = \frac{\sqrt{a^2+b^2+2ab\cos\alpha}}{\sin\alpha}$ if B lies inside the given angle or inside the angle vertical

to it; $|AB| = \frac{\sqrt{a^2+b^2-2ab\cos\alpha}}{\sin\alpha}$ in the remaining cases.

178. $2\arcsin\frac{h_a h_b}{l(h_a+h_b)}$. 179. $\frac{3\sqrt{3}}{5\pi-3}$.

180. Since EF is perpendicular to CO (O the point of intersection of the diagonals), and the conditions of the problem imply that AC is the bisector of the angle A which is equal to 60°, we have: $|AE| = |AF| = |EF|$. If K is the midpoint of EF, then $|AO| = 2a\frac{\sqrt{3}}{3}$, $|CO| = a\frac{\sqrt{3}}{3}$, $|CK|\cdot|OK| = |EK|^2 = \frac{1}{3}|AK|^2$.

Answer: $\frac{a^2\sqrt{3}}{4}$ and $2a^2\sqrt{3}$.

181. $\frac{3}{4}h$. 182. Denote: $\angle BAC = \angle BDC = \alpha$, $\angle CBA = \angle BCD = \beta$, $\angle BAM = \varphi$. Then

$$\frac{|BM|+|MC|}{|AM|+|MD|} = \frac{\sin\varphi+\sin(\alpha-\varphi)}{\sin(\beta+\alpha-\varphi)+\sin(\beta+\varphi)} = \frac{\sin\frac{\alpha}{2}\cos\left(\frac{\alpha}{2}-\varphi\right)}{\sin\left(\beta+\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}-\varphi\right)} = \frac{\sin\alpha}{\sin(\beta+\alpha)+\sin\beta} = \frac{c}{a+b}.$$

183. There is always a chord parallel to the base of the triangle. The chord is divided by the lateral sides into three equal parts (undoubtedly, $0 < a < 2$). Its length is $\frac{3a}{2a^2+1}$ In addition, if $a < 1/\sqrt{2}$, then there exists one more chord, which is not parallel to the base and possesses the same property. The length of this chord is $3/\sqrt{9-2a^2}$.

184. Let BC and AC intersect MN at points P and Q, respectively. Setting $\frac{|MC|}{|CN|} = x$, we have: $\frac{|MP|}{|PN|} = \frac{S_{BMC}}{S_{BNC}} = \frac{|MB|\cdot|MC|}{|BN|\cdot|CN|} = \frac{3x}{4}$. Hence, $|MP| = \frac{3x}{3x+4}$. Analogously, $|MQ| = \frac{x}{x+1}$. For x we get the equation $\frac{x}{x+1} - \frac{3x}{3x+4} = a$, $3ax^2 + (7a-1)x + 4a = 0$. Since $D \geqslant 0$ and $0 < a < 1$, the greatest value of a is equal to $7 - 4\sqrt{3}$.

185. The equality $S_{ABN} = S_{CDM}$ implies that $S_{MBN} = S_{MCN}$ since MN is a median of the triangles ABN and CDM. Hence $BC \parallel MN$ and $AD \parallel MN$, that is, $ABCD$ is a trapezoid with bases AD and BC.

Answer: $\frac{5k-2 \pm 2\sqrt{2k(2k-1)}}{2-3k}$.

186. We have: $|AD| \geqslant |DM| - |AM| = 2$. On the other hand, $|AD| \leqslant \frac{|BD|}{\sin 60^\circ} = 2$. Consequently, $|AD| = 2$, AD is the larger base, and the point M lies on the line AD.

Answer: $\sqrt{7}$.

187. Let BD denote an angle bisector in a triangle ABC, A_1 and C_1 the midpoints of the sides BC and AB, $|DA_1| = |DC_1|$. Two cases are possible: (1) $\angle BA_1D = \angle BC_1D$ and (2) $\angle BA_1D + \angle BC_1D = 180^\circ$. In the first case $|AB| = |BC|$. In the second case, we rotate the triangle AC_1D about D through the angle C_1DA_1 to carry C_1 into A_1. We get a triangle with sides $\frac{ba}{a+c}$,

$\frac{a+c}{2}$, $\frac{bc}{a+c}$ (a, b, and c the sides of $\triangle ABC$), which is similar to the triangle ABC. Consequently, $\frac{ba}{a+c} : a = \frac{a+c}{2} \; b = \frac{bc}{a+c} : c$, so $a + c = b\sqrt{2}$. Since $a \neq c$, at least one of the two inequalities $b \neq a$, $b \neq c$ is true. Let $b \neq c$, then $b + c = a\sqrt{2}$, $b = a$, and we get a triangle with sides a, a, $a(\sqrt{2} - 1)$, possessing this property. Thus, there are two classes of triangles satisfying the conditions of the problem: regular triangles and triangles similar to that with sides $1, 1, \sqrt{2} - 1$.

188. If α is the angle between the sides a and b, then we have: $a + b \sin \alpha \leqslant b + a \sin \alpha$, $(a - b) \times (\sin \alpha - 1) \geqslant 1$, $\sin \alpha \geqslant 1$. Hence, $\alpha = 90°$.

Answer: $\sqrt{a^2 + b^2}$.

189. Prove that of all the quadrilaterals circumscribed about the given circle, square has the least area. (For instance, we may take advantage of the inequality $\tan \alpha + \tan \beta \geqslant 2 \tan[(\alpha + \beta)/2]$ where α and β are acute angles.) On the other hand,

$$S_{ABCD} \leqslant \frac{1}{2}(|MA|\cdot|MB| + |MB|\cdot|MC| + |MC|\cdot|MD| + |MD|\cdot|MA|) \leqslant \frac{1}{4}(|MA|^2 + |MB|^2) + \frac{1}{4}(|MB|^2 + |MC|^2) + \frac{1}{4}(|MC|^2 + |MD|^2) + \frac{1}{4}(|MD|^2 + |MA|^2) = 1.$$

Consequently, $ACBD$ is a square whose area is 1.

190. Let us denote: $|BM| = x$, $|DM| = y$, $|AM| = l$, $\angle AMB = \varphi$. Suppose that M lies on the line segment BD. Writing the law of cosines for the triangles AMB and AMD and eliminating $\cos \varphi$, we get: $l^2(x + y) + xy(x + y) = a^2y + d^2x$. Analogously, we get the relationship $l^2(x+y)+$

$xy(x+y) = b^2y + c^2x$. Thus, $(a^2 - b^2)y = (c^2 - d^2)x$.

Answer: $\left|\frac{a^2-b^2}{c^2-d^2}\right|$.

191. If the vertices of the rectangle lie on the concentric circles (two opposite vertices on the circles of radii R_1 and R_2, and the other two on the circles of radii R_3 and R_4), then the equality $R_1^2 + R_2^2 = R_3^2 + R_4^2$ must be fulfilled. Let us prove this. Let A denote the centre of the circles, the vertices K and M of the rectangle $KLMN$ lie on the circles of radii R_1 and R_2, respectively, and L and N on the circles of radii R_3 and R_4, respectively. In the triangles AKM and ALN, the medians emanating from the vertex A are equal, the sides KM and LN are also equal. This means that our statement is true.

Let the second side of the rectangle be x, $x > 1$. The radii R_1, R_2, R_3, R_4 are equal, in some order, to the numbers 1, x, $\sqrt{x^2+1}$, $\frac{1}{2}\sqrt{x^2+1}$. Checking various possibilities of the order, we find: $x^2 = 7$, $R_1 = 1$, $R_2 = 2\sqrt{2}$, $R_3 = \sqrt{2}$, $R_4 = \sqrt{7}$.

Consider the square $K_1L_1M_1N_1$ with side y whose vertices lie on the circles of radii $R_1 = 1$, $R_3 = \sqrt{2}$, $R_2 = 2\sqrt{2}$, $R_4 = \sqrt{7}$. Denote: $\angle AK_1L_1 = \varphi$, then $\angle AK_1N_1 = 90^\circ \pm \varphi$ or $\varphi \pm 90^\circ$. Writing the law of cosines for the triangles AK_1L_1 and AK_1N_1, we get

$$\begin{cases} 1 + x^2 - 2x\cos\varphi = 2, \\ 1 + x^2 \pm 2x\sin\varphi = 7, \end{cases} \Rightarrow \begin{cases} 2x\cos\varphi = x^2 - 1, \\ \pm 2x\sin\varphi = x^2 - 6. \end{cases}$$

Squaring the last two equalities and adding the results, we get: $2x^4 - 10x^2 + 37 = 0$, $x^2 = 5 \pm \frac{1}{2}\sqrt{26}$.

Answer: $\sqrt{5 \pm 2\sqrt{26}}$.

192. Let us first prove the following statement. If the perpendiculars to AB and BC at their midpoints intersect AC at points M and N so that $|MN| = \lambda |AC|$, then either $\tan A \tan C = 1 - 2\lambda$ or $\tan A \tan C = 1 + 2\lambda$. Let us denote: $|AB| = c$, $|BC| = a$, $|AC| = b$. If the segments of the perpendiculars from the midpoints of the sides to the points M and N do not intersect, then

$$|MN| = b - \frac{c}{2\cos A} - \frac{a}{2\cos C} = \lambda b \Rightarrow 2(1 - \lambda)\sin B \cos A \cos C = \frac{1}{2}(\sin 2C + \sin 2A) \Rightarrow$$

$$2(1 - \lambda)\sin(A + C)\cos A \cos C = \sin(A + C)\cos(A - C) \Rightarrow 2(1 - \lambda)\cos A \times \cos C = \cos A \cos C + \sin A \sin C \Rightarrow \tan A \times \cos C = 1 - 2\lambda.$$

And if these segments intersect, then $\tan A \tan C = 1 + 2\lambda$. In our case $\lambda = 1$, that is, either $\tan A \times \tan C = -1$ or $\tan A \tan C = 3$. For the angles B and C we get ($\lambda = 1/2$) either $\tan B \tan C = 0$ (this is impossible) or $\tan B \tan C = 2$. The system

$$\begin{cases} \tan A \tan C = -1, \\ \tan B \tan C = 2, \\ A + B + C = \pi \end{cases}$$

has no solution. Hence, $\tan A \tan C = 3$. Solving the corresponding system, we find: $\tan A = 3$, $\tan B = 2$, $\tan C = 1$.

Answer: $\pi/4$.

193. Let R denote the radius of the circle circumscribed about $\triangle ABC$, O its centre, N the median point of the triangle BCM. The perpendicularity of ON and CM implies the equality $|CN|^2 - |MN|^2 = |CO|^2 - |OM|^2$. Let $|AB| = 1$, $|MB| = x$, $|CM| = y$, then $|MN|^2 = \frac{1}{9}(2y^2 + 2x^2 - k^2)$, $|CN|^2 = \frac{1}{9}(2y^2 + 2k^2 - x^2)$,

$$|CO|^2 = R^2, \quad |OM|^2 = R^2 \cos^2 C + \left(x - \frac{1}{2}\right)^2.$$

We get equation for x: $2x^2 - 3x + k^2 = 0$.

Answer: $\dfrac{3 \pm \sqrt{9-8k^2}}{4}$ $\left(\text{if } 1 < k < \dfrac{3\sqrt{2}}{4}\right.$, then both points are found inside the line segment AB).

194. If O is the midpoint of AC, then $|AB|^2 = |BO|^2 + |AO|^2 = |BK|^2 - |KO|^2 + |AO|^2 = |BK|^2 + (|AO| - |AK|)(|AO| + |AK|) = |BK|^2 + |AK| \cdot |CK| = b^2 + bd$.

Answer: $\sqrt{b^2+bd}$.

195. (1) The length of a broken line of three segments is equal to the line segment joining its end points. This is possible only if all of its vertices lie on this segment. $x = \dfrac{2ab}{a+b\sqrt{3}}$, $y = \dfrac{2ab}{a\sqrt{3}+b}$.

(2) x, y, z are the sides of a triangle whose altitudes are a, b, and c. Such a triangle must not be obtuse-angled. To find x, y, z, let us take advantage of the fact that a triangle whose sides are inversely proportional to the altitudes of the given triangle is similar to the latter.

$x = \dfrac{1}{2as}$, $y = \dfrac{1}{2bs}$, $z = \dfrac{1}{2cs}$, where

$s = \sqrt{p\left(p - \dfrac{1}{a}\right)\left(p - \dfrac{1}{b}\right)\left(p - \dfrac{1}{c}\right)}$, $2p = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$. The problem has a solution if $\dfrac{1}{a^2} + \dfrac{1}{b^2} \geqslant \dfrac{1}{c^2}$, $\dfrac{1}{b^2} + \dfrac{1}{c^2} \geqslant \dfrac{1}{a^2}$, $\dfrac{1}{c^2} + \dfrac{1}{a^2} \geqslant \dfrac{1}{b^2}$.

(3) Consider the points $A(a, b)$, $B(x, 0)$, $C(0, y)$ in a rectangular coordinate system. It follows from the given system that ABC is an equilateral triangle. When rotated about A through an angle of 60° in the appropriate direction, the

point B goes into C. We can find the equation of the straight line into which the x-axis is carried by this rotation. (In particular, the slope is equal to $\pm\sqrt{3}$.)

Answer: $x=-a\pm b\sqrt{3}$, $y=-b\pm a\sqrt{3}$.

(4) If $x>0$, $y>0$, $z>0$, then x, y, z are the distances to the vertices of a right triangle ABC in which the legs BC and CA are a and b, respectively, from such a point M inside it from which all of its sides can be observed at an angle of 120°. To determine the sum $x+y+z$, let us rotate the triangle CMA about C through an angle of 60° in the direction external with respect to the triangle ABC. As a result, M and A go into M_1 and A_1, respectively. Then BMM_1A_1 is a straight line and, consequently, $x+y+z=|BM|+|CM|+|AM|=|BA_1|=\sqrt{a^2+b^2+ab\sqrt{3}}$. Analogously, we consider the case when one of the variables is negative (generally speaking, not any of them can be negative) and other cases.

Answer: $\pm\sqrt{a^2+b^2\pm ab\sqrt{3}}$.

196. Let x be the distance from the centre of the square to the straight line l, φ the acute angle formed by one of the diagonals of the square and the line l. The distances from the vertices of the square to l are equal to (in the order of traverse):

$$x+a\frac{\sqrt{2}}{2}\sin\varphi,\quad x+a\frac{\sqrt{2}}{2}\cos\varphi,$$

$\left|x-a\frac{\sqrt{2}}{2}\sin\varphi\right|$, $\left|x-a\frac{\sqrt{2}}{2}\cos\varphi\right|$. By hypothesis, $\left|x^2-\frac{a^2}{2}\sin^2\varphi\right|=\left|x^2-\frac{a^2}{2}\cos^2\varphi\right|$, whence either $\tan^2\varphi=1$, which is impossible, or $x^2=a^2/4$.

Answer: $a/2$.

197. From the condition $\angle B=2\angle C$ there follows the relationship for the sides of the triangle: $b^2 = c^2 + ac$. Looking over $b = 2c$, $a = 2c$, $b = 2a$, and $a = 2b$, we choose $a = 2c$ since in other cases the triangle inequality is not fulfilled.

Answer: $\angle C = \pi/6$, $\angle B = \pi/3$, $\angle A = \pi/2$.

198. Let D be the midpoint of BC. We have: $b^2 = |BM|^2 = (|BD| + |DN|)(|BD| - |DN|) = |BD|^2 - |DN|^2 = |AB|^2 - |AD|^2 - |DN|^2 = (a+b)^2 - |AD|^2 - |DN|^2$. Hence, $|AN|^2 = |AD|^2 + |DN|^2 = (a+b)^2 - b^2 = a^2 + 2ab$.

Answer: $\sqrt{a^2 + 2ab}$.

199. We take on BC a point N such that the triangle ABN is similar to the triangle ADL. Then $\angle NMA = \angle MAK + \angle KAD = \angle MAB + \angle DAL = \angle MAN$. Consequently, $|MN| = |AN| = k|AL|$.

Answer: $\dfrac{a}{k} + b$.

200. $2\sqrt{pq}$.

201. (a) $\dfrac{a}{R}\sqrt{(R \pm x)(R \pm y)}$, the plus sign corresponding to external tangency of the circles, the minus sign to internal. (b) $\dfrac{a}{R}\sqrt{(R+x)(R-y)}$.

202. Let $|AM| : |MC| = k$. The equality of the radii of the circles inscribed in the triangles ABM and BCM means that the ratio of their areas is equal to the ratio of their perimeters. Hence, since the ratio of the areas is k, we get $|BM| = \dfrac{13k - 12}{1 - k}$. It follows from this equality, in particular, that $12/13 < k < 1$. Writing for the triangles ABM and BCM the laws of cosines (with respect to the angles BMA and BMC) and eliminating the cosines of the angles from those equations, we get for k a quadratic equation with roots $2/3$ and $22/23$. Taking into account the limitations for k, we get $k = 22/23$

203. Let ABC denote the given triangle, O, K, H the centres of the circumscribed and inscribed circles, and the intersection point of the altitudes of the triangle ABC, respectively. Let us take advantage of the following fact: in an arbitrary triangle the bisector of any of its angles makes equal angles both with the radius of the circumscribed circle and with the altitude emanating from the same vertex (the proof is left to the reader). Since the circle passing through O, K, and H contains at least one vertex of the triangle ABC (say, the vertex A), it follows that $|OK| = |KH|$. The point K is situated inside at least one of the triangles OBH and OCH. Let it be the triangle OBH. The angle B cannot be obtuse. In the triangles OBK and HBK, we have: $|OK| = |HK|$, KB is a common side, $\angle OBK = \angle HBK$. Hence, $\triangle OBK = \triangle HBK$, since otherwise $\angle BOK + \angle BHK = 180°$ which is impossible (K is inside the triangle OBH). Consequently, $|BH| = |BO| = R$. The distance from O to AC equals $0.5\,|BH| = 0.5R$ (Problem 20 of Sec. 1), that is, $\angle B = 60°$ ($\angle B$ is acute), $|AC| = R\sqrt{3}$. If now A_1, B_1, and C_1 are the points of tangency of the sides BC, CA, and AB to the inscribed circle, respectively, then $|BA_1| = |BC_1| = r\sqrt{3}$, $|CA_1| + |AC_1| = |CB_1| + |B_1A| = |AC| = R\sqrt{3}$. The perimeter of the triangle is equal to $2\sqrt{3}\,(R + r)$. It is now easy to find its area.

Answer: $\sqrt{3}\,(R + r)\,r$.

204. Let P be the projection of M on AB, $|AP| = a + x$. Then $|PB| = a - x$, $|MP| =$

$$y = \sqrt{a^2 - x^2}, \quad |AN| = (a+x)\frac{a\sqrt{2}}{a\sqrt{2}+y}, \quad |NB| =$$

$$2a - (a+x)\frac{a\sqrt{2}}{a\sqrt{2}+y} = \frac{a\sqrt{2}\,(a-x+y\sqrt{2})}{a\sqrt{2}+y},$$

$$|AL| = \frac{a\sqrt{2}\,(a+x+y\sqrt{2})}{a\sqrt{2}+y}.$$

Hence

$$|AL|^2+|NB|^2 = \frac{4a^2}{(a\sqrt{2}+y)^2}(a^2+2\sqrt{2}\,ay+2y^2+x^2) = \frac{4a^2}{(a\sqrt{2}+y)^2}(a^2+2\sqrt{2}\,ay+2y^2+(a^2-y^2)) = 4a^2.$$

205. Let x denote the side of the triangle, and the sides emanating from the common point of the circles form with the straight line passing through the centres angles α and β; $\alpha \pm \beta = 60°$, then $\cos\alpha = \frac{x}{2R}$, $\cos\beta = \frac{x}{2r}$ (or vice versa). Finding $\sin\alpha$ and $\sin\beta$ from the equation $\cos(\alpha \pm \beta) = \frac{1}{2}$, we determine the side of the regular triangle:

$$\frac{Rr\sqrt{3}}{\sqrt{R^2+r^2-Rr}}.$$

206. We draw a straight line BA and denote by D the second point of intersection with the smaller circle. Consider the arcs AB and AD (each less than a semicircle). Since the common tangent to the circle at A forms equal angles with AB and AD, the central angles corresponding to these arcs are also equal. Consequently, $\frac{|AD|}{|AB|} = \frac{r}{R}$, $AD = a\frac{r}{R}$, $|BC| = \sqrt{|BD|\cdot|BA|} = a\sqrt{\frac{R+r}{R}}$.

207. Let O_1, O_2, and O denote the centres of the circles (the first two touching AB), x, y, and R their radii, respectively. The common tangents to the circles centred at O_1 and O_2, O_1 and O, O_2 and

O are equal to $2\sqrt{xy}$, $2\sqrt{Rx}$, $2\sqrt{Ry}$, respectively. By hypothesis, $2\sqrt{xy} = a$. Consider the right triangle O_1MO_2 with the right angle at the vertex M; O_1M is parallel to BC, $|O_1O_2| = x + y$, $|O_2M| = 2R - (x + y)$, $|O_1M| = |2\sqrt{Rx} - 2\sqrt{Ry}|$ (O_1M being equal to the difference between the common tangents to the circles with centres O, O_1 and O, O_2). Thus, $(x + y)^2 = (2R - x - y)^2 + (2\sqrt{Rx} - 2\sqrt{Ry})^2$, whence $R = 2\sqrt{xy} = a$.

208. Note that $O_1O_2O_3O_4$ is a parallelogram with angles α and $\pi - \alpha$ ($O_1O_4 \perp AC$ and $O_2O_3 \parallel AC$, hence, $O_1O_4 \parallel O_2O_3$, etc.). If K is the midpoint of AM, L the midpoint of MC, then $|O_3O_4| = \frac{|KL|}{\sin\alpha} = \frac{|AC|}{2\sin\alpha}$. Analogously, $|O_2O_3| = \frac{BD}{2\sin\alpha}$; consequently, $S_{O_1O_2O_3O_4} = \frac{|AC|\cdot|BD|\sin\alpha}{4\sin^2\alpha} = \frac{S_{ABCD}}{2\sin^2\alpha}$.

Answer: $2\sin^2\alpha$.

209. When intersecting, the angle bisectors of the parallelogram form a rectangle whose diagonals are parallel to the sides of the parallelogram and are equal to the difference of the sides of the parallelogram. Consequently, if a and b are the sides of the parallelogram and α the angle between them, then $S = ab\sin\alpha$, $Q = \frac{1}{2}(a - b)^2\sin\alpha$,

$$\frac{S}{Q} = \frac{2ab}{(a-b)^2}.$$

Answer: $\frac{S+Q+\sqrt{Q^2+2QS}}{S}$.

210. Let x denote the area of the triangle OMN, y the area of the triangle CMN, then

$\frac{|ON|}{|OA|}=\frac{x}{S_1}=\frac{S_3}{S_2}$, $x=\frac{S_1S_3}{S_2}$, $\frac{|AM|}{|MC|}=\frac{S_1+x}{y}=\frac{S_1+S_2}{S_3+x+y}$. The sought-for area is equal to $\frac{S_1S_3(S_1+S_2)(S_3+S_2)}{S_2(S_2^2-S_1S_3)}$.

211. Let in the triangle ABC the angle C be a right one, M the median point, O the centre of the inscribed circle, r its radius, $\angle B=\alpha$; then $|AB|=r\left(\cot\frac{\alpha}{2}+\cot\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\right)=\frac{r\sqrt{2}}{\sin\frac{\alpha}{2}\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}$, $|CM|=\frac{1}{3}|AB|$,

$|CO|=r\sqrt{2}$, $|OM|=r$, $\angle OCM=\alpha-\frac{\pi}{4}$. Writing the law of cosines for the triangle COM, we get $1=2+\frac{8}{9(2x-\sqrt{2})^2}-\frac{8x}{3(2x-\sqrt{2})}$, where $x=\cos\left(\frac{\pi}{4}-\alpha\right)$, whence $x=\frac{4\sqrt{6}-3\sqrt{2}}{6}$.

Answer: $\frac{\pi}{4}\pm\arccos\frac{4\sqrt{6}-3\sqrt{2}}{6}$.

212. Let each segment of the median be equal to a. We denote by x the smallest of the line segments into which the side corresponding to the median is divided by the point of tangency. Now, the sides of the triangle can be expressed in terms of a and x. The sides enclosing the median are $a\sqrt{2}+x$, $3a\sqrt{2}+x$, the third side is $2a\sqrt{2}+2x$. Using the formula for the length of a median (see Problem 11, Sec. 1), we get $9a^2=\frac{1}{4}[2(a\sqrt{2}+x)^2+2(3a\sqrt{2}+x)^2-(2a\sqrt{2}+2x)^2]$, whence $x=a\sqrt{2}/4$.

Answer: 10 : 5 : 13.

213. Let $|BC| = a$, $\angle C > \angle B$, D and E be the midpoints of AB and AC. The quadrilateral $EMDN$ is an inscribed one (since $\angle MEN = \angle MDN = 90°$), $|MN| = a$, $|ED| = a/2$, MN is a diameter of the circle circumscribed about $MEND$. Consequently, $\angle DME = 30°$, $\angle CAB = 90° - \angle EMD = 60°$, $\angle CBA = \angle EDN = \angle EMN = \angle EMD/2 = 15°$, $\angle ACB = 105°$.

Answer: $\angle A = 60°$, $\angle B = 15°$, $\angle C = 105°$ or $\angle A = 60°$, $\angle B = 105°$, $\angle C = 15°$.

214. We denote by K and M the points of intersection of the straight line EF with AD and

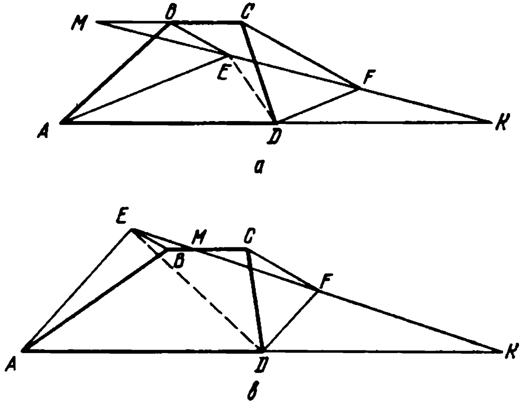


Fig. 1

BC, respectively. Let M lie on the extension of BC beyond the point B. If $|AD| = 3a$, $|BC| = a$, then from the similarity of the corresponding triangles, it follows that $|DK| = |AD| = 3a$, $|MB| = |BC| = a$ (Fig. 1, *a*).

In addition, $|ME| = |EF| = |FK|$. If h is the altitude of the trapezoid, then the distance from E to AD is equal to $\frac{2}{3}h$, $S_{EDK} = ah$, $S_{EDF} = \frac{1}{2}S_{EDK} = \frac{ah}{4} = \frac{1}{4}S$.

If the line EF intersects the base BC at a point M, then $|BM| = \frac{1}{3}a$ (Fig. 1, b). In this case $\frac{|EK|}{|MK|} = 2 : \frac{5}{3} = \frac{6}{5}$ and the distance from E to AD equals $\frac{6}{5}h$, so that $S_{EFD} = \frac{1}{2}S_{EDK} = \frac{1}{4}\cdot 3a\cdot\frac{6}{5}h = \frac{9}{20}S$.

Answer: $\frac{1}{4}S$ or $\frac{9}{20}S$.

215. Let O be the centre of the inscribed circle, M the midpoint of BC, K, L and N the points of tangency of the inscribed circle with the sides AC, AB, and BC of the triangle, respectively. We denote: $|AK| = |AL| = x$, $|CK| = |CN| = y$, $|BL| = |BN| = z$, $y + z = a$. By hypothesis, $|OM| = \frac{a}{2} - r$. Consequently, $|NM| = \sqrt{|OM|^2 - |ON|^2} = \sqrt{\frac{a^2}{4} - ar}$ and one of the line segments, either y or z, is equal to $\frac{a}{2} - \sqrt{\frac{a^2}{4} - ar}$, and the other to $\frac{a}{2} + \sqrt{\frac{a^2}{4} - ar}$. Equate the expressions for the area of the triangle by Hero's formula and $S = pr$: $\sqrt{(x+y+z)\,xyz} = (x+y+z)\,r \Rightarrow xar = (x+a)\,r^2 \Rightarrow x = \frac{ar}{a-r}$. Thus, the sought-for area is equal to $\left(\frac{ar}{a-r} + a\right) r = \frac{a^2r}{a-r}$.

216. Let us prove that if C_1 and C_2 (Fig. 2) are situated on the other side of BC than the vertex A, then the centre of the circle circumscribed about the triangle CC_1C_2 is found at the point O on the side AB, and $|BO| = \frac{1}{4}|AB|$. Drawing the altitude CM from the vertex C, we obtain the quadrilateral BC_1CM which is a rectangle. Hence

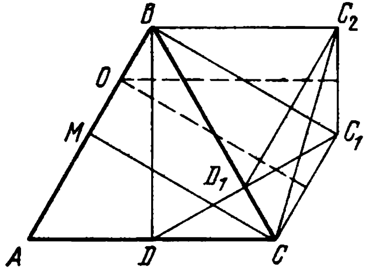


Fig. 2

the perpendicular drawn to CC_1 at its midpoint passes through O. Taking into consideration that $C_1C_2 \parallel BD$ and $|C_1C_2| = \frac{1}{2}|BD|$, we see that the middle perpendicular to C_1C_2 also passes through O. Now, we find easily the desired radius: it is equal to $\sqrt{|CM|^2 + |MO|^2} =$

$$\sqrt{\frac{3a^2}{4} + \frac{a^2}{16}} = \frac{a}{4}\sqrt{13}.$$

217. Consider two cases: (1) the feet of the perpendiculars are found on the sides of the parallelogram, and (2) one of the perpendiculars does not intersect the side on which it is dropped. In the first case we arrive at a contradiction, while in the second case we obtain $\cos\alpha = \frac{2ab}{a^2 + b^2}$,

where α is the acute angle of the given parallelogram.

218. Expressing the angle PQN in terms of the angles of the triangle and bearing in mind that $\angle PMN + \angle PQN = 180°$, we find: $\angle PMN = 60°$; hence $\angle NPQ = \angle QMN = 30°$, $\angle PNQ = \angle PMQ = 30°$, that is PQN is an isosceles triangle with angles at the side PN of $30°$, $|PQ| = |QN| = 1/\sqrt{3}$.

219. It follows from the conditions that $ABCD$ is a trapezoid, $BC \parallel AD$, and AC is the bisector of the angle BAD; hence $|AB| = |BC|$, analogously, $|BC| = |CD|$. Let $|AB| = |BC| = |CD| = a$, $|AD| = b$. The distance between the midpoints of the diagonals is $2r$, consequently $\frac{b-a}{2} = 2r$. We draw the altitude BM from the point B on AD and we get that $|AM| = \frac{b-a}{2} = 2r$, $|BM| = 2r$. Consequently, $a = |AB| = 2r\sqrt{2}$, $b = 4r + 2r\sqrt{2}$.

Answer: $4r^2(\sqrt{2}+1)$.

220. Let us denote the angles A, B, and C by α, β, and γ, respectively. Let H be the point of intersection of the altitudes, O the centre of the circle passing through A, H, and C. Then $\angle HOC = 2\angle HAC = 2(90° - \gamma)$, $\angle HOA = 2\angle HCA = 2(90° - \alpha)$. But $\angle AOC = 180° - \beta$ (since $BAOC$ is an inscribed quadrilateral), $2(90° - \gamma) + 2(90° - \alpha) = 180° - \beta$, $360° - 2\alpha - 2\gamma = 180° - \beta$, $2\beta = 180° - \beta$, $\beta = 60°$, $|AC| = 2R\sin\beta = \sqrt{3}$.

221. Denoting the ratio $\frac{|AM|}{|MC|} = \lambda$, we have: $S_{MCP} = \frac{T}{\lambda}$, $S_{CPN} = \lambda Q$, $S_{MCP} = \lambda S_{CPN}$; consequently, $(T/Q) = \lambda^3$, $S_{ABC} = \frac{|AC|}{|MC|}\cdot\frac{|BC|}{|CN|}S_{CMN} =$

$\frac{(\lambda+1)^2}{\lambda}\left(\frac{T}{\lambda}+\lambda Q\right)=\frac{(\lambda+1)^2}{\lambda^2}(T+\lambda^2 Q)=$
$(\lambda+1)^3 Q=(T^{1/3}+Q^{1/3})^3$.

222. If O is the centre of the circle, then the area of $\triangle OMN$ is $\frac{a}{a-R}$ times the area of $\triangle KMN$. If $\angle MON=\alpha$, then $\frac{R^2}{2}\sin\alpha=\frac{a}{a-R}S$, $\sin\alpha=\frac{2aS}{R^2(a-R)}$, $|MN|=2R\sin\frac{\alpha}{2}=R\sqrt{1-\cos\alpha}=R\sqrt{1\pm\sqrt{1-\frac{4a^2S^2}{R^4(a-R)^2}}}$. The problem has a solution if $S\leqslant\frac{R^2(a-R)}{2a}$.

223. If $\angle BAC=\angle BCA=2\alpha$, then by the law of sines, we find: $|AE|=\frac{2m\sin 2\alpha}{\sin 3\alpha}$, $|AF|=\frac{|AE|}{\cos\alpha}=\frac{2m\sin 2\alpha}{\sin 3\alpha\cos\alpha}$. Thus, $\frac{9}{4}m=\frac{2m\sin 2\alpha}{\sin 3\alpha\cos\alpha}$, whence $\cos 2\alpha=\frac{7}{18}$, $S_{ABC}=m^2\tan 2\alpha=\frac{5m^2\sqrt{11}}{7}$.

224. The points C, M, D, and L lie on a circle, consequently, $\angle CML=\angle CDL=30°$. In similar fashion $\angle CMK=30°$; thus, $\angle LMK=60°$ and $\triangle LMK$ is regular, $|KM|=2/\sqrt{5}$. By the law of cosines, we find: $\cos\angle LCK=-3/5$. Since $\angle DCB=\angle LCK-120°$, we have: $|DB|=\frac{2-\sqrt{3}}{\sqrt{5}}$.

225. Let A be the point of intersection of the straight lines BC and KM. The quadrilateral

$ONBC$ is an inscribed one ($\angle OCB = \angle ONB = 90°$), consequently, $\angle OBC = \angle ONC = \alpha/2$. Similarly, $CMAO$ is also an inscribed quadrilateral and $\angle CAO = \angle CMO = \alpha/2$, that is, OAB is an isosceles triangle. Thus, $|CB| = |AC| = |CO| \times \cot\frac{\alpha}{2} = \sqrt{R^2 + b^2 - 2Rb\cos\frac{\alpha}{2}}\cot\frac{\alpha}{2}$.

226. The points E, M, B, and Q lie on a circle of diameter BE, and the points E, P, D, and N

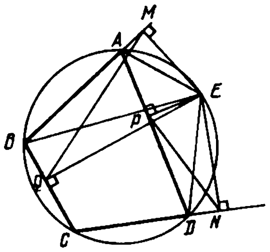


Fig. 3

on a circle of diameter ED (Fig. 3). Thus, $\angle EMQ = \angle EBQ = 180° - \angle EDC = \angle EDN = \angle EPN$, analogously, $\angle EQM = \angle ENP$, that is, the triangle EMQ is similar to the triangle EPN with the ratio of similitude of $\sqrt{k}$. (For completeness of solution, it is necessary to consider other cases of the arrangement of the points.)

Answer: $d\sqrt{k}$.

227. Extending the non-parallel sides of the trapezoid to their intersection, we get three similar triangles, the ratio of similitude of the middle to larger triangle and of the smaller to middle one being the same. Let us denote this ratio by λ, the larger base by x, the radius of the larger circle by R. Then the line segments parallel to the larger base are, respectively, equal to λx and $\lambda^2 x$, the

larger lateral side of the lower trapezoid to $2R\frac{d}{c}$, the second radius to λR. Hence, $R + \lambda R = \frac{c}{2}$. By the property of an circumscribed quadrilateral, $x + \lambda x = 2R + 2R\,\frac{d}{c}$. And finally, dropping from the end point of the smaller base of the entire trapezoid a perpendicular on the larger base, we get a right triangle with legs c, $x - \lambda^2 x$, and hypotenuse d. Thus, we have the system

$$\begin{cases} x(1+\lambda) = 2R\,\dfrac{c+d}{c}, \\ x(1-\lambda^2) = \sqrt{d^2-c^2}, \\ R(1+\lambda) = c/2, \end{cases}$$

whence $\lambda = \dfrac{d-\sqrt{d^2-c^2}}{c}$.

Answer: the bases are equal to $\dfrac{d-\sqrt{d^2-c^2}}{c}$ and $\dfrac{d+\sqrt{d^2-c^2}}{c}$

228. Let us draw perpendiculars from the centres of the circles to one of the sides and draw through the centre of the smaller circle a straight line parallel to this side. In doing so, we obtain a right triangle with hypotenuse $R + r$, one of the legs $R - r$ and an acute angle α at this leg equal to the acute angle at the base of the trapezoid. Thus $\cos\alpha = \frac{R-r}{R+r}$. The larger base is equal to $2R\cot\frac{\alpha}{2} \quad 2R\sqrt{\frac{R}{r}}$. The smaller base is equal to $2r\tan\frac{\alpha}{2} = 2r\sqrt{\frac{r}{R}}$.

229. Let us take on the side AB a point K such that $|BK| = |BD|$, and on the extension of AC a point E such that $|CE| = |CD|$. Let us show that the triangle ADK is similar to the triangle ADE. If A, B, and C are the sizes of the interior angles of the $\triangle ABC$, then, $\angle DKA = 180° - \angle DKB = 180° - (90° - \angle B/2) = 90° + \angle B/2$, $\angle ADE = 180° - \angle CED - \angle A/2 = 180° - \frac{1}{2}(\angle A + \angle C) = 90° + \angle B/2$. Thus, $\angle AKD = \angle ADE$. In addition, by hypothesis, $\angle DAE = \angle DAK$.

Answer: $\sqrt{ab}$.

230. Using the notation of the preceding problem, we have:

$$|AD|^2 = (|AC| + |CD|)(|AB| - |BD|) = |AC|\cdot|AB| - |CD|\cdot|BD| + (|AB|\cdot|CD| - |AC|\cdot|BD|).$$

But the term in the parentheses is equal to zero since $\frac{|AB|}{|AC|} = \frac{|BD|}{|CD|}$ (see Problem 9 in Sec. 1).

231. Let us extend BN and CN to intersect the second circle for the second time at points K and L, respectively; $|MN| = |NK|$ since $\angle ANB = 90°$ and MK is a chord of the circle centred at A. Since the corresponding arcs are equal, we have $\angle LNK = \angle BNC = \angle BND$. Thus, $|LN| = |ND| = b$, $|MN|\cdot|NK| = |MN|^2 = ab$, $|MN| = \sqrt{ab}$.

232. Note that PQ is perpendicular to CB. Let T be the point of intersection of MN and PQ, and L and K the feet of the perpendiculars dropped from C and B on the straight line MN (L and K lie on the circles constructed on CN and BM as diameters). Using the properties of intersecting chords in circles, we get: $|PT|\cdot|TQ| = |NT| \times |LT|$, $|PT|\cdot|TQ| = |MT|\cdot|TK|$. But $|LT| = |CD|$, $|TK| = |DB|$ (since $CLKB$ is a rectangle and PQ is perpendicular to CB). Thus,

$$|NT|\cdot|CD| = |MT|\cdot|DB|,\ \frac{|MT|}{|NT|} = \frac{|CD|}{|DB|},$$
that is, the straight line PQ divides CB and MN in the same ratio, hence, PQ passes through the point A, and D is the foot of the altitude.

Answer: $|BD| : |DC| = 1 : \sqrt{3}$.

233. Let $\angle BOC = 2\alpha$, $\angle BOL = 2\beta$. Then $|AC| = 2R\cos\alpha$, $|CL| = 2R\sin(\alpha+\beta)$, $|CM| = |CL|\cos(90° - \beta) = 2R\sin(\alpha+\beta)\times \sin\beta$, $|AM| = |AC| - |CM| = 2R\,(\cos\alpha - \sin(\alpha+\beta)\sin\beta) = 2R\cos\beta\cos(\alpha+\beta)$, and, finally, $|AN| = a = |AM|\cos\alpha = 2R\cos\alpha \times \cos\beta\cos(\alpha+\beta)$. On the other hand, if K, P, and Q are the midpoints of AO, CO, and CL, respectively, then $|KP| = \frac{1}{2}|AC| = R\cos\alpha$. Further $|PQ| = R/2$, $\angle KPQ = \angle KPO + \angle OPQ = \alpha + 180° - \angle COL = 180° - \alpha - 2\beta$, and, by the law of cosines, $|KQ|^2 = \frac{R^2}{4} + R^2\cos^2\alpha + R^2\cos\alpha\cos(\alpha+2\beta) = R^2/4 + 2R^2\cos\alpha\cos\beta\cos(\alpha+\beta) = \frac{R^2}{4} + Ra$.

Answer: $\sqrt{\frac{R^2}{4}+Ra}$.

234. It follows from the similarity of the triangles MAB and MBC that
$$\frac{|MA|}{|MC|} = \frac{|MA|}{|MB|}\cdot\frac{|MB|}{|MC|} = \frac{|BA|^2}{|BC|^2} = k^2.$$

235. From Problem 234 in Sec. 1, it follows that $\frac{|AM|^2}{|MB|^2} = \frac{|AC|}{|BC|}$, $\frac{|AN|^2}{|NB|^2} = \frac{|AD|}{|BD|}$. If K is the point of intersection of MN and AB, then
$$\frac{|AK|}{|KB|} = \frac{S_{AMN}}{S_{BMN}} = \frac{|AM|\cdot|AN|\sin\angle MAN}{|MB|\cdot|NB|\sin\angle MBN} =$$
$$\sqrt{\frac{|AC|}{|BC|}\cdot\frac{|AD|}{|BD|}} = \sqrt{\frac{\alpha\beta}{(\alpha-1)(\beta-1)}}$$

236. Let K, L, M, and N be the points of tangency of the sides AB, BC, CD, and DA with the circle, respectively. Let P denote the point of intersection of AC and KM. If $\angle AKM = \varphi$, then $\angle KMC = 180° - \varphi$. Thus, $\frac{|AP|}{|PC|} = \frac{S_{AKM}}{S_{KMC}} =$

$$\frac{\frac{1}{2}|AK|\cdot|KM|\sin\varphi}{\frac{1}{2}|KM|\cdot|MC|\sin(180°-\varphi)} = \frac{|AK|}{|MC|} = \frac{a}{b}.$$

But in the same ratio AC is divided by the straight line NL. Hence, the lines AC, KM, and NL meet in the same point. Considering the diagonal BD and reasoning in the same way, we prove that BD also passes through the point P. The sought-for ratio is equal to a/b.

237. Let P and Q be the intersection points of BK and AC, and AB and DC, respectively. The straight line QP intersects AD at a point M, and BC at a point N. Using the similarity of the corresponding triangles we get: $\frac{|AM|}{|MD|} = \frac{|BN|}{|NC|} = \frac{|MK|}{|AM|} = \frac{|AK|-|AM|}{|AM|}$. If $|AM| = x\,|AD|$, then $\frac{|AM|}{|MD|} = \frac{|AM|}{|AD|-|AM|} = \frac{x}{1-x}$, $\frac{x}{1-x} = \frac{\lambda - x}{x}$, whence $x = \frac{\lambda}{\lambda+1}$.

Answer: $\frac{\lambda}{\lambda+1}$

If $\lambda = \frac{1}{n}$, then $|AM| = \frac{1}{n+1}|AD|$. Thus, taking first K to be coincident with D ($\lambda = 1$), we get the midpoint of AB as M_1; taking K to be coincident with M_1, we find that M_2 is 1/3 distant from AD, and so forth.

238. Let $|KM| = |KN| = x$, $|AD| = y$, and $|DB| = z$. Then $|CD| = \sqrt{yz}$, $y + z = c$. The radius of the circle inscribed in the triangle AKB is equal to $\frac{1}{2}|CD| = \frac{1}{2}\sqrt{yz}$. Express the area of the triangle AKB by Hero's formulas and $S = pr$. We get the equation

$$\sqrt{(x+y+z)\,xyz} = (x+y+z)\,\frac{1}{2}\sqrt{yz}.$$

Knowing that $y + z = c$, we find $x = c/3$.

239. Through the point A_2, draw a straight line parallel to AC. Let R be the point of intersection of this line and AB. Bearing in mind that $\frac{|AR|}{|RC_1|} = \frac{|B_1A_2|}{|A_2C_1|} = \frac{1}{k}$, $\frac{|AC_1|}{|C_1B|} = k$, we find: $\frac{|AR|}{|AB|} = \frac{k}{(k+1)^2}$. In similar fashion, drawing through C_2 a straight line parallel to AC to intersect BC at a point S, we obtain that $\frac{|CS|}{|CB|} = \frac{k}{(k+1)^2}$. Therefore the points R, A_2, C_2, and S lie on a straight line parallel to AC. Thus, the sides of the triangles ABC and $A_2B_2C_2$ are correspondingly parallel. Now it is easily obtained that $|A_2C_2| = |RS| - |RA_2| - |C_2S| = |AC| \times \left(1 - \frac{3k}{(k+1)^2}\right)$, therefore the ratio of similitude is equal to $\frac{k^2-k+1}{(k+1)^2}$.

240. Let us use the following formula for the area of a triangle: $S = 2R^2 \sin A \sin B \sin C$, where A, B, and C are its angles. Then the area of the triangle $A_1B_1C_1$, where A_1, B_1, and

C_1 are the intersection points of the angle bisectors of the triangle ABC with the circumscribed circle, will be equal to $S_1 = 2R^2 \times \sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{C+A}{2} = 2R^2\cos\frac{C}{2}\times \cos\frac{A}{2}\cos\frac{B}{2}$, and $\frac{S}{S_1} = 8\sin\frac{A}{2}\sin\frac{B}{2}\times \sin\frac{C}{2}$. On the other hand, $|BC| = 2R\sin A$, $r\times \left(\cot\frac{B}{2}+\cot\frac{C}{2}\right) = 2R\sin A$, and $r = 4R\sin\frac{A}{2}\times \sin\frac{B}{2}\sin\frac{C}{2}$. Thus, $\frac{S}{S_1} = \frac{2r}{R}$

241. Let O be the centre of similarity of the inscribed and circumscribed triangles, M_1 and M_2 two similar vertices (M_1 lies on the side AB), and let the line segment OA intersect the inscribed triangle at a point K. Then $S_{OM_1K} = \lambda S_1$, $S_{OM_2A} = \lambda S_2$, $\frac{S_{OM_1A}}{S_{OM_2A}} = \frac{|OM_1|}{|OM_2|} = \sqrt{\frac{S_1}{S_2}}$, whence $S_{OM_1A} = \lambda\sqrt{S_1S_2}$, where $\lambda = \frac{S_{OM_1K}}{S_1}$. Considering six such triangles and adding together their areas, we get: $S_{ABC} = \sqrt{S_1S_2}$.

242. Let O denote the centre of the circumscribed circle, H the intersection point of the altitudes of the triangle ABC. Since the straight line OH is perpendicular to the bisector of the angle A, it intersects the sides AB and AC at points K and M such that $|AK| = |AM|$. Thus, $\angle AOB = 2\angle C$ (we assume the angle C to be acute); $\angle OAK = 90° - \angle C = \angle HAM$. Hence, $\triangle OAK = \triangle HAM$, and $|OA| = |HA| = R$ (R the radius of the circumscribed circle). If D is the foot of the perpendicular dropped from O on BC, then $|OD| = |AH|/2 = R/2$. Consequently, $\cos A = \cos\angle DOC = 1/2$, $\angle A = 60°$.

243. Prove that the triangle will be acute, right, or obtuse according as the distance between the centre of the circumscribed circle and the point of intersection of the altitudes is less than, equal to, or greater than half the greatest side.

Answer: 90°, 60°, and 30°.

244. The condition $S_{\triangle BDM} = S_{\triangle BCK}$ means that $|BD|\cdot|BM| = |BK|\cdot|BC|$, that is,

$$(|BA| + |AC|)\,|BM| = |BK|\cdot|BC|. \qquad (1)$$

Through M, draw a straight line parallel to AC; let L be the point of intersection of this line and BA. Prove that $|LM| = |KL|$; hence it follows that the desired $\angle BKM = \frac{1}{2}\angle BAC = \frac{\alpha}{2}$. Since the triangle BLM is similar to the triangle BAC, we have $|LM| = \frac{|BM|}{|BC|}\cdot|AC|$, $|BL| = \frac{|BM|}{|BC|}\times |AB|$. Now, we find $|BK|$ from (1) and compute: $|KL| = |BK| - |BL| = \frac{|BA|+|AC|}{|BC|}\cdot|BM| - \frac{|BM|}{|BC|}\cdot|AB| = \frac{|BM|}{|BC|}\cdot|AC|$, whence $|LM| = |KL|$.

245. Let $|AD| = a$, $|BC| = b$. Drop from O a perpendicular OK on AB. We now find: $|BK| = \sqrt{ab}\,\frac{b}{b+a}$, $|BE| = \sqrt{ab}\,\frac{b}{a-b}$ $|MK| = \frac{\sqrt{ab}}{2} - \sqrt{ab}\,\frac{b}{b+a} = \sqrt{ab}\,\frac{a-b}{2(a+b)}$, $|EK| = |BE| + |BK| = \sqrt{ab}\,\frac{2ab}{(a-b)(a+b)}$, $|OK| = \frac{ab}{a+b}$. It is easy to check that $|OK|^2 = |EK|\cdot|MK|$.

Answer: 90°.

246. Note that the points A, M, N, and O lie on the same circle (see Fig. 4). Consequently, $\angle NMO = \angle OAN = 90° - \angle AON$. Hence, with OA rotated about O through an angle φ, the straight line NM rotates through the same angle φ (in the opposite direction), and when A displaces along

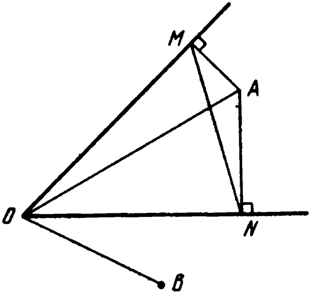


Fig. 4

OA, the line NM displaces parallel to itself. Hence it follows that the desired angle is equal to α.

247. If O_1 is the centre of the smaller circle and $\angle BOA = \varphi$, then $\angle BAO = 90° - \frac{\varphi}{2}$, $\angle CO_1A = 90° + \varphi$, $\angle CAO_1 = 45° - \frac{\varphi}{2}$. Thus, $\angle BAC = \angle BAO - \angle CAO_1 = 45°$.

248. Construct a regular triangle ABK on AB inside the square. Then $\angle KAB = 60°$, $\angle KCD = 15°$, that is, K coincides with M.

Answer: 30°.

249. Let M_1 be symmetric to M with respect to BC and CB is the bisector of the angle MCM_1. Since $\angle M_1CA = 60°$ and $|AC| = \frac{1}{2}|CM_1|$ we have that $\angle M_1AC = 90°$, hence AB is the bisector

of the angle M_1AC. In addition, CB is the bisector of the angle M_1CM, that is, B is equidistant from the straight lines M_1C and M_1A and lies on the bisector of the angle adjacent to the angle AM_1C. Thus, $\angle BMC = \angle BM_1C = 75°$.

Answer: 75°.

250. If $\angle BAC = 2\alpha$, then we readily find that $\angle KMC = \angle MKC = 30° + \alpha$, that is, $|MC| = |KC|$. Let us extend MK to intersect the circle at a point N; $\triangle KMC$ is similar to $\triangle KAN$, hence, $|AN| = |KN| = R$, i.e., to the radius of the circle (since $\angle AMN = 30°$). The points A, K, and O lie on a circle centred at N, $\angle ANO = 60°$, consequently, $\angle AKO = 30°$ or 150° depending on whether the angle AMC is obtuse or acute.

Answer: 30° or 150°.

251. (a) Draw the bisector of the angle A and extend BM to intersect the bisector at a point N (Fig. 5). Since $|BN| = |NC|$, $\angle BNC = 120°$,

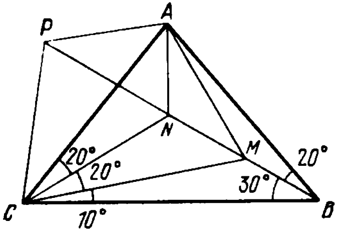


Fig. 5

hence each of the angles BNA and CNA is also equal to 120°, $\angle NCA = \angle NCM = 20°$, that is, $\triangle NMC = \triangle NCA$, $|MC| = |AC|$. Consequently, the triangle AMC is isosceles, and $\angle AMC = 70°$.

(b) The points M, P, A, and C lie on the same circle (the point M from Item (a)); $\angle PAC = \angle PMC = 40°$.

252. Describe a circle about the triangle MCB (Fig. 6) and extend BN to intersect this circle at a point M_1; $|CM_1| = |CM|$ since the sum of the angles subtended by them (80° and 100°) is equal to 180°; $\angle M_1CM = \angle M_1BM = 20°$, that

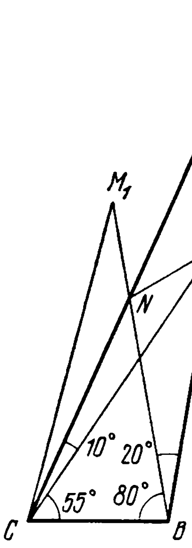


Fig. 6

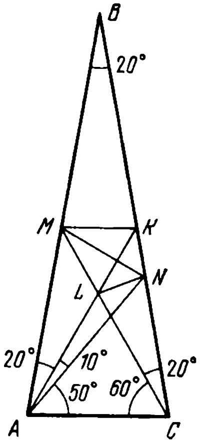


Fig. 7

is, NC is the bisector of the angle M_1CM and $\triangle M_1CN = \triangle NCM$, $\angle NMC = \angle NM_1C = \angle CMB = 25°$.

253. On BC, let us take a point K (Fig. 7) such that $\angle KAC = 60°$, $MK \parallel AC$. Let L be the intersection point of AK and MC; ALC is a regular triangle, ANC is an isosceles triangle (the reader is invited to determine the angles). Hence, LNC

is also an isosceles triangle, $\angle LCN = 20°$. Let us now find the angles NLM and MKN—each of them is equal to $100°$. Since MKL is a regular triangle, each of the angles KLN and NKL is equal to $40°$, i.e., $|KN| = |LN|$ and $\triangle MKN = \triangle MLN$, $\angle NML = \angle KMN = 30°$.

254. Let us take a point K (Fig. 8) such that $\angle KBC = \angle KCB = 30°$ and denote by L the intersection point of the straight lines MC and BK.

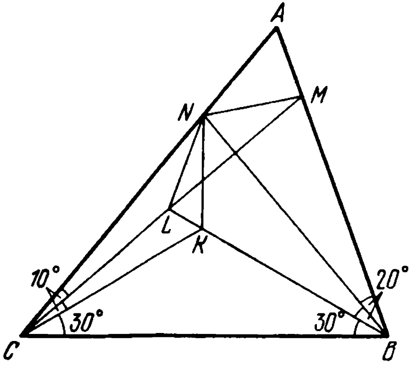


Fig. 8

Since $\triangle BNC$ is isosceles ($\angle NBC = \angle NCB = 50°$), $\angle KNC = 40°$. L is the intersection point of the angle bisectors of the triangle NKC (LK and LC are angle bisectors). Consequently, NL is also the bisector of the angle KNC and $\angle LNB = 60°$; BN, in turn, is the bisector of the angle MBL; in addition, BN is perpendicular to ML; hence, BN bisects ML, and $\angle MNB = \angle BNL = 60°$ and $\angle NMC = 30°$.

255. Let O be the centre of the inscribed circle; the points C, O, K, and M lie on the same circle ($\angle COK = \angle A/2 + \angle C/2 = 90° - \angle B/2 =$

$\angle KMB = 180° - \angle KMC$; if the point K lies on the extension of NM, then $\angle COK = \angle CMK$). Thus, $\angle OKC = \angle OMC = 90°$.

256. If P lies on the arc AB, Q on the arc AC, then, denoting the angle PAB by φ, and the angle QAC by ψ, we get two relationships:

$$\begin{cases} \sin^2 (C - \varphi) = \sin \varphi \sin (B + C - \varphi), \\ \sin^2 (B - \psi) = \sin \psi \sin (B + C - \psi). \end{cases}$$

Writing out the difference of these equalities and transforming it, we get: $\sin (B + C - \varphi - \psi) \times \sin [(B - C) + (\varphi - \psi)] = \sin (B + C - \varphi - \psi) \times \sin (\varphi - \psi)$, whence (since $0 < B + C - \varphi - \psi < \pi$) $B - C + \varphi - \psi = \pi - (\varphi - \psi)$ and we get the answer.

Answer: $\dfrac{\pi - \alpha}{2}$

257. Let us prove that the triangle CMN is similar to the triangle CAB (Fig. 9). We have:

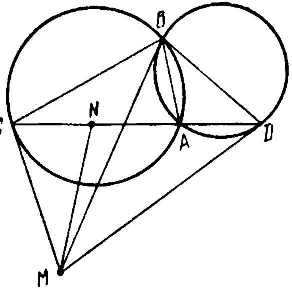


Fig. 9

$\angle MCN = \angle CBA$. Since the quadrilateral $CBDM$ is an inscribed one, we have $\dfrac{|CM|}{|CB|} = \dfrac{\sin \angle CBM}{\sin \angle CMB} =$
$\dfrac{\sin \angle CDM}{\sin \angle CDB} = \dfrac{\sin \angle DBA}{\sin \angle ADB} = \dfrac{|AD|}{|AB|} = \dfrac{|CN|}{|AB|}$.

Hence, $\angle CMN = \angle BCA$, that is, the desired angle is equal to either $\frac{\alpha}{2}$ or $\pi - \frac{\alpha}{2}$

258. Let $\angle ABC = 120°$, and BD, AE, and CM the angle bisectors of the triangle ABC. We are going to show that DE is the bisector of the angle BDC, and DM the bisector of the angle BDA. Indeed, BE is the bisector of the angle adjacent to the angle ABD, that is, for the triangle ABD, E is the intersection point of the bisectors of the angle BAD and the angle adjacent to the angle ABD; hence, the point E is equidistant from the straight lines AB, BD, AD; thus, DE is the bisector of the angle BDC. Exactly in the same way, DM is the bisector of the angle BDA.

259. Denote: $\angle ABD = \alpha$, $\angle BDC = \varphi$. By hypothesis, $\angle DAC = 120° - \alpha$, $\angle BAC = 30° + \alpha$, $\angle ADB = 30° - \alpha$, $\angle DBC = 60° + \alpha$. By the law of sines for the triangles ABC, BCD, ACD, we get $\frac{|BC|}{|AC|} = \frac{\sin(30° + \alpha)}{\sin(60° + 2\alpha)} = \frac{1}{2\cos(30° + \alpha)}$, $\frac{|DC|}{|BC|} = \frac{\sin(60° + \alpha)}{\sin\varphi}$, $\frac{|AC|}{|DC|} = \frac{\sin(30° - \alpha + \varphi)}{\sin(120° - \alpha)}$. Multiplying these equalities, we have: $\sin(30° - \alpha + \varphi) = 2\cos(30° + \alpha)\sin\varphi \Rightarrow 2\cos(60° + \alpha) \times \sin(30° - \varphi) = 0$; thus $\angle BDC = \varphi = 30°$.

260. In Problem 17 of Sec. 1 we derived the formula for the bisector of an interior angle of a triangle ABC. In the same way it is possible to prove that the bisector of the exterior angle A is computed by the formula $l_A = \frac{2bc\sin\frac{A}{2}}{|b - c|}$ ($|AB| = c$, $|BC| = a$, $|CA| = b$). We then find $\sin\frac{A}{2} : \sin\frac{A}{2} = \sqrt{\frac{1}{2}\left(1 - \cos A\right)} =$

$$\sqrt{\frac{1}{2}\left(1-\frac{b^2+c^2-a^2}{2bc}\right)}=\sqrt{\frac{(a+b-c)(a+c-b)}{4bc}}.$$

Finding in the same way $l_C \sin\frac{A}{2}$ and $\sin\frac{C}{2}$ in terms of the sides of the triangle, and equating l_A to l_C, we get $\frac{\sqrt{c(a+b-c)}}{|b-c|}=\frac{\sqrt{a(b+c-a)}}{|b-a|}$. By hypothesis, $b=2$, $c=1$. Hence, a must satisfy the equation $\sqrt{a+1}=\frac{\sqrt{a(3-a)}}{|a-2|}\Rightarrow(a-1)\times(a^2-a-4)=0$. But $a\neq 1$, consequently, $|BC|=a=\frac{1+\sqrt{17}}{2}$

261. If O and O_1 are the centres of the circles circumscribed about the triangles ABC and ADB, respectively, then the triangle AOO_1 is similar to the triangle ACD.

Answer: aR.

262. If K is the midpoint of the arc AB, O the centre of the circle, $|AB|=2R=c$, then $|CM|^2=|CD|^2+|DM|^2=|CD|^2+|DK|^2=|AD|\cdot|DB|+R^2+|DO|^2=(R+|DO|)\times(R-|DO|)+R^2+|DO|^2=2R^2=c^2/2$.

Answer: $c\sqrt{2}/2$.

263. Let KM be a line segment parallel to BC, and N and L the points at which the inscribed circle touches the sides AC and BC. As is known (see Problem 18 ir Sec. 1), $|AN|=|AL|=p-a$, where p is the semiperimeter of the triangle ABC. On the other hand, $|AN|=|AL|$ is the semiperimeter of the triangle AKM which is similar to the triangle ABC. Consequently, $\frac{p-a}{p}=\frac{b}{a}$, $p=\frac{a^2}{a-b}$.

Answer: $\frac{2a^2}{a-b}$.

264. If a, b, c are the sides of the given triangle, then the perimeters of the cut-off triangles are $2(p-a)$, $2(p-b)$, $2(p-c)$, where p is the semiperimeter of the given triangle. Consequently, if R is the radius of the circumscribed circle, then

$$R_1+R_2+R_3=\left(\frac{p-a}{p}+\frac{p-b}{p}+\frac{p-c}{p}\right)R=R.$$

Answer: $R_1+R_2+R_3$.

265. If $\angle A=\alpha$, then $|AM|=\frac{|AC|}{\sin\alpha}$, $|AN|=\frac{|AB|}{\sin\alpha}$, that is, $|AM|:|AN|=|AC|:|AB|$; thus, $\triangle AMN$ is similar to $\triangle ABC$ with the ratio of similitude $\frac{1}{\sin\alpha}$, therefore $|MN|=\frac{|BC|}{\sin\alpha}=2R$.

266. Let O_1 and O_2 be the centres of the intersecting circles. We denote their radii by x and y, respectively, $|OA|=a$. Since, by hypothesis, the triangles AOO_1 and AOO_2 are equivalent, expressing their areas by Hero's formula and bearing in mind that $|O_1A|=x$, $|OO_1|=R-x$, $|O_2A|=y$, $|OO_2|=R-y$, after transformations we get: $(R-2x)^2=(R-2y)^2$, whence (since $x\neq y$) we obtain: $x+y=R$.

Answer: R.

267. Let AB and CD be the given chords and M the point of their intersection.

(a) The sum of the arcs AC and BD is equal to $180°$ (semicircle); consequently, $|AC|^2+|BD|^2=4R^2$, thus, $|AM|^2+|MC|^2+|MB|^2+|MD|^2=|AC|^2+|BD|^2=4R^2$.

Answer: $4R^2$.

(b) $|AB|^2+|CD|^2=(|AM|+|MB|)^2+(|CM|+|MD|)^2=4R^2+2|AM|\ |MB|+2|CM|\cdot|MD|=4R^2+2(R^2-a^2)=6R^2-2a^2$.

Answer: $6R^2-2a^2$.

268. If M is the second point of intersection of BC and the smaller circle, then $|BM|=|PC|$

(M between B and P), $|BP| = |MP| + |BM|$, $|PA|^2 + |PB|^2 + |PC|^2 = |PA|^2 + (|PB| - |PC|)^2 + 2|PB|\cdot|PC| = |PA|^2 + |MP|^2 + 2|PB||PC| = 4r^2 + 2(R^2 - r^2) = 2(R^2 + r^2)$.

269. Let us denote the lengths of the segments of the chords as in Fig. 10 and the diameter by $2r$.

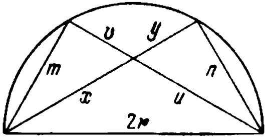


Fig. 10

Taking advantage of the fact that the angles based on the diameter are right ones, and $xy = uv$, we get $x(x + y) + u(u + v) = (u + v)^2 + x^2 - v^2 = (u + v)^2 + m^2 = 4r^2$.

270. If α, β, γ, δ are the arcs corresponding to the sides a, b, c, and d, then the equality to be proved corresponds to the trigonometric equality $\sin\frac{\alpha}{2}\cos\frac{\gamma}{2} + \cos\frac{\alpha}{2}\sin\frac{\gamma}{2} = \sin\frac{\beta}{2}\cos\frac{\delta}{2} + \cos\frac{\beta}{2}\sin\frac{\delta}{2}$, or $\sin\frac{\alpha+\gamma}{2} = \sin\frac{\beta+\delta}{2}$.

271. Let $ABCD$ be an inscribed quadrilateral. AB and CD intersect at a point P, A and D lie on the line segments BP and CP, respectively. BC and AD intersect at a point Q, while C and D lie on the line segments BQ and AQ. Let us circumscribe a circle about the triangle ADP and denote by M the intersection point of this circle and the straight line PQ. (Prove that M lies on the line segment PQ.) We have: $\angle DMQ = \angle DAP = \angle BCD$. Consequently, $CDMQ$ is an inscribed quadrilateral. Since, by hypothesis, the tangents drawn from P and Q to the original circle are equal to a and b, respectively, we have $|QM|\cdot|QP| =$

$|QD|\cdot|QA| = b^2$, $|PM|\cdot|PQ| = |PD| \times |PC| = a^2$. Adding together these equalities, we get $|PQ|^2 = a^2 + b^2$.

Answer: $\sqrt{a^2 + b^2}$.

272. The line segment QP is equal to $\sqrt{(b^2 - R^2) + (c^2 - R^2)} = \sqrt{b^2 + c^2 - 2R^2}$ (see the preceding problem). Let $ABCD$ be the given quadrilateral, Q the intersection point of AB and CD (A lies on the line segment BQ). To find the length of PQ, we circumscribe a circle about the triangle QCA and denote the point of intersection of QP with this circle by N. Since $\angle ANP = \angle ACQ = \angle ABP$, the points A, B, N, and P also lie on a circle. We have $|QP|\cdot|QN| = |QA|\cdot|QB| = b^2 - R^2$, $|PN|\cdot|PQ| = |CP| \times |PA| = R^2 - a^2$. Subtracting the second equality from the first one, we get $|QP|^2 = b^2 + a^2 - 2R^2$. Analogously, $|PM|^2 = c^2 + a^2 - 2R^2$.

Answer: $|QM|$ $\sqrt{b^2 + c^2 - 2R^2}$, $|QP| = \sqrt{b^2+a^2-2R^2}$, $|PM| = \sqrt{c^2+a^2-2R^2}$.

273. The radius of the inscribed circle is contained between the values of the radii for the two limiting cases. It cannot be less than the radius of the circle inscribed in the triangle with sides $a + b$, $b + c$, $c + a$ which is equal to S/p, where S is the area and p the semiperimeter of the triangle; thus $r > \frac{S}{p} = \frac{\sqrt{(a+b+c)\,abc}}{a+b+c} = \sqrt{\frac{abc}{a+b+c}}$. On the other hand, r is less than the radius of the circle shown in Fig. 11 (on this figure, the opposite tangents are parallel, and the point C tends to infinity). Since for the angles α, β, and γ marked in the figure the following equality is fulfilled: $\alpha + \beta + \gamma = \pi/2$, $\tan\alpha = c/\rho$, $\tan\beta = a/\rho$, $\tan\gamma = b/\rho$, where ρ is the radius of the shown circle, $\tan(\alpha + \beta) = \cot\gamma$,

or $\dfrac{(c+p)\rho}{\rho^2 - ac} = \dfrac{\rho}{b}$, whence $\rho = \sqrt{ab+bc+ca}$.

Thus, $\sqrt{\dfrac{abc}{a+b+c}} < r < \sqrt{ab+bc+ca}$.

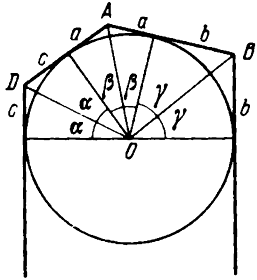


Fig. 11

274. Let M be the point of intersection of the straight line CB and the lines of centres of the given circles. Let us denote: $|AM| = x$, $\angle ACB = \varphi$; $|AB|^2 = 2rx$, $|AC|^2 = 2Rx$, $\sin\varphi = \dfrac{x}{|AC|}$. If ρ is the radius of the circle circumscribed about $\triangle ABC$, then $\rho = \dfrac{|AB|}{2\sin\varphi} = \dfrac{|AB|\cdot|AC|}{2x} = \sqrt{Rr}$.

Answer: $\sqrt{Rr}$.

275. Let O_1, O_2 be the centres of the circles and A the point of their intersection most remote from BC, $\angle O_1AO_2 = \varphi$. Let us show that $\angle BAC = \varphi/2$. $\left(\text{For the other point the angle is } 180° - \dfrac{\varphi}{2}.\right)$ Indeed, $\angle BAC = 180° - \angle ABC - \angle BCA = 180° - (90° - \angle ABO_1) - (90° - \angle ACO_2) = \angle ABO_1 + \angle ACO_2 = \angle BAO_1 + \angle CAO_2 = \varphi - \angle BAC)$. Let $|O_1O_2| = a$. Drawing $O_2M \parallel BC$

(M on O_1B), we get $|BC| = |O_2M| = \sqrt{a^2 - (R-r)^2}$. From the triangle O_1AO_2 we find that $\cos\varphi = \frac{R^2 + r^2 - a^2}{2Rr}$; thus, the radius of the circle circumscribed about the triangle ABC is equal to $\frac{|BC|}{2\sin\frac{\varphi}{2}} = \frac{\sqrt{a^2-(R-r)^2}}{\sqrt{2}\sqrt{1-\frac{R^2+r^2-a^2}{2Rr}}} = \sqrt{Rr}$.

Answer: $\sqrt{Rr}$ (for both triangles).

276. DO and CO are the bisectors of the angles ADC and DCB. Let α, β, and γ denote the corre-

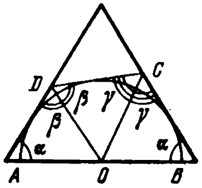


Fig. 12

sponding angles (Fig. 12). But $\alpha + 2\beta + 2\gamma + \alpha = 2\pi$; hence, $\alpha + \beta + \gamma = \pi$; hence it follows that $\angle DOA = \gamma$, $\angle COB = \beta$, and the triangle AOD is similar to the triangle COB; consequently, $|AD|\cdot|CB| = |AO|\cdot|OB| = |AB|^2/4$.

Answer: $a^2/4b$.

277. It follows from the conditions of the problem that the bisectors of the angles C and D intersect on the side AB. Let us denote this point of intersection by O. Circumscribe a circle about the triangle DOC. Let K be a second point of intersection of this circle with AB. We have: $\angle DKA = \angle DCO = \frac{1}{2}\angle DCB = \frac{1}{2}\times$

$(180^\circ \quad \angle DAK) \quad \frac{1}{2}(\angle DKA + \angle ADK)$. Hence, $\angle DKA = \angle ADK$ and $|AD| = |AK|$. Similarly, $|BC| = |BK|$; consequently, $|AD| + |CB| = |AB|$.

Answer: $a - b$.

278. On the ray MC, we take a point N such that $|AN| = |AB| = |AD|$. Since
$$\frac{\sin \angle MNA}{\sin \angle MCA} = \frac{|AC|}{|AN|} = \frac{|AC|}{|AD|} = \frac{\sin \angle ADC}{\sin \angle ACD}$$
and $\angle MCA = \angle ACD$, we have: $\sin \angle MNA = \sin \angle ADC = \sin \angle ABM$, that is, the angles ABM and MNA are either congruent or their sum totals to 180°. But M is inside the triangle ABN, hence, $\angle ABM = \angle MNA$. Now, we can prove that $\triangle ABM = \triangle AMN$; $\angle NAC = \angle MNA - \angle NCA = \angle ADC - \angle ACD = \varphi$.

Answer: $\frac{\alpha + \varphi}{2}$.

279. Let K and L denote the points of tangency of the first and second circles with one of the sides of the angle, and M and N the other points of intersection of the straight line AB with the first and second circles, respectively. Let O denote the centre of the second circle. Since A is the centre of similarity of the given circles, $\frac{|AK|}{|AL|} = \frac{|AM|}{|AB|} = \frac{|AB|}{|AN|} = \lambda$, whence $|AK| \cdot |AL| = \lambda |AL|^2 = \lambda |AB| \cdot |AN| = |AB|^2$. On the other hand, from the similarity of the triangles AKC and ALO we have: $|AK| \cdot |AL| = |AC| \cdot |AO|$. Consequently, $|AC| \cdot |AO| = |AB|^2$; hence, the triangles ABC and AOB are similar.

Answer: $\frac{\alpha}{2}$ or $\pi - \frac{\alpha}{2}$.

280. Let $\angle BAF = \varphi$, $\angle DBA = \alpha$, $\angle DAB = 2\alpha$ (by hypothesis, it follows that the points A, E, and F lie on the same side of BD, and $\angle BDA <$

$90°$, that is, $\alpha > 30°$). By the law of sines, for the triangles DEA, DAB, and BAF we have: $\frac{|DE|}{|AD|} = \frac{\sin(120° - 2\alpha)}{\sin(30° + \alpha)} = 2\cos(30° + \alpha)$; $\frac{|AD|}{|AB|} = \frac{\sin\alpha}{\sin 3\alpha} =$

$$\frac{1}{4\cos(30° + \alpha)\cos(30° - \alpha)}, \quad \frac{|AB|}{|BF|} = \frac{\cos(\alpha - \varphi)}{\sin\varphi}.$$

Multiplying the equalities, we find: $\frac{\cos(\alpha - \varphi)}{\sin\varphi} = 2\cos(\alpha - 30°)$, whence $\angle BAF = \varphi = 30°$.

281. Consider two cases.

(1) The line segment BK intersects AC. From the condition that $\angle BKC = \frac{3\angle A - \angle C}{2}$ it follows that $\angle C = 90°$ ($\angle BCK = \angle B + \angle C$, $\angle CBK = \frac{\angle B}{2}$, $\frac{3\angle A - \angle C}{2} + (\angle B + \angle C) + \frac{\angle B}{2} = 180°$, etc.). Consequently, the point O is found on AB, and the sum of the distances from O to AC and AB is equal to $\frac{1}{2}|BC|$; thus, $|BC| = 4 > 2 + \sqrt{3} = |AC| + |AB| > |AB|$, that is, a leg is greater than the hypotenuse which is impossible. Thus, we have arrived at a contradiction.

(2) The line segment BK does not intersect AC. In this case, $\angle CBK = 180° - \frac{\angle B}{2}$, $\angle BCK = \angle A$, $\angle BKC = \frac{3\angle A - \angle C}{2}$ (by hypothesis); hence,

$$\left(180° - \frac{\angle B}{2}\right) + \angle A + \frac{3\angle A - \angle C}{2} = 180°,$$

whence $\angle A = 30°$.

Again, two cases are possible.

(2a) The centre of the circumscribed circle O is inside the triangle ABC. Let the perpendicular dropped from O on AB intersect AB at N, and AC at K, and let the perpendicular drawn to AC intersect AC at M and AB at L. Let us denote: $|OM| = x$, $|ON| = y$; $x + y = 2$ (by hypothesis), $|OK| = 2x/\sqrt{3}$, $|MK| = x/\sqrt{3}$, $|AK| = 2|NK| = 2y + 4x/\sqrt{3}$, $|AM| = |AK|$ $|MK| = 2y + x/\sqrt{3}$. Similarly, we find: $|AN| = 2x + y\sqrt{3}$. By hypothesis, $|AN| + |AM| = \frac{1}{2}(|AB| + |AC|) = \frac{1}{2}(2 + \sqrt{3})$. On the other hand, $|AN| + |AM| \quad (2 + \sqrt{3}) \times (x + y) = 2(2 + \sqrt{3})$, which is a contradiction.

(2b) The point O is outside the triangle ABC. We can show that $\angle B$ is obtuse. Otherwise, if $\angle C > 90°$, then $\frac{3\angle A - \angle C}{2} < 0$, thus, O is found inside the line segment AC not containing B; however, this does not affect the answer. Using the notation of the preceding item, we have: $|AM| = 2y - x\sqrt{3}$, $|AN| = y\sqrt{3} - 2x$ From the system $y + x = 2$, $|AM| + |AN| = (2 + \sqrt{3})\,y - (2 + \sqrt{3})\,x = \frac{2 + \sqrt{3}}{2}$ we find: $x = \frac{3}{4}$, $y = \frac{5}{4}$, $|AM| = \frac{5}{2} - \frac{3\sqrt{3}}{4}$, the radius of the circle is $\sqrt{|AM|^2 + |MO|^2} = 1/2\sqrt{34 - 15\sqrt{3}}$.

282. If C_1 is a point symmetric to C with respect to AB, and B_1 is symmetric to B with respect to AC, then (as usually, a, b, c are the sides of $\triangle ABC$, S its area) $|C_1B_1|^2 = b^2 + c^2 - 2bc\cos 3A = a^2 + 2bc(\cos A - \cos 3A) = a^2 + 8bc\sin^2 A \times \cos A = a^2 + 16(b^2 + c^2 - a^2)\frac{S^2}{b^2c^2}$. Thus, we get

the system of equations:

$$\begin{cases} a^2b^2c^2 + 16S^2(b^2 + c^2 - a^2) = 8b^2c^2, \\ a^2b^2c^2 + 16S^2(a^2 + b^2 - c^2) = 8a^2b^2, \\ a^2b^2c^2 + 16S^2(c^2 + a^2 - b^2) = 14c^2a^2. \end{cases}$$

Subtracting the second equation from the first one and bearing in mind that $a \neq c$, we find: $4S^2 = b^2$. Replacing S^2 by $b^2/4$, we get:

$$\begin{cases} a^2c^2 + 4(b^2 - c^2 - a^2) = 0, \\ a^2b^2c^2 + 4b^2c^2 + 4b^2a^2 - 4b^4 - 14a^2c^2 = 0, \\ b^2 = 4S^2. \end{cases}$$

Denoting $a^2c^2 = x$, $a^2 + c^2 = y$, we have:

$$\begin{cases} 4y - x = 4b^2, \\ x(b^2 - 14) + 4b^2y = 4b^4. \end{cases}$$

Multiplying the first equation of the latter system by b^2 and subtracting the result from the second equation, we find: $x(2b^2 - 14) = 0$, whence $b = \sqrt{7}$.

Answer: $1, \quad \sqrt{7}, \sqrt{8}$ or $\sqrt{\dfrac{21 - \sqrt{217}}{2}}$, $\sqrt{7}, \sqrt{\dfrac{21 + \sqrt{217}}{2}}$

283. Prove that $\tan \alpha = \dfrac{|b^2 - c^2|}{2S}$, where S is the area of the triangle (prove this for the other angles in a similar way).

Answer: $\arctan |\tan \alpha \pm \tan \beta|$.

284. Let us find the cotangent of the angle between the median and the side of the triangle ABC. If $\angle A_1AB = \varphi$ (AA_1 a median of the triangle ABC, a, b, c the sides of the triangle, m_a, m_b, m_c its medians, S the area), then $\cot \varphi = \dfrac{2c - a\cos\beta}{h_c} = \dfrac{2c^2 - ac\cos B}{2S} = \dfrac{3c^2 + b^2 - a^2}{4S}$.

Let M be the median point of the triangle ABC; the straight lines perpendicular to the medians emanating from the vertices A and B intersect at C_1; $\angle MC_1B = \angle MAB = \varphi$($MAC_1B$ is an inscribed quadrilateral). Consequently, $S_{MBC_1} = \frac{1}{2}\left(\frac{2}{3}m_b\right)^2 \times \cot\varphi = \frac{(2a^2+2c^2-b^2)(3c^2+b^2-a^2)}{72S}$. The area of the required triangle is the sum of the areas of the six triangles, each area being found in a similar way. Finally, we get $\frac{(a^2+b^2+c^2)^2}{12S} = \frac{27\,(R^2-d^2)^2}{4S}$ (the equality $a^2+b^2+c^2 = 9\,(R^2-d^2)$ is left to the reader).

Answer: $\frac{27}{4}\,(R^2-d^2)^2$.

285. 60°.

286. First note that $|MN|$ is equal to the common external tangent to the circles with centres at O_1 and O_2 (Problem 142, Sec. 1). Consequently, if the radii of these circles are x and y and $x+y = 2R-a$, then $|MN| = \sqrt{a^2-(x-y)^2}$. Let φ denote the angle formed by AB with O_1O_2, L the point of intersection of AB and O_1O_2. We have: $|O_1L| = \frac{xa}{x+y} = \frac{xa}{2R-a}$, $\sin\varphi = \frac{x}{|O_1L|}$ $\frac{2R-a}{a}$, $|OL|$ $|x + |O_1L| - R|$ $\frac{R}{2R-a}\,|2x+a-2R| = \frac{R}{2R-a}\,|x-y|$, $|AB|$ $2\sqrt{R^2-|OL|^2\sin^2\varphi} = \frac{2R}{a}\sqrt{a^2-(x-y)^2} = \frac{2R}{a}\,|MN|$.

Answer: $\frac{2R}{a}$ (in both cases).

287. The angle AKB is equal to 90° (see Problem 255, Sec. 1). Let R be the point of intersection of BK and AC, Q a point on BK such that $NQ \parallel AC$. Using the usual notation, we have: $|AR| = |AB| = c$, $|MR| = c-(p-a) = p-b = |NB|$,

$$\frac{|MK|}{|KN|} = \frac{|MR|}{|QN|} = \frac{|CB|}{|RC|} = \frac{a}{b-c} \quad (b > c).$$

Since $|MN|$ $2(p-c)\sin\frac{\alpha}{2}$, $|MK| = a\sin\frac{\alpha}{2}$. Other line segments are considered in a similar way. The desired triangle is similar to the triangle ABC, the ratio of similitude being equal to $\sin(\alpha/2)$. Its area equals $S\cdot\sin^2(\alpha/2)$.

288. Let $|AM| = x$, $|CN| = y$, $x + y = a$, where a is the side of the square. We denote by E and F the points of intersection of MD and DN with AC. The line segments $|AE|$, $|EF|$, $|CF|$ are readily computed in terms of a, x, y, whereupon it is possible to check the equality $|EF|^2 = |AE|^2 + |FC|^2 - |AE|\cdot|FC|$.

289. Let P be the point of intersection of the straight line DE with AB, K a point on AB such that KD is parallel to AC, AKD is an isosceles triangle ($\angle KDA = \angle DAC = \angle DAK$). Hence, KD is a median in the right triangle, and $|MN| = \frac{1}{2}|KD| = \frac{1}{4}|AP| = \frac{1}{4}|AE| = \frac{1}{4}a$.

290. Let A_1 be another point of intersection of the circles circumscribed about tha $\triangle ABC$ and $\triangle AB_1C_1$. It follows from the hypothesis that $|BB_1| = |CC_1|$, in addition, $\angle ABA_1 = \angle ACA_1$ and $\angle AB_1A_1 = \angle AC_1A_1$. Consequently, $\triangle A_1BB_1 = \triangle A_1CC_1$. Hence, $|A_1B| = |A_1C|$. Let $\angle ABC = \beta$, $\angle ACB = \gamma$, $\angle ABA_1 = \angle ACA_1 = \varphi$. Since $\triangle A_1BC$ is isosceles, we have $\angle A_1BC = \angle A_1CB$, i.e., $\beta + \varphi = \gamma - \varphi$, $\varphi = \frac{1}{2}(\gamma - \beta)$ and if the radius of the circle circumscribed about the $\triangle ABC$ is R, then $|AA_1| = 2R\sin\frac{\gamma - \beta}{2}$; but $|AB|$ $|AC| = 2R(\sin\gamma - \sin\beta) = 4R\sin\frac{\gamma-\beta}{2}\cos\frac{\beta+\gamma}{2}$

$2\,|AA_1|\sin\frac{\alpha}{2}$; consequently, $|AA_1| = \frac{a}{2\sin\frac{\alpha}{2}}$.

291. Note that the points A, O, M, B lie on the same circle ($\angle AMB$ is measured by one-half of the sum of the arc AB and the arc symmetric to AB with respect to OC, that is, $\angle AMB = \angle AOB$). We lay off on AM a line segment MK equal to MB; then the triangle AKB is similar to the triangle OMB.

Answer: $|AB| = 2a$.

292. Let $|AB| = 2r$, $|BC| = 2R$, O_1 the midpoint of AB, O_2 the midpoint of BC, O_3 the midpoint of AC, O the centre of the fourth circle whose radius is x. From the conditions of the problem it follows that $|O_1O_3| = R$, $|O_2O_3| = r$, $|O_1O| = r + x$, $|O_2O| = R + x$, $|O_3O| = R + r - x$. Equating the expressions for the areas of the triangles O_1OO_3 and O_1OO_2 obtained by Hero's formula and as one-half of the product of the corresponding base and altitude, we get two equations:

$$\begin{cases} \sqrt{(R+r)\,r\,(R-x)\,x} = \frac{1}{2}Rd. \\ \sqrt{(R+r+x)\,Rrx} = \frac{1}{2}(R+r)\,d, \end{cases}$$

Squaring each of them and subtracting one from the other, we find: $x = d/2$.

Answer: $d/2$.

293. Let P be the foot of the perpendicular dropped from N on the straight line MB, then $|MP| = R\cos\alpha$; consequently, $|MP|$ is equal to the distance from the centre O to AB. But the distance from the vertex of a triangle to the point of intersection of its altitudes is twice the distance from the centre of the circumscribed circle to the opposite side (Problem 20, Sec. 1), i.e., $|MP| =$

$\frac{1}{2}$ $|MK|$. Hence, it follows that if M is located on the major arc, that is, $\angle AMB = \alpha$, then $|NK| = R$; and if $\angle AMB = 180° - \alpha$ (that is, M is found on the minor arc of the circle), then $|NK|^2 = R^2(1 + 8\cos^2\alpha)$.

Answer: $|NK| = R$ if M is on the major arc and $|NK| = R\sqrt{1 + 8\cos^2\alpha}$ if M lies on the minor arc of the circle.

294. Let ABC be the given triangle, CD its altitude, O_1 and O_2 the centres of the circles inscribed in the triangles ACD and BDC, respectively, K and L the points of intersection of the straight lines DO_1 and DO_2 with AC and CB, respectively. Since the triangle ADC is similar to the triangle CDB, and KD and LD are the bisectors of the right angles of these triangles, O_1 and O_2 divide, respectively, KD and LD in the same ratio. Hence, KL is parallel to O_1O_2. But $CKDL$ is an inscribed quadrilateral ($\angle KCL = \angle KDL = 90°$). Consequently, $\angle CKL = \angle CDL = \pi/4$, $\angle CLK = \angle CDK = \pi/4$. Thus, the straight line O_1O_2 forms an angle of $\pi/4$ with each of the legs. If M and N are the points of intersection of O_1O_2 with CB and AC, then the triangle CMO_2 is congruent to the triangle CDO_2 (CO_2 is a common side, $\angle O_2CD = \angle O_2CM$, $\angle CDO_2 = \angle CMO_2$). Hence, $|CM| = |NC| = h$.

Answer: the angles of the triangle are $\pi/4$, $\pi/4$, $\pi/2$, and its area is $h^2/2$.

295. For designation see Fig. 13. $CKDL$ is a rectangle. Since $\angle LKA = 90° + \alpha$, $\angle LBA = 90° - \alpha$, $BLKA$ is an inscribed quadrilateral,

$$\tan\varphi = \frac{|LC|}{|CA|} = \frac{h\cos\alpha}{\dfrac{h}{\sin\alpha}} = \frac{1}{2}\sin 2\alpha. \tag{1}$$

If R is the radius of the circle, then

$$R = \frac{|KL|}{2\sin\varphi} = \frac{h}{2\sin\varphi}. \tag{2}$$

Since $\angle LOK = 2\varphi$, we have: $|ON| = R\cos\varphi = \frac{h}{2\tan\varphi} = \frac{h}{\sin 2\alpha}$ (we have used the equalities (1) and (2)), $|OM| = |ON|\sin(90° - 2\alpha) =$

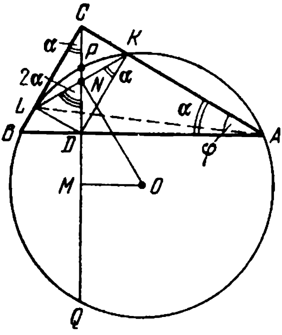


Fig. 13

$h\frac{\cos 2\alpha}{\sin 2\alpha} = h\cot 2\alpha$, and, finally, we get the expression $\frac{1}{2}|PQ| = |QM| = \sqrt{R^2 - |OM|^2} =$

$$\sqrt{\frac{h^2}{4\sin^2\varphi} - h^2\cot^2 2\alpha} =$$

$$h\sqrt{\frac{1}{4}(1+\cot^2\varphi) - \cot^2 2\alpha} =$$

$$h\sqrt{\frac{1}{4}\left(1+\frac{4}{\sin^2 2\alpha}\right) - \cot^2 2\alpha} = \frac{h\sqrt{5}}{2},$$

$|PQ| = h\sqrt{5}$. If now the segments $|PD|$ and $|DQ|$ of the chord are denoted by x and y, then $x+y = h\sqrt{5}$, $xy = h^2$, whence the desired line segments are equal to $\frac{\sqrt{5}+1}{2}h$, $\frac{\sqrt{5}-1}{2}h$.

296. Let (Fig. 14) P and Q be the points of tangency of the tangents drawn from E. Prove that

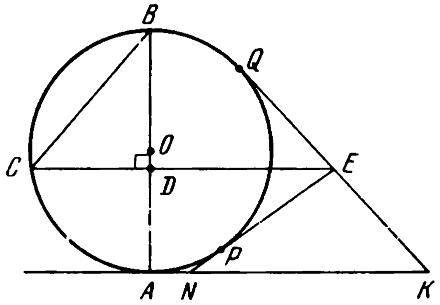


Fig. 14

$|EP| = |EQ| = |BD|$. Indeed, $|EP|^2 = (|ED| + |DC|)(|ED| - |DC|) = |ED|^2 - |DC|^2 = |BC|^2 - |DC|^2 = |BD|^2$ (by hypothesis, $|ED| = |BC|$). Denote $|KN| = x$, $|PN| = |NA| = y$. $|EQ| = |EP| = |BD| = z$. Then $|KE| = x + y - z$. We have: $S_{KEN} = \frac{1}{2}x(2R - z)$; on the other hand, $S_{KEN} = S_{KON} + S_{KOE} - S_{EON} = \frac{1}{2}R(x + x + y - z - y - z) = R(x - z)$. Thus, $\frac{1}{2}x(2R - z) = R(x - z)$, $x = 2R$.

Answer: $2R$.

297. First, find $\lim\limits_{\alpha \to 0} \frac{|AO|}{|OC|}$. Denote: $\angle C = \beta$.

We have:

$$\frac{|AO|}{|OC|}=\frac{S_{ABD}}{S_{BDC}}=\frac{\frac{1}{2}ab\sin\alpha}{\frac{1}{2}(p-a)(p-b)\sin\beta} \qquad (1)$$

But by the law of cosines, $a^2+b^2-2ab\cos\alpha=(p-a)^2+(p-b)^2-2(p-a)(p-b)\cos\beta \Rightarrow \cos\beta=\frac{p(p-a-b)+ab\cos\alpha}{(p-a)(p-b)}$, whence

$$\sin\beta=\sqrt{1-\cos^2\beta}=\sqrt{(1-\cos\beta)(1+\cos\beta)}$$
$$=\frac{\sqrt{ab(1-\cos\alpha)(2p^2-2ap-2bp+ab+ab\cos\alpha)}}{(p-a)(p-b)}. \qquad (2)$$

If $\alpha\to 0$, then $\cos\alpha\to 1$; consequently, $\frac{\sin\alpha}{\sqrt{1-\cos\alpha}}=\sqrt{2}\cos\frac{\alpha}{2}\to\sqrt{2}$ as $\alpha\to 0$. Taking this into account, we obtain from (1) and (2): $\lim\limits_{\alpha\to 0}\frac{|AO|}{|OC|}=\sqrt{\frac{ab}{(p-a)(p-b)}}$. Since $|AC|\to p$,

$$\lim_{\alpha\to 0}|AO|=\frac{p\sqrt{ab}}{\sqrt{ab}+\sqrt{(p-a)(p-b)}}$$

Section 2

1. Prove that if D is the projection of M on AB, then $|AD|^2-|DB|^2=|AM|^2-|MB|^2$.

2. Suppose that there is such a point (let us denote it by N), then the straight line MN is perpendicular to all the three sides of the triangle.

3. If M is the point of intersection of the perpendiculars from A_1 and B_1 on BC and AC, then (see Problem 1 in Sec. 2) $|MB|^2-|MC|^2=|A_1B|^2-|A_1C|^2$, $|MC|^2-|MA|^2=|B_1C|^2-|B_1A|^2$; adding together these equalities and taking into consideration the conditions of the problem, we get: $|MB|^2 - |MA|^2 = |C_1B|^2 - |C_1A|^2$, that is, M lies on the perpendicular drawn to AB through C_1.

4. It follows from the result of the preceding problem that the condition of intersecting at one point for the perpendiculars dropped from A_1, B_1, and C_1 on the sides BC, CA, and AB is the same as that for the perpendiculars from A, B, and C on B_1C_1, C_1A_1, and A_1B_1, respectively.

5. We note that the perpendiculars dropped from A_1, B_1, C_1 on BC, CA, AB, respectively, intersect at one point D and then use the result obtained in the preceding problem.

6. The next problem proves a more general fact. From the reasoning of that problem it will follow that the centre of the circle lies on the straight line AB.

7. We introduce the rectangular coordinate system. If the coordinates of the points $A_1, A_2, \ldots, A_n$ are, respectively, $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ and those of the point M are (x, y), then the locus is given by the equation $a(x^2 + y^2) + bx + cy + d = 0$, where $a = k_1 + k_2 + \ldots + k_n$; hence, there follows our statement.

8. If B is the point of tangency and O the centre of the given circle, then $|OM|^2 - |AM|^2 = |OM|^2 - |BM|^2 = |OB|^2 = R^2$. Hence, M lies on the straight line perpendicular to OA (see Problem 1 of Sec. 2).

9. The condition defining the set of points M is equivalent to the condition $|AM|^2 - k^2|BM|^2 = 0$, that is, this is a circle (see Problem 7 in Sec. 2). This circle is called *Apollonius' circle*; obviously, its centre lies on the straight line AB.

10. Since MB is the bisector of the angle AMC, $\frac{|AM|}{|MC|} = \frac{|AB|}{|BC|}$. Consequently, the bisector of the exterior angle with respect to the angle AMC intersects the line AC at a constant point K:

$\frac{|AK|}{|KC|} = \frac{|AB|}{|BC|}$, and the sought-for set of points M is the arc of the circle constructed on BK as diameter and enclosed between the straight lines perpendicular to the line segment AC and passing through the points A and C.

11. Let O_1 and O_2 be the centres of the given circles, r_1 and r_2 their radii, M a point of the desired set, MA_1 and MA_2 tangents. By hypothesis, $|MA_1| = k\,|MA_2|$. Consequently, $|MO_1|^2 - k^2\,|MO_2|^2 = r_1^2 - k^2r_2^2$. Hence (see Problem 6 of Sec. 2), for $k \neq 1$, the sought-for set of points M is a circle with centre on the straight line O_1O_2, while for $k = 1$, the desired set is a straight line perpendicular to O_1O_2.

12. Let (Fig. 15) K and L be the points of intersection of the tangent to the second circle

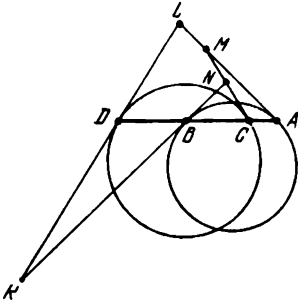


Fig. 15

passing through D and the tangents to the first circle passing through B and A, and M and N two other points. It is obvious that $\angle DKB = \angle CMA$ (either of these angles is equal to one-half of the difference between the angles corresponding to the arcs AB and CD). Therefore (in the figure) $\angle LMN + \angle LKN = 180°$. Consequently, $KLMN$

is an inscribed quadrilateral. Further, we have $\frac{|DK|}{|KB|}=\frac{\sin\angle DBK}{\sin\angle BDK}=\frac{\sin\frac{1}{2}\smile AB}{\sin\frac{1}{2}\smile DC}$. The ratios of the lengths of the tangents drawn through the points L, M, and N are found in a similar way. All these ratios are equal; hence, the centre of the circle circumscribed about $KLMN$ lies on the straight line passing through the centres of the given circles (see Problem 6 in Sec. 2).

13. Expressing the distances from the vertices of the triangle to the points of tangency, check the fulfillment of the conditions of Problem 3 in Sec. 2.

14. Let $|AM_1|:|BM_1|:|CM_1| = p:q:r$. Then the set of points M such that $(r^2-q^2)|AM|^2+(p^2-r^2)|BM|^2+(q^2-p^2)|CM|^2=0$ is a straight line passing through M_1, M_2, and the centre of the circle circumscribed about the triangle ABC (see Problem 7 in Sec. 2).

15. Points M_1 and M_2 belong to the set of points M for which $5|MA|^2-8|MB|^2+3|MC|^2=0$. This set is a straight line, and, obviously, the centre of the circumscribed circle satisfies the condition that defines this set (see Problem 7 of Sec. 2).

16. Let $|AA_1|=a$, $|BB_1|=b$, $|CC_1|=c$, $|A_1B_1|=x$, $|B_1C_1|=y$, $|C_1A_1|=z$. Then $|AB_1|^2=a^2+x^2$, $|B_1C|^2=c^2+y^2$ and so forth. Now, it is easy to check the conditions of Problem 3, Sec. 2.

17. Let $|AD|=x$, $|BD|=y$, $|CD|=z$, $|AB|=a$. Let A_2, B_2, C_2 denote the points of tangency of the circles inscribed in the triangles BCD, CAD, ABD, respectively, with sides BC, CA, AB. The perpendiculars drawn through the points A_1, B_1, C_1 to the sides BC, CA, and AB coincide with those drawn to the same sides at the points A_2, B_2, C_2. But $|BA_2|=\frac{a+y-z}{2}$,

$|A_2C| = \frac{a+z-y}{2}$; $|AC_2|$, $|C_2B|$, $|AB_2|$, $|B_2C|$ are found in a similar way. Now, it is easy to check the conditions of Problem 3, Sec. 2.

18. Apply the conditions of Problem 3 in Sec. 2, taking the centres of the circles as the points A, B, and C, and each one of the two intersection points of the circles as the points A_1, B_1, C_1 (A_1 is one of the points of intersection of the circles with centres B and C, and so on).

19. Take the third circle with diameter BC. The altitudes of the triangle drawn from the vertices B and C are common chords of the first and third, and also the second and third circles. Consequently (see Problem 18 in Sec. 2), the common chord of the given circles also passes through the intersection point of the altitudes of the triangle ABC.

20. Let O denote the centre of the given circle, R its radius, MC a tangent to the circle. We have $|MO|^2 - |MN|^2 = |MO|^2 - |MB|\cdot|MA| = |MO|^2 - |MC|^2 = R^2$, that is, the point M lies on the straight line perpendicular to the straight line ON (see Problem 1 in Sec. 2). It can be easily shown that all the points of this line belong to the set.

21. Let O denote the centre of the circle, r the radius of the circle, $|OA| = a$, BC a chord passing through A, and M the point of intersection of the tangents. Then

$$|OM|^2 = |BM|^2 + r^2,$$

$$|AM|^2 = |BM|^2 - \frac{1}{4}|BC|^2 + \left(\frac{1}{2}|BC| - |BA|\right)^2 =$$

$$|BM|^2 - |BC|\cdot|BA| + |BA|^2 =$$

$$|BM|^2 - |BA|\cdot|AC| = |BM|^2 - r^2 + a^2.$$

Thus, $|OM|^2 - |AM|^2 = 2r^2 - a^2$, that is (see Problem 1 of Sec. 2) the required set of points is a straight line perpendicular to OA. This line is called the *polar of the point A with respect to the given circle.*

22. Show that if M_1 and M_2 are two distinct points belonging to the set, then any point M of the segment of the straight line M_1M_2 enclosed inside the triangle also belongs to this set. To this end, let us denote by x_1, y_1, and z_1 the distances from M_1 to the sides of the triangle, and by x_2, y_2, z_2 the distances from M_2. Then we can express the distances x, y, z from M to the sides of the triangle in terms of those quantities and the distances between M_1, M_2, M. For instance, if $|M_1M| = k|M_1M_2|$ and the directions of M_1M and M_1M_2 coincide, then $x = (1-k)x_1 + kx_2$, $y = (1-k)y_1 + ky_2$, $z = (1-k)z_1 + kz_2$. Hence, it follows that if the equality is true for three noncollinear points inside the triangle, then it is true for all the points of the triangle.

Remark. The statement of the problem remains true for an arbitrary convex polygon. Moreover, we may consider all the points in the plane, but the distances to the straight line from the points situated on opposite sides of the line must be taken with opposite signs.

23. For the distances x, y, z to be the sides of a triangle, it is necessary and sufficient that the inequalities $x < y + z$, $y < z + x$, $z < x + y$ be fulfilled. But the set of points for which, for instance, $x = y + z$ is a line segment with the end points lying at the feet of the angle bisectors (at the foot of the angle bisector two distances are equal, the third being equal to zero; consequently, the equality is true; and from the preceding problem it follows that this equality is true for all points of the line segment).

Answer: the sought-for locus consists of points situated inside the triangle with vertices at the feet of the angle bisectors.

24. Since the perpendiculars from A_2, B_2, and C_2 on B_1C_1, C_1A_1, and A_1B_1, respectively, are concurrent, the perpendiculars from A_1, B_1, and C_1 on B_2C_2, C_2A_2, and A_2B_2, respectively, are also concurrent (see Problem 4 of Sec. 2).

25. Let a_1 and a_2 denote the distances from A to the straight lines l_2 and l_3, respectively, b_1 and b_2 the distances from B to the straight lines l_3 and l_1, respectively, c_1 and c_2 the distances from C to the straight lines l_1 and l_2, respectively, x, y, and z the distances from A_1, B_1, and C_1 to l, respectively. For the perpendiculars drawn respectively from A, B, and C on B_1C_1, C_1A_1, and A_1B_1, it is necessary and sufficient that the following equality be true (see Problem 3 of Sec. 2): $|AB_1|^2 - |B_1C|^2 + |CA_1|^2 - |A_1B|^2 + |BC_1|^2 - |C_1A|^2 = 0$ or $(a_1^2 + y^2) - (c_2^2 + y^2) + (c_1^2 + x^2) - (b_2^2 + x^2) + (b_1^2 + z^2) - (a_2^2 + z^2) = 0$ which leads to the condition $a_1^2 - a_2^2 + b_1^2 - b_2^2 + c_1^2 - c_2^2 = 0$, independent of x, y, z.

26. It suffices to check the fulfillment of the condition (see Problem 3 of Sec. 2) $|AB_2|^2 - |B_2C|^2 + |CA_2|^2 - |A_2B|^2 + |BC_2|^2 - |C_2A|^2 = 0$. Note that the triangles BB_2C_1 and AA_2C_1 are similar, hence, $|AC_1|\cdot|C_1B_2| = |BC_1|\cdot|C_1A_2|$; in addition, $\angle AC_1B_2 = \angle BC_1A_2$, consequently, $|AB_2|^2 - |BA_2|^2 = (|AC_1|^2 - |C_1B|^2) + (|C_1B_2|^2 - |A_2C_1|^2)$. By writing the corresponding equalities for $|CA_2|^2 - |AC_2|^2$ and $|BC_2|^2 - |CB_2|^2$ and adding them together, we see that the sum of the difference in the first parentheses yields zero (apply the conditions of Problem 3 of Sec. 2 to the triangles ABC and $A_1B_1C_1$; we get zero since the altitudes intersect at one point). It is easy to prove that AA_2, BB_2, and CC_2 pass through the centre of the circle circumscribed about ABC, that is, the sum of the differences in the second parentheses is also zero.

32. Through K and L, draw straight lines parallel to BC to intersect the median AD at points N and S. Let $|AD| = 3a$, $|MN| = xa$, $|MS| = ya$. Since $\frac{|LS|}{|NK|} = \frac{|AS|}{|AN|}$, $\frac{|LS|}{|NK|} = \frac{|MS|}{|MN|}$, we have $\frac{|AS|}{|AN|} = \frac{|MS|}{|MN|}$, $\frac{(2+y)\,a}{(2-x)\,a} = \frac{y}{x}$, $y =$

$\frac{x}{1-x}$. The equality $\frac{1}{|MK|}=\frac{1}{|ML|}+\frac{1}{|MP|}$ is equivalent to $\frac{1}{|MN|}=\frac{1}{|MS|}+\frac{1}{|MD|}$ $\frac{1}{ax}=\frac{1}{ay}+\frac{1}{a}$. Substituting $y=\frac{x}{1-x}$, we get a true equality.

34. Let O be the point of intersection of the diagonals AC and BD; taking advantage of the similarity of the appropriate triangles, we get

$$\frac{|OK|}{|OC|}=\frac{|OK|}{|OB|}\cdot\frac{|OB|}{|OC|}=\frac{|OA|}{|OD|}\cdot\frac{|OM|}{|OA|}=\frac{|OM|}{|OD|},$$

which was to be proved.

35. Let F and D denote the points of intersection of EN and EM with AB and BC, respectively. Prove that the triangles AFN and MDC are similar. Using the similarity of various triangles and equality of the opposite sides of the parallelogram, we have: $\frac{|NF|}{|FA|}=\frac{|NF|}{|FB|}\cdot\frac{|FB|}{|FA|}=\frac{|BD|}{|DM|}\times$ $\frac{|ED|}{|FA|}=\frac{|BD|}{|DM|}\cdot\frac{|DC|}{|FE|}=\frac{|BD|}{|DM|}\cdot\frac{|DC|}{|BD|}=$ $\frac{|DC|}{|DM|}$, that is, the triangle AFN is similar to the triangle MDC.

36. The statement of the problem becomes obvious from the following two facts:

(1) If, on the sides of the quadrilateral $ABCD$, points K, L, M, and N are taken so that the sides AB, BC, CD, and DA are divided by them in the same ratio $\left(\frac{|BK|}{|KA|}=\frac{|CM|}{|MD|}=\frac{|BL|}{|LC|}=\frac{|AN|}{|ND|}\right)$, then the line segments KM and LN are also divided in the same ratio by the point P of their intersection.

Indeed, from the fact that the straight lines KL and NM are parallel to the diagonal AC it follows that $\frac{|KP|}{|PM|}=\frac{|KL|}{|NM|}=\frac{|KL|}{|AC|}\cdot\frac{|AC|}{|NM|}=\frac{|BK|}{|BA|}\times$ $\frac{|AD|}{|ND|}=\frac{|BK|}{|BA|}\cdot\frac{|BA|}{|KA|}=\frac{|BK|}{|KA|}$.

2) If, on the sides AB and CD of the quadrilateral, points K_1 and K, M_1 and M are taken so that $\frac{|K_1K|}{|AB|}=\frac{|M_1M|}{|CD|}=\frac{1}{m}$, $|AK_1|=|KB|$, $|DM_1|=CM$, then the area of the quadrilateral K_1KMM_1 is $\frac{1}{m}$ of the area of the quadrilateral $ABCD$. Indeed, $S_{BKC}=\frac{|BK|}{|BA|}S_{ABC}$, $S_{AM_1D}=$ $\frac{|M_1D|}{|CD|}S_{ACD}=\frac{|BK|}{|BA|}S_{ACD}$. Consequently, $S_{AKCM_1}=\left(1-\frac{|BK|}{|BA|}\right)S_{ABCD}=\frac{|AK|}{|BA|}S_{ABCD}$. Similarly, $S_{K_1KMM_1}=\frac{|K_1K|}{|AK|}S_{AKCM_1}$. Thus, $S_{K_1KMM_1}=\frac{|K_1K|}{|AB|}S_{ABCD}=\frac{1}{m}S$.

37. Let K be the midpoint of DB, L that of AC, $S_{ANM}=S_{CNM}$ (since $|AL|=|LC|$). In similar fashion, $S_{BNM}=S_{DNM}$, whence there follows the statement of the problem.

38. If M is the midpoint of DC, N that of BC, K and L are the points of intersection of DN with AM and AB, respectively, then $\frac{|KM|}{|AK|}=$ $\frac{|DM|}{|AL|}=\frac{1}{4}$, that is, $|AK|=\frac{4}{5}|AM|$; consequently, $S_{ADK}=\frac{4}{5}S_{ADM}=\frac{4}{5}\cdot\frac{1}{4}S=\frac{1}{5}S$ (S the

area of the parallelogram). Thus, the area of the sought-for figure is $S - 4S_{ADK} = \frac{1}{5}S$.

39. Let Q, N, and M be the midpoints of AD, BC, and DC; K, P, and R the points of intersection of DN and AM, QC and DN, and QC and AM, respectively. Then $|DK| = \frac{2}{5}|DN|$,

$|DP| = |PN|$, $|QP| = |PC|$, $|QR| = \frac{1}{3}|QC|$,

$$\frac{S_{RPQ}}{S_{QPD}} \quad \frac{|RP|}{|QP|} \cdot \frac{|KP|}{|DP|} = \frac{1}{3} \times \frac{1}{5} = \frac{1}{15},$$

$S_{RPK} = \frac{1}{15} \times \frac{S}{8} = \frac{S}{120}$.

Consequently, from the quadrilateral considered in the preceding problem, four triangles, each having an area of $\frac{S}{120}$ are thus cut off, the area of the desired octagon being $\frac{S}{5} - \frac{4S}{120} = \frac{S}{6}$.

40. Let the straight line HC intersect AB and LM at points T and N, respectively, the straight line AL intersect ED at a point K, and the straight line BM intersect PG at a point P. We have: $S_{ACDE} = S_{ACHK} = S_{ATNL}$, $S_{BCFG} = S_{BCHP} = S_{BMNT}$; thus, $S_{ACDE} + S_{BCFG} = S_{ABML}$.

41. Let Q denote the area of the pentagon, s_1, s_2, and s_3 the areas of the triangles adjoining one of the lateral sides, the smaller base, and the other lateral side, respectively; x the area of the triangle enclosed between the triangles of areas s_1 and s_2, and y the area of the triangle enclosed between the triangles having areas s_2 and s_3. Then

$$s_1 + x + s_2 = s_2 + y + s_3 = \frac{1}{2}(x + y + s_2 + Q)$$

and, thus, $s_1 + x + s_2 + s_2 + y + s_3 = x + y + s_2 + Q \Rightarrow s_1 + s_2 + s_3 = Q$.

42. If S is the area of the parallelogram, then $S_{ABK} + S_{KCD} = \frac{1}{2} S$. On the other hand, $S_{DBC} = S_{EKC} + S_{KCD} = \frac{1}{2} S$, hence, $S_{ABK} = S_{EKC}$. Analogously, $S_{AKD} = S_{KCP}$; adding together the last two equalities, we get: $S_{ABKD} = S_{CEKF}$.

43. We have: $\dfrac{|AC_1|}{|C_1B|} = \dfrac{S_{ACC_1}}{S_{CC_1B}} =$

$$\frac{\frac{1}{2}|AC|\cdot|CC_1|\sin\angle ACC_1}{\frac{1}{2}|CC_1|\cdot|CB|\sin\angle O_1CB} = \frac{|AC|}{|BC|}\cdot\frac{\sin\angle ACC_1}{\sin\angle C_1CB}.$$

Having obtained similar equalities for the ratios $\dfrac{|BA_1|}{|A_1C|}$ and $\dfrac{|CB_1|}{|B_1A|}$ and multiplying them, we get the required statement.

44. Let us show that if the straight lines intersect at the same point (let M denote this point), then $R^* = 1$ (and consequently, $R = 1$; see Problem 43, Sec. 2). By the law of sines for the triangle AMC we have: $\dfrac{\sin\angle ACC_1}{\sin\angle A_1AC} = \dfrac{|AM|}{|MC|}$. Writing out similar equalities for the triangles AMB and BMC and multiplying them, we get the required assertion. Conversely: if $R = 1$, and all the points A_1, B_1, C_1 lie on the sides of the triangle (or only one of them), then, drawing the straight lines AA_1 and BB_1, we denote the point of their intersection by M_1; let the straight line CM_1 intersect AB at a point C_2. Taking into consideration the conditions of the problem and that the necessary condition $R = 1$ is proved, we have: $\dfrac{|AC_1|}{|C_1B|} = \dfrac{|AC_2|}{|C_2B|}$, both of the points C_1 and C_2 lying either on the line segment AB or outside it. Consequently, C_1 and C_2 coincide.

45. Let A_1, B_1, C_1 be collinear. Through C, we draw a straight line parallel to AB and denote the point of its intersection with the straight line A_1B_1 by M. From the similarity of appropriate triangles, we get: $\frac{|BA_1|}{|A_1C|} = \frac{|BC_1|}{|CM|}$, $\frac{|CB_1|}{|B_1A|} = \frac{|CM|}{|AC_1|}$ Replacing the corresponding ratios in the expression for R (see Problem 43 of Sec. 2) with the aid of those equalities, we get: $R = 1$. The converse is proved much in the same way as it was done in the preceding problem (we draw the straight line B_1A_1, denote the point of its intersection with AB by C_2, and so forth).

46. Check the following: if for the given straight lines $R^* = 1$, then for the symmetric lines the same is true. If the straight line passing, say, through the vertex A intersects the side BC, then the line symmetric to it with respect to the bisector of the angle will also intersect the side BC (see Problems 43 and 44 in Sec. 2).

47. If A_0, B_0, C_0 are the midpoints of the line segments AO, BO, CO, respectively, then the constructed straight lines turn out to be symmetric to the lines A_0O, B_0O, C_0O with respect to the angle bisectors of the triangle $A_0B_0C_0$ (see the preceding problem).

48. (a) Let the straight line BM intersect AC at a point B', and the line CK intersect AB at a point C'. Through M, we draw a straight line parallel to AC and denote by P and Q the points of its intersection with AB and BC, respectively. Obviously, $\frac{|AB'|}{|B'C|} = \frac{|PM|}{|MQ|}$. Drawing through K a straight line parallel to AB and denoting by E and F the points of its intersection with CA and CB, respectively, we have: $\frac{|BC'|}{|C'A|} = \frac{|FK|}{|KE|}$. We carry out a similar construction for the point L. Replacing the ratios entering the expression for R

(see Problem 43 of Sec. 2) with the aid of that equality we take into account that for each line segment in the numerator there is an equal segment in the denominator, for instance: $|PM| = |KE|$.

(b) Let, for the sake of definiteness, the line l intersect the line segments C_0A, CA_0 and form an acute angle φ with OK. The straight line A_1L divides the line segment MK in the ratio $\frac{S_{LMA_1}}{S_{LKA_1}}$ (starting from the point M). The ratios in which the sides KL and LM of the triangle KLM are divided can be found in a similar way. We have to prove that there holds the equality $R = 1$ (see Problem 43, Sec. 2). Let us replace the ratios of the line segments by the ratio of the areas of the corresponding triangles. Then R will contain S_{LMA_1} in the numerator and S_{KMC_1} in the denominator. Prove that $\frac{S_{LMA_1}}{S_{KMC_1}} = \frac{\sin C}{\sin A}$, where A and C are angles of the triangle ABC. Obviously $\frac{S_{B_0OA_0}}{S_{B_0OC_0}} = \frac{\sin C}{\sin A}$. In addition, $\angle A_1B_0A_0 = \angle C_0B_0A_0 + \angle A_1B_0C_0 = 90° - \frac{\angle B}{2} + \varphi$ (this follows from the fact that the circle of diameter AO passes through B_0, C_0 and A_1) and $\angle B_0A_1O = \angle B_0AO = \frac{\angle A}{2}$ In similar fashion $\angle B_0C_1O = \frac{\angle C}{2}$ and $\angle C_1B_0C_0 = \left(90° - \frac{\angle B}{2}\right) + \angle C_1OL = \left(90° - \frac{\angle B}{2}\right) + (180° - \angle C - \angle B_0OC_1) = 90° - \frac{\angle B}{2} + (\angle B_0OA_1 - \angle C) = 90° - \angle B/2 + (180° - \angle A - \angle C - \varphi) = 90° + \angle B/2 - \varphi$, i.e.,

$\sin\angle A_1B_0A_0 = \sin\angle C_1B_0C_0$. Thus, $\frac{S_{A_1B_0A_0}}{S_{C_1B_0C_0}} =$

$$\frac{|B_0A_1|\cdot|B_0A_0|}{|B_0C_1|\cdot|B_0C_0|} = \frac{\sin\frac{C}{2}\cdot\cos\frac{C}{2}}{\sin\frac{A}{2}\cdot\cos\frac{A}{2}} = \frac{\sin C}{\sin A}.$$

Let r denote the radius of the inscribed circle $|OL| = |OK| = |OM| = a$. We have:

$$\frac{S_{LMA_1}}{S_{KMC_1}} = \frac{S_{LOM}+S_{LOMA_1}}{S_{KOM}+S_{KOMC_1}}$$

$$= \frac{\frac{a^2}{r^2}S_{A_0OB_0}+\frac{a}{r}S_{A_0OB_0A_1}}{\frac{a^2}{r^2}S_{C_0OB_0}+\frac{a}{r}S_{C_0OB_0C_1}}$$

$$= \frac{\frac{a}{r}S_{A_0OB_0}+(S_{A_0B_0A_1}-S_{A_0OB_0})}{\frac{a}{r}S_{C_0OB_0}+(S_{C_0B_0C_1}-S_{C_0OB_0})}$$

$$= \frac{\left(\frac{a}{r}-1\right)S_{A_0OB_0}+S_{A_0B_0A_1}}{\left(\frac{a}{r}-1\right)S_{C_0OB_0}+S_{C_0B_0C_1}} = \frac{\sin C}{\sin A}.$$

(The latter of the equalities follows from the fact that $\frac{S_{A_0OB_0}}{S_{C_0OB_0}} = \frac{S_{A_0B_0A_1}}{S_{C_0B_0C_1}} = \frac{\sin C}{\sin A}$.)

In similar fashion, we single out in the numerator and denominator of the expression for R, two more pairs of magnitudes whose ratios are equal to $\frac{\sin A}{\sin B}$ and $\frac{\sin B}{\sin C}$, respectively. Hence, $R=1$. It remains only to prove that the number of points of intersection of the straight lines LA_1, KC_1, and MB_1 with the line segments KM, ML, and LK, respectively, is odd.

49. Consider the triangle ACE through whose vertices the straight lines AD, CF, and EB are drawn. The sines of the angles formed by these lines with the sides of the triangle ACE are proportional to the chords they are based on; consequently, the condition $R = 1$ (see Problem 44 of Sec. 2) is equivalent to the condition given in the problem.

50. Find out whether the equality $R = 1$ is fulfilled (in Item (b) use the result obtained in Problem 234 of Sec. 1) and all the three points lie on the extensions of the sides of the triangle. Thus, our statement follows from Menelaus' theorem (see Problem 45 of Sec. 2).

51. By the property of the secants drawn from an exterior point to a circle, or by the property of the segments of the chords of a circle passing through the same point, we have: $|BC_1|\cdot|BC_2| = |BA_1|\cdot|BA_2|$, $|CB_1|\cdot|CB_2| = |CA_1|\cdot|CA_2|$, $|AB_1|\cdot|AB_2| = |AC_1|\cdot|AC_2|$. Now, it is easy to check that if the assertion in Ceva's theorem (the equality $R = 1$) is true for the points A_1, B_1, C_1, then it is also true for the points A_2, B_2, C_2. It follows from the statement of the problem that either all the three points A_2, B_2, C_2 lie on the corresponding sides of the triangle or only one of them (see Problem 44 of Sec. 2).

52. Writing out the equality $R = 1$ (according to Ceva's and Menelaus' theorems—see problems 44 and 45 in Sec. 2) for the points A_1, B_1, C_1; A_1, B_1, C_2; A_1, B_2, C_1; and A_2, B_1, C_1, we get that for the points A_2, B_2, C_2 the equality $R = 1$ is also true. Now, it remains only to prove that either all the three points A_2, B_2, and C_2 lie on the extensions of the sides of the triangle (that is the case when the points A_1, B_1, C_1 are found on the sides of the triangle) or only one lies on the extension (if only one of the points A_1, B_1 and C_1 is on

the sides of the triangle) and use Menelaus' theorem (see Problem 45 of Sec. 2).

53. Make use of Menelaus' theorem (see Problem 45 of Sec. 2). As the vertices of the given triangle, take the midpoints of the sides of the triangle ABC on whose sides and their extensions the points under consideration lie.

54. If a is the length of the side of the pentagon $MKLNP$, b the length of the side of the pentagon with one side on AB, c the length of the side of the pentagon whose one side is on AC, then $\frac{|BA_1|}{|C_1B|} = \frac{a}{b}$, $\frac{|AC_1|}{|B_1A|} = \frac{b}{c}$, $\frac{|CB_1|}{|A_1C|} = \frac{c}{a}$. Multiplying these equalities, we find $R = 1$ and then use Ceva's theorem (Problem 44 of Sec. 2).

55. Check to see that the points A_1, A_2, A_3 and B_1, B_2, B_3 are found either on the sides of the triangle $O_1O_2O_3$ (O_1, O_2, O_3 centres of the circles) or on their extensions, and the ratio of the distances from each of these points to the corresponding vertices of the triangle $O_1O_2O_3$ is equal to the ratio of the radii of the corresponding circles. Further, make use of Menelaus' theorem (see Problem 45 of Sec. 2) for each of these three points.

56. The statement of the problem follows from Problems 43 and 44 of Sec. 2.

58. Make use of the equality $\frac{\sin \angle B_1AA_2}{\sin \angle A_2AC_1} = \frac{|AC_1|}{|AB_1|} \frac{|B_1A_2|}{|A_2C_1|}$ Obtaining similar equalities for the other angles and multiplying them, we get our statement on the strength of the results of Problems 43 and 44 of Sec. 2.

59. We apply Menelaus' theorem to the triangles ABD, BDC, and DCA (Problem 45* in Sec. 2, Remark): $\frac{AL}{LB} \cdot \frac{BQ}{QD} \cdot \frac{DP}{PA} = -1$, $\frac{BM}{MC} \cdot \frac{CR}{RD} \times \frac{DQ}{QB} = -1$, $\frac{AP}{PD} \cdot \frac{DR}{RC} \cdot \frac{CN}{NA} = -1$ (L, M, and N

the points of intersection of AB and PQ, BC and QR, AC and PR, respectively). Multiplying these equalities, we get: $\frac{CN}{NA} \cdot \frac{AL}{LB} \cdot \frac{BM}{MC} = -1$, that is, the points L, M and N are collinear.

60. Consider the coordinate system whose axes are the given lines (this is the affine system of coordinates). The equation of a straight line in this system, in the usual fashion, has the form $ax + by + c = 0$. We shall first prove the necessary condition. Let the point N have coordinates (u, v) and the point M the coordinates $(\lambda u, \lambda v)$. The equations of the straight lines A_1B_1, A_2B_2, A_3B_3, A_4B_4 have the form: $y - v = k_1(x - u)$, $y - v = k_2(x - u)$, $y - \lambda v = k_3(x - \lambda u)$, $y - \lambda v = k_4(x - \lambda u)$, respectively. Then the points A_1, A_2, A_3, A_4 situated on the x-axis have, respectively, the coordinates on this axis: $u - \frac{1}{k_1} v$, $u - \frac{1}{k_2} v$, $\lambda u - \frac{\lambda}{k_3} v$, $\lambda u - \frac{\lambda}{k_4} v$, while the points B_1, B_2, B_3, B_4 situated on the y-axis have the coordinates $v - k_1u$, $v - k_2u$, $\lambda v - k_3\lambda u$, $\lambda v - k_4\lambda u$, respectively. Now, it is easy to check the equality given in the hypothesis. Sufficiency, in usual fashion, can be proved by contradiction.

61. In Items (a) and (c), make use of Ceva's and Menelaus' theorems (Problems 44 and 45 of Sec. 2, Remark). In Item (b), in addition, use the result of the preceding problem; here, it is convenient, as in Problem 60, to consider the affine coordinate system whose axes are the straight lines AB and AC, and the points B and C have the coordinates (0, 1) and (1, 0).

62. Let S denote the point of intersection of the straight lines A_1M, B_1L, and C_1K. Applying Menelaus' theorem (Problem 45 in Sec. 2, Remark) to the triangles SMK, SKL, and SLM, we get $\frac{KL_1}{L_1M} \cdot \frac{MA_1}{A_1S} \cdot \frac{SC_1}{C_1K} = -1$, $\frac{LM_1}{M_1K} \cdot \frac{KC_1}{C_1S} \cdot \frac{SB_1}{B_1L} =$

-1, $\frac{MK_1}{K_1L}\cdot\frac{LB_1}{B_1S}\cdot\frac{SA_1}{A_1M}=-1$. Multiplying these equalities we get:

$$\frac{KL_1}{L_1M}\cdot\frac{LM_1}{M_1K}\cdot\frac{MK_1}{K_1L}=-1. \tag{1}$$

Equality (1) is a necessary and sufficient condition for the lines A_1M, B_1L, and C_1K to intersect at a point. The necessity has been already proved. The sufficiency is proved, as usually, by contradiction. (Let us denote by S' the point of intersection of A_1M and B_1L, draw $S'C_1$, denote by K' the point of its intersection with the given straight line, and prove that K and K' coincide.) Since the equality (1) goes over into itself with K, L, M replaced by K_1, L_1, M_1, respectively, and vice versa, the assertion of the problem has been proved.

63. Applying Ceva's theorem (Problem 44* in Sec. 2, Remark) to the triangles ABD, BDC and CDA, we get: $\frac{AP}{PB}\cdot\frac{BF}{FD}\cdot\frac{DE}{EA}=1$, $\frac{BQ}{QC}\cdot\frac{CG}{GD}\cdot\frac{DF}{FB}=1$, $\frac{CR}{RA}\times\frac{AE}{ED}\cdot\frac{DG}{GC}=1$. Multiplying these equalities, we get: $\frac{AP}{PB}\cdot\frac{BQ}{QC}\cdot\frac{CR}{RA}=1$, that is, the straight lines AQ, BR and CP intersect at a point. Let us denote it by N. Let T be the point of intersection of PG and DN. By Menelaus' theorem we have $\frac{DT}{TN}\cdot\frac{NP}{PC}\cdot\frac{CG}{GD}=-1$, whence $\frac{DT}{TN}=-\frac{PC}{NP}\times\frac{GD}{CG}=-\frac{CP}{PN}\cdot\frac{GD}{CG}$. If $\frac{AE}{ED}=\alpha$, $\frac{BF}{FD}=\beta$, $\frac{CG}{GD}=\gamma$, then $\frac{AP}{PB}=\frac{\alpha}{\beta}$, $\frac{CR}{RA}=\frac{\gamma}{\alpha}$, $\frac{CN}{NP}=-\frac{BA}{PB}\cdot\frac{RC}{AR}=\frac{\alpha+\beta}{\beta}\cdot\frac{\gamma}{\alpha}$, $\frac{CP}{PN}=-\left(1+\frac{CN}{NP}\right)=$

$=\dfrac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta}$. Thus, $\dfrac{DT}{TN}=\dfrac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}$. The line segment DN is divided in the same ratio by the other straight lines.

64. Let us first consider the limiting case when the point N is found at infinity; then the straight lines AN, BN, and CN are parallel to the straight line l. Let the distances from the points A, B, and C to the line l be equal to a, b, and c. (For convenience, let us assume that A, B, and C are on the same side of l.) The straight lines parallel to l and passing through A, B, and C intersect the straight lines B_1C_1, C_1A_1, and A_1B_1 at points A_2, B_2, C_2, respectively. It is easy to see that $\dfrac{|A_1C_2|}{|C_2B_1|}=\dfrac{a+c}{c+b}$, $\dfrac{|B_1A_2|}{|A_2C_1|}=\dfrac{b+a}{a+c}$, $\dfrac{|C_1B_2|}{|B_2A_1|}=\dfrac{c+b}{b+a}$. Multiplying these equalities, we make sure that the statement of Menelaus' theorem (Problem 45 of Sec. 2) is fulfilled (it is necessary also to make sure that an odd number of points from among A_2, B_2, C_2 are found on the extensions of the sides of the triangle $A_1B_1C_1$). Hence, the points A_2, B_2, C_2 are collinear.

The general case can be reduced to the considered one if, for instance, the given arrangement of the triangles is projected from a point in space on another plane. In choosing this point, we should get that the symmetry of the triangles is not violated, and the point N tends to infinity. It is also possible not to resort to spatial examinations. Let us introduce a coordinate system with the straight line l as the x-axis and the origin at N. We carry out the transformation $x'=1/x$, $y'=y/x$. As a result of this transformation, the points of the x-axis ($y=0$) go into the straight line $y'=0$; the points symmetric about the x-axis go into the points symmetric with respect to the line $y'=0$; also straight lines go into straight lines; straight lines passing through the origin go into straight lines parallel to the line $y' = 0$ (this transformation is, in essence, the projection mentioned above). When this transformation is carried out, we get the arrangement we considered.

65. We assume the given lines to be parallel. This can be achieved by projecting or transforming the coordinates (see the solution of Problem 64

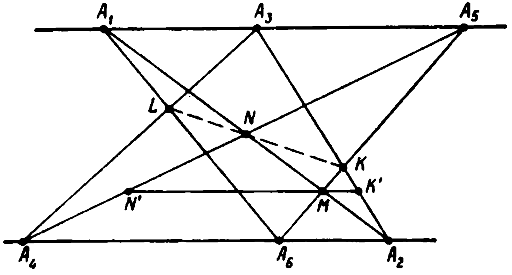


Fig. 16

of Sec. 2). Apply Menelaus' theorem (Problem 45 of Sec. 2) to the triangle A_1A_6M (Fig. 16, $N'K'$ is parallel to the given straight lines). We have

$$\frac{|A_1L|}{|LA_6|}\cdot\frac{|A_6K|}{|KM|}\cdot\frac{|MN|}{|NA_1|}=\frac{|A_1A_3|}{|A_4A_6|}\cdot\frac{|A_6A_2|}{|K'M|}\times\frac{|MN'|}{|A_5A_1|}=\frac{|A_1A_3|}{|K'M|}\cdot\frac{|MN'|}{|A_4A_6|}\cdot\frac{|A_6A_2|}{|A_5A_1|}=$$
$$\frac{|A_1A_2|}{|A_2M|}\cdot\frac{|MA_5|}{|A_5A_6|}\cdot\frac{|A_6A_2|}{|A_5A_1|}=\frac{|A_1M|}{|A_2M|}\cdot\frac{|MA_5|}{|A_5M|}\times\frac{|A_2M|}{|MA_1|}=1.$$

Thus, the points L, N, and K are collinear. According to the Remark to Problems 44 and 45 of Sec. 2, we could consider the ratios $\frac{A_1L}{LA_6}$ and others instead of $\frac{|A_1L|}{|LA_6|}$ and others. In this case the product of the appropriate ratios is equal to (-1).

67. The desired locus consists of two straight lines passing through the point symmetric to the point A with respect to the straight line l and forming angles of 60° with l.

68. The required set is the arc BC of the circle circumscribed about the triangle ABC corresponding to a central angle of 120°.

69. If N is the point of intersection of the straight lines PQ and AB, then $\frac{|CN|}{|AN|} = \frac{|PC|}{|AQ|} = \frac{|CB|}{|AC|}$, that is, N is a fixed point. The required set is a circle with diameter CN. If now M is a fixed point, then D lies on the straight line parallel to the line MN and passing through a fixed point L on the straight line AB such that $\frac{|AL|}{|LB|} = \frac{|AN|}{|CN|}$, L being arranged relative to the line segment AB in the same manner as N with respect to the line segment AC.

70. Let φ denote the angle between BD and AC; $S_{APK} = \frac{1}{2}|AK|\cdot|PD|\sin\varphi$, $S_{BPC} = \frac{1}{2}|BP|\times|DC|\sin\varphi = \frac{1}{2}|BP|\cdot|AD|\sin\varphi$. Since $S_{APK} = S_{BPC}$, $|AK|\cdot|PD| = |BP|\cdot|AD|$, or $\frac{|AK|}{|AD|}\times\frac{|PD|}{|bP|} = 1$, but by Menelaus' theorem for the triangle BDK (see Problem 45 of Sec. 2). $\frac{|AK|}{|AD|}\cdot\frac{|DP|}{|PB|}\cdot\frac{|BM|}{|MK|} = 1$ (M the point of intersection of AP and BK), consequently $|BM| = |MK|$, that is, the required locus is the midline of the triangle ABC parallel to the side AC (if the points P and K are taken on the straight lines AC and BD, then we get a straight line parallel to the side AC passing through the midpoints of the line segments AB and BC).

71. Let C denote the vertex of the given angle, and β its size. We drop perpendiculars OK and OL from O on the sides of the angle (Fig. 17, *a*). A circle can be circumscribed about the quadrilateral $OKAM$. Consequently, $\angle KMO = \angle KAO$. Analogously, $\angle OML = \angle OBL$. Hence, $\angle KML = \angle KAO + \angle OBL = \alpha + \beta$, that is, M lies on

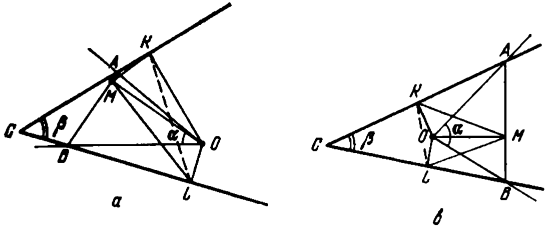


Fig. 17

an arc of the circle passing through K and L and containing the angle $\alpha + \beta$, all the points of this arc belonging to the set. If $\alpha \leqslant \beta$, then there are no other points in the set. And if $\alpha > \beta$, then added to the set are points M located on the other side of the straight line KL for which $\angle KML = \alpha - \beta$ (Fig. 17, *b*). In this case, the set of points is a pair of arcs whose end points are determined by the limiting positions of the angle AOB. If the rays of the fixed angle β and movable angle α are extended, and instead of the angles, the pairs of straight lines are considered, then the desired set is a pair of circles (containing both arcs mentioned above).

72. Consider the quadrilateral $DEPM$ in which $\angle DEM = \angle DPM = 90°$, consequently, this is an inscribed quadrilateral. Hence, $\angle DME = \angle DPE = 45°$. The required locus is the straight line DC.

73. Consider the case when the point B lies inside the given angle. First of all we note that

all possible triangles BCD (Fig. 18) are similar since $\angle BCD = \angle BAD$, $\angle BDC = \angle BAC$. Therefore, if N is the midpoint of CD, then the angles BNC and BND are constant. Let us circumscribe a circle about the triangle BNC and let K be the second point of intersection of this circle and AC.

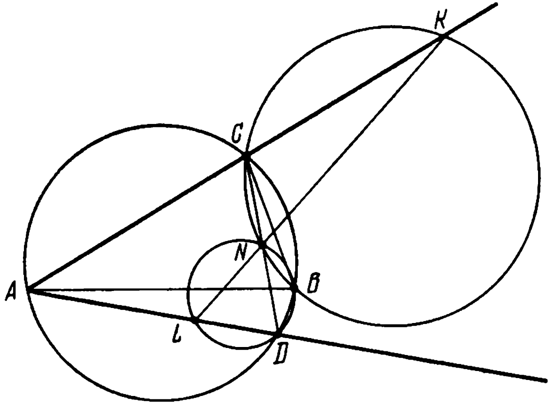


Fig. 18

Since $\angle BKA = 180° - \angle BNC$, K is a fixed point. Analogously, also fixed is L, the second point of intersection of the circle circumscribed about the triangle BND and the straight line AD. We have: $\angle LNK = \angle LNB + \angle BNK = 180° - \angle BDA + \angle BCK = 180°$, that is, N lies on the straight line LK. The set of points N is the line segment LK, and the locus of centres of mass of the triangle ACD is the line segment parallel to LK dividing AK in the ratio 2 : 1 (obtained with the aid of a homothetic transformation with centre at A and the ratio of similitude equal to 2/3).

74. If O is the vertex of the angle, $ABCD$ is a rectangle (A fixed), then the points A, B, C, D and

O lie on the same circle. Consequently, $\angle COA = 90°$, that is, the point C lies on the straight line perpendicular to OA and passing through O.

75. Note that all the triangles ABC obtained are similar. Consequently, if we take in each triangle a point K dividing the side BC in the same ratio, then, since $\angle AKC$ remains unchanged, the point K describes a circle. Hence, the point M dividing AK in a constant ratio also describes a circle which is obtained from the first circle by a homothetic transformation with centre at A and the ratio of similitude $k = |AM|/|AK|$. This reasoning is used in all the items: (a), (b), and (c).

76. Let K denote the midpoint of AB, and M the foot of the perpendicular dropped from K on AC. All the triangles AKM are similar (by two congruent angles), consequently, all the triangles ABM are similar. Now, it is easy to get that the desired locus is a circle with a chord BC, the angles based on this chord being equal either to the angle AMB or to its complementary angle. (The minor arc of this circle lies on the same side of BC as the minor arc of the original circle.)

77. If M, N, L, and K are the given points (M and N lie on opposite sides of the rectangle as

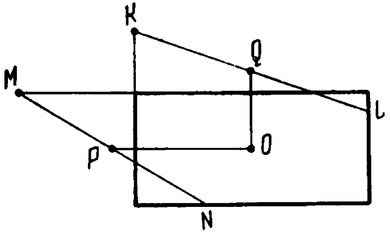


Fig. 19

do L and K), P is the midpoint of MN, Q the midpoint of KL, O the intersection point of the diagonals of the rectangle (Fig. 19), then $\angle POQ =$

$90°$. Consequently, the desired locus is the circle constructed on PQ as diameter.

78. Let R and r denote the radii of the given circles ($R > r$), D the point of tangency of the chord BC and the smaller circle. Let K and L be the points of intersection of the chords AC and AB with the smaller circle, and, finally, let O be the centre of the circle inscribed in the triangle ABC. Since the angular measures of the arcs AK and AC are equal, $|AK| = rx$, $|AC| = Rx$; hence, we get $|DC|^2 = |AC| \cdot |CK| = (R-r)Rx^2$. Similarly, $|AB| = Ry$, $|DB|^2 = (R-r)Ry^2$; consequently, $\frac{|CD|}{|DB|} = \frac{x}{y} = \frac{|AC|}{|AB|}$, that is, AD is the bisector of the angle BAC. Further, we have:

$$\frac{|AO|}{|OD|} = \frac{|AC|}{|CD|} = \frac{Rx}{\sqrt{(R-r)Rx}} = \sqrt{\frac{R}{R-r}}.$$

Thus, the desired locus is a circle touching internally the two given circles at the same point A with radius

$$\rho = r\frac{|AO|}{|AD|} = \frac{r\sqrt{R}}{\sqrt{R}+\sqrt{R-r}}.$$

79. Let O_1 and O_2 denote the centres of the given circles, the straight line O_1O_2 intersect the circles at points A, B, C, and D (in succession). Consider two cases:

(a) The rectangle $KLMN$ is arranged so that the opposite vertices K and M lie on one circle while L and N on the other. In this case, if P is the point of intersection of the diagonals (Fig. 20, a), then $|O_1P|^2 - |O_2P|^2 = (|O_1K|^2 - |KP|^2) - (|O_2L|^2 - |LP|^2) = |O_1K|^2 - |O_2L|^2 = R_1^2 - R_2^2$, where R_1 and R_2 are the radii of the circles, that is, the point P lies on a common chord of the circles; the midpoint of the common chord and its end points are excluded, since in this case the rectangle degenerates.

(b) Two neighbouring vertices of the rectangle $KLMN$ lie on one circle, and two others on the

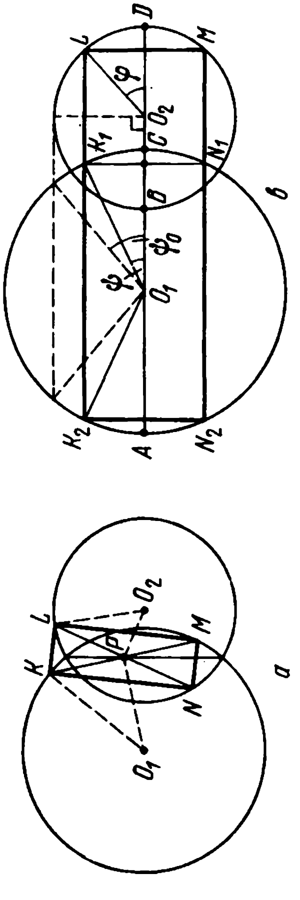


Fig. 20

other circle. Since the perpendiculars from O_1 on KN and from O_2 on LM must bisect them, the straight line O_1O_2 is the axis of symmetry for the rectangle $KLMN$.

Let R_2 be less than R_1 and the radius O_2L form an angle φ with the line of centres. We draw through L a straight line parallel to O_1O_2. This line intersects the circle O_1 at two points K_1 and K_2, and to the point L there will correspond two rectangles: K_1LMN_1 and K_2LMN_2 (Fig. 20, *b*). With φ varying from 0 to $\pi/2$, the angle ψ formed by the radius O_1K_1 and the ray O_1O_2 varies from 0 to a certain value ψ_0. With a further change in φ (from $\pi/2$ to π), ψ decreases from ψ_0 to 0. Meanwhile, the centres of rectangles K_1LMN_1 will trace a line segment from the midpoint of CD to the midpoint of BC excluding the extreme points and the point of intersection of this line segment with the common chord. Analogously, the centres of rectangles K_2LMN_2 will fill in the interval with end points at the midpoints of AB and AD (the end points of the interval are not contained in the locus).

If three vertices of the rectangle and, hence, the fourth one lie on a circle, then the centre of the rectangle coincides with the centre of the corresponding circle.

Thus, the locus is the union of three intervals: the end points of the first interval—the midpoints of AB and AD, respectively, the end points of the second interval—the midpoints of BC and CD, the end points of the third interval—the points of intersection of the circles, the midpoint of the common chord being excluded.

80. If B and C are the first and second points of reflection, O the centre, then BO is the bisector of the angle CBA. The path of the ball is symmetric with respect to the diameter containing C, therefore A lies on this diameter. If $\angle BCO = \angle CBO = \varphi$, then $\angle ABO = \varphi$, $\angle BOA = 2\varphi$; applying the law of sines to the triangle ABO

($|BO| = R$, $|OA| = a$), we get: $\dfrac{R}{\sin 3\varphi} = \dfrac{a}{\sin \varphi}$, whence $\cos 2\varphi = \dfrac{R-a}{2a}$, and for $a > \dfrac{R}{3}$ we can find φ.

Answer: points situated outside the circle of radius $R/3$ centred at the centre of the billiards.

81. The required locus are two straight lines perpendicular to the given lines.

82. If the line AB is not parallel to l, then there are two circles passing through A and B and touching l. Let O_1 and O_2 denote their centres. The sought-for locus is the straight line O_1O_2 excluding the interval (O_1O_2). If AB is parallel to l, then the desired locus consists of one ray perpendicular to l.

83. (a) Let A (Fig. 21) be a vertex of a triangle. Extend the line segment AM beyond M such that

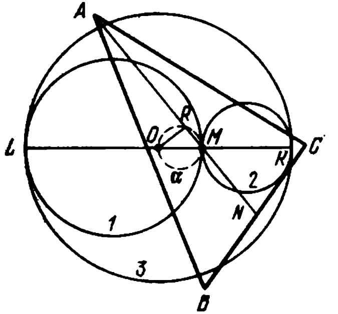


Fig. 21

the extension has a magnitude $|MN| = \dfrac{1}{2}|AM|$. The point N is the midpoint of the side opposite the vertex A, consequently, N must lie inside the circumscribed circle, that is, inside the circle of radius $|OA|$ centred at O. Drop a perpendicular OR from O on AN. There must be fulfilled the inequality $|AR| > |RN|$. If $\angle AMO \geqslant 90°$, then this inequality is fulfilled automatically. And if $\angle AMO < 90°$, then $|AM| - |MR| > |MN| + |MR| \Rightarrow |AM| - \frac{1}{2}|AM| > 2|MR| \Rightarrow |AM| > 4|MR|$. But R lies on the circle α of diameter OM, hence A must be located outside the circle which is homothetic to the circle α with the ratio of similitude equal to 4 and centre at M. Further, the point N must not get on the circle α since otherwise the side of the triangle whose midpoint it is, being perpendicular to ON, would lie on the straight line AN, that is, all the vertices of the triangle would be located on a straight line. Consequently, A must not lie on the circle which is homothetic to α with centre of similitude M and the ratio of 2. Thus, if we take on the straight line OM points L and K such that $|LO| : |OM| : |MK| = 3 : 1 \ \ 2$, and construct on LM as diameter the circle *1*, on MK the circle *2*, then the required locus is represented by all the points outside the circle *1* excluding the points of the circle *2* except the point K (the point K belongs to the locus).

(b) If O is the centre of the circumscribed circle, M the centre of mass of the triangle, then K (see Item (a)) is the intersection point of the altitudes of the triangle (see Problem 20 in Sec. 1). But the distance from the centre of the circle circumscribed about an obtuse triangle to the point of intersection of the altitudes is greater than the radius of the circumscribed circle. Consequently, the vertices of the obtuse triangle are found inside the circle *3*, constructed on LK as diameter, outside the circle *1* excluding the points of the circle *2* (the vertices of obtuse angles lying inside the circle *2*).

84. Let ABC (Fig. 22) be the original regular triangle, $A_1B_1C_1$ an arbitrary triangle with $A_1C_1 \parallel$

AC, $A_1B_1 \parallel AB$, O the centre of the circle, O_1 the intersection point of the altitudes of the triangle $A_1B_1C_1$. Let $\angle BOB_1 = \varphi$. Since $O_1B_1 \parallel OB$, we have $\angle OB_1O_1 = \varphi$; since $\angle C_1O_1B_1 = \angle C_1OB_1 = 120°$, the quadrilateral $C_1O_1OB_1$ is inscribed in a circle, and, hence, $\angle O_1OC_1 = \angle O_1B_1C_1 = 30° - \varphi$. Thus $\angle O_1OB = \varphi + 120° + 30° - \varphi = 150°$, that is, the straight line OO_1 is parallel to

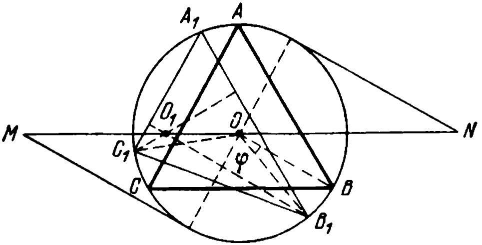


Fig. 22

CB. To find the path which can be "covered" by the point O_1, while moving along this straight line, note that to determine the position of the point O_1, we draw through the variable point B_1 a straight line parallel to OB to intersect the straight line passing through O parallel to CB. Obviously, the most remote points are obtained for the end points of the diameter perpendicular to OB. Thus, MN (the segment of the line parallel to CB, whose length is $4R$ with the midpoint at O) is a part of the locus, the entire locus consisting of three such line segments (with the end points of the segments excluded).

85. If ABC (Fig. 23) is the given triangle, and the vertex of the circumscribed rectangle $AKLM$ coincides with A (B on KL, C on LM), then L belongs to the semicircle of diameter BC, the angles ABL and ACL being obtuse, that is, L has two

extreme positions: L_1 and L_2, $\angle L_1CA = \angle L_2BA = 90°$, while the centre O describes an arc homothetic to the arc L_1L_2 with the centre of similitude at A and ratio 1/2.

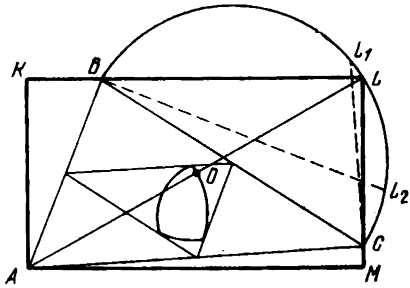


Fig. 23

Answer: if the triangle is acute, then the desired set is a curvilinear triangle formed by the arcs of the semicircles constructed on the midlines as diameters and faced inside the triangle formed by the midlines; if the triangle is not acute, then the required set consists of two arcs of the semicircles constructed on two smaller midlines in the same fashion.

86. If the first square is rotated about the point M through an angle of 60° (see Fig. 24) either clockwise or anticlockwise, then it must be entirely inside the second square. Conversely, to each square situated inside the larger square, and congruent to the smaller one, whose sides form angles of 30° and 60° with the sides of the larger square, there corresponds a point M possessing the needed property. (This square is shown in the figure by a dashed line.) This point is the centre of the rotation through an angle of 60° carrying the square $ABCD$ into the square $A_1B_1C_1D_1$; this point can be obtained from O_1 by rotating about O in the needed direction through an angle of 60°. Consider the extreme positions of squares $A_1B_1C_1D_1$ (when two vertices are found on the sides of the

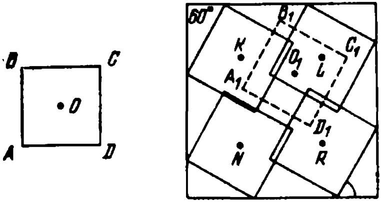


Fig. 24

larger square). Their centres serve as vertices of the square $KLRN$ whose side is equal to $b - \frac{1}{2} a \times (\sqrt{3} + 1)$ (the sides of the square $KLRN$ are parallel to the sides of the given squares, the centre coinciding with the centre of the larger square). The centres of another family of squares forming angles of 30° and 60° with the sides of the larger square also fill up the square $KLRN$. Thus, the required locus consists of the union of two squares one of which is obtained from the square $KLRN$ by rotating the latter about O through an angle of 60° in one direction, and the other by rotating through an angle of 60° in the opposite direction.

The problem has a solution if $b \geqslant \frac{a}{2} (\sqrt{3} + 1)$ (the points P and Q may be located on the boundary of the squares).

87. There is only one such point, viz. the centre of mass of the triangle (the median point). It is easily seen, that in this case for any point N on the boundary of the triangle we may take one of the vertices of the triangle as a point P. Let us

take some other point M_1. We assume that this point is found either inside the triangle AMD or on its boundary, where M is the centre of mass of the triangle ABC, D the midpoint of AC. We draw through M_1 a straight line parallel to BD and take the point of intersection of this line and AD as N, denoting its intersection point with AM by M_2. Obviously, for any point P inside the triangle or on its boundary the area of the triangle M_1NP does not exceed the area of one of the triangles AM_2N, M_2NC, M_2NB. It is also obvious that $S_{AM_2N} < S_{AMD} = \frac{1}{6}S$. Further, if $|AD| = |DC| = a$, $|ND| = x$, then $\frac{S_{M_2NC}}{S_{MDC}} = \frac{|M_2N|}{|MD|} \cdot \frac{|NC|}{|DC|} = \frac{a^2 - x^2}{a^2} \leqslant 1$. Finally, $\frac{S_{M_2NB}}{S_{AMD}} = \frac{|M_2N|}{|MD|} \times \frac{|ND|}{|AD|} = \frac{(a-x)x}{a^2} < 1$.

88. If A, B, C are the angles of the triangle ABC, then the angles of the triangle ABI are equal to $\frac{A}{2}$, $\frac{B}{2}$, $90° + \frac{C}{2}$ (Fig. 25); consequently, the sought-for locus is a pair of triangles two sides of which are line segments, the third being an arc which is a part of the segment constructed on AI and containing an angle $\alpha/2$.

89. We erect a perpendicular to BM at the point M; let P denote the point of intersection of this perpendicular and the perpendicular erected to the original straight line at the point B. Let us show that the magnitude $|PB|$ is constant. Let $\angle MBC$ be φ; K and L denote the feet of the perpendiculars from A and C on MB. By hypothesis, $\frac{|MK|}{|KA|} + \frac{|LM|}{|LC|} = k$, but $|LC| = |BC| \sin\varphi$, $|AK| = |BA| \sin\varphi$. Hence,

$$\frac{|MK|}{|BA|\sin\varphi}+\frac{|LM|}{|BC|\sin\varphi}=k \Leftrightarrow \frac{|BM|\pm|BK|}{|BA|\sin\varphi}+\frac{|BM|\mp|BL|}{|BC|\sin\varphi}=k \Leftrightarrow \frac{|BM|}{\sin\varphi}\left(\frac{1}{|BA|}+\frac{1}{|BC|}\right)\pm\left(\frac{|BK|}{|BA|\sin\varphi}-\frac{|BL|}{|BC|\sin\varphi}\right)=k \Leftrightarrow \frac{|BM|}{\sin\varphi}=\frac{k|BA|\cdot|BC|}{|BA|+|BC|} \Leftrightarrow |PB|=\frac{k|BA|\cdot|BC|}{|BA|+|BC|},$$

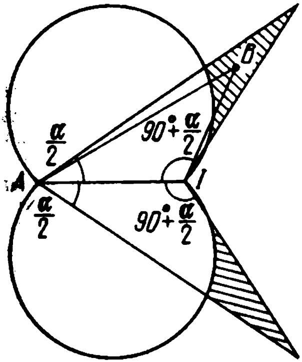


Fig. 25

which was to be proved. Consequently, the sought-for locus consists of two circles touching the straight line AC at a point B and whose diameters are equal to $\frac{k|BA|\cdot|BC|}{|BA|+|BC|}$.

90. Extend AQ beyond the point Q and take on this ray a point M such that $|QM|=\frac{1}{2}|AQ|$ and a point A_1 such that $|MA_1|=|AM|$; M is the midpoint of the side BC of the triangle ABC; $\angle CBA_1=\angle BCA$, $\angle ABA_1=180°-\angle BAC$.

Consequently, if we construct circles on AM, MA_1, and AA_1 as diameters, then the sought-for locus consists of points situated outside the first two circles and inside the third one.

91. Consider four cases: either the triangle ABC is acute, or one of the angles A, B or C is obtuse. In all the cases, it is possible to express the angles of the triangle ABH in terms of the angles of the triangle ABC.

92. If the end points of the rays do not coincide, then the required locus is formed by the parts of the following lines: the bisectors of the two angles formed by the straight lines containing the given rays, the midperpendicular to the line segment joining the end points of the rays, and two parabolas (the parabola is a locus of points equidistant from a given point and a given straight line). If the end points coincide, then the desired locus consists of both the bisector of the angle formed by the rays and the part of the plane enclosed inside the angle formed by the perpendiculars erected at the end points of the rays.

93. Let A denote the vertex of the angle. It is possible to prove that the centre of the circle circumscribed about the triangle MON coincides with the point of intersection of the angle bisector AO and the circle circumscribed about AMN. Let α be the size of the angle, r the radius of the circle, K the midpoint of AO. On the angle bisector AO, we take points L and P such that $|AL| = \dfrac{r}{\sin\frac{\alpha}{2}\left(1+\sin\frac{\alpha}{2}\right)}$, $|AP| = \dfrac{r}{\sin\frac{\alpha}{2}\left(1-\sin\frac{\alpha}{2}\right)}$. The sought-for locus consists of the line segment KL (K not belonging and L belonging to this set) and the ray lying on the angle bisector with origin at P.

94. Let O_1, O_2 denote the centres of the circles, r_1, r_2 their radii, M the midpoint of AB, O the midpoint of O_1O_2. We have (by the formula for the

length of a median, Problem 11 of Sec. 1) $|O_1M|^2 = \frac{1}{4}(2r_1^2 + 2|O_1B|^2 - |AB|^2)$, $|O_2M|^2 = \frac{1}{4}(2r_2^2 + 2|O_2A|^2 - |AB|^2)$, $|O_1B|^2 = \frac{1}{2}(|O_1O_2|^2 + 4|OB|^2 - 2r_2^2)$, $|O_2A|^2 = \frac{1}{2}(|O_1O_2|^2 + 4|OA|^2 - 2r_1^2)$. Thus, $|O_1M|^2 - |O_2M|^2 = r_1^2 - r_2^2$, that is (Problem 1 of Sec. 2) points M lie on the perpendicular to O_1O_2. If the circles have different radii and do not intersect, then the sought-for locus consists of two line segments obtained in the following way: from the line segment with end points at the midpoints of the common external tangents, we remove the points situated between the midpoints of the common internal tangents (if M is a point on the line segment with end points at the midpoints of the common internal tangents, then the straight line passing through M perpendicular to OM does not intersect the circle). In the remaining cases (the circles intersect or are equal) the sought-for locus is the entire line segment with end points at the midpoints of the common external tangents.

95. (a) Since $\angle FNB = 90°$, $\angle CNM = 135°$, $\angle FNM = 45°$ (we suppose that $|AM| > |MB|$), $\angle FNC = 90°$ and C, N, and B are collinear, and so forth.

(b) We consider the right isosceles triangle ABK with hypotenuse AB (K lying on the other side of AB than the squares). The quadrilateral $ANBK$ is an inscribed one, $\angle ANK = \angle ABK = 45°$, that is, NK passes through M.

The desired locus is the midline of the triangle ALB, where L is a point symmetric to the point K with respect to AB.

96. Let N denote the point of intersection of the middle perpendicular and the tangent; O the centre of the circle, R its radius. We have: $|ON|^2 -$

$|NA|^2 = R^2 + |MN|^2 - |NA|^2 = R^2$. Thus, the required locus is a straight line perpendicular to OA (Problem 1 of Sec. 2).

97. If O_1 and O_2 are the centres of the given circles, Q_1 and Q_2 are the centres of the circles circumscribed about the triangles ABC_1 and AB_1C, then $O_1Q_1O_2Q_2$ is a parallelogram. The straight line Q_1Q_2 passes through the midpoint of the line segment O_1O_2 (the point D). The second point of intersection of the circles circumscribed about the triangles ABC_1 and AB_1C is symmetric to the point A with respect to the straight line Q_1Q_2. The sought-for locus is a circle of radius $|AD|$ centred at the point D.

98. Let O_1 and O_2 denote the centres of the given circles, r_1 and r_2 their radii. Consider two right isosceles triangles O_1O_2O and O_1O_2O' with hypotenuse O_1O_2. The desired locus is two annuli with centres at O and O' and the following radii: external $\frac{\sqrt{2}}{2}(r_1 + r_2)$ and internal $\frac{\sqrt{2}}{2}|r_1 - r_2|$. Let us prove this. Let M be a point on the circle O_1, N on the circle O_2. If M is fixed, and N traverses the second circle, then the vertices of the right angles of the right isosceles triangles describe two circles of radius $\frac{\sqrt{2}}{2} r_2$, which are obtained from the circle O_2 by rotating about M through an angle of 45° (both clockwise and anticlockwise) followed by a homothetic transformation with centre of similitude at M and the ratio $\sqrt{2}/2$. Let O_M be the centre of one of those circles. The point O_M is obtained from O_2 by rotating the latter about M in the appropriate direction followed by a homothetic transformation with centre of similitude at M and the ratio $\sqrt{2}/2$. But O_M can be obtained by corresponding rotation and a homothetic transformation with the centre of similitude at O_2. Consequently, when M describes

the circle O_1, O_M describes a circle of radius $\frac{\sqrt{2}}{2} r_1$ with centre at O or O'.

99. The union of the three constructed parallelograms represents the parallelogram circumscribed about the given triangle separated into four smaller parallelograms. It is easy to express the relationships in which each of the diagonals under consideration is divided by the other diagonal in terms of the segments of the sides of the larger parallelogram.

If the parallelograms are rectangles, then, on having translated two of the three considered diagonals, we get a triangle congruent to the given one, and this means that the angles between them either equal the corresponding angles of the triangle or supplement them to 180°. The sought-for locus is a circle passing through the midpoints of the sides of the given triangle.

100. We prove that $\frac{|AM|}{|AD|} = \cos \angle BAC$, where D is the point of intersection of AM with the circle. Let O denote the centre of the circle, P the midpoint of BC, K the midpoint of AH. The triangles DOA and MKA are similar. Hence, $\frac{|MA|}{|AD|} = \frac{|AK|}{|DO|} = \frac{|OP|}{|OB|} = |\cos \angle BAC|$. The desired locus consists of two arcs belonging to two distinct circles.

101. Let B_0 and C_0 be the midpoints of the sides AC and AB, BB_1 and CC_1 the altitudes, K the midpoint of DE (Fig. 26), GK and C_0N perpendicular to AB, B_0M perpendicular to AC. Then $\frac{|ML|}{|NM|} = \frac{|GC_1|}{|C_0C_1|} = \frac{|KP|}{|C_0C_1|} = \frac{|DC|}{|BC|}$ (the last equality follows from the similarity of the triangles DCE and ABC, K, P and C_0, C_1 being the corresponding points in those triangles). In similar fashion, the middle perpendicular to DF intersects MN at a point L_1 such that $\frac{|NL_1|}{|NM|} = \frac{|BD|}{|BC|}$, that is, the points L and L_1 coincide.

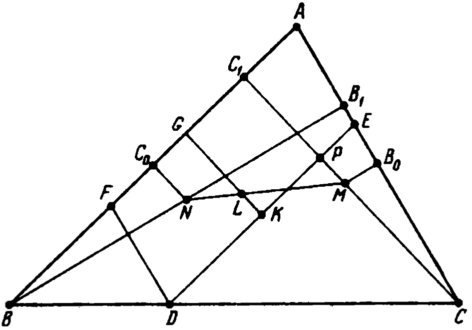


Fig. 26

The sought-for locus is the straight line MN.

102. It is obvious that any point of any of the altitudes of the triangle ABC belongs to the required locus. We show that there are no other points. Let us take a point M not lying on the altitudes of the triangle ABC. Let the straight line BM intersect the altitudes dropped from the vertices A and C at points M_1 and M_2, respectively. If the conditions of the problem were fulfilled for all the three points M_1, M_2, and M, then the equalities $\angle MAM_1 = \angle MCM_1$, $\angle MAM_2 = \angle MCM_2$ would hold, and then the five points A, M, M_1, M_2 and the point C_1 symmetric to C with respect to the straight line BM would lie on one circle, which is impossible.

103. Note that if a straight line l possessing the required property passes through M, then there exists either a straight line l_1 passing through M

and a vertex of the triangle or a straight line l_2 passing through M and perpendicular to a side of the triangle and possessing the same property. Indeed, let the line l intersect the sides AB and CB of the triangle ABC at points C_0 and A_0, and let there be a point B_1 symmetric to B with respect to l inside the triangle ABC. We rotate l about M so that B_1, moving in the arc of the corresponding circle, approaches AB or BC until the point C_0 or B_0 coincides with the vertex C or A (and we get the line l_1) or until B_1 gets on the corresponding side (and we get the line l_2). Let α denote the set of the points of our triangle situated inside the quadrilateral bounded by the angle bisectors drawn to the smallest and largest sides of the triangle and the perpendiculars erected at their midpoints. (If the given triangle is isosceles, then α is empty. In all other cases α is a quadrilateral or a pentagon.) The sought-for locus consists of all the points of the triangle excluding the interior points of α.

105. We have: $|MB|^2 = a^2 + c^2\cos^2 A = a^2 + c^2 - c^2\sin^2 A = a^2 + c^2 - a^2\sin^2 C = c^2 + a^2\cos^2 C = |NB|^2$.

107. Prove that the point symmetric to the intersection point of the altitudes of a triangle with respect to a side of the triangle lies on the circumscribed circle.

109. Let H denote the intersection point of the altitudes of the triangle ABC, AD the altitude, K, L, M, and N the projections of D on AC, CH, HB, and BA, respectively. Take advantage of the fact that K and L lie on the circle of diameter CD, L and M on the circle of diameter HD, and M and N on the circle of diameter DB.

111. Prove that the radius of the circle circumscribed about the triangle under consideration is equal to the radius of the given circles, and these circles are symmetric to the circumscribed one with respect to the sides of the triangle.

112. Let $ABCD$ denote the given rectangle, and let the points K, L, M, and N lie on the straight lines AB, BC, CD, and DA, respectively. Let P_1 be the second point of intersection of the straight line LN with the circle circumscribed about the given rectangle (the first point is P). Then $BP_1 \parallel KN$, $P_1D \parallel LM$, and $\angle B_1D = 90°$. Hence, $KN \perp LM$. In addition, $LN \perp KM$; thus, N is the intersection point of the altitudes of the triangle KLM. Let now, for definiteness, L and N be on the sides BC and DA. Denote: $|AB| = a$, $|BC| = b$, $|KP| = x$, $|PN| = y$. The straight line KN divides BD in the ratio $\frac{(a+y)x}{(b-x)y}$, counting from the vertex B. The straight line LM divides BD in the same ratio.

113. The line segments $|AP|$, $|BQ|$, and $|CR|$ can be expressed in terms of sides of the triangle, for instance: $|AP| = \frac{bc}{b+c}$.

114. Let M denote the midpoint of AD. Check to see that $|BF|^2 + |FM|^2 = |BM|^2$.

115. Draw through D a straight line perpendicular to the bisector of the angle A, then denote the points of its intersection with AB and AC by K and M, respectively, and prove that $|AK| = |AM| = \frac{b+c}{2}$. Since $|AC_1| = |AB_1| = p - a$, $|AC_2| = |BC_2| = p$ (p the semiperimeter of the triangle ABC, and a, b, c its sides), the points K and M are the midpoints of the line segments C_1C_2 and B_1B_2.

116. Prove that l forms with AD the same angles as the straight line BC touching the circle. Hence it follows that the other tangent to the circle passing through D is parallel to l.

117. We construct the circle touching the straight lines MN, AC and BC so that the points of tangency P and Q with the lines AC and BC lie outside the line segments CM and CN (this is a circle escribed in the triangle MCN). If R is the

point of tangency of MN with the circle, then $|MP| = |MR|$, $|NQ| = |NR|$, consequently, $|MN| = |MP| + |NQ|$; but we are given that $|MN| = |MA| + |NB|$. Thus, one of the points P or Q lies on the corresponding side, while the other on its extension. We have: $|CP| = |CQ| = \frac{1}{2}(|CP| + |CQ|) = \frac{1}{2}(|AC| + |CB|)$, that is, the constructed circle is constant for all the straight lines.

118. If O is the centre of the circle circumscribed about the triangle ABC, D the midpoint of CB, H the point of intersection of the altitudes, L the midpoint of AN, then $|AL| = |OD|$ and, since AL is parallel to OD, OL bisects AD, that is, L is symmetric to O with respect to the midpoint of AD.

119. Let BD denote the altitude of the triangle, and $|BD| = R\sqrt{2}$, where R is the radius of the circumscribed circle, K and M are the feet of the perpendiculars dropped from D on AB and BC, respectively, O is the centre of the circumscribed circle. If the angle C is acute, then $\angle KBO = 90° - \angle C$. Since $BMDK$ is an inscribed quadrilateral, $\angle MKD = \angle DBM = 90° - \angle C$. Hence, $\angle MKB = 180° - 90° - (90° - \angle C) = \angle C$; consequently, BO is perpendicular to KM. But $S_{BKM} = \frac{1}{2}|BD|^2 \sin A \sin B \sin C = R^2 \sin A \sin B \sin C = \frac{1}{2} S_{ABC}$. (We have used the formula $S = 2R^2 \sin A \sin B \sin C$.) On the other hand, if h_1 is the altitude of the triangle BKM drawn from the vertex B, then $\frac{1}{2} S = \frac{1}{4}|AC|\cdot|BD| = S_{BKM} = \frac{1}{2}|KM|\ h_1$ $\frac{1}{2}|BD|h_1 \sin B$, hence, $h_1 = \frac{|AC|}{2\sin B} = R$; bear-

ing in mind that $BO \perp KM$, we get that the point O lies on KM.

120. Note that the triangles ADK and ABK are similar since $|AK|^2 = |AC|^2 = |AD| \times |AB|$. If O is the centre of the circle circumscribed about the triangle ABK, then $\angle OAD + \angle ADK = 90° - \angle AKB + \angle ADK = 90°$ ($\angle AKB$ was assumed to be acute; if $\angle AKB$ is obtuse, our reasoning is analogous).

121. Prove that the straight line parallel to BC and passing through E divides the bisector of the angle A in the same ratio as it is divided by the bisector of the angle C.

122. If O is the vertex of the angle, A a point on the angle bisector, B_1 and B_2 the intersection points of one circle with the sides of the angle, C_1 and C_2 (B_1 and C_1 on the same side) the intersection points of the other circle, then $\triangle AB_1C_1 = \triangle AB_2C_2$.

123. Take advantage of the fact that the common chord of the two circles passing through A, A_1 and B, B_1 passes through the point D (Problem 18 of Sec. 2).

125. If O is the centre of the circle circumscribed about the triangle AMB, then $\angle MAB = 90° - \angle OMB = \angle BMC - 180°$. The angle MAC has the same size.

126. It is easy to prove that the circles under consideration intersect at one point. Let us denote this point by P. If the points are arranged as in Fig. 27, then $\angle PB_2M = 180° - \angle BB_2P = \angle PC_1B = 180° - \angle PC_1A = \angle PB_1A = \angle PA_2A = 180° - \angle PA_2M$, that is, the points P, B_2, M, and A_2 lie on one circle. In similar fashion, we prove that the points P, B_2, M, C_2 lie on one circle, consequently, the five points P, M, A_2, B_2, C_2 lie on one and the same circle.

127. Prove that the sides of the triangle $A_1B_1C_1$ are parallel to the corresponding sides of the triangle ABC.

128. Prove that as the straight line KL dis-

places, the centre of the circle circumscribed about KLB_1 describes a straight line.

129. Prove that any two line segments are bisected by the point of their intersection.

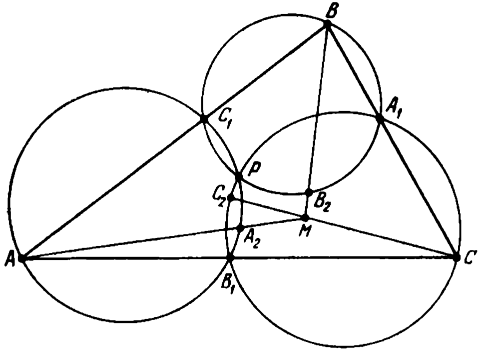


Fig. 27

130. If KN is a perpendicular from K on AB, $\angle CAB = \alpha$, then $\frac{|KN|}{|OM|} = \frac{|AK|}{|AO|} = \frac{|AO| - |KO|}{|AO|} =$

$$\frac{|AO| - 2\,|OM| \sin \frac{\alpha}{2}}{|AO|} = \frac{|AO| - 2\,|AO| \sin^2 \frac{\alpha}{2}}{|AO|} =$$

$\cos \alpha = \frac{|CD|}{|CB|}$. Since the triangles ACB and ACD are similar, it follows that KN is equal to the radius of the circle inscribed in the triangle ACD, and since K lies on the bisector of the angle A, K is the centre of the circle inscribed in the triangle ACD. The proof for L is carried out in a similar way.

131. Denote by C_1 and A_1 the midpoints of AB and BC, by B' and A' the points of tangency of the inscribed circle to AC and BC. Let, for definite-

ness, $c \geqslant b$ (c and b sides of the triangle ABC), then the bisector of the angle A intersects the extension of C_1A_1 at a point K such that $|A_1K| = \frac{c-b}{2}$, and the straight line $B'A'$ must pass through the same point K since the triangles KA_1A' and $A'B'C'$ are isosceles, $|A'C| = |B'C|$, $|A_1K| = |A_1A'|$, $\angle A'A_1K = \angle A'CB'$

132. Consider the angle at vertex A. Three points B_1, B_2, B_3 are taken on one side of the angle and three points C_1, C_2, C_3 on the other side. From Menelaus' theorem (Problem 45 of Sec. 2, Remark) it follows that for the straight lines B_1C_1, B_2C_2, B_3C_3 to meet in the same point, it is necessary and sufficient that the following equality is fulfilled:

$$\frac{AB_2}{B_2B_1} \cdot \frac{C_1C_2}{C_2A} = \frac{AB_3}{B_3B_1} \cdot \frac{C_1C_3}{C_3A} \tag{1}$$

(the ratios are understood in the sense indicated in Remark). Indeed, if the equality (1) is fulfilled, then it follows from Menelaus' theorem that the straight lines B_2C_2 and B_3C_3 intersect the side B_1C_1 of the triangle AB_1C_1 at one point.

133. Through A, draw a line parallel to BC and denote by K and L the points of its intersection with A_1C_1 and A_1B_1, respectively. We have: $\frac{|KA|}{|BA_1|} = \frac{|AC_1|}{|C_1B|}$, $\frac{|CB_1|}{|B_1A|} = \frac{|A_1C|}{|AL|}$. And, by Ceva's theorem (Problem 44 in Sec. 2), $\frac{|AC_1|}{|C_1B|} \times \frac{|BA_1|}{|A_1C|} \cdot \frac{|CB_1|}{|B_1A|} = 1$, hence, $|KA| = |AL|$. But if AA_1 is the bisector of the angle KA_1L, then, since $|KA| = |AL|$, AA_1 is perpendicular to KL, that is, AA_1 is the altitude of the triangle ABC.

134. Let K be the point of intersection of AA_1 and BB_1, H the intersection point of the altitudes

of the triangle ABC. The points A, K, H, and B lie on a circle (the angles AKB and AHB are either equal to each other or their sum yields 180° according as the points K and H are located either on the same side of the straight line AB or on different sides). The radius of this circle is equal to the radius R of the circle circumscribed about the triangle ABC. If φ is the angle between AA_1 and AH, then $|KH| = 2R \sin \varphi$.

135. Let H denote the intersection point of the altitudes of the triangle $A_1B_1C_1$. The points A_1, H, B_1 and C lie on the same circle, the points B_1, H, C_1, and A also lie on a circle, the radii of these circles being equal; the angles HB_1C and HB_1A are either equal or supplement each other to 180°. Consequently, $|HA| = |HC|$. The converse is false. For each point A_1 on the straight line BC there exist, generally speaking, two triangles: $A_1B_1C_1$ and $A_1B_1'C_1'$ (B_1 and B_1' lying on AC, C_1 and C_1' on AB), for which the points of intersection of the altitudes coincide with the centre of the circle circumscribed about the triangle ABC, one of them being similar to the triangle ABC, the other not. For instance, if ABC is a regular triangle, A_1 the midpoint of BC, then we may take the midpoints of AC and AB as B_1 and C_1, and, the points on the extensions of AC and AB beyond C and B, as B_1' and C_1', $|CB_1'| = |CB|$, $|BC_1'| = |BC|$. The converse is true provided that the points A_1, B_1, and C_1 are situated on the sides of the triangle ABC, but not on their extensions.

136. We prove that the centre of the desired circle coincides with the orthocentre (the intersection point of the altitudes). Let BD denote the altitude, H the intersection point of the altitudes, and K and L the midpoints of the constructed line segments emanating from the vertex B, $|BK| = |BL| = l$, M the midpoint of BD. Then $|KH|^2 = |LH|^2 = |MH|^2 + |KM|^2 = l^2 \quad |BM|^2 + |MH|^2 = l^2 - \frac{|BD|^2}{4} +$

$\left(|BH| - \frac{|BD|}{2}\right)^2 = l^2 + |BH|^2 - |BH| \times |BD| = l^2 - |BH| \cdot |HD|$. It remains to prove that the products of the segments of the altitudes into which each of them is divided by the point of their intersection are equal. We draw the altitude AE. Since the triangles BHE and AHD are similar, we have: $|BH| \cdot |HD| = |AH| \cdot |HE|$, which was to be proved.

137. We denote (Fig. 28): $|BC| = a$, $|CA| = b$, $|AB| = c$. Through the centre of the inscribed

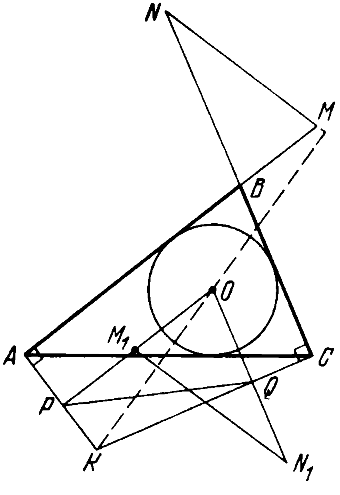


Fig. 28

circle, we draw straight lines parallel to AB and BC to intersect AK and KC at points P and Q. In the triangle OPQ we have: $\angle POQ = \angle ANC$, $|OQ| = p - c$, $|OP| = p - a$, where p is the semiperimeter of the triangle ABC. But, by hypothesis, $\angle NBM = \angle ABC$, $|NB|$ $p - a$, $|MB| = p - c$. Consequently, $\triangle POQ = \triangle NBM$. If we take on the straight line OP a point M_1 such that $|OM_1| = |OQ|$ and on OQ a point N_1 such that $|ON_1| = |OP|$, then $\triangle ON_1M_1 = \triangle NBM$, and its corresponding sides turn out to be parallel, i.e., $BM \parallel OM_1$ and $BN \parallel ON_1$. Hence, $N_1M_1 \parallel NM$. Let us prove that OK is perpendicular to N_1M_1. Since two opposite angles are right ones, in the quadrilateral $OPKQ$, the latter is an inscribed quadrilateral, consequently, $\angle OKP = \angle OQP$. Further, $\angle KOP + \angle OM_1N_1 = \angle KOP + \angle OQP = \angle KOP + \angle OKP = 90°$, and this means that $OK \perp M_1N_1$.

138. Let, for definiteness, P lie on the arc AC. The points A, M, P, and N lie on one and the same circle, hence, $\angle NMP = \angle NAP$. Analogously, the points P, M, Q, and C are located on one and the same circle, $\angle PMQ = 180° - \angle PCQ = 180° - \angle PAN = 180° - \angle PMN$.

139. Let ABC be the given triangle (Fig. 29), H the point of intersection of its altitudes. Note that

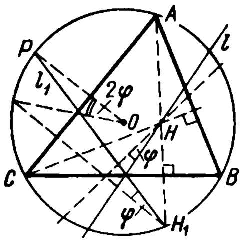


Fig. 29

the points symmetric to H with respect to its sides lie on the circle circumscribed about the triangle ABC (see Problem 107 in Sec. 2). If H_1 is a point symmetric to H with respect to the side BC, then the straight line l_1 symmetric to l with respect to the same side passes through H_1. With l rotated about H through an angle φ, the line l_1 rotates about H_1 through the same angle φ in the opposite direction. Consequently, if P is the second intersection point of the line l_1 with the circumscribed circle, then the radius OP (O the centre of the circumscribed circle) rotates about O through an angle 2φ in the appropriate direction. The same reasoning holds true for the two other straight lines symmetric to l. But if l coincides with an altitude of the triangle, then the statement of the problem is obvious (the point P coincides with the corresponding vertex of the triangle). Consequently, this stat^ment is always true.

140. Let the points A, B, C, and M have the following coordinates in the rectangular Cartesian system: (x_1, y_1), (x_2, y_2), (x_3, y_3), (x, y), respectively, and let the coordinates of the point G are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$ Then the validity of the assertion follows from the identity $3\left(x-\frac{x_1+x_2+x_3}{3}\right)^2=(x-x_1)^2+(x-x_2)^2+(x-x_3)^2-\frac{1}{3}((x_1-x_2)^2+(x_2-x_3)^2+(x_3-x_1)^2)$ and analogous relationship for the ordinates.

141. Consider the case when the point M (Fig. 30) lies inside the triangle ABC. Rotate the triangle ABM about A through an angle of 60° to bring B into C. We get the triangle AM_1C which is congruent to the triangle ABM; the triangle AMM_1 is equilateral, consequently, the sides of the triangle CMM_1 are equal to the line segments MA, MB, MC. The points M_2 and M_3 are obtained in a similar way. The area of the hexagon $AM_1CM_3BM_2$ is twice the area of the triangle ABC, that is, equals $a^2\sqrt{3}/2$. On the other hand, the area of this hexagon is expressed as the sum of the areas of three equilateral triangles AMM_1, CMM_3, and BMM_2 and the three triangles congruent to the desired one. Consequently, $3S + (|MA|^2 + |MB|^2 + |MC|^2)\frac{\sqrt{3}}{4} = a^2\frac{\sqrt{3}}{2}$. Using the result of Problem 140 of Sec. 2, we get $3S + (3d^2 +$

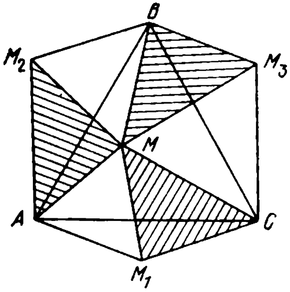


Fig. 30

$a^2)\frac{\sqrt{3}}{4} = a^2\frac{\sqrt{3}}{4}$, whence $S = \frac{\sqrt{3}}{12}(a^2 - 3d^2)$. Other cases of arrangement of the point M can be considered in a similar way.

142. Use the results of Problems 141 and 6 in Sec. 2. Generally speaking, the sought-for locus consists of a straight line and a circle.

143. Let (Fig. 31,a) O be the centre of the circumscribed, and I the centre of the inscribed circle. From O and I, we drop perpendiculars ON, OP, IL, and IQ on AB and BC. If a, b, c denote the corresponding lengths of the sides BC, CA, and AB, and p the semiperimeter of the triangle ABC, then $|BK| = |c - b|$, $|BM| = |a - b|$, $|BN| = c/2$, $|BP| = a/2$, $|BL| = |BQ| = p - b$, $|NL| = \frac{1}{2}|a - b|$, $|PQ| =$

$\frac{1}{2}|c-b|$ (see Problem 18 in Sec. 1). Consequently, if we draw, through O, straight lines parallel to the sides AB and BC to intersect the perpendiculars dropped from I, then we get the triangle ORS similar to BKM with the ratio of similitude of 1/2. But the circle constructed on OI as diameter is circumscribed in the triangle ORS. Conse-

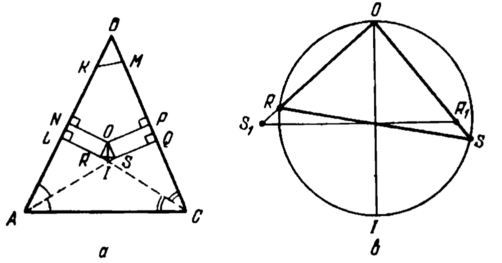


Fig. 31

quently, the radius of the circle circumscribed about the $\triangle BKM$ is equal to OI. To prove the second part of the problem, we note that if a line segment OR_1 equal to OR is laid off on the straight line OS, and a line segment OS_1 equal to OS—on the line OR, then the line S_1R_1 is parallel to KM (Fig. 31,*b*); but $\angle OR_1S_1 + \angle IOR_1 = \angle ORS + \angle IOS = 90°$, that is, $S_1R_1 \perp OI$.

144. Using the notation of the preceding problem, we draw through A a straight line perpendicular to OI and denote by D the point of its intersection with the straight line BC. Prove that the difference between the radii of the circles circumscribed about the triangles ABD and ACD is equal to the radius of the circle circumscribed about the triangle BKM.

145. Let the sides of the triangle be equal to a, b, and c, and $b = (a + c)/2$.

(a) From the equality $pr = \frac{1}{2}bh_b$ (p the semi-perimeter, r the radius of the inscribed circle, h_b the altitude drawn to the side b), we get: $\frac{1}{2}(a + b + c) = \frac{1}{2}bh_b$; but $a + c = 2b$, hence, $h_b = 3r$.

(b) This assertion follows from the fact that $r = \frac{1}{3}h_b$ and the median point divides each median in the ratio 2 1.

(c) Extend the angle bisector BD to intersect the circumscribed circle at a point M. If we prove that O, the centre of the inscribed circle, bisects BM, then thereby our statement is proved. (We draw the diameter BN, then the line joining the centres of the inscribed and circumscribed circles is parallel to NM, and $\angle BMN = 90°$.) But the triangle COM is isosceles since $\angle COM = \angle OCM = \frac{1}{2}(\angle C + \angle B)$. Hence, $|CM| = |OM|$. From the condition $b = (a + c)/2$, by the property of an angle bisector, we get: $|CD| = a/2$. Let K be the midpoint of CB; $\triangle CKO = \triangle CDO$ ($|CK| = |CD|$, $\angle KCO = \angle OCD$); hence it follows: $\angle BKO = \angle CDM$; in addition, $\angle DCM = \angle OBK = \angle B/2$, $|CD| = |BK|$ that is, $\triangle BKO = \triangle CDM$, $|CM| = |BO|$, hence $|BO| = |OM|$ which was to be proved.

(d) We take any point on the angle bisector. Let the distances to the sides BC and BA be equal to x, while to the side AC to y. We have: $\frac{1}{2} \times (ax + cx + by) = S_\Delta \Rightarrow b(2x + y) = 2S_\Delta \Rightarrow 2x + y = h_b$.

(e) If L is the midpoint of BA, then the desired quadrilateral is homothetic to the quadrilateral $BCMA$ with the ratio 1/2 (see Item (c)).

146. Let N denote the intersection point of the common tangent with BC. It suffices to check that $|FN|\ \ |NG| = |KN|\ \ |NM| = |DN|\cdot|NE|$. All the line segments are readily computed, since $|BD| = |CE| = p-b$, $|DE| = |b-c|$, $\frac{|DN|}{|NE|} = \frac{r}{r_a} = \frac{p-a}{p}$ (r_a the radius of the circle touching the side BC and the extensions of the sides AB and AC), and so on.

147. Through the vertices of the triangle ABC, we draw straight lines parallel to the opposite sides to form a triangle $A_1B_1C_1$ which is similar to the triangle ABC. It is obtained from the triangle ABC by a homothetic transformation with centre at the centre of mass, common for the triangles ABC and $A_1B_1C_1$, the ratio of similitude being equal to -2. The intersection point of the altitudes of the triangle ABC is the centre of the circle circumscribed about the triangle $A_1B_1C_1$. Consequently, the points O (the centre of the circumscribed circle), G (the centre of mass), and H (the intersection point of the altitudes of the triangle ABC) lie on a straight line, and $|OG| = \frac{1}{2}|GH|$, G lying on the line segment OH.

148. In an acute triangle, Euler's line intersects the largest and the smallest sides. In an obtuse triangle—the largest and the middle sides.

150. Show that the required property is possessed by such a point P on Euler's line for which $|PO| = |OH|$ (O the centre of the circumscribed circle, H the intersection point of the altitudes); in this case, for each triangle the distance from the centre of mass to the opposite vertex of the original triangle is equal to $\frac{4}{3}R$, where R is the radius of the circle circumscribed about the triangle ABC, and the straight line passing through the centre of mass of this triangle and the opposite vertex of the original triangle passes through the point O.

151. Let C_1 denote the centre of the circle circumscribed about the triangle APB, and C_2 the point symmetric to C_1 with respect to AB. Similarly, for the triangles BPC and CPA we determine the points A_1 and A_2, B_1 and B_2, respectively. Since the triangles AC_1B, AC_2B, BA_1C, BA_2C, CB_1A, CB_2A are isosceles with vertex angles of 120°, the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are regular (see Problem 296 of Sec. 2). Computing the angles of the quadrilaterals with vertices P, A_2, B_2, and C_2, we can prove that they lie on the same circle. Further, if H is the intersection point of the altitudes of the triangle APB, then, since $|PH| = |C_1C_2|$ and, hence, PHC_2C_1 is a parallelogram, the straight line C_1H (Euler's line of the triangle APB) passes through the midpoint of PC_2. But PC_2 is a chord of the circle with centre at C_1, consequently, C_1H is perpendicular to PC_2. Thus, the three Euler's lines coincide with the midperpendiculars of the line segments PC_2, PB_2 and PA_2, and since the points P, A_2, B_2, C_2 lie on the same circle, those lines intersect at its centre which is the centre of the regular triangle $A_2B_2C_2$. It follows from the result of Problem 296 of Sec. 2 that these three Euler's lines intersect at the median point of the triangle ABC.

152. Let ABC be the given triangle whose sides are a, b, and c ($a \geqslant b \geqslant c$), A_1, B_1, C_1 the points of tangency of the inscribed circle, I the centre of the inscribed circle, O the centre of the circumscribed circle. Since, with respect to the triangle $A_1B_1C_1$, I is the centre of the circumscribed circle, it suffices to prove that the straight line IO passes through the intersection point of the altitudes of the triangle $A_1B_1C_1$. Lay off on the rays AC and BC line segments AK and BL ($|AK| = |BL| = c$), and on the rays AB and CB line segments AM and CN ($|AM| = |CN|) = b$). As is known (see Problem 143 in Sec. 2), the line IO is perpendicular to LK and MN, hence, $LK \parallel MN$. Denote: $\angle KLC = \angle BNM = \varphi$. By the law of sines for

the triangles KLC and BNM, we have:

$$\frac{|LC|}{|KC|}=\frac{a-c}{b-c}=\frac{\sin(\varphi+C)}{\sin\varphi}, \tag{1}$$

$$\frac{|BN|}{|BM|}=\frac{a-b}{b-c}=\frac{\sin(B-\varphi)}{\sin\varphi}. \tag{2}$$

Now, in the triangle $A_1B_1C_1$, we draw the altitude to the side B_1C_1. Let Q be the point of its intersection with the straight line IO. We have to prove that Q is the intersection point of the altitudes of the triangle $A_1B_1C_1$. But the distance from I to B_1C_1 is $|IA_1|\cos A_1 = r\sin\frac{A}{2}$. Hence, the equality $|A_1Q| = 2r\sin\frac{A}{2}$ must be true. The angles of the triangle QIA_1 can be expressed in terms of the angles of the triangle ABC and φ, namely, $\angle QIA_1 = 180° - \varphi$, $\angle QA_1I = \frac{\angle B-\angle C}{2}$. We have to prove that $2\sin\frac{A}{2}=\frac{\sin\varphi}{\sin\left(\varphi-\frac{B-C}{2}\right)} \Leftrightarrow$ $\sin(\varphi+C)-\sin(B-\varphi)=\sin\varphi$. The last equality follows from (1) and (2).

153. When carrying out the proof, we make use of the fact that if perpendiculars PK and PL are dropped from a point P on the straight lines intersecting at a point M, then the points P, K, L, and M lie on the same circle.

154. Use the result of Problem 246, Sec. 1.

156. The distance between the projections of M on AC and BC is equal to $|CM|\sin C$. If K and L are the projections of M on AB and BC, then the projection of AB on the straight line KL (this is just Simson's line) is equal to $|AB| \times |\cos\angle BKL| = |AB|\,|\cos\angle BML| = |AB|\sin\angle CBM = |CM|\sin C$.

157. Prove that the sides of the triangles $A_1B_1C_1$, $A_2B_2C_2$, and $A_3B_3C_3$ are correspondingly parallel.

158. Prove that Simson's line corresponding to A_1 is perpendicular to B_1C_1 (the same for the other points). Further it is possible to prove that Simson's line corresponding to the point A_1 passes through the midpoint of A_1H, where H is the point of intersection of the altitudes of the triangle ABC (see also the solution of Problem 166 of Sec. 2). Consequently, Simson's lines are the altitudes of the triangle whose vertices are the midpoints of the line segments A_1H, B_1H, C_1H. **Remark.** We can prove that Simson's lines of arbitrary points A_1, B_1, C_1 with respect to the triangle ABC form a triangle similar to the triangle $A_1B_1C_1$, the centre of the circle circumscribed about it coinciding with the midpoint of the line segment joining the points of intersection of the altitudes of the triangles ABC and $A_1B_1C_1$.

159. First of all, we check the validity of the following statement: if the perpendiculars drawn to the sides (or their extension) of the triangle at the points of intersection with a straight line meet at a point M, then M lies on the circle circumscribed about the triangle. (This statement is the converse of the statement of Problem 153.) Consider the parabola $y = ax^2$. An arbitrary tangent to it has the form: $y = kx - \frac{k^2}{4a}$ (the tangent has only one common point with the parabola, hence, the discriminant of the equation $ax^2 = kx + b$ is equal to zero). This tangent intersects the x-axis at the point $x = k/4a$. The perpendicular to the tangent at this point is represented by the straight line $y = -\frac{1}{k}\left(x - \frac{k}{4a}\right) = -\frac{x}{k} + \frac{1}{4a}$. Consequently, all such perpendiculars pass through the point

$\left(O; \frac{1}{4a}\right)$ (the focus of the parabola). Now we use the remark at the beginning of the solution.

160. Let ABC denote the given triangle, H the point of intersection of its altitudes, A_1, B_1, C_1 the midpoints of the line segments AH, BH, and CH, respectively; AA_2 the altitude, A_3 the midpoint of BC. We assume, for convenience, that ABC is an acute triangle. Since $\angle B_1A_1C_1 = \angle BAC$ and $\triangle B_1A_2C_1 = \triangle B_1HC_1$, we have $\angle B_1A_2C_1 = \angle B_1HC_1 = 180° - \angle B_1A_1C_1$, that is, the points A_1, B_1, A_2, and C_1 lie on the same circle. It is also easy to see that $\angle B_1A_3C_1 = \angle B_1HC_1 = 180° - \angle B_1A_1C_1$, that is, the points A_1, B_1, A_3, and C_1 also lie on one (that is, on the same) circle. Hence it follows that all the nine points, mentioned in the hypothesis, lie on one and the same circle. The case of an obtuse triangle ABC is considered in similar fashion. Note that the nine-point circle is homothetic to the circumscribed circle with centre of similitude at H and the ratio of 1/2. (The triangles ABC and $A_1B_1C_1$ are arranged just in such a manner.) On the other hand, the nine-point circle is homothetic to the circumscribed circle with centre of similitude at the median point of the triangle ABC and the ratio of —1/2. (The triangle ABC and the triangle with vertices at the midpoints of its sides are arranged exactly in such a way.)

161. Our statement follows from the fact that D lies on the nine-point circle, and this circle is homothetic to the circumscribed circle with centre of similitude at H and the ratio of 1/2 (see Problem 160 of Sec. 2).

162. Our statement follows from the fact that E lies on the nine-point circle, and this circle is homothetic to the circumscribed circle with centre of similitude at M and the ratio of —1/2 (see Problem 160 of Sec. 2).

163. This distance is half the sum of the distances to BC from the intersection point H of the altitudes and the centre of the circumscribed circle, the latter being equal to half $|HA|$.

164. Let M_0 be the midpoint of HP, A_0 the midpoint of HA, and the points A_0, A_1, and M_0 lie on the nine-point circle. Consequently, M also lies on this circle since the hypothesis implies the equality $|M_0H|\cdot|HM| = |A_0H|\cdot|HA_1|$, and H is simultaneously either inside or outside each of the line segments M_0M and A_0A_1.

165. We prove that M and N lie on the corresponding midlines of the triangle ABC. If P is the midpoint of AB, then $\angle MPA = 2\angle ABM = \angle ABC = \angle APL$. Let, for the sake of definiteness, ABC be an acute triangle, $\angle C \geqslant \angle A$, then $\angle MNK = 180° - \angle KNB = \angle KCB = \angle MLK$ (we have taken advantage of the facts that the points K, N, B, and C lie on the same circle and that ML is parallel to BC). Hence, the points M, L, N, and K lie on the same circle. Further

$\angle LMK \quad \angle PMB + \angle NMK = \frac{1}{2}\angle B +$

$\angle BMK = \frac{1}{2}\angle B + \angle A$. If O is the centre of the circle circumscribed about the triangle LMK, then $\angle LOK = 2\angle LMK = \angle B + 2\angle A = 180° - \angle C + \angle A = 180° - \angle LPK$ ($\angle LPK = \angle APK - \angle APL = 180° - 2\angle A - \angle B = \angle C - \angle A$), that is, O lies on the circle passing through the points L, P, and K, and this is just the nine-point circle.

166. Since the midpoint of FN lies on the nine-point circle (see Problem 160 in Sec. 2), it suffices to show that Simson's line corresponding to the point F also bisects FH. Let K be the projection of F on a side of the triangle, D the foot of the altitude drawn to the same side, H_1 the point of intersection of this altitude and the circumscribed circle, $|H_1D| = |HD|$ (see the solution of Problem 107 in Sec. 2), L the point of intersection of Simson's line with the same altitude, and, finally, M the point on the straight line HH_1 for which $FM \parallel KD$; then $\triangle FMH_1 = \triangle KDL$ ($|FM| = |KD|$), both of them being right-angled, and $\angle DLK = \angle MH_1F$ since the altitude of the triangle is the Simson line corresponding to the vertex it emanates from, and we may use the statement of Problem 154 of Sec. 2. It is also easy to show that the directions of $\overrightarrow{H_1M}$ and $\overrightarrow{DL}$ coincide, that is, $FKHL$ is a parallelogram whence there follows our statement.

167. In Fig. 32: O is the centre of the circumscribed circle, A_1, B_1, C_1 the midpoints of the sides,

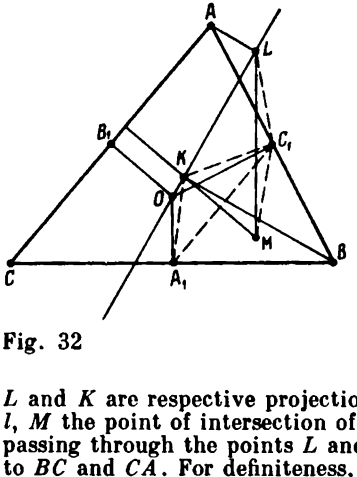


Fig. 32

L and K are respective projections of A and B on l, M the point of intersection of the straight lines passing through the points L and K perpendicular to BC and CA. For definiteness, the triangle ABC is acute-angled. First, we prove that C_1 is the centre of the circle circumscribed about the triangle KLM. The points A_1, O, K, C_1, and B lie on the same circle. Consequently, $\angle C_1KL = \angle OA_1C_1 = 90° - \angle C$; in similar fashion, $\angle C_1LK = 90° - \angle C$. Hence, $|KC_1| = |C_1L|$,

$\angle LC_1K = 2\angle C$, and since $\angle KML = \angle C$, our statement has been proved. Further, KM is perpendicular to A_1C_1, $|KC_1| = |C_1M|$, hence, $\angle C_1MA_1 = \angle C_1KA_1 = 180° - \angle B$, that is, M lies on the circle circumscribed about $A_1B_1C_1$.

168. Let H denote the intersection point of the altitudes of the triangle ABC, and A_2, B_2, C_2 the midpoints of the line segments AH, BH, CH, respectively. Note that the triangles AB_1C_1, A_1BC_1, A_1B_1C are similar (the corresponding vertices being denoted by the same letters), A_2, B_2, and C_2 denoting the corresponding centres of the circles circumscribed about them. First, we prove the following assertion: three straight lines passing through the points A_2, B_2, and C_2 and occupying the same positions relative to the triangles AB_1C_1, A_1BC_1, A_1B_1C meet in a point on the nine-point circle. Note that the straight lines A_2B_1, B_2B, and C_2B_1 are equally arranged with respect to the triangles AB_1C_1, A_1BC_1, and A_1B_1C and intersect at the point B_1 lying on the nine-point circle. Since the points A_2, B_2, C_2 lie on the nine-point circle, it is obvious that the three lines obtained from the straight lines A_2B_1, B_2B, and C_2B_1 by rotating them about the points A_2, B_2, and C_2, respectively, through the same angle, also intersect at one point located on the nine-point circle. Let now P be the intersection point of the Euler lines of the triangles $AB_1C_1, A_1BC_1, A_1B_1C$. Denote: $\angle PA_2A = \varphi$. For the sake of convenience, we assume that ABC is an acute triangle, and the point P lies on the arc B_1A_2 of the nine-point circle (see Fig. 33). Then $\angle PA_2A_1 = 180° - \varphi$, $\angle PA_2B_1 = 180° - \varphi - \angle B_1A_2A_1 = 180° - \varphi - \angle B_1C_1A_1$ $2\angle C - \varphi$, $\angle PA_2C_1 = 180° - \varphi + 180° - 2\angle B = 360° - \varphi - 2\angle B$. Since the chords PA_1, PB_1, and PC_1 are proportional to the sines of the angles subtended by them, it remains to prove that one of the three quantities: $\sin\varphi$, $\sin(2C - \varphi)$, $-\sin(2B + \varphi)$, (in our case the first one) is equal to the sum of the two others, that is, $\sin\varphi =$

$\sin(2C-\varphi)-\sin(2B+\varphi)$. But in the triangle AA_2H_1: $|AA_2|=R$, $|AH_1|=2B\cos A$ (R the radius of the circumscribed circle, $R\cos A$ the distance from the centre of the circumscribed circle A_2 to B_1C_1), $\angle H_1AA_2=\angle A+2\angle B-180°$.

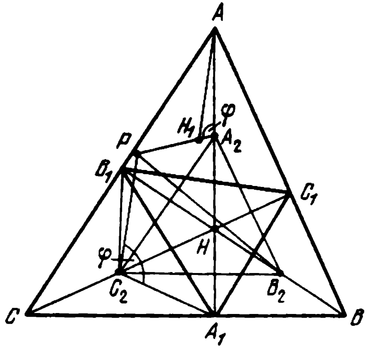


Fig. 33

By the law of sines for the $\triangle AA_2H_1$, we have: $\frac{2\cos A}{\sin\varphi}\ \frac{1}{\sin(2B+A+\varphi)}\Rightarrow-\sin(2B+2A+\varphi)-\sin(2B+\varphi)=\sin\varphi\Rightarrow\sin(2C-\varphi)-\sin(2B+\varphi)=\sin\varphi$, which was required to be proved. Thus, we have proved the statement for an acute triangle. The case of an obtuse triangle ABC can be considered exactly in the same way.

169. Let ABC be the given triangle, A_1, B_1, and C_1 the midpoints of the corresponding sides. Prove that the circle passing, for instance, through the vertex A and satisfying the conditions of the problem passes through the points of intersection of the bisectors of the internal and external angle A and the midline B_1C_1. Hence, for all points M of this circle the equality $|B_1M|:|C_1M|=|B_1A|:|C_1A|=b:a$ is fulfilled (see

Problem 9 in Sec. 2). Thus, if M_1 and M_2 are intersection points of two such circles, then $|A_1M_1| : |B_1M_1| : |C_1M_1| = a : b : c$ (the same for the point M_2), therefore M_1 and M_2 belong to a third circle. In addition, M_1 and M_2 belong to a straight line for all points M of which the equality $(c^2 - b^2)\,|A_1M|^2 + (a^2 - c^2)\,|B_1M|^2 + (b^2 - a^2)\,|C_1M|^2 = 0$ is fulfilled (see Problem 14 in Sec. 2 and its solution). This line passes through the centre of the circle circumscribed about the triangle $A_1B_1C_1$ and through the point of intersection of its medians (check this, expressing the lengths of the medians in terms of the lengths of the sides), that is, it coincides with the Euler line of the triangle $A_1B_1C_1$, and, hence, with that of the triangle ABC.

170. (a) As it was done in the preceding problem, we can prove that these three circles intersect at two points M_1 and M_2, and $|AM_1|$ $|BM_1| : |CM_1| = bc : ac \quad ab$ (the same for the point M_2).

(b) Follows from (a) and Problem 14 of Sec. 2.

(c) Prove that if M is inside the triangle ABC, then $\angle AM_1C = 60° + \angle B$, $\angle BM_1A = 60° + \angle C$, $\angle CM_1B = 60° + \angle B$ (for this purpose, use Bretschneider's theorem—Problem 236 of Sec. 2).

171. Take on BC a point A_1 and on BA a point C_1 such that $|BA_1| = |BA|$, $|BC_1| = |BC|$ (the triangle A_1BC_1 is symmetric to the triangle ABC with respect to the bisector of the angle B). Obviously, BK bisects A_1C_1. We construct two parallelograms BA_1MC_1 and $BCND$ (the corresponding sides of the parallelograms are parallel, the points B, K, M, and N are collinear); $|CN| \quad |AA_1| \dfrac{|BC|}{|BA_1|} = \dfrac{|BC|^2}{|BA|}$, consequently,

$$\frac{|AK|}{|KC|} = \frac{|AB|}{|CN|} = \frac{|AB|^2}{|BC|^2}$$

172. We have (Fig. 34) $\angle FE_1A = \angle EDF = \angle A$, hence, $|AF| = |E_1F|$, $\angle FE_1N =$

$\angle FDB = \angle C$, $\angle E_1FN = \angle A$. Consequently, $\triangle E_1FN$ is similar to $\triangle ABC$, $\frac{|AF|}{|FN|} = \frac{|E_1F|}{|FN|} = \frac{|AC|}{|AB|}$, $\angle AFN = 180° - \angle A$. Now, we can show

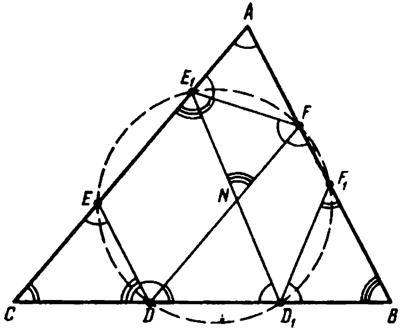


Fig. 34

that AN is symedian. To this end, consider the parallelogram ACA_1B; AA_1 bisects BC, the triangle ACA_1 is similar to the triangle AFN, hence $\angle NAF = \angle A_1AC$.

173. The Apollonius circle passing through the vertex B of the triangle ABC is the locus of points M for which $\frac{|AM|}{|MC|} = \frac{|AB|}{|BC|}$ (Problem 170, of Sec. 2, Solution). Consequently, if D is the point of intersection of this Apollonius circle and the circle circumscribed about the triangle ABC, then the straight line BD divides AC in the ratio $\frac{S_{BAD}}{S_{BCD}} = \frac{|AB| \cdot |AD|}{|CB| \cdot |CD|} = \frac{|AB|^4}{|CB|^2}$.

174. Let N denote the point of intersection of BQ and CD, O the centre of the circle, R its radius. Note that $\angle NBC = \frac{1}{2}\angle PMQ$. (If Q lies on the line segment NB, then $\angle NBC = 90°$ $\angle QBP = 90° - \frac{1}{2}\angle QOP = \frac{1}{2}PMQ$.) Hence, the triangles NBC and POM are similar, $|CN| = |BC| \frac{R}{|PM|} = R\frac{|PD|}{|PM|} = R\frac{|BP|}{|AB|} = \frac{1}{2}|BP| = \frac{1}{2}|CD|$.

175. Let H be the intersection point of the altitudes, O the centre of the circumscribed circle, B_1 the midpoint of CA. The straight line MN passes through K, which is the midpoint of BH, $|BK| = |B_1O|$. Prove that the line MN is parallel to OB (if $\angle C > \angle A$, then $\angle MKN = 2\angle MBN = \angle C - \angle A = \angle OBH$).

176. Let the straight line AM intersect for the second time the circle passing through B, C, and M at a point D. Then $\angle MDB = \angle MBA = \angle MAC$, $\angle MDC = \angle MBC = \angle MAB$. Consequently, $ABCD$ is a parallelogram.

177. From the solution of Problem 234 of Sec. 2 it follows that $\frac{|LM|}{|MK|} = \frac{|LN|}{|NK|}$. We may assume that l passes through N. Appyling the law of sines to the triangle NKP and replacing the ratio of sines by the ratio of the corresponding chords, we have: $|NP| = \frac{|NK| \sin \angle NKP}{\sin \angle KPN} = \frac{|NK| \sin \angle NKM}{\sin \angle KMA} = \frac{|NK|}{|KM|}|NM|$ and so forth.

178. Let O denote the centre of the inscribed circle, K and L the points of tangency with the sides AC and AB. The straight line passing through N parallel to BC intersects the sides AB and AC at points R and M. The quadrilateral $OKMN$ is an inscribed one ($\angle ONM = \angle OKM = 90°$); consequently, $\angle OMN = \angle OKN$, analogously, $\angle ORN = \angle OLN$, but $\angle OLN = \angle OKN$, hence $\angle ORN = \angle OMN$, and the triangle ORM is isosceles, ON is its altitude; thus $|RN| = |NM|$.

179. If $|BC| = a$, $|CA| = b$, $|AB| = c$, then, as is known (see Problem 18 in Sec. 1), $|MC| = \frac{a+b-c}{2}$. We draw through K a straight line parallel to AC, and denote its intersection points with AB and BC by A_1 and C_1, respectively. The circle inscribed in the triangle ABC is an escribed one for the triangle A_1BC_1 (it touches A_1C_1 and the extensions of BA_1 and BC_1). But the triangle A_1BC_1 is similar to the triangle ABC. Consequently, the circle escribed in ABC will touch AC at a point N; let R and L denote the points of tangency of the circle with the extensions of BA and BC, respectively. We have: $|BR| = |BL| = \frac{1}{2}(a + b + c)$, hence $|AN| = |AR| = |RB| - |BA| = \frac{a+b-c}{2} = |MC|$.

180. Draw through K a straight line parallel to BC. Let L and Q denote the points of intersection of the tangent at P with the line BC and the line constructed parallel to it, and N the point of intersection of AK and BC. Since $|CN| = |BM|$ (see Problem 179 of Sec. 2), it suffices to prove $|NL| = |LM|$; but $|PL| = |LM|$, hence, have to prove that $|PL| = |NL|$. Since the triangle PLN is similar to the triangle PQK, in which $|PQ| = |QK|$ we have $|PL| = |NL|$ and $|CL| = |LB|$.

181. Let M and N denote the points of intersection of the straight line LK and the straight lines l and CD. Then $|AM|^2 = |ML|\cdot|MK|$. From the similarity of the triangles KMB and DKN it follows that $|MK| = \frac{|KN|\cdot|MB|}{|DN|}$.

Since the triangles CNL and MLB are similar, we have: $|ML| = \frac{|LN|\cdot|MB|}{|CN|}$.

Thus, $|MK|\ |ML| = \frac{|KN|\cdot|LN|}{|CN|\cdot|DN|} \times |MB|^2 = |MB|^2$, that is $|MA|^2 = |MB|^2$, $|MA| = |MB|$.

182. Let B be a second common point of the circles, C the point on the straight line AB from which the tangents are drawn, and, finally, K the point of intersection of the straight lines MN and

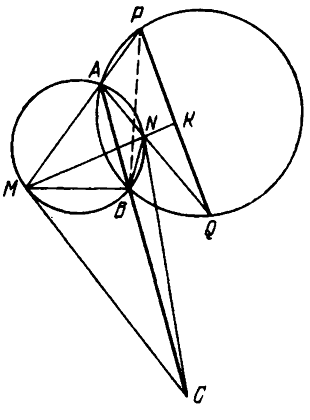


Fig. 35

PQ (Fig. 35). Making use of the law of sines and the result of Problem 234 in Sec. 1, we get: $\frac{|PM|}{|MA|} = \frac{|PM|}{\sin\angle PBM}\cdot\frac{\sin\angle PBM}{|MA|} = \frac{|BM|}{\sin\angle BPM}\times\frac{\sin\angle PBM}{|MA|} = \sqrt{\frac{|CB|}{|CA|}}\cdot\frac{\sin\angle PBM}{\sin\angle BPM}$. Thus, denoting the angle AMB by α and the angle APB

by β (α and β constant) we get: $\frac{|PM|}{|MA|}=\sqrt{\frac{|CB|}{|CA|}\cdot\frac{\sin(\alpha+\beta)}{\sin\beta}}$. Analogously, we find: $\frac{|AN|}{|NQ|}=\sqrt{\frac{|CA|}{|CB|}\cdot\frac{\sin\beta}{\sin(\alpha+\beta)}}$. But, by Menelaus' theorem (see Problem 45 in Sec. 2), $\frac{|PM|}{|MA|}\times\frac{|AN|}{|NQ|}\cdot\frac{|QK|}{|KP|}=1$. Hence, $|QK|/|KP|=1$.

183. Through the point M, we draw a straight line parallel to AC to intersect the straight lines BA and BC at points A_1 and C_1. We have: $\angle A_1KM = 90° - \angle DKM = 90° - \angle KBD = \angle BAD = \angle KA_1M$; consequently, KMA_1 is an isosceles triangle, and $|A_1M| = |MK|$. Analogously, $|MC_1| = |ML|$; but $|KM| = |ML|$, hence $|A_1M| = |MC_1|$, that is, the straight line BM bisects AC.

184. Let M denote the point of intersection of ND and AB, and P the point of intersection of the tangents to the circle at the points A and D.

Since the straight lines NC, AB, and PD are parallel, from the similarity of the corresponding triangles we get:

$$|AM| = |DP|\cdot\frac{|AN|}{|NP|}, \tag{1}$$

$$\frac{|MB|}{|NC|}=\frac{|MD|}{|ND|}=\frac{|AP|}{|ND|}, \tag{2}$$

$$|MB| = |NC|\frac{|AP|}{|NP|};$$

but $|DP| = |AP|$, $|NC| = |AN|$. Consequently, the right-hand sides of the expressions (1) and (2) are equal to each other, that is, $|AM| = |MB|$.

185. We assume that D is the midpoint of CB, and AD intersects the circle for the second time at a point K. Let us prove that the tangents to the circle at the points B and C intersect on the straight line MK.

Consider the quadrilateral $CMBK$. For intersection point of the tangents to the circle at the points C and B to lie on the diagonal MK, it is necessary and sufficient (see Problem 234 of Sec. 1) that $\frac{|CM|}{|CK|} = \frac{|MB|}{|BK|}$; but $\frac{|CM|}{|CK|} = \frac{|AB|}{|CK|} = \frac{|BD|}{|DK|} = \frac{|CD|}{|DK|} = \frac{|AC|}{|BK|} = \frac{|MB|}{|BK|}$. (In the first and last equalities we have used the fact that $|CM| = |AB|$, $|AC| = |MB|$ since AM is parallel to CB, in the second and fourth equalities—that the triangle ABD is similar to the triangle CDK, and the triangle ADC to the triangle KDB, in the third, the fact that AD is a median.)

186. Let O denote the centre of the circle, N_1, M_1, P_1, R_1 the points symmetric to the points N, M, P, R with respect to the straight line OA, respectively, K the point of intersection of the straight lines N_1R_1 and QS. We have to prove that the points R_1, S, and K coincide. The points N_1, M_1, and B lie on the same straight line symmetric to the straight line NMC; the points N_1, P_1, R_1 also lie on a straight line symmetric to the straight line NPR (Fig. 36). The points B, N_1, Q, and K lie on one circle since $\angle BN_1K = \angle M_1N_1P_1 = \angle MNP = \angle PQM = \angle BQK$. The points B, N_1, Q, and R_1 are also on one circle since $\angle N_1R_1B = \angle N_1P_1P = \angle N_1QP = \angle N_1QB$. Consequently, the five points B, N_1, Q, R_1, and K are located on the same circle; but the points N_1, R_1, and K are collinear, hence R_1 and K coincide.

187. Let us confine ourselves to the case when ABC is an acute triangle. Consider the parallelogram A_1MON (M and N on A_1B_1 and A_1C_1, respectively). Since A_1O forms with A_1C_1 and A_1B_1

angles of $(90° - \angle B)$ and $(90° - \angle C)$, we have

$$\frac{|A_1M|}{|A_1N|} = \frac{|A_1M|}{|MO|} = \frac{\cos B}{\cos C} = \frac{|A_1L|}{|A_1K|}.$$

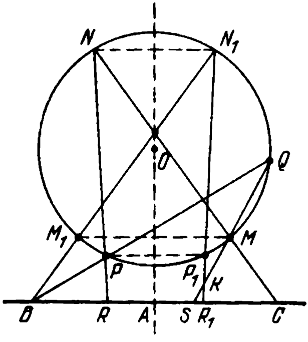


Fig. 36

188. The statements of the problem follow from the fact: if a circle is constructed on each side of the triangle so that the sum of the angular values of their arcs (located on the same side with the triangle) is equal to 2π, then these circles have a common point.

189. Take the points E_1 and F_1 symmetric to the points E and F with respect to AB. Then the problem is reduced to a particular case of Problem 186, Sec. 2.

190. On the extension of AC beyond the point C, we take a point M such that $|CM| = |CB|$; then E is the centre of the circle circumscribed about the triangle AMB ($|AE| = |BE|$, $\angle AEB = \angle ACB = 2\angle AMB$). Hence it follows that F is the midpoint of AM, and DF bisects the perimeter of the triangle ABC. In addition, DF is parallel to BM, and BM is parallel to the bisector of the angle C of the triangle ABC that is, DF

is the bisector of the angle D of the triangle DKL, where K and L are the midpoints of AC and CB, respectively.

191. Let the straight line intersect the sides AC and AB of the triangle ABC at points M and N. Denote: $|AM| + |AN| = 2l$. The radius of the circle with centre on MN touching AC and AB is equal to S_{AMN}/l, and, by hypothesis, $S_{AMN}/l = S_{ABC}/p = r$, where p is the semiperimeter and r the radius of the circle inscribed in the triangle ABC.

192. Prove that in the homothetic transformation with centre at M and the ratio of similitude of $-1/2$ the point N goes into I (obviously, this homothetic transformation carries the point I into S). Let ABC be the given triangle, A_0, B_0, and C_0 the midpoints of the sides BC, CA, and AB, respectively, A_1 a point on the side BC such that AA_1 divides the perimeter into two equal parts. It is easy to see that A_1 is the point of tangency with the side BC of the escribed circle which also touches the extensions of the sides AB and AC, A_2 the point of tangency of the inscribed circle with the side BC. We have: $|BA_2| = |CA_1|$. We erect at point A_2 a perpendicular to BC and denote by D the point of its intersection with AA_1. Repeating the reasoning for the solution of Problem 179 of Sec. 2, we prove that $|A_2I| = |ID|$. Consequently, the straight line A_0I is parallel to AA_1. If we carry out the homothetic transformation mentioned at the beginning, then the straight line AA_1 goes into the line A_0I. In similar fashion, two other straight lines bisecting the perimeter go into B_0I and C_0I, respectively. Hence, all these three lines intersect at such a point N which goes into I in this transformation. This implies the statement of the problem.

193. (a) Using the formulas $r = \frac{S}{p}$, $R = \frac{abc}{4S}$, $S = \sqrt{p(p-a)(p-b)(p-c)}$, where S is the area

of the triangle ABC, we easily prove the given relationship.

(b) Use Leibniz's formula (Problem 140 in Sec. 2), taking the centre of the circumscribed circle as M.

(c) Use Leibniz's formula (Problem 140 of Sec. 2), taking the centre of the inscribed circle as M. To compute, for instance, $|MA|^2$, we drop a perpendicular MK on AB; we have: $|MK| = r$ $|AK| = p - a$; hence, $|AM|^2 = (p - a)^2 + r^2$, $|MB|^2$ and $|MC|^2$ are computed in a similar way. For simplifying the right-hand side, use the result of Item (a).

(d) Let M denote the intersection point of the bisector of the angle B and the circumscribed circle. If $|IO| = d$, then $|BI|\,|IM| = R^2 - d^2$. The triangle ICM is isosceles ($|IM| = |CM|$) since $\angle CIM = \frac{1}{2}(\angle B + \angle C)$ and $\angle ICM = \frac{1}{2}(\angle B + \angle C)$. Consequently, $R^2 - d^2 = |BI|\cdot|IM| = |BI|\cdot|CM| = \frac{r}{\sin\frac{B}{2}}\cdot 2R\sin\frac{B}{2} = 2Rr$.

(e) Can be proved in much the same way as Item (d).

(f) The distance between the projections of I and I_a on AC is a. We take a point K such that $IK \parallel AC$, $I_aK \perp AC$. In the right triangle IKI_a, we have: $\angle KII_a = \frac{1}{2}\angle A$, $|IK| = a$, $|I_aK| = r_a - r$. Thus, $|II_a|^2 = \frac{|IK|^2}{\cos^2\frac{A}{2}} = \frac{a}{\sin A}2\,|IK| \times \tan\frac{A}{2} = 4R(r_a - r)$.

194. Through the point O, we draw straight lines parallel to AB and AC and denote by L and K the intersection points of these lines with the

perpendiculars dropped from I_a on AB and AC, respectively. Let us prove that the triangles AB_1C_1 and OLK are similar. We have: $\angle B_1AC_1 = \angle LOK$, $|AB_1| = \frac{bc}{c+a}$ $|AC_1| = \frac{bc}{c+a}$, $|OL| = p - \frac{c}{2} = \frac{1}{2}(a+b)$, $|OK| = p - \frac{b}{2} = \frac{1}{2}(a+c)$; thus, $\frac{|AB_1|}{|OL|} = \frac{|AC_1|}{|OK|} = \frac{2bc}{(c+a)(b+a)}$. But OI_a is the diameter of the circle circumscribed about the triangle OLK. Consequently, $|B_1C_1| = \frac{2bc}{(c+a)(b+a)} |LK| = \frac{2bc}{(c+a)(b+a)} |OI_a| \sin A = \frac{abc}{(c+a)(b+a)R} \cdot |OI_a|$.

196. Prove that the area Q_a of the triangle with vertices at the points of tangency of the escribed circle centred at I_a can be computed by the formula $Q_a = S_{ABC}\frac{r_a}{2R} = \frac{S^2_{ABC}}{2R(p-a)}$, where the notation is the same as in Problem 193 of Sec. 2. Analogous formulas can be obtained for the areas of other triangles. (See the solution of Problem 240 of Sec. 1.)

197. Let O be the centre of the circle circumscribed about the triangle ABC, B_1 the midpoint of AC, N the point of tangency of the inscribed circle with AC. Then $|AN| = p - a$, $|CN| = p - c$ (see Problem 18 in Sec. 1), $|ON|^2 = |OB_1|^2 + |B_1N|^2 = |AO|^2 - |AB_1|^2 + |B_1N|^2 = R^2 - \frac{b^2}{4} + \left(p - a - \frac{b}{2}\right)^2 \quad R^2$ $(p-a)(p-c)$. We then determine the squares of the distances to the other points of tangency and add them together to get the desired sum; it is equal to $3R^2 - (p-a)(p-c) - (p-c) \times (p-b) - (p-b)(p-a) = 3R^2 - M$. Making use of Hero's formula for the area of a triangle and the formulas $S = pr$ and $S = abc/4R$, we get:

$r^2 = (p-a)(p-b)(p-c)/p$, $4Rr = abc/p$. Adding together the last equalities and using the identity $(p-a)(p-b)(p-c) + abc = p((p-a) \times (p-b) + (p-b)(p-c) + (p-c)(p-a)) = pM$, we find $M = 4Rr + r^2$.

Answer: $3R^2 - 4Rr - r^2$.

198. The product of the lengths of the line segments from the vertex A of the triangle ABC to the points of intersection of the side AB with the given circle is equal to the product for the side AC. Each of these line segments can be readily expressed in terms of the sides of the triangle and the chords under consideration. Thus, we obtain a system of three equations enabling us to express the chords in terms of the sides of the triangle. To avoid the looking over of variants, it is convenient to choose a certain direction of traversing the triangle and regard the line segments to be directed and their lengths to be arbitrary real numbers.

199. Let K_1 and L_1 be points on BC and BA, respectively, such that $K_1K \parallel L_1L \parallel B_1B$. It suffices to prove that the triangles BK_1K and BL_1L are similar, that is, $\frac{|BK_1|}{|K_1K|} = \frac{|BL_1|}{|L_1L|}$. We have: $\frac{|BK_1|}{|BA_1|} = \frac{|B_1K|}{|B_1A_1|}$, $\frac{|K_1K|}{|BB_1|} = \frac{|A_1K|}{|B_1A_1|}$, and by the property of an angle bisector (Problem 9 in Sec. 1), $\frac{|BK_1|}{|K_1K|} = \frac{|B_1K|}{|A_1K|} \cdot \frac{|BA_1|}{|BB_1|} = \frac{|CB_1|}{|CA_1|} \times \frac{|BA_1|}{|BB_1|} = \frac{c}{b} \frac{|CB_1|}{|BB_1|} = \frac{ca}{(c+a)|BB_1|}$. The last expression is symmetric with respect to a and c, and, hence, it is also equal to $\frac{|BL_1|}{|L_1L|}$.

200. Let $\angle KAL = \angle KLA = \varphi$, $\angle KCL = \angle LKC = \psi$. Then $\angle BKL = 2\varphi$, $\angle BLK = 2\psi$, $2\varphi + 2\psi = 180° - \angle B$. If Q is the point of intersection of AL and KC, then $\angle AQC = 180° - (\varphi + \psi) = 90° + \frac{1}{2}\angle B$. Through M, we draw a

straight line parallel to BC to intersect KC at a point N, then MQ is the bisector of the angle AMN and $\angle AQN = 90° + \frac{1}{2}\angle B$. Hence it follows that Q is the intersection point of the angle bisectors of the triangle AMN (see Problem 46 in Sec. 1); hence the triangle AMN is similar to the triangle KBL, and the triangle KMN is similar to the triangle KBC. Let $|AK| = |KL| = |LC| = x$, $|AM|$ y, $|MN|$ z. Then $\frac{z}{a-x} = \frac{y}{c-x}$, $\frac{y-x}{c-x} = \frac{z}{a}$, whence $y = a$.

201. Let B_1 be the midpoint of AC. Extend the angle bisector to intersect the perpendicular, erected at the point B_1 to AC, at a point B_2. The point B_2 lies on the circumscribed circle. Through the point M, we draw a perpendicular to AC; let L be the point of its intersection with AC, K that with BB_1, then $|KM| = |ML|$. We draw through the point K a straight line parallel to AC to intersect the straight lines AB and BC at points D and E, respectively. If G and F are the projections of D and E, respectively, on AC, then M is the centre of the rectangle $GDEF$, the triangle DME being similar to the triangle AB_2C (the triangle DME is obtained from the triangle AB_2C by means of a homothetic transformation with centre at B). We have: $\cot \angle MCL = \frac{|LC|}{|ML|} = \frac{|LF|}{|ML|} + \frac{|FC|}{|ML|} = \frac{|AB_1|}{|B_1B_2|} + 2\frac{|FC|}{|EF|} = \cot\frac{B}{2} + 2\cot C$. If now B' is the foot of the angle bisector, P and T are, respectively, the projection of N and B' on BC, then $\cot \angle NCB = \frac{|PC|}{|NP|} = \frac{|PT|}{|NP|} + \frac{|TC|}{|NP|} = \frac{|BP|}{|NP|} + 2\frac{|TC|}{|B'T|} = \cot\frac{B}{2} + 2\cot C$, that is, $\angle MCA = \angle NCB$.

202. (a) This well-known problem has many proofs. Consider one of them based on the following test for the congruence of triangles. Two triangles are congruent by one equal side, an equal angle opposite to this side, and an equal bisector of this angle. Let us prove this test. Consider two triangles ACB and ACB_1 in which $\angle B = \angle B_1$ (B and B_1 lying on the same side of AC). These triangles have a common circumscribed circle. We may assume that B and B_1 lie on the same side of the diameter of this circle which is perpendicular to AC. Let the bisector of the angle B intersect AC at a point D, and the bisector of the angle B_1 at a point D_1, M the midpoint of AC, N the midpoint of the arc AC not containing the points B and B_1. The points B, D, and N are collinear, as well as B_1, D_1, and N. Let B and B_1 be non-coincident, and, hence, D and D_1 are also non-coincident. Suppose that $|MD| > |MD_1|$; then $|BN| < |B_1N|$, $|DN| > |D_1N|$. Consequently, $|B_1D_1| = |B_1N| - |ND_1| > |BN| - |ND| = |BD|$ which is a contradiction. Let now the bisector AA_1 in the triangle ABC be equal to the bisector CC_1. Apply the test just proved to the triangles BAA_1 and BCC_1.

(b) If both bisectors of the external angles A and C of the triangle ABC are found inside the angle B, then the proof can be carried out just in the same manner as in Item (a).

Let these bisectors be situated outside the angle B. We shall assume that $|BC| > |BA|$. Take on CB a point B_1 such that $|CB_1| = |AB_1|$. Let $\angle B_1AC = \angle BCA = \alpha$, $\angle B_1AB = \varphi$, L the intersection point of the bisector of the external angle C and AB, M the intersection point of the bisector of the external angle A and CB. The rest of the notations are clear from Fig. 37. By hypothesis, $|CL| = |AM|$, in addition, $|CL_1| = |AM_1|$, since B_1AC is an isosceles triangle, $|CM_1'| = |AM|$ since the triangles CL_1M_1' and AM_1M are congruent. Further, $|CM_1''| > |CM_1'|$, since

$\angle M_1''M_1'C > \angle M_1'CA > 90°$. On the other hand, the points C, A, L, and M_1'' lie on the same circle in which the acute angle subtended by LC ($\angle LAC$) is greater than the acute angle subtended by $M_1''C$. Hence, $|AM| = |CM_1'| < |CM_1''| < |CL|$. But this is a contradiction.

In the general case, the equality of the bisectors of the external angles does not imply that the

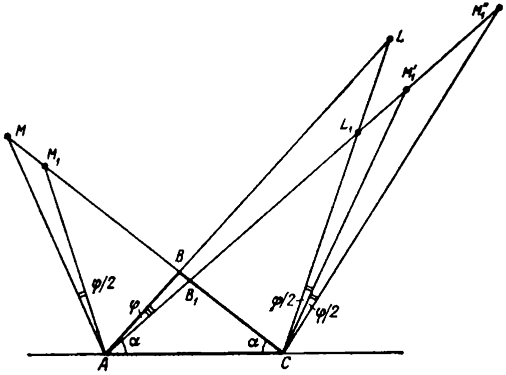


Fig. 37

triangle is isosceles. Problem 256 of Sec. 1 gives an example of such a triangle.

203. Let ABC be the given triangle, AA_1, BB_1, CC_1 the angle bisectors. If $|A_1B_1| = |A_1C_1|$, then either $\angle A_1B_1C = \angle A_1C_1B$ (in this case the $\triangle ABC$ is isosceles) or $\angle A_1B_1C + \angle A_1C_1B = 180°$. In the second case, we rotate the triangle A_1B_1C about the point A_1 through an angle $B_1A_1C_1$. As a result, the triangles A_1C_1B and A_1B_1C turn out to be applied to each other and form a triangle similar to the triangle ABC. If the sides of the triangle ABC are a, b, and c, then the sides of the

obtained triangle are equal to $\frac{ac}{b+c}$, $\frac{ab}{b+c}$ and $\frac{ac}{a+b}+\frac{ab}{a+c}$. Bearing in mind that the triangles are similar, we get:

$$\frac{c}{a+b}+\frac{b}{a+c}=\frac{a}{b+c}$$
$$\Leftrightarrow b^3+c^3-a^3+b^2c+b^2a+c^2b+c^2a-a^2b-a^2c+abc=0. \qquad (1)$$

Let us denote $\cos\angle BAC=x$. By the law of cosines, $b^2+c^2-a^2=2bcx$. Multiplying the last equality, in succession, by a, b, and c and subtracting it from (1), we get:

$$2x(a+b+c)+a=0 \Leftrightarrow a=-\frac{2(b+c)x}{2x+1}.$$

Since $0<a<b+c$, we have

$$-\frac{1}{4}<x<0. \qquad (2)$$

Expressing a in the law of cosines in terms of b, c, and x and denoting $b/c=\lambda$, we obtain for λ the equation $(4x+1)\lambda^2-2\lambda(4x^3+8x^2+x)+4x+1=0$. For this equation to have a solution ($\lambda>0$, $\lambda\neq 1$) under the conditions (2), the following inequalities must be fulfilled:

$$4x^3+8x^2+x>0, \qquad (3)$$

$$\frac{1}{4}D=(4x^3+8x^2+x^2)-(4x+1)^2$$
$$=(2x+1)^2(x+1)(2x-1)(2x^2+5x+1)>0, \qquad (4)$$

where D is the discriminant of the quadratic equation. The system of inequalities (2), (3), (4) is true for $-\frac{1}{4}<x<\frac{\sqrt{17}-5}{4}$.

Thus, the original triangle is not necessarily isosceles. But it has been proved that it can be isosceles if one of the angles of the original triangle is obtuse and its cosine lies in the interval $\left(-\frac{1}{4}, \frac{\sqrt{17}-5}{4}\right)$, which corresponds approximately to an angle from 102°40′ to 104°28′ If $x = -1/4$, then the constructed triangle degenerates; for $x = \frac{\sqrt{17}-5}{4}$ we have: $\angle A_1B_1C = \angle A_1C_1B = 90°$, that is, the two cases considered at the beginning of the solution coincide for this size of the angle.

204. Let M denote the point of intersection of AD and KL:

$$\frac{|KM|}{|ML|} = \frac{S_{AKD}}{S_{ALD}} = \frac{\frac{1}{2}|AK|\cdot|AD|\sin\angle KAD}{\frac{1}{2}|DL|\cdot|AD|\sin\angle ADL}$$

$$= \frac{|AK|\cdot|CD|}{|DL|\cdot|AF|}.$$

(We have used the fact that the sines of the inscribed angles are proportional to the chords.) Analogously, if M_1 is the point of intersection of BE and KL, then we get: $\frac{|KM_1|}{|M_1L|} = \frac{|BK|\cdot|EF|}{|LE|\cdot|BC|}$. But from the similarity of the triangles AKF and BKC, and CLD and FLE, we have $\frac{|AK|}{|AF|} = \frac{|BK|}{|BC|}$, $\frac{|CD|}{|DL|} = \frac{|FE|}{|LE|}$; multiplying these equalities, we get: $\frac{|KM|}{|ML|} = \frac{|KM_1|}{|M_1L|}$, that is, M and M_1 coincide. **Remark.** We can show that the statement of the problem is retained if A, B, C, D, E, and F are six arbitrary points on the circle. Usually, Pascal's theorem is formulated as follows: if A, B, C, D, E, F are points on a circle, then the three intersection points of pairs of straight lines AB and DE, BC and EF, CD and FA lie on a straight line.

205. Let N be the point of intersection of the straight line A_2A_1 and the circle, N being distinct from A_2. Apply Pascal's theorem to the hexagon $ABCC_2NA_2$ which is possibly self-intersecting (Problem 204 in Sec. 2). Intersection points of two pairs of straight lines AB and C_2N, BC and NA_2 (the point A_1), CC_2 and AA_2 (the point M) lie on one straight line. Consequently, AB and C_2N intersect at a point C_1.

206. Let the given mutually perpendicular straight lines be the x- and y-axes of a rectangular coordinate system. Then the altitudes of the triangle lie on the lines $y = k_i x$ $(i = 1, 2, 3)$; in this case the sides of the triangle must have slopes equal to $-\frac{1}{k_i}$, and given the condition that the vertices (x_i, y_i) belong to the altitudes we find the ratios of absolute terms c_i in the equations of the sides $k_i y + x = c_i$: $c_1 = k_1 y_3 + x_3$, $c_2 = k_2 y_3 + x_3$, $y_3 = k_3 x_3 \Rightarrow \frac{c_1}{c_2} = \frac{k_1k_3+1}{k_1k_3+1}$, etc. With a properly chosen unit of length, we may take $c_i = \frac{k_i}{k+k_i}$, where $k = k_1k_2k_3$. The points of intersection of the line $k_i y + x = \frac{k_i}{k+k_i}$ with axes: $\left(0, \frac{1}{k+k_i}\right)$ and $\left(\frac{k_i}{k+k_i}, 0\right)$, the midpoint (P_i) of the line segment between them: $\left(\frac{k_i}{2(k+k_i)}, \frac{1}{2(k+k_i)}\right)$. The slope of the straight line P_1P_2 is equal to $\left(\frac{1}{2(k+k_2)} - \frac{1}{2(k+k_1)}\right) \div \left(\frac{k}{2(k+k)_2} - \right.$

$\left.\frac{k_1}{2(k+k_1)}\right) = (k_1 - k_2) \div (kk_2 - kk_1) = -\frac{1}{k}$ The slopes of the lines P_2P_3 and P_3P_1 are just the same. Therefore the points P_1, P_2, P_3 lie on a straight line (its equation: $ky + x = 1/2$).

Remark 1. Joining the point H of intersection of the altitudes of the triangle to the points P_1, P_2, and P_3 with straight lines, we get an intersecting consequence. Let α_1, α_2, and α_3 be the angles of the triangle enumerated anticlockwise, a_1, a_2, and a_3 the straight lines containing the sides opposite these angles; three straight lines p_1, p_2, and p_3 pass through the point H so that the angles between the pairs p_2 and p_3, p_3 and p_1, p_1 and p_3 (measured anticlockwise) are equal to α_1, α_2, α_3. Then the points of intersection of the pairs p_1 and a_1, p_2 and a_2, p_3 and a_3 lie on a straight line. The particular cases of this theorem are left to the reader (many of these geometrical facts being elegant, and far from obvious).

Remark 2. In our problem, instead of the midpoints of the line segments cut out on the sides of the triangle, we might have taken the points dividing them in the same ratios. These points will also turn out to be collinear.

207. To determine the angles of the triangle $A_1B_1C_1$, take advantage of the fact that the points P, A_1, B_1, and C_1 lie on a circle (the same is true for the other fours of points). If the point P lies inside the triangle ABC, then $\angle A_1C_1B_1 = \angle A_2C_2B_2 = \angle APB - \angle ACB$. For a scalene triangle ABC there exist eight distinct points P such that the corresponding triangles $A_1B_1C_1$ and $A_2B_2C_2$ are similar to the triangle ABC (the triangle $A_2B_2C_2$ being congruent to it). Of these eight points, six lie inside the circle circumscribed about the triangle ABC, and two outside it.

208. The straight lines under consideration are the middle perpendiculars to the sides of the triangle $A_1B_1C_1$.

209. Notation: ABC is the given triangle, M the point situated at a distance d from the centre of the circle circumscribed about the triangle ABC, A_1, B_1, and C_1 the feet of the perpendiculars dropped from M on BC, CA, and AB; A_2, B_2, C_2 the intersection points of AM, BM, CM with the circle circumscribed about the triangle ABC, respectively, a, b, and c the sides of the triangle ABC, a_1, b_1, c_1 and a_2, b_2, c_2 the sides of the triangles $A_1B_1C_1$ and $A_2B_2C_2$, respectively; S, S_1, and S_2 the areas of those triangles, respectively. We have:

$$a_1 = |AM| \sin A = |AM| \frac{a}{2R}. \tag{1}$$

The sides b_1 and c_1 are found in a similar way. From the similarity of the triangles B_2MC_2 and BMC, we get:

$$\frac{a_2}{a} = \frac{|B_2M|}{|CM|} = \frac{|C_2M|}{|BM|}. \tag{2}$$

Analogous ratios are obtained for $\frac{b_2}{b}$ and $\frac{c_2}{c}$. The triangles $A_1B_1C_1$ and $A_2B_2C_2$ are similar (see Problem 207 of Sec. 2); in addition,

$$\frac{S_2}{S} = \frac{a_2b_2c_2}{abc}. \tag{3}$$

Bearing all this in mind, we have:

$$\left(\frac{S_1}{S}\right)^3 = \frac{S_1^3}{S_2^3} \cdot \frac{S_2^3}{S^3} = \frac{a_1^2b_1^2c_1^2}{a_2^2b_2^2c_2^2} \cdot \frac{a_2^3b_2^3c_2^3}{a^3b^3c^3}$$

$$= \left(\frac{1}{4R^2}\right)^3 \frac{|AM|^2\,|BM|^2\,|CM|^2\,a^2b^2c^2}{a^3b^3c^3} \cdot a_2b_2c_2$$

$$= \left(\frac{1}{4R^2}\right)^3 |AM|^2\,|BM|^2\,|CM|^2$$

$$\times \frac{|B_2M|}{|CM|} \frac{|C_2M|}{|AM|} \frac{|A_2M|}{|BM|} = \left(\frac{1}{4R^2}|R^2-d^2|\right)^3$$

(In the second equality we have used the similarity of the triangles $A_1B_1C_1$ and $A_2B_2C_2$ and the equality (3), in the third the formulas (1), in the fourth the formulas (2).) **Remark.** For $d = R$ the area of the triangle formed by the feet of the perpendiculars turns out to be equal to zero, that is, these feet are situated on a straight line. This line is Simson's line (see Problem 153 in Sec. 2).

210. The statement follows from a more general fact: if on the sides of the triangle circles are constructed so that their arcs located outside the triangle are totally equal to 4π or 2π, then those circles have a common point (in our case, as such a triangle, we may take the triangle with vertices at the midpoints of the sides of the triangle ABC and prove that the three circles passing through the midpoints of AB, AC, and AD; BA, BC, and BD; CA, CB, and CD have a common point).

211. The statement is based on the following fact. Let an arbitrary circle intersect the sides of the angle with vertex N at points A, B and C, D; the perpendiculars erected at the points A and D to the sides of the angle intersect at a point K, and the perpendiculars erected at the points B and C intersect at a point L. Then the straight lines NK and NL are symmetric with respect to the bisector of this angle. Indeed, $\angle ANK = \angle ADK$ (the points A, K, D, and N lying on the same circle). In similar fashion, $\angle LNC = \angle LBC$. Then $\angle ADK = 90° - \angle ADN = 90° - \angle NBC = \angle LBC$. (The quadrilateral $ABCD$ was supposed to be non-self-intersecting.)

212. Let A, B, C, and D be the given points, D_1 the point of intersection of the straight lines which are symmetric to AD, BD, and CD with respect to the corresponding angle bisectors of the triangle ABC. It was proved in the preceding problem that the pedal circles of the points D and D_1 with respect to the triangle ABC coincide. Let the straight lines symmetric to BA, CA, and DA with respect to the corresponding angle bisectors of the triangle BCD intersect at a point A_1. It is easy to prove that the points A_1 and D_1 are symmetric with respect to the straight line CB. Consequently, the pedal circles of the points D (or D_1) with respect to the triangle ABC as well as the points A (or A_1) with respect to the triangle BCD pass through the midpoint of D_1A_1. On having determined the points B_1 and C_1 in a similar way, we see that each of the pedal circles under consideration passes through the midpoints of the corresponding line segments joining the points A_1, B_1, C_1, and D_1. Thus, our problem has been reduced to Problem 210 of Sec. 2.

213. Let B_2 and C_2 be the points diametrically opposite to the points B and C, M the second point of intersection of B_2B_1 and the circle circumscribed about the triangle ABC, C_1' the point of intersection of AB and C_2M. By Pascal's theorem in Problem 204 in Sec. 2 applied to the hexagon AB_2CMBC_2, the points O (the centre of the circle), B_1 and C_1' lie on one straight line, that is, C_1' coincides with C_1. But $\angle BMB_1 = \angle BMB_2 = 90°$, $\angle CMC_1 = \angle CMC_2 = 90°$; hence, M is one of the intersection points of the circles with the diameters BB_1 and CC_1. Let N be the second point of intersection of those circles. Their common chord MN contains the point H of intersection of the altitudes of the triangle ABC (Problem 19 in Sec. 2). If BB_0 is the altitude of the triangle ABC, then $|MH|\cdot|HN| = |BH|\cdot|HB_0|$. Hence (see Problem 164 in Sec. 2), N lies on the nine-point circle of the triangle ABC.

218. Let the radius of the circle be r, and the angles between the neighbouring radii drawn to the points of tangency, in the order of traverse, are equal to 2α, 2β, 2γ, 2δ $(\alpha + \beta + \gamma + \delta = \pi)$. Then

$$S = r^2 (\tan \alpha + \tan \beta + \tan \gamma + \tan \delta). \qquad (1)$$

The sides of the quadrilateral (we are going to find one of them) are equal to $r(\tan \alpha + \tan \beta) =$

$r\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$ and so forth. Since $\sin(\alpha+\beta)=\sin(\gamma+\delta)$, $\sin(\beta+\gamma)=\sin(\alpha+\delta)$, the formula given in the hypothesis is reduced to

$$S=r^2\frac{\sin(\alpha+\beta)\sin(\beta+\gamma)\sin(\gamma+\alpha)}{\cos\alpha\cos\beta\cos\gamma\cos\delta} \tag{2}$$

It remains to prove the equality of the right-hand members of (1) and (2) provided that $\alpha+\beta+\gamma+\delta=\pi$.

219. Prove that $S_{BNA}=S_{BMC}+S_{AMD}$. If $\frac{|AM|}{|AB|}=\frac{|CN|}{|ND|}=\lambda$, then $S_{BMC}=(1-\lambda)S_{BAC}$, $S_{AMD}=\lambda S_{BAD}$. On the other hand, denoting the distances from C, D, and N by h_1, h_2, and h, respectively, we find that $h=\lambda h_1+(1-\lambda)h_2$. Consequently, $S_{ABN}=\frac{1}{2}|AB|\cdot h=\lambda\frac{1}{2}|AB|h_1+(1-\lambda)\frac{1}{2}|AB|h_2=\lambda S_{ABD}+(1-\lambda)S_{BAC}=S_{AMD}+S_{BMC}$.

221. The angles between the sides and also between the sides and diagonals of the quadrilateral Q_2 are expressed in terms of the angles between the sides and between the sides and diagonals of the quadrilateral Q_1. (The diagonals of the quadrilateral Q_2 are perpendicular to the corresponding diagonals of the quadrilateral Q_1 and pass through their midpoints.)

222. Consider the parallelograms $ABMK$ and $DCML$ and prove that KL divides DA in the same ratio as the point N, and the straight line MN is the bisector of the angle KML.

223. First of all, prove that the diagonals of the given quadrilateral are bisected by the point of intersection, that is, that the quadrilateral is a parallelogram. Let $ABCD$ be the given quadrilateral, O the point of intersection of the diagonals. Suppose that $|BO|<|OD|$, $|AO|\leqslant|OC|$; consider

the triangle OA_1B_1 symmetric to the triangle OAB with respect to the point O; obviously, the radius of the circle inscribed in the triangle OA_1B_1 is less than the radius of the circle inscribed in the triangle OCD, while, by hypothesis, they are equal. Thus, O is the midpoint of both diagonals. We prove that all the sides of the quadrilateral are equal. We use the formula $S = pr$ (S the area, p the semiperimeter, r the radius of the circle inscribed in the triangle). Since the areas and the radii of the circles inscribed in the triangles ABO and BOC are equal, their perimeters are also equal, that is, $|AB| = |BC|$.

224. Using the solution of the preceding problem, prove that the diagonals of the quadrilateral are bisected by the point of their intersection.

225. The hypothesis implies that $ABCD$ (Fig. 38) is a convex quadrilateral. Consider the

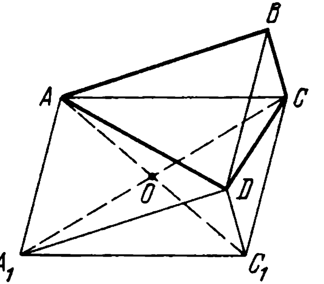


Fig. 38

parallelogram ACC_1A_1 in which the sides AA_1 and CC_1 are equal to each other and parallel to the diagonal BD. The triangles ADA_1, CDC_1, and C_1DA_1 are congruent to the triangles ABD, BCD, and ABC, respectively. Consequently, the line segments joining D to the vertices A, C, C_1, and A_1 separate the parallelogram into four triangles in which the radii of the inscribed circles are equal. If O is the intersection point of the diagonals of the parallelogram ACC_1A_1, then D must coincide with O (for instance, if D is inside the triangle COC_1, then the radius of the circle inscribed in the triangle ADA_1 is greater than the radius of the circle inscribed in the triangle AOA_1, and the more so in the triangle CDC_1). Thus, $ABCD$ is a parallelogram, but, in addition, it follows from Problem 223 of Sec. 2 that ACC_1A_1 is a rhombus, that is, $ABCD$ is a rectangle.

226. The necessary and sufficient condition for all four items to be fulfilled is the equality $|AB|\cdot|CD| = |AD|\cdot|BC|$. For Items (a) and (b) it follows from the theorem on the bisector of an interior angle of a triangle, for Items (c) and (d) from the result of Problem 234 of Sec. 1.

227. Let $ABCD$ be the given quadrilateral. We assume that the angles A and D are obtuse, B and

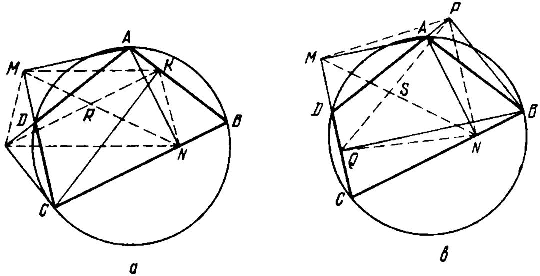


Fig. 39

C are acute. Denote the feet of the perpendiculars dropped from the vertex A by M and N, and from the vertex C by K and L (Fig. 39, a), R the point of intersection of MN and LK. Note that the points A, K, N, C, L, and M lie on one and the same circle of diameter AC. Let us show that

$MK \parallel LN$: $\angle MKL = \angle MAL = 90° - \angle B = \angle KCB = \angle KLN$. Thus, $\frac{|MR|}{|RN|} = \frac{|MK|}{|LN|} = \frac{\sin \angle MCK}{\sin \angle LAN} = \frac{\sin(\angle C + \angle B - 90°)}{\sin(\angle A + \angle B - 90°)} = \frac{\cos(\angle A - \angle B)}{\sin(\angle A + \angle B - 90°)}$. Let now P and Q be the feet of the perpendiculars dropped from the vertex B, and S is the point of intersection of MN and PQ (Fig. 39, b). Since $\angle PNB = \angle PAB = \angle C$, PN is parallel to DC, that is, $MQNP$ is a trapezoid ($ANBP$ is an inscribed quadrilateral with diameter AB). Thus $\frac{|MS|}{|SN|} = \frac{|MQ|}{|PN|} = \frac{|AB| \cos(\angle A + \angle D - 180°)}{|AB| \sin(\angle B + \angle A - 90°)} = \frac{\cos(\angle A - \angle B)}{\sin(\angle A + \angle B - 90°)}$. (We have used the fact that MQ is the projection of AB on DC; the angle between AB and DC is equal to $\angle A + \angle D - 180°$.) Thus the points R and S divide MN in the same ratio, that is, they coincide; hence, the three straight lines intersect at one point. Now, it is easy to show that all the four straight lines intersect at the same point.

228. Let us find the ratio in which BC divides MN. This ratio is equal to the ratio $\frac{S_{MCB}}{S_{CBN}} = \frac{|MC| \cos \angle BCD}{|BN| \cos \angle CBA}$. Analogously, the ratio in which AD divides MN is equal to $\frac{|AM| \cos \angle BAD}{|ND| \cos \angle ADC}$. But these ratios are equal to each other since $\angle BCD = \angle BAD$, $\angle CBA = \angle CDA$, and the triangle AMC is similar to the triangle DNB.

229. Take M_1 such that $BCMM_1$ is a parallelogram; M_1 lies on the circle passing through the points B, M, and A. Since $|AM_1| = |DM|$ ($ADMM_1$ is also a parallelogram), the triangles CDM and BAM_1 are congruent, that is, the radius

of the circle circumscribed about the triangle CDM is equal to R. The radius of the circle circumscribed about the triangle ADM is also equal to R.

230. Let K and L denote the points of tangency of the given circle with the straight lines AB and AD. Let, for definiteness, K and L be situated inside the line segments AB and AD. On the straight line CB, we take a point P such that $|BP| = |BK|$, B lying between P and C, and on the line CD a point Q such that $|DQ| = |DL|$, D lying between C and Q. We have: $|CP|$ $|CB| + |BK| = |CB| + |AB| - |AK|$ $|CQ|$. The circle passing through the points P and Q and touching the lines CB and CD intersects BD at such points M_1 and N_1 for which the equalities $|BM_1|\cdot|BN_1| = |BM|\cdot|BN|$; $|CN_1|\cdot|CM_1| = |CN|\cdot|CM|$ are valid. These equalities imply that M_1 and N_1 must coincide with M and N, respectively. The other cases of arrangement of the points are considered much in the same way. It is possible to avoid looking over alternate versions by specifying positive directions on the lines AB, BC, CD, and DA and considering directed segments on these lines.

231. For definiteness, we assume that the points B and D lie inside the circle. Let P and Q denote the points of intersection of the straight line BD and the circle (P is the nearest to B), L the point of intersection of CB and the circle, l the tangent to the circle passing through the point C.

Consider the triangle PCN from whose vertices the straight lines PQ, NM, and l emanate. With the aid of Ceva's theorem (Problem 44 of Sec. 2) reasoning in the same way as in Problem 49 of Sec. 2, we get that for the lines PQ, NM, and l to intersect at one point, it is necessary and sufficient that the following equality be fulfilled:

$$\frac{|PM|}{|MC|}\cdot\frac{|CQ|}{|QN|}\cdot\frac{|NC|}{|CP|}=1. \qquad (1)$$

On the other hand, in the hexagon $ALPMCQ$ the diagonals AM, LC, and PQ intersect at one point. Hence (see Problem 49 in Sec. 2)

$$|AL|\cdot|PM|\cdot|CQ|=|LP|\cdot|MC|\cdot|QA|. \quad (2)$$

Obviously, $|NC| = |AL|$, $|QN| = |LP|$, $|CP| = |QA|$. Thus, from the validity of the equality (2) there follows the validity of the equality (1).

232. 1. Since O_1 is the centre of the circle inscribed in the triangle ABC, we have: $\angle BO_1A = 90° + \frac{1}{2}\angle BCA$ (Problem 46 of Sec. 1). Hence, $\angle BO_1A = \angle BO_4A$, and ABO_1O_4 is an inscribed quadrilateral (see Fig. 40, *a*); consequently, the angle adjacent to the angle BO_1O_4 is equal to $\angle BAO_4 = \frac{1}{2}\angle BAD$. Similarly, the angle adjacent to $\angle BO_1O_2$ is equal to $\frac{1}{2}\angle BCD$. But $\frac{1}{2}(\angle BAD+\angle BCD)=90°$; hence, $O_4O_1O_2=90°$.

2. To prove the second part of the statement, let us first show that the distance from a vertex of the triangle to the point of intersection of the altitudes is completely determined by the size of the angle at this vertex and the length of the opposite side, namely (Fig. 40, *b*): $|CH| = |CB| \times \frac{\cos\alpha}{\sin\angle CAB} = \frac{|AB|}{\sin\alpha}\cos\alpha = |AB|\cot\alpha$. Since $ABCD$ is an inscribed quadrilateral, $|AH_3| = |BH_2|$ and AH_3 is parallel to BH_2; hence, ABH_2H_3 is a parallelogram. Thus, the point of intersection of AH_2 and BH_3 bisects these line segments. Considering the other parallelograms, we see that the line segments H_2A, H_3B, H_4C, and H_1D intersect at the same point (M) and are bisected by this point, that is, the quadrilaterals $ABCD$ and $H_1H_2H_3H_4$ are centrally symmetric with respect to the point M (Fig. 40, *c*).

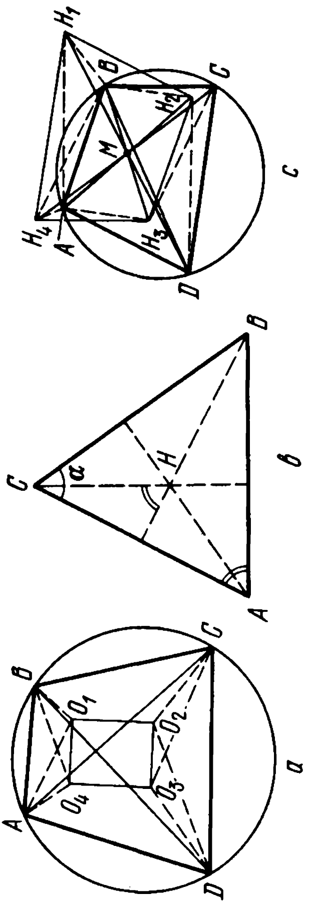


Fig. 40

233. If the sides of the triangle ABC, opposite the vertices A, B, and C, are respectively equal to a, b, and c, and the angles ADB, BDC, and CDA are, respectively, equal to α, β, and γ (we assume that $\alpha + \beta + \gamma = 2\pi$), then the distances from the point D to the intersection points of the altitudes of the triangles ADB, BDC, and CDA are equal to the magnitudes of $c \cot \alpha$, $a \cot \beta$, $b \cot \gamma$, respectively (see the solution of Problem 232 of Sec. 2). It is easy to make sure that the area of the triangle with vertices at the intersection points of the altitudes of the triangles ADB, BDC, and CDA is equal to $\frac{1}{2} c \cot \alpha \cdot a \cot \beta \sin B + \frac{1}{2} \times$ $a \cot \beta \cdot b \cot \gamma \sin C + \frac{1}{2} b \cot \gamma \cdot c \cot \alpha \sin A =$ $S_{ABC} (\cot \alpha \cot \beta + \cot \beta \cot \gamma + \cot \gamma \cot \alpha) = S_{ABC}$ since the expression in the parentheses is equal to 1. (Prove this taking into account that $\alpha + \beta + \gamma = 2\pi$). Analogously, we consider other cases of location of the point D (when one of the angles α, β, γ is equal to the sum of two others).

234. (a) Let $ABCD$ be the given quadrilateral, R and Q the points of tangency of the circles inscribed in the triangles ABC and ACD, respectively, with the straight line AC. Then (see Problem 18 of Sec. 1) $|RQ| = ||AQ| - |AR|| = \frac{1}{2} |(|AB| + |AC| - |BC|) - (|AD| + |AC| - |CD|)| = \frac{1}{2} ||AB| + |CD| - |AD| - |BC||$. Since $ABCD$ is a circumscribed quadrilateral, $|AB| + |CD| = |AD| + |BC|$, that is, $|RQ| = 0$.

(b) If K, L, M, N are the points of tangency of the circle with the sides of the quadrilateral, and K_1, L_1, M_1, and N_1 the points of tangency of the circles inscribed in the triangles ABC and ACD (Fig. 41), then $N_1K_1 \parallel NK$, and $M_1L_1 \parallel ML$. Let us prove that $K_1L_1 \parallel KL$ and $N_1M_1 \parallel NM$. Since the circles inscribed in the triangles ACB and ACD

touch each other on the diagonal at a point P, we have: $|AN_1| = |AP| = |AM|$, that is, $N_1M_1 \parallel NM$. Consequently, $K_1L_1M_1N_1$, as well as $KLMN$, is an inscribed quadrilateral.

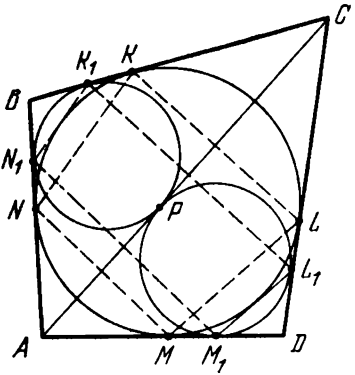


Fig. 41

235. Let O_1, O_2, O_3, O_4 denote the centres of the circles inscribed in the triangles ABC, BCD,

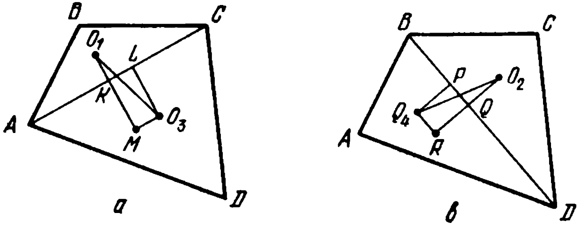


Fig. 42

CDA, and DAB, respectively, (Fig. 42, *a*, *b*). Since $O_1O_2O_3O_4$ is a rectangle (see Problem 232 in Sec. 2), we have: $|O_1O_3| = |O_2O_4|$. If K and L are the points of tangency with AC of the circles inscribed in the triangles ABC and ACD, then $|KL| = \frac{1}{2}||AB| + |CD| - |BC| - |AD||$ (see the solution of Problem 234 in Sec. 2). Analogously, if P and Q are the points of tangency of the corresponding circles with BD, then $|PQ| = |KL|$. Through O_3, we draw a straight line parallel to AC to intersect the extension of O_1K. We get the triangle O_1O_3M; we then construct the triangle O_2O_4R in a similar way. These two right triangles are congruent, since in them: $|O_1O_3| = |O_2O_4|$, $|O_3M| = |KL| = |PQ| = |O_4R|$. Hence, $|O_1M| = |O_2R|$; but $|O_1M|$ equals the sum of the radii of the circles inscribed in the triangles ABC and ACD, and $|O_2R|$ is equal to the sum of the radii of the circles inscribed in the triangles ACD and BDA (see also Problem 315 in Sec. 2).

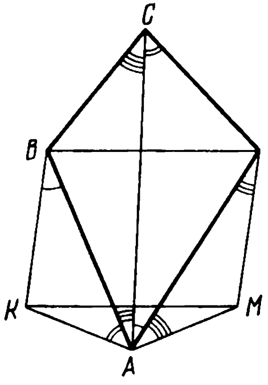


Fig. 43

236. In the quadrilateral $ABCD$ (Fig. 43): $|AB| = a$, $|BC| = b$, $|CD| = c$, $|DA| = d$, $|AC| = m$, $|BD| = n$. We construct externally

on the side AB a triangle AKB similar to the triangle ACD, where $\angle BAK = \angle DCA$, $\angle ABK = \angle CAD$, and on the side AD we construct the triangle AMD similar to the triangle ABC, where $\angle DAM = \angle BCA$, $\angle ADM = \angle CAB$. From the corresponding similarity we get: $|AK| = \frac{ac}{m}$, $|AM| = \frac{bd}{m}$, $|KB| = |DM| = \frac{ad}{m}$. In addition, $\angle KBD + \angle MDB = \angle CAD + \angle ABD + \angle BDA + \angle CAB = 180°$, that is, the quadrilateral $KBDM$ is a parallelogram. Hence, $|KM| = |BD| = n$. But $\angle KAM = \angle A + \angle C$. By the law of cosines for the triangle KAM, we have: $n^2 = \left(\frac{ac}{m}\right)^2 + \left(\frac{bd}{m}\right)^2 - 2\left(\frac{ac}{m}\right)\left(\frac{bd}{m}\right)\cos(A+C)$, whence $m^2n^2 = a^2c^2 + b^2d^2 - 2abcd\cos(A + C)$.

237. The statement of Ptolemy's theorem is a corollary of Bretschneider's theorem (see Problem 236 of Sec. 2), since for an inscribed quadrilateral $\angle A + \angle C = 180°$.

238. If MB is the greatest of the line segments $|MA|$, $|MB|$, and $|MC|$, then, applying Bretschneider's theorem (Problem 236 of Sec. 2) to the quadrilateral $ABCM$, we get: $|MB|^2 = |MA|^2 + |MC|^2 - 2|MA|\cdot|MC|\cos(\angle AMC + 60°)$, that is, $|MB| < |MA| + |MC|$ since $\angle AMC \neq 120°$.

239. Replacing in the expression

$$t_{\alpha\beta}t_{\gamma\delta} + t_{\beta\gamma}t_{\delta\alpha} = t_{\alpha\gamma}t_{\beta\delta} \tag{1}$$

the segments of the tangents with the aid of the formulas obtained when solving Problem 201 of Sec. 1, we make sure that if the relationship (1) is fulfilled for some circles α, β, γ, and δ touching the given circle at points A, B, C, and D, then it is fulfilled for any such circles. It remains to check the validity of the relationship (1) for some particular case. If α, β, γ, and δ are circles of zero radii, then we get an ordinary Ptolemy's theorem (Problem 237 of Sec. 2). In order not to refer to

Ptolemy's theorem, we may take the circles α and δ having a zero radius, and the circles β and γ touching both the circle circumscribed about the quadrilateral $ABCD$ and the chord AD. In this case, the validity of the relationship (1) is readily verified. Hence, in accordance with the remark made, we get the validity of (1) in all the cases (thereby we have simultaneously proved Ptolemy's theorem itself).

240. When proving our statement, we shall use the method of "extension" of circles. The essence of this method consists in the following. Let two circles, say α and β, touch externally some circle Σ. Consider the circles α', β', and Σ' which are concentric with α, β, and Σ, respectively. If the radius of the circle Σ' is greater than the radius of the circle Σ by a quantity χ and the radii of the circles α' and β' are less than those of the circles α and β by the same quantity χ which is sufficiently small, then the circles α' and β' touch the circle Σ' externally, and the length of the common external tangent to the circles α' and β' is equal to the length of the common external tangent to the circles α and β. The case when α and β touch the circle Σ internally is considered in the same way. And if one of the circles α and β touches Σ externally, and the other internally, then, with an increase in the radius of Σ, the radius of the first circle decreases and the radius of the second circle increases, the length of the common internal tangent to the circles α' and β' remaining unchanged.

For the sake of definiteness, consider the case when in the equality (*) (see the statement of the problem) there appear only the segments of the common external tangents. (Note that none of the circles can be found inside the other.) Let us prove that the circles α, β, γ, and δ touch a certain circle Σ in the same manner, all of them either externally or internally. Let not all of the circles α, β, γ, and δ have equal radii (the case of equal radii is readily considered separately), and, for definiteness, let r_α, the radius of the circle α, be the smallest. Consider the circles α', β', γ', δ', where α' is a circle of zero radius, that is, a point coinciding with the centre of the circle α and β', γ', δ' circles concentric with the circles β, γ, δ with radii reduced by the quantity r_α. For further reasoning, let us take advantage of the following assertion which is marked by (T):

If β', γ', δ' are three circles none of which lies inside another and at least one of them has a non-zero radius, then there are exactly two circles Σ_1 and Σ_2 each of which touches the circles β', γ', and δ' in the same manner. We shall return to this assertion at the end of the solution.

On the circles Σ_1 and Σ_2, take points α_1 and α_2 such that $\dfrac{t_{\alpha_1\beta'}}{t_{\alpha_1\delta'}}=\dfrac{t_{\alpha_2\beta'}}{t_{\alpha_2\delta'}}=\dfrac{t_{\alpha'\beta'}}{t_{\alpha'\delta'}}=\lambda$, α_1 and α_2 lying on the arcs not containing the point of tangency of the circle γ'. For three fours of circles $(\alpha', \beta', \gamma', \delta')$, $(\alpha_1, \beta', \gamma', \delta')$, $(\alpha_2, \beta', \gamma', \delta')$ the relationship (*) is fulfilled: for the first four circles, this is the assertion of the problem, for two other fours—on the basis of the assertion of Problem 239 of Sec. 2 (α', α_1, α_2 are circles of zero radius). Consequently, $\dfrac{t_{\alpha_1\beta'}}{t_{\alpha_2\gamma'}}=\dfrac{t_{\alpha_2\beta'}}{t_{\alpha_2\gamma'}}=$

$$\frac{t_{\alpha'\beta'}}{t_{\alpha'\gamma'}}=\mu.$$

But the locus of points M for which the ratio of tangents to two fixed circles is constant is a circle (see Problem 11 in Sec. 1). Hence, α_1, α_2, and α' belong both to the locus of points for which the ratio of the tangents drawn to the circles β' and δ' is equal to λ and to the locus of points for which the ratio of the tangents drawn to the circles β' and γ' is equal to μ. And this means that α' must coincide either with α_1 or α_2.

Let α_1 and α_2 coincide. Prove that in this case the circles defined by the parameters λ and μ touch each other. Let us take $\tilde{\lambda} \neq \lambda$, but sufficiently close to λ. Then $\tilde{\lambda}$ defines on Σ_1 and Σ_2 two points $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ for which $\frac{t_{\tilde{\alpha}_1\beta'}}{t_{\tilde{\alpha}_1\delta'}} = \frac{t_{\tilde{\alpha}_2\beta'}}{t_{\tilde{\alpha}_2\delta'}} = \tilde{\lambda}$. We find:

$$\tilde{\mu} = \frac{t_{\tilde{\alpha}_1\beta'}}{t_{\tilde{\alpha}_1\gamma'}} = \frac{t_{\tilde{\alpha}_2\beta'}}{t_{\tilde{\alpha}_2\gamma'}}.$$ Hence, the circles corresponding to the parameters $\tilde{\lambda}$ and $\tilde{\mu}$ have a common chord $\tilde{\alpha}_1\tilde{\alpha}_2$. If $\tilde{\lambda} \to \lambda$, then $\tilde{\mu} \to \mu$, $|\tilde{\alpha}_1\alpha_2| \to 0$, that is, the circles corresponding to the parameters λ and μ touch each other at a point $\alpha_1 = \alpha_2$. Thus, α', β', γ', and δ' touch either Σ_1 or Σ_2. "Extending" Σ_1 or Σ_2 by the quantity $\pm r_\alpha$, we get that α, β, γ, and δ touch a circle or a straight line (Σ_1 or Σ_2 may turn out to be a straight line) or have a common point.

If in the equality (*) some of the line segments are segments of common internal tangents, then we have to prove the existence of a circle Σ touching α, β, γ, and δ and such that those of the circles α, β, γ, δ for which in the equality (*) there appears a common internal tangent touch Σ in different ways. The assertion (T) must change accordingly.

Let us return to the assertion (T). By means of "extension", we can reduce the assertion to the case when one of the circles β', γ', and δ' has a zero radius, i.e. is a point. The reader familiar with the notion of inversion can easily prove that the assertion (T) now turns out to be equivalent to the assertion that any two circles not lying one inside the other have exactly two common external tangents (see Appendix). **Remark.** If three of the four given circles α, β, γ, δ have a zero radius (they are points), the proof can be considerably simplified.

Do this independently. Henceforward (see Problem 287 of Sec. 2), we shall need just this particular case.

241. Show that each of these conditions is both necessary and sufficient for a circle inscribed in the quadrilateral $ABCD$ to exist (see also Problem 19 in Sec. 1).

242. Show that each of these conditions is both necessary and sufficient for a circle, touching the lines AB, BC, CD, and DA, whose centre is outside the quadrilateral $ABCD$, to exist.

243. Let $ABCD$ be a circumscribed quadrilateral, O the centre of the inscribed circle, M_1 the midpoint of AC, M_2 the midpoint of BD, r the radius of the circle (the distances from O to the sides are equal to r each), x_1, y_1, z_1, and u_1 the distances from M_1 to AB, BC, CD, DA, respectively; x_2, y_2, z_2, and u_2 the distances from M_2 to the same sides, respectively. Since $|AB| + |CD| = |BC| + |DA|$, we have: $|AB|r - |BC|r + |CD|r - |DA|r = 0$. In addition, $|AB|x_1 - |BC|y_1 + |CD|z_1 - |DA|u_1 = 0$, $|AB|x_2 = |BC|y_2 + |CD|z_2 = |DA|u_2 = 0$, and this just means that the points O, M_1, and M_2 lie on a straight line (see the remark to Problem 22 of Sec. 2). Other cases of the arrangement of the points A, B, C, and D and the centre of the circle are considered exactly in the same way. Here, use the relationships occurring among the line segments $|AB|$, $|BC|$, $|CD|$, $|DA|$ (see Problems 241 and 242 in Sec. 2), and, as is said in the remark to Problem 22 of Sec. 2 assign unlike signs to corresponding distances if any two points turn out to be located on both sides of a straight line.

244. Let L and P denote the points of intersection of the straight lines AM and AN with the circle, respectively. As there follows from Problem 204 of Sec. 2, the straight lines BL, DP, and MN meet at one point. But, being diameters, BL and DP intersect at the centre of the circle, consequently, MN passes through the centre of the circle.

245. Make use of Pascal's theorem (Problem 204 in Sec. 2).

246. Let P denote the point of intersection of the diagonals, and K, L, M, and N the feet of the perpendiculars from P on AB, BC, CD, and DA, respectively, (Fig. 44). Since $PKBL$ is an inscribed

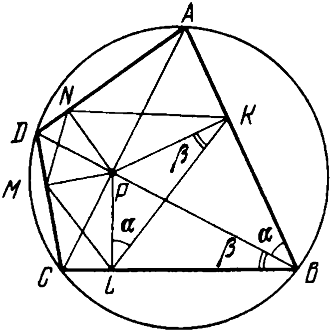


Fig. 44

quadrilateral, we have: $\angle PKL = \angle PBC$, analogously, $\angle PKN = \angle PAD$; but $\angle PBC = \angle PAD$ since they are subtended by the same arc. Consequently, KP is the bisector of the angle NKL; hence, the bisectors of the angles of the quadrilateral $KLMN$ intersect at the point P which is just the centre of the circle inscribed in the quadrilateral $KLMN$. Let now AC and BD be mutually perpendicular, R the radius of the given circle, d the distance from P to its centre, $|AP|\cdot|PC| = R^2 - d^2$.

The radius r of the sought-for circle is equal, in particular, to the distance from P to KL. Denoting $\angle KLP = \angle ABP = \alpha$, $\angle PBC = \beta$, we find:

$$r = |PL| \sin\alpha = |PB| \sin\beta \sin\alpha = |PB| \frac{|PC|}{|BC|} \times$$
$$\frac{|AP|}{|AB|} = (R^2 - d^2) \frac{|PB|\ |AC|}{|BC|\ \ |AB| \sin(\alpha+\beta)} \times$$
$$\frac{\sin(\alpha+\beta)}{|AC|} = (R^2 - d^2) \frac{2S_{ABC}}{2S_{ABC}} \cdot \frac{1}{2R} = \frac{R^2 - d^2}{2R}.$$

Answer: $\dfrac{R^2-d^2}{2R}$.

247. Let $ABCD$ be the given quadrilateral, P the point of intersection of the diagonals, K the midpoint of BC, L the midpoint of AD (Fig. 45). Let

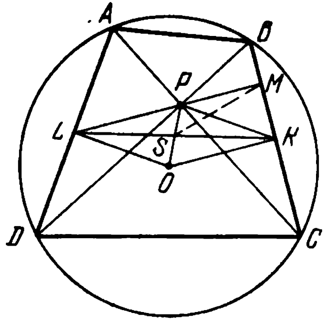


Fig. 45

us prove that the straight line LP is perpendicular to BC. Denoting the point of intersection of LP and BC by M, we have: $\angle BPM = \angle LPD = \angle ADP = \angle PCB$. Consequently, PM is perpendicular to BC. Hence, OK is parallel to LP. Similarly, PK is parallel to LO, and $KOLP$ is a parallelogram, $|LK|^2 + |PO|^2 = 2(|LP|^2 + |PK|^2) = 2\left(\frac{|AD|^2}{4} + \frac{|BC|^2}{4}\right) = 2R^2$. (If the chords AD and BC are brought to a position in which they have a common end point and the corresponding arcs continue each other, then a right triangle is formed with legs $|AD|$ and $|BC|$ and hypotenuse $2R$, hence, $|AD|^2 + |BC|^2 = 4R^2$.) Consequently, $|LK|^2 = 2R^2 - d^2$, and the points L and K lie on the circle with centre at S (the midpoint of PO) and the radius $1/2\sqrt{2R^2-d^2}$. But LMK is a right triangle, MS is its median, $|MS| = \frac{1}{2}|LK| = \frac{1}{2}\sqrt{2R^2-d^2}$, that is, M lies on the same circle.

Answer: $1/2\sqrt{2R^2-d^2}$.

248. From Problems 246 and 247 it follows that if the diagonals of the inscribed quadrilateral are mutually perpendicular, then the projections of the intersection point of the diagonals of this quadrilateral on its sides serve as vertices of a quadrilateral which can be inscribed in a circle and about which a circle can be circumscribed. The radii of the inscribed and circumscribed circles and the distance between their centres are completely determined by the radius of the circle circumscribed about the original quadrilateral and the distance from its centre to the intersection point of the diagonals of the quadrilateral inscribed in it. Consequently, when the diagonals of the original quadrilateral are rotated about the point of their intersection, the quadrilateral formed by the projections of this point rotates remaining inscribed in one and the same circle and circumscribed about one and the same circle. Taking into consideration the expressions for the radii of the inscribed and circumscribed circles obtained in the two previous problems, it is easy to show that the relationship to be proved is fulfilled for such quadrilaterals.

To complete the proof, it remains to prove that any "inscribed-circumscribed" quadrilateral can be obtained from an inscribed quadrilateral with mutually perpendicular diagonals using the above method. Indeed, if $KLMN$ is an "inscribed-circumscribed" quadrilateral, P the centre of the inscribed circle, then drawing the lines perpendicular to the angle bisectors KP, LP, MP, and NP and passing through the points K, L, M, and N, respectively, we get the quadrilateral $ABCD$ (see Fig. 44). In this case, $\angle BPK = \angle KLB = 90° - \frac{1}{2} \angle MLK$ (here, we have used the fact that in the quadrilateral $PKBL$ the opposite angles are right ones and, consequently, it is an inscribed quadrilateral). Similarly, $\angle KPA = \angle KNA = 90° - \frac{1}{2} \angle MNK$, and, hence, $\angle BPA = \angle BPK + \angle KPA = 180° - \frac{1}{2} (\angle MLK + \angle MNK) = 90°$. Thus, all the angles BPA, APD, DPC, and CPB are right ones, P is the intersection point of the diagonals of the quadrilateral $ABCD$, the diagonals themselves being mutually perpendicular. It is easy to show that $ABCD$ is an inscribed quadrilateral since

$$\angle ABC + \angle ADC = \angle PBL + \angle PBK + \angle PDN + \angle PDM = \angle PKL + \angle PLK + \angle PMN + \angle PNM = \frac{1}{2} (\angle NKL + \angle KLM + \angle LMN + \angle MNK) = 180°.$$

Note:: see also Problem 319, Sec. 2.

249. The midpoints of the sides of the quadrilateral form a parallelogram whose diagonals are parallel to the line segments joining the centres of mass of the opposite triangles. The other parallelogram is formed by the four altitudes of the triangles in question emanating from the vertices of the quadrilateral. The sides of the first parallelogram are parallel to the diagonals of the quadrilateral, while those of the second parallelogram are perpendicular to them. In addition, the sides of the second parallelogram are cot α times greater than the

corresponding sides of the first one (α is an acute angle between the diagonals of the quadrilateral).

250. We prove that both assertions (BD is the bisector of the angle ANC, and AC is the bisector of the angle BMD) are equivalent to the equality $|AB|\cdot|CD| = |AD|\cdot|BC|$. On the arc BAD we take a point A_1 such that $|DA_1| = |AB|$. The conditions of the problem imply that the straight line A_1C passes through N, the midpoint of BD, that is, the areas of the triangles DA_1C and A_1BC are equal, whence $|DA_1|\cdot|DC| = |BA_1|\cdot|BC|$, that is $|AB|\cdot|CD| = |AD| \times |BC|$.

251. The perpendicularity of the angle bisectors is proved quite easily. Let us prove the second assertion. Let M denote the midpoint of AC, and N the midpoint of BD. From the similarity of the triangles AKC and BKD, it follows that $\angle MKA = \angle NKD$ and $\frac{|MK|}{|KN|} = \frac{|AC|}{|BD|}$, that is, the bisector of the angle BKC is also the bisector of the angle MKN and divides the line segment MN in the ratio $\frac{|MK|}{|KN|} = \frac{|AC|}{|BD|}$. Obviously, the bisector of the angle ALB divides the line segment MN in the same ratio.

252. Let $ABCD$ be the given quadrilateral, O the centre of the circle circumscribed about the triangle ABC, O_1 and O_2 the centres of the circles circumscribed about the triangles DAB and BCD, K and L the midpoints of the sides AB and BC, respectively. The points O_1 and O_2 lie on OK and OL, respectively, and $\frac{|OO_1|}{|O_1K|} = \frac{|OO_2|}{|O_2L|}$. This follows from the fact that O_1O_2 is perpendicular to DB and, consequently, parallel to LK (LK is parallel to AC). Hence, the straight lines AO_1 and CO_2 divide OB in the same ratio. (We apply Menelaus' theorem (Problem 45 in Sec. 2) to the triangles OKB and OLD.)

253. Let R denote the radius of the circle, and a, b, and c the distances from P, Q, and M to its centre, respectively. Then (Problem 272 of Sec. 1) $|QP|^2 = a^2 + b^2 - 2R^2$, $|QM|^2 = b^2 + c^2 - 2R^2$, $|PM|^2 = c^2 + a^2 - 2R^2$. If O is the centre of the circle, then for QO to be perpendicular to PM, it is necessary and sufficient that the inequality $|QP|^2 - |QM|^2 = |OP|^2 - |OM|^2$ or

$$(a^2 + b^2 - 2R^2) - (b^2 + c^2 - 2R^2) = a^2 - c^2$$

(Problem 1 of Sec. 2). The perpendicularity of the other line segments is checked in a similar way.

254. If M, N, P, and Q are the points of tangency of the sides AB, BC, CD, and DA with the circle, respectively, then, as it follows from the solution of Problem 236 of Sec. 1, MP and NQ meet at the point of intersection of AC and BD. In similar fashion, we prove that the lines MN and PQ meet at the point of intersection of the straight lines AC and KL, and the straight lines MQ and NP at the point of intersection of the lines KL and BD. Now, we use the result of the preceding problem for the quadrilateral $MNPQ$.

255. Denote: $\angle DAN = \angle MAB = \varphi$. Let L be the point of intersection of AM and NB, P the point of intersection of AN and DM, Q the point of intersection of AK and MN. By Ceva's theorem (Problem 44 of Sec. 2), for the triangle AMN we have:

$$\frac{|NQ|}{|QM|} = \frac{|AL|}{|LM|} \cdot \frac{|NP|}{|PA|} = \frac{S_{NAB}}{S_{NMB}} \cdot \frac{S_{DNM}}{S_{DAM}} =$$

$$\frac{\frac{|AN|}{2}\frac{|AM|}{\cos\varphi}\sin\angle NAB\,\frac{|AN|}{2}\,|NM|\tan\varphi\cos\angle ANM}{\frac{|AM|}{2}|NM|\tan\varphi\cos\angle AMN\frac{|AN|}{2\cos\varphi}\,|AM|\sin\angle MAD}$$

$$= \frac{|AN| \cos \angle ANM}{|AM| \cos \angle AMN},$$

that is, Q divides NM in the same ratio as the altitude drawn from A on NM.

257. First, prove the following additional assertion: if A, B, and C are collinear points, M is an arbitrary point in the plane, then the centres of the circles circumscribed about the triangles MAC, MBC, MCA and the point M lie on one and the same circle. Then use the result of Problem 256, Sec. 2.

258. Let A, B, C, D, P, and Q denote the intersection points of the straight lines (the points are arranged in the same way as in the solution of Problem 271 of Sec. 1); O the centre of the circle passing through A, B, C, and D; R its radius; a and b the tangents drawn to the circle from P and Q, respectively. The fact that M lies on PQ was proved when we were solving Problem 271 of Sec. 1. In addition, it was proved that $|PM| \times |PQ| = a^2$, $|QM| \cdot |QP| = b^2$, $|QP|^2 = \sqrt{a^2+b^2}$. Thus, $|PM| = \frac{a^2}{\sqrt{a^2+b^2}}$, $|QM| = \frac{b^2}{\sqrt{a^2+b^2}}$ In addition $|PO| = \sqrt{a^2-R^2}$, $|QO| = \sqrt{b^2-R^2}$. Consequently, $|PO|^2 - |QO|^2 = a^2 - b^2 = |PM|^2 - |QM|^2$. And this means that OM is perpendicular to PQ. To complete the proof, we have to consider the case when (using the same notation) the points A, C, P, and Q are found on the circle (see also Problem 253 in Sec. 2 and its solution).

259. If one of the straight lines is displaced parallel to itself, then Euler's line of the triangle one of whose sides is the line displaced moves parallel to itself. Taking this into account, we can easily

reduce the problem to the following. Let A, C, and D be three collinear points, and B an arbitrary point in the plane. If Euler's line of the triangle ABC is parallel to BD, then Euler's line of the triangle CBD is parallel to AB (Fig. 46). Let us prove this. We denote: $\angle BCD = \varphi$ (we assume that C lies between A and D, $\varphi \leqslant 90°$), O_1 and H_1 the centre of the circumscribed circle and the intersection point of the altitudes of the triangle

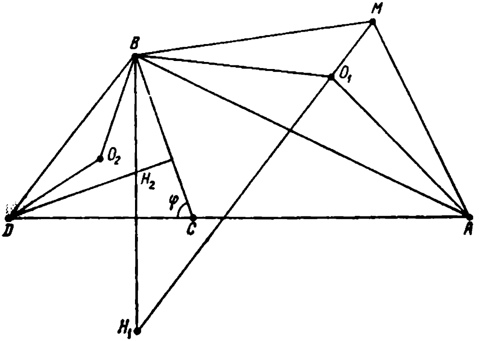


Fig. 46

ABC, respectively, O_2 and H_2 the centre of circumscribed circle and the intersection point of the altitudes of the triangle BCD. Describe a circle about ABH_1 to intersect O_1H_1 at a point M. Let us prove that the quadrilaterals O_1AMB and O_2DH_2B are similar. First of all, the triangles O_1AB and O_2DB are similar isosceles triangles, and $\angle MAB = \angle MH_1B = \angle H_1BD = \angle H_2BD$ (BD is parallel to O_1H_1), $\angle MBA = \angle MH_1A = \angle H_2DB$ (AH_1 and DH_2 are perpendicular to CB). The similarity of the quadrilaterals has been proved. Further: $\angle O_2H_2B = \angle O_1MA = \angle H_1MA = \angle H_1BA = \angle H_2BA$, that is, H_2O_2 is parallel to AB.

260. It follows from the result of Problem 19 in Sec. 2 that the common chord of the circles with the diameters AE and DC (and also DC and BF, BF and AE) contains the intersection points of the altitudes of the triangles ABC, BDE, DAF, and CEF. Further, let K denote the point of intersection of AE and DC and L the point of intersection of AE and BF. By Menelaus' theorem (Problem 45 in Sec. 2), for the triangles BEA and EAC we have: $\frac{|AK|}{|KE|}\cdot\frac{|EC|}{|CB|}\cdot\frac{|BD|}{|DA|}=1$, $\frac{|AL|}{|LE|}\cdot\frac{|EB|}{|BC|}\cdot\frac{|CF|}{|FA|}=1$. Dividing these equalities, one by the other, termwise and bearing in mind that $\frac{|CE|}{|EB|}\ \frac{|BD|}{|DA|}\cdot\frac{|AF|}{|FC|}=1$, we get: $\frac{|AK|}{|AL|}=\frac{|KE|}{|LE|}$ Consider the circle with diameter AE. For all points P of this circle the ratio $\frac{|PK|}{|PL|}$ is constant (see Problem 9 of Sec. 2). The same is true for the circles with diameters DC and BF. Thus, these three circles intersect at two points P_1 and P_2 such that the ratios of the distances from P_1 and P_2 to K, L, and M for them are equal. Now, we can use the result of Problem 14, Sec. 2.

261. The statement follows from the result of the preceding problem.

262. Let $l\,(ABC)$ denote the midperpendicular to the line segment joining the point of intersection of the altitudes to the centre of the circle circumscribed about the triangle ABC. Let a straight line intersect the sides BC, CA and AB of the triangle ABC at points D, E, and F, respectively. Let us first prove that as the straight line DEF displaces parallel to itself, the point M of intersection of the lines $l\,(DFB)$ and $l\,(DEC)$ describes a straight line. Let the points D_1, E_1, F_1; D_2, E_2,

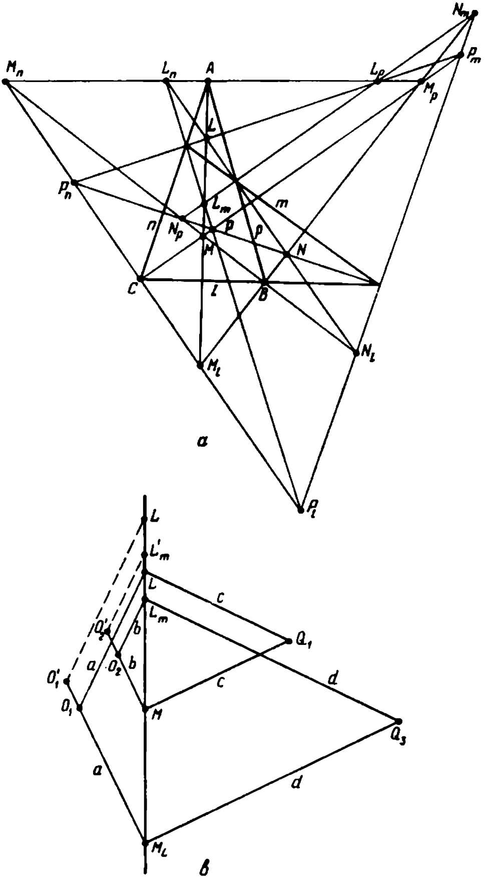
F_2; D_3, E_3, F_3 correspond to three positions of this line. The lines $l\,(D_iF_iB)$ and $l\,(D_iE_iC)$, where $i = 1, 2, 3$, meet at M_i and intersect the straight line BC at points N_i and K_i. It is easily seen that the point N_2 divides the line segment N_1N_3 in the same ratio in which the point K_2 divides the line segment K_1K_3. This ratio is equal to the ratio in which D_2 divides D_1D_3, E_2 divides E_1E_3, and F_2—F_1F_3. Since the straight lines $l\,(D_iF_iB)$ are parallel, and the straight lines $l\,(D_iE_iC)$ are also parallel, the line $l\,(D_2F_2B)$ divides the line segment M_1M_3 in the same ratio as the line $l\,(D_2E_2C)$, that is, M_2 lies on the line segment M_1M_3.

Let us now show that the point M describes a straight line $l\,(ABC)$. To this end, it suffices to prove that for two positions of the straight line DEF the corresponding point M lies on $l\,(ABC)$. Consider the case when this line passes through A (the points E and F coincide with A). We introduce a coordinate system in which the points A, B, C, and D have the following coordinates: $A\ (0, a)$, $B\ (b, 0)$, $C\ (c, 0)$, $D\ (d, 0)$. We then find the equation of the straight line $l\,(ABC)$. The intersection point of the altitudes of the triangle ABC has the coordinates $\left(0, -\frac{bc}{a}\right)$, the centre of the circumscribed circle the coordinates $\left(\frac{b+c}{2}, \frac{1}{2}\left(a+\frac{bc}{a}\right)\right)$ Let us write the equation of the straight line $l\,(ABC)$:

$$x(b+c)+y\left(a+\frac{3bc}{a}\right)=\frac{a^2+b^2+c^2}{4}+bc-\frac{3b^2c^2}{4a^2}.$$

Replacing c by d in this equation, we get the equation of the line $l\,(ABD)$, and replacing b by d the equation of the line $l\,(ACD)$.

We can verify that all the three straight lines have a common point $Q\,(x_0, y_0)$, where $x_0 =$



N_m
P_m
M_n
L_n
A
L_p
M_p
L
P_n
L_m
m
n
N_p
P
p
M
N
C
l
B
N_l
M_l
a
P_l
L
L'_m
L
c
L_m
O'_2
b
Q_1
O'_1
a
O_2
b
d
c
O_1
M
Q_3
a
d
M_L
б

Fig. 47

$\frac{1}{4}(b+c+d)-\frac{3bcd}{4a^2}$, $y_0=\frac{1}{4a}(a^2-bc-cd-db)$.
And this is the end of the proof since the case when the line DEF passes through B or C is equivalent to the above case.

263. Let l, m, n, and p be the straight lines which form the triangles (Fig. 47, a). Let us introduce the following notation: P is the centre of the circle inscribed in the triangle formed by the lines l, m, and n, and P_l is the centre of the escribed circle for the same triangle which touches the side lying on the line l. The notations L, M_p, N_m, etc. have the same sense.

L	N	M_l	P_n	O_1
M	P	L_m	N_p	O_2
P_m	M_p	N_m	L_p	O_3
N_l	L_n	P_l	M_n	O_4
Q_1	Q_2	Q_3	Q_4	

In the above table, the four points forming a row or a column lie on the same circle, the centres of the circles corresponding to the rows lying on one straight line (q_1), while the centres corresponding to the columns on the other (q_2); q_1 and q_2 are mutually perpendicular and intersect at Michell's point (Problem 256 in Sec. 2). Let us prove this. The fact that the indicated fours lie on the same circle is proved easily. Let O_i, Q_i ($i=1, 2, 3, 4$) denote the centres of the corresponding circles. Let us prove that O_1O_2 is perpendicular to Q_1Q_3 and Q_2Q_4. If in the triangle (l, m, n) the angle between l and m is equal to α, then $\angle LNM_l=\angle L_mPM=90^\circ+\frac{\alpha}{2}$; consequently, $\angle LO_1M_l=$

$\angle L_mO_2M = 180° - \alpha$. In similar fashion, $\angle LP_mM = \angle L_mP_lM_l = \alpha/2$, $\angle LQ_1M = L_mQ_3M_l = \alpha$. The triangles LO_1M_l, L_mO_2M, LQ_1M, $L_mO_3M_l$ are isosceles ones, their lateral sides being respectively perpendicular (for instance, O_1L and LQ_1). Further (Fig. 47, *b*) $|Q_1O_1|^2 - |O_1O_3|^2 = (a^2+c^2) - (a^2+d^2) = (b^2+c^2) - (b^2+d^2) = |O_2Q_1|^2 - |O_2Q_3|^2$. Consequently, O_1O_2 and Q_1Q_3 are mutually perpendicular. In similar fashion, we prove that O_1O_2 and Q_2O_4 are also mutually perpendicular (consider the straight line on which the points N, P, N_p, and P_n are located). Therefore Q_1Q_3 and Q_2Q_4 are parallel (if these points do not lie on one and the same straight line). In similar fashion, Q_1Q_4 and Q_3Q_2 are also parallel (they are perpendicular to O_1O_3), Q_1Q_2 is parallel to Q_3Q_4 (they are perpendicular to O_1O_4), and this means that the points Q_1, Q_2, Q_3, Q_4 are collinear, they lie on the straight line q_2; the points O_1, O_2, O_3, O_4 are also collinear, they lie on the line q_1. Obviously, q_1 and q_2 are mutually perpendicular.

We shall displace the straight line m parallel to itself. Let L', L'_m, O'_1, O'_2 correspond to the straight line m' The ratio $\frac{|O_1O'_1|}{|O_2O'_2|} = \frac{|LL'|}{|L_mL'_m|}$ is constant $\left(\text{it is equal to } \frac{|AL|}{|AL_m|}\right)$. This means that when the line m is displaced, the line O_1O_2, that is, q_1 passes through a fixed point. The straight line q_2 also passes through a fixed point. Since q_1 and q_2 are mutually perpendicular, the point of their intersection describes a circle. But when m passes through A (and also B or C), the points L and L_m coincide with A, the lines O_1O_2 and Q_1Q_3, that is, q_1 and q_2 pass through A (correspondingly, B or C). Thus, the point of intersection of q_1 and q_2 traverses the circle circumscribed about the triangle ABC. Displacing the other lines (l, n, p), we prove that the point of intersection of q_1 and q_2 belongs to any circle circumscribed about one of the triangles formed by the lines l, m, n, p, that is, the lines q_1 and q_2 meet at the intersection point of the circles circumscribed about those triangles, that is, at Michell's point.

Note that we have proved at the same time that the four circles circumscribed about the four triangles formed by four straight lines in the plane intersect at one point (Problem 256 of Sec. 2).

266. Let C denote one of the intersection points through which the straight line passes. Let B_1, B_2, B_3 be the feet of the perpendiculars dropped respectively from O_1, O_2, O_3 on the straight line, and K and M the points of intersection of the straight lines, parallel to A_1A_2 and passing through O_1 and O_2, with O_2B_2 and O_3B_3. Since B_1 and B_2 are the midpoints of the chords A_1C and CA_2, we have: $|B_1B_2| = |A_1A_2|/2$. If α is the angle between the straight lines A_1A_3 and O_1O_3, then $\frac{|A_1A_2|}{|O_1O_2|} = \frac{2|B_1B_2|}{|O_1O_2|} = 2\frac{|O_1K|}{|O_1O_2|} = 2\cos\alpha$; in similar fashion, $\frac{|A_2A_3|}{|O_2O_3|} = 2\cos\alpha$.

268. Let O_1 and O_2 be the centres of the circles, R_1 and R_2 their radii, $|O_1O_2| = a$, M the point of intersection of the common internal tangents. A circle of diameter O_1O_2 passes through the points of intersection of the common external and internal tangents. In the homothetic transformation with the centre of similitude at the point M and the ratio of $\frac{a-R_1-R_2}{a}$ this circle goes into the circle with the centre on the straight line O_1O_2 which is tangent to the given circles externally.

269. Let M be one of the points of intersection of the circles; then MA and MC are the bisectors of the (exterior and interior) angle M of the triangle BMD since the circle of diameter AC is the locus of points M for which $\frac{|MA|}{|MC|} = \frac{|MB|}{|MD|}$

(see Problem 9 in Sec. 2). Using the relationships between the angles of the right triangle AMC and the triangle BMD, make sure that the radii of the circumscribed circles drawn from the vertex M are mutually perpendicular.

271. Note (Fig. 48, a) that the triangle APM is similar to the triangle AMQ, APL to AKQ, and AKN to ALN; from these facts of similarity we get: $\frac{|PM|}{|MQ|}=\frac{|AM|}{|AQ|}$, $\frac{|QK|}{|PL|}=\frac{|AQ|}{|AL|}$, $\frac{|LN|}{|NK|}=\frac{|AL|}{|AN|}$. Multiplying these equalities and taking into consideration that $|AM|=|AN|$, we get that $\frac{|PM|}{|MQ|}\cdot\frac{|QK|}{|PL|}\cdot\frac{|LN|}{|NK|}=1$, and this (see Problem 49 in Sec. 2) is just a necessary and sufficient condition for the straight lines MN, PK, and QL to meet in one point.

The method of constructing tangent lines by a ruler only is clear from Fig. 48, b. The numbers 1, 2, indicate the succession in which the lines are drawn.

272. The desired set is a straight line which is the polar of the point with respect to the given circle (see Problem 21 in Sec. 2).

273. The angles AMN and BNM can be expressed in terms of the central angle corresponding to the arc AB of the given circle (consider various cases of location of the point N); this done, it is possible to determine the angle AMB. The sought-for locus is a circle.

274. Take advantage of the results of Problems 271 and 21 in Sec. 2. The obtained locus coincides with the locus in Problem 21 of Sec. 2, that is, this is the polar of the point A with respect to the given circle.

275. Let O denote the point of intersection of AM and DC (Fig. 49). Through the point B, we draw a tangent to the second circle and denote the point of its intersection with AC by K (as in the

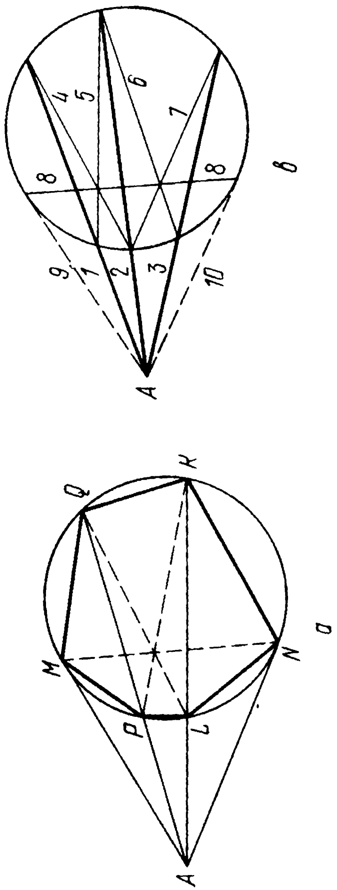


Fig. 48

hypothesis). Obviously, the statement of the problem is equivalent to the assertion that KO is parallel to CM. Let the angle subtended by the arc AB in the first circle be α, in the second β, then $\angle BCM = \angle BAC$, $\angle BDM = \angle BAD$, $\angle DMC = 180° - \angle BDM - \angle BCM = 180° - \angle BAD - \angle BAC = 180° - \angle DAC$; consequently, $ADMC$ is an inscribed quadrilateral, $\angle AMC = \beta$. Further, if the tangent BK intersects DM at a

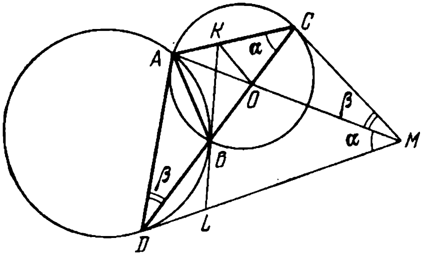


Fig. 49

point L, then $\angle KBO = \angle LBD = \angle BDL = \angle CAM$; hence, $KABO$ is also an inscribed quadrilateral, and $\angle KOA = \angle KBA = \beta$, that is, KO is parallel to CM (the cases of other relative positions of the points D, B, and C are considered in similar fashion).

276. Since the circle with diameter CD passes through a fixed point A on MN ($MN \perp CD$), the quantity

$$|CN|\cdot|ND| = |NA|^2 \qquad (1)$$

is constant. Denote the point of intersection of PQ and MN by K. Let us show that $\frac{|MK|}{|KN|}$ is a constant. Note that $\angle PNQ = 180° - \angle PMQ$; hence,

$$\frac{|MK|}{|KN|}=\frac{S_{PMQ}}{S_{PQN}}=\frac{|PM|\cdot|MQ|}{|PN|\cdot|NQ|}=\frac{|MN|}{|CN|}\cdot\frac{|MN|}{|ND|}=$$
$\frac{|MN|^2}{|AN|^2}$ (we have used Equality (1) and the fact that the triangle MNP is similar to the triangle MNC, and the triangle MNQ to the triangle MND).

277. The equality $\angle O_1AO_2 = \angle MAN$ follows from the result of Problem 279 of Sec. 1, the equality $\angle O_1AO_2 = 2\angle CAE$ was proved when solving Problem 275, Sec. 1.

278. Let O and O_1 denote the centres of the two circles under consideration (O the midpoint of AB), K the point of tangency of the circles (K on the straight line OO_1), N the point of contact of the circle O_1 with the straight line CD, M the point of intersection of AB and CD. Since O_1N is parallel to AB, and the triangles KO_1N and KOA are isosceles and similar, the points K, N, and A are collinear. Let t denote the tangent to the circle O_1 from the point A (the circle O_1 is assumed to lie inside the segment CBD). We have: $t^2 = |AN| \times |AK| = |AN|^2 + |AN|\cdot|NK| = |AM|^2 + |MN|^2 + |CN|\cdot|ND| = |AM|^2 + |MN|^2 + (|CM| - |MN|)(|CM| + |MN|) = |AM|^2 + |CM|^2 = |AK|^2$, which was to be proved.

279. Let A be the midpoint of the arc of the given circle not contained by the segment, and let the tangents from A to the circles inscribed in the segment be equal (Problem 278 in Sec. 2). This means that A lies on the straight line MN since $|AO_1|^2 - |AO_2|^2 = |O_1M|^2 - |O_2M|^2$, where O_1 and O_2 are the centres of the circles.

280. Consider the general case of arbitrary circles. Let the points F and F' be arranged as in Fig. 50. The notations are clear from the figure. Prove that there is a circle inscribed in the quadrilateral $AKBM$, and then use the result of Problem 55 of Sec. 2. To this end, it suffices to prove that (see Problems 241 and 242 of Sec. 2)

$$|BF| + |BF'| = |AF'| + |AF|. \qquad (1)$$

Bearing in mind that $|BL| = |BT|$, and $|FS| = |FT|$, we get: $|BF| = |BL| - |FS|$, and similarly, $|FA| = |FQ| - |AE|$, $|BF'| = |F'P| - |BL|$, $|F'A| = |AE| - |F'R|$. Substituting these expressions into (1), we get:

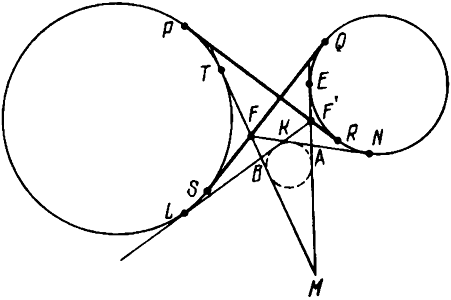


Fig. 50

$|BL| - |FS| + |F'P| - |BL| = |AE| - |F'R| + |FQ| - |AE| \Rightarrow |F'R| + |F'P| = |FQ| + |FS| \Rightarrow |PR| = |SQ|$. The remaining cases of arrangement of the points F and F' on the tangents are considered exactly in the same way (the results of Problems 241 and 242 of Sec. 2 being taken into account). Since each tangent is divided into four parts by the points of tangency and the point of intersection, we have $1/2 \times 4^2 = 8$ cases.

To prove the second part of the problem, we note that the midpoints of AB, FF' and the centre of the third circle O_3, inscribed in $AKBM$, lie on a straight line (see Problem 243 in Sec. 2). But since the radii of the given circles are equal, AB is parallel to O_1O_2 (O_1, O_2 the centres of the given circles); A and B lie on the straight lines O_1O_3 and

O_2O_3, respectively. Hence, the straight line passing through O_3 and the midpoint of AB bisects O_1O_2.

281. Let M be the point of intersection of the tangents l_1, m_1, and n_1, N the point of intersection of l_2 and m_2 (Fig. 51). Through N, we draw a straight line n_2', touching α, distinct from l_2. In the same way, as it was done in Problem 280 of

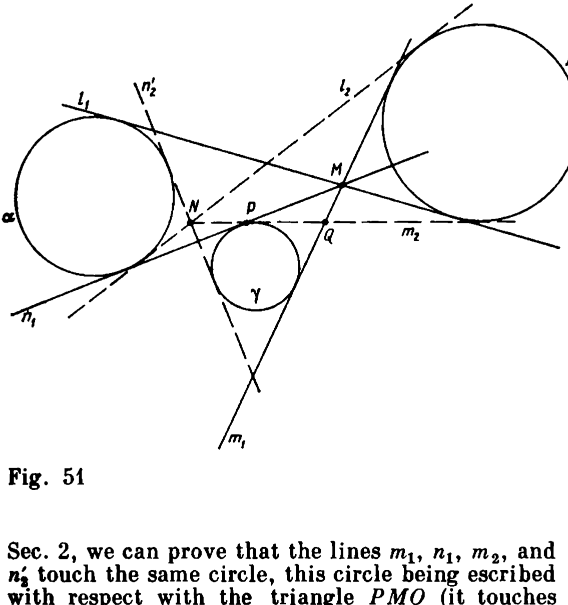


Fig. 51

Sec. 2, we can prove that the lines m_1, n_1, m_2, and n_2' touch the same circle, this circle being escribed with respect with the triangle PMQ (it touches the side PQ), that is, coincides with γ. **Remark.** Figure 51 corresponds to the general case of the arrangement of the circles satisfying the conditions of the problem.

282. Prove that the straight line D_1C passes through O, the centre of the arc AB, and the straight line DC_1 through O_1, the centre of the arc AB_1 (Fig. 52). DAD_1 is a regular triangle, $|DC| = |AC|$, consequently, D_1C is perpendicular to DA, and D_1C passes through O. Analogously,

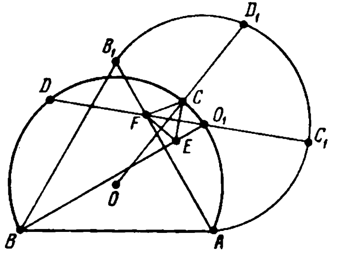


Fig. 52

DC_1 is perpendicular to D_1A. The point O_1 lies on the arc AB since it is obtained from O by rotating the latter about the point A through an angle of $\pi/3$. Let both arcs be measured by the quantity 6α (for convenience, $\alpha > \pi/6$). Then, $\angle AO_1C_1 = 2\alpha$, $\angle O_1C_1A = \pi/2 - \alpha$, $\angle FAC_1 = 2\alpha$. Consequently, $\angle AFC_1 = \pi - 2\alpha - \left(\frac{\pi}{2} - \alpha\right) = \frac{\pi}{2} - \alpha = \angle FC_1A$, that is, $|AF| = |AC_1| = |AC|$. Let us prove that the triangles FAC and EDC are congruent. We have: $|AF| = |AC| = |DC| = |DE|$, $\angle CDE = \angle CDB - \angle BDE = \pi - 2\alpha - (\pi - 2\angle DBE) = -2\alpha + 2\left(2\alpha - \frac{\pi}{6}\right) = 2\alpha - \frac{\pi}{3} = \angle FAC$; thus, $|FC| = |CE|$. Further, we find $\angle DCE = \frac{2\pi}{3} - \alpha$, $\angle B_1FD = \frac{\pi}{2} - \alpha$ (measured by half the sum of the corresponding arcs), $\angle B_1FC = \pi - \angle CFA = \frac{\pi}{3} + \alpha$, $\angle DFC = \frac{5}{6}\pi$, $\angle DCF = \pi -$

$\frac{5}{6}\pi - \alpha + \frac{\pi}{6} = \frac{\pi}{3} - \alpha$ and, finally, $\angle FCE = \left(\frac{2\pi}{3} - \alpha\right) - \left(\frac{\pi}{3} - \alpha\right) = \frac{\pi}{3}$.

283. Consider two cases: (1) the triangle ABC is circumscribed about the given circle; (2) the given circle touches the extensions of the sides AB and AC.

In the first case, we consider the circle touching both the sides of the angle at points M, N and the circle circumscribed about the triangle ABC internally. Let a, b, c be the sides of the triangle ABC, r the radius of the given circle, $\angle A = \alpha$, $|AM| = |AN| = x$. Let us make use of Ptolemy's generalized theorem (Problem 239 in Sec. 2): $xa = (b - x)c + (c - x)b$, whence $x = \frac{2bc}{a+b+c} = \frac{4S_{ABC}}{(a+b+c)\sin\alpha} = \frac{2r}{\sin\alpha}$, that is, x is constant. (It is possible to prove that MN passes through the centre of the given circle.)

In the second case, we have to take the circle touching externally both the sides of the angle and the circle circumscribed about the triangle ABC.

284. Denote the sides of the triangle ABC in a usual way: a, b, c; let $|BD| = d$, $|AD| = b_1$, $|AM| = x$. Use Ptolemy's generalized theorem (Problem 239 in Sec. 2): $xa + (d - b_1 + x)b = (b - x)c$ whence

$$x = \frac{b(c + b_1 - d)}{a+b+c}. \tag{1}$$

Take on AB a point N such that MN is parallel to BD. We have:

$$|MN| = \frac{x}{b_1}d, \quad |AN| = \frac{x}{b_1}c,$$

$$S_{AMN}=\left(\frac{x}{b_1}\right)^2 S_{ABD}=\left(\frac{x}{b_1}\right)^2 \frac{b_1}{b} S_{ABC}$$
$$=\frac{x^2}{b_1 b} S_{ABC}. \qquad (2)$$

Let r be the radius of the circle touching MN and the extensions of AN and AM. Then from (1) and (2) it follows that

$$r=\frac{2S_{AMN}}{|AM|+|AN|-|MN|}=\frac{2x^2 S_{ABC}}{bx(b_1+c-d)}$$
$$=\frac{2S_{ABC}}{a+b+c},$$

that is, r is equal to the radius of the circle inscribed in the triangle ABC, which was to be proved.

285. Let M and K denote the points of tangency of the circles, with centres at O_1 and O_2, and AC, respectively. It follows from the result of the preceding problem that $\angle O_1DM = \angle OKD = \frac{\varphi}{2}$, $\angle O_2DK = \angle OMD = 90°-\frac{\varphi}{2}$ We extend OK and OM to intersect O_1M and O_2K at points L and P, respectively (Fig. 53). In the trapezoid $LMKP$ with bases LM and PK we have: $\frac{|MO_1|}{|O_1L|}=\frac{|MD|}{|DK|}=\frac{|PO_2|}{|O_2K|}$. Consequently, O_1O_2 passes through the intersection point of the diagonals of the trapezoid—the point O. In addition,

$$\frac{|O_1O|}{|OO_2|}=\frac{|LM|}{|PK|}=\frac{|MK|\tan\frac{\varphi}{2}}{|MK|\cot\frac{\varphi}{2}}=\tan^2\frac{\varphi}{2}.$$

286. The statement of the problem follows from the results of Problems 285 and 232 Sec. 2.

287. The statement of this problem can be proved with the aid of the result of Problem 240, more precisely, of its particular case, when the three circles have a zero radius, that is, they are

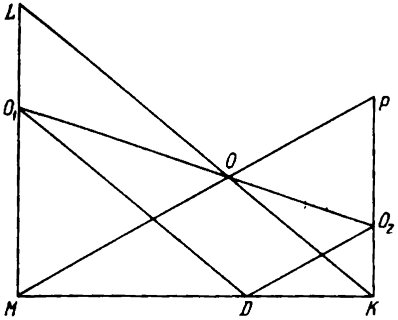


Fig. 53

points. In this case, these points are the midpoints of the sides of the triangle.

288. The statement of this problem follows from Feuerbach's theorem (see Problem 287 in Sec. 2) and from the fact that the triangles ABC, AHB, BHC, and CHA have the same nine-point circle (the proof is left to the reader).

289. Let in the triangle ABC, for definiteness, $a \leqslant b \leqslant c$. Denote the midpoints of the sides BC, CA, and AB by A_1, B_1, and C_1, respectively, and the points of tangency of the inscribed and escribed circles and the nine-point circle of the triangle ABC by F, F_a, F_b, F_c, respectively. We have to prove that in the hexagon $C_1F_cFA_1F_aF_b$ (the points taken in the indicated order form a hexagon since $a \leqslant b \leqslant c$) the diagonals C_1A_1, F_cF_a, and FF_a meet at a point. To this end, it suffices to

prove (see Problem 49 of Sec. 2) that

$$|C_1F_c|\cdot|FA_1|\,|F_aF_b| = |F_cF| \times |A_1F_a|\,|F_bC_1|. \quad (1)$$

Using the formulas obtained in Problem 201 of Sec. 1, we find

$$|C_1F_c| = \frac{b-a}{2}\sqrt{\frac{R}{R+2r_c}},$$

$$|FA_1| = \frac{c-b}{2}\sqrt{\frac{R}{R-2r}},$$

$$|F_aF_b| = \frac{(a+b)R}{\sqrt{R+2r_a}\cdot\sqrt{R+2r_b}},$$

$$|F_cF| = \frac{(b-a)R}{\sqrt{R-2r}\cdot\sqrt{R+2r_c}},$$

$$|A_1F_a| = \frac{c-b}{2}\sqrt{\frac{R}{R+2r_a}},$$

$$|F_bC_1| = \frac{a+b}{2}\sqrt{\frac{R}{R+2r_b}}.$$

Now, the equality (1) can be readily checked.
Remark. It is possible to prove that the intersection points of the opposite sides of the quadrilateral whose vertices are the points of tangency of the inscribed and escribed circles of the given triangle with its nine-point circle lie on the extensions of the midlines of this triangle.

290. Using the formulas of Problems 193, 194, and 289 in Sec. 2 (in the last problem, see its solution), we find: $\frac{|F_bF_c|}{|B_1C_1|} = \frac{(a+b)(b+c)(c+a)R^3}{abc\cdot|OI_a|\cdot|OI_b|\cdot|OI_c|}$. The ratios of the other corresponding sides of the triangles $F_aF_bF_c$ and $A_1B_1C_1$ are the same. The similarity of the other pairs of triangles is proved in similar fashion. For $|A_1B_2|$ and the other

quantities, we derive formulas similar to that of Problem 194, Sec. 2.

291. Prove that $\triangle ABP = \triangle ACQ$. For this purpose, it suffices to prove that $\triangle KBP = \triangle ABC$ and $\triangle FCQ = \triangle ABC$ (by two sides and the angle between them): $\angle QAP = \angle CAB + \angle CAQ + \angle BAP = \angle CAB + \angle CAQ + \angle CQA = \angle CAB + 180° - \angle QCA = \angle CAB + 90° - \angle QCF = 90°$ (it was assumed that $\angle CAB \leqslant 90°$; the case $\angle CAB > 90°$ is considered in a similar way).

292. Since $\angle FE_1E = \angle FCE = 90°$, FE_1EC is an inscribed quadrilateral, $\angle FCE_1 = \angle FEE_1 = 60°$. Analogously, FE_1AD is an inscribed quadrilateral, and $\angle E_1DF = \angle E_1AF = 60°$, that is, DE_1C is an equilateral triangle. In similar fashion, we prove that BF_1C is also an equilateral triangle.

293. Let P, Q, and R denote the points of intersection of LB and AC, AN and BC, LB and AN, respectively. Let $|BC| = a$, $|AC| = b$. It suffices to show that $S_{ACQ} = S_{APB}$ (both of these areas differ from the areas under consideration by the area of the triangle APR). By the similarity of the corresponding triangles we get $|CQ| = |PC| = \frac{ab}{a+b}$. Consequently, $S_{ACQ} = \frac{1}{2}|AC| \times |CQ| = \frac{ab^2}{2(a+b)}$, $S_{APB} = S_{ACB} - S_{PCB} = \frac{1}{2}ab - \frac{a^2b}{2(a+b)} = \frac{ab^2}{2(a+b)}$.

295. Prove that the area of the triangle with vertices at the centres of the squares constructed on the sides of the given triangle and located outside it and the area of the triangle with vertices at the centres of the squares constructed on the same sides inside the given triangle are respectively equal to $S + \frac{1}{8}(a^2 + b^2 + c^2)$ and $\Big| S - \frac{1}{8} \times$

$(a^2+b^2+c^2)\Big|$, where a, b, and c are the sides and S the area of the given triangle.

296. Denote: $\angle A_1BC=\alpha$, $\angle A_1CB=\beta$; then AA_1 divides BC in the ratio equal to $\frac{S_{ABA_1}}{S_{ACA_1}}=$

$$\frac{\frac{1}{2}|AB|\cdot|BA_1|\sin(\angle B+\alpha)}{\frac{1}{2}|AC|\cdot|CA_1|\sin(\angle C+\beta)}=\frac{c}{b}\,\frac{\sin\beta}{\sin\alpha}\times$$

$\frac{\sin(\angle B+\alpha)}{\sin(\angle C+\beta)}$. Having carried out similar computations for the other sides of the triangle ABC, use Ceva's theorem (Problem 44 of Sec. 2).

297. Let KL be the arc contained inside the triangle ABC. Extending the sides AB and BC

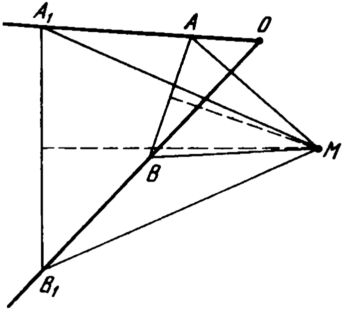


Fig. 54

beyond the point B, we get the arc MN symmetric to the arc KL with respect to the diameter parallel to AC. Since $\angle B$ is measured by the arc equal to $\frac{1}{2}(\smile KL+\smile MN)=\smile KL$, the arc KL has a constant length, and a central angle equal to the angle B corresponds to it.

298. Let O be the intersection point of the straight lines, A and A_1 two positions of the point on one line of different instants, B and B_1 the positions of the other point on the other line at the same instants (Fig. 54). Erect perpendiculars at the midpoints of AB and A_1B_1 and denote the point of their intersection by M: $\triangle AA_1M = \triangle BB_1M$ since they have three equal sides: one is obtained from the other by rotation through the angle AOB with centre at M. This rotation makes a point on AO go into the corresponding position of a point on OB so that the point M possesses the required property.

299. (a) Let A and B denote the points of intersection of the circles, A the starting point of the cyclists, M and N the positions of the cyclists at a certain instant of time. If M and N are on the same side of AB, then $\angle ABM = \angle ABN$, if they are on both sides, then $\angle ABM + \angle ABN = 180°$, that is, the points B, M, and N lie on a straight tine. If L and K are two points of the circles diametrically opposite to B (L and K are fixed), then, since $\angle LNM = \angle NMK = 90°$, the point P which is the midpoint of LK is equidistant from N and M. We can make sure that P is symmetric to the point B with respect to the midpoint joining the centres of the circles (Fig. 55, *a*).

(b) Let O_1 and O_2 denote the centres of the circles. Take a point A_1 such that $O_1AO_2A_1$ is a parallelogram. It can be easily seen that the triangle MO_1A_1 is congruent to the triangle NO_2A_1 since $|MO_1| = |O_1A| = |O_2A_1|$, $|O_1A_1| = |O_2A| = |NO_2|$, $\angle MO_1A_1 = \varphi + \angle AO_1A_1 = \varphi + \angle AO_2A_1 = \angle NO_2A_1$, where φ is the angle corresponding to the arcs covered by the cyclist (Fig. 55, *b*). Thus, the sought-for points are symmetric to the points of intersection of the circles with respect to the midpoint of the line segment O_1O_2. **Remark.** In Item (a) we could proceed just

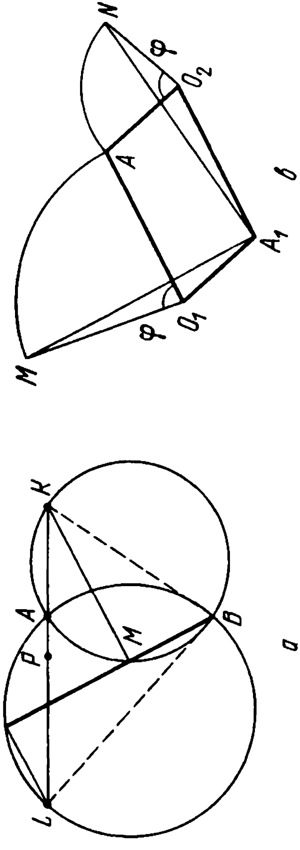


Fig. 55

in the same way as in Item (b). Namely, taking the point P so that $\triangle O_1PO_2 = \triangle O_1AO_2$ (A and P are on the same side of O_1O_2 and do not coincide), it is easy to prove that the corresponding triangles are congruent.

300. (b) Use the result of Item (a). Replace the rotation about O_1 by two axial symmetry mappings, taking the straight line O_1O_2 as the axis of symmetry for the second mapping and the rotation about the point O_2 by two symmetry mappings, taking the straight line O_1O_2 as the axis of the symmetry for the first mapping. **Remark.** If $\alpha + \beta = 2\pi$, then the application of the given rotations in succession, as it is easy to make sure, is equivalent to a translation.

Answer: if $\alpha + \beta < 2\pi$, then the angles are equal to $\frac{\alpha}{2}$, $\frac{\beta}{2}$, $\pi - \frac{\alpha+\beta}{2}$, and if $\alpha + \beta > 2\pi$, then the angles are equal to $\pi - \frac{\alpha}{2}$, $\pi - \frac{\beta}{2}$, $\frac{\alpha+\beta}{2}$.

301. Let us carry out three successive rotations in the same direction about the points K, L, and M (or about K_1, L_1, and M_1) through the angles α, β, and γ. Since $\alpha + \beta + \gamma = 2\pi$, the transformation obtained in a translation (see Problem 300 in Sec. 2). But since one of the vertices of the original triangle remains fixed in these rotations, all the points of the plane must remain fixed.

Thus, the centre of the third rotation (the point M) must coincide with the centre of the rotation resulting from application in succession of the first two rotations: about the points K and L. Now, take advantage of the result of the preceding problem.

302. Denote: $\angle BOC = 2\alpha$, $\angle DOE = 2\beta$, $\angle FOA = 2\gamma$. Let K, M, and L be, respectively, the intersection points of the circles circumscribed about the triangles BOC and AOF, BOC and DOE, AOF and DOE. The point K lies inside the triangle AOB, and $\angle BKO = 180° - \angle BCO = 90° + \alpha$, $\angle AKO = 90° + \gamma$, and since $\alpha + \beta + \gamma = 90°$, $\angle AKB = 90° + \beta$. Similarly, L lies inside the triangle FOE, and $\angle OLF = 90° + \gamma$, $\angle OLE = 90° + \beta$, $\angle FLE = 90° + \alpha$. Hence, $|OL| = |AK|$, $\angle KOL = 2\gamma + \angle KOA + \angle LOF = 2\gamma + \angle KOA + \angle KAO = 90° + \gamma = \angle AKO$; thus, the triangles KOL and AKO are congruent, that is, $|KL| = |AO| = R$. We then prove in a similar way that each of the two other sides of the triangle KLM is equal to R.

303. Let $ABCD$ denote the given quadrilateral, O_1, O_2, O_3, O_4 the centres of the rhombi constructed on AB, BC, CD, DA, respectively; K and L the midpoints of the sides AB and BC, respectively, M the midpoint of the diagonal AC. The triangles O_1KM and O_2LM are congruent $\left(|O_1K| = \frac{1}{2}|AB| \quad |LM|, \quad |KM| \quad \frac{1}{2}|BC| = |O_2L|, \ \angle O_1KM = \angle O_2LM\right)$ If $\angle ABC + \alpha < \pi$, then these triangles are located inside the triangle O_1MO_2, and if $\angle ABC + \alpha > \pi$, then they are found outside the triangle O_1MO_2 (the angles of the rhombi with vertex at B are equal to α). Thus, $|O_1M| = |O_2M|$, $\angle O_1MO_2 = \pi - \alpha$. In similar fashion, $|O_3M| = |O_4M|$, $\angle O_3MO_4 = \pi - \alpha$. Consequently, the triangles O_1MO_3 and O_2MO_4 are congruent, and one is obtained from the other by a rotation about M through the angle $\pi - \alpha$. Hence, there follows the statement of the problem.

304. Let ABC be the given triangle, $A_1B_1C_1$ the triangle Δ, $A_2B_2C_2$ the triangle δ (A_1 and A_2 the centres of the triangles constructed on BC), a, b, c the sides of the triangle ABC.

(a) The fact that $A_1B_1C_1$ and $A_2B_2C_2$ are regular triangles follows, for instance, from the result of Problem 301, Sec. 2.

(b) Let us prove a more general assertion. If on the sides of the triangle ABC there constructed externally (or internally) similar triangles A_1BC, B_1CA, C_1AB so that $\angle A_1BC = \angle B_1CA = \angle C_1AB$, $\angle A_1CB = \angle B_1AC = \angle C_1BA$, then the median points of the triangles ABC and $A_1B_1C_1$ coincide. First note that if M is the point of intersection of the medians of the triangle ABC, then $\overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC} = 0$, and, conversely, if this equality is fulfilled, then M is the median point of the triangle ABC. It remains to check that $\overrightarrow{MA_1} + \overrightarrow{MB_1} + \overrightarrow{MC_1} = 0$ or $(\overrightarrow{MA} + \overrightarrow{AC_1}) + (\overrightarrow{MB} + \overrightarrow{BA_1}) + (\overrightarrow{MC} + \overrightarrow{CB_1}) = 0$. But $\overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC} = 0$. In addition, $\overrightarrow{AC_1} + \overrightarrow{BA_1} + \overrightarrow{CB_1} = 0$ since each of the vectors $\overrightarrow{AC_1}$, $\overrightarrow{BA_1}$, $\overrightarrow{CB_1}$ is obtained from the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$, respectively, by rotating the latter through the same angle ($\angle A_1BC$) and multiplying by the same number.

(c) Consider a more general case. The isosceles triangles A_1BC, B_1CA, C_1BA and $A_1'BC$, $B_1'CA$, $C_1'BA$ in which the ratio of the length of the altitude drawn to the base to the length of the base is equal to k are constructed on the sides of the triangle ABC externally and internally as on bases. Let O denote the centre of the circle circumscribed about the triangle ABC; a, b, c its sides; A_0, B_0, C_0 the midpoints of BC, CA, AB, respectively. For definiteness, we assume ABC to be acute triangle. Then, $S_{A_1OC_1} = \frac{1}{2} |A_1O| \cdot |C_1O| \sin B = \frac{1}{2}(|OA_0| + ka)(|OC_0| + kc) \sin B = \frac{1}{2} |OA_0| \times$

$|OC_0|\sin B+\frac{1}{2}k^2ac\sin B+\frac{k}{2}(a|OC_0|+c|OA_0|)\times$ $\sin B=k^2S_{ABC}+S_{A_0OC_0}+\frac{k}{4}b^2$. Obtaining similar relationships for the triangles A_1OB_1 and B_1OC_1 and adding them together, we find: $S_{A_1B_1C_1}=$ $\left(3k^2+\frac{1}{4}\right)S_{ABC}+\frac{k}{4}(a^2+b^2+c^2)$ (this equality is also valid for an obtuse triangle ABC). For the triangle $A_1'B_1'C_1'$ we have: $S_{A_1'B_1'C_1'}=\left|\frac{k}{4}(a^2+b^2+c^2)-\left(3k^2+\frac{1}{4}\right)S_{ABC}\right|$. Consequently, if $\frac{k}{4}\times$ $(a^2+b^2+c^2)-\left(3k^2+\frac{1}{4}\right)S_{ABC}\geqslant 0$, then $S_{A_1B_1C_1}-S_{A_1'B_1'C_1'}=\left(6k^2+\frac{1}{2}\right)S_{ABC}$, and if $\frac{k}{4}\times$ $(a^2+b^2+c^2)-\left(3k^2+\frac{1}{4}\right)S_{ABC}<0$, then $S_{A_1B_1C_1}-S_{A_1'B_1'C_1'}=\frac{k}{2}(a^2+b^2+c^2)$. We can prove that always $a^2+b^2+c^2\geqslant 4\sqrt{3}\,S_{ABC}$ (in Problem 362 of Sec. 2, a stronger inequality is proved), and this means that for $k=\frac{1}{2\sqrt{3}}$ the difference between the areas of the triangles $A_1B_1C_1$ and $A_1'B_1'C_1'$ is equal to S_{ABC}.

305. Let the three given points form a triangle ABC. Two families of regular triangles circumscribed about the triangle ABC are possible. The first family is obtained in the following way. Let us construct circles on the sides of the triangle so that the arcs of these circles lying outside the triangle are measured by the angle of $4\pi/3$. We take an arbitrary point A_1 on the circle constructed on BC. The straight line A_1B intersects the circle constructed on BA for the second time at a point

C_1, and the straight line A_1C intersects the circle constructed on CA at a point B_1. The triangle $A_1B_1C_1$ is one of the triangles belonging to the first family. Let E, F, and G denote the intersection points of the angle bisectors of the triangle $A_1B_1C_1$ and the circles constructed on the sides of the given triangle. The points E, F, and G are fixed (E the midpoint of the arc of the circle constructed on BC and situated on the same side of BC with the triangle ABC). The points E, F, and G are the centres of the equilateral triangles constructed on the sides of the triangle ABC internally. The triangle EFG is a regular one (see Problem 304 in Sec. 2), its centre coinciding with the median point of the triangle ABC. The centre of the triangle $A_1B_1C_1$ lies on the circle circumscribed about the triangle EFG; the square of the radius of this circle being equal to $\frac{1}{9}\left(\frac{a^2+b^2+c^2}{2}-2S\sqrt{3}\right)$, where a, b, and c are the sides and S the area of the triangle ABC (see the solution of Problem 304 of Sec. 2).

The second family of equilateral triangles circumscribed about the triangle ABC is obtained if on the sides of the triangle ABC circles are constructed whose arcs located outside the triangle ABC are equal to $2\pi/3$ (each).

The required locus consists of two concentric circles whose centres coincide with the median point of the triangle ABC, and the radii are equal to $\frac{1}{3}\sqrt{\frac{1}{2}(a^2+b^2+c^2)\pm 2S\sqrt{3}}$.

306. Prove that the triangles CB_1A_2 and CA_1B_2 are obtained one from the other by rotation about the point C through an angle of 90°. Indeed, $\triangle CAA_1 = \triangle CBB_1$ ($|BB_1| = |AC|$, $|BC| = |AA_1|$, $\angle CBB_1 = \angle CAA_1$), and since $AA_1 \perp BC$ and $BB_1 \perp AC$, we have: $B_1C \perp A_1C$. Similarly, A_2C and B_2C are equal to each other and mutually perpendicular.

307. Prove that the tangents to the circle drawn from the vertices between which one of the vertices of the polygon is located are equal to each other. Hence, it follows that for a polygon with an odd number of sides the points of tangency are the midpoints of the sides.

308. Note that if we consider the system of vectors whose initial points lie at the centre of the regular n-gon and whose terminal points are at its vertices, then the sum of these vectors equals zero. Indeed, if all of these vectors are rotated through an angle of $2\pi/n$, then their sum remains unchanged, and on the other hand, the vector equal to their sum rotates through the same angle. Hence, the sum of the projections of these vectors on any axis is also equal to zero.

Let us return to our problem. If φ is the angle between the given straight line (let us denote it by l) and one of the vectors, then the remaining vectors form the angles $\varphi+\frac{2\pi}{n}$, $\varphi+2\frac{2\pi}{n}$, $\varphi+(n-1)\frac{2\pi}{n}$. The square of the distance from the kth vertex to l is equal to $\sin^2\left(\varphi+k\frac{2\pi}{n}\right)^2=\frac{1}{2}\left(1-\cos\left(2\varphi+k\frac{4\pi}{n}\right)\right)$ But the quantities $\left(2\varphi+k\frac{4\pi}{n}\right)$ can be regarded as projections on l of the system of n vectors forming angles $2\varphi+k\frac{4\pi}{n}$ ($k=0, 1, \quad n-1$) with l. If n is odd, these vectors form a regular n-gon, if n is even, then they yield an $\frac{n}{2}$-gon repeated twice.

Answer: $\frac{n}{2}$.

309. (a) If the side of the polygon is equal to a, S is its area, $x_1, x_2, \ldots, x_n$ are distances from a certain point inside the polygon to its sides, then the statement of the problem follows from the equality $S = (ax_1 + ax_2 + \ldots + ax_n)/2$.

(b) Consider the regular polygon containing the given one whose sides are parallel to the sides of the given polygon. The sum of distances from an arbitrary point inside the given polygon to the sides of the regular polygon is constant (Item (a)) and differs from the sum of the distances to the sides of the given polygon by a constant.

310. Let $B_1, B_2, \ldots, B_{n+1}$ denote the points symmetric to $A_1, A_2, \ldots, A_{n+1}$ with respect to the diameter A_0A_{2n+1}, C_k and C'_k the points of intersection of the straight line A_kA_{2n+1-k} with OA_n and OA_{n+1}. Let D_{k-1} and D_k be the points of intersection of the straight lines A_kB_{k-1} and A_kB_{k+1} with the diameter. Obviously, the same points are the points of intersection of the straight lines B_kA_{k-1} and B_kA_{k+1} with the diameter. It is also obvious that the triangle $D_{k-1}A_kD_k$ is congruent to the triangle $C_kOC'_k$. Thus, the sum of the line segments $C_kC'_k$ is equal to the sum of the line segments $D_{k-1}D_k$ $(k = 1, \ldots, n)$, $D_0 = A_0$, $D_n = 0$, that is, equals the radius.

311. Let A (Fig. 56) be the given point, A_k a vertex of the $2n$-gon, B_{k-1} and B_k the feet of the perpendiculars dropped from the point A on the sides enclosing A_k, and α_k and β_k the angles formed by the straight line AA_k with those sides ($\beta_k = \angle AA_kB_{k-1}$, $\alpha_k = \angle AA_kB_k$). Since a circle can be circumscribed about the quadrilateral $AB_{k-1}A_kB_k$, we have: $\angle AB_{k-1}B_k = \alpha_k$, $\angle AB_kB_{k-1} = \beta_k$ (or supplement these angles to 180°); thus, by the law of sines, $\dfrac{|AB_{k-1}|}{|AB_k|} = \dfrac{\sin\beta_k}{\sin\alpha_k}$, $\dfrac{|AB_{k-1}|\,|AB_{k+1}|}{|AB_k|^2} = \dfrac{\sin\beta_k \sin\alpha_{k+1}}{\sin\alpha_k \sin\beta_{k+1}}$.
Multiplying those equalities for $k = 2, 4, \ldots,$

$2n$ and replacing the index $2n+1$ by 1, we get the desired result.

312. Prove that if O_k and O_{k+1} are the centres of the circles touching the given circle at points A_k and A_{k+1}; B the point of their intersection lying on the chord A_kA_{k+1}; r_k, r_{k+1} their radii, then $r_k + r_{k+1} = r$, $\angle A_kO_kB = \angle A_{k+1}O_{k+1}B = \angle A_kOA_{k+1}$ (r the radius of the given circle, O

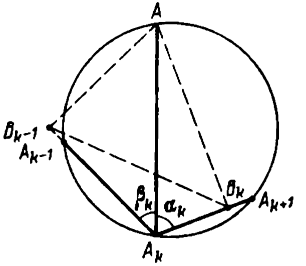


Fig. 56

its centre). Hence it follows the equality of every other radii, which for an odd n means that all of them are equal to $r/2$. In addition, $\smile A_kB + \smile BA_{k+1} = \smile A_kA_{k+1}$ (the minor arcs of the corresponding circles are taken).

313. (a) Let A be an arbitrary point of the circle (A on the arc A_1A_{2n+1}). Let a denote the side of the polygon, and b the length of the diagonal joining every other vertex. By Ptolemy's theorem (Problem 237 in Sec. 2), for the quadrilateral $AA_kA_{k+1}A_{k+2}$ we have: $|AA_k|a + |AA_{k+2}|a = |AA_{k+1}|b$ $(k = 1, 2, \ldots, 2n-1)$. Similar relationships can be written for the quadrilaterals $A_{2n}A_{2n+1}AA_1$ and $A_{2n+1}AA_1A_2$:

$$|AA_1|a + |AA_{2n+1}|b = |AA_{2n}|a,$$

$$|AA_{2n+1}|a + |AA_1|b = |AA_2|a.$$

Adding together all these equalities and leaving even vertices on the right and odd vertices on the left, we get the required statement.

(b) Our statement follows from Item (a) and the result of the Problem 206 of Sec. 1 (A similar formula can be obtained for the case of internal tangency.)

314. (a) Let l intersect AC and BC at points K and N, respectively, and touch the circle at a point M (Fig. 57). Let us denote: $|AC| =$

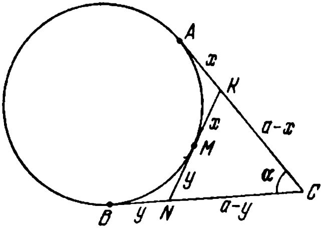


Fig. 57

$|BC| = a$, $|AK| = |KM| = x$, $|BN| = |NM| = y$. Obviously, $\dfrac{w^2}{uv} = \dfrac{(a-x)(a-y)}{xy}$, but, by the law of cosines, for the triangle CKN the following equality holds true: $(x+y)^2 = (a-x)^2 + (a-y)^2 - 2(a-x)(a-y)\cos\alpha \Rightarrow$ $\sin^2\dfrac{\alpha}{2} = \dfrac{xy}{(a-x)(a-y)}$. Thus, $\dfrac{uv}{w^2} = \sin^2\dfrac{\alpha}{2}$. (Other cases of arrangement of the line l are considered in a similar way.)

(b) Let us use the result of Item (a). Multiplying the corresponding equalities for all the angles of the n-gon, we get the square of the sought-for ratio, and the ratio itself turns out to be equal to $1\Big/\left(\sin\dfrac{\alpha_1}{2}\sin\dfrac{\alpha_2}{2} \quad \sin\dfrac{\alpha_n}{2}\right)$ where α_1, $\alpha_2, \ldots, \alpha_n$ are the angles of the polygon.

(c) We use the result of Item (a). We denote the points of tangency of the sides A_1A_2, A_2A_3, $A_{2n-1}A_{2n}$, $A_{2n}A_1$ with the circle by B_1, B_2, . . ., B_{2n-1}, B_{2n}, respectively; the distances from A_1, A_2, . . ., A_{2n} to l by x_1, x_2, . . ., x_{2n-1}, x_{2n}, respectively; the distances from B_1, B_2, . . ., B_{2n} to l by y_1, y_2, y_{2n}, respectively. Then we get:

$$\frac{x_1^2}{y_{2n}y_1}=\frac{1}{\sin^2\frac{\alpha_1}{2}},\quad \frac{x_2^2}{y_1y_2}=\frac{1}{\sin^2\frac{\alpha_2}{2}},$$

$$\frac{x_{2n}^2}{y_{2n-1}y_{2n}}=\frac{1}{\sin^2\frac{\alpha_{2n}}{2}},$$

where α_1, α_2, . . ., α_{2n} are the angles of the polygon. Multiplying the equalities containing x_1, x_3, . . ., x_{2n-1} and dividing them by the product of the remaining equalities, we get:

$$\left(\frac{x_1x_3\ \ldots\ x_{2n-1}}{x_2x_4\ \ \ \ x_{2n}}\right)^2 = \left(\frac{\sin\frac{\alpha_2}{2}\sin\frac{\alpha_4}{2}\ \ \ \ \sin\frac{\alpha_{2n}}{2}}{\sin\frac{\alpha_1}{2}\sin\frac{\alpha_3}{2}\ \ \ \ \sin\frac{\alpha_{2n-1}}{2}}\right)^2$$

315. The statement of the problem can be proved by induction. The beginning step of the proof, $n = 4$, is considered in Problem 235 of Sec. 2.

However, we can suggest another way of solution based on the following equality. Let in the triangle ABC the angle A be the greatest, r and R the radii of the inscribed and circumscribed circles, respectively, d_a, d_b, and d_c the distances from the centre of the circumscribed circle to the corresponding sides of the triangle. Then

$$r+R = d_a + d_b + d_c \qquad (1)$$

for an acute triangle and

$$r + R = -d_a + d_b + d_c \qquad (2)$$

for an obtuse one (for a right triangle, $d_a = 0$ and for it any of the above relationships holds true).

Proof. Let ABC be an acute triangle; A_0, B_0, C_0 the midpoints of the sides BC, CA, AB, respectively; O the centre of the circumscribed circle. By Ptolemy's theorem (Problem 237 in Sec. 2), for the quadrilateral AB_0OC_0 we have: $\frac{b}{2}d_c + \frac{c}{2}d_b = \frac{a}{2}R$. Writing two more similar relationships for the quadrilaterals BC_0OA_0 and CB_0OA_0 and adding them together, we get:

$$\left(\frac{a}{2}+\frac{b}{2}\right)d_c + \left(\frac{a}{2}+\frac{c}{2}\right)d_b + \left(\frac{b}{2}+\frac{c}{2}\right)d_a = \frac{1}{2}(a+b+c)R = pR,$$

whence $p(d_a + d_b + d_c) - \frac{1}{2}(cd_c + bd_b + ad_a) = pR$. Since $\frac{1}{2}(cd_c + bd_b + ad_a) = S = pr$, after reducing by p, we get the equality (1). The case $\angle A > 90°$ is considered in a similar way.

The statement of the problem follows from the relationships (1) and (2). To this end, let us write the corresponding equalities for all the triangles of the partition. Note that each of the diagonals serves as a side for the two triangles. Consequently, the distance to the chosen diagonal enters the relationships, corresponding to these triangles, with opposite signs. Hence, adding together all these equalities, we get (provided that the centre of the circle lies inside the polygon): $\sum r + R = d_1 + d_2 + \ldots + d_n$, where $d_1, d_2, \ldots, d_n$ are the distances from the centre of the circle to the sides of the polygon. If the centre of the circle is outside the polygon, then the distance to the greatest side should be taken with the minus sign.

316. Consider, for definiteness, the case when the point M is found inside the polygon. Let u and v denote the distances from M to A_1A_2 and A_1A_n, respectively; x and y the projections of A_1M on A_1A_2 and A_1A_n (x and y should be assumed to be positive, if these projections are situated on the rays A_1A_2 and A_1A_n, and negative otherwise). $|A_1B_1| = |A_1B_n| = a$, $\angle A_2A_1A_n = \alpha$. The distances u and v can be expressed in terms of x and y: $u = \frac{y}{\sin\alpha} - x\frac{\cos\alpha}{\sin\alpha}$, $v = \frac{x}{\sin\alpha} - y\frac{\cos\alpha}{\sin\alpha}$; hence $u + v = (x + y)\frac{1 - \cos\alpha}{\sin\alpha} = (x + y)\tan\frac{\alpha}{2} = (x + y)\frac{r}{a}$. We now have:

$$\begin{aligned}(|MB_1|^2 + |MB_n|^2)\,a &= ((x - a)^2 + u^2 + (y - a)^2 + v^2)\,a\\ &= ((x - a)^2 + (u - r)^2 + (y - a)^2 + (v - r)^2 + 2r(u + v) - 2r^2)\,a\\ &= 2d^2a + 2ra(u + v) - 2r^2a = 2d^2a + 2r^2(x + y) - 2r^2a.\end{aligned}$$

Writing similar equalities for each of the vertices and adding them together, we get the statement of the problem.

317. Consider three triangles ABC, ACD, and ADB having a common vertex A. Denote the projections of M on AB, AC, and AD by B_1, C_1, and D_1, respectively. The straight lines B_1C_1, C_1D_1, and D_1B_1 are Simson's lines of the point M with respect to the triangles ABC, ACD, and ADB. But the points A, M, B_1, C_1, and D_1 lie on the same circle (AM being its diameter). Consequently, the projections of the point M on B_1C_1,

C_1D_1, and D_1B_1 lie on a straight line which is the Simson line of the point M with respect to the triangle $B_1C_1D_1$. Considering then the projections of the point on Simson's lines corresponding to the three triangles with a common vertex B, we get that those three projections also lie on a straight line, hence, the four projections are collinear.

The passage by induction from n to $n + 1$ is performed exactly in the same way.

318. Let, for definiteness, B_1 lie on the arc A_1A_2 which bounds the segment not containing the circle

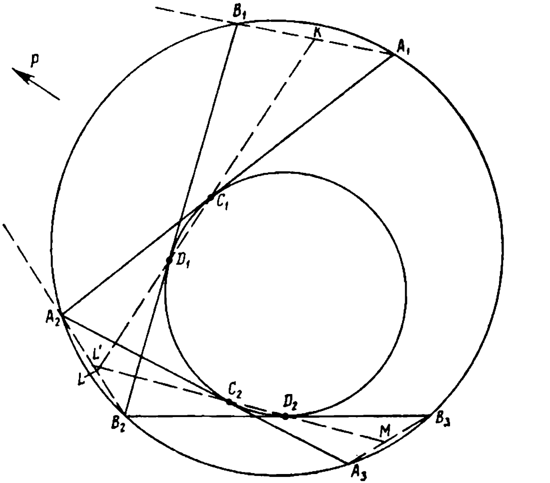


Fig. 58

β. Let C_1, C_2,   denote the points of tangency of A_1A_2, A_2A_3,   with the circle β, respectively; D_1, D_2,   . the points of tangency of B_1B_2, B_2B_3,   with the same circle (Fig. 58); K, L, and P the points of tangency of D_1C_1 and A_1B_1, D_1C_1 and A_2B_2, A_1B_1 and A_2B_2.

In the triangles A_1KC_1 and D_1LB_2, we have: $\angle KC_1A_1 = \angle LD_1B_2$, $\angle C_1A_1K = \angle D_1B_2L$; hence, $\angle C_1KA_1 = \angle D_1LB_2$, that is, KPL is an isosceles triangle, $|KP| = |PL|$.

Consider the circle γ touching KP and PL at points K and L, respectively. The centre of this circle is found on the straight line passing through the centres of α and β (see Problem 12 in Sec. 2).

Let the line D_2C_2 intersect A_2B_2 and A_3B_3 at points L' and M, respectively. As in the preceding case, let us prove that there is a circle γ' with centre on the straight line passing through the centres of α and β and touching A_2B_2 and A_3B_3 at points L' and M, respectively. Let us prove that γ and γ' coincide. To this end, it suffices to prove the coincidence of L and L' We have: $\frac{|A_2L|}{|LB_2|} =$

$$\frac{S_{A_2C_1D_1}}{S_{B_2C_1D_1}} = \frac{\frac{1}{2}|D_1C_1|\cdot|A_2C_1|\sin\angle A_2C_1D_1}{\frac{1}{2}|D_1C_1|\cdot|B_2D_1|\sin\angle B_2D_1C_1} =$$

$\frac{|A_2C_1|}{|B_2D_1|}$. Similarly $\frac{|A_2L'|}{|L'B_2|} = \frac{|A_2C_2|}{|B_2D_2|} = \frac{|A_2C_1|}{|B_2D_1|}$, that is, L and L' coincide. **Remark.** It follows from our reasoning that in the case under consideration the points of tangency of γ with the straight lines A_1B_1, A_2B_2, . ., are found inside the line segments A_1B_1, A_2B_2, . . .

319. Using the notation of the preceding problem, the statement is reduced to the following: if A_{n+1} coincides with A_1, then B_{n+1} coincides with B_1. Suppose the contrary. Then A_1B_1 and A_1B_{n+1} touch the circle γ, A_1A_2 intersects γ, and B_1 and B_{n+1} lie on the arc A_1A_2 corresponding to the segment not containing β. The points of tangency of A_1B_1 and A_1B_{n+1} with γ lie inside the line segments A_1B_1 and A_1B_{n+1}. Thus, we have obtained that two tangents are drawn from A_1 to

γ, the points of their contact with γ being located on the same side of the secant A_1A_2. But this is impossible.

320. Let us consider the triangle B_0XC_0. The straight line XR is the bisector of the angle C_0XB_0. It is readily checked that $\angle C_0RB_0 = \frac{\pi}{2} + \frac{1}{2}\angle C_0XB_0$. Hence, it follows that C_0R and B_0R are the bisectors of the angles XC_0B_0 and XB_0C_0, respectively (see Problem 46 in Sec. 1). In similar fashion, in the triangles C_0YA_0 and A_0ZB_0 the points P and Q are the points of intersection of the angle bisectors. Hence, taking into consideration that $\angle PA_0Q = \angle A/3$, $\angle QB_0R = \angle B/3$, $\angle RC_0P = \angle C/3$, we get the statement from which Morley's theorem follows.

321. When solving the problem, we use the following assertions which can be easily proved.

(a) If a point N is taken on the bisector of the angle M of the triangle KLM (inside this triangle) so that $\angle KNL = \frac{1}{2}(\pi + \angle KML)$, then N is the intersection point of the angle bisectors of the triangle KLM (see Problem 46 of Sec. 1).

(b) If a point N is taken inside the angle KML and outside the triangle KLM on the extension of the bisector of the interior angle M so that $\angle KNL = \frac{1}{2}(\pi - \angle KML)$, then N is the intersection point of the bisector of the angle M and the bisectors of the exterior angles K and L.

(c) If a point N is taken inside the angle KML and on the bisector of the exterior angle K of the triangle KML so that $\angle MNL = \frac{1}{2}\angle MKL$, then N is the intersection point of the bisector of the angle M and the bisectors of the exterior angles K and L.

We carry out the proof of the assertion for all possible values of i, j, k (all in all, seven cases) according to one scheme. Each time we formulate and prove the corresponding converse assertion equivalent to the considered case of Morley's theorem. The preceding problem is an example of following such a scheme. In order to avoid repetition, let us first single out the general part of reasoning. Consider the regular triangle PQR. Constructed on its sides as bases are isosceles triangles PXQ, QYR, RZP (what triangles and how they are constructed is explained for each of the seven cases). Let A_0 denote the point of intersection of the straight lines ZP and YQ, B_0 the point of intersection of XQ and ZR, and C_0 the point of intersection of YR and XP. Then we prove for each case that the triangle $A_0B_0C_0$ is similar to the triangle ABC, and that the rays A_0P and A_0Q, B_0Q and B_0R, C_0R and C_0P are its angle trisectors of the corresponding kind.

Let us now indicate what triangles and how they should be constructed on the sides of the triangle PQR in each case.

(1) $i = j = k = 1$; $\angle PXQ = \frac{1}{3}(\pi + 2\angle A)$, $\angle QYR = \frac{1}{3}(\pi + 2\angle B)$, $\angle RZP = \frac{1}{3}(\pi + 2\angle C)$. All the triangles are arranged externally with respect to the triangle PQR.

(2) $i = 1$, $j = k = 2$, $\angle PXQ = \frac{1}{3}(\pi - 2\angle A)$, $\angle QYR = \pi - \frac{2\angle B}{3}$, $\angle RZP = \pi - \frac{2\angle C}{3}$. All the triangles are arranged externally with respect to the triangle PQR. (We assume that $\angle A < \pi/2$. If $\angle A > \pi/2$, then the triangle PXQ is "turned out" on the other side of the triangle PQR, $\angle PXQ = \frac{1}{3}(2\angle A - \pi)$. If $\angle A = \pi/2$,

then the triangle PXQ turns to a pair of parallel straight lines. This note should be borne in mind when considering the further cases.)

(3) $i = j = 1$, $k = 3$; $\angle PXQ = \frac{1}{3}(\pi - 2\angle A)$,

$\angle QYR = \frac{1}{3}(\pi - 2\angle B)$, $\angle RZP = \frac{1}{3}(\pi + 2\angle C)$. The triangles PXQ and QYR are arranged externally and RZP internally with respect to the triangle PQR (see Item (2)).

(4) $i = j = k = 2$; $\angle PXQ = \frac{1}{3}(\pi - 2\angle A)$,

$\angle QYR = \frac{1}{3}(\pi - 2\angle B)$, $\angle RZP = \frac{1}{3}(\pi - 2\angle C)$. All the triangles and the triangle PQR itself are arranged on the same side of the corresponding sides of the triangle PQR, (see Item (2)).

(5) $i = 1$, $j = 2$, $k = 3$; $\angle PXQ = \frac{1}{3}(\pi + 2\angle A)$,

$\angle QYR = \frac{1}{3}(\pi - 2\angle B)$, $\angle RZP = \pi - \frac{2\angle C}{3}$ The triangle PXQ is constructed externally with respect to the triangle PQR, while the other two internally (see Item 2)).

(6) $i = 2$, $j = k = 3$; $\angle PXQ = \pi - \frac{2\angle A}{3}$,

$\angle QYR = \frac{1}{3}(\pi + 2\angle B)$, $\angle RZP = \frac{1}{3}(\pi + 2\angle C)$. The triangle PXQ is arranged externally and the two others internally with respect to the triangle PQR.

(7) $i = j = k = 3$; $\angle PXQ = \pi - \frac{2\angle A}{3}$, $\angle QYR = \pi - \frac{2\angle B}{3}$, $\angle RZP = \pi - \frac{2\angle C}{3}$. All the triangles are arranged inside the triangle PQR.

Item (1) was proved in Problem 320, Sec. 2.

Let us, for example, prove Item (2).

Let $\angle A < \pi/2$. Consider the triangle B_0XC_0 in which XR is the bisector of the angle B_0XC_0. In addition, $\angle B_0RC_0 = \frac{1}{2}(\pi + \angle B_0XC_0)$. In accordance with the assertion (a), R is the intersection point of the angle bisectors of this triangle (if $A > \pi/2$, then B_0R and C_0R are the bisectors of the exterior angles of the triangle B_0XC_0). Further, in the triangle C_0YA_0 we have: YP is the bisector of the exterior angle Y, $\angle A_0PC_0 = \frac{1}{2}\angle AYC_0$ (this can be readily checked). In accordance with the assertion (c), P is the intersection point of the bisector of the angle C_0A_0Y and the bisectors of the exterior angles A_0C_0Y and C_0YA_0 of the triangle C_0YA_0. In similar fashion, the point Q with respect to the triangle A_0ZB_0 is the intersection point of the bisector of the angle ZA_0B_0 and the bisectors of the exterior angles A_0ZB_0 and A_0B_0Z. (This implies that the triangle PQR, with respect to the triangle $A_0B_0C_0$, is formed by the intersection of the trisectors of the first kind of the angle A_0 with the trisectors of the second kind of the angles B_0 and C_0 (Item (2) is meant).) The triangle $A_0B_0C_0$ itself is similar to the triangle ABC.

In all the remaining items (from 3 to 7) we reason in a similar way varying only the assertion used ((a), (b), (c)).

Interchanging the indices i, j, k, we note that to Item 5 there correspond six regular triangles, to each of Items (2), (3), and (6) three regular triangles, to each of Items (1), (4), and (7) one equilateral triangle. Thus, the total number of regular triangles obtained is eighteen.

Now, in each case we choose the dimensions of the triangle PQR so that the corresponding triangle $A_0B_0C_0$ is equal to the triangle ABC. We superimpose the eighteen obtained drawings by turns so that the triangles ABC are brought into coincidence.

It should be done in the following succession: first, we take the drawing corresponding to Item (1), then the three drawings corresponding to Item (3), then the six drawings corresponding to Item (5), then the three drawings corresponding to Item (2), and, finally, the three drawings from Item (6), one from Item (4) and one from Item (7). In each successive superposition, at least one of the vertices of the corresponding regular triangle must coincide with one of the vertices of the triangles already superimposed. If we count the angles we can see that five vertices of two equilateral triangles, having a common vertex, lie on two straight lines passing through this common vertex. Thus, the vertices of all the eighteen equilateral triangles "must" be arranged, without fail, as in Fig. 59. (In this figure, $\alpha_1\beta_1'$ denotes the point of intersection of the trisectrices α_1 and β_1', etc.).

322. For the equilateral triangle with side equal to 1 the radius of each of Malfatti's circles is equal to $\frac{\sqrt{3}-1}{4}$. The sum of the areas of the corresponding circles equals $\frac{3\pi(2-\sqrt{3})}{8}$. And the sum of the three circles one of which is inscribed in this triangle and each of the two others touches this circle and two of the sides of the triangle is equal to $\frac{11\pi}{108} > \frac{3\pi(2-\sqrt{3})}{8}$.

323. Use the equality $Rr=\frac{abc}{4p}$ and inequality $2p=a+b+c \geqslant 3\sqrt[3]{abc}$ (the mean-value theorem).

324. If p_1 is the semiperimeter of the triangle with its vertices at the feet of the altitudes of the given triangle; p, S, r, and R the semiperimeter, the area, the radii of the inscribed and circumscribed circles, respectively, then $S=pr$ and, in addition, $S=p_1R$ (the latter follows from the fact

that the radius of the circumscribed circle drawn into the vertex of the triangle is perpendicular to the line segment joining the feet of the altitudes dropped on the sides emanating from this vertex). Consequently, $p_1 = p\frac{r}{R} \leqslant \frac{1}{2}p$.

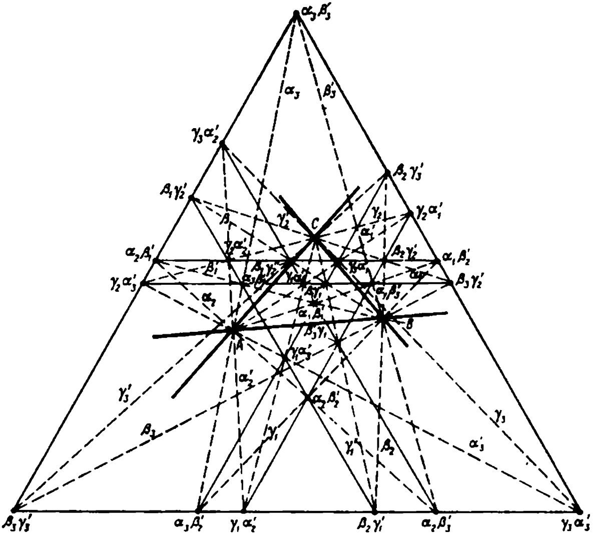


Fig. 59

325. Let m_a be the greatest of the medians. If we use the relationship $m_a^2 > m_b^2 + m_c^2$, following from the hypothesis, and replace the medians by the sides a, b, and c of the triangle (Problem 11 of Sec. 1), we get: $5a^2 < b^2 + c^2$, whence $\cos A > \frac{2(b^2+c^2)}{5bc} = \frac{2}{5}\left(\frac{b}{c}+\frac{c}{b}\right) \geqslant \frac{4}{5} > \frac{\sqrt{2}}{2}$.

326. Let O denote the intersection point of the diagonals of the quadrilateral $ABCD$. Suppose that all the angles indicated in the hypothesis are greater than $\pi/4$. Then, on the line segments OB and OC, we can take, respectively, points B_1 and C_1 such that $\angle B_1AO = \angle OB_1C_1 = \pi/4$. Let $\angle BOA = \alpha > \pi/4$. We have:

$$|OC| > |OC_1| = \frac{|OB_1|}{\sqrt{2}\sin\left(\alpha - \frac{\pi}{4}\right)} =$$

$$\frac{|OA|}{2\sin\left(\alpha-\frac{\pi}{4}\right)\sin\left(\alpha+\frac{\pi}{4}\right)} = -\frac{|OA|}{\cos 2\alpha} \geqslant |OA|.$$

In similar fashion, we prove the inequality $|OA| > |OC|$. Thus, we have arrived at a contradiction.

327. Let the sides in the triangle ABC be related by the inequalities $c \leqslant b \leqslant a$. We take on CB a point M such that $\angle CAM = \frac{1}{2}\angle C$. Now, we have to prove that $|CM| \leqslant \frac{a}{2}$. By the law of sines, for the triangle CAM we have: $|CM| =$

$$\frac{b\sin\frac{C}{2}}{\sin\frac{3C}{2}} = \frac{b}{2\cos C + 1} = \frac{ab^2}{a^2+ab+b^2-c^2} \leqslant \frac{a}{2}.$$

328. Let D denote the midpoint of AC. We erect at D a perpendicular to AC and denote the point of its intersection with BC by M. AMC is an isosceles triangle, hence, $\angle MAC = \angle BCA$. By hypothesis, ABD is also an isosceles triangle, $\angle ABD = \angle BDA$, $\angle ABM > 90°$ (by hypothesis), $\angle ADM = 90°$, hence, $\angle MBD > \angle MDB$, and $|MD| > |BM|$. Hence it follows that $\angle MAD > \angle MAB$ (if B is mapped symmetrically with respect to the straight line AM, then we get a point B_1

inside the angle MAD since MD is perpendicular to AD and $|MD| > |MB| = |MB_1|$); thus $\angle C > \angle A - \angle C$, $\angle C > \frac{1}{2}\angle A$.

329. If the circle touches the extensions of the sides AB and AC of the triangle ABC, and its centre is O, then it is easy to find that $\angle BOC = 90° - \frac{1}{2}\angle A$. Thus, $\angle BOC + \angle A = 90° + \frac{1}{2}\angle A \neq 180°$.

330. Let AD denote the altitude, AL the angle bisector, AM the median. We extend the angle bisector to intersect the circle circumscribed about the triangle at a point A_1. Since MA_1 is parallel to AD, we have: $\angle MA_1A = \angle LAD$.

Answer: if $\angle A < 90°$, then the angle between the median and angle bisector is less than the angle between the angle bisector and altitude. If $\angle A > 90°$, then vice versa; if $\angle A = 90°$, then the angles are equal.

331. If AD is the altitude, AN the median, M the median point then $\cot B + \cot C = \frac{|DB|}{|AD|} + \frac{|CD|}{|AD|} = \frac{|CB|}{|AD|} \geqslant \frac{|CB|}{|AN|} = \frac{|CB|}{3|MN|} = \frac{2}{3}$.

332. From the fact that $S_{BAM} = S_{BCM}$, $|BC| > |BA|$, and $|CM| > |MA|$ it follows that $\sin \angle BAM > \sin \angle BCM$. Hence, if the angles are acute, then $\angle BAM > \angle BCM$; only the angle BAM can be obtuse. Thus, we always have: $\angle BAM > \angle BCM$.

333. If $|OA| = a$, R the radius of the circle, K the point of intersection of OA and DE, then it is easy to find that $|OK| = a - \frac{a^2 - R^2}{2a} = \frac{a^2 + R^2}{2a} > R$.

334. The notation is given in Fig. 60. In the first case (Fig. 60, *a*), $|AB| < |AA_1| + |A_1B_1| + |B_1B| = |AA_1| + |A_1C| + |B_1D| + |BB_1| = |AC| + |BD|$. In the second case (Fig. 60, *b*),

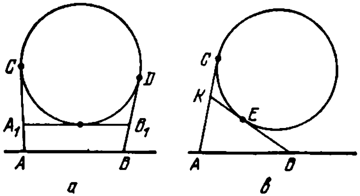


Fig. 60

$|AB| > |BK| - |AK| > |BE| - |AC|$. The converse can be readily proved by contradiction.

335. Let K, L, and M denote the points at which the drawn lines intersect AC; we further denote: $|AC| = b$, $|BC| = a$, $|AB| = c$, $|BL| = l$. By the theorem on the bisector of an interior angle, we find: $|LC| = \frac{ab}{a+c}$; applying this theorem once more to the triangle BCL, we find $|LM| = \frac{ba}{a+c} \cdot \frac{l}{l+a} = \frac{ba}{a+c}\left(1 - \frac{a}{a+l}\right)$; but $\angle BLA = \frac{1}{2}\angle B + \angle C = \frac{\pi - \angle A + \angle C}{2} > \angle A$ (since $\angle C > 3\angle A - \pi$). Hence, $c > l$ and $|LM| < \frac{ba}{a+c}\left(1 - \frac{a}{a+c}\right) = b\frac{ac}{(a+c)^2} \leqslant \frac{b}{4}$.

336. Let $ABCD$ be the given quadrilateral. Consider the quadrilateral AB_1CD, where B_1 is symmetric to B with respect to the midperpendicular to the diagonal AC. Obviously, the areas $ABCD$ and AB_1CD are equal to each other, the sides of the quadrilateral AB_1CD, in the order of traverse, are equal to b, a, c, d. For this quadrilat-

eral, the inequality $S \leqslant \frac{1}{2}(ac + bd)$ is obvious, the equality occurring if $\angle DAB_1 = \angle B_1CD = 90°$, that is, AB_1CD is an inscribed quadrilateral with two opposite angles of 90° each; hence, the quadrilateral $ABCD$ is also inscribed (in the same circle), and its diagonals are mutually perpendicular.

337. Consider two cases.

(1) The given triangle (ABC) is acute. Let $\angle B$ be the greatest: $60° \leqslant \angle B < 90°$. Since the bisectors of the angles A and C are less than 1, the altitudes of these angles h_A and h_C are also less than 1. We have: $S_{ABC} = \frac{h_A h_C}{2 \sin B} < \frac{\sqrt{3}}{3}$.

(2) If one of the angles of the triangle, say B, is not acute, then the sides containing this angle are less than the corresponding angle bisectors, that is, less than 1, and the area does not exceed 1/2.

338. Let c be the greatest side lying opposite the vertex C. If $a^2 + b^2 + c^2 - 8R^2 > 0$, then $a^2 + b^2 > 8R^2 - c^2 \geqslant c^2$ (since $c \leqslant 2R$), that is, the triangle is acute. Conversely, let the triangle be acute, then $a^2 + b^2 + c^2 = 2m_c^2 + \frac{3}{2}c^2$ (m_c the median to the side c); therefore, the less the median, the less the sum $a^2 + b^2 + c^2$. But the median is maximal if C is the midpoint of the arc and its length decreases as C displaces in the arc. When the triangle becomes right-angled, the sum $a^2 + b^2 + c^2 - 8R^2$ is equal to 0.

339. Replacing R and r by the formulas $R = \frac{abc}{4S}$, $r = \frac{S}{p}$, for computing S make use of Hero's formula and the equality

$$4S^2\left(p - \frac{abc}{2S} - \frac{S}{p}\right)\left(p + \frac{abc}{2S} + \frac{S}{p}\right)$$
$$= \frac{1}{8}(a^2 + b^2 - c^2)(a^2 - b^2 + c^2)(-a^2 + b^2 + c^2).$$

340. Let us assume the contrary, for instance, $c \geqslant a$; then $2c \geqslant c + a > b$; squaring the inequalities and adding them together, we get: $5c^2 > a^2 + b^2$, which is a contradiction.

341. The bisector of the angle B is the bisector of $\angle OBH$, and the bisector of the angle A is the bisector of $\angle OAH$. Further, $\angle BAH = 90° - \angle B < 90° - \angle A = \angle ABH$; hence, $|AH| > |BH|$. If K and M are the intersection points of the bisectors of the angles A and B with OH, then $\frac{|HK|}{|KO|} = \frac{|AH|}{|AO|} = \frac{|AH|}{R} > \frac{|BH|}{R} = \frac{|BH|}{|OB|} = \frac{|HM|}{|MO|}$. Thus, $|HK| > |HM|$, and the point of intersection of the bisectors is found inside the triangle BOH.

342. Denote: $|AB| = |BC| = a$, $|AM| = c$, $|MC| = b$, $|MB| = m$, $\angle BMO = \psi$, $\angle MBO = \varphi$. We have to prove that $|OB| > |OM|$ or $\psi > \varphi$ or $\cos\psi < \cos\varphi$. By the law of cosines for the triangles MBA and MBC, we get:

$$\cos\varphi - \cos\psi = \frac{m^2 + a^2 - c^2}{2ma} - \frac{m^2 + b^2 - a^2}{2mb}$$

$$= \frac{m^2(b-a) - a(b^2 - a^2) + b(a^2 - c^2)}{2mab}.$$

But $a - c = b - a$; hence,

$$\cos\varphi - \cos\psi = \frac{(b-a)(m^2 - ab - a^2 + ab + bc)}{2mab}$$

$$= \frac{(b-a)(m^2 - a^2 - b(2a-b))}{2mab}$$

$$= \frac{(b-a)(m+b-a)(m-a+b)}{2mab} > 0,$$

which was to be proved.

343. Through the point M, we draw a straight line parallel to AC to intersect AB at a point K.

We easily find: $|AK| = |CM|\cdot\frac{|AB|}{|CB|}$, $|MK| = |MB|\cdot\frac{|AC|}{|CB|}$. Since $|AM| \leqslant |AK| + |KM|$, replacing $|AK|$ and $|KM|$, we get $|AM| \leqslant \frac{|CM|\cdot|AB|}{|BC|} + \frac{|MB|\cdot|AC|}{|CB|} \Rightarrow (|AM| - |AC|) \times |BC| \leqslant (|AB| - |AC|)\,|MC|$, which was to be proved.

344. The minimum is equal to $\frac{a^2+b^2+c^2}{3}$ and is reached if M is the centre of mass of the triangle ABC. (This can be proved, for instance, using the method of coordinates or Leibniz' theorem—see Problem 140 in Sec. 2).

345. Let us "rectify" the path of the ball. To this end, instead of "reflecting" the ball from the

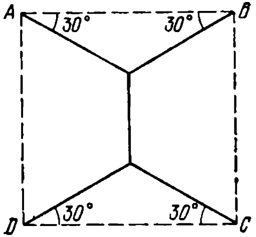


Fig. 61

side of the billiards, we shall specularly reflect the billiards itself with respect to this side. As a result, we obtain a system of rays with a common vertex; any two neighbouring rays form an angle α. The maximal number of rays in the system which can be intersected by a straight line is just the maximal number of reflections of the ball. This number is equal to $\left[\frac{\pi}{\alpha}\right]+1$ if $\frac{\pi}{\alpha}$ is not a whole number ($[x]$ the integral part of the number x); if $\frac{\pi}{\alpha}$ is a whole number, then it is equal to the maximal number of reflections.

346. If the roads are constructed as is shown in Fig. 61 (A, B, C and D denote the villages, and the roads are shown by continuous lines), then their total length is $2+2\sqrt{3}<5.5$. It is possible to show that the indicated arrangement of the roads realizes the minimum of their total length.

347. If one of the sides of the triangle through A forms an angle φ with the straight line perpendicular to the given parallel straight lines, then the other side forms an angle of $180°-\varphi-\alpha$; on having found these sides, we get that the area of the triangle is equal to $-\frac{ab\sin\alpha}{2\cos\varphi\cos(\varphi+\alpha)}=$ $-\frac{ab\sin\alpha}{\cos\alpha+\cos(\alpha+2\varphi)}$. This expression is minimal if $\alpha+2\varphi=180°$.

Answer: $S_{\min}=ab\cot\frac{\alpha}{2}$.

348. We have: $S_{ACBD}=\frac{|AB|}{|MO|}S_{OCD}=$ $2(k+1)S_{OCD}$. Consequently, S_{ABCD} is the greatest if the area of the triangle OCD is the greatest. But OCD is an isosceles triangle with lateral side equal to R, hence, its area is maximal when the sine of the angle at the vertex O reaches its maximum. Let us denote this angle by φ. Obviously $\varphi_0\leqslant\varphi<\pi$, where φ_0 corresponds to the case when AB and CD are mutually perpendicular. Consequently, if $\varphi_0\leqslant\pi/2$, then the maximal area of the triangle OCD corresponds to the value $\varphi_1=\pi/2$, and if $\varphi_0>\pi/2$, then to the value $\varphi_1=\varphi_0$.

Answer: if $k\leqslant\sqrt{2}-1$, then $S_{\max}=(k+1)R^2$; if $k>\sqrt{2}-1$, then $S_{\max}=2R^2\sqrt{k(k+2)}/(k+1)$.

349. Let the straight line BC satisfy the condition: $|BP| = |MC|$ (the order in which the points follow is B, P, M, C). We are going to prove that the area of the quadrilateral $ABNC$ is the smallest. We draw another straight line intersecting the sides of the angle at points B_1 and C_1. Let the point B lie between the points A and B_1, then the point C_1 lies between A and C. We have to prove that $S_{BB_1N} > S_{CC_1N}$. This inequality is equivalent to the inequality $S_{BB_1P} > S_{CC_1P}$, since $\frac{S_{BB_1P}}{S_{BB_1N}} = \frac{S_{CC_1P}}{S_{CC_1N}} = \frac{|AP|}{|AN|}$. Adding S_{BPC_1} to both sides of the last inequality, we get: $S_{BB_1P} + S_{BPC_1} = S_{BB_1PC_1} = S_{C_1CB_1}$ (follows from the equality $|BP| = |MC|$) for the left-hand member and $S_{CC_1P} + S_{BPC_1} = S_{C_1CB}$ for the right-hand member. But, obviously, $S_{C_1CB_1} > S_{C_1CB}$. The case when the point B_1 lies between A and B is considered in a similar way.

Construction. It suffices to draw a straight line to intersect the sides of the given angle and the straight lines AN and AM at points B_0, P_0, M_0, and C_0, respectively, so that $|B_0P_0| = |M_0C_0|$ and then to draw through M a straight line parallel to B_0C_0. Consider the parallelogram AB_0DC_0; let K and L denote the points of intersection of the straight lines AP_0 and AM_0 with B_0D and C_0D, respectively. It follows from the equality $|B_0P_0| = |M_0C_0|$ that $S_{AB_0K} = S_{AC_0L}$. The problem is reduced to constructing two equivalent triangles AB_0K and AC_0L all of whose angles are known. Taking B_0 arbitrarily, we construct the triangle AB_0K. We then take on AB_0 a point E such that $\angle B_0KE = \angle ALC_0$ and construct the line segment AC_0 equal to $\sqrt{|B_0E|\,|B_0A|}$. B_0C_0 is the required straight line.

Remark. Consider the following problem. Through a point M lying inside a given angle draw a straight line intersecting the sides of the angle at points B and C so that the line segment BC is the smallest. It follows from the above problem that BC will be the smallest line segment if $|BP| = |MC|$, where P is the projection of the vertex of the given angle on BC. (It follows even a stronger assertion, namely, if the line segment BC possesses the indicated property, then for any other straight line passing through M and intersecting the sides of the angle at points B_1 and C_1 the projection of the line segment B_1C_1 on the line segment BC is greater than $|BC|$.) However, it is not always possible to construct such a line segment by means of a pair of compasses and a ruler.

350. Let M_1 and N_1 be two other points on the sides of the angle (Fig. 62). Then $\angle M_1AN_1 = \beta$,

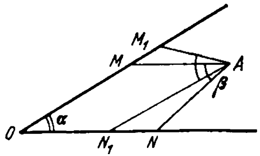


Fig. 62

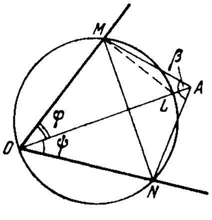


Fig. 63

$\angle AM_1M \quad 360° - \alpha - \beta - \angle ON_1A > 180° - \angle ON_1A = \angle AN_1N$. Hence, bearing in mind that $\angle MAM_1 = \angle NAN_1$, we get that $|M_1A| < |N_1A|$, and, hence, $S_{M_1AM} < S_{N_1AN}$; thus, $S_{OM_1AN_1} < S_{OMAN}$.

351. Taking into account the results of the preceding problem, we have to find out on what conditions we can find on the sides of the angle points M and N such that $\angle MAN = \beta$ and $|MA| = |AN|$. Circumscribe a circle about the triangle MON (Fig. 63). Since $\varphi + \psi + \beta < 180°$, the

point A is located outside this circle. If L is the point of intersection of the straight line OA and the circle, then the following inequalities must be fulfilled: $\angle AMN = 90° - \frac{\beta}{2} > \angle LMN = \angle LON$ and $\angle ANM = 90° - \frac{\beta}{2} > \angle LOM$. Thus, if $\varphi < 90° - \frac{\beta}{2}$ and $\psi < 90° - \frac{\beta}{2}$, then it is possible to find points M and N such that $|MA| = |AN|$ and $\angle MAN = \beta$. If the conditions are not fulfilled, then such points cannot be found. In this case, the quadrilateral of the maximal area degenerates into a triangle (either M or N coincides with O).

352. Let us take a point A_1 on BC (Fig. 64). The quadrilateral OMA_1N is equivalent to the

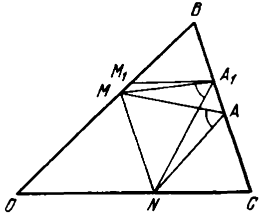


Fig. 64

quadrilateral $OMAN$, $\angle MA_1N < \angle MAN$; consequently, if we take on OB a point M_1 such that $\angle M_1A_1N = \angle MAN$, then $S_{OM_1A_1N} > S_{OMAN}$; hence, the area of the quadrilateral corresponding to the point A_1, which, taking into consideration the results of the two previous problems, proves the statement.

353. Let, for definiteness, $\sin\alpha \geqslant \sin\beta$; on the extension of AB, we take a point K such that $\angle BKC = \beta$. Since $\angle CBK = \angle ADC$ (since

$ABCD$ is an inscribed quadrilateral), the triangle KBC is similar to the triangle ACD. But $|BC| \geqslant |CD|$, consequently, $S_{BCK} \geqslant S_{ADC}$ and $S_{AKC} \geqslant S_{ABCD}$. But $S_{AKC} = \frac{a^2 \sin(\alpha+\beta)\sin\alpha}{2\sin\beta}$, hence, $S_{ABCD} \leqslant \frac{a^2 \sin(\alpha+\beta)\sin\alpha}{2\sin\beta}$ In similar fashion, we can prove that $S_{ABCD} \geqslant \frac{a^2 \sin(\alpha+\beta)\sin\beta}{2\sin\alpha}$.

354. Consider the other positions of the points M_1 and N_1 ($\angle M_1AN_1 = \beta$) and, bearing in mind the condition $\alpha + \beta > 180°$, show that the "added" triangle has a greater area than the triangle by which the area is reduced (similar to the solution of Problem 350 of Sec. 2).

355. Taking into account the result of the preceding problem and reasoning exactly as in Problem 351 in Sec. 2, we get: if $\varphi > 90° - \frac{\beta}{2}$ and $\psi > 90° - \frac{\beta}{2}$, then a quadrilateral of the smallest area exists and for it $|MA| = |AN|$. If this condition is not fulfilled, then the desired quadrilateral degenerates (one of the points M or N coincides with the vertex O).

356. We take the point A for which the conditions of the problem are fulfilled and some other point A_1. Drawing through A_1 straight lines parallel to AM and AN and which intersect the sides at points M_1 and N_1, we make sure that $S_{OM_1A_1N_1} < S_{OMAN}$ and, consequently, the more so, the area of the minimal quadrilateral corresponding to the point A_1 is less than the area of the quadrilateral $OMAN$ which is the minimal quadrilateral corresponding to A.

357. The radius of the largest circle is equal to $2R/\sqrt{3}$, that is, to the radius of the circle circumscribed about the regular triangle with side $2R$. (Let us take such a triangle and, on its sides as

diameters, construct the circles.) For any circle of a greater radius, provided it is coverable by the given circles, there is an arc of at least 120° covered by one circle, but such an arc contains a chord greater than $2R$. Thus, we have arrived at a contradiction.

In the general case, if there is an acute triangle with sides $2R_1$, $2R_2$, $2R_3$, then the radius of the circle circumscribed about this triangle is the required one. In all other cases, the radius of the greatest circle is equal to the greatest of the numbers R_1, R_2, R_3.

358. It is possible. Figure 65 shows three unit squares covering a square 5/4 on a side.

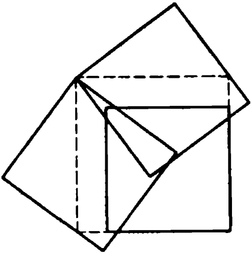


Fig. 65

359. Let us first note that the side of the smallest regular triangle covering the rhombus with side a and acute angle of 60° is equal to $2a$. Indeed, if the vertices of the acute angles M and N of the rhombus lie on the sides AB and BC of the regular triangle ABC and $\angle BNM = \alpha$, $30° \leqslant \alpha \leqslant 90°$, then, using the law of sines for finding $|BN|$ from the triangle BNM and $|CN|$ from the triangle KNC (K the vertex of the obtuse angle of the rhombus which may be assumed to lie on the side AC), we get after transformations: $|BC| =$

$2a \dfrac{\cos(60° - \alpha)}{\cos 30°}$. Taking into account that $30° \leqslant \alpha \leqslant 90°$, we find that $|BC| \geqslant 2a$. It is easy to see that an equilateral triangle 3/2 on a side can be covered by three regular triangles with side 1. To this end, we place each of the unit triangles so that one of its vertices is brought into coincidence with one of the vertices of the triangle to be covered, while the midpoint of the opposite side coincides with the centre of the covered triangle.

Let us now show that it is impossible to cover an equilateral triangle with side $b > 3/2$ with three equilateral triangles of unit area. If such a covering were possible, then the vertices A, B, and C would be covered by different triangles, and each of the sides AB, BC, and CA would be covered by two triangles. Let A belong to the triangle I, B to the triangle II, C to the triangle III, the centre O of the triangle belonging, say, to the triangle I. Let us take on AB and AC points M and N, respectively, such that $|AM| = |AN| = \frac{1}{3} b$. Since $|BM| = |CN| = \frac{2}{3} b > 1$, the points M and N also belong to the triangle I and, consequently, the rhombus $AMON$ is entirely covered by the triangle whose side is less than $2|AM|$ $(2|AM| > 1)$, which is impossible.

360. Denote the ratios $\frac{|AM|}{|MC|}$, $\frac{|CN|}{|NB|}$ and $\frac{|ML|}{|LM|}$ by α, β, and γ. Then (see the solution of Problem 221 in Sec. 1) $P = Q\alpha\beta\gamma$, $S = Q(\alpha+1) \times (\beta+1)(\gamma+1)$. Finally, take advantage of the inequality $(\alpha+1)(\beta+1)(\gamma+1) \geqslant (\sqrt[3]{\alpha\beta\gamma}+1)^3$.

361. Let $\cot\alpha = x$, $\cot\beta = y$, then $\cot\gamma = \dfrac{-xy+1}{x+y} = \dfrac{x^2+1}{x+y} - x$, $a^2 \cot\alpha + b^2 \cot\beta +$

$$c^2 \cot \gamma = (a^2 - b^2 - c^2)\, x + b^2 (x+y) + c^2 \frac{x^2+1}{x+y}.$$

The minimum of the expression $b^2 (x + y) + c^2 \frac{x^2+1}{x+y}$ with x fixed and $x+y>0$ is reached for such an y for which the following equality is fulfilled: $b^2 (x+y) = c^2 \frac{x^2+1}{x+y} = \frac{x+y}{\sqrt{x^2+1}} = \frac{c}{b}$.

Thus, $\frac{c}{b} = \frac{x+y}{\sqrt{x^2+1}} = \frac{\sin \gamma}{\sin \beta}$. Hence, the least value of the given expression is reached for such α, β, and γ whose sines are proportional to the sides a, b, and c, that is, when the triangles under consideration are similar. But in this case an equality occurs (it is readily checked).

362. Denote: $p - a = x$, $p - b = y$, $p - c = z$ (p the semiperimeter). Leaving $4S\sqrt{3}$ in the right-hand side of the inequality, we get, after transforming the left-hand side (for instance, $a^2 - (b - c)^2 = 4(p - b)(p - c) = 4yz$) and replacing S by Hero's formula, the inequality $xy + yz + zx \geqslant \sqrt{3(x + y + z)\,xyz}$. Dividing both sides of the inequality by $\sqrt{xyz}$ and making the substitutions $u = \sqrt{(xy)/z}$, $v = \sqrt{(yz)/x}$, $w = \sqrt{(zx)/y}$ ($x = uw$, $y = vu$, $z = wv$), we get the inequality $u + v + w \geqslant \sqrt{3(uv + vw + wu)}$, which, on squaring, is reduced to the known inequality $u^2 + v^2 + w^2 \geqslant uv + vw + wu$.

363. There are two families of regular triangles circumscribed about the given triangle (see Problem 305 in Sec. 2). On the sides of the triangle ABC, we construct externally the triangles ABC_1, BCA_1, and CAB_1 and circumscribe circles about them. The vertices of the triangles of the first family lie on these circles (one per each circle). Let $O_1O_2O_3$ denote the centres of those circles ($O_1O_2O_3$ is a regular triangle, see Problem 304 in Sec. 2). The triangle whose sides are parallel to the sides of

the triangle $O_1O_2O_3$ has the greatest area (the secant passing through the point of intersection of the two circles has the greatest length when it is parallel to the line of centres; in this case its length is twice the distance between the centres). The area of the greatest triangle is $S_0 = 4S_{O_1O_2O_3} = \frac{\sqrt{3}}{3}\left(\frac{a^2+b^2+c^2}{2}+2S\sqrt{3}\right)$, where S is the area of the given triangle (see the solution of Problem 305 in Sec. 2). The area of the greatest triangle belonging to the second family is less. Among the regular triangles inscribed in the given one, the triangle whose sides are parallel to the sides of the greatest circumscribed triangle has the smallest area. This follows from the result of Problem 241 of Sec. 1. Its area is equal to $S_1 = S^2/S_0$. Thus, the area of the greatest circumscribed regular triangle is $S_0 = \frac{\sqrt{3}(a^2+b^2+c^2)}{6} + 2S$, and the area of the smallest inscribed triangle equals $S_1 = \frac{S^2}{S_0}$, where S is the area of the given triangle.

364. Circumscribe a circle about the triangle AMC. All the triangles A_1MC obtained as M displaces in the arc AC are similar, consequently, the ratio $\frac{|CM|}{|A_1M|}$ is the same for them. Therefore, if M is the point of minimum of the expression $f(M) = \frac{|BM|\cdot|CM|}{|A_1M|}$, then BM must pass through the centre of the circle circumscribed about the triangle AMC, otherwise we can reduce $|BM|$ leaving the ratio $\frac{|CM|}{|A_1M|}$ unchanged. Let now B_1 and C_1 be, respectively, the points of intersection of the straight lines BM and CM with the circle circumscribed about the triangle ABC, then

$$\frac{|BM|\cdot|CM|}{|A_1M|}=\frac{|CM|\cdot|AM|}{|B_1M|}=\frac{|AM|\cdot|BM|}{|C_1M|}.$$

Consequently, the straight lines AM and CM must also pass through the centres of the circles circumscribed about the triangles BMC and AMB, respectively. Thus, the point M is the centre of the inscribed circle (see Problem 125 of Sec. 2). In addition, in this case A_1 is the centre of the circle circumscribed about the triangle CMB, $\sin\angle MBC=\frac{r}{|MB|}$, $\frac{CM}{\sin\angle MBC}=2|A_1M|$; hence, $\frac{|BM|\cdot|CM|}{|A_1M|}=2r$.

Let us return to the question of the least value for the function $f(M)$. One of the theorems of mathematical analysis states that a function, continuous on a closed set, always reaches its greatest and least values on that set. In particular, this theorem is true for a function of two variables defined on a polygon. But the theorem is not applicable directly to this problem, since the function $f(M)$ is not defined at the vertices of the triangle ABC. But cutting away from the triangle its small corners, we get a hexagon on which $f(M)$ becomes a continuous function and has, consequently, its least value. It is possible to prove that near the boundary of the triangle $f(M)>2r$. Therefore, if the cut-away corners are sufficiently small, then the function $f(M)$ reaches its least value on the hexagons, and hence, on the triangle, when M is the centre of the inscribed circle, this least value being equal to $2r$. On the other hand, the function $f(M)$ does not attain its greatest value although it is bounded. Prove that $f(M)<l$, where l is the length of the greatest side of the triangle ABC, for all the points of the triangle with the exception of the vertices, and that $f(M)$ can take on values arbitrarily close to l.

365. On the rays MB and MC, we take points C_1 and B_1, respectively, such that $|MC_1|=$

$|MC|$, $|MB_1| = |MB|$ (the triangle MC_1B_1 is symmetric to the triangle MBC with respect to the bisector of the angle BMC), C_2 and B_2 are the projections of C_1 and B_1 on the straight line AM, respectively. We have: $|BM| \sin \angle AMC + |CM| \sin \angle AMB = |B_1M| \sin \angle AMC + |C_1M| \times \sin \angle AMB = |B_1B_2| + |C_1C_2| \geqslant |B_1C_1| = a$. Writing two more such inequalities and adding them together, we prove the statement of the problem. It is easy to check that if M coincides with the centre of the inscribed circle, then the inequality turns into an equality.

366. (a) Let us first solve the following problem. Let M be a point on the side AB of the triangle ABC; the distances from M to the sides BC and AC are equal to u and v, respectively; h_1 and h_2 are the altitudes drawn to BC and AC, respectively. Prove that the expression $\frac{h_1}{u} + \frac{h_2}{v}$ reaches the least value when M is the midpoint of AB. We denote, as usually: $|BC| = a$, $|AC| = b$, S the area of the triangle ABC. We have: $au + bv = 2S$, $v = \frac{2S - au}{b}$. Substituting v into the expression $\frac{h_1}{u} + \frac{h_2}{v} = t$, we get: $atu^2 - 2Stu + 2h_1S = 0$. The discriminant of this equation is nonnegative, $S^2(t^2 - 4t) \geqslant 0$, whence $t \geqslant 4$. The least value $t = 4$ is reached for $u = S/a = h_1/2$, $v = h_2/2$. It follows from this problem that the least value of the left-hand member of the inequality of Item (a) is attained when M is the median point. The inequalities of Items (b) and (c) are proved in a similar way. In Item (b) we have to determine for what point M on the side AB the product uv reaches its greatest value. In Item (c), we first divide both sides of the inequality by uvw and solve the problem on the minimum of the function $(h_1/u - 1) \times (h_2/v - 1)$ for the point M on AB.

367. Let for the acute triangle ABC the inequal-

ity $|AC| \leqslant |AB| \leqslant |BC|$ be fulfilled; BD the altitude, O the centre of the circumscribed and I the centre of the inscribed circle of the triangle ABC, E the projection of I on BD. Since $|ED| = r$, we have to prove that $|BE| \geqslant R = |BO|$. But BI is the bisector of the angle EBO (BI is the bisector of the angle ABC and $\angle ABD = \angle OBC$), $\angle BEI = 90°$, $\angle BOI \geqslant 90°$ (the latter follows from the fact that the projection of CI on BC does not exceed $|BC|/2$). Consequently, $|BE| \geqslant |BO|$ (we map BO symmetrically with respect to BI).

368. Since the area of the triangle formed by the medians of the other triangle is 3/4 of the area of the original triangle, and for any triangle $abc = 4RS$, we have to prove that for an acute triangle the following inequality holds true:

$$m_a m_b m_c > \frac{5}{8} abc. \qquad (1)$$

Let, for the convenience of computations, one of the sides be equal to $2d$, and the median drawn to this side be m. Since the triangle is acute-angled, we have: $m > d$. Let t denote the cosine of the acute angle formed by this median and the side $2d$, $0 \leqslant t < d/m$ ($t < d/m$ is the condition for a triangle to be acute-angled). Expressing the sides and median in terms of d, m, and t and substituting the found expressions into the inequality (1), we get after transformations: $m^2(9d^2 + m^2)^2 - 25d^2(d^2 + m^2)^2 > t^2d^2m^2(64m^2 - 100d^2)$. The left-hand member of the inequality is reduced to the form: $(m^2 - 4dm + 5d^2) \times (m^2 + 4dm + 5d^2)(m^2 - d^2)$. For $m > d$ this expression is positive. In addition, if $m = d$ (the triangle is right-angled), then the left-hand member of the inequality is no less than the right-hand member (equality for $t = 0$). Further, if $d < m \leqslant \frac{5}{4} d$, then the right-hand member of the inequality is nonpositive, and the inequality holds true. Let $m > \frac{5}{4}d$. In this case, the right-hand member of the inequality is less than the value obtained for $t = d/m$. But for $t = d/m$ the original triangle is right-angled, and for right triangles the validity of a slack inequality has been already proved. (It suffices to repeat the same reasoning with respect to the other side of the triangle.) Thus, it has been proved that the inequality (1) is valid for any nonobtuse triangles except for isosceles right triangles; for the latter an equality occurs.

369. Let M lie inside ABC at distances x, y, and z from the sides BC, CA, and AB, respectively. The problem is to find the minimum of $x^2 + y^2 + z^2$ provided that $ax + by + cz = 2S_{ABC}$. Obviously, this minimum is reached for the same values of x, y, z as the minimum of $x^2 + y^2 + z^2 - 2\lambda(ax + by + cz) = (x - \lambda a)^2 + (y - \lambda b)^2 + (z - \lambda c)^2 - \lambda^2(a^2 + b^2 + c^2)$, where λ is an arbitrary fixed number (also provided that $ax + by + cz = 2S_{ABC}$). Taking $\lambda = \frac{2S_{ABC}}{a^2 + b^2 + c^2}$ (λ is found from the equations $x = \lambda a$, $y = \lambda b$, $z = \lambda c$, $ax + by + cz = 2S_{ABC}$), we see that the minimum of the last expression is reached for $x = \lambda a$, $y = \lambda b$, $z = \lambda c$. Let now the point M be at distances λa, λb, and λc from BC, CA, and AB, respectively, and the point M_1 symmetric to M with respect to the bisector of the angle A. Since $S_{AM_1C} = S_{AM_1B}$, M_1 lies on the median emanating from A, and this means that M lies on the symedian of this angle (see Problem 171 in Sec. 2).

370. Let M be a point inside the triangle ABC whose greatest angle is less than 120°. We rotate the triangle AMC about the point A through an angle of 60° externally with respect to the triangle ABC. As a result, the point C goes into the point C_1, and the point M into the point M_1. The sum $|AM| + |BM| + |CM|$ is equal to the

broken line BMM_1C. This line is the smallest when the points M and M_1 lie on the line segment BC_1. Hence, there follows the statement of the problem.

371. Let ABC be the given acute triangle, A_1 a point on the side BC, B_1 a point on the side CA, C_1 a point on the side AB; A_2 and A_3 points symmetric to A_1 with respect to the sides AB and AC, respectively. The broken line $A_2C_1B_1A_3$ is equal to the perimeter of the triangle $A_1B_1C_1$; consequently, with the point A_1 fixed, this perimeter is the smallest and equals $|A_2A_3|$ when the points C_1 and B_1 lie on the line segment A_2A_3. But AA_2A_3 is an isosceles triangle, $\angle A_2AA_3 = 2\angle BAC$, $|A_2A| = |A_3A| = |AA_1|$. Hence, $|A_2A_3|$ is the smallest if AA_1 is the altitude of the triangle BAC. In similar fashion, BB_1 and CC_1 must also be altitudes.

372. If the greatest angle of the triangle is less than 120°, then the sum of the distances takes on the least value for the point from which the sides can be observed at an angle of 120° (see Problem 370 in Sec. 2). This sum is equal to $|BC_1|$ (using the notation of Problem 370 of Sec. 2). The square of this sum is equal to $a^2 + b^2 - 2ab\cos(\angle C + 60°) = \frac{1}{2}(a^2 + b^2 + c^2) + 2S\sqrt{3}$.

But it follows from Problem 362 of Sec. 2 that $a^2 + b^2 + c^2 \geqslant 4S\sqrt{3}$. It remains to prove the inequality $S \geqslant 3\sqrt{3}r^2$. It is proved in a rather simple way; it implies that among all the triangles circumscribed about a given circle the equilateral triangle has the smallest area (for this triangle the equality is fulfilled). To complete the proof, it is necessary to check whether the inequality $a + b \geqslant 6r$ is true, since for a triangle with an angle exceeding 120° the least value is reached by the sum of the distances to the vertices at the vertex of the obtuse angle.

373. Let us prove the right-hand member of the inequality. Let, for definiteness, $b \geqslant c$.

(1) If $a \leqslant b$, then $2p = a+b+c = (b-a)+c+2a < 2c+2a \leqslant 2\frac{bc}{a}+2a = 2\frac{bc+a^2}{a}$.

(2) If $a \geqslant b \geqslant c$, then $a < 2b$ and $2p = a+b+c = (b+c-a)+2a \leqslant c+2a < \frac{2bc}{a}+2a = 2\frac{bc+a^2}{a}$.

The left-hand member of the inequality follows from the right-hand member and the identity $(b+c)(p-a) - bc\cos A = a\left(\frac{bc+a^2}{a} - p\right)$

374. We have: $\frac{|BN|}{|NC|} = \frac{|AM|}{|MC|} = \frac{|AL|}{|LD|} = \frac{|BK|}{|KD|}$, that is, KN is parallel to CD, the quadrilateral $KLMN$ is a parallelogram. Let $|AK| = a$, $|KC| = b$, $|BK| = x$, $|KD| = y$, $\frac{x}{y} \geqslant \frac{a}{b}$; then

$$S_{KLM} = S_{ALM} - S_{AKL} = \left(\frac{x}{x+y}\right)^2 S_{ADC} - \frac{x}{x+y}\cdot\frac{a}{a+b} S_{ADC} = \frac{x}{x+y}\cdot\left(\frac{x}{x+y} - \frac{a}{a+b}\right)\frac{y}{y+x} S_{ABCD} < \frac{x^2y}{(x+y)^3} S_{ABCD}.$$

We denote: $y/x = t$. It is easy to prove that the greatest value $4/27$ is attained by the function $t/(1+t)^3$ for $t = 1/2$ (for instance, by taking the derivative of this function). Thus, $S_{KLMN} = 2S_{KLM} < \frac{8}{27} S_{ABCD}$.

375. Let a, b and c denote the sides of the triangle ABC, I the centre of the inscribed circle. The following vector equality holds true (it follows

from the property of the angle bisector, see Problem 9 in Sec. 1):

$$\overrightarrow{IA} \cdot a + \overrightarrow{IB} \cdot b + \overrightarrow{IC} \cdot c = 0. \tag{1}$$

In addition, $|IB| < c$, $|IC| < b$. These inequalities follow from the fact that the angles AIB and AIC are obtuse. Let us take a point A_1 sufficiently close to the point A so that the inequalities are fulfilled as before: $|I_1B| < c$, $|I_1C| < b$, where I_1 is the centre of the circle inscribed in the triangle A_1BC. The sides of the triangle A_1BC are equal to a, b_1, c_1. The same as for the triangle ABC, we write the equality

$$\overrightarrow{I_1A_1} \cdot a + \overrightarrow{I_1B} \cdot b_1 + \overrightarrow{I_1C} \cdot c_1 = 0. \tag{2}$$

Subtract (1) from (2):

$$a(\overrightarrow{I_1A_1} - \overrightarrow{IA}) + \overrightarrow{I_1B} \cdot b_1 - \overrightarrow{IB} \cdot b + \overrightarrow{I_1C} \cdot c_1 - \overrightarrow{IC} \cdot c = 0. \tag{3}$$

Note that

$$\overrightarrow{I_1A_1} - \overrightarrow{IA} = \overrightarrow{I_1I} + \overrightarrow{AA_1}, \tag{4}$$

$$\overrightarrow{I_1B} \cdot b_1 - \overrightarrow{IB} \cdot b - \overrightarrow{I_1B}(b_1 - b) + \overrightarrow{I_1I} \cdot b, \tag{5}$$

$$\overrightarrow{I_1C} \cdot c_1 - \overrightarrow{IC} \cdot c + \overrightarrow{I_1C}(c_1 - c) + \overrightarrow{I_1I} \cdot c. \tag{6}$$

Replacing in (3) the corresponding differences by the formulas (4), (5), (6), we get

$$\overrightarrow{I_1I}(a + b + c) + \overrightarrow{AA_1} \cdot a + \overrightarrow{I_1B}(b_1 - b) + \overrightarrow{I_1C}(c_1 - c) = 0.$$

Since $|\overrightarrow{I_1B}| < c$, $|\overrightarrow{I_1C}| < b$, $|b_1 - b| < |A_1A|$, $|c_1 - c| < |\overrightarrow{A_1A}|$, we have: $|I_1I| = \frac{1}{a+b+c} \times$ $|\overrightarrow{AA_1} \cdot a + \overrightarrow{I_1B}(b_1 - b) + \overrightarrow{I_1C}(c_1 - c)| < |AA_1| \times$

$\frac{a+b+c}{a+b+c} = |AA_1|$, whence we can derive the statement of the problem for any position of A_1. **Remark.** We have actually differentiated the equality (1) and proved that $|V_A| > |V_I|$, where V_A and V_I are the velocities of displacement of the points A and I, respectively.

376. Circumscribe circles about the triangles ABF, BCD, and CAE. They have a common point M. Since the angles of the triangle DEF are constant, $\angle D = \gamma$, $\angle E = \alpha$, $\angle F = \beta$, the constructed circles and point M are independent of φ. The side DF (and, consequently, EF and ED) is the smallest when DF is perpendicular to BM. Let φ_0 be the angle corresponding to this position. Then $\angle MBC = \angle MCA = \angle MAB = 90° - \varphi_0$. Extend CM to intersect the circle circumscribed about the triangle AMB at a point F_1. We can find that $\angle F_1BA = \alpha$, $\angle F_1AB = \beta$; F_1B turns out to be parallel to AC. From F_1 and B, we drop perpendiculars F_1N and BL, respectively, on AC. Since $|F_1N| = |BL|$, we have: $\tan\varphi_0 = \cot(90° - \varphi_0) = \frac{|CN|}{|F_1N|} = \frac{|AN|}{|F_1N|} + \frac{|AL|}{|BL|} + \frac{|CL|}{|BL|} = \cot\beta + \cot\alpha + \cot\gamma$. Thus, $\tan\varphi_0 = \cot\alpha + \cot\beta + \cot\gamma$. **Remark.** The angle $\omega = 90° - \varphi_0$ is called the *Brocard angle*, and the point M the *Brocard point*. There are two Brocard points for each triangle. The position of the second point M_1 is determined by the condition: $\angle M_1BA = \angle M_1AC = \angle M_1CB$.

377. Set: $\frac{|AC_1|}{|AB|} = x$, $\frac{|BA_1|}{|BC|} = y$, $\frac{|CB_1|}{|CA|} = z$.
We assume that $x \leqslant 1/2$. Suppose that the areas of the triangles AB_1C_1, BC_1A_1, and CA_1B_1 are greater than the area of the triangle $A_1B_1C_1$. Then $z \leqslant 1/2$ (otherwise $S_{AC_1B_1} \leqslant S_{A_1C_1B_1}$) and $y \leqslant 1/2$. The areas of all the triangles under consideration are readily expressed in terms of S_{ABC} and x, y,

z, for instance: $S_{AB_1C_1} = x(1 - z) S_{ABC}$. The inequality $S_{A_1B_1C_1} < S_{AB_1C_1}$ is reduced to the form $1 - x(1 - z) - y(1 - x) - z(1 - y) < x(1 - z)$. Adding three such inequalities together, we get: $3 - 4x(1 - z) - 4y(1 - x) - 4z(1 - y) < 0$. The last inequality is linear with respect to x, y, z. If it were fulfilled for certain x, y, z between 0 and 1/2, it should also be fulfilled for a set of the extreme values of the variables, that is, when each variable is equal either to 0 or 1/2. But it is possible to check to see that this is not so. The obtained contradiction proves our statement.

378. Let Q denote the midpoint of OH. As is known, Q is the centre of the nine-point circle (see Problem 160 in Sec. 2). We have: $|OH|^2 + 4|QI|^2 = 2|OI|^2 + 2|HI|^2$. Since $|QI| = R/2 - r$ (by Feuerbach's theorem, Problem 287 of Sec. 2), $|OI|^2 = R^2 - 2Rr$ (Euler's formula, Problem 193 of Sec. 2), and bearing in mind that $R \geqslant 2r$, we get:

$$|OH|^2 = 2|IH|^2 + R^2 - 4r^2 \geqslant 2|IH|^2.$$

379. An elegant idea for proving inequalities of such a type was suggested by Kazarinoff (Michigan Mathematical Journal, 1957, No. 2, pp. 97-98). Its main point consists in the following. Take points B_1 and C_1 on the rays AB and AC, respectively. It is obvious that the sum of the areas of the parallelograms constructed on AB_1 and AM and on AC_1 and AM is equal to the area of the parallelogram one of whose side is B_1C_1, the other being parallel to AM and equal to $|AM|$ (see also Problem 40 of Sec. 2). Consequently,

$$|AC_1|\, v + |AB_1|\, w \leqslant |B_1C_1|\, x. \qquad (1)$$

(a) Let us take the points B_1 and C_1 coinciding with the points B and C; then the inequality (1) yields the inequality $bv + cw \leqslant ax$. Adding together three such inequalities, we get the required inequality.

(b) If $|AB_1| = |AC|$, $|AC_1| = |AB|$, then the inequality (1) will yield $cv + bw \leqslant ax$ or $x \geqslant \frac{c}{a}v + \frac{b}{a}w$. Adding together three such inequalities, we get:

$$x+y+z \geqslant \left(\frac{b}{c}+\frac{c}{b}\right)u+\left(\frac{c}{a}+\frac{a}{c}\right)v$$
$$+\left(\frac{b}{a}+\frac{a}{b}\right)w \geqslant 2(u+v+w).$$

(c) In Item (a), we proved the inequality $ax \geqslant bv+cw$, whence $xu \geqslant \frac{b}{a}uv+\frac{c}{a}wu$. In similar fashion, $yv \geqslant \frac{a}{b}uv+\frac{c}{b}wv$, $zw \geqslant \frac{a}{c}uw+\frac{b}{c}vw$. Adding together these three inequalities, we get:

$$xu+yv+zw \geqslant \left(\frac{a}{b}+\frac{b}{a}\right)uv+\left(\frac{b}{c}+\frac{c}{b}\right)vw+$$
$$\left(\frac{a}{c}+\frac{c}{a}\right)wu \geqslant 2(uv+vw+wu).$$

(d) Let A_1, B_1, and C_1 denote, respectively, the projections of the point M on the sides BC, CA, and AB of the triangle ABC. On the rays MA, MA_1, MB, MB_1, MC, MC_1, take, respectively, points A', A_1', B', B_1', C', C_1' such that $|MA| \times |MA'| = |MA_1|$ $|MA_1'| = |MB|\cdot|MB'| = |MB_1|$ $|MB_1'| = |MC|\cdot|MC'| = |MC_1| \times |MC_1'| = d^2$*. It is possible to prove that the points A', B', C' lie on the straight lines $B_1'C_1'$, $C_1'A_1'$, $A_1'B_1'$, respectively, MA', MB', MC' being respectively perpendicular to these lines. Thus, in the triangle $A_1'B_1'C_1'$, the distances from M

* This transformation is called *inversion*. See the Remark to the solution of Problem 240, Sec. 2, and also Appendix.

to the vertices are equal to $\frac{d^2}{u}$, $\frac{d^2}{v}$, $\frac{d^2}{w}$, and to the opposite sides to $\frac{d^2}{x}$, $\frac{d^2}{y}$, $\frac{d^2}{z}$. Applying the inequality of Item (b), we get the required inequality.

(e) Let us take in the inequality (1) $b_1=c_1=l$; then $a_1=2l\sin\frac{A}{2}$. We have $x\geqslant\frac{1}{2\sin\frac{A}{2}}(u+v)$. On having obtained similar inequalities for y and z, and multiplying them, we get:

$$xyz\geqslant\frac{1}{8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}(u+v)(v+w)(w+u)$$
$$=\frac{R}{2r}(u+v)(v+w)(w+u)$$

$\left(\text{the equality } \sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}=\frac{r}{4R}\text{ was proved when solving Problem 240 in Sec. 1}\right)$.

(f) From the inequality of the preceding item it follows: $xyz\geqslant\frac{R}{2r}2\sqrt{uv}\cdot2\sqrt{vw}\cdot2\sqrt{wu}=\frac{4R}{r}\times uvw$.

(g) Dividing the inequality of Item (d) by the inequality of Item (f), we get the required inequality.

Remark. In the inequality of Item (a), equality is achieved for any acute triangle when M coincides with the intersection point of the altitudes of the triangle. In Items (b), (c), (d), and (g), equality is achieved for an equilateral triangle, when M is the centre of this triangle. In Items (e) and (f), equality is achieved in any triangle, when M is the centre of the inscribed circle.

380. Consider the class of similar triangles.

As a representative of this class we choose such a triangle ABC in which $|AB| = v$, $|BC| = u$, $|AC| = 1$, $u \leqslant v \leqslant 1$. Thus, to each class of similar triangles there corresponds a point B inside the curvilinear triangle CDE, where D is the midpoint of the arc AC, the arc EC is an arc of the circle with centre at A and the radius of 1, ED being perpendicular to AC (Fig. 66). The trian-

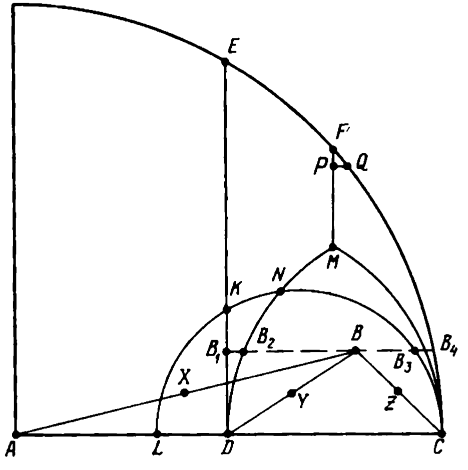


Fig. 66

gle ABD will be called a "left-hand" triangle, the triangle BDC a "right-hand" triangle. Consider the process described in the hypothesis; in doing so, at each step we shall leave only the triangles similar to which we have not met before. For each triangle we shall take the representative of the class described above. Let X, Y, Z be midpoints of AB, DB, CB, respectively; $m = |DB|$, h the altitude of the triangle ABC. For "right-hand" triangles, the following three cases are possible.

(1) $u \leqslant 1/2$, $m \leqslant 1/2$ or $u \leqslant m$, $1/2 \leqslant m$, that is, the greatest is the side DC or BD. This case occurs if B is located inside the figure $DMFC$, where DM is an arc of the circle of radius 1/2 centred at the point C, FC the right-hand part of the arc EC, $|DM| = |MC| = 1/2$, DC and FM line segments, $FM \perp DC$. In this case, the arc MC (centred at D) separates the domain for which DC is the greatest side in the triangle DBC from the domain for which the greatest side is DM. In this case, the representative of the triangle DBC has an altitude equal to $2h$ if DC is the greatest side, or $\frac{h}{2m^2} \geqslant \frac{h}{2|DB_4|^2} = \frac{h}{\frac{5}{2} - 2\sqrt{1-h^2}} = q_1(h)\, h$, $q_1(h) > 1$ if $h < \sqrt{7}/4$.

(2) $u > m$, $u > 1/2$, $v > 2m$. Note that the equality $v = 2m$ occurs for the circle with diameter LC, where $|AL| = 1/3$. Inside this circle $v > 2m$. This case takes place if the point B is inside the curvilinear triangle DKN (KN and ND arcs, DK a line segment). Since the triangle DZC is similar to the original triangle ABC, we consider only the triangle DZB. Its greatest side is DZ equal to $v/2$. Its representative has the altitude equal to $\frac{h}{4(v/2)^2} = \frac{h^2}{v^2} \geqslant \frac{h}{|AB_2|^2} \geqslant \frac{h}{|AB_3|^2} = \frac{h}{5/9 + (4/3)\sqrt{1/9 - h^2}} = q_2(h)\, h_1$, $q_2(h) > 1$.

(3) $u \geqslant 1/2$, $u \geqslant m$, $v \leqslant 2m$. In this case, the greatest side in the triangle BZD is BD equal to m, and there is no need to consider the parts of the triangle BDC since the triangle BYZ is similar to the triangle BDC, and the triangle DYZ is similar to the triangle ABD (we do not consider the triangle DZC any longer).

For "left-hand" triangles, two cases are possible, they are analogous to Cases 2 and 3 for "right-hand" triangles.

(2′) If B is inside the figure $DKNC$, then the triangle DXB, congruent to the triangle DZB, is left for further consideration; its representative has an altitude no less than $q_2(h)\,h$.

(3′) If B is outside the figure $DKNC$, then further consideration of parts of the triangle ABD is ceased.

Note that, with an increase in h, the coefficient $q_2(h)$ increases, while $q_1(h)$ decreases and becomes equal to 1 at the point F, $h = \sqrt{7}/4$. Let us take points P and Q on FM and the arc FC, respectively, sufficiently close to F. Inside the figure $B_1KNMPQB_4$, the inequalities $q_1(h) \geqslant q_0$, $q_2(h) \geqslant q_0$, and $q_0 > 1$ are fulfilled. Consequently, in all cases the rate of increase of h is no less than q_0, and in a finite number of steps or for all the triangles under consideration either Case 3 will occur or the vertex of the triangle will be located inside the curvilinear triangle PFQ. The case when the point B is inside the triangle PFQ involves no difficulties and is considered separately. In that case, "right-hand" triangles should be considered. It suffices to meet the condition $|FP| \leqslant |FM| = \dfrac{\sqrt{7}-\sqrt{3}}{4}$. In the triangle BDC, the side BD equal to m is the greatest, $h^2 \leqslant 7/16$. We can show that to the representative of the class of triangles similar to the triangle BDC, there will correspond a point lying outside the curvilinear triangle PFQ. And since the altitude is not decreased in this case, Case 3 will occur for both parts of the triangle BDC. The proof of the first part has been thereby completed.

The second part follows from the result of Problem 327 of Sec. 2 and also from the fact that all the triangles which are considered after the first division have a representative whose altitude is no less than h, and, consequently, the smallest angle is no less than $\angle B_4AC > \frac{1}{2}\angle B_1AC \geqslant \frac{1}{2}\angle BAC$.

381. Let us formulate and prove the result obtained by M. D. Kovalev which is stronger than it is required by the hypothesis. Among all the convex figures covering any triangle with sides not exceeding unity, the smallest area is possessed by the triangle ABC in which $\angle A = 60°$, $|AB| = 1$, and the altitude drawn to AB is equal to cos 10°. The area of this triangle equals $\frac{1}{2}\cos 10° \approx 0.4924$.

(1) Note that it suffices to find a triangle covering any isosceles triangle whose lateral sides are equal to 1, the angle φ between them not exceeding 60°. This follows from the fact that any triangle with sides not exceeding 1 can be covered by an isosceles triangle of the indicated type.

(2) Let us prove that any isosceles triangle mentioned in Item (1) can be covered by the triangle ABC. We construct a circle of radius 1 and centred at the point C. Let K, L, M, and N be the successive points of its intersection with CB, BA, and AC (L and M are found on BA), $\angle LCM = \angle MCN = 20°$. Hence, isosceles triangles with the angle $0 \leqslant \varphi \leqslant 20°$ are coverable by the sector CMN, whereas triangles in which $20° < \varphi \leqslant \angle C$ are covered by the triangle ABC if the end points of the base are taken on the arcs KL and MN and the third vertex at the point C. Let us now construct a circle of unit radius with centre at the point A. This circle passes through the point B, again intersects BC at a point P, intersects the side AC at a point Q. We get: $\angle PAB = 180° - 2\angle B < \angle C$, since B is the greatest angle of the triangle ABC. Hence, taking the vertex of the isosceles triangle at the point A and the

end points of the base at the point B and the arc PQ, we can cover any isosceles triangle for which $\angle C < \varphi \leqslant 60°$ (even $180° - 2\angle B \leqslant \varphi \leqslant 60°$).

(3) Let us prove that whatever the arrangement (in the plane) of the isosceles triangle DEF in which $\angle DEF = 20°$, $|DE| = |EF| = 1$ and the equilateral triangle XYZ with side 1, the area of the smallest convex figure containing the triangles DEF and XYZ is no less than 0.5 cos 10°. First note that the side of the regular triangle containing DEF is equal to $\frac{2}{\sqrt{3}} \cos 10°$. (The following statement is true: if one triangle can be placed inside the other, then it can be arranged so that two of its vertices are found on the sides of the larger triangle. We are not going to prove this general statement. It suffices to check to see its validity in the case when one of them is the triangle DEF, the other being a regular triangle. This can be done easily.) Now, consider the smallest regular triangle $X_1Y_1Z_1$ with sides parallel to those of the triangle XYZ, and containing the triangles DEF and XYZ. The side of $\triangle X_1Y_1Z_1$ is no less than $(2/\sqrt{3})\cos 10°$, and the altitude is no less than $\cos 10°$. The vertices of the triangle DEF must lie on the sides of the triangle $X_1Y_1Z_1$ not containing the sides of the triangle XYZ. Consequently, the sum of the distances from the vertices of the triangle DEF which are outside the triangle XYZ to the corresponding sides of the triangle XYZ must be at least $\cos 10° - \sqrt{3}/2$, and the area of the smallest convex polygon containing the triangles DEF and XYZ is no less than $0.5(\cos 10° - \sqrt{3}/2) + \sqrt{3}/4 = 0.5 \cos 10°$.

(M. D. Kovalev also proved that the smallest (by area) convex cover found for triangles with sides exceeding unity is unique.)

Appendix: Inversion

Definitions

Consider in the plane a circle α of radius R centred at a point O. For any point A, distinct from O, let us define the point A' in the following way. The point A' is located on the ray OA so that $|OA'|\cdot|OA| = R^2$. Thus, for all points in the plane, except for the point O, a transformation is assigned which is called the *inversion with respect to the circle* α. This transformation is also called a *symmetry with respect to a circle*, the points A and A' being said to be *symmetric with respect to the circle* α. (If a straight line is assumed to be a circle of infinite radius, then the symmetry with respect to a straight line can be represented as a limiting case of symmetry with respect to a circle.) The point O is called the *centre of inversion*, the quantity $k = R^2$, the *power of inversion*. Obviously, the points A and A' are interchanged: A goes into A', and A' goes into A. All the points of the circle α, and only those points, remain fixed. The interior points of the circle α become exterior, and vice versa.

We can "supplement" the plane with a point at infinity (∞) and assume that as a result of the inversion the point O goes into ∞, and ∞ into O.

Henceforward, the points into which the points A, B, C, . . . go as a result of the inversion are denoted by A', B', C',

Basic Properties of Inversion

Let us consider the basic properties of an inversion leaving the simplest and obvious properties unproved and outlining a scheme for reasoning in the rest of the cases. (Completing the reasoning with missing links, considering various configurations, as well as carrying out computations and making drawings are left to the reader.)

1. A straight line passing through the centre of inversion goes into itself.

2. If the points O, A, and B are not collinear, then the triangles OAB and $OB'A'$ are similar. The vertices A and B', B and A' are similar. In addition, $|A'B'| = (k\,|AB|)/|OA|\cdot|OB|$.

Note that the last equality is also true if the points O, A, and B are collinear.

3. A straight line not passing through the centre of inversion O, goes into a circle passing through O. In this case, if l is a given line, A the foot of the perpendicular from O on l, then l goes into a circle of diameter OA'.

Let us take an arbitrary point B on l. From the similarity of the triangles OAB and $OB'A'$ (Property 2) it follows that $\angle OB'A' = \angle OAB = 90°$.

4. A circle ω passing through the centre of inversion O, goes into a straight line perpendicular to the straight line passing through O and the centre of the circle ω.

5. If a straight line l and a circle ω go into each other in an inversion with centre at O, then the tangent to ω at the point O is parallel to l.

6. A circle ω not passing through O goes into the circle ω' which does not contain O either. In this case, O is the external centre of similitude of the circles ω and ω'.

To prove this property, let us draw a straight line through O and denote by A and B the points of its intersection with the circle (in particular, we may assume A and B to be diametrically opposite points on ω). Suppose that B lies on the line segment OA. Then A' belongs to the line segment OB'. If C is an arbitrary point of the circle, then, taking into account the similarity of appropriate triangles (Property 2), we have: $\angle A'C'B' = \angle OC'B' - \angle OC'A' = \angle OBC - \angle OAC = \angle ACB$.

Since the number of intersection points of two lines remains unchanged in inversion, we have:

7. Depending on the position of the centre of inversion, two touching circles go into:

(a) two touching circles (if O lies on neither of them);

(b) a circle and a line tangent to this circle (O lies on one of the circles, but does not coincide with the point of tangency);

(c) a pair of parallel lines (O coincides with the point of tangency).

The Angle Between Circles

The angle between two intersecting circles is defined as the angle between the tangents to the circles passing through one of the points of their intersection. The angle between a circle and a straight line intersecting this circle is defined as the angle between that line and the tangent to the circle passing through one of the points of intersection. Here, we may assume that the angle between the lines does not exceed 90°.

Obviously, the choice of the point of intersection is of no importance for determining the angle between two circles. It is also obvious that the angle between the circles is equal to the angle between their radii drawn to the point of intersection.

8. The inversion retains the angle between straight lines, i.e., the angle between straight lines is equal to the angle between their images.

If the centre of inversion coincides with the point of intersection of the lines, then the assertion is trivial. And if this centre does not coincide with the point of intersection of the lines, then it follows from Property 5 and the definition of the angle between two circles or between a circle and a straight line.

9. In inversion. the angle between two circles is equal to the angle between their images.

Consider the case when the centre of inversion does not lie on given circles. Let A be one of the intersection points of the circles ω_1 and ω_2, l_1 and l_2 the tangents to ω_1 and ω_2, respectively, passing through A. Let us also assume that the centre of inversion O does not lie on the straight lines l_1 and l_2. In the inversion with centre O, the circles ω_1 and ω_2 go into ω_1' and ω_2', respectively, and the lines l_1 and l_2 into the circles l_1' and l_2' touching ω_1' and ω_2' at the point A' of their intersection (Property 7), that is, the angle between l_1' and l_2' is equal to the angle between ω_1' and ω_2', and since the angle between l_1' and l_2' is equal to the angle between l_1 and l_2 (Property 8), the angle between ω_1' and ω_2' is equal to the angle between ω_1 and ω_2.

10. If the circles α and ω are orthogonal, that is, the angle between them is equal to 90°, then in inversion with respect to α the circle ω goes into itself. And conversely, if in inversion with respect to the circle α the circle ω not coinciding with α goes into itself, then α and ω are orthogonal.

Obviously, the last property is symmetric with respect to α and ω. The radii of the circles α and ω are, respectively, equal to the tangents drawn from the centre of one circle to the other circle.

On the basis of Property 10, the inversion can be defined in the following way. All the points of the circle α go into themselves. If A does not belong to α and does not coincide with its centre, then the image of the point A is represented by the point A' which is the second point of intersection of any two circles orthogonal to α and passing through A. Now, the sense of the synonymic name for inversion—symmetry with respect to a circle—becomes clearer. From this definition and the property of inversion to preserve the angle between two intersecting circles, it follows that:

11. For any circle ω and two points A and B going into each other in the inversion with respect to ω their images in the inversion with respect to the circle α whose centre does not belong to ω are represented by the circle ω' and points A'

and B' which go into each other in the inversion with respect to ω'. If the centre of α lies on ω, then ω goes into the straight line l, and the points A and B into the points A' and B', symmetric with respect to l.

The Radical Axis of Two Circles

Solve the following problem.

Given two non-concentric circles ω_1 and ω_2. Find the locus of points M for which the tangents drawn to the circles ω_1 and ω_2 are equal.

Solution. Let O_1 and O_2 denote the centres of the circles ω_1 and ω_2, r_1 and r_2 their radii, A_1 and A_2 the points of tangency, respectively. We have $|MO_1|^2 - |MO_2|^2 = (|MA_1|^2 + r_1^2) - (|MA_2|^2 + r_2^2) = r_1^2 - r_2^2$. Thus, all the points belong to one and the same straight line perpendicular to O_1O_2. This line is called the *radical axis* of the circles ω_1 and ω_2. To complete the solution of the problem, it remains to determine which points of the found line satisfy its conditions. It is possible to show that if the circles do not intersect, then all the points of the radical axis are suitable. If ω_1 and ω_2 intersect, then the radical axis contains their common chord; but all the points of the common chord are not contained in the required locus of points. Therefore, if ω_1 and ω_2 touch each other, then the point of tangency is excluded.

Consider the circle α with centre M on the radical axis of the circles ω_1 and ω_2 and radius equal to the length of the tangent drawn from M to ω_1 or ω_2. (M is assumed to be located outside ω_1 and ω_2.) The circle α is orthogonal to the circles ω_1 and ω_2. Thus, the points of the radical axis situated outside the circles which intersect or touch each other constitute the locus of centres of the circles orthogonal simultaneously to ω_1 and ω_2, and there is an inversion that carries each of them into itself.

Now, let us prove one more property of the inversion.

12. If the circles ω_1 and ω_2 do not intersect, then there is an inversion carrying them into concentric circles.

Let us take a circle α orthogonal to ω_1 and ω_2 with centre on the straight line l containing the centres of ω_1 and ω_2. Since the circles ω_1 and ω_2 do not intersect, such a circle α is existent. Let O be one of the intersection points of the circle α and the line l. In the inversion with centre O, the line l goes into itself, and the circle α into the straight line p. The lines l and p intersect and are orthogonal to the circles ω_1' and ω_2' which are the images of ω_1 and ω_2 in the inversion with respect to α. Hence it follows that the centres of ω_1' and ω_2' coincide with the point of intersection of the lines l and p, that is, ω_1' and ω_2' are concentric circles. (Prove that if a straight line is orthogonal to a circle, then the former passes through its centre.)

Here, we should like to note that any circle orthogonal to the concentric circles ω_1' and ω_2' is a straight line, that is, a circle of infinite radius. Hence, in the inversion with respect to the circle α all the circles, orthogonal to the circles ω_1 and ω_2 must go into straight lines. Consequently, all the circles orthogonal to ω_1 and ω_2 intersect the line l at two fixed points.

13. For any two circles ω_1 and ω_2, there exists at least one inversion which carries them into each other. The circle defining this inversion is called the *middle circle* of ω_1 and ω_2.

Theorem 13 should be formulated more exactly in the following way. If ω_1 and ω_2 intersect, then there exist exactly two inversions in which ω_1 goes into ω_2, and vice versa. If ω_1 and ω_2 touch each other or do not intersect, then there is only one such inversion.

Let us first consider the case of intersecting circles ω_1 and ω_2. Apply an inversion I with centre in one of the points of their intersection; as a result, ω_1 and ω_2 go into intersecting straight lines l_1 and l_2. The lines l_1 and l_2 have two bisectors with respect to which l_1 and l_2 are symmetric. Consequently (Property 11), in the inversion I those bisectors go into two circles with respect to which ω_1 and ω_2 are symmetric.

If ω_1 and ω_2 do not intersect, then there is an inversion I (Property 12) carrying them into concentric circles ω_1' and ω_2'. Let O denote the centre of ω_1' and ω_2', and r_1 and r_2 their radii. Inversion with respect to the circle α' with centre at O and radius $\sqrt{r_1 r_2}$ carries ω_1' and ω_2' into each other. In the inversion I applied, the circle α' goes into the required circle α with respect to which ω_1 and ω_2 are symmetric.

To conclude this section, let us give the definition of the radical centre of three circles. Consider three circles ω_1, ω_2, and ω_3 whose centres do not lie on a straight line. It is possible to prove that three radical axes corresponding to three pairs of those circles intersect at a point M. This point is called the *radical centre* of the circles ω_1, ω_2, and ω_3. The tangents drawn from M to the circles ω_1, ω_2, and ω_3 are equal to one another. Hence, there is an inversion with centre M that carries each of the circles ω_1, ω_2, and ω_3 into itself.

Problems and Exercises

1. Find the image of a square in the inversion with respect to the circle inscribed in the square.

2. Given a triangle ABC. Find all points O such that the inversion with centre O carries the straight lines AB, BC, and CA into circles of the same radius.

3. Let A', B', and C' denote the images of the points A, B, and C, respectively, in the inversion with centre at a point O. Prove that:

(a) if O coincides with the centre of the circle circumscribed about the triangle ABC, then the triangle $A'B'C'$ is similar to the triangle ABC;

(b) if O coincides with the centre of the inscribed circle, then the triangle $A'B'C'$ is similar to the triangle whose vertices lie at the centres of the escribed circles;

(c) if O coincides with the intersection point of the altitudes of the triangle ABC, then the triangle $A'B'C'$ is similar to the triangle with vertices at the feet of the altitudes of the triangle.

4. Points A and A' are symmetric with respect to a circle α, M is an arbitrary point of the circle. Prove that $|AM|/|A'M|$ is constant.

5. Two mutually perpendicular diameters are drawn in a circle α. The straight lines joining the end points of one of the diameters to an arbitrary point of the circle α intersect the second diameter and its extension at points A and A'. Prove that A and A' are symmetric with respect to the circle α.

6. Prove that if a circle ω passes through the centre of a circle α, then the image of ω in the inversion with respect to α is their radical axis.

7. Given a circle and two points A and B on it. Consider all possible pairs of circles touching the given circle at the points A and B and touching each other at a point M. Find the locus of points M.

8. Given two touching circles. An arbitrary circle touches one of them at point A and the other at B. Prove that the straight line AB passes through a fixed point in the plane. (In the case of equal circles AB is parallel to the straight line passing through their centres.)

9. Given three circles α_1, α_2, α_3, passing through the same point. The straight line passing through the points of intersection of the circles α_1 and α_2 contains the centre of the circle α_3; the straight line passing through the points of intersection α_2 and α_3 contains the centre of the circle α_1. Prove that the straight line passing through the points of intersection α_3 and α_1 contains the centre of the circle α_2.

10. Given two circles ω_1 and ω_2. Consider two arbitrary circles which touch the given circles at some points and also each other at a point M. Find the locus of points M.

11. Prove that by inversion any two circles can be carried into two equal circles.

12. Prove that by inversion any four points A, B, C, D, not lying on a straight line can be carried into the vertices of a parallelogram.

13. The inversion with respect to a circle with centre O and radius R carries the circle with centre A and radius r into the circle of radius r' Prove that $r' = (rR^2)/|| OA |^2 - r^2 |$.

14. Four points A, B, C, and D are given in a plane. Prove that $| AB | \cdot | CD | + | AD | \cdot | BC | \geqslant | AC | \cdot | BD |$.

15. In a triangle ABC, the side AC is the greatest. Prove that for any point M the following inequality holds: $| AM | + | CM | \geqslant | BM |$.

16. Prove that all the circles passing through a given point A and intersecting a circle α at diametrically opposite points contain one more fixed point distinct from A.

17. Given four points A, B, C, and D. Prove that the angle between the circles circumscribed about the triangles ABC and BCD is equal to the angle between the circles circumscribed about the triangles CDA and DAB.

18. A circle ω passes through the centre of a circle α. A is an arbitrary point of the circle ω. The straight line passing through A and the centre of the circle α intersects a common chord of the circles α and ω at a point A'. Prove that A and A' are symmetric with respect to the circle α.

19. Given two non-intersecting circles, which do not contain each other, and a point A lying outside the circles. Prove that there are exactly four circles (straight lines can also occur among them) passing through A and touching the given circles.

20. Let s denote the area of the circle whose

centre is found at a distance a from the point O. The inversion with respect to the circle with centre O and radius R carries the given circle into the circle of area s'. Prove that $s' = s \cdot R^4/(a^2 - R^2)^2$.

21. Given two circles tangent to each other. Consider two other circles tangent to the given circles and to each other. Let r_1 and r_2 denote the radii of the last two circles, and d_1 and d_2 the distances from their centres to the straight line passing through the centres of the given circles. Prove that $\left|\frac{d_2}{r_2} - \frac{d_1}{r_1}\right| = 2$ or $\frac{d_2}{r_2} + \frac{d_1}{r_1} = 2$.

22. Let ω_1 and ω_2 be two circles tangent to each other. Consider the sequence of distinct circles $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$, each of which touches ω_1 and ω_2, and, in addition, the circle α_{k+1} touches the circle α_k. Denote the radii of the circles $\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots$, by $r_0, r_1, \ldots, r_n, \ldots$, and the distances from their centres to the straight line passing through the centres of ω_1 and ω_2 by $d_0, d_1, \ldots, d_n$, Express d_n in terms of r_n if:

(a) $d_0 = 0$ (this case is possible if ω_1 and ω_2 touch each other internally);

(b) $d_0 = kr_0$.

23. Let α_1 and α_2 denote two intersecting circles, A and B the points of their intersection, ω an arbitrary circle touching α_1 and α_2, r the radius of the circle ω, and d the distance from its centre to the straight line AB. Prove that the ratio r/d can take on only two distinct values.

24. Given two non-intersecting circles α_1 and α_2 and a collection of circles $\omega_1, \omega_2, \ldots, \omega_n$, touching α_1 and α_2, where ω_2 touches ω_1, ω_3 touches $\omega_2, \ldots, \omega_n$ touches ω_{n-1}. We say that the system of circles $\omega_1, \omega_2, \ldots, \omega_n$ forms a chain if ω_n and ω_1 touch each other. Prove that if for the circles α_1 and α_2 there exists at least one chain consisting of n circles, then there are infinitely many chains. In this case, for any point A on either α_1 or α_2 there is a chain for which A is the point

of tangency of one of the circles of the chain.

25. Prove that if for the circles α_1 and α_2 there exists a chain of n non-intersecting circles (see the preceding problem), then $(R \pm r)^2 - d^2 = 4Rr \tan^2 (\pi/n)$, where R and r are the radii of the circles α_1 and α_2 and d is the distance between their centres. (The minus sign is taken if one circle is located inside the other, and the plus sign if otherwise.)

26. Consider three circles each of which touches three escribed circles of a triangle, each of those circles touching one of the escribed circles internally and the two other escribed circles externally. Prove that the three circles intersect at one point.

27. Let $d_1, d_2, \ldots, d_n$ denote the distances from a point M lying on the arc A_1A_n of the circle circumscribed about the regular n-gon $A_1A_2 \ldots A_n$ to the vertices $A_1, A_2, \ldots, A_n$. Prove that $\frac{1}{d_1d_2} + \frac{1}{d_2d_3} + \ldots + \frac{1}{d_{n-1}d_n} = \frac{1}{d_1d_n}$.

28. Let $a_1, a_2, \ldots, a_{n-1}, a_0$ denote the sides $A_1A_2, A_2A_3, \ldots, A_{n-1}A_n, A_nA_1$ of the n-gon $A_1A_2, \ldots, A_n$; $p_1, p_2, \ldots, p_{n-1}, p_0$ the distances from an arbitrary point M on the arc A_nA_1 of the circle to the straight lines $A_1A_2, A_2A_3, A_{n-1}A_n, A_nA_1$.

Prove that $\frac{a_0}{p_0} = \frac{a_1}{p_1} + \frac{a_2}{p_2} + \ldots + \frac{a_{n-1}}{p_{n-1}}$.

Hints and Solutions

2. There are four points possessing the required property: the centre of the circle inscribed in the triangle and the centres of the three escribed circles.

3. (b) Prove that the triangles OAB and OI_bI_a are similar. Now from Property 2 it follows that the straight lines $A'B'$ and I_aI_b are parallel.

7. Let α and β be circles touching the given circle ω at points A and B. In the inversion with centre at A, the circles ω and α go into parallel straight lines l and p, the circle β into the circle β' which touches l at a fixed point B' and the straight line p at a point M'. Thus, M' lies on the straight line passing through B' perpendicular to l. The required locus is a circle passing through A and B and orthogonal to ω. (The points A and B themselves a e excluded.) Its centre is found at the point of intersection of the tangents to ω passing through A and B.

8. Let O denote the point of tangency of the given circles. In the inversion with centre at O those circles go into a pair of parallel straight lines containing the points A' and B', the line segment $A'B'$ being perpendicular to them. The straight line AB goes into the circle circumscribed about the triangle $A'B'O$; this circle, obviously, passes through the point P symmetric to the point O with respect to a straight line equidistant from the obtained parallel lines.

9. Let O be the point of intersection of the circles $\alpha_1, \alpha_2, \alpha_3$; and A_1, A_2, A_3, respectively, the points of intersection, distinct from O, of the circles α_2 and α_3, α_3 and α_1, α_1 and α_2. The inversion with centre at O carries the circles $\alpha_1, \alpha_2, \alpha_3$ into the straight lines forming the triangle $A_1'A_2'A_3'$. From the hypothesis and Property 3 it follows that $A_3'O \perp A_1'A_2'$, $A_1'O \perp A_2'A_3'$. Hence, O is the intersection point of the altitudes of the triangle $A_1'A_2'A_3'$ and $A_2'O \perp A_3'A_1'$.

10. If ω_1 and ω_2 intersect, then the desired locus consists of two circles—the middle circles ω_1 and ω_2 (Theorem 13) excluding the points of intersection of ω_1 and ω_2 themselves. If they touch each other, then it consists of one middle circle, excluding the point of tangency. To prove this, it suffices to apply an inversion with centre at a common point of the circles ω_1 and ω_2. If ω_1 and ω_2 have no points in common, then the entire mid-

dle circle is the locus. In this case, we have to apply the inversion carrying ω_1 and ω_2 into concentric circles.

11. Any inversion with centre on the middle circle possesses the desired property since this inversion carries the middle circle into a straight line with respect to which the images of the given circles are symmetric.

12. Consider two cases.

(1) The points A, B, C, and D lie on the same circle ω. The given points may be regarded as the successive vertices of the inscribed quadrilateral. Let O be the point of intersection of the circle orthogonal to ω and passing through A and C with the circle orthogonal to ω and passing through B and D. In the inversion with centre O the quadrilateral $ABCD$ goes into the inscribed quadrilateral $A'B'C'D'$ whose diagonals are diameters, that is, $A'B'C'D'$ is a rectangle.

(2) A, B, C, and D do not lie on the same circle. Let ω_A, ω_B, ω_C, ω_D denote the circles circumscribed about the triangles BCD, CDA, DAB, ABC, respectively. We take the middle circle for ω_B and ω_D separating the point B from the point D and the middle circle for ω_A and ω_C separating the points A and C. Let O denote the point of their intersection. (Prove that those circles intersect.) In the inversion with centre O, the given points go into the vertices of a convex quadrilateral $A'B'C'D'$ each of whose diagonals separates it into two triangles with equal circumscribed circles (see Problem 11); consequently, the opposite angles of the quadrilateral are equal, hence it follows that $A'B'C'D'$ is a parallelogram (prove it).

13. Let the line OA intersect the circle with centre at A at points B and C. Then $|B'C'| = 2r'$ Now, we can use the formula given in Item 2.

14. We apply the inversion with centre at A. We have $|B'C'| + |C'D'| \geqslant |B'D'|$. Then use the formula given in Item 2.

15. It follows from the preceding problem that $|AC|\cdot|BM| \leqslant |AB|\cdot|CM| + |BC|\cdot|AM|$. Since AC is the largest side, $|BM| \leqslant \frac{|AB|}{|AC|}\cdot$ $|CM| + \frac{|BC|}{|AC|}\cdot|AM| \leqslant |AM| + |MC|$.

16. Let A' be obtained from A by inversion with respect to the circle α; A_1 is symmetric to A' about the centre of the circle α. Prove that all the mentioned circles pass through A_1.

17. We apply the inversion with centre at A. The first angle is equal to the angle between the straight line $B'C'$ and the circle circumscribed about $B'C'D'$, the second—to the angle between the lines $D'C'$ and $D'B'$.

18. The inversion with respect to the circle α carries the straight line AB into ω.

19. We apply the inversion with centre at A. Then the statement of the problem is equivalent to the statement that two circles arranged outside each other have exactly four tangent lines.

20. Let the straight line passing through the centre of the inversion and the centre of the given circle intersect the given circle at points whose coordinates are x_1 and x_2 (the origin lying at the point O). Then $s' = \frac{\pi}{4}\left(\frac{R^2}{x_1} - \frac{R^2}{x_2}\right)^2 =$ $\frac{\pi}{4}(x_1 - x_2)^2 \frac{R^4}{(x_1x_2)^2} = s\frac{R^4}{(a^2 - R^2)^2}$.

21. Note that in the inversion with centre at O, for any straight line l passing through O the following equality is true: $d/r = d'/r'$ for an arbitrary circle, where r and r' are the radii of the given circle and its image, respectively, d and d' are the distances from their centres to the line l, respectively. This follows from the fact that O is the external centre of similitude of both circles (Property 6).

Let us return to our problem. We apply the inversion with centre at the point of tangency of the given circles. The given circles go into a pair of parallel straight lines, the line l passing through the centres of the given circles is perpendicular to them. The circles with the radii r_1 and r_2 go into a pair of circles of the same radius r' which touch each other and also a pair of parallel lines obtained. Now it is obvious that if the centres of the last two circles lie on the same side of l, and, for definiteness, $d'_2 > d'_1$, then $\frac{d'_2}{r'}-\frac{d'_1}{r'}=\frac{d'_1+2r'}{r'}-\frac{d'_1}{r'}=2$. If on both sides, then $\frac{d'_2}{r'}+\frac{d'_1}{r'}=2$.

22. Use the result of the preceding problem. We get in Case (a) $d_n = 2nr_n$; in Case (b) two answers are possible: $d_n = (2n + k) r_n$ and $d_n = |k - 2n| r_n$.

23. We apply the inversion with centre at A; the circles α_1 and α_2 go into the straight lines l_1 and l_2 intersecting at the point B' situated on the straight line AB. As was proved when solving Problem 21, $r/d = r'/d'$. But $r'/d'$is the ratio of the radius of the circle touching the lines l_1 and l_2 to the distance from its centre to the fixed straight line passing through the point of intersection of l_1 and l_2. Hence, r'/d' takes on only two values depending on which of the two pairs of the vertical angles formed by l_1 and l_2 the circle is located.

24. We apply the inversion carrying α_1 and α_2 into concentric circles (see Theorem 12). This done, the assertion of the problem becomes obvious. This theorem is called *Steiner's porism*.

25. If α_1 and α_2 are concentric circles with radii R and r, then the validity of the equality $(R - r)^2 = 4Rr\cdot\tan^2 (\pi/n)$ $(d = 0)$ is readily obtained from the obvious relationship $R - r = (R + r) \sin (\pi/n)$, $R > r$. We apply the inversion whose centre is at a distance a from the common centre of the circles α_1 and α_2. Let, for definiteness, $a > R$. The circles α_1 and α_2 will go into the circles α'_1 and α'_2, α'_2 inside α'_1. In this case, by

the formula from Problem 13, we have $R' = \dfrac{R\rho^2}{a^2 - R^2}$, $r' = \dfrac{r\rho^2}{a^2 - r^2}$, where ρ^2 is the power of inversion. To find d' (the distance between the centres of the circles α_1' and α_2') we draw a straight line through the centre of the inversion and the centres of α_1 and α_2; the segment of this line enclosed between the first two points of intersection with the circles α_1 and α_2 is equal to the width of the annulus $(R - r)$. The inversion carries this segment into the segment of length $b = \dfrac{(R-r)\,\rho^2}{(a-r)\,(a-R)}$ (see Item 2), consequently, $d' = \mid R' - r' - b \mid = \left| \dfrac{R\rho^2}{a^2 - R^2} - \dfrac{r\rho^2}{a^2 - r^2} - \dfrac{(R-r)\,\rho^2}{(a-r)\,(a-R)} \right| = \dfrac{a\,(R^2 - r^2)\,\rho^2}{(a^2 - r^2)\,(a^2 - R^2)}$. Further, replacing R' and r' with the aid of the formulas derived above, we get $R' - r' = \dfrac{(R-r)\,(a^2 + Rr)\,\rho^2}{(a^2 - r^2)\,(a^2 - R^2)}$. We have to verify the validity of the equality $(R' - r')^2 - (d')^2 = 4R'r' \tan^2 (\pi/n)$. Expressing all the quantities entering this equality in terms of R, r, a, and ρ and simplifying the result obtained, we lead to the equality $(R - r)^2 (a^2 + Rr)^2 - (R - r)^2\, a^2 (R + r)^2 = 4Rr\,(a^2 - r^2) \times (a^2 - R^2) \tan^2 (\pi/n)$. But $(R - r)^2 = 4Rr \tan (\pi/n)$. Hence, we have to check to see that $(a^2 + Rr)^2 - a^2 (R + r)^2 = (a^2 - r^2)\,(a^2 - R^2)$. This can be done easily.

The case $a < R$ is identic to the above. And if $r < a < R$, then α_1' and α_2' are located outside each other, and in the given formula the plus sign should be taken.

26. We apply the inversion with centre in the radical centre of the escribed circles in which the escribed circles go into themselves. This inversion carries the straight lines containing the sides of the triangle into the circles mentioned in the

hypothesis. All the three circles pass through the radical centre of the escribed circles of the triangle.

27. We apply the inversion with centre at M and of power 1. As a result, the points $A_1, A_2, \ldots, A_n$ go into the points $A_1', A_2', \ldots, A_n'$ situated on a straight line. Let the side of the n-gon be equal to a. From the formula of Item 2 it follows that

$$|A_1'A_2'| = \frac{1}{d_1d_2}a; \quad |A_2'A_3'| = \frac{1}{d_2d_3}a;$$

$|A_{n-1}'A_n'| = \frac{1}{d_{n-1}d_n}a$; $|A_1'A_n'| = \frac{1}{d_1d_n}a$. Substituting these expressions into the obvious relationship $|A_1'A_n'| = |A_1'A_2'| + |A_2'A_3'| + \ldots + |A_{n-1}'A_n'|$, we get the desired result.

28. We apply the inversion with centre at M. The vertices of the given n-gon go into n points lying on a straight line, and

$$|A_1'A_n'| = |A_1'A_2'| + |A_2'A_3'| + \ldots + |A_{n-1}'A_n'|. \qquad (*)$$

Let p' denote the length of the perpendicular from the point M on the straight line $A_1'A_n'$. From the similarity of the triangles A_1MA_2 and $A_1'MA_2'$ (Property 2) it follows that $\frac{|A_1A_2|}{|A_1'A_2'|} = \frac{p_1}{p'}$, $|A_1'A_2'| = \frac{a_1}{p_1}p'$. Similarly,

$$|A_2'A_3'| = \frac{a_2}{p_2}p', \qquad |A_{n-1}'A_n'| = \frac{a_{n-1}}{p_{n-1}}p',$$

$$|A_1'A_n'| = \frac{a_0}{p_0}p'$$

Substituting these expressions into the relationship (*) and reducing by p', we get the required equality.